Introducing the Forster-Warmuth Nonparametric Counterfactual Regression

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- Nonparametric Series Regression in Statistics

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- Simulation Studies and an Application

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Nonparametric Regression in Statistics

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- Series or Sieve estimation approximates unknown conditional mean based on linear combination of K basis functions $\overline{\phi}_{K}^{\mathsf{T}}\left(x\right) = \left[\phi_{1}\left(x\right), ...\phi_{K}\left(x\right)\right]$ and where K may grow with sample size n.

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  - Balancing bias and variance gives best rate for $k(n)$ in minimax sense.

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- In this vein, given data $\{X_i, Y_i : i = 1, ..., n\}$, we aim to estimate $m^*(x) = E[Y|X = x]$ using LS series estimator defined by:

$$\widehat{m}(x) = \overline{\phi}_K^{\mathsf{T}}(x)\,\widehat{\beta} \text{ where } \widehat{\beta} = \left(\Phi_K^{\mathsf{T}}\Phi_K\right)^{-1}\Phi_k^{\mathsf{T}}\mathbf{Y}$$

where Φ_k is the $n \times k$ matrix $\left[\overline{\phi}_k(X_1), ..., \overline{\phi}_k(X_n)\right]^{\mathsf{T}}$ and $\mathbf{Y} = (Y_1, ..., Y_n)^{\mathsf{T}}$.

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- LS series estimator extensively studied in literature; several papers provide sufficient conditions for consistency, corresponding convergence rates, and asymptotic normality in case of polynomial series, regression splines, fourier series, Wavelets series, local polynomial partition series for example,see Chen (2007), Newey (1997), Györfi et al. (2002), Huang (2003), Cattaneo and Farrell (2013). Belloni et al. (2015).

Nonparametric Regression in Statistics

- These prior works established that under certain conditions on the basis functions $\overline{\phi}_k$,for X d-dimensional and m^* in Holder Space $\mathcal{H}\left(\beta_m, C_m\right)$ with $\lfloor\beta_m\rfloor$ bounded derivatives; setting $K_{opt} = n^{\frac{d}{2\beta_m+d}}$:

$$\|m^* - \widehat{m}_{LS}\|_2 = \sqrt{\int \left(m^*(x) - \widehat{m}_{LS}(x)\right)^2 dF(x)} \lesssim_P n^{-\frac{\beta_m}{2\beta_m+d}}$$

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- Specifically, this is known to hold for local basis functions including Fourier, Wavelets, Splines (Newey, 1997, Huang, 2003, Belloni et al, 2015) and local polynomial partition series, Cattaneo and Farrell (2013) for all $\beta_m, d > 0$.

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- In contrast, for polynomial series only known to hold only for $\beta_m \geq d/2$ (Belloni et al, 2015).

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- These results critically rely on "size" of basis functions, as well as its ability to approximate smooth functions.
- Specifically, above optimality of LS estimator requires that

$$\sup_{x \in \mathcal{X}} ||\overline{\phi}_K(x)|| \lesssim \sqrt{K} \tag{1}$$

and that for any K, there exist a vector β such that

$$\left\| m^* - \overline{\phi}_K^{\mathsf{T}} \beta \right\|_{L_2(\mu)} \lesssim K^{-2\beta_m/d} \tag{2}$$

which essentially limits results to local basis functions.

Forster-Warmuth nonparametric Regression in Statistics

- We have developed a new type of series regression estimator that in principle can attain well-established minimax nonparametric rates of estimation in settings where covariates and outcomes are fully observed, under weaker conditions compared to existing literature (e.g. Belloni et al. (2015)) on the distribution of covariates and basis functions.

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- The approach builds on an estimator we refer to as Forster–Warmuth series estimator (FW-estimator) originating in the online learning literature, which is obtained via a careful modification of the renowned non-linear Vovk–Azoury–Warmuth forecaster (Vovk, 2001; Forster and Warmuth, 2002).

