APS1022

Financial Engineering II

Course Project 2

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Yuyang Shao - 1002961636

Heng Li - 1003894690

Hua Zhong - 1004065744

UNIVERSITY OF TORONTO

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Summary

Options are financial contracts which can provide investors the rights to buy or sell an asset within a specified time period. Call options allow investors to buy the asset at a predetermined price within a specific time and put options allow investors to sell the asset at a predetermined price within a specific time. The purpose of this project is to price some common types of Exotic options and American options using Monte Carlo Simulation and Lattice approach. Exotic options are hybrids of American and European options and will often fall somewhere in between these other two styles. There are many types of exotic options in financial market but only the following types of exotic options are included in this project:

1. Asian options: An option type where the payoff depends on the average price of the underlying asset over a certain period as opposed to standard options (American or European options)
2. Lookback and Float Lookback options: An option allows the holder to "look back" over time to determine the payoff. A Lookback option with floating strike is called Float Lookback options.
3. American options (standard options): An option allows holders to exercise their rights at any time before and including the expiration date.

In this project, we assume the risk-free rate $r_f = 2\%$, current price $S_0 = \$100$, and volatility $\delta = 25\%$ with no dividend payments. All options have a strike price $k = \$105$, maturity $= 2\ months = \frac{1}{6} year$ and simulation period $dt \approx \frac{1}{48} year$. At last, the simple size of the Monte Carlo is $n = 2000$ and number of weeks until maturity for each path is $m = 8$.

Methods

A. Monte Carlo Simulation

Monte Carlo simulation is used to predict the probability of different outcomes of an event when the intervention of random variables exists. It can be used to estimate the price of various options. Specifically, it generates a particular number of simulated price paths for an option and the average payoff calculated is the price of the option.

Pricing of exotic options(path-dependent options)

The payoff of path-dependent options are shown below:

Option type	Payoff
Asian call	$(\overline{S} - K)^+$
Asian put	$(K - \overline{S})^+$
Lookback call	$(S_{max} - K)^+$
Lookback put	$(K - S_{min})^+$
Floating lookback call	$(S_T - S_{min})^+$
Floating lookback put	$(S_{max} - S_T)^+$

The growth factor for each period can be calculated as:

$$S_{t_j} = S_{t_{j-1}} e^{\left(r - \frac{\sigma^2}{2}\right)\delta t + \sigma\sqrt{\delta t} Z_j} \qquad Z_j \sim N(0,1) \qquad j = 1, \ldots, m \qquad (3.1)$$

The algorithm to price these options are as follows:

(1) For each simulated path i = 1 to n(number of simulated paths):

For each time step j = 1 to m(number of time periods till maturity:8):

Simulate the current stock price according to formula(3.1)

Append the newest simulated price to a list.

Append the price over m periods of a single path to a 2d list.

(2) Compute the mean, maximum, and minimum price as well as the end price for each path n.

(3) Compute the payoff for each option according to the payoff table above.

(4) Average the payoffs and discount them to the current price to get the price for each exotic option.

(5) Compute the 95%confidence interval for each option([price-1.96*s/n^1/2, price+1.96*s/n^1/2]).

Pricing of an American put option

The formula to calculate $E[S_T|S_t]$, d1 and d2 are shown below:

$$\mathbb{E}[S_T \mid S_t] = \begin{cases} S_t N(d_1) - K e^{-r(T-t)} N(d_2) & \text{for a call option} \\ K e^{-r(T-t)} N(-d_2) - S_t N(-d_1) & \text{for a put option} \end{cases} \quad (6.2)$$

$$d_1 = \frac{\log(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t} \qquad (6.3)$$

The algorithm to calculate the price of an American put option is as follows:

(1) For each simulated path i = 1 to n(number of simulated paths):

For each time step j = 1 to m(number of time periods till maturity:8):

(i) Calculate the time remaining until maturity T-t. (maturity - j * 1/48)

(ii) Compute d1,d2 according to formula (6.3)

(iii) Compute the current stock price S_t according to formula(3.1)

(iv) Compute $E[S_T|S_t]$(the price of a European option at time t) using formula(6.4)

(v) Compute payoff Z_t from exercise the option at time t.

