To prove Lemma 2, we show Lemma 7 and Lemma 8 first.

Lemma 7. Under Assumption 1, there exists a constant $M > 0$ that satisfies the following conditions:

1. For arbitrary $M' > M$, $e = (u, v) \in E$, $t \in T$, and $k \in \{0, 1, 2, \dots, |T|\}$, the following holds:
$$\begin{aligned}(M' + w_{et} - k\ell)p_{vt}(M')\\ -(M + w_{et} - k\ell)p_{vt}(M) < 0,\end{aligned} \tag{8}$$
where $\ell := \max_{v\in V, t\in T} \left\{p_{vt}^{-1}\left(\frac{1}{|E||T|}\right)\right\} + \max_{e\in E, t\in T}\{w_{et}\}$.
2. For all $v \in V, t \in T$,
$$p_{vt}(M) \leq \frac{1}{|E||T|}. \tag{9}$$
3. For all $e \in E, t \in T$,
$$M + w_{et} > 0. \tag{10}$$

Proof. Since p_{vt} is monotonically decreasing and bijective from Assumption 1, we have $\lim_{x\to\infty} p_{vt}(x) = 0$ for all $v \in V$ and $t \in T$. Therefore, condition 2 is satisfied for sufficiently large M. Condition 3 is satisfied for $M > -\min_{e\in E, t\in T}\{w_{et}\}$. Therefore, Lemma 7 is satisfied if we show that condition 1 holds when M is sufficiently large.

First, we show $p'_{vt}(x) < 0$ for all $v \in V, t \in T$, and $x \in \mathbb{R}$. Since $p'_{vt}(x) \leq 0$ from Assumption 1, we assume there exist (v, t) and $b \in \mathbb{R}$ that satisfy $p'_{vt}(b) = 0$, and then derive a contradiction. First, from the definition,
$$p_{vt}(x) > 0. \tag{11}$$
Then, the following holds for arbitrary $x \leq b$:
$$0 \geq p'_{vt}(x)/p_{vt}(x) \geq p'_{vt}(b)/p_{vt}(b) = 0 \tag{12}$$
Here, the first inequality holds from (11) and $p'_{vt}(x) \leq 0$. The second inequality holds since $p'_{vt}(x)/p_{vt}(x)$ is monotonically non-increasing from Assumption 1. From (12), we have $p'_{vt}(x) = 0$ for any $x \leq b$, which contradicts that p_{vt} is bijective from Assumption 1. Therefore, $p'_{vt}(x) < 0$ for all $v \in V, t \in T, x \in \mathbb{R}$.

Here, let $h_{cvt}(x) := (x + c)p_{vt}(x)$ for $c \in \mathbb{R}, v \in V, t \in T$. Then,
$$\begin{aligned}h'_{cvt}(x) &= p_{vt}(x) + (x + c)p'_{vt}(x)\\ &= p'_{vt}(x)\left(\frac{p_{vt}(x)}{p'_{vt}(x)} + x + c\right).\end{aligned} \tag{13}$$
The last equality holds since $p'_{vt}(x) < 0$ (that is, $p'_{vt}(x) \neq 0$) for all $v \in V, t \in T$, and $x \in \mathbb{R}$. From Assumption 1, $p_{vt}(x)/p'_{vt}(x)$ is monotonically non-decreasing, so if we take a sufficiently large M, then $p_{vt}(x)/p'_{vt}(x) + x + c > 0$ for any $x > M$. Therefore, since $p'_{vt}(x) < 0$, it yields that $h'_{cvt}(x) < 0$ for $x > M$, i.e., $h_{cvt}(x)$ is monotonically decreasing for $x > M$. Hence, when M is sufficiently large, (8) holds for arbitrary $M' > M$, $e = (u, v) \in E, t \in T$, and $k \in \{0, 1, 2, \dots, |T|\}$ by letting $c := w_{et} - k\ell$. □

Lemma 8. Suppose that Assumption 1 holds. Let M be a constant that satisfies the conditions of Lemma 7, and let $L := \max_{e\in E, t\in T} w_{et}$. Then, Algorithm 2 works when the input $(\boldsymbol{x}, \boldsymbol{z})$ is a feasible solution for (PA). Moreover, the following hold for Algorithm 2:

Algorithm 2: IMPROVE

Input: $\boldsymbol{x} \in \mathbb{R}^{V\times T}, \boldsymbol{z} \in \mathbb{R}^{E\times T}, i \in V, j \in T$

1: **if** $x_{ij} < -L$ **:**

2: $\quad \hat{x}_{vt} \leftarrow \begin{cases}-L & \text{if } (v,t) = (i,j)\\ x_{vt} & \text{otherwise}\end{cases}$

3: $\quad \hat{z}_{et} \leftarrow \begin{cases}0 & \text{if } (e,t) \in \delta(i) \times \{j\}\\ z_{et} & \text{otherwise}\end{cases}$

4: **else if** $x_{ij} > M$ **:**

5: $\quad \hat{x}_{vt} \leftarrow \begin{cases}M & \text{if } (v,t) = (i,j)\\ x_{vt} & \text{otherwise}\end{cases}$

6: $\quad$ Select an edge, $f = (u_f, i), \in \delta(i)$

7: $\quad \hat{z}_{et} \leftarrow \begin{cases}z_{fj} + p_{ij}(M) - p_{ij}(x_{ij}) & \text{if } (e,t) = (f,j)\\ z_{et} & \text{otherwise}\end{cases}$

8: $\quad$ **for** $t \in T$ **:**

9: $\quad\quad$ **if** $\sum_{e\in\delta(u_f)} \sum_{t':0\leq t-t'<c_{et'}} \hat{z}_{et'} > 1$ **:**

10: $\quad\quad\quad$ Select $(g, r) \in E \times T$ satisfying $g \in \delta(u_f), 0 \leq t - r < c_{gr}, z_{gr} > \frac{1}{|E||T|}$

11: $\quad\quad\quad \hat{z}_{gr} \leftarrow z_{gr} - p_{ij}(M) + p_{ij}(x_{ij})$

12: **else if** $-L \leq x_{ij} \leq M$ **:**

13: $\quad \hat{\boldsymbol{x}} \leftarrow \boldsymbol{x}, \hat{\boldsymbol{z}} \leftarrow \boldsymbol{z}$

14: **return** $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$

(I) The output $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ is a feasible solution for (PA).

(II) The objective value of (PA) of the output $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ is greater than or equal to that of the input $(\boldsymbol{x}, \boldsymbol{z})$.

(III) For the output $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$,
$$\hat{x}_{vt} = x_{vt}, \quad \forall (v, t) \in V \times T \setminus (i, j),$$
and
$$\hat{x}_{ij} = \begin{cases}x_{ij}, & \text{if} \quad -L \leq x_{ij} \leq M,\\ M, & \text{if} \quad x_{ij} > M,\\ -L, & \text{if} \quad x_{ij} < -L.\end{cases}$$

Proof. Since $(\boldsymbol{x}, \boldsymbol{z})$ is a feasible solution for (PA), we have
$$\sum_{e\in\delta(v)} z_{et} \leq p_{vt}(x_{vt}), \quad \forall v \in V, \forall t \in T, \tag{14}$$
$$\sum_{e\in\delta(u)} \sum_{t':0\leq t-t'<c_{et'}} z_{et'} \leq 1, \quad \forall u \in U, \forall t \in T, \tag{15}$$
$$0 \leq \boldsymbol{z} \leq 1. \tag{16}$$

Here, we consider the case where $x_{ij} > M$. Then Algorithm 2 performs lines 5–11. Let $\tilde{T} \subseteq T$ be the set of $t \in T$ that satisfy the condition of line 9 through running Algorithm 2. Let $(g(t), r(t))$ be (g, r) selected in line 10 at $t \in \tilde{T}$. Here,
$$\sum_{e\in\delta(i)} z_{ej} \leq p_{ij}(x_{ij}) \leq p_{ij}(M) \leq \frac{1}{|E||T|},$$
where the first inequality holds from (14), the second inequality holds since $x_{ij} > M$ and p_{vt} is monotonically decreasing from Assumption 1, and the last inequality holds

from (9). Therefore, there does not exist z_{ej} satisfying $z_{ej} > \frac{1}{|E||T|}$ for $e \in \delta(i)$. Hence, from the condition of line 10,

$$(g(t), r(t)) \notin \delta(i) \times \{j\}, \quad \forall t \in \tilde{T}. \tag{17}$$

This equation always holds in the case where $x_{ij} > M$.