Forster-Warmuth Nonparametric Regression in Statistics

- We define the FW-regression as followed:

$$\widehat{\beta} = \underbrace{(1 - h_n)}_{(a)} \overline{\phi}_K^{\mathsf{T}}(x) \left(\Phi_K^{\mathsf{T}} \Phi_K + \underbrace{\overline{\phi}_K(x) \overline{\phi}_K^{\mathsf{T}}(x)}_{(b)} \right)^{-1} \Phi_k^{\mathsf{T}} \mathbf{Y}$$

where

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- (b) incorporates incorporates a fake observation at x to design matrix, while (a) adjusts for the leverage of this observation if extreme relative to observed basis functions. This effectively controls the size of the projection matrix at the value of interest x regardless of the basis function.

Forster-Warmuth Nonparametric Regression in Statistics

Theorem

(Yang, Kuchibhotla, TT, 2023) Suppose that $E(Y^2|X)$ is bounded almost surely and the basis functions $\overline{\phi}_K^{\mathsf{T}}$ satisfy the optimal approximation property that for each K, there exist a β such that

$$\left\| m^* - \overline{\phi}_K^{\mathsf{T}} \beta \right\|_{L_2(\mu)} \lesssim K^{-2\beta_m/d}$$

then letting $K_{opt} = n^{\frac{d}{2\beta_m + d}}$

$$E\left(m^*(X) - \widehat{m}_{FW}(X)\right)^2 \lesssim n^{-\frac{2\beta_m}{2\beta_m + d}}$$

regardless of the basis functions used.

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- This theorem establishes that the FW-learner is minimax rate optimal (wrt MSE) for any basis system that constitutes a full approximation set in the sense of Lorentz (1966) and Yang and Barron (1999), including polynomial basis for all $\beta_m, d > 0$.

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- This theorem establishes that the FW-learner is minimax rate optimal (wrt MSE) for any basis system that constitutes a full approximation set in the sense of Lorentz (1966) and Yang and Barron (1999), including polynomial basis for all $\beta_m, d > 0$.
- Corrections (a) and (b) in the FW-learner are key to this result and obviate the need to control the size of the basis function. Note also that (a) and (b) become negligible in large samples and $\widehat{m}_{FW}(X) \approx \widehat{m}_{LS}(X)$ essentially does away with the size control.

Nonparametric Counterfactual Nonparametric Regression

- In many practical applications in health and social sciences it is not unusual for an outcome to be missing on some subjects, either by design, say in two-stage sampling studies where the outcome can safely be assumed to be missing at random with known non-response mechanism, or by happenstance, in which case the outcome might be missing not at random.

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- An example of the former type might be a study (Cornelis et al., 2009) in which one aims to develop a polygenic risk prediction model for type-2 diabetes based on stage 1 fully observed covariate data on participants including high dimensional genotype (i.e., SNPs), age, and gender,

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- While costly manual chart review by a panel of physicians yield reliable type-2 diabetes labels on a subset of subjects with known selection probability based on stage-1 covariates.

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- In contrast, an example of the latter type might be a household survey in Zambia (Marden et al., 2018) in which eligible household members are asked to test for HIV, however, nearly 30% decline the test and thus have missing HIV status.

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- In contrast, an example of the latter type might be a household survey in Zambia (Marden et al., 2018) in which eligible household members are asked to test for HIV, however, nearly 30% decline the test and thus have missing HIV status.
- The concern here might be that participants who decline to test might not be a priori exchangeable with participants who agree to test for HIV with respect to key risk factors for HIV infection, even after adjusting for fully observed individual and household characteristics collected in the household survey.

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- The concern here might be that participants who decline to test might not be a priori exchangeable with participants who agree to test for HIV with respect to key risk factors for HIV infection, even after adjusting for fully observed individual and household characteristics collected in the household survey.
- Any effort to build an HIV risk regression model that generalizes to the wider population of Zambia requires carefully accounting for HIV status possibly missing not at random for a non-negligible fraction of the sample.

