Online Machine Learning Homework

Assignment 4

Student: Lei Feng, `please-help@each.other`
Student ID: 123456
Lecturer:

Problem 1: Programming: Adaptive Learning Rates

Recall in programming HW#1, part 2(c), you implemented the OGD algorithm with a constant learning rate η and used it to train a linear support-vector machine on a small spam-classification task. Now you will solve the same problem, but using adaptive per- coordinate learning rates. In particular, the update will be computed separately for each coordinate $i \in \{1, 2, \dots\}$ based on the rule

$$w_{t+1,i} = w_{t,i} - \eta_{t,i} g_{t,i} \tag{1}$$

where the learning rates have the form

$$\eta_{t,i} = \frac{\alpha}{\sqrt{1 + \sum_{s=1}^{t} g_{s,i}^2}}$$

Here α is a parameter you will choose, and $g_{s,i} \in \mathbb{R}$ is the ith coordinate of the $g_s \in \partial f_s(w_s)$, a subgradient of the sth loss function at w_s. In addition to your code, you will produce a plot showing the average per-round loss as a function of t for $t = 1, \dots, 4601$, with three lines corresponding to $\alpha \in \{0.2\alpha_0, \alpha_0, 5.0\alpha_0\}$ with $\alpha_0 = 7.2$ We have chosen these values so that $\alpha = \alpha_0$ should produce the lowest average per-round loss on the final round; since both a somewhat lower and higher value of α produce worse loss, this is a good indication we have done a good job picking α. For a real application, you would want to try a larger range of αs, and plot the final cumulative loss as a function of α — you should see a nice, U-shaped curve. We did this in order to choose the value α_0, see Figure 1.

For comparison, again solve the problem with fixed learning-rate OGD, where the update is just

$$w_{t+1} = w_t - \eta g_t.$$

Plot three lines for constant-learning rate OGD for $\eta \in \{0.2\eta_0, \eta_0, 5.O\eta_0\}$ with $\eta_0 = 0.22$.

Recall that the loss function for a linear SVM is the hinge loss, defined as

$$f_t(w) = \max\{0, 1 - y_t w^T x_t\}$$

where $x_t, w_t \in \mathbb{R}^d$ and $y_t \in \{-1, +1\}$. Note that while we can view OGD as FTRL on linearized loss functions $\hat{f}_t(w) = g_t \cdot w$ for $g_t \in \partial f_t(w_t)$ (which drops constant terms), when computing the average per-round loss, it is critical you use the *original* true loss functions f_t, not the linearized functions $\hat{f}_t$. (You should think about why this is the case, but you do **not** need to write up your answer.)

Comment: In order for regret bounds of the form $BG\sqrt{T}$ to hold, where the L_2 norm of the post-hoc comparator u is less than B, technically we should use the update that first applies the per-coordinate gradient update of Eq.1, and then *projects* that point into the feasible set $\mathcal{W}$ (usually an L_∞ ball when using per-coordinate rates). However, in practice

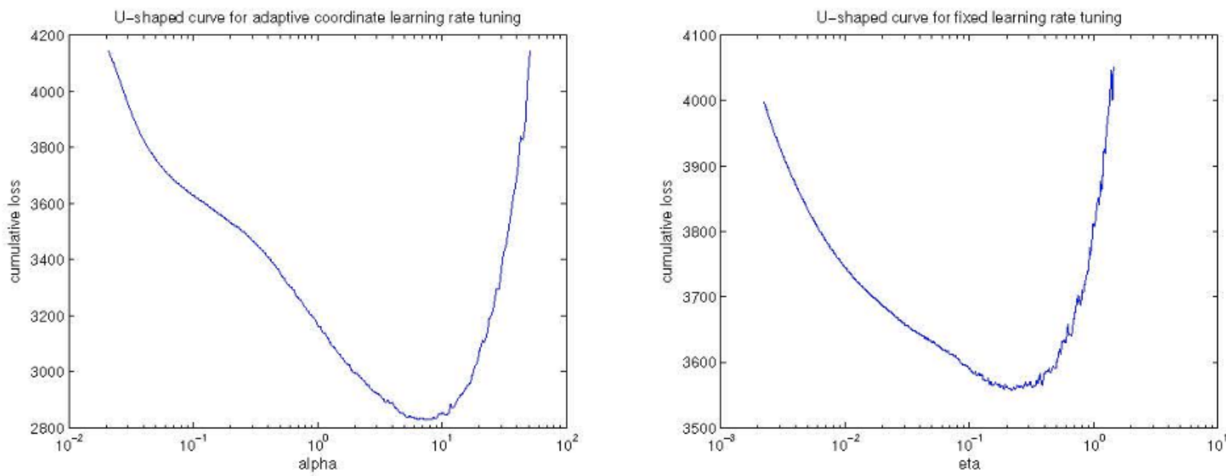

Figure 1: Learning-rate tuning plots. The left plot has α plotted on a log-scale, and the right plot has n plotted on a log scale.

this is often unnecessary, and requires tuning an extra parameter (the radius of the feasible set), and so we will not implement this here.

