Calculus Crash Course

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September 2023

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§1 Sept 18 - Limit

§1.1 Limit

§1.1.1 Definition of Limit

Question 1.1. Why do we invent *limit*?

Example 1.2 (Achilles and the tortoise)

In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead. - *Aristotle, Physics VI:9, 239b15*

In the paradox of Achilles and the tortoise, Achilles is in a footrace with the tortoise. Achilles allows the tortoise a head start of 100 meters, for example. Suppose that each racer starts running at some constant speed, one faster than the other. After some finite time, Achilles will have run 100 meters, bringing him to the tortoise's starting point. During this time, the tortoise has run a much shorter distance, say 2 meters. It will then take Achilles some further time to run that distance, by which time the tortoise will have advanced farther; and then more time still to reach this third point, while the tortoise moves ahead. Thus, whenever Achilles arrives somewhere the tortoise has been, he still has some distance to go before he can even reach the tortoise.

It seems to be counter-intuitive but convincing, but how do you know that it is actually a paradox?

Definition 1.3 (An Intuitive Definition of Limit)**.** We say $\lim_{x \to a} f(x) = L$ if $f(x)$ approaches L when x approaches a.

But what does it mean by *approaches* but not *reach*? This is an ambiguous definition of limit, even though this is what we studied in IB. Here we also introduce another rigorous definition of limit known as *the* $\delta - \varepsilon$ *definition of a limit.*

Definition 1.4 (The $\delta - \varepsilon$ Definition of A Limit)**.** Let $f(x)$ be a function defined on an open interval around x_0 $\big(f(x_0)$ need not be defined$\big)$. We say that the limit of $f(x)$ as x approaches x_0 is L, i.e.

$$\lim_{x \to x_0} f(x) = L,$$

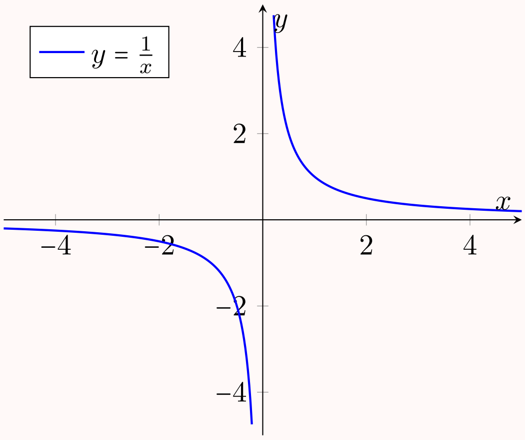
if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$$

This is a better definition of limit. Why? It does not need complicated $\lim_{x \to x_0^+} f(x)$ or $\lim_{x \to x_0^-} f(x)$ as already included in the absolute value, and also it gives a correct definition of *approaches but not reaches.* Also, it implies that $L \in \mathbb{R}$ since that if L is not a real number, the difference between $f(x)$ and L cannot be compared.

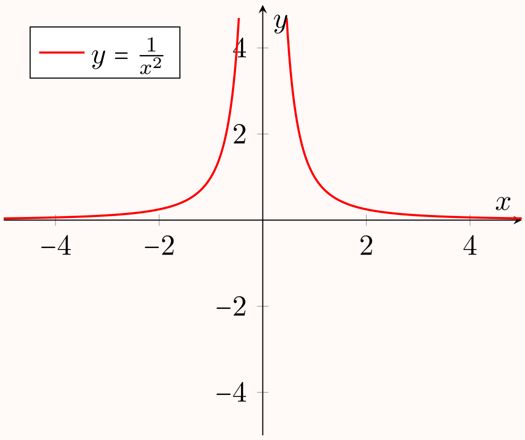
Example 1.5 (An example that shows the reason of using $\delta - \varepsilon$ definition)

Consider $f(x) = \dfrac{1}{x}$.



When x is approaching 0 from the right side, marked with $\lim\limits_{x\to 0^+} \dfrac{1}{x}$ should be $+\infty$ and $\lim\limits_{x\to 0^-} \dfrac{1}{x}$ should be $-\infty$ (even though as stated before, it is not that rigorous because we do not typically state that a limit is infinity). By $\lim\limits_{x\to 0^+} \dfrac{1}{x} \neq \lim\limits_{x\to 0^-} \dfrac{1}{x}$, we know that it does not exist.

However, what about $g(x) = \dfrac{1}{x^2}$?



If we just consider the intuitive definition, we say $\lim\limits_{x\to 0^+} \dfrac{1}{x^2} = \lim\limits_{x\to 0^-} \dfrac{1}{x^2} = +\infty$, and this cause $\lim\limits_{x\to 0} \dfrac{1}{x^2} = +\infty$ but it's actually not true according to $\delta - \varepsilon$ definition since we cannot define a constant value $+\infty - f(x)$ as ε.

Remark 1.6. When the domain of a function $f(x)$ is (a, b), $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} f(x)$

§1.1.2 Existence of Limit

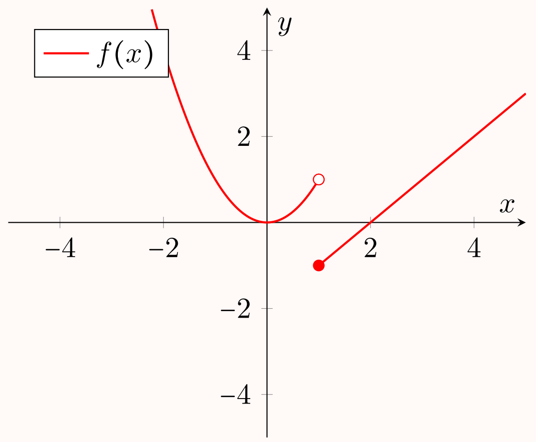
According to the definition of limit (either the intuitive one or $\delta - \varepsilon$ description), a few conditions can obviously considered where the limit does not exist.

The first condition is shown in Example 2.5, for both $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$. We say there exists a **break** when $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ does not exist, specifically when they approach $\pm\infty$.

When the left and right limit does exist, but are equal to different values, this is called a **jump**.

Example 1.7 (Example of Jump)

We define $h(x) = \begin{cases} x^2, & x < 1 \\ x-2, & x \geq 1 \end{cases}$, and the diagram of $h(x)$ is shown below.

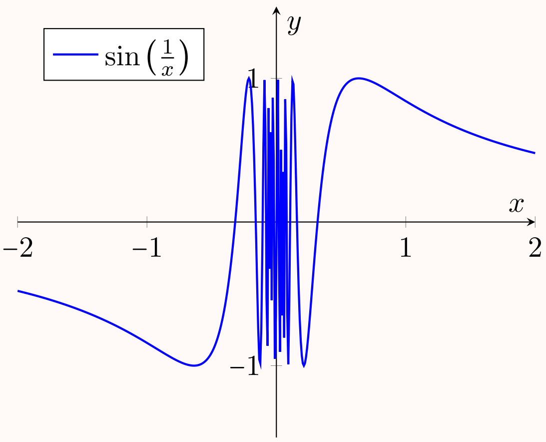


Here in the diagram, we know that $\lim_{x\to 1^+} h(x) = -1$ and $\lim_{x\to 1^-} h(x) = 1$ and they are different, therefore $\lim_{x\to 1} h(x)$ does not exist.

The 3rd possibility is **oscillation**

Example 1.8 (Oscillation)

Here we provide a function $k(x) = \sin\left(\frac{1}{x}\right)$, known as the *Notorious Oscillating Function* for its difficulty of plotting.



Intuitively, we know that $\lim_{x\to 0} k(x)$ does not exist.

Question 1.9. Can you explain it using $\delta - \varepsilon$ definition of limit?

§1.1.3 Limit Laws

There exist a few limit laws.

Remark 1.10. For $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$, when l and m equals to ∞ or 0 at the same time, *L' Hospital Theorem* should be used.

Consider function $f(x)$ and $g(x)$ for which $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, where $a, l, m \in \mathbb{R}$.

- $\lim_{x\to a}(f(x) \pm g(x)) = l \pm m$
- $\lim_{x\to a} f(x)g(x) = lm$
- $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{l}{m}$ when provided $m \neq 0$

§1.2 Continuity

Example 1.11 (The Meaning of Continuity - A Classic Mathematics Modelling Question)

When a normal desk, with 4 feet and equal length is placed on uneven ground, it normally stands with 3 feet. However, it can be moved to where it can stand with 4 feet, why is that? Can you prove it?