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- This is because under the consistency assumption the observed outcome for subjects who actually received treatment matches their potential outcome under treatment, while their potential outcome under no treatment is missing, and vice-versa for the untreated.

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  - (ii) has conditional expectation given covariates that matches the counterfactual regression of interest if as for an oracle, nuisance functions were known.

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- The proposed approach recovers in specific cases such as the CATE under unconfoundedness, previously proposed forms of pseudo-outcomes (Kennedy, 2020), while offering new pseudo-outcome constructions in several new, more challenging counterfactual prediction pbs.

Nonparametric Counterfactual Nonparametric Regression

- We describe FW-Counterfactual Regression for estimation of the CATE under unconfoundedness

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- By consistency $Y=AY^{a=1}+\left(1-A\right)Y^{a=0}$, that is $Y^{a=0}$ is missing for treated patients, and $Y^{a=1}$ is missing for untreated patients.
- $m^{*}\left(x\right)$ is central to precision medicine as not only does it quantify effect heterogeneity by x, i.e. the extent to which X predicts a person's causal effect, it also can be used to decide whether or not a patient should be treated, i.e. if $m^{*}\left(x\right)>0$ if larger values of Y are desirable.

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- An oracle with access to $Y^{a=1} - Y^{a=0}$ could in principle perform FW-regression to obtain a minimax rate optimal estimator of the CATE $m^*(x) = E\left(Y^{a=1} - Y^{a=0} | X = x\right)$.

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- Note that in a randomized experiment, where $A \sim$Bernoulli$(\frac{1}{2})$ is independent of $\left(Y^{a=1}, Y^{a=0}, X\right)$, one can match the oracle rate by replacing $Y^{a=1} - Y^{a=0}$ with the oracle pseudo-outcome $H = 2(-1)^{1-A} Y$ and running the FW-regression using H, upon noting that $E(H|X = x) = E\left(Y^{a=1} - Y^{a=0} | X = x\right)$.

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- Neither approach is feasible in observational setting where A is not randomized. Therefore, the most one can hope for is to construct an estimator $\widehat{H}$ of the pseudo-outcome H with "small bias".

Nonparametric Counterfactual Nonparametric Regression

- Constructing a pseudo outcome requires a causal identification condition. The most common assumption is that of no unmeasured confounding given X, That is A is independent of $\left(Y^{a=1}, Y^{a=0}\right)$ conditional on X.

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  - g-computation: $\widehat{H} = \widehat{E}(Y|A=1,X) - \widehat{E}(Y|A=0,X)$ where $\widehat{E}(Y|A=a,X)$ is NP estimator in separate sample
- Unfortunately these natural choices do not lead to small bias, as $E\left(\widehat{H}|X\right) - E(H|X)$ dominated by bias of $\widehat{f}(A|X)$ for IPTW and by $\widehat{E}(Y|A=a,X)$ for g-computation, therefore Oracle minimax rate of the CATE may not be attainable. The next result gives a generic approach to debias any pseudo-outcome.

Nonparametric Counterfactual Nonparametric Regression

Theorem

(Yang, Kuchibhotla, TT, 2023) Let $n^*(x;\eta) = E\left(r\left(\eta\right)|x;\eta\right) = m^*\left(x\right)$ *for an observed data based initial pseudo-outcome* $r\left(\eta\right)$ *identifying a counterfactual regression* $m^*\left(x\right)$ *where* η *is a possibly infinite dimensional nuisance parameter* $\eta \in \mathbb{B}$ *(for a normed metric space* $\mathbb{B}$ *with norm* $\|\cdot\|_2$*), under a semiparametric model* $\mathcal{M}$*. Furthermore, suppose that there exists a function* $R\left(\eta, n^*(\eta)\right)$ *in* L_2 *such that for any regular parametric submodel* η_t *of* $\mathcal{M}$ *with score* S*, we have that*

