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Matrix Weights, Maximal Operators, Calderón–Zygmund Operators, and Besov–Triebel–Lizorkin-Type Spaces — A Survey

Fan Bu, Dachun Yang*, Wen Yuan and Yuze Zhao

Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China

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Dedicated to Professor Donggao Deng for His 90th Birth Anniversary

Abstract. The primary purpose of this survey is threefold. First, the authors recall some histories and present some recent developments of matrix weights, in which the authors not only improve some known results on the intrinsic properties of matrix weights, but also establish some new ones. Then the authors summarize matrix-weighted inequalities associated with various operators, such as Hardy–Littlewood-type maximal operators and Calderón–Zygmund operators. Finally, the authors overview matrix-weighted function spaces, including matrix-weighted Sobolev, BMO, and Besov–Triebel–Lizorkin-type spaces. Several open questions on these subjects are also presented.

Key Words: Sobolev space, BMO, Besov–Triebel–Lizorkin-type space, matrix weight, Hardy–Littlewood maximal operator, Calderón–Zygmund operator.

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1 Introduction

Throughout the whole article, we mainly work in $\mathbb{R}^n$ and, unless necessary, we will not explicitly specify this underlying space.

The study of weighted norm inequalities in harmonic analysis began with the work of Muckenhoupt [112], who introduced the scalar A_p weights as a necessary and sufficient condition for the boundedness of the Hardy–Littlewood maximal operator on weighted Lebesgue spaces. This foundational result was soon extended by Hunt et al. [78] to the Hilbert transform. Coifman and Fefferman [40] later simplified Muckenhoupt's arguments and introduced the concept of A_∞ weights, along with reverse Hölder's inequality. Several equivalent characterizations of the A_∞ condition have since been established; see, for example, [57, 69, 96, 113, 134].

*Corresponding author. *Email addresses:* fanbu@mail.bnu.edu.cn (F. Bu), dcyang@bnu.edu.cn (D. Yang), wenyuan@bnu.edu.cn (W. Yuan), yuzezhao@mail.bnu.edu.cn (Y. Zhao)

Over the past three decades, increasing attention has been devoted to quantitative weighted inequalities. Buckley [31] established the sharp bound for the Hardy–Littlewood maximal operator and provided a quantitative weighted inequality for Calderón–Zygmund operators. However, the sharp bound for the latter remained an open problem at that time. This problem, known as the A_2 conjecture, was completely solved by Hytönen [80], and one of its important applications can be found in [4]. A simpler proof of the A_2 conjecture was later given by Lerner [101]. Several other quantitative weighted inequalities have been studied; see, for example, [83, 98, 99, 104, 105].

To study the prediction theory of multivariate stochastic processes, Wiener and Masani [162, Section 4] introduced the matrix-weighted Lebesgue space $\mathscr{L}^2(W)$ over $[0,2\pi]$. For any $p\in(0,\infty)$ and any matrix weight W on $\mathbb{R}^n$, the *matrix-weighted Lebesgue space* $\mathscr{L}^p(W)$ on $\mathbb{R}^n$ is defined to be the set of all measurable vector-valued functions $\vec{f}\colon\mathbb{R}^n\to\mathbb{C}^m$ such that

$$\left\|\vec{f}\right\|_{\mathscr{L}^p(W)}:=\left[\int_{\mathbb{R}^n}\left|W^{\frac{1}{p}}(x)\vec{f}(x)\right|^p\,dx\right]^{\frac{1}{p}}<\infty.$$

Motivated by problems arising from both the multivariate random stationary process and the Toeplitz operator, Treil and Volberg [148] introduced matrix $\mathscr{A}_2$ weights on $\mathbb{R}$ and proved that

$$\text{the Hilbert transform is bounded on } \mathscr{L}^2(W) \text{ over } \mathbb{R} \text{ if and only if } W\in\mathscr{A}_2. \tag{1.1}$$

Subsequently, Nazarov and Treil [118] and Volberg [155] independently introduced matrix $\mathscr{A}_p$ weights on $\mathbb{R}$ and extended (1.1) to the full range $p\in(1,\infty)$ via different methods. The underlying space in both works is still $\mathbb{R}$. Since then, the study of $\mathscr{L}^p(W)$ on $\mathbb{R}^n$ has attracted increasing attention. For further studies on the properties of matrix weights, see, for example, [18, 100, 155, 156].

The matrix analogue of the A_∞ condition has been relatively scarce, with notable references including [27, 52, 87, 88, 155]. A central difficulty in this area comes from the lack of a universally accepted definition that adequately characterizes this class in the matrix setting. A prominent advancement in this direction is the introduction of $\mathscr{A}_{p,\infty}$-matrix weights on $\mathbb{R}$ by Volberg [155], which can be interpreted as a refined matrix analogue of the scalar A_∞ weights. A comprehensive investigation of $\mathscr{A}_{p,\infty}$-matrix weights on $\mathbb{R}^n$ was carried out by Bu et al. [27], where they also introduced dimensions of weights and presented a detailed analysis of associated properties, such as equivalent characterizations, the reverse Hölder's inequality, and the self-improvement property.

The study of matrix-weighted inequalities is significantly more challenging than that in the scalar setting. The matrix-weighted maximal operator was introduced by Christ and Goldberg (also called the Christ–Goldberg maximal operator) in [38] and its boundedness on $\mathscr{L}^p(W)$ was later showed by Goldberg in [67]. The sharp quantitative matrix-weighted inequality for the fractional matrix-weighted maximal operator was proved by Isralowitz and Moen [90]. In contrast, the situation is markedly different for Calderón–Zygmund operators. The quantitative matrix-weighted inequality associated with Calderón–Zygmund operators was first considered by Bickel et al. [13]. Later, Nazarov et al. [117] employed the convex body-valued sparse domination to obtain the best improved estimate, that is, for any $W\in\mathscr{A}_2$ and any Calderón–Zygmund operator T,

$$\|T\|_{\mathscr{L}^2(W)\to\mathscr{L}^2(W)}\lesssim[W]_{\mathscr{A}_2}^{\frac{3}{2}}, \tag{1.2}$$

where $\|T\|_{\mathcal{L}^2(W)\to\mathcal{L}^2(W)}:=\sup_{\|\vec{f}\|_{\mathcal{L}^2(W)}=1}\|T\vec{f}\|_{\mathcal{L}^2(W)}$ and the implicit positive constant is independent of T and W. Recently, Domelevo et al. [55] further demonstrated that the matrix $\mathscr{A}_2$ conjecture does not hold as in the scalar case by proving that (1.2) is sharp, completing the study of this problem. This is quite surprising and indicates that the extension from the scalar setting to the matrix setting is not just simply enhancing the dimension of function spaces. For the case $p\neq 2$, Cruz-Uribe et al. [42] derived a general constant involving $[W]_{\mathscr{A}_p}$. These results underscore the inherent complexities of matrix-weighted settings and highlight the difference between matrix and scalar settings.