Definition 1.12 (Continuity). We say a function is **continuous** when $\lim_{x\to a} f(x) = f(x)$.

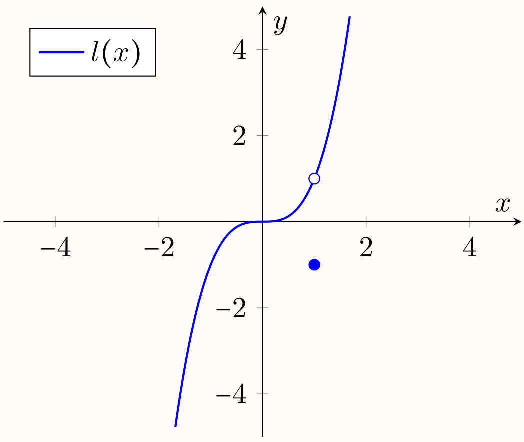
Directly from the definition can we derive

Claim 1.13 — $\lim_{x\to a} f(x)$ DNE $\implies$ $f(x)$ is not continuous.

When $\lim_{x\to a} f(x)$ does exist but is not equal to $f(x)$, the discontinuity is defined as a **hole**.

Example 1.14 (Example of Hole)

We define $l(x) = \begin{cases} x^3, & x \neq 1 \\ -1, & x = 1 \end{cases}$



Both $\lim_{x\to 1^+} l(x)$ and $\lim_{x\to 1^-} l(x)$ are 1 while $l(1) = -1$.

Hole discontinuity is **removable**, which means can be changed to a continuous function just by changing $l(x)$ to $l'(x) = x^3, x \in \mathbb{R}$, while other discontinuity that is caused by undefined limit is irremovable.

§1.3 Two Special Limit

§1.3.1 $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (x in radians)

Theorem 1.15 (Sandwich Theorem (Squeeze Theorem))
Let I be an interval containing the point a. Let g, f and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have

$$g(x) \le f(x) \le h(x)$$

and also suppose that

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

then $\lim_{x \to a} f(x) = L$.

First, try a few numerical values:

x	$\sin x$
1	0.84
0.8	0.72
0.6	0.56
0.4	0.39
0.2	0.199

it seems that when x approaches 0, $\sin x$ and x are closer and closer.

Proposition 1.16
When x approaches 0, $\dfrac{\sin x}{x}$ approaches 1, equivalent to

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

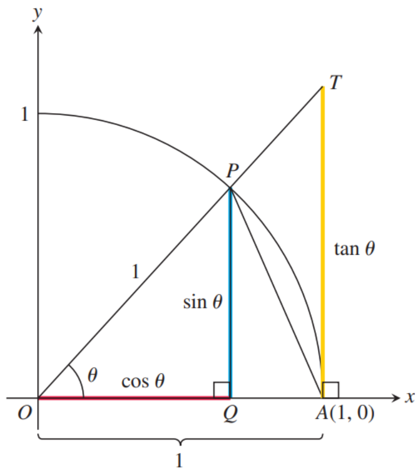


Figure 1: $\sin x/x$ proof using Sandwich Theorem

Proof. First, consider the Sandwich Theorem, we compute the areas of $\triangle OAP$, sector OAP, and $\triangle OAT$:

$$S_{\triangle OAP} = \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin\theta = \frac{1}{2} \cdot \sin\theta$$
$$S_{Sector\ OAP} = \frac{1}{2} \cdot 1 \cdot 1 \cdot \theta = \frac{\theta}{2}$$
$$S_{\triangle OAT} = \frac{1}{2} \cdot 1 \cdot \tan\theta = \frac{1}{2} \cdot \tan\theta$$

Thus, referring to the diagram, we know $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$.

By dividing $\sin\theta$ on every part of the inequality and then taking reciprocals, we can get a new one with $\frac{\sin\theta}{\theta}$.

Since $\theta \to 0^+$, $\cos\theta < \frac{\sin\theta}{\theta} < 1$.

Since $\lim_{\theta\to 0^+}\cos x = 1$, $\lim_{x\to 0}\frac{\sin x}{x} = 1$ by Sandwich (Squeeze) Theorem. □

Remark 1.17. The proof of $\lim_{x\to 0}\frac{\sin x}{x} = 1$ cannot use the L' Hospital Theorem, it will cause circular proof indeed.

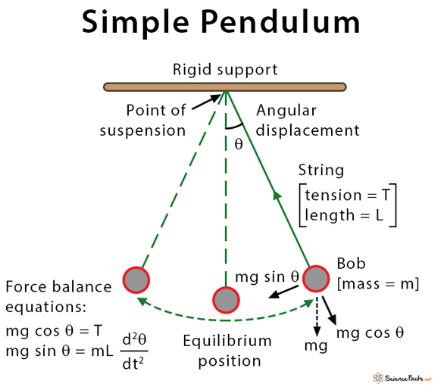


Figure 2: Application (Simple harmonic motion)

Exercise 1.18. $\lim_{x\to 0}\frac{\sin Ax}{\sin Bx} = \frac{\sin Ax}{Ax}\cdot\frac{Bx}{\sin Bx}\cdot\frac{A}{B} = A/B$ (A and B are constants not equal to 0)

$\lim_{x\to 0}\frac{\tan x}{5x} = \frac{\sin x}{\cos x\cdot 5x} = \frac{1}{5}$

§1.3.2 $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$

Definition 1.19. The number e, also known as Euler's Number, is an irrational number, with a numerical value of 2.718281828459...

Example 1.20

A little story: Suppose you put 1 dollar in a bank. The annual interest rate is 100%, but if you take the money twice a year, the interest rate becomes 50%, and so on... Can you have infinite money?

n	$(1+\frac{1}{n})^n$
2	2.25
5	2.49
10	2.59
20	2.65
100	2.70

Your money will approach a value, which is e.

Remark 1.21. Another explanation of e:

$$e = \left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!} + \dots$$

Proof. Binomial expansion is used in the proof.

Theorem 1.22 (Binomial Expansion)

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0 b^n$$

where $\binom{n}{r} = C_r^n = \dfrac{n!}{r!(n-r)!}$.

Notice that the Euler's Number e is the key to continuity! □

Exercise 1.23. Compute $\lim_{n\to\infty}\left(1 + \frac{1}{kn}\right)^n = \left(1 + \frac{1}{kn}\right)^{kn/k}$.

Solution.

$$\begin{aligned}\lim_{n\to\infty}\left(1 + \frac{1}{kn}\right)^n &= \left(1 + \frac{1}{kn}\right)^{kn/k} \\ &= \boxed{e^{\frac{1}{k}}}\end{aligned}$$

□

Exercise 1.24. Compute $\lim_{x\to\infty}\left(\frac{x+7}{x-7}\right)^x = \left(1 + \frac{14}{x-7}\right)^x$.

Solution.

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x+7}{x-7}\right)^x &= \left(1 + \frac{14}{x-7}\right)^x \\ &= \left(1 + \frac{14}{x-7}\right)^{\frac{x-7}{14}\cdot\frac{14}{x-7}\cdot x} \\ &= e^{\lim_{x\to\infty}\frac{14x}{x-7}} \\ &= \boxed{e^{14}}\end{aligned}$$

□

References and Extended Reading Materials

[1] Mathematics: Analysis and Approaches HL - Haese Mathematics

[2] Thomas' Calculus

[3] Epsilon-delta definition of continuity - Serlo

[4] Squeeze Theorem Wikipedia

[5] Why proving $\lim_{x\to 0}\frac{\sin x}{x} = 1$ using L' Hospital is circular

[6] Simple Pendulum

[7] Binomial Expansion

§2 October 8 - Derivative

§2.1 Derivative Function

Definition 2.1 (Derivative Function)**.** Gradient function, gradient of the tangent for the original function, of $y = f(x)$ is called its derivative function and is labelled $f'(x)$ or $\dfrac{dy}{dx}$

Exercise 2.2. What is the derivative function of $y = 3$ and $y = 2x$?