(vi) Check if $Z_t \geq E[S_T|S_t]$, and record the stop time $\tau_i = t_j$ as the optimal stop time for this path.

(vii) If the optimal stop time wasn't found, $\tau_i = T$

(viii) calculate the present value of payoff from exercise at time τ_i, $h_i = Z_{\tau i}$

(2) Compute the average optimal stopping time, its variance and standard deviation.

(3) Compute the average present value of payoffs at optimal stopping time as the option price, its variance and standard deviation.

(4) Compute the 95% confidence interval for the American put option([price-1.96*s/n^1/2, price+1.96*s/n^1/2]) and for the optimal stopping time([stop time-1.96*v/n^1/2, stop time+1.96*v/n^1/2]), where s is the standard deviation of the payoffs and v is the standard deviation of the optimal stopping time.

B. Lattice Approach

The Lattice method is a powerful tool that we could use to value options from a risk-neutral perspective. Suppose the initial price of a stock is S_0. After each time step, the stock's price either rises to S_0u or drops to S_0d with risk-neutral probability p and $1-p$ respectively. Following this method, we would get a binary tree(or lattice) given several time steps. The lattice would have a structure like *{{0},{u,d},{uu,ud,dd}.{uuu,uud,udd,ddd},...}*. We use *f* to represent the value of the option on the tree, for example, f_{dd} means the future value at node dd.

