Due June 9, 15h

Send your assignment in groups of 2 students to `marino.arroyo@upc.edu` with the words "CSM assignment 3" in the subject. Send a concise but complete report, including selected figures as needed, and including the pieces of code that you have modified in an appendix.

1. **Kirchhoff Saint-Venant material model.**

   Isotropic linear elasticity can be derived from balance of linear momentum, the linearized strain displacement relation $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$, and the stored elastic energy function

$$W(\boldsymbol{\varepsilon}) = \frac{\lambda}{2}(\mathrm{tr}\ \boldsymbol{\varepsilon})^2 + \mu\ \mathrm{tr}(\boldsymbol{\varepsilon}^2) = \frac{\lambda}{2}(\varepsilon_{ii})^2 + \mu\ \varepsilon_{jk}\varepsilon_{jk}.$$

   (1) Check that, the stress tensor obtained from $\boldsymbol{\sigma} = \partial W / \partial \boldsymbol{\varepsilon}$ agrees with the usual linear elasticity expression.

   Since the linearization of the Green-Lagrange strain tensor $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{Id})$ is the small strain tensor $\boldsymbol{\varepsilon}$, it is natural to extend isotropic elasticity to nonlinear elasticity as

$$W(\boldsymbol{E}) = \frac{\lambda}{2}(\mathrm{tr}\ \boldsymbol{E})^2 + \mu\ \mathrm{tr}(\boldsymbol{E}^2). \tag{1}$$

   This hyperelastic model is called Kirchhoff Saint-Venant material model.

   (2) According to the definition we gave in class about isotropy in nonlinear elasticity, is this model isotropic?

   (3) Derive the second Piola-Kirchhoff stress $\boldsymbol{S}$.

   (4) For a uniform deformation of a rod aligned with the X axis ($x = \Lambda X$, $y = Y$, $z = Z$, where $\Lambda > 0$ is the stretch ratio along the X direction) derive the relation between the nominal normal stress P (the xX component of the first Piola-Kirchhoff stress) and the stretch ratio Λ, $P(\Lambda)$, and plot it.

   (5) Is the relation $P(\Lambda)$ monotonic? If not, derive the critical stretch Λ_{crit} at which the model fails with zero stiffness. Does this critical stretch depend on the elastic constants? Show that the material does not satisfy the growth conditions

$$W(\boldsymbol{E}) \longrightarrow +\infty \quad \text{when} \quad J \longrightarrow 0^+.$$

   Discuss your answers.

   (6) Consider now the modified Kirchhoff Saint-Venant material model:

$$W(\boldsymbol{E}) = \frac{\lambda}{2}(\ln J)^2 + \mu\ \mathrm{tr}(\boldsymbol{E}^2).$$

   Does this model circumvent the drawbacks of the previous model?

(7) Implement the material model in Eq. (1) in the Matlab code. Perform the consistency test to check your implementation. Try to demonstrate the *material* instabilities of this model with a numerical example.

2. Implementation of line-search.

Implement a line-search algorithm to be used in combination with Newton's method. For this, I suggest you resort to Matlab's function `fminbd`, which performs 1D nonlinear minimization with bounds. You need to define a function `Ener_1D` that evaluates the energy along the line that passes through `x` in the direction of `p` (the descent search direction). The function `LineSearch` may include lines like the ones suggested next:

```
t=1;
opts=optimset('TolX',options.TolX,'MaxIter',options.n_iter_max_LS);
t = fminbnd(@(t) Ener_1D(t,x_short,p),0,2,opts);
x_short=x_short+t*p;
```

Test the code with the examples where you expect buckling (the compression of the beams, or the deflection of the arch), and compare the results with and without line-search.

3. Implementation of a material model.

The code you are given implements a plane-strain finite element method for finite deformation elasticity. A compressible Neo-Hookean material is already in place (modeling a slightly porous rubber for instance), whose strain energy density (or hyper-elastic potential) is

$$W(\boldsymbol{C}) = \frac{1}{2}\lambda_0(\ln J)^2 - \mu_0 \ln J + \frac{1}{2}\mu_0(\text{trace } \boldsymbol{C} - 3)$$

This constitutive model is isotropic. Note that, since we are considering plane strain, we can use a 2×2 reduced right Cauchy-Green deformation tensor and replace trace $\boldsymbol{C} - 3$ by trace $\boldsymbol{C} - 2$ in the above equation.

We want to consider now an anisotropic material, more specifically, a transversely isotropic material. We consider a material constitutive law for a rubber reinforced by fibers, all aligned in the same direction in such a way that perpendicular to the fibers, the material remains isotropic. The orientation of the fibers is given in the reference configuration by a unit vector $\boldsymbol{N}^{fib}$. Such a model depends on the principal invariants of $\boldsymbol{C}$, and additionally by the fourth invariant

$$I_4(\boldsymbol{C}) = \boldsymbol{N}^{fib} \cdot \boldsymbol{C} \cdot \boldsymbol{N}^{fib} = C_{IJ} N_I^{fib} N_J^{fib}.$$

More specifically,

$$W(\boldsymbol{C}) = \frac{1}{2}\mu_0(\text{trace } \boldsymbol{C} - 3) - \mu_0 \ln J + \kappa \mathcal{G}(J) + c_0 \left\{ \exp\left[c_1 (\sqrt{I_4(\boldsymbol{C})} - 1)^4) \right] - 1 \right\},$$

where μ_0, κ, c_0 and c_1 are material parameters, and $\mathcal{G}(J)$ provides the volumetric response of the material. We consider

$$\mathcal{G}(J) = \frac{1}{4}(J^2 - 1 - 2 \ln J).$$

The last term in the strain energy function specifies the contribution to the deformation energy of the fibers, and as typical in biological fibers, with this model these become stiffer the more deformed they are.

(a) Implement this material model into the code. The code is prepared for this model (`material=2`), including the definition of the material parameters in `preprocessing.m`.

(b) Check the correctness (consistency test) of your implementation by running the script `Check_Derivatives.m` with `material=2`. This script checks the gradient of the energy (out-of-balance forces) and the Hessian of the energy (tangent stiffness matrix) by numerical differentiation. Check also that when solving a mechanical problem with this mode, Newton's method converges quadratically.

(c) Solve `example=0`, a dead load applied on an elastic block in tension, with a few representative orientations of the fibers. Consider $\theta = 0$ (fibers aligned with the loading direction), $\theta = \pi/6$, $\theta = \pi/4$ and $\theta = \pi/2$ (fibers perpendicular to the loading direction), where

$$\boldsymbol{N}^{fib} = [\cos\theta, \sin\theta]^T .$$

Explain the results from a mechanical viewpoint.