In [63, 138], Frazier and Roudenko investigated the properties of matrix-weighted Besov spaces. The duality of matrix-weighted Besov spaces was further explored in [139]. Based on these foundations, Frazier and Roudenko in [65] studied matrix-weighted Triebel–Lizorkin spaces and established their φ-transform characterizations; they also proved the equivalence $\dot{F}^0_{p,2}(W)=\mathcal{L}^p(W)$, where $p\in(1,\infty)$ and $W\in\mathscr{A}_p$. These contributions highlight the significance of matrix-weighted function spaces in advancing the understanding of harmonic analysis in vector-valued settings. A series of work by Bu et al. [24–28] extended the results of Frazier and Roudenko to Besov–Triebel–Lizorkin-type spaces; moreover, they introduced the concept of the dimensions of weights and applied it to investigate the real-variable characterizations of Besov–Triebel–Lizorkin-type spaces, such as φ-transform characterizations, various decomposition characterizations, and the boundedness of almost diagonal operators et al. Recently, Bu. et al. [30, 167] further extended the results of Bu et al. [28] to matrix-weighted Besov–Triebel–Lizorkin spaces of optimal scale. Furthermore, a critical advancement in the study of matrix-weighted function spaces involves their extension to infinite-dimensional Banach spaces (see, for example, [2, 99]).

This comprehensive survey is designed to fulfill three primary objectives. First, we recall some histories and present some recent developments of matrix weights, including their structural properties and some nontrivial examples of matrix Muckenhoupt weights, in which we not only improve some known results, but also establish some new ones. Then we summarize matrix-weighted inequalities associated with various operators, such as the Hardy–Littlewood maximal operator and its variants, Calderón–Zygmund operators, and fractional integral operators. Finally, we overview matrix-weighted function spaces, including matrix-weighted Sobolev spaces, matrix-weighted BMO spaces, and matrix-weighted Besov–Triebel–Lizorkin-type spaces. Additionally, throughout the article, we present several open questions.

To be precise, the remainder of this article is organized as follows.

In Section 2, we begin with showing examples that illustrate the intrinsic properties of Muckenhoupt scalar and matrix weights. We also present the equivalent characterization of A_∞ weights and prove some new corresponding properties for $\mathscr{A}_{p,\infty}$-matrix weights. We then revisit the fundamental properties of scalar Muckenhoupt weights and their matrix counterparts. Additionally, we recall the concept of the dimension of weights and establish the relation between the dimension and the sharp indices of weights.

In Section 3, we review the recent development of weighted inequalities associated with the Hardy–Littlewood-type maximal operators and the Calderón–Zygmund operators. We recall their progress in both scalar and matrix settings in order and present the best results so far. We further recall the boundedness of other operators in both scalar and matrix settings, such as fractional integral operators and fractional maximal operators.

In Section 4, we present an overview of matrix-weighted function spaces, including Sobolev spaces, bounded mean oscillation (BMO) spaces, and Besov–Triebel–Lizorkin-type spaces. We also recall some famous theorems in the matrix setting, such as Poincaré's inequality, the boundedness of commutators in matrix-weighted BMO spaces, and a series of real-variable characterizations of matrix-weighted Besov-type and Triebel–Lizorkin-type spaces. Besides, we conclude with remarks on very recent work on Bourgain–Morrey spaces and modulation spaces. Additionally, we review some new work on the extension from matrix weights to Banach-valued weights.

At the end of this section, we make some conventions on symbols. The *ball* B of $\mathbb{R}^n$, centered at $x\in\mathbb{R}^n$ with radius $r\in(0,\infty)$, is defined by setting

$$B:=\{y\in\mathbb{R}^n:\ |x-y|<r\}=:B(x,r);$$

moreover, for any $\lambda\in(0,\infty)$, $\lambda B:=B(x,\lambda r)$. Also, a *cube* Q of $\mathbb{R}^n$ is not required to be open or closed but its edges are parallel to coordinate axes. For any cube Q of $\mathbb{R}^n$, let c_Q be its center and $\ell(Q)$ its edge length. For any $\lambda\in(0,\infty)$ and any cube Q of $\mathbb{R}^n$, let λQ be the cube with the same center of Q and the edge length $\lambda\ell(Q)$. In addition, for any $s\in\mathbb{R}$, we use $t\to s^+$ (resp. $t\to s^-$) to denote that there exists some $c\in(s,\infty)$ [resp. $c\in(-\infty,s)$] such that $t\in(s,c)$ [resp. $t\in(c,s)$] and $t\to s$. For any $r\in\mathbb{R}$, r_+ is defined as $r_+:=\max\{0,r\}$ and r_- is defined as $r_-:=\max\{0,-r\}$. For any $t\in(0,\infty)$, $\log_+ t:=\max\{0,\log t\}$. For any $a,b\in\mathbb{R}$, $a\wedge b:=\min\{a,b\}$ and $a\vee b:=\max\{a,b\}$. The symbol C denotes a positive constant which is independent of the main parameters involved, but may vary from line to line. The symbol $A\lesssim B$ or $B\gtrsim A$ means $A\le CB$. If both $A\lesssim B$ and $B\lesssim A$, we write $A\sim B$. For any $p\in(1,\infty)$, let $1=\frac{1}{p}+\frac{1}{p'}$. Let $\mathbb{N}:=\{1,2,\ldots\}$, $\mathbb{Z}_+:=\mathbb{N}\cup\{0\}$, and $\mathbb{Z}_+^n:=(\mathbb{Z}_+)^n$. We use $\mathbf{0}$ to denote the *origin* of $\mathbb{R}^n$. For any set $E\subset\mathbb{R}^n$, we use $\mathbf{1}_E$ to denote its *characteristic function*. Let $\mathscr{M}$ denote the set of all measurable functions on $\mathbb{R}^n$. For any $p\in(0,\infty]$, the *Lebesgue space* L^p has the usual meaning, and the *local Lebesgue space* L^p_{loc} is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that, for any bounded measurable set E, $\|f\|_{L^p(E)}:=\|f\mathbf{1}_E\|_{L^p}<\infty$. For any measurable function w on $\mathbb{R}^n$ and any measurable set $E\subset\mathbb{R}^n$, let $w(E):=\int_E w(x)\,dx$ and, for any measurable set $E\subset\mathbb{R}^n$ with $|E|\in(0,\infty)$, let

$$w_E:=\fint_E w(x)\,dx:=\frac{1}{|E|}\int_E w(x)\,dx.$$

The *Hardy–Littlewood maximal operator* M is defined by setting, for any $f\in L^1_{\mathrm{loc}}$ and $x\in\mathbb{R}^n$,

$$M(f)(x):=\sup_{\mathrm{ball}\,B\ni x}\fint_B|f(y)|\,dy.$$

Finally, in all proofs, we consistently retain the symbols introduced in the original theorem (or related statement).

