The scripts of this directory is to compute numerical values of g_n in [1, eq. (1.2)] or $\mathcal{I}_n = g_n - \frac{1}{n}$. As was shown by Bettin–Conrey, it holds that

$$g_n = \frac{1}{n(n+1)} + 2b_n + 2\sum_{j=0}^{n-2} \binom{n-1}{j} b_{j+2} \tag{$\clubsuit$}$$

for $n \in \mathbb{Z}_{\geq 2}$, where

$$b_k = \frac{B_k \zeta(k)}{k},$$

B_k is the k-th Bernoulli number and $\zeta(s)$ is the Riemann zeta-function. According to [1, the first displayed formula in §6], we have

$$b_{2l} = (-1)^{l-1} \frac{(2\pi)^{2l} B_{2l}^2}{4l \cdot (2l)!}$$

for $l \in \mathbb{Z}_{\geq 1}$. Since B_k vanishes for odd $k \geq 3$, we find

$$|b_k| = \frac{(2\pi)^k B_k^2}{2k \cdot k!}$$

for $k \in \mathbb{Z}_{\geq 2}$.

To investigate significant digits of precision needed for computation of g_n, we set

$$G_n^{(1)} = \frac{1}{n(n+1)} + 2|b_n| + 2\sum_{j=0}^{n-2} \binom{n-1}{j} |b_{j+2}|, \qquad G_n^{(\infty)} = 2 \max_{0 \leq j \leq n-2} \binom{n-1}{j} |b_{j+2}|.$$

The script `SDP.gp` is to compute $G_n^{(\infty)}$ and $G_n^{(1)}$. After start Pari/GP, do `\r SDP.gp`. To compute $G_{1001}^{(\infty)}$ (*resp.* $G_{1001}^{(1)}$), type `Ginf(1001)` (*resp.* `Gone(1001)`). When we calculate g_n up to 1001, we set significant digits of precision to much larger than the number of digits of $G_{1001}^{(1)}$ or $1001 \times G_{1001}^{(\infty)}$.

By ($\clubsuit$) we find

$$\mathcal{I}_n = -\frac{1}{n+1} + 2b_n + 2\sum_{j=0}^{n-2} \binom{n-1}{j} b_{j+2}$$

for $n \in \mathbb{Z}_{\geq 2}$. Since b_n vanishes for odd $n \geq 3$, we notice

$$\mathcal{I}_{2l} = -\frac{1}{2l+1} + 4l \times b_{2l} + \sum_{j=0}^{l-2} \binom{2l-1}{2j} b_{2(j+1)}, \tag{1}$$

$$\mathcal{I}_{2l+1} = -\frac{1}{2l+2} + \sum_{j=0}^{l-1} \binom{2l}{2j} b_{2(j+1)} \tag{2}$$

for $l \in \mathbb{Z}_{\geq 1}$. The script `main.gp` outputs numerical values of I_n for $2 \leq n \leq 1001$ as a vector. Since $G_{1001}^{(1)} = 7.14\ldots \times 10^{1773}$, we compute g_n with 2000 decimal digits.

References

[1] H. Akatsuka and Y. Murakami, An asymptotic property on a reciprocity law for the Bettin–Conrey cotangent sum, `arXiv:2402.14216`.