$$\frac{\partial E\left(r\left(\eta_t\right)|x\right)}{\partial_t} = E\left[R\left(\eta, n^*(\eta)\right) S|X = x\right]$$

then $\left\|E\left\{R\left(\eta', n^*(\eta')\right) + r\left(\eta'\right) - n^*(X;\eta)|X\right\}\right\|_2 = O\left(\left\|\eta' - \eta\right\|_2^2\right)$*. Furthermore,* $R\left(\eta, n^*(\eta)\right) + r\left(\eta\right) - \psi$ *is an influence function of the marginal functional* $\psi = E\left\{n^*(x;\eta)\right\}$ *on* $\mathcal{M}$*.*

Nonparametric Counterfactual Nonparametric Regression

- Theorem 2 formally establishes that a pseudo-outcome for a given counterfactual regression $m^{*}(x)$ can be obtained by effectively deriving an influence function of the marginal functional $\psi = E\{n^{*}(x;\eta)\}$ under a given semiparametric model $\mathcal{M}$.

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- Theorem 2 formally establishes that a pseudo-outcome for a given counterfactual regression $m^*(x)$ can be obtained by effectively deriving an influence function of the marginal functional $\psi = E\{n^*(x;\eta)\}$ under a given semiparametric model $\mathcal{M}$.
- The resulting influence function is given by $R(\eta, n^*(\eta)) + r(\eta) - \psi$ and the oracle pseudo-outcome may appropriately be defined as $R(\eta, n^*(\eta)) + r(\eta)$.

Nonparametric Counterfactual Nonparametric Regression

- Theorem 2 is quite general as it applies to the most comprehensive class of non-parametric counterfactual regressions studied to date. The result thus provides a unified solution to the problem of counterfactual regression, recovering several existing methods, and more importantly, providing a number of new results.

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- Namely, the theorem provides a formal framework for deriving a pseudo-outcome which by construction is guaranteed to satisfy so-called "Neyman Orthogonality" property, i.e. that the bias incurred by estimating nuisance functions is at most of second order (Chernozhukov et al., 2017).

FW-Learner of the CATE under unconfoundedness

- Applying Theorem 2 to the CATE with initial pseudo outcome $r(\eta) = E(Y|A=1,X;\eta) - E(Y|A=0,X;\eta)$ produces the corrected pseudo outcome

$$\begin{aligned} & R(\eta, n^*(\eta)) + r(\eta) \\ = \quad & (-1)^{1-A} Y / f(A|X;\eta) + E(Y|A=1,X;\eta) - E(Y|A=0,X;\eta) \end{aligned}$$

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- The corresponding empirical pseudo-outcome

$$\widehat{H} = (-1)^{1-A} Y / \widehat{f}(A|X) + \widehat{E}(Y|A=1,X) - \widehat{E}(Y|A=0,X)$$

can then be used as an outcome to construct the FW-learner of the CATE.

FW-Learner of the CATE under unconfoundedness

- Note that the above results continue to apply if as often the case we aim to evaluate CATE with respect to a subset of covariate V of X.

$$m^{*}(v) = E\left(Y^{a=1} - Y^{a=0} | V = v\right)$$

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- As confounding adjustment requires accounting for all of X, the empirical pseudo-outcome remains

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however the FW-learner regresses $\widehat{H}$ on basis functions $\overline{\phi}_K(v)$.