§2.2 First principle

Question 2.3. What is the gradient of a line if A $(a, f(a))$ and B $(a+h, f(a+h))$ are on the line?

Claim 2.4 — When A and B gets infinitely close, the gradient is the gradient of the tangent for $y = f(x)$ where $x = a$.

Definition 2.5 (First principle)**.** The derivative function is defined as: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

Exercise 2.6. Compute $y = 2x$, $y = 3x^2$ using first principle.

Exercise 2.7. Prove that $\dfrac{d}{dx}x^n = nx^{n-1}$ using first principle.

Exercise 2.8. Prove that if $f(x) = cu(x)$, then $f'(x) = cu'(x)$ using first principle.

Exercise 2.9. Prove that if $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$ using first principle.

§2.3 Differentiability

Definition 2.10. If the limit $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ exists, $f(x)$ is differentiable at $x = a$.

Claim 2.11 — If f is differentiable at $x = a$, then f is also continuous at $x = a$.

Proof.

$$\begin{aligned}
&\lim_{h\to 0} f(a+h) - f(a) \\
=&\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \times h \\
=&\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \times \lim_{h\to 0} h && \{\text{by the limit laws, since both limits exist}\} \\
=&f'(a) \times 0 \\
=&0
\end{aligned}$$

Therefore, $\lim_{h\to 0} f(a+h) = f(a)$

Letting $x = a + h$, this is equivalent to $\lim_{x\to a} f(x) = f(a)$.

Therefore, f is continuous at $x = a$. □

So we can conclude the way to test for differentiability:

Proposition 2.12 (Test for Differentiability)
A function f with domain D is **differentiable at $x = a$**, $a \in D$, if:

1. f is continuous at $x = a$, and
2. $f'_-(a) = \lim_{h\to 0^-} \dfrac{f(a+h) - f(a)}{h}$ and $f'_+(a) = \lim_{h\to 0^+} \dfrac{f(a+h) - f(a)}{h}$ both exist and are equal.

§2.4 Fundamental rules of differentiation

We have learned from former exercise that if $f(x) = cu(x)$, then $f'(x) = cu'(x)$, and if $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$.

Then we can start thinking about the $f'(x)$ when $f(x) = u(x)v(x)$ or $f(x) = \dfrac{u(x)}{v(x)}$. Try to deduce the formula by using first principle.

Theorem 2.13 (The Product Rule)
If $f(x) = u(x)v(x)$, then $f'(x) = u'(x)v(x) + u(x)v'(x)$. Alternatively, if $y = uv$ where u and v are functions of x, then

$$\frac{dy}{dx} = u'v + uv' = \frac{du}{dx}v + u\frac{dv}{dx}$$

Theorem 2.14 (The Quotient Rule)
If $Q(x) = \dfrac{u(x)}{v(x)}$, then $Q'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$. Alternatively, if $y = \dfrac{u}{v}$ where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$

The rules about calculations between simple functions are all listed and the next and maybe the most important rule is the chain rule.

Definition 2.15 (Chain rule)**.** Version 1: If $y = g(u)$ where $u = f(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Version 2: If $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$

Proof.

$$\begin{aligned}
\frac{dy}{du} &= \lim_{\delta x\to 0} \frac{\delta y}{\delta u}\frac{\delta u}{\delta x} \\
&= (\lim_{\delta x\to 0} \frac{\delta y}{\delta u})(\lim_{\delta x\to 0} \frac{\delta u}{\delta x}) \\
&= (\lim_{\delta u\to 0} \frac{\delta y}{\delta u})(\lim_{\delta x\to 0} \frac{\delta u}{\delta x}) \\
&= \frac{dy}{du}\frac{du}{dx}
\end{aligned}$$

□

§2.5 Derivative of different functions

§2.5.1 Derivative of logarithmic functions

Exercise 2.16. Prove that $(\log_a(x))' = \dfrac{1}{x \ln a}$ by using first principle.

Proof.

$$
\begin{aligned}
(\log_a(x))' &= \lim_{\delta x \to 0} \frac{\log_a(x+\delta x) - \log_a(x)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\log_a(\frac{x+\delta x}{x})}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\log_a(1+\frac{\delta x}{x})}{x} \cdot \frac{x}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\log_a(1+\frac{\delta x}{x})^{\frac{x}{\delta x}}}{x} \\
&= \frac{\log_a(e)}{x} \\
&= \frac{1}{x \cdot \ln a}
\end{aligned}
$$

□

Exercise 2.17. Show that $(\ln f(x))' = \dfrac{f'(x)}{f(x)}$

§2.5.2 Derivative of exponential functions

Exercise 2.18. Using $x = \ln e^x$, find $(e^x)'$

Exercise 2.19. Show that $(a^x)' = \ln a \cdot a^x$

Exercise 2.20. Compute $(x^x)'$

§2.5.3 Derivative of trigonometric functions

Exercise 2.21. Show that $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

Proof.

$$
\begin{aligned}
(\sin x)' &= \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\sin x \cos \delta x + \sin \delta x \cos x - \sin x}{\delta x} \\
&= \cos x \\
(\cos x)' &= \lim_{\delta x \to 0} \frac{\cos(x+\delta x) - \cos x}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\cos x \cos \delta x - \sin \delta x \sin x - \cos x}{\delta x} \\
&= -\sin x
\end{aligned}
$$

□

Try to prove the following derivatives by using product rule and quotient rule:

$$
\begin{aligned}
(\sin x)' &= \cos x \\
(\cos x)' &= -\sin x \\
(\tan x)' &= \sec^2 x \\
(\cot x)' &= -\csc^2 x \\
(\sec x)' &= \tan x \cdot \sec x \\
(\csc x)' &= -\cot x \cdot \csc x
\end{aligned}
$$

Proof.

$$
\begin{aligned}
(\tan x)' &= (\frac{\sin x}{\cos x})' \\
&= \frac{(\sin x)'\cos x - (\cos x)'\sin x}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}
$$

□

§2.5.4 Derivative of inverse trigonometric functions

Exercise 2.22. Show that $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$, $(\arctan x)' = \frac{1}{1+x^2}$

Proof.

$$
\begin{aligned}
y = \arcsin x, &x = \sin y \\
\frac{dx}{dy} &= \cos y \\
\frac{dy}{dx} &= \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1-\sin y^2}} \\
&= \frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$

□

§3 October 16 - Applications of Derivative

§3.1 Sketching Graph by Derivative

Definition 3.1. Suppose S is an interval in the domain of $f(x)$ such that $f(x)$ is defined for all x in S

- $f(x)$ is increasing on S $\longleftrightarrow f(a) \leqslant f(b)$ for all $a, b \in S$ and $a < b \longleftrightarrow f'(x) \geqslant 0$
- $f(x)$ is decreasing on S $\longleftrightarrow f(a) \geqslant f(b)$ for all $a, b \in S$ and $a < b \longleftrightarrow f'(x) \leqslant 0$

Example 3.2

Prove that lnx is an increasing function when $x > 0$

Traditional Way:

$$\forall x_1 > x_2 > 0$$
$$f(x_1) - f(x_2) = lnx_1 - lnx_2 = ln\frac{x_1}{x_2} > 0$$
$$\therefore f(x_1) > f(x_2)$$
$$\therefore lnx \text{ is an increasing function when } x > 0$$

Using Derivative:

$$(lnx)' = 1/x$$
$$\because x > 0 \therefore 1/x > 0$$
$$\therefore lnx \text{ is an increasing function when } x > 0$$

Theorem 3.3 (Fermat's Theorem)

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Example 3.4

Find the maximum and minimum value of $sinx + cos2x$

Traditional way:

$$\sin x + \cos 2x = \sin x + (1 - 2\sin^2 x) = -2\sin^2 x + \sin x + 1 = -2(\sin x - \tfrac{1}{4})^2 + \tfrac{9}{8}$$
$$\because -1 \leqslant \sin x \leqslant 1$$
$$\therefore -2 \leqslant -2(sinx - \tfrac{1}{4})^2 + \tfrac{9}{8} \leqslant \tfrac{9}{8}$$

Using Derivative:

$$(\sin x + \cos 2x)' = \cos x - 2\sin 2x$$

The original function $f(x)$ reaches its maximum when $(\sin x + \cos 2x)' = 0$, solving the equation and we can get $\sin x = \frac{1}{4}$, $\cos 2x = 1 - 2 \times \frac{1}{4}^2 = \frac{7}{8}$ or $\cos x = 0, x = \pi/2 + k\pi (k \in \mathbb{Z})$. Therefore, the maximum of function is $\frac{9}{8}$ while the minimum of function is -2.