Problem 2: Theory: Adaptive Regret Bounds for Strongly Convex Functions

Recall we proved the following theorem, using the Strong FTRL Lemma and some results from convexity theory:

Theorem 1. *Consider the FTRL algorithm that plays according to*

$$w_{t+1} = arg\min_{w} f_{1:t}(w) + r_{0:t}(w) \tag{2}$$

where the proximal regularizers $r_t(w) \geq 0$ for $t \in \{0, 1, \ldots, T\}$, and $r_t(w_t) = 0$, and the functions $f_t : R^d \to R$ are convex. Let $h_0 = r_0$, and $h_t = r_t + f_t$ for $t \geq 1$. Then, further suppose the r_t are chosen such that $h_{0:t}$ is 1-strongly-convex w.r.t. some norm $\| \cdot \|_{(t)}$ for $w \in dom r_{0:t}$. Then, choosing any $g_t \in \partial f_t(w_t)$ on each round, for any $u \in \mathbb{R}^d$,

$$Regret(u) \leq r_{0:T}(u) - \sum_{t=1}^{T} \|g_t\|^2_{(t),*}. \tag{3}$$

We will use this theorem to prove a regret bound for the Follow-The-Leader algorithm on strongly-convex functions, which plays

$$w_{t+1} = arg\min_{w} f_{1:t}(w). \tag{4}$$

Suppose each f_t is 1-strongly convex w.r.t a fixed norm $\| \cdot \|$, and let $G_T = \max_{t \in \{1,\ldots,T\}} \|g_t\|_*$. (Typically in order to provide such a guarantee on the g_t in advance, we would have to constrain $w_t \in W$ for some bounded feasible set, but we won't worry about that for this problem.) You will prove the regret bound

$$Regret(u) \leq G_T^2(1 + logT).$$

which holds simultaneously for all T:

a) Define regularizers such that the update of Eq.4 is equal to that of Eq.2 (this is trivial).

b) Prove that $\|w\|_{(t)} = \sqrt{t}\|w\|$ can be used in Theorem 1, and further that $\|g\|_{(t),*} = \frac{1}{\sqrt{t}}\|g\|_*$. Prove the first fact from the definition of strong convexity, and the second from the definition of the dual norm (see the lecture 5 notes for both definitions). You don't need to prove that $\|w\|_{(t)}$ is actually a norm (though you might want to check this for yourself).

c) Plug the definition of r_t and $\|\cdot\|_{(t),*}$ into Eq.3, and simplify using the definition of G_T, and the fact that $\sum_{t=1}^{T} \frac{1}{t} \leq 1 + \log T$.

Observe that this $\log T$ regret bound is significantly better than the $\sqrt{T}$ bounds achievable for general convex functions. The key is that the strongly-convex functions are essentially self-regularizing.

(a) Set $r_t(w) = 0 \quad$ for all $t \geq 1$, thus reducing FTRL update to the FTL update:

$$w_{t+1} = \arg\min_{w} f_{1:t}(w).$$

(b) **1.** Prove that $\|w\|_{(t)} = \sqrt{t}\|w\|$ can be used in Theorem 1, i.e., that $\|w\|_{(t)}$ satisfies the strong convexity condition.

Proof. From the definition of strong convexity, $h_{0:t}$ is 1-strongly convex with respect to $\|\cdot\|_{(t)}$ if for all $w, u \in \mathrm{dom}(h_{0:t})$, i.e.,

$$h_{0:t}(w) \geq h_{0:t}(u) + \nabla h_{0:t}(u)^\top (w-u) + \frac{1}{2}\|w-u\|_{(t)}^2.$$

Since each f_t is 1-strongly convex with respect to $\|\cdot\|$, their sum $f_{1:t}$ is t-strongly convex with respect to $\|\cdot\|$. Scaling $\|\cdot\|$ by $\sqrt{t}$ gives:

$$\|w-u\|_{(t)}^2 = t\|w-u\|^2.$$

Thus, $\|w\|_{(t)} = \sqrt{t}\|w\|$ satisfies the strong convexity condition. □

2. Prove that $\|g\|_{(t),*} = \frac{1}{\sqrt{t}}\|g\|_*$.

Proof. $\|g\|_{(t),*} = \frac{1}{\sqrt{t}}\|g\|_*$: The dual norm $\|\cdot\|_*$ is defined as:

$$\|g\|_* = \sup_{\|w\| \leq 1} g^\top w.$$

For the scaled norm $\|\cdot\|_{(t)} = \sqrt{t}\|\cdot\|$, the corresponding dual norm is:

$$\|g\|_{(t),*} = \sup_{\|w\|_{(t)} \leq 1} g^\top w = \sup_{\sqrt{t}\|w\| \leq 1} g^\top w = \frac{1}{\sqrt{t}} \sup_{\|w\| \leq 1} g^\top w = \frac{1}{\sqrt{t}}\|g\|_*.$$

□

(c) Start from the regret bound in Theorem 1:

$$\text{Regret}(u) \le r_{0:T}(u) - \sum_{t=1}^{T} \|g_t\|_{(t),*}^2.$$

First, for FTL, $r_t(w) = 0$ for $t \ge 1$, so $r_{0:T}(u) = r_0(u)$.

Then, using $\|g_t\|_{(t),*} = \frac{1}{\sqrt{t}}\|g_t\|_*$, we have:

$$\|g_t\|_{(t),*}^2 = \frac{1}{t}\|g_t\|_*^2.$$

Thus:

$$\sum_{t=1}^{T} \|g_t\|_{(t),*}^2 = \sum_{t=1}^{T} \frac{1}{t}\|g_t\|_*^2 \le G_T^2 \sum_{t=1}^{T} \frac{1}{t},$$

where $G_T = \max_{t \in \{1,\ldots,T\}} \|g_t\|_*$.

Additionally, using the inequality $\sum_{t=1}^{T} \frac{1}{t} \le 1 + \log T$, we get:

$$\sum_{t=1}^{T} \|g_t\|_{(t),*}^2 \le G_T^2(1 + \log T).$$

Now we simplify the regret bound in Theorem.1,

$$\text{Regret}(u) \le r_{0:T}(u) - \sum_{t=1}^{T} \|g_t\|_{(t),*}^2 \le r_0(u) + G_T^2(1 + \log T).$$