Simulation study

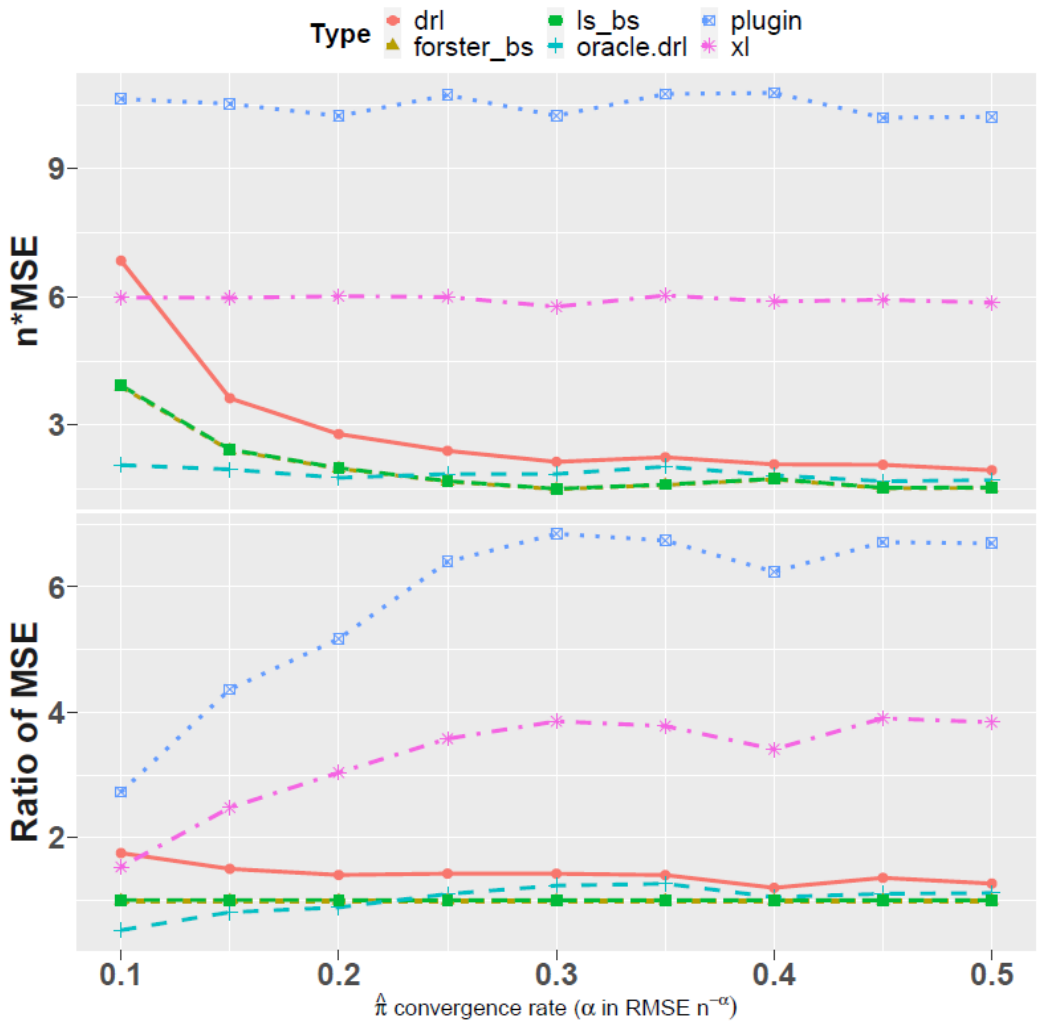

Figure 2: A comparison between different estimators, sample size $n = 2000$—-Top figure shows $n \times$ MSE of each estimator; The bottom plot shows the ratio of MSE of different estimators compared to the proposed Forster–Warmuth estimator with basic splines (baseline). The MSE is averaged over 500 simulations.