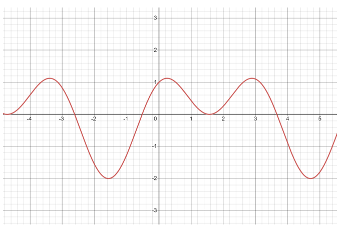


Figure 3: $\sin x + \cos 2x$

Definition 3.5. The second derivative, or the second-order derivative, of a function f is the derivative of the derivative of f. It can be written as:

$$\frac{d^2y}{dx^2} = f''(x)$$

Definition 3.6. If the graph of lies above all of its tangents on an interval , then it is called **concave upward on ($f''(x) > 0$).** If the graph of lies below all of its tangents on I,it is called **concave downward on ($f''(x) < 0$).**

This is because $f''(x)$ represents rate of change of$f'(x)$, namely the slope of a function.

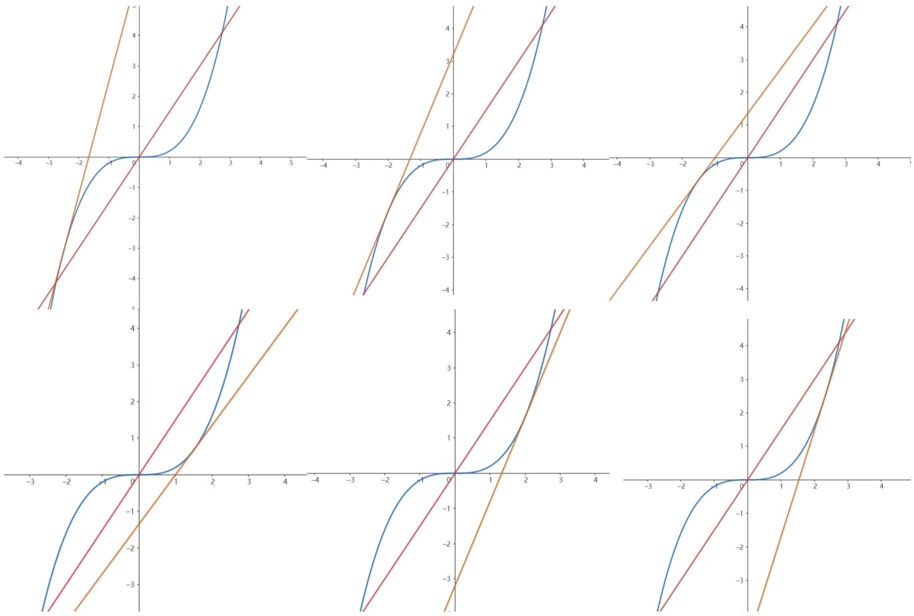


Figure 4: Example of how rate of change of slope effect function's shape

Definition 3.7. A point P on a curve $f(x)$ is called **an inflection point** if $f(x)$ is continuous there, the curve changes from concave upward to concave downward or from concave downward to concave upward at P ($f'' = 0$).

Remark 3.8. Are the gradient of a function at an inflection point necessarily equal to 0?
The answer is NO. There is no relationship between $y'' = 0$ and $y' = 0$.

Theorem 3.9 (The Second Derivative Test)
For $f(x)$ continuous near a:
If $f'(a) = 0$ and $f''(a) > 0$, $f(x)$ has a local minimum at a.
If $f'(a) = 0$ and $f''(a) < 0$, $f(x)$ has a local maximum at a.

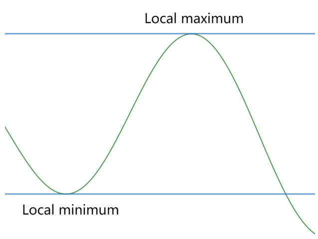


Figure 5: Example of the second derivative test

Now we can sketch almost all elementary functions. Let's try!

Exercise 3.10. Sketch the graph of $y = x^4 - 3x^3 + 1$.

$$\begin{aligned} y' &= 4x^3 - 9x^2, &\quad \text{when } y' = 0, x = 0 \text{ or } \frac{9}{4}, \\ y'' &= 12x^2 - 18x, &\quad \text{when } y' = 0,\ x = 0 \text{ or } \frac{3}{2}. \end{aligned}$$

Exercise 3.11. Sketch the graph of $y = \dfrac{x^2}{\sqrt{x+1}}$

Exercise 3.12. Sketch the graph of $y = \sin(2x) + \cos(x)$

Answers are in the shared GeoGebra File.

§3.2 Indeterminate Forms and L' Hopital Rule

§3.2.1 Indeterminate Form 0/0

Theorem 3.13 (L' Hopital[a] Rule)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

[a] L' Hopital should be pronounced as *lowpeetal* as its original pronunciation in French.

§3.2.2 Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an indeterminate form like $\infty/\infty, \infty \cdot 0, \infty - \infty$ instead of 0/0. We first consider the form ∞/∞.

When we are trying to calculate $\lim\limits_{x \to a} f(x)/g(x)$ while $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$$

and since $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, $1/f(x) \to 0$ and $1/g(x) \to 0$, therefore, we can apply L'Hopital Rule to it.

Similarly, for the $0 \cdot \infty$ case, just transform the ∞ to $1/0$ and therefore, the $0 \cdot \infty$ indeterminate case turns into $0/0$ form.

For the $\infty - \infty$ case, turn $f(x) - g(x)$ into fractional form, an example here will be more clear:

Example 3.14

Find the limit of this $\infty - \infty$ form:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

Solution.

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) &= \lim_{x\to 0}\frac{x-\sin x}{x\sin x} && \frac{0}{0} \\
&= \lim_{x\to 0}\frac{(x-\sin x)'}{(x\sin x)'} \\
&= \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} && \text{Still } \frac{0}{0} \\
&= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.
\end{aligned}$$

□

§3.2.3 Extension - Proof of L'Hopital Rule

Theorem 3.15 (The Rolle's Theorem)

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Proof. This is intuitively easy and is related to the local/global minima/maxima and interior points. Can you sketch a proof for it by yourself? This is left as an exercise for reader. □

Theorem 3.16 (The Mean Value Theorem)

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof. We picture the graph of f and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$. The secant line can be expressed by

$$g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

with point-slope equation. The vertical difference between the graphs of f and g at x is

$$\begin{aligned}
h(x) &= f(x) - g(x) \\
&= f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)
\end{aligned}$$

According to Rolle's Theorem, we know that there must exist at least one point c such that $h'(c) = 0$.
We differentiate both sides of the equation with respect to x and set $x = c$:

$$\begin{aligned}
h'(x) &= f'(x) - \frac{f(b)-f(a)}{b-a} \\
h'(c) &= f'(c) - \frac{f(b)-f(a)}{b-a} \\
0 &= f'(c) - \frac{f(b)-f(a)}{b-a} \\
f'(c) &= \frac{f(b)-f(a)}{b-a}
\end{aligned}$$

and therefore we are done. □

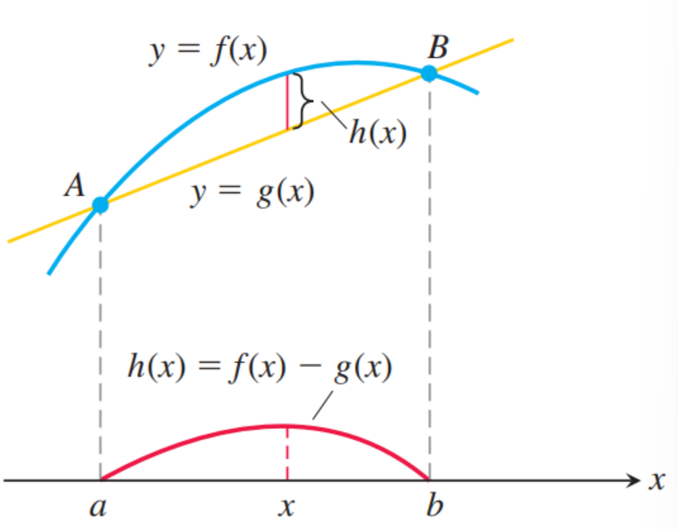


Figure 6: The secant AB is the graph of the function $g(x)$. THe function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x.