2 Properties of Scalar and Matrix Muckenhoupt Weights

A function on $\mathbb{R}^n$ is called a *scalar weight* if it is positive almost everywhere and locally integrable. Now, we recall the concept of Muckenhoupt weights (see, for instance, [69, Definitions 7.1.1, 7.1.3, and 7.3.1]).

Definition 2.1. Let $p\in[1,\infty]$. A scalar weight w is called an A_p*-scalar weight* if w satisfies that, when $p=1$,

$$[w]_{A_1}:=\sup_{\text{cube}\,Q}\fint_Q w(x)\,dx\left\|w^{-1}\right\|_{L^\infty(Q)}<\infty$$

or that, when $p\in(1,\infty)$,

$$[w]_{A_p}:=\sup_{\text{cube}\,Q}\fint_Q w(x)\,dx\left\{\fint_Q [w(x)]^{-\frac{p'}{p}}\,dx\right\}^{\frac{p}{p'}}<\infty$$

or that, when $p=\infty$,

$$[w]_{A_\infty}:=\sup_{\text{cube}\,Q}\fint_Q w(x)\,dx\,\exp\left(\fint_Q \log\left([w(x)]^{-1}\right)dx\right)<\infty.$$

Remark 2.1. The definition of A_p-scalar weights in Definition 2.1 can be equivalently given with cube $Q\subset\mathbb{R}^n$ therein replaced by ball $B\subset\mathbb{R}^n$.

There is another commonly used equivalent definition of A_1 weights (see, for instance, [70, Defintion 8.2.5]).

Proposition 2.1. Let w be a scalar weight. Then $w\in A_1$ if and only if there exists a positive constant C such that, for almost every $x\in\mathbb{R}^n$, $M(w)(x)\le Cw(x)$.

Now, to better understand their intrinsic connections among A_1, $\{A_p\}_{p\in(1,\infty)}$, and A_∞, we recall the following well-known results (see, for instance, [60, p. 187]).

Proposition 2.2. Let Q be a cube in $\mathbb{R}^n$. Then the following assertions hold.

(i) If $f\in L^r(Q)\cap L^\infty(Q)$ for some $r\in(0,\infty)$, then, for any $q\in(r,\infty)$, one has $f\in L^q(Q)$ and

$$\lim_{q\to\infty}\left[\fint_Q |f(x)|^q\,dx\right]^{\frac{1}{q}}=\|f\|_{L^\infty(Q)}.$$

(ii) If $f\in L^r(Q)$ for some $r\in(0,\infty)$, then, for any $q\in(0,r)$, one has $f\in L^q(Q)$ and

$$\lim_{q\to0^+}\left[\fint_Q |f(x)|^q\,dx\right]^{\frac{1}{q}}=\exp\left(\fint_Q \log|f(x)|\,dx\right).$$

The following proposition is a discrete version of Proposition 2.2, which helps us understand the latter one.

Proposition 2.3. Let $N\in\mathbb{N}$ and the sequence $\{a_i\}_{i=1}^N$ be in $(0,\infty)$. Then the following hold.

(i)

$$\lim_{q\to\infty}\left(\frac{1}{N}\sum_{i=1}^N a_i^q\right)^{\frac{1}{q}}=\max_{i\in\{1,\dots,N\}}a_i. \tag{2.1}$$

(ii)

$$\lim_{q\to0^+}\left(\frac{1}{N}\sum_{i=1}^N a_i^q\right)^{\frac{1}{q}}=\left(\prod_{i=1}^N a_i\right)^{\frac{1}{N}}=\exp\left(\frac{1}{N}\sum_{i=1}^N \log a_i\right). \tag{2.2}$$

Although (2.1) obviously holds, (2.2) is a well-known non-trivial conclusion (see, for instance, [35, p.176]), which shows that the limit of the power mean is the geometric mean. Moreover, observe that, in (2.2), if we take $N=2$, $a_1=1$, and $a_2=e$, then this inequality reduces to

$$\lim_{q\to0^+}\left(\frac{1+e^q}{2}\right)^{\frac{1}{q}}=e^{\frac{1}{2}}. \tag{2.3}$$

We can prove (2.3) as follows. Indeed,

$$\begin{aligned}\lim_{q\to0^+}\left(\frac{1+e^q}{2}\right)^{\frac{1}{q}}&=\lim_{s\to\infty}\left(\frac{1+e^{\frac{1}{s}}}{2}\right)^s=\lim_{s\to\infty}\left(1+\frac{e^{\frac{1}{s}}-1}{2}\right)^s\\&=\lim_{s\to\infty}\left(1+\frac{e^{\frac{1}{s}}-1}{2}\right)^{\frac{2}{e^{\frac{1}{s}}-1}\frac{e^{\frac{1}{s}}-1}{\frac{1}{s}}\frac{1}{2}}=e^{\frac{1}{2}},\end{aligned}$$

where, in the last step, we used Euler's limit

$$\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t=e. \tag{2.4}$$

Thus, (2.3) holds. This explains that the underlying reason for the presence of the exponential function in Proposition 2.2(ii) is Euler's limit (2.4).

To be more precise, applying Proposition 2.2 with f replaced by w^{-1}, we obtain, for any cube Q in $\mathbb{R}^n$,

$$\lim_{p\to1^+}\left\{\fint_Q[w(x)]^{-\frac{p'}{p}}\,dx\right\}^{\frac{p}{p'}}=\lim_{q\to\infty}\left\{\fint_Q[w(x)]^{-q}dx\right\}^{\frac{1}{q}}=\left\|w^{-1}\right\|_{L^\infty(Q)}$$

and

$$\lim_{p\to\infty}\left\{\fint_Q [w(x)]^{-\frac{p'}{p}}\,dx\right\}^{\frac{p}{p'}}=\lim_{q\to 0^+}\left\{\fint_Q [w(x)]^{-q}\,dx\right\}^{\frac{1}{q}}=\exp\left(\fint_Q \log\left([w(x)]^{-1}\right)dx\right).$$

These observations imply the intrinsic connections among A_1, $\{A_p\}_{p\in(1,\infty)}$, and A_∞. Indeed, Grafakos [69, Proposition 7.1.5(7)] and Sbordone and Wik [145, Theorem 1] proved that, for any scalar weight w,

$$\lim_{p\to 1^+}[w]_{A_p}=[w]_{A_1}\quad\text{and}\quad \lim_{p\to\infty}[w]_{A_p}=[w]_{A_\infty}.$$

Next, we recall the concept of reverse Hölder classes (see, for instance, [45, pp. 2941–2942]).