Simulation Study

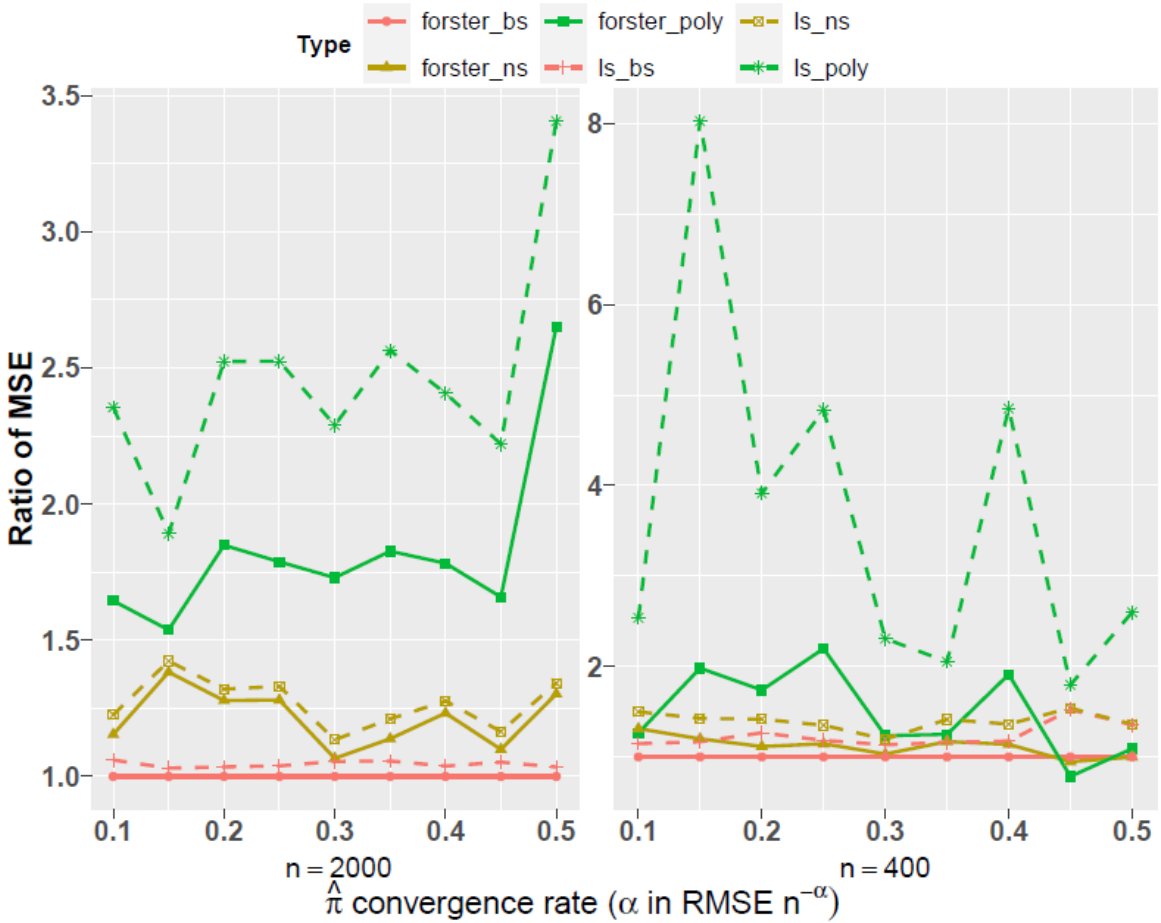

Figure 3: A comparison between FW and LS estimators with different basis for X with heavy-tailed distribution, baseline method is the FW-Learner with basic splines (FW_bs); Left: sample size $n = 2000$; Right: $n = 400$. The MSE is averaged over 500 simulations.