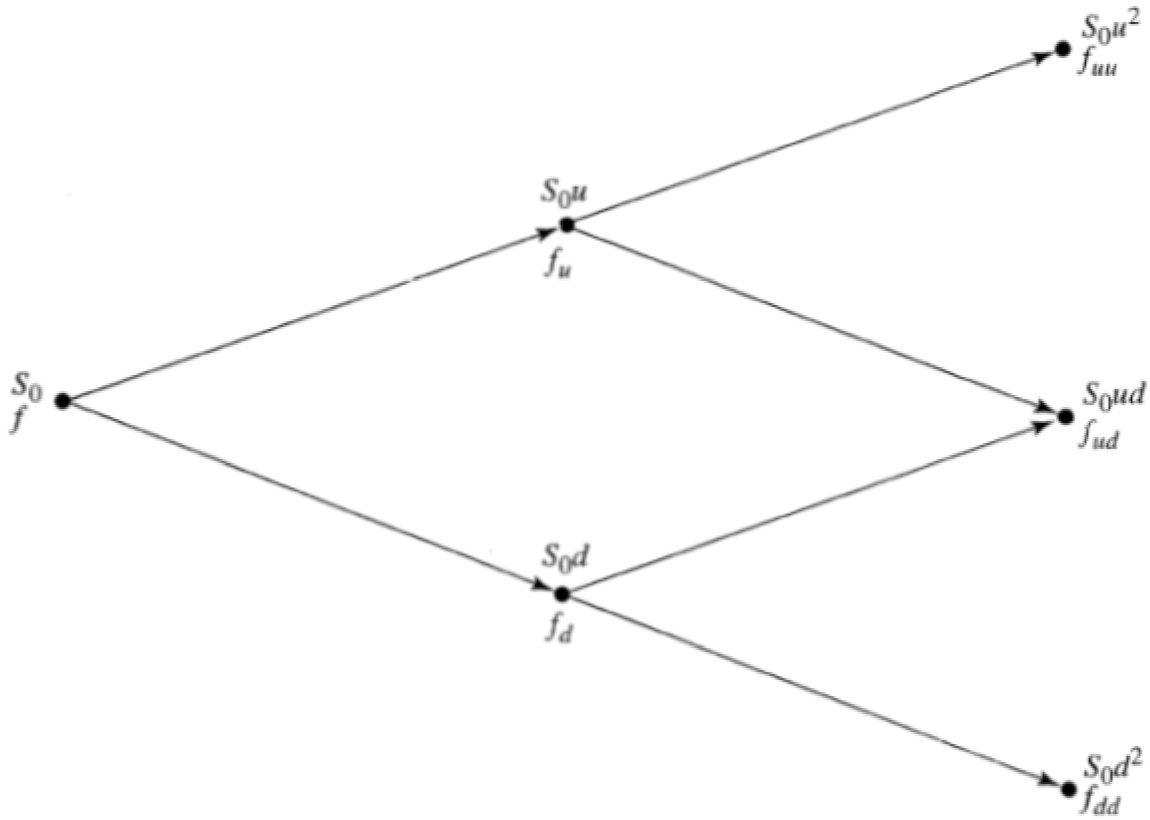

Figure 1. Stock and option prices in a two-step binary tree(from John Hull *Options, Futures, and Other Derivatives 8th Ed* Pg.262)

The first step in the Lattice approach is to determine the parameter we need to build the binary tree, i.e. *u,d,p*. Given the stock's volatility σ, time step Δt, and risk-free rate r_f, we have the following formulas to compute the parameters for the lattice:

$$p = \frac{a-d}{u-d}$$

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$a = e^{(r-q)\Delta t}$$

Pricing of exotic options(path-dependent options)

For path-dependent options, we need the price information on the whole path to price the option. Therefore, we simply iterate over all the possible paths and calculate the options' payoff for each path. Then the value of an option would be the average payoff for all the possible paths.

Pricing of American options

As for American options, because of their independence on their paths, we could compute their value by working backwards from the end of the lattice tree.

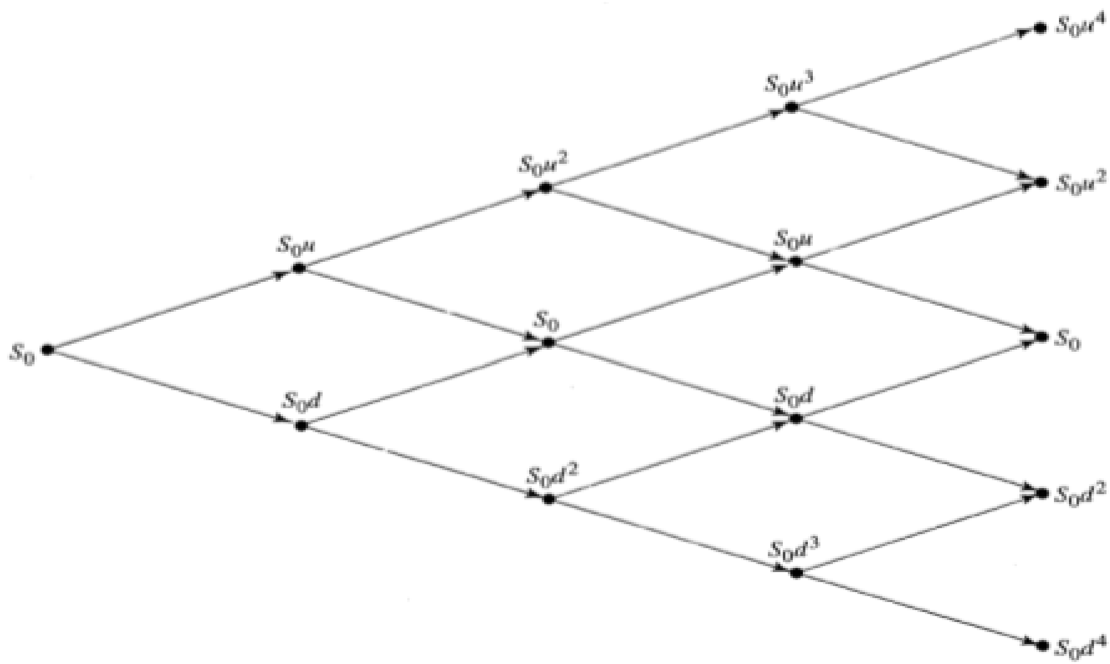

Figure 2. Tree used to value American options(John Hull *Options, Futures, and Other Derivatives 8th Ed* Pg.430)