Definition 2.2. Let $q\in(1,\infty]$ and w be a scalar weight. We say that w satisfies the *reverse Hölder condition*, denoted by RH_q, if w satisfies that, when $q\in(1,\infty)$,

$$[w]_{RH_q}:=\sup_{\mathrm{cube}\,Q}\left[\fint_Q w(x)\,dx\right]^{-1}\left\{\fint_Q [w(x)]^q\,dx\right\}^{\frac{1}{q}}<\infty$$

or that, when $q=\infty$,

$$[w]_{RH_\infty}:=\sup_{\mathrm{cube}\,Q}\left[\fint_Q w(x)\,dx\right]^{-1}\|w\|_{L^\infty(Q)}<\infty.$$

Remark 2.2. It is well known that

$$A_\infty=\bigcup_{p\in[1,\infty)}A_p=\bigcup_{q\in(1,\infty]}RH_q$$

(see, for instance, [40, Theorem V and Lemmas 3 and 5]).

In the 1990s, motivated by problems related to multivariate random stationary processes and Toeplitz operators, a series work by Nazarov et al. [118, 148, 155] extended the theory of scalar Muckenhoupt weights to matrix Muckenhoupt weights.

Before recalling the concept of matrix Muckenhoupt weights, we first recall some concepts about matrices. For any $m,n\in\mathbb{N}$, let $M_{m,n}(\mathbb{C})$ be the set of all $m\times n$ complex matrices and $M_m(\mathbb{C}):=M_{m,m}(\mathbb{C})$. The zero matrix in $M_{m,n}(\mathbb{C})$ is denoted by $O_{m,n}$ and $O_{m,m}$ is simply denoted by O_m. For any $A\in M_m(\mathbb{C})$, let

$$\|A\|:=\sup_{\vec{z}\in\mathbb{C}^m,|\vec{z}|=1}|A\vec{z}|. \tag{2.5}$$

Let A^* denote the *conjugate transpose* of A. A matrix $A\in M_m(\mathbb{C})$ is said to be *positive definite* if, for any $\vec{z}\in\mathbb{C}^m\setminus\{\vec{0}\}$, $\vec{z}^*A\vec{z}>0$, and A is said to be *nonnegative definite* if, for any $\vec{z}\in\mathbb{C}^m$, $\vec{z}^*A\vec{z}\geq 0$. The matrix $A\in M_m(\mathbb{C})$ is called a *unitary matrix* if $A^*A=I_m$, where I_m is the identity matrix.

If $A\in M_m(\mathbb{C})$ is a positive definite matrix, then A has positive eigenvalues $\{\lambda_i\}_{i=1}^m$. Using [77, Theorem 2.5.6(c)], we find that there exists an unitary matrix $U\in M_m(\mathbb{C})$ such that

$$A=U\operatorname{diag}(\lambda_1,\ldots,\lambda_m)U^*$$

and hence, for any $\alpha\in\mathbb{R}$, we define

$$A^\alpha:=U\operatorname{diag}(\lambda_1^\alpha,\ldots,\lambda_m^\alpha)U^*;$$

see [76, (6.2.1)] or [74, Definition 1.2]. From [76, p. 408], we deduce that A^α is independent of the choices of the order of $\{\lambda_i\}_{i=1}^m$ and U, and hence A^α is well-defined. Now, we present the following result, which is precisely [136, Lemma 2.3.5].

Lemma 2.1. *Let $W\colon \mathbb{R}^n\to M_m(\mathbb{C})$ satisfy that all entries of W are measurable and, for any $x\in\mathbb{R}^n$, $W(x)$ is Hermitian. Then there exists a matrix-valued function $U\colon \mathbb{R}^n\to M_m(\mathbb{C})$ whose entries are all measurable such that, for any $x\in\mathbb{R}^n$, $U(x)$ is unitary and $[U(x)]^*W(x)U(x)$ is diagonal.*

Recall that a matrix-valued function $W\colon \mathbb{R}^n\to M_m(\mathbb{C})$ is called a *matrix weight* if W is positive definite almost everywhere and all entries of W are locally integrable. By Lemma 2.1, we obtain, for any $\alpha\in\mathbb{R}$ and any matrix weight W, all entries of W^α are measurable.

Now, we recall the concept of matrix Muckenhoupt weights. Corresponding to A_p-scalar weights, Nazarov and Treil [118] and Volberg [155] originally independently introduced $\mathscr{A}_p$-matrix weights on $\mathbb{R}$ with $p\in(1,\infty)$. Subsequently, Roudenko [138] gave an equivalent definition of $\mathscr{A}_p$-matrix weights with $p\in(1,\infty)$. Later, Frazier and Roudenko [63] introduced $\mathscr{A}_p$-matrix weights with $p\in(0,1]$. The following version of $\mathscr{A}_p$-matrix weights is based on the definition originally introduced by Roudenko in [138, Corollary 3.3] and Frazier and Roudenko in [63, Definition in p. 1227].

Definition 2.3. Let $p\in(0,\infty)$. A matrix weight W on $\mathbb{R}^n$ is called an $\mathscr{A}_p(\mathbb{R}^n,\mathbb{C}^m)$*-matrix weight* if W satisfies that, when $p\in(0,1]$,

$$[W]_{\mathscr{A}_p(\mathbb{R}^n,\mathbb{C}^m)}:=\sup_{\text{cube}\,Q}\ \operatorname*{ess\,sup}_{y\in Q}\fint_Q\left\|W^{\frac1p}(x)W^{-\frac1p}(y)\right\|^p dx<\infty$$

or that, when $p\in(1,\infty)$,

$$[W]_{\mathscr{A}_p(\mathbb{R}^n,\mathbb{C}^m)}:=\sup_{\text{cube}\,Q\subset\mathbb{R}^n}\fint_Q\left[\fint_Q\left\|W^{\frac1p}(x)W^{-\frac1p}(y)\right\|^{p'}dy\right]^{\frac{p}{p'}}dx<\infty.$$

When $p\in(0,\infty)$, it is easy to show $\mathscr{A}_p(\mathbb{R}^n,\mathbb{C})=A_{\max\{1,p\}}$. For convenience, we denote $\mathscr{A}_p(\mathbb{R}^n,\mathbb{C}^m)$ simply by $\mathscr{A}_p$. The following equivalent characterization of $\mathscr{A}_p$ is exactly [24, Proposition 2.11].

Proposition 2.4. If $p\in(0,1]$, then $W\in\mathscr{A}_p$ if and only if

$$[W]_{\mathscr{A}_p}^*:=\sup_{\text{cube}\,Q}\fint_Q\operatorname*{ess\,sup}_{y\in Q}\left\|W^{\frac1p}(x)W^{-\frac1p}(y)\right\|^p dx<\infty.$$

Moreover, there exists a positive constant C, depending only on m and p, such that, for any $W\in\mathscr{A}_p$, $[W]_{\mathscr{A}_p}\le[W]_{\mathscr{A}_p}^*\le C[W]_{\mathscr{A}_p}$.