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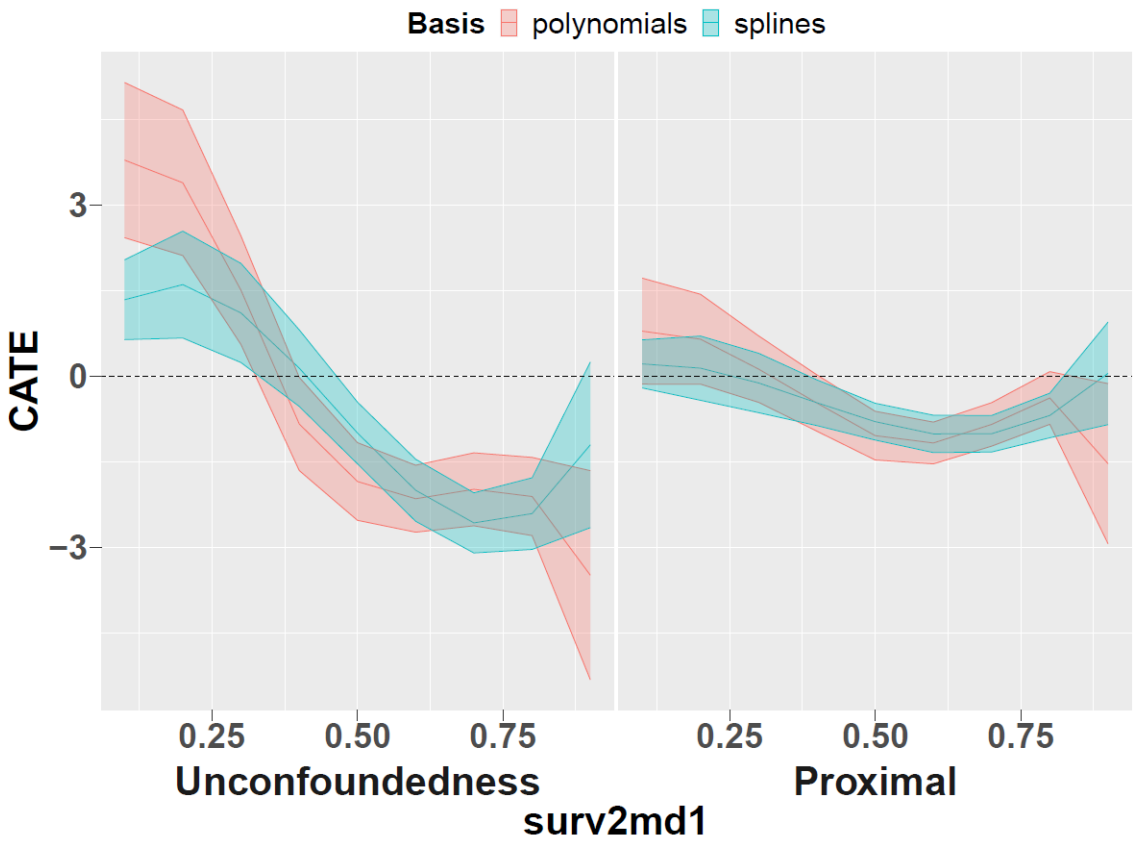
- RHC was performed in 2184 patients within the initial 24 hours of ICU stay, while 3551 patients were managed without RHC
- Study collected rich patient information encoded in 73 covariates, including demographics (such as age, sex, race, education, income, and insurance status), baseline estimated probability of survival, comorbidity, vital signs, physiological status, and functional status. The outcome of interest is the number of days between admission and death or censoring at 30 days (Y).

RHC Application

- We aim to estimate the CATE conditional only on V =baseline estimated probability of survival.



Conclusion

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  - Conditional Quantile Treatment Effect:

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  - Conditional Causal Effect for GLMs:

$$\begin{aligned} m^*(x) &= g^{-1}\left\{E\left(Y|A=1,X\right)\right\} - g^{-1}\left\{E\left(Y|A=1,X\right)\right\} \\ H &= \frac{(-1)^{1-A}\left[Y - E\left(Y|A,X\right)\right]}{f\left(A|X\right) g'\left(g^{-1}\left\{E\left(Y|A,X\right)\right\}\right)} \\ &\quad + F^{-1}_{Y_{a=1|x}}\left(q|x\right) - F^{-1}_{Y_{a=0|x}}\left(q|x\right) \end{aligned}$$

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  - Nonparametric regression under MAR and MNAR conditions using a shadow variable for the latter.

Acknowledgments

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- Yang, Y., Kuchibhotla, A.K. and Tchetgen, E.T., 2023. Forster-Warmuth Counterfactual Regression: A Unified Learning Approach. arXiv preprint arXiv:2307.16798.