Suppose the lattice shown above is the lattice we work on. The payoff of an American put option with strike price K at node *uuuu* would be

$$f_{uuuu}=max(K-S_0u^4,0)$$

while the payoff at node *uuuk* would be

$$f_{uuuk}=max(K-S_0u^2,0)$$

because not exercising the option would result in expiration. However, in node *uuu*, we could choose to exercise or keep it because we still have opportunities to exercise it in the future. Therefore, the value of the option at node *uuu* would be

$$f_{uuu}=max(K-S_0u^3,0,(p*f_{uuuu}+(1-p)*f_{uuuk})*e^{-rf\Delta t}).$$

Actually, this backwards recurrence relation holds for every node in the lattice except the nodes in the last layer. Following this backwards recurrence relation, we could compute the value of this option at node *0*.

Explanation of Major Functions in Code

Monte Carlo simulation

1. A growth factor is defined so that according to formula(3.1) S_{j-1}*growth factor = S_j.

2. A discount factor is defined as np.exp(-rf * maturity), so that we can convert the future value of the price of an option to its present value.

3. Formula(6.2) requires using cumulative distribution functions. We used the cdf function from the scipy.norm library.

4.american_put_simulation() function simulates the prices for American put options using the parameters defined at the beginning of the code. This method follows strictly the pricing method defined in the methodology part for the American put options. It calculates the estimated price of the American option, its variance and standard deviation. It also computes the average optimal stop time, its variance and standard deviation. This method prints out the calculated price of the American option with its 95% confidence interval and the average optimal stop time with its 95% confidence interval. It also returns the estimated price as a reference.

Lattice Approach

1. Lattice(T,s0,u,d) function takes the number of time steps *T*, the initial price *s0*, and parameters *u* and *d* as inputs and generates a *T*-step binary tree. This *T*-step binary tree is a nested list in Python in format as [*[S0],[S0*u,S0*d],[S0*uu,S0*ud,dd].[S0*uuu,S0*uud,S0*udd,S0*ddd]*,...]. However, in order to speed up the computation in the later steps, the sublist is stored as a NumPy array.

2. As pricing the path-dependent options, we need to iterate over all the possible paths. The total number of possible paths is 2^T and we represent each path as a T-digit binary number. For

example, 11111111 means the stock rising at all 8 time steps. Therefore, to iterate over all possible paths, we simply create a python for loop using i in range(2**T) and transform i into binary number to represent each path.

3.In the American option pricing part, we follow the method that we introduced in previous section. We create a lattice pricing_lat that has the same structure with Lattice(T,s0,u,d) to store the valuation data. First, we compute the value of options of the nodes at the ending layer(period) using the formula $max(0,K-S_t)$. Then we iterate over layers backwards using the recurrence relation we discussed before. And the value of this option would be the value of the first node of pricing_lat.

Visualization

Figure 3 shows the resulting price of options of lattice approach and Monte Carlo Simulation. The price of Lookback put options is always higher than other options using both Monte Carlo Simulation and lattice approach. On the other hand, the price of Lookback call options is the lowest among all other options. Also, only the Floating Lookback option has the stable call option and put option price, because the initial price is lower than the strike price which leads to the fact that there are many 0 payoff paths in call option. Monte Carlo Simulation always generated lower prices than the Lattice approach except the Asian put options. Figure 4 shows the simulated paths of Monte Carlo simulation. This plot looks very messy since there are 2000 paths shown in this figure, but we can still observe that initial stock price $s_0 = 100$, maturity T = 8 weeks and the approximately range of the end of each path is between 70-140.