Theorem 3.17 (L' Hopital Rule)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

Proof. We first establish the limit equation for the case $x \to a^+$. The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.

Suppose that x lies to the right of a. Then $g'(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from a to x. This step produces a number c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

But $f(a) = g(a) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

As x approaches a, c approaches a because it always lies between a and x. Therefore,

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{c\to a^+} \frac{f'(c)}{g'(c)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

which establishes L'Hopital's Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy's Mean Value Theorem to the closed interval $[x, a], x < a$. □

§4 November 19 - Taylor Series and Prerequisites

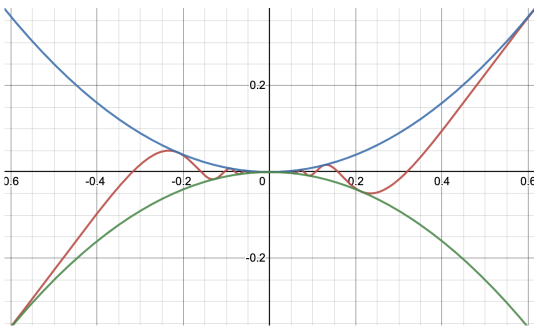
§4.1 Infinitesimals

Definition 4.1.
- Suppose the function f defined on $U^\circ(x_0)$ satisfies the limit $\lim_{x\to x_0} f(x) = 0$. Then, f is the infinitesimal when $x \to x_0$. (The infinitesimal when x goes to ∞, or reaches a right or left limit is defined analogously.)
- We call g a bounded quantity when $x \to x_0$ if the function g is bounded on some $U^\circ(x_0)$.

Example 4.2 (Miscellaneous Calculations)
Prove that

- The sum, difference, and product of two infinitesimal amounts of the same type are all infinitesimal amounts.
- The product of an infinitesimal quantity and a bounded quantity is an infinitesimal quantity.

Proof to the second observation:
Suppose that $\forall x \in U^\circ(x_0)$, $|g(x)| < M$. Then, we must have $|f(x)g(x) - 0| \le |f(x)||g(x)| < M|f(x)|$. By the definition of the infinitesimal, we know that $\forall \epsilon' > 0$, $\exists \delta > 0$ such that $|f(x)| < \epsilon' = \frac{\epsilon}{M}$ for all $x \in U^\circ(x_0, \delta)$ by the arbitrary nature of ϵ'. This naturally ends the proof.



$x^2 \sin \frac{1}{x}$

Next, we would like to introduce a method to compare the speed at which infinitesimals approach x_0.

Definition 4.3. If $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$, then f is an infinitesimal of higher order when $x \to x_0$. We note this as

$$f(x) = o(g(x)) \ (x \to x_0).$$

Specifically, $f(x) = o(1)\ (x \to x_0)$ stands for "f is the infinitesimal as $x \to x_0$."

Example 4.4
It is obvious that $x^{k+1} = o(x^k),\ x \to 0$ for $k \in \mathbb{Z}^+$.

Example 4.5
Show that $1 - \cos x = o(\sin x),\ x \to 0$.

Note: it should be clarified that $o(g(x))$ refers to $\{f | \lim_{x\to x_0} \frac{f(x)}{g(x)} = 0\}$. The following definitions address other outcomes of convergence speed comparison. (We will not cover them in the lecture.)

Definition 4.6.
- If $\exists K, L > 0$ such that
$$K \le \left| \frac{f(x)}{g(x)} \right| \le L,$$
then f and g are infinitesimals of the same order when $x \to x_0$. Specifically, $\lim_{x \to x_0} \frac{f(x)}{g(x)} = c \neq 0$ ensures the holding of this condition.
- If $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$, f and g are equivalent infinitesimals noted as $f(x) \sim g(x)$, $(x \to x_0)$.

The readers should be able to prove that equivalent infinitesimals can be used interchangeably in limit computations.

§4.2 Differentials

Example 4.7
The side length (x_0) of a square is increased by Δx. What is the square's change in area ΔS?
$$\Delta S = 2x_0 \Delta x + (\Delta x)^2$$

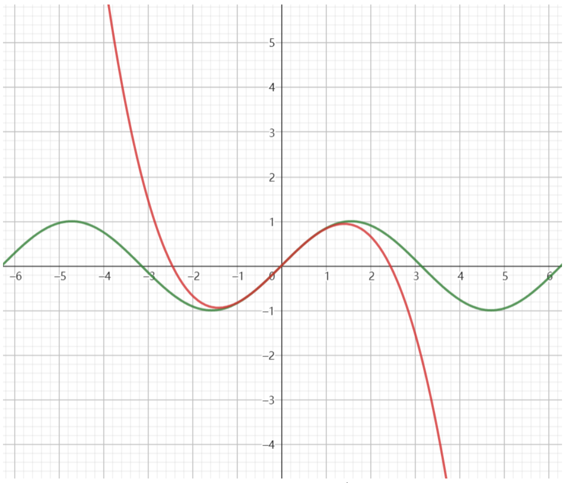
Remark 4.8. By Example 1.4, $(\Delta x)^2 = o(\Delta x)$.

Definition 4.9. Some function f defined near x_0 is differentiable at x_0 if $\exists A \in \mathbb{C}$ such that
$$\Delta y = A \Delta x + o(\Delta x).$$

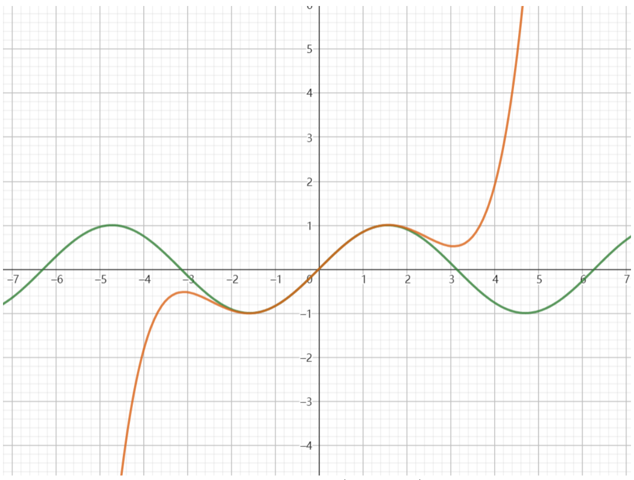
Remark 4.10. After taking the limit as $x \to x_0$, we see that $A = f'(x_0)$.

§4.3 Taylor Series

Example 4.11
Let's look at an example of how polynomials approximate function $f(x) = \sin x$.
We start from $x = 0$. Suppose the polynomial to be $p(x)$. We need $f(0) = p(0)$, so $p(x) = 0$.
To approximate the trend of $f(x)$, we need $f'(0) = p'(0)$, so $p(x) = 0 + x$.
To approximate the concavity and convexity of $f(x)$, we need $f''(0) = p''(0)$, so $p(x) = 0 + x + 0 \cdot x^2$.
Following the same pattern, we need $f'''(0) = p'''(0)$, so $p(x) = x - \frac{1}{3!}x^3$.
Following the same pattern, we get $p(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{(-1)^{m-1}}{(2m-1)!} + \ldots$.



(a) $p(x) = x - \frac{1}{3!}x^3$



(b) $p(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

Example 4.12

We derive from the definition of differentials that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0).$$

This equation $f(x) = f(x_0) + f'(x_0)(x - x_0)$ can provide a decent approximation for $f(x)$ at x_0. However, in real-world approximations, $o(x - x_0)$ often prove to be not accurate enough: we would like to have an error of $o((x - x_0)^n)$ for any positive integer n. To do so, we consider the polynomial

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n.$$

Compute $p_n(x_0)$, $p_n'(x_0)$, ..., $p_n^{(n)}(x_0)$. What do you notice?
Solution: $p_n(x_0) = a_0$; $p_n'(x_0) = a_1$; ... $p_n^{(k)}(x_0) = k!a_k$; ...; $p_n^{(n)}(x_0) = n!a_n$.
We thus substitute these derivations into $p_n(x)$ and deduce that

$$p_n(x) = p_n(x_0) + p_n'(x_0)(x - x_0) + \frac{p_n''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{p_n^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Definition 4.13. For a general function f that is n^{th}-order differentiable at x_0, define its Taylor Series at x_0 as

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

We easily observe that $T_n(x_0) = f(x_0)$, $f^{(k)}(x_0) = T^{(k)}(x_0)$, $(k = 0, 1, \ldots, n)$.