In the following, we divide the proof into four parts: proving that Algorithm 2 works, and proving that conditions (I), (II), and (III) hold.

Proof that Algorithm 2 works: A necessary and sufficient condition for Algorithm 2 to work is that, in line 10, $(g, r) \in E \times T$ must exist such that $g \in \delta(u_f), 0 \le t - r < c_{gr}, z_{gr} > \frac{1}{|E||T|}$. Here, $\{(g, r) \mid g \in \delta(u_f), 0 \le t - r < c_{gr}\} \ne \emptyset$ since $\delta(u_f) \ne 0$ and $c_{gr} \ge 1$ from the definition. In addition, $(f, j) \ne (g(t), r(t))$ for all $t \in \tilde{T}$ from (17). Then, whenever line 10 is executed, there exists $(g, r) \in \{(g, r) \mid g \in \delta(u_f), 0 \le t - r < c_{gr}\}$ satisfying

$$z_{gr} > \frac{1}{|T||E|}. \tag{18}$$

Otherwise,

$$\begin{aligned}
&\sum_{e \in \delta(u_f)} \sum_{t' : 0 \le t - t' < c_{et'}} \hat{z}_{et'} \\
&\le \sum_{(e,t') \in \{e \in \delta(u_f), 0 \le t - t' < c_{et'}\} \setminus (f,j)} z_{et'} \\
&\quad + z_{fj} + p_{ij}(M) - p_{ij}(x_{ij}) \\
&\le \sum_{(e,t') \in \{e \in \delta(u_f), 0 \le t - t' < c_{et'}\} \setminus (f,j)} z_{et'} + p_{ij}(M) \\
&\le \sum_{(e,t') \in \{e \in \delta(u_f), 0 \le t - t' < c_{et'}\} \setminus (f,j)} \frac{1}{|T||E|} + \frac{1}{|T||E|} \\
&\le |T||E| \frac{1}{|T||E|} = 1,
\end{aligned}$$

which contradicts the condition of line 9. The first inequality holds since $-p_{ij}(M) + p_{ij}(x_{ij}) < 0$ and then $\hat{z}_{gr} = z_{gr} - p_{ij}(M) + p_{ij}(x_{ij}) < z_{gr}$ in the operation of line 11. The second inequality holds since $z_{fj} \le \sum_{e \in \delta(i)} z_{ej} \le p_{ij}(x_{ij})$ from (14) and $f \in \delta(i)$. The third inequality holds from (9). Therefore, whenever line 10 is executed, $(g, r) \in E \times T$ exists such that $g \in \delta(u_f), 0 \le t - r < c_{gr}, z_{gr} > \frac{1}{|E||T|}$. Hence, Algorithm 2 works.

Proof of condition (I): At the end of Algorithm 2, condition (I) is satisfied if the following hold:

(i) $\sum_{e \in \delta(v)} \hat{z}_{et} \le p_{vt}(\hat{x}_{vt})$ for all $v \in V,\ t \in T$,

(ii) $\sum_{e \in \delta(u)} \sum_{t' : 0 \le t - t' < c_{et'}} \hat{z}_{et'} \le 1$ for all $u \in U, t \in T$,

(iii) $0 \le \hat{\boldsymbol{z}} \le 1$.

Here, when $-L \le x_{ij} \le M$, $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}) = (\boldsymbol{x}, \boldsymbol{z})$ from Algorithm 2 and condition (I) holds since $(\boldsymbol{x}, \boldsymbol{z})$ is a feasible solution for (PA). Therefore, we show that (i), (ii), and (iii) hold in the two cases of **(a)** $x_{ij} < -L$ and **(b)** $x_{ij} > M$.

(a) The case where $x_{ij} < -L$:
(i) From line 2–3 of Algorithm 2, $p_{vt}(\hat{x}_{vt}) = p_{vt}(x_{vt}) \ge \sum_{e \in \delta(v)} z_{et} = \sum_{e \in \delta(v)} \hat{z}_{et}$ for all $(v, t) \in V \times T \setminus (i, j)$ and $p_{ij}(\hat{x}_{ij}) = p_{ij}(-L) \ge 0 = \sum_{e \in \delta(i)} \hat{z}_{ej}$.
(ii) Line 3 does not increase z_{et} for all $(e, t) \in E \times T$ since $\boldsymbol{z} \ge 0$ from (16). Therefore, for all $(u, t) \in U \times T$,

$$\sum_{e \in \delta(u)} \sum_{t' : 0 \le t - t' < c_{et'}} \hat{z}_{et'} \le \sum_{e \in \delta(u)} \sum_{t' : 0 \le t - t' < c_{et'}} z_{et'} \le 1.$$

Here, the second inequality holds from (15).
(iii) From line 3, $\hat{z}_{et} = 0$ for all $(e, t) \in \delta(i) \times \{j\}$ and $\hat{z}_{et} = z_{et} \in [0, 1]$ for all $(e, t) \in E \times T \setminus \delta(i) \times \{j\}$. Therefore, $0 \le \hat{\boldsymbol{z}} \le 1$.

(b) The case where $x_{ij} > M$:
(i) For $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ at the end of line 7, the following holds:

$$\begin{aligned}
p_{ij}(\hat{x}_{ij}) &= p_{ij}(M) \\
&\ge p_{ij}(M) - p_{ij}(x_{ij}) + \sum_{e \in \delta(i)} z_{ej} \\
&= \sum_{e \in \delta(i)} \hat{z}_{ej}.
\end{aligned}$$

The inequality holds from (14). Also, $p_{vt}(\hat{x}_{vt}) = p_{vt}(x_{vt}) \ge \sum_{e \in \delta(v)} z_{et} = \sum_{e \in \delta(v)} \hat{z}_{et}$ for all $(v, t) \in V \times T \setminus (i, j)$. Therefore, at the end of line 7, $p_{vt}(\hat{x}_{vt}) \ge \sum_{e \in \delta(v)} \hat{z}_{et}$ for all $v \in V,\ t \in T$. Line 11 does not increase z_{gr} since $-p_{ij}(M) + p_{ij}(x_{ij}) \le 0$. Thus, at the end of Algorithm 2, $p_{vt}(\hat{x}_{vt}) \ge \sum_{e \in \delta(v)} \hat{z}_{et}$ for all $v \in V,\ t \in T$.
(ii) For all $(u, t) \in \{U \setminus u_f\} \times T$,

$$\sum_{e \in \delta(u)} \sum_{t' : 0 \le t - t' < c_{et'}} \hat{z}_{et'} = \sum_{e \in \delta(u)} \sum_{t' : 0 \le t - t' < c_{et'}} z_{et'} \le 1.$$

Therefore, if we show that $\sum_{e \in \delta(u_f)} \sum_{t' : 0 \le t - t' < c_{et'}} \hat{z}_{et'} \le 1$ for all $t \in T$, then (ii) holds. Here,

$$\hat{z}_{et} \le z_{et}, \quad \forall (e, t) \in E \times T \setminus (f, j), \tag{19}$$

since $-p_{ij}(M) + p_{ij}(x_{ij}) \le 0$ and $\hat{z}_{gr} = z_{gr} - p_{ij}(M) + p_{ij}(x_{ij}) \le z_{gr}$ in line 11. Then, for u_f and $t \in \tilde{T}$,

$$\begin{aligned}
&\sum_{e \in \delta(u_f)} \sum_{t' : 0 \le t - t' < c_{et'}} \hat{z}_{et'} \\
&= \sum_{\{e \in \delta(u_f), t' : 0 \le t - t' < c_{et'}\} \setminus \{(f,j) \cup (g(t), r(t))\}} \hat{z}_{et'} + \hat{z}_{fj} + \hat{z}_{g(t)r(t)} \\
&\le \sum_{\{e \in \delta(u_f), t' : 0 \le t - t' < c_{et'}\} \setminus \{(f,j) \cup (g(t), r(t))\}} z_{et'} \\
&\quad + z_{fj} + p_{ij}(M) - p_{ij}(x_{ij}) \\
&\quad + z_{g(t)r(t)} - p_{ij}(M) + p_{ij}(x_{ij}) \\
&= \sum_{e \in \delta(u_f)} \sum_{t' : 0 \le t - t' < c_{et'}\}} z_{et'} \\
&\le 1.
\end{aligned}$$

The first equality holds from (17) and the first inequality holds from (19). Moreover, $\sum_{e \in \delta(u_f)} \sum_{t' : 0 \le t - t' < c_{et'}} \hat{z}_{et'} \le 1$ for $t \notin \tilde{T}$ from the condition of line 9 since line 11 does not increase

$\hat{z}_{g(t)r(t)}$ for any $t \in \tilde{T}$. Thus, at the end of Algorithm 2, $\sum_{e\in\delta(u_f)}\sum_{t':0\leq t-t'<c_{et'}} \hat{z}_{et'} \leq 1$ for all $t \in T$.

(iii) Since line 7 only raises z_{fj} by $p_{ij}(M) - p_{ij}(x_{ij}) > 0$, $\hat{\boldsymbol{z}} \geq 0$ from (16) at the end of line 7. Next, at the end of line 11 for all $t \in \tilde{T}$,

$$
\begin{aligned}
\hat{z}_{g(t)r(t)} &= z_{g(t)r(t)} - p_{ij}(M) + p_{ij}(x_{ij}) > z_{g(t)r(t)} - p_{ij}(M). \\
&> \frac{1}{|T||E|} - \frac{1}{|T||E|} = 0.
\end{aligned}
$$

The first inequality holds since $p_{ij}(x_{ij}) > 0$, and the second inequality holds from the condition of line 10 ($z_{gr} > \frac{1}{|E||T|}$) and (9). Therefore, $0 \leq \hat{\boldsymbol{z}}$ at the end of Algorithm 2. Moreover, $\hat{\boldsymbol{z}} \leq 1$ at the end of Algorithm 2 since (ii) holds.

Consequently, in both cases (a) and (b), the output $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ of Algorithm 2 satisfies condition (I).

Proof of condition (II): When $-L \leq x_{ij} \leq M$, condition (II) holds since $(\boldsymbol{x}, \boldsymbol{z}) = (\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ from Algorithm 2. Therefore, we show that condition (II) holds in the two cases: **(a)** $x_{ij} < -L$ and **(b)** $x_{ij} > M$.

(a) The case where $x_{ij} < -L$:
Line 2–3 increases the objective value by

$$
\begin{aligned}
-\sum_{e\in\delta(i)}(x_{ij} + w_{ej})z_{ej} &\geq -\sum_{e\in\delta(i)}(-L + \max_{e\in E, t\in T} w_{et})z_{ej} \\
&= 0.
\end{aligned}
$$

Here, the inequality follows from (16) and $x_{ij} < -L$, and the equality follows from the definition of L.

(b) The case where $x_{ij} > M$:
Let $\ell := \max_{v\in V, t\in T} p_{vt}^{-1}(\frac{1}{|E||T|}) + \max_{e\in E, t\in T}\{w_{et}\}$. Then, for all $\tilde{t} \in \tilde{T}$,

$$
\begin{aligned}
\ell &= \max_{v\in V, t\in T} p_{vt}^{-1}(\frac{1}{|E||T|}) + \max_{e\in E, t\in T}\{w_{et}\} \\
&\geq p_{v(g(\tilde{t}))r(\tilde{t})}^{-1}(\frac{1}{|E||T|}) + w_{g(\tilde{t})r(\tilde{t})} \\
&\geq x_{v(g(\tilde{t}))r(\tilde{t})} + w_{g(\tilde{t})r(\tilde{t})},
\end{aligned}
\tag{20}
$$

where $v(g(\tilde{t}))$ is $v \in V$ incident to $g(\tilde{t})$. The second inequality holds because p_{vt}^{-1} is monotone decreasing for all $v \in V$ and $t \in T$ from Assumption 1 and $p_{v(g(\tilde{t}))r(\tilde{t})}(x_{v(g(\tilde{t}))r(\tilde{t})}) \geq \sum_{e\in\delta(v(g(\tilde{t})))} z_{er(\tilde{t})} \geq z_{g(\tilde{t})r(\tilde{t})} > \frac{1}{|E||T|}$ from (14). Here, $(f, j) \neq (g(t), r(t))$ from (17) for all $t \in \tilde{T}$. Then, line 5-7 and line 8-11 increase the objective value by

$$
\begin{aligned}
&\sum_{e=(u,v)\in\delta(i)\setminus\{f\}}(M + w_{ej})z_{ej} \\
&+ (M + w_{fj})(z_{fj} + p_{ij}(M) - p_{ij}(x_{ij})) \\
&- \sum_{e=(u,v)\in\delta(i)}(x_{ij} + w_{ej})z_{ej} \\
&- \sum_{t\in\tilde{T}}(x_{v(g(t))r(t)} + w_{g(t)r(t)})(p_{ij}(M) - p_{ij}(x_{ij})) \\
&\geq (M - x_{ij})\sum_{e=(u,v)\in\delta(i)} z_{ej} \\
&\quad + (M + w_{fj})(p_{ij}(M) - p_{ij}(x_{ij})) \\
&\quad - |\tilde{T}|\ell(p_{ij}(M) - p_{ij}(x_{ij})) \\
&\geq (M - x_{ij})p_{ij}(x_{ij}) \\
&\quad + (M + w_{fj})(p_{ij}(M) - p_{ij}(x_{ij})) \\
&\quad - |\tilde{T}|\ell(p_{ij}(M) - p_{ij}(x_{ij})) \\
&= (M + w_{fj} - |\tilde{T}|\ell)p_{ij}(M) \\
&\quad - (x_{ij} + w_{fj} - |\tilde{T}|\ell)p_{ij}(x_{ij}) \\
&\geq 0.
\end{aligned}
$$

The first inequality holds from $p_{ij}(M) - p_{ij}(x_{ij}) > 0$ and (20). The second inequality holds from (14) and $M - x_{ij} < 0$. The third inequality holds from (8). Therefore, condition (II) holds.

Consequently, in both cases (a) and (b), the output $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ of Algorithm 2 satisfies condition (II).

Proof of condition (III): It is clearly satisfied from line 2, line 5, and line 13 in Algorithm 2. □

Then, we show Lemma 1 from Lemma 8.

Proof. Let $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ be the output of Algorithm 3 for an arbitrary feasible solution $(\boldsymbol{x}, \boldsymbol{z})$ of (PA). Then, from Lemma 8, $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ satisfies the following conditions: (i) $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ is a feasible solution of (PA), (ii) the objective value of (PA) of $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ is greater than or equal to that of $(\boldsymbol{x}, \boldsymbol{z})$, and (iii) $-L \leq \hat{x}_{et} \leq M$ for all $e \in E, t \in T$. Therefore, (PA) with the constraint "$-L \leq \hat{x}_{et} \leq M$ for all $e \in E, t \in T$" is equivalent to (PA). Here, the problem with the additional constraints has an optimal solution from the Extreme-Value Theorem [Royden and Fitzpatrick 1988, Section 1.6] since it has a non-empty bounded closed set as the feasible region and a continuous function as the objective function. Therefore, (PA) has an optimal solution. □

Algorithm 3: Generate bounded solution

Input: $\boldsymbol{x} \in \mathbb{R}^{V\times T}, \boldsymbol{z} \in \mathbb{R}^{E\times T}$

```
x̂, ẑ ← x, z
for i ∈ V :
    for j ∈ T :
        x̂, ẑ ← IMPROVE(x̂, ẑ, i, j)
return x̂, ẑ
```

References

Royden, H. L.; and Fitzpatrick, P. 1988. *Real analysis*. Macmillan New York.