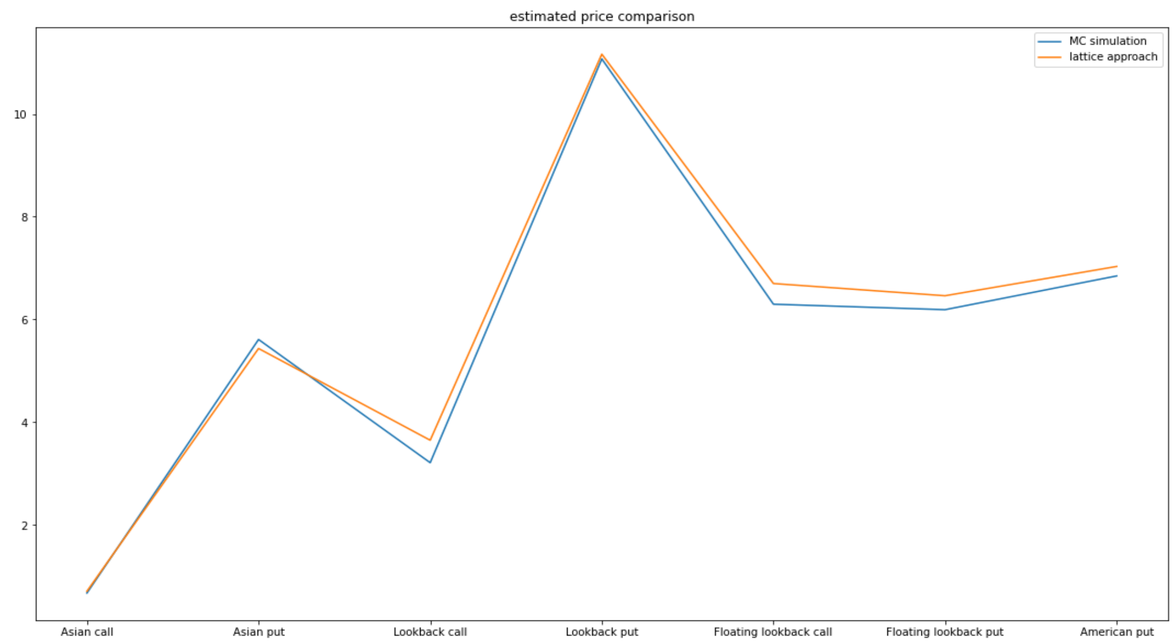

Figure 3 Estimated Price with MC Simulation and Lattice approach

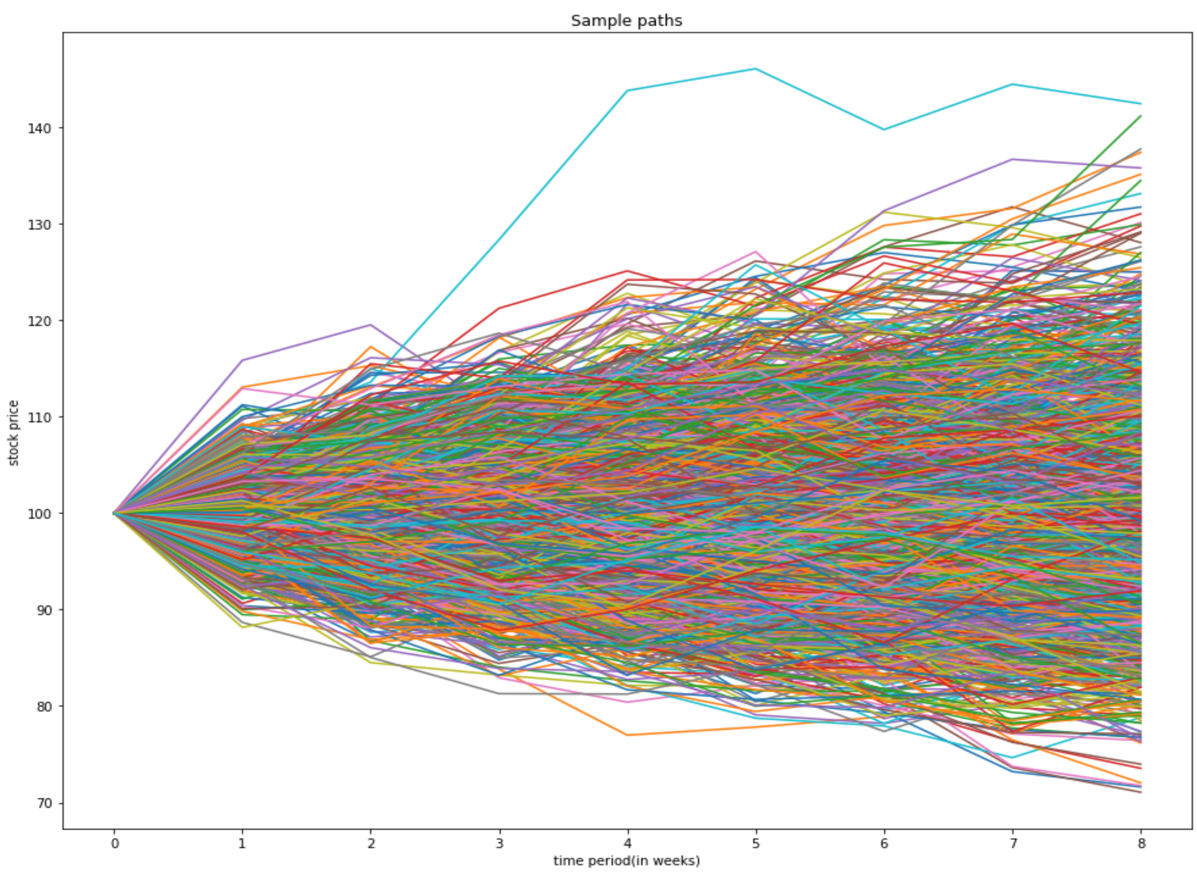

Figure 4 Simulation Paths of Monte Carlo Simulation

Comparison and Discussion

Monte Carlo Simulation uses stochastic processes to generate the random paths of the underlying asset through simulation. Because of the random walks, it takes a wide range of possibilities to help us reduce uncertainty parameters. Monte Carlo also allows us to vary risk assumptions under all parameters and thus model a range of possible outcomes. However, Monte Carlo Simulation still has some limitations. The first limitation is that Monte Carlo Simulation assumes the market returns are normally distributed, even though we know that the actual market returns are not normally distributed. So, Monte Carlo will lose accuracy if we cannot guarantee that the market returns are normally distributed. The other limitation is that it underestimates the risk. It is not safe to use Monte Carlo Simulation to estimate the probability of extreme bear events like a financial crisis. Similar to Monte Carlo Simulation, Lattice approach employs binomial tree paths to provide accurate pricing of options. Although Monte Carlo Simulation and Lattice approach have very similar estimated prices of options based on the observation from Figure 1.1, we can still conclude that some differences between these two methods exist. When dealing with path-dependent options, since the Lattice model builds binomial trees to generate the price of options and calculate the payoff for each node, the computations scale