Theorem 4.14

For a general function f satisfying the properties in the previous definition,

$$f(x) = T_n(x) + o((x - x_0)^n).$$

Proof. Let $R_n(x) = f(x) - T_n(x)$, $Q_n(x) = (x - x_0)^n$. We aim to show that $\lim_{x \to x_0} \frac{R_n(x)}{Q_n(x)} = 0$. By our previous observations,

$$R_n(x_0) = R_n'(x_0) = \cdots = R_n^{(n)}(x_0) = 0,$$
$$Q_n(x_0) = \cdots = Q_n^{(n-1)}(x_0) = 0,\ Q_n^{(n)}(x_0) = n!.$$

Since $f^{(n)}(x_0)$ exists, $f^{(n-1)}(x_0)$ exists $\forall x \in U^\circ(x_0)$. Therefore, L'Hopital's rule may be applied up to $n-1$ times. We thus have

$$\begin{aligned}\lim_{x\to x_0}\frac{R_n(x)}{Q_n(x)} = \lim_{x\to x_0}\frac{R_n^{(n-1)}(x)}{Q_n^{(n-1)}(x)} &= \lim_{x\to x_0}\frac{f^{(n-1)}(x)-f^{(n-1)}(x_0)-f^{(n)}(x_0)(x-x_0)}{n!(x-x_0)} \\ &= \frac{1}{n!}\lim_{x\to x_0}\left[\frac{f^{(n-1)}(x)-f^{n-1}(x_0)}{x-x_0}-f^{(n)}(x_0)\right] \\ &= 0.\end{aligned}$$

□

Remark 4.15. $o((x-x_0)^n)$ is the Peano remainder of the Taylor Series. We will introduce another type of remainder next week. (Lagrange's remainder)

Example 4.16

Some common Taylor Series:

$$e^x = 1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+o(x^3)$$

$$\ln(x+1) = x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)$$

$$\sin x = x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+o(x^5)$$

$$\cos x = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+o(x^4)$$

$$\frac{1}{1-x} = 1+x+x^2+x^3+o(x^3)$$

$$(1+x)^a = 1+\frac{a}{1!}x+\frac{a(a-1)}{2!}x^2+\frac{a(a-1)(a-2)}{3!}x^3+o(x^3)$$

§5 December 13 - Integration and Its Application

§5.1 What is Integration?

In this section, we will first find the relationship between area and definite integral. Then, we will delve into indefinite integral and some calculating techniques related to it. Finally, we will go back to some complex indefinite integration and its applications.

§5.1.1 Riemann Sums

Definition 5.1. A Riemann sum is an approximation of a region's area, obtained by adding up the areas of multiple simplified slices of the region.

The Riemann sums can be described in this formula:

$$S_n = \sum_{k=1}^{n} f(a + k \cdot \frac{b-a}{n})(\frac{b-a}{n})$$

where we can express $\frac{b-a}{n}$ as Δx

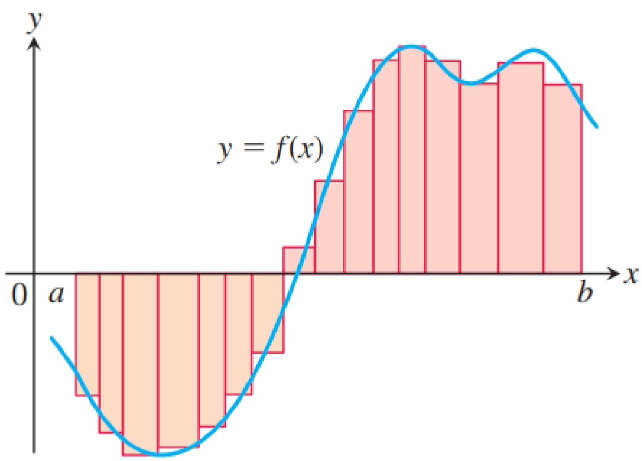


Figure 7: An Example of Riemann Sum

§5.1.2 Definite Integral

Definition 5.2.

- If limit $J = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(a + k\Delta x) \cdot \Delta x$ exists, we say that the definite integral exists.
- We express definite integral as $\int_a^b f(x)dx$, where a is the lower limit, b is the upper limit, $f(x)$ is the integrand, and dx is the variable of the integration.

Definition 5.3. If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, the definite integral $\int_a^b f(x)dx$ exists and f is the integrable over $[a, b]$.

§5.1.3 Definite Integral and Area

Example 5.4

The definite integral, or Riemann sum, of $f(x) = x^2$ over $[0, 2]$ can be calculated through these procedures:

$$\begin{aligned} A_{\text{smaller}} &= \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(k-1) \cdot \Delta x \\ &= \lim_{\Delta x \to 0} \sum_{k=1}^{n} \left((k-1)\frac{2}{n} \right)^2 \cdot \frac{2}{n} \\ &= 8 \lim_{\Delta x \to 0} \sum_{k=1}^{n} (k-1)^2 \cdot \left(\frac{1}{n} \right)^3 \\ &= 8 \lim_{n \to \infty} \frac{(n-1)(n)(2n-1)}{6n^3} \\ &= \frac{4}{3} \lim_{n \to \infty} \left(2 - \frac{3}{n} + \frac{1}{n^2} \right) \\ &= \frac{8}{3} \end{aligned}$$

$$\begin{aligned} A_{\text{larger}} &= \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(k) \cdot \Delta x \\ &= \lim_{\Delta x \to 0} \sum_{k=1}^{n} \left((k\frac{2}{n} \right)^2 \cdot \frac{2}{n} \\ &= 8 \lim_{\Delta x \to 0} \sum_{k=1}^{n} k^2 \cdot \left(\frac{1}{n} \right)^3 \\ &= 8 \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3} \\ &= \frac{4}{3} \lim_{n \to \infty} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) \\ &= \frac{8}{3} \end{aligned}$$

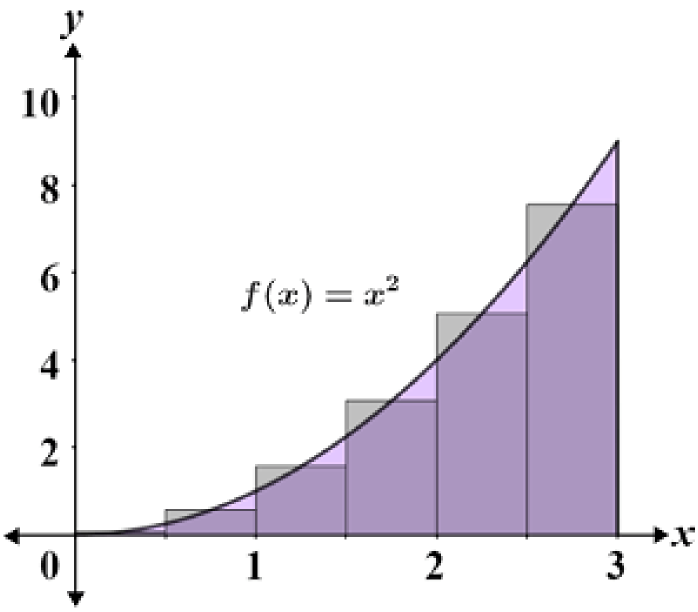


Figure 8: $y = x^2$ Diagram

But how is this related to the derivative we discussed before? In fact, integration is the inverse operation of derivative, and applying integration can help us to calculate the value of definite integral in a very simple way.

