The file `coefficients.gp` calculates $\widetilde{C}_l$ in [1, eq. (5.2)] up to 100. To explain the script, we simply recall the setting. We set

$$\lambda(y) = \frac{1}{e^y - 1} - \frac{1}{y} + \frac{1}{2} = \sum_{k=1}^{\infty} \frac{B_k}{(k+1)!} y^k,$$

where B_k is the k-th Bernoulli number. We define $P_k(T)$ by the generating function via

$$e^{-T\lambda(y)} = \sum_{k=0}^{\infty} P_k(T) y^k.$$

We note $P_0(T) = 1$. According to [1, Corollaries 5.4 and 5.5], we know $P_k(T) \in \mathbb{Q}[T]$ with degree k and $T^k P_k(T) \in T^2\mathbb{Q}[T^2]$ for any $k \in \mathbb{Z}_{\geq 1}$. We also set

$$\begin{aligned}\widetilde{C}_l(T) &= \sum_{\substack{j,k\geq 0\\ j+k=l}} (k+1, j) T^k P_k(T) 2^{-2j}\\ &= \sum_{k=0}^{l} (k+1, l-k) T^k P_k(T) 2^{-2(l-k)},\end{aligned}$$

where $(k+1, j)$ is the Hankel symbol. We notice $\widetilde{C}_0(T) = 1$. For $l \geq 1$ we write

$$\widetilde{C}_l(T) = (1, l) 2^{-2l} + \sum_{k=1}^{l} (k+1, l-k) T^k P_k(T) 2^{-2(l-k)}.$$

We easily see $\widetilde{C}_l(T) \in \mathbb{Q}[T^2]$ and $\deg_{\mathbb{Q}[T]} \widetilde{C}_l(T) = 2l$. The numbers $\widetilde{C}_l$ are given by $\widetilde{C}_l = \widetilde{C}_l(2\pi i)$.

To run the script, we firstly change the directory `./coefficients/`. Then we start Pari/GP and do `\r coefficients.gp`. Note that the script uses 128 MB. As a result, output the following three files:

- `tCkTout`: save $\widetilde{C}_l(T) \in \mathbb{Q}[T]$ for $1 \leq l \leq 100$ as a vector.
- `tCkout_pi`: save $\widetilde{C}_l(T)$ as $\mathbb{Q}$-linear combinations of $1, \pi^2, \ldots, \pi^{2l}$ for $1 \leq l \leq 100$ as a vector. Here `p`$= \pi$.
- `tCkout_num`: save numerical values of $\widetilde{C}_l$ for $1 \leq l \leq 100$ as a vector.

References

[1] H. Akatsuka and Y. Murakami, An asymptotic property on a reciprocity law for the Bettin–Conrey cotangent sum, `arXiv:2402.14216`.