exponentially. For example, if the number of weeks until maturity is 1000, there will be 2^{1000} paths need to be calculated. Compared to the Monte Carlo Simulation, Lattice approach will be inefficient if we have a very large number of periods. When dealing with American options, Lattice approach takes advantage of the backwards recurrence relation and requires much less computations. In this project, the maturity is 8 weeks which means that both methods are suitable to use in calculating the price of Exotic options or American options.

Lesson Learned

Both Monte Carlo simulation and Lattice approach are reliable techniques to estimate exotic options and American options. For Monte Carlo simulation, we have to simulate large enough times in order to get the reliable estimated price and its confidence intervals. For Lattice approach, we have to iterate m*2^m times(m: number of time periods till maturity) and running time would grow exponentially when pricing path-dependent options. Therefore, when pricing path-dependent options with a large number of periods, it is computationally efficient to use Monte Carlo simulation. However, when pricing American options, the Lattice method can provide relatively accurate results by iterating over every node backwards while the number of nodes only grows quadratically, assuming that n>m(number of paths > number of time periods till maturity). Thus, we may choose Lattice approach when pricing American options.

Reference

Hull, J. (2012) Options, Futures and Other Derivatives. 8th Edition, Prentice Hall

Mcwilliams, N. (2022). Pricing American Options Using Monte Carlo Simulation.