Remark 5.5. If f is continuous over $[a,b]$ and F is any antiderivative of f on $[a,b]$ ($F'(x) = f(x)$), then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof. Suppose $f(t)$ is continuous over $[a,b]$. Looking at this $y = f(t)$ at $t = x$, we want to find the rate of the increasing rate of the Area $A(x)$ under the curve if an infinitely small amount $h = dt$ is added to x.

$$\frac{dA(x)}{dt} = \frac{f(x)\cdot h}{dt} = f(x)$$

Since integration is the inverse operation of derivative, we conclude that $\int_a^x f(t)dx = F(x) - F(a)$,where $F'(x) = f(x)$ □

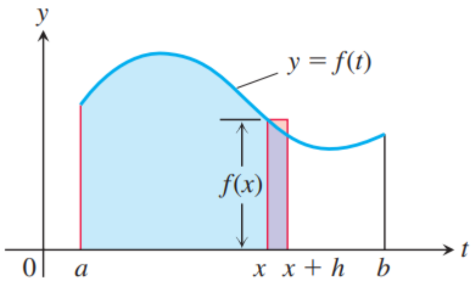


Figure 9: $y = f(t)$ at $t = x$

Example 5.6

We can apply this approach to compute the area under the curve of $f(x) = x^2$ over $[0,2]$:

$$\int_0^2 x^2dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$$

We can see this aligns with the result we obtain in Example 1.4.

Example 5.7

Compute the area under the curve $f(x) = \sin x$ over $[0,\pi]$:

$$\int_0^\pi \sin x dx = [-\cos x]_0^\pi = 1 + 1 = 2$$

And the area under the curve $f(x) = \sin x$ over $[0,2\pi]$:

$$\int_0^{2\pi} \sin x dx = [-\cos x]_0^{2\pi} = -1 + 1 = 0$$

How could the area become zero? This is because the approach considers the area under x-axis as negative, so it cancels out.

Example 5.8

Compute the definite integral

$$\int_0^{\frac{12}{13}} \frac{\sec^2(\frac{1}{2}\arcsin x)}{2\sqrt{1-x^2}}dx.$$

Solution. Let $u = \sin x$; then $du = \cos x dx$, and $u|_{x=\frac{12}{13}} = \arcsin \frac{12}{13}$.

$$I = \int_0^{\arcsin \frac{12}{13}} \frac{\sec^2\left(\frac{1}{2}u\right)}{2}du = \tan\left(\frac{1}{2}\arcsin\frac{12}{13}\right) = \frac{12}{18} = \frac{2}{3}.$$

□

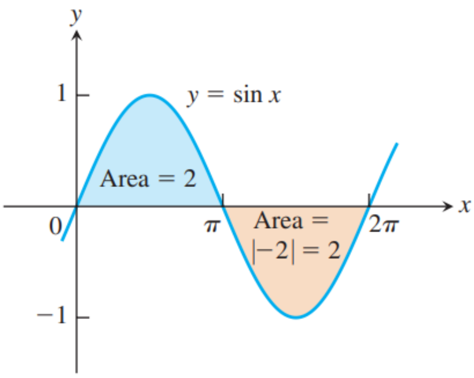


Figure 10: $y = \sin x$

§5.1.4 Properties of Definite Integral

Some definite integral rules are listed below.

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$
- $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

§5.2 Indefinite Integral

Definition 5.9. We defined the **indefinite integral** of the function with respect to x as the set of all antiderivatives of , symbolized by $\int (x)dx$. We need to add a constant because the derivative of a constant is 0:

$$\int (x)dx = F(x) + C$$

§5.3 Integration Techniques

§5.3.1 Substitution

Suppose

$$\frac{dF}{du} = f(u)$$

$$\int f(u)du = F(u) + c$$

Recall chain rule

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx}$$

$$\int f(u)\frac{du}{dx}dx = F(u) + c$$

comparing the two integration

$$\int f(u)\frac{du}{dx}dx = \int f(u)du$$

Example 5.10

Compute

$$\int_1^2 \frac{\ln x}{x}dx$$

Solution. Let $u = \ln x$, then $du = \frac{1}{x}dx, u|_{x=1} = 0, u|_{x=2} = \ln 2$

$$\int_1^2 \frac{\ln x}{x}dx = \int_0^{\ln 2} udu = \frac{(\ln 2)^2}{2}$$

□

§5.3.2 Integration by Parts

Recall product rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x),$$

in the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + f'(x)g(x)]dx = f(x)g(x)$$

rearranging the equation, it becomes

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

This is the **formula for integration by parts**.

Example 5.11

Compute

$$\int_0^{\pi/2} e^x \sin x dx$$

Solution.

$$\int e^x \sin x dx = e^x(-\cos x) - \int e^x(-\cos x)dx = -e^x\cos x + \int e^x \cos x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$$

$$\int_0^{\pi/2} e^x \sin x dx = \left[\frac{-e^x\cos x + e^x \sin x}{2}\right]_0^{\pi/2} = \frac{1+e^{\pi/2}}{2}$$

□

§5.3.3 Trig Substitution

Recall basic trigonometric formula:

$$1 - \sin^2(x) = \cos^2(x)$$

$$1 + \tan^2(x) = \sec^2(x)$$

Thus we can have the following trigonometric substitution:

for	$\sqrt{a^2 - x^2}$	we can have	$x = a\sin\theta$
	$\sqrt{a^2 + x^2}$		$x = a\tan\theta$
	$\sqrt{x^2 - a^2}$		$x = a\sec\theta$

Table 1: Trigonometric substitution

Example 5.12

Compute

$$\int \frac{\sqrt{9-x^2}}{x^2}dx$$

Solution. Let $x = 3\sin\theta$, then $dx = 3\cos\theta d\theta, \theta = \arcsin\frac{x}{3}$

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{9\cos^2\theta}{9\sin^2\theta}d\theta = \int(\csc^2\theta - 1)d\theta = -\cot\theta - \theta + c = -\frac{\sqrt{9-x^2}}{x} - \arcsin\frac{x}{3} + c$$

□

§5.3.4 Partial Fractions

$$\int \frac{dx+e}{ax^2+bx+c}dx = \int(\frac{A}{mx+n} + \frac{B}{px+q})dx = \frac{A}{m}\ln|mx+n| + \frac{B}{p}\ln|px+q| + c$$

Example 5.13

Compute

$$\int \frac{7x-2}{x^2-x-2}dx$$

Solution:

$$\int \frac{7x-2}{x^2-x-2}dx = \int (\frac{4}{x-2} + \frac{3}{x+1})dx = 4\ln|x-2| + 3\ln|x+1| + c$$

§5.4 Common Applications of Integration

§5.4.1 Area between Curves

Consider the region S shown in the figure that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$.

We divide S into n strips of equal width and then we approximate the ith strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$, we could take all of the sample points to be right endpoints, in which case $x_i^* = x_i$. The Riemann sum

$$\sum_{i=1}^{n}[f(x_i^*) - g(x_i^*)]\Delta x$$

is therefore an approximation to what we intuitively think of as the area of S.

When we divide the area to $n \to \infty$ pieces,

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i*) - g(x_i*)]\Delta x$$

We recognize the limit is equivalent to the definite integral of $f - g$, therefore, the area formula is

$$A = \int_a^b [f(x) - g(x)]dx$$

Example 5.14

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution. It is trivial that they intersect at $(0, 0)$ and $(1, 1)$, so the total area is

$$\begin{aligned} A &= \int_0^1 (2x - 2x^2)dx = 2\int_0^1 (x - x^2)dx \\ &= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right] = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \boxed{\frac{1}{3}} \end{aligned}$$

□

§5.4.2 Volumes

Consider a random non-cylindrical geometry S.

We can cut it into n slices by approximating each slice as a column. Use a plane to intersect the geometric body S to obtain a planar region called the plane region of the cross section of S. Let $A(x)$ be the area of cross section of S on plane P_x that lies perpendicular to the x-axis that passes through point x which $a \leq x \leq b$.

We can use planes P_{x1}, P_{x2}, ... to divide S into equal width of slices. We approximate the ith slice by a cylindrical with base $A(\overline{x_i})$ and height Δx.

The Riemann sum

$$\sum_{i=1}^{n} A(x_i)\Delta x$$

is therefore an approximation to the volume of S.