Corresponding to A_∞-scalar weight, Volberg [155, (2.2)] originally introduced $\mathscr{A}_{p,\infty}$-matrix weights on $\mathbb{R}$ with $p\in(1,\infty)$. Recently, Bu et al. [27, Definition 3.1] gave an equivalent definition of $\mathscr{A}_{p,\infty}$ as follows.

Definition 2.4. Let $p\in(0,\infty)$. A matrix weight W on $\mathbb{R}^n$ is called an $\mathscr{A}_{p,\infty}(\mathbb{R}^n,\mathbb{C}^m)$*-matrix weight* if W satisfies that, for any cube $Q\subset\mathbb{R}^n$, $\log_+(\fint_Q\|W^{\frac1p}(x)W^{-\frac1p}(\cdot)\|^p dx)\in L^1(Q)$ and

$$[W]_{\mathscr{A}_{p,\infty}(\mathbb{R}^n,\mathbb{C}^m)}:=\sup_{\mathrm{cube}\,Q}\exp\left(\fint_Q\log\left(\fint_Q\left\|W^{\frac1p}(x)W^{-\frac1p}(y)\right\|^p dx\right)dy\right)<\infty.$$

When $p\in(0,\infty)$, it is easy to show $\mathscr{A}_{p,\infty}(\mathbb{R}^n,\mathbb{C})=A_\infty$. If there is no confusion, for convenience, we denote $\mathscr{A}_{p,\infty}(\mathbb{R}^n,\mathbb{C}^m)$ simply by $\mathscr{A}_{p,\infty}$.

The remainder of this section is organized as follows. In Subsection 2.1, we present several nontrivial examples of both scalar and matrix Muckenhoupt weights. These examples illustrate intrinsic properties of Muckenhoupt weights and show the sharpness of certain results. At the end of this subsection, we also provide some methods for constructing new examples of matrix Muckenhoupt weights based on existing weights. In Subsection 2.2, we recall some classical properties of scalar weights, while in Subsection 2.3, we present the analogous results in the setting of matrix weights. In Subsection 2.4, we recall the recently introduced concept of the dimensions of weights and their properties. Moreover, we further discuss the relationship between these dimensions and the sharp indices of weights.

2.1 Examples of Scalar and Matrix Muckenhoupt Weights

Let us begin with recalling the concept of critical indices, which plays an important role in the study of weighted spaces (see, for example, [21, 164, 174]). For any $w\in A_\infty$, let

$$r_s(w):=\inf\{r\in(1,\infty):\ w\in A_r\}$$

be the *critical self-improvement index* of w and

$$r_h(w):=\sup\{r\in(1,\infty):\ w\in RH_r\}$$

the *critical reverse Hölder index* of w. One of the most classical examples of Muckenhoupt weights is the class of power weights (see, for instance, [69, Example 7.1.7]).

Example 2.1. Let $a\in\mathbb{R}$ and $w_a(\cdot):=|\cdot|^a$. Then the following statements hold.

(i) $w_a\in A_1$ if and only if $a\in(-n,0]$.

(ii) If $p\in(1,\infty)$, then $w_a\in A_p$ if and only if $a\in(-n,n(p-1))$.

(iii) $w_a\in A_\infty$ if and only if $a\in(-n,\infty)$.

However, power weights sometimes fail to reveal the finer properties of Muckenhoupt weights, and more refined examples are required. Before presenting such examples, we first recall a lemma, which is precisely [24, Lemma 2.40] and [27, Lemma 7.8].

Lemma 2.2. *Let $a\in(-n,\infty)$ and $b\in\mathbb{R}$. For any $x\in\mathbb{R}^n$, let*

$$w_{a,b}(x):=|x|^a[\log(2+|x|)]^b \quad \text{and} \quad \widetilde{w}_{a,b}(x):=|x|^a\left[\log\left(2+|x|^{-1}\right)\right]^b.$$

Then, for any $x_0\in\mathbb{R}^n$ and $r\in(0,\infty)$,

$$\fint_{B(x_0,r)} w_{a,b}(x)\,dx\sim(|x_0|+r)^a[\log(2+|x_0|+r)]^b$$

and

$$\fint_{B(x_0,r)} \widetilde{w}_{a,b}(x)\,dx\sim(|x_0|+r)^a\left[\log\left(2+(|x_0|+r)^{-1}\right)\right]^b,$$

where the positive equivalence constants depend only on n, a, and b.

Next, we present a more refined example, which is used to prove the sharpness of Lemmas 2.35 and 2.40.

Example 2.2. Let $a,b\in\mathbb{R}$. Assume that $w_{a,b}$ and $\widetilde{w}_{a,b}$ are as in Lemma 2.2. Then the following assertions hold.

(i) If $a\neq 0$, then $w_{a,b}\in A_1$ if and only if $a\in(-n,0)$.

(ii) $w_{0,b}\in A_1$ if and only if $b\in(-\infty,0]$.

(iii) If $p\in(1,\infty)$, then $w_{a,b}\in A_p$ if and only if $a\in(-n,n(p-1))$.

(iv) $w_{a,b}\in A_\infty$ if and only if $a\in(-n,\infty)$.

(v) If $a\in(-n,\infty)$, then $r_s(w_{a,b})=1+\frac{a_+}{n}$.

(vi) If $a\neq 0$, then $\widetilde{w}_{a,b}\in A_1$ if and only if $a\in(-n,0)$.

(vii) $\widetilde{w}_{0,b}\in A_1$ if and only if $b\in[0,\infty)$.

(viii) If $p\in(1,\infty)$, then $\widetilde{w}_{a,b}\in A_p$ if and only if $a\in(-n,n(p-1))$.

(ix) $\widetilde{w}_{a,b}\in A_\infty$ if and only if $a\in(-n,\infty)$.

(x) If $a\in(-n,\infty)$, then $r_s(\widetilde{w}_{a,b})=1+\frac{a_+}{n}$.

Proof. By similarity, we only prove assertions (i) through (v). From (iii), it follows that (iv) and (v) hold and hence we only need to prove assertions (i) through (iii). Notice that $a\in(-n,\infty)$ if and only if $w_{a,b}$ is locally integrable. Therefore, without loss of generality, we may assume $a\in(-n,\infty)$.

We first consider (i). If $a\in(0,\infty)$, then

$$\left\|w_{a,b}^{-1}\right\|_{L^\infty(B(\mathbf{0},1))}=\sup_{r\in(0,1)} r^{-a}[\log(2+r)]^{-b}=\infty$$

and hence $w_{a,b}\notin A_1$. This finishes the proof of the necessity of (i). Now, we prove the sufficiency of (i) by considering two cases for b.

Case (1) $b\in(-\infty,0]$. In this case, using [24, Lemma 2.39], we find that $w_{a,b}\in A_1$. This finishes the proof of the necessity of (i) in this case.

Case (2) $b\in(0,\infty)$. In this case, for any $t\in[0,\infty)$, let $f(t):=t^{-a}[\log(2+t)]^{-b}$. Applying Lemma 2.2, we obtain, for any ball $B\subset\mathbb{R}^n$,

$$\fint_B w_{a,b}(x)\,dx\sim(|c_B|+r_B)^a[\log(2+|c_B|+r_B)]^b=[f(|c_B|+r_B)]^{-1}. \tag{2.6}$$

By the definition of $\|\cdot\|_{L^\infty(B)}$, we conclude that, for any ball $B\subset\mathbb{R}^n$,

$$\left\|w_{a,b}^{-1}\right\|_{L^\infty(B)}=\sup_{x\in B}|x|^{-a}[\log(2+|x|)]^{-b}\le\max_{t\in[0,|c_B|+r_B]}t^{-a}[\log(2+t)]^{-b}. \tag{2.7}$$

From $a\in(-n,0)$, we infer that, for any $t\in(e^{-\frac{b}{a}},\infty)$,

$$\begin{aligned}f'(t)&=-at^{-a-1}[\log(2+t)]^{-b}-\frac{bt^{-a}[\log(2+t)]^{-b-1}}{2+t}\\&=t^{-a-1}[\log(2+t)]^{-b}\left[-a-\frac{bt}{(2+t)\log(2+t)}\right]\\&>t^{-a-1}[\log(2+t)]^{-b}\left(-a+a\frac{t}{2+t}\right)>0\end{aligned}$$

and hence f is increasing on $(e^{-\frac{b}{a}},\infty)$. Using this, (2.6), and (2.7), we conclude that, if $|c_B|+r_B\in(e^{-\frac{b}{a}},\infty)$, then

$$\begin{aligned}\fint_B w_{a,b}(x)\,dx\left\|w_{a,b}^{-1}\right\|_{L^\infty(B)}&\lesssim[f(|c_B|+r_B)]^{-1}\max\left\{\max_{t\in[0,e^{-\frac{b}{a}}]}f(t),f(|c_B|+r_B)\right\}\\&\le\max\left\{\left[f\left(e^{-\frac{b}{a}}\right)\right]^{-1}\max_{t\in[0,e^{-\frac{b}{a}}]}f(t),1\right\}<\infty\end{aligned} \tag{2.8}$$

and, if $|c_B|+r_B\in(0,e^{-\frac{b}{a}}]$, then

$$\begin{aligned}\fint_B w_{a,b}(x)\,dx\left\|w_{a,b}^{-1}\right\|_{L^\infty(B)}&\lesssim(|c_B|+r_B)^a\left[\log\left(2+e^{-\frac{b}{a}}\right)\right]^b\max_{t\in[0,|c_B|+r_B]}t^{-a}[\log(2+t)]^{-b}\\&\le\left[\log\left(2+e^{-\frac{b}{a}}\right)\right]^b(\log 2)^{-b},\end{aligned}$$

which, together with (2.8), further implies that $[w_{a,b}]_{A_1}<\infty$ and hence $w_{a,b}\in A_1$. This finishes the proof of the sufficiency of (i), which completes the proof of (i).

Next, we consider (ii). From [59, Lemma 2.3(iv)], we deduce that, if $b\in(-\infty,0]$, then $w_{0,b}\in A_1$. Applying Lemma 2.2 and (2.7), we conclude that, if $b\in(0,\infty)$, then

$$\fint_{B(\mathbf{0},r)}w_{0,b}(x)\,dx\left\|w_{0,b}^{-1}\right\|_{L^\infty(B(\mathbf{0},r))}\sim\left[\frac{\log(2+r)}{\log 2}\right]^b\to\infty$$

as $r\to\infty$ and hence $w_{0,b}\notin A_1$. This finishes the proof of (ii).

Now, we prove (iii). Using [24, Lemma 2.39(ii)], we find that, if $a\in(-n,n(p-1))$, then $w_{a,b}\in A_p$. On the other hand, from Lemma 2.2, we infer that, if $a\in[n(p-1),\infty)$, then $a\frac{p'}{p}\ge n(1-\frac{1}{p})p'=n$ and hence

$$\begin{aligned}\fint_{B(\mathbf{0},1)}[w_{a,b}(x)]^{-\frac{p'}{p}}\,dx&\ge\min\left\{(\log 2)^{-b\frac{p'}{p}},(\log 3)^{-b\frac{p'}{p}}\right\}\fint_{B(\mathbf{0},1)}|x|^{-a\frac{p'}{p}}\,dx\\&\ge\min\left\{(\log 2)^{-b\frac{p'}{p}},(\log 3)^{-b\frac{p'}{p}}\right\}\fint_{B(\mathbf{0},1)}|x|^{-n}\,dx=\infty,\end{aligned}$$

which further implies that $w_{a,b}\notin A_p$. This finishes the proof of (iii) and hence Example 2.2. $\square$

Corresponding to Example 2.2, we obtain the following example about the reverse Hölder classes.

Example 2.3. Let $a,b\in\mathbb{R}$. Assume that $w_{a,b}$ and $\widetilde{w}_{a,b}$ are as in Lemma 2.2. Then the following statements hold.

(i) If $a\neq 0$, then $w_{a,b}\in RH_\infty$ if and only if $a\in(0,\infty)$.

(ii) $w_{0,b}\in RH_\infty$ if and only if $b\in[0,\infty)$.

(iii) If $q\in(1,\infty)$, then $w_{a,b}\in RH_q$ if and only if $a\in(-\frac{n}{q},\infty)$.

(iv) If $a\in(-n,\infty)$, then $r_h(w_{a,b})=\frac{n}{a_-}$, where $\frac{n}{0}:=\infty$.

(v) If $a\neq 0$, then $\widetilde{w}_{a,b}\in RH_\infty$ if and only if $a\in(-n,0)$.

(vi) $\widetilde{w}_{0,b}\in RH_\infty$ if and only if $b\in(-\infty,0]$.

(vii) If $q\in(1,\infty)$, then $\widetilde{w}_{a,b}\in RH_q$ if and only if $a\in(-\frac{n}{q},\infty)$.

(viii) If $a\in(-n,\infty)$, then $r_h(\widetilde{w}_{a,b})=\frac{n}{a_-}$.

Proof. The proof of (iii) is similar to that of Example 2.2(iii). For the convenience of the reader, we provide the details here. Let $q\in(1,\infty)$. If $a\in(-\frac{n}{q},\infty)$, then, by Lemma 2.2, we find that, for any ball $B\subset\mathbb{R}^n$,

$$\left[\fint_B w_{a,b}(x)\,dx\right]^{-1}\left\{\fint_B[w_{a,b}(x)]^q\,dx\right\}^{\frac{1}{q}}\sim 1$$

and hence $w\in RH_q$. Conversely, if $w\in RH_q$, then

$$\infty>\fint_{B(\mathbf{0},1)}[w_{a,b}(x)]^q\,dx\ge\min\left\{(\log 2)^{bq},(\log 3)^{bq}\right\}\fint_{B(\mathbf{0},1)}|x|^{aq}\,dx$$

and hence $aq>-n$, which is equivalent to $a\in(-\frac{n}{q},\infty)$. This finishes the proof of (iii).

Applying an argument similar to that used in the proof of Example 2.2, we conclude that all remaining statements of Example 2.3 hold. This finishes the proof of Example 2.3. $\square$

The following lemma is exactly [27, Lemma 3.2].

Lemma 2.3. *Let $p\in(0,\infty)$ and $d\in[0,\infty)$. Let w be a scalar weight and $W:=wI_m$, where I_m is the identity matrix. Then following assertions hold.*

(i) *$W\in\mathscr{A}_p$ if and only if $w\in A_{\max\{1,p\}}$.*

(ii) *$W\in\mathscr{A}_{p,\infty}$ if and only if $w\in A_\infty$.*

With the help of Lemma 2.3, we can easily construct matrix weights from scalar weights. However, such trivial examples can not reveal the finer properties of matrix weights. To address this, we present a nontrivial example. Before that, we recall a well-known result.

Lemma 2.4. *All the norms of a given finite dimensional Banach space are equivalent.*

Using Lemma 2.4, we find that, for any $A:=(a_{ij})_{m\times m}\in M_m(\mathbb{C})$,

$$\|A\|\sim\sum_{i=1}^m\sum_{j=1}^m|a_{ij}| \tag{2.9}$$

with the positive equivalence constants independent of A because the right-hand side of (2.9) defines another norm of $M_m(\mathbb{C})$.

We are now ready to present the example.

Example 2.4. Let $m\geq 2$, $\alpha\in(0,\infty)$, and $\beta\in\mathbb{R}$. For any $x\in\mathbb{R}^n$, let

$$W_{\alpha,\beta}(x):=[U_\alpha(x)]^*\Lambda_\beta(x)U_\alpha(x),$$

where

$$U_\alpha(x):=\begin{pmatrix}\begin{matrix}\cos\theta_\alpha(x) & -\sin\theta_\alpha(x)\\ \sin\theta_\alpha(x) & \cos\theta_\alpha(x)\end{matrix} & O_{2,m-2}\\ O_{m-2,2} & I_{m-2}\end{pmatrix}$$

with

$$\theta_\alpha(x):=\begin{cases}\frac{1}{2}|x|^\alpha & \text{if } x\in B(\mathbf{0},1)\bigcap(0,\infty)^n,\\ -\frac{1}{2}|x|^\alpha & \text{if } x\in B(\mathbf{0},1)\setminus(0,\infty)^n,\\ 2\pi & \text{otherwise}\end{cases}$$

and

$$\Lambda_\beta(x):=\operatorname{diag}(\lambda_\beta(x),1,\dots,1)\ \text{ with }\ \lambda_\beta(x):=\begin{cases}|x|^\beta & \text{if } x\in B(\mathbf{0},1),\\ 1 & \text{otherwise.}\end{cases}$$

Then the following statements hold.

(i) For any $p\in(1,\infty)$, $W_{\alpha,\beta}\in\mathscr{A}_p$ if and only if $\beta\in(-n,n(p-1))$ and $\alpha\in[\frac{|\beta|}{p},\infty)$.

(ii) For any $p\in(0,1]$, $W_{\alpha,\beta}\in\mathscr{A}_p$ if and only if $\beta\in(-n,0]$ and $\alpha\in[-\frac{\beta}{p},\infty)$.

(iii) For any $p\in(0,\infty)$, $W_{\alpha,\beta}\in\mathscr{A}_{p,\infty}$ if and only if $\beta\in(-n,\infty)$ and $\alpha\in[\frac{|\beta|}{p},\infty)$.

When $n=1$ and $m=2$, Example 2.4(i) reduces to the example constructed by Bownik [18, Proposition 5.3]. To the best of our knowledge, the statements (ii) and (iii) of Example 2.4 are new. Surprisingly, as in [18, Remark 5.4], Example 2.4 shows that matrix Muckenhoupt weights do not possess the self-improvement property.

Proof of Example 2.4. We claim that $W_{\alpha,\beta}$ is a matrix weight if and only if $\beta\in(-n,\infty)$. By some fundamental calculations, we find that, for any $x\in\mathbb{R}^n$,

$$U_\alpha(x) \text{ is a unitary matrix,} \tag{2.10}$$

which, combined with the definition of Λ_β, further implies that, for any $x\in\mathbb{R}^n\setminus\{\mathbf{0}\}$, $W_{\alpha,\beta}(x)$ is positive definite. Therefore, it suffices to consider the local integrability of all entries of $W_{\alpha,\beta}$. Applying the definition of $W_{\alpha,\beta}$, we obtain, for any $x\in\mathbb{R}^n$,

$$W_{\alpha,\beta}(x)=\begin{pmatrix} \begin{matrix} w_{1,1}(x) & w_{1,2}(x)\\ w_{2,1}(x) & w_{2,2}(x)\end{matrix} & O_{2,m-2}\\ O_{m-2,2} & I_{m-2}\end{pmatrix},$$

where

$$\begin{aligned} &w_{1,1}(x):=\cos^2\theta_\alpha(x)\lambda_\beta(x)+\sin^2\theta_\alpha(x),\\ &w_{1,2}(x):=w_{2,1}(x):=-\sin\theta_\alpha(x)\cos\theta_\alpha(x)\lambda_\beta(x)+\sin\theta_\alpha(x)\cos\theta_\alpha(x),\end{aligned}$$

and

$$w_{2,2}(x):=\cos^2\theta_\alpha(x)+\sin^2\theta_\alpha(x)\lambda_\beta(x).$$

Notice that the only singularity in the integrals of all entries of $W_{\alpha,\beta}$ occurs at $\mathbf{0}$ and, as $x\to\mathbf{0}$,

$$w_{1,1}(x)\sim|x|^\beta+|x|^{2\alpha},\ \ |w_{1,2}(x)|=|w_{2,1}(x)|\lesssim\left(1+|x|^\beta\right)|x|^\alpha,\ \text{ and }\ w_{2,2}(x)\sim1+|x|^{2\alpha+\beta}.$$

Thus, $W_{\alpha,\beta}$ is locally integrable if and only if $\beta\in(-n,\infty)$. This finishes the proof of the above claim. Therefore, without loss of generality, we may assume $\beta\in(-n,\infty)$.

By the definition of $W_{\alpha,\beta}$, we find that, for any $x,y\in\mathbb{R}^n\setminus B(\mathbf{0},1)$,

$$\left\|W_{\alpha,\beta}^{\frac1p}(x)W_{\alpha,\beta}^{-\frac1p}(y)\right\|=1. \tag{2.11}$$

Using the definition of θ_α, we conclude that, for any $x,y\in B(\mathbf{0},1)$, $|\theta_\alpha(x)-\theta_\alpha(y)|\in[0,1)$ and hence

$$|\cos(\theta_\alpha(x)-\theta_\alpha(y))|\sim1\ \text{ and }\ |\sin(\theta_\alpha(x)-\theta_\alpha(y))|\sim|\theta_\alpha(x)-\theta_\alpha(y)|.$$

From these, (2.10), [77, Theorem 2.1.4(g)], and (2.9), we infer that, for any $x,y\in B(\mathbf{0},1)\setminus\{\mathbf{0}\}$,

$$\begin{aligned}\left\|W_{\alpha,\beta}^{\frac{1}{p}}(x)W_{\alpha,\beta}^{-\frac{1}{p}}(y)\right\|^p&=\left\|\Lambda_\beta^{\frac{1}{p}}(x)U_\alpha(x)[U_\alpha(y)]^*\Lambda_\beta^{-\frac{1}{p}}(y)\right\|^p\\&\sim\lambda_\beta(x)\lambda_{-\beta}(y)\left|\cos(\theta_\alpha(x)-\theta_\alpha(y))\right|^p\\&\quad+[\lambda_\beta(x)+\lambda_{-\beta}(y)]\left|\sin(\theta_\alpha(x)-\theta_\alpha(y))\right|^p\\&\quad+\left|\cos(\theta_\alpha(x)-\theta_\alpha(y))\right|^p+m-2\\&\sim\lambda_\beta(x)\lambda_{-\beta}(y)+[\lambda_\beta(x)+\lambda_{-\beta}(y)]\left|\theta_\alpha(x)-\theta_\alpha(y)\right|^p+1\\&=|x|^\beta|y|^{-\beta}+\left(|x|^\beta+|y|^{-\beta}\right)\left|\theta_\alpha(x)-\theta_\alpha(y)\right|^p+1,\end{aligned}\tag{2.12}$$

where, in the last equality, we used the definition of λ_β.

Next, we show the sufficiency of (i), that is, for any $p\in(1,\infty)$, if

$$\beta\in(-n,n(p-1))\ \text{ and }\ \alpha\in\left[\frac{|\beta|}{p},\infty\right),\tag{2.13}$$

then $W_{\alpha,\beta}\in\mathscr{A}_p$. Let $p\in(1,\infty)$ be fixed. Notice that, for any ball $B\subset B(\mathbf{0},1)$, $|c_B|+r_B<2$. This, together with (2.12), the definition of θ_α, (2.13), and Lemma 2.2, further implies that, for any ball $B\subset B(\mathbf{0},1)$ and any $y\in B\setminus\{\mathbf{0}\}$,

$$\begin{aligned}\fint_B\left\|W_{\alpha,\beta}^{\frac{1}{p}}(x)W_{\alpha,\beta}^{-\frac{1}{p}}(y)\right\|^p dx&\lesssim\fint_B|x|^\beta|y|^{-\beta}\,dx+\fint_B\left(|x|^\beta+|y|^{-\beta}\right)\left(|x|^{\alpha p}+|y|^{\alpha p}\right)dx+1\\&\sim(|c_B|+r_B)^\beta|y|^{-\beta}+(|c_B|+r_B)^{\alpha p+\beta}+(|c_B|+r_B)^\beta|y|^{\alpha p}\\&\quad+(|c_B|+r_B)^{\alpha p}|y|^{-\beta}+|y|^{\alpha p-\beta}+1\\&\sim\left[(|c_B|+r_B)^\beta+(|c_B|+r_B)^{\alpha p}\right]|y|^{-\beta}+(|c_B|+r_B)^{\alpha p+\beta}+1\\&\sim\left[(|c_B|+r_B)^\beta+(|c_B|+r_B)^{\alpha p}\right]|y|^{-\beta}+1.\end{aligned}\tag{2.14}$$

By $\beta\in(-n,n(p-1))$, we have

$$-\beta\frac{p'}{p}=-\beta\frac{1}{p-1}\in\left(-n,\frac{n}{p-1}\right).\tag{2.15}$$

For any ball $B\subset\mathbb{R}^n$, define

$$I_1(B):=\fint_B\left[\fint_B\left\|W_{\alpha,\beta}^{\frac{1}{p}}(x)W_{\alpha,\beta}^{-\frac{1}{p}}(y)\right\|^p dx\right]^{\frac{p'}{p}}dy.$$

Using (2.14), (2.15), Lemma 2.2, and $\alpha\in[\frac{|\beta|}{p},\infty)$, we find that, for any $r\in(0,1]$,

$$I_1(B(\mathbf{0},r))\lesssim\fint_{B(\mathbf{0},r)}\left(r^\beta+r^{\alpha p}\right)^{\frac{p'}{p}}|y|^{-\beta\frac{p'}{p}}\,dy+1\sim1+r^{(\alpha p-\beta)\frac{p'}{p}}\lesssim1.$$

From this, the definition of $W_{\alpha,\beta}$, (2.5), (2.15), and $\beta\in(-n,n(p-1))$, we infer that, for any $r\in(1,\infty)$,

$$\begin{aligned}
I_1(B(\mathbf{0},r)) &\sim r^{-n(1+\frac{p'}{p})}\left\{\int_{B(\mathbf{0},1)}\left[\int_{B(\mathbf{0},1)}\left\|W_{\alpha,\beta}^{\frac{1}{p}}(x)W_{\alpha,\beta}^{-\frac{1}{p}}(y)\right\|^p dx\right]^{\frac{p'}{p}} dy\right.\\
&\quad+\int_{B(\mathbf{0},1)}\left[\int_{B(\mathbf{0},r)\setminus B(\mathbf{0},1)}\cdots\right]^{\frac{p'}{p}}+\int_{B(\mathbf{0},r)\setminus B(\mathbf{0},1)}\left[\int_{B(\mathbf{0},1)}\cdots\right]^{\frac{p'}{p}}\\
&\quad\left.+\int_{B(\mathbf{0},r)\setminus B(\mathbf{0},1)}\left[\int_{B(\mathbf{0},r)\setminus B(\mathbf{0},1)}\cdots\right]^{\frac{p'}{p}}\right\}\\
&\lesssim r^{-n(1+\frac{p'}{p})}\left\{1+\int_{B(\mathbf{0},1)}(r^n-1)^{\frac{p'}{p}}\left\|W_{\alpha,\beta}^{-\frac{1}{p}}(y)\right\|^{p'}dy\right.\\
&\quad\left.+(r^n-1)\left[\int_{B(\mathbf{0},1)}\left\|W_{\alpha,\beta}^{