When we divide S to $n \to \infty$ slices,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x$$

Thus, the volume formula is

$$V = \int_a^b A(x)dx$$

Example 5.15

Derive the formula $V = \frac{1}{3}a^2h$ for the volume of a pyramid with a square base.

Solution. According to figure below, $A(x)$ would be $A(x) = s^2 = (\frac{ax}{h})^2$

So the volume is

$$V = \int_0^h A(x)dx = \frac{a^2}{h^2}\int_0^h x^2dx = [\frac{a^2}{h^2}(\frac{1}{3}x^3)]_0^h = \frac{1}{3}a^2h$$

□

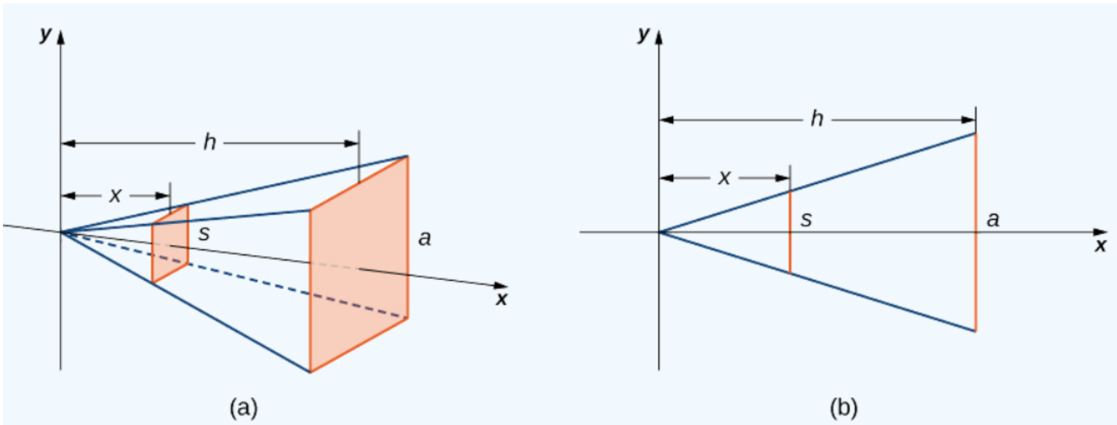


Figure 11: (a)A pyramid with a square base (b) A two-dimensional view of the pyramid

§5.4.3 Volumes by Cylindrical Shells

Again, consider the region S shown in the figure that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$.

Now suppose this region S rotate around the y-axis and consider the shape V obtained by the revolution.

Again, we divide V into n rings with equal width and we approximate the ith ring by a cylindrical shell with mean radius $\overline{x_i}$, height $f(\overline{x_i}) - g(\overline{x_i})$, and width Δx.

The Riemann sum

$$\sum_{i=1}^{n} 2\pi\overline{x_i}[f(\overline{x_i}) - g(\overline{x_i})]\Delta x$$

is therefore an approximation to the volume of V.

When we divide V to $n \to \infty$ pieces,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\overline{x_i}[f(\overline{x_i}) - g(\overline{x_i})]\Delta x$$

Therefore, the volume formula is

$$V = \int_a^b 2\pi x[f(x) - g(x)]dx$$

With the same method, we can also deduct the following formula:
When the region is rotating around the x-axis:

$$V = \int_a^b \pi[f(x)^2 - g(x)^2]dx$$

Example 5.16

Find the volume of a sphere, radius of 1.

Solution. In this case, $f(x) = \sqrt{1 - x^2}$ and $g(x) = 0$.
So the volume is

$$V = \int_{-1}^1 \pi(\sqrt{1 - x^2})^2 dx = \pi[x - \frac{1}{3}x^3]_{-1}^1 = \frac{4}{3}\pi$$

□

§5.4.4 Arc Length

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$.

We divide the curve into n segments with equal width and we approximate the ith segment by a line segment with horizontal change Δx and vertical change $\Delta y_i = f(x_i + \Delta x) - f(x_i)$.

By the Pythagorean theorem, the length of the line segment is

$$\sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{1 + (\frac{\Delta y_i}{\Delta x})^2}\Delta x$$

By the Mean Value Theorem, there is a point x_i^* which $x_i + \Delta x \geq x_i^* \geq x_i$ such that $f'(x_i^*) = \dfrac{\Delta y_i}{\Delta x}$
Then the length of the line segment is given by

$$\sqrt{1 + (f'(x_i^*))^2}\Delta x$$

The Riemann sum

$$\sum_{i=1}^n \sqrt{1 + (f'(x_i^*))^2}\Delta x$$

is therefore an approximation of the curve length.

Thus, the arc length formula is

$$l = \int_a^b \sqrt{1 + (f'(x_i^*))^2}dx$$

Example 5.17

Find the perimeter of a circle, radius of 1. We first consider one half of the circle, so the function is:

$$f(x) = \sqrt{1 - x^2}$$

$$f'(x) = \frac{x}{\sqrt{(1 - x^2)}}$$

Applying the formula:

$$l = \int_{-1}^{1} \sqrt{1 + \frac{x^2}{(1 - x^2)}} dx = \int_{-1}^{1} \sqrt{\frac{1}{1 - x^2}} dx = \arcsin x|_{-1}^{1} = \pi$$

(Note: This is just a display of this method's application. It is actually a circular proof.)

§5.4.5 Area of a Surface of Revolution

Again, consider $f(x)$ as a smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis.

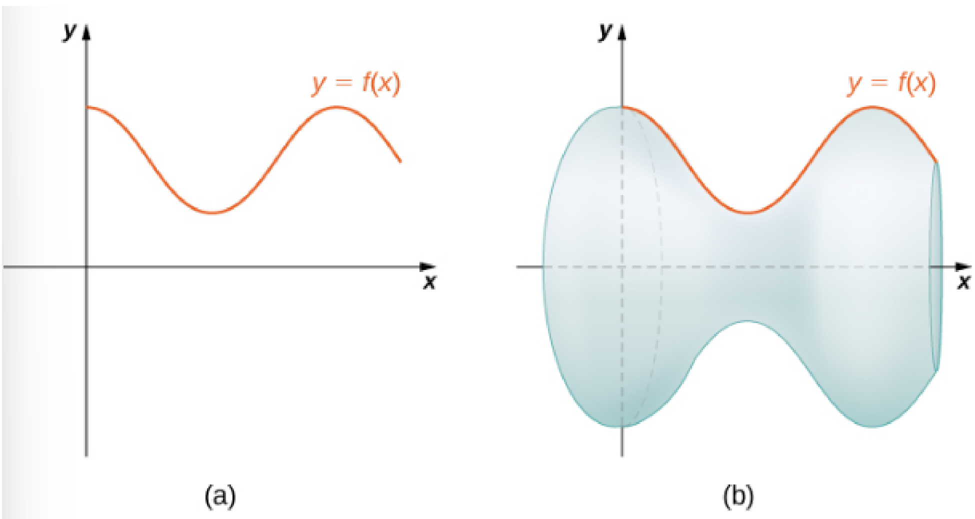


Figure 12: (a)A curve representing f(x).(b)The surface of revolution formed by revolving the graph of $f(x)$ around the x-axis

As we have done many times, we are going to divide it into n slices with equal width and we approximate the ith slice by a ring. We can unfold the ring and calculate the rectangle area with length $2\pi f(x_i^*)$ and width $\sqrt{1 + (f'(x_i^*))^2}\Delta x$.

The Riemann sum

$$\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + (f'(x_i^*))^2}\Delta x$$

is therefore an approximation of the area of the surface of revolution.

Thus, the area formula is

$$A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2} dx$$

Example 5.18

Find the surface area of a sphere, radius of 1.

Solution. The curve $f(x)$ would be $y = \sqrt{1-x^2}$ in this case, and its derivative is $f'(x) = \dfrac{-x}{\sqrt{1-x^2}}$.

Thus the surface area would be $S = \int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{1+(\dfrac{-x}{\sqrt{1-x^2}})^2}dx = [2\pi x]_{-1}^{1} = 4\pi$ $\square$

References and Extended Reading Materials

[1] Thomas' Calculus (George B. Thomas, Joel R. Hass etc.)

[2] Calculus (J. Stewart, D. Clegg, S. Watson) Metric Version — 9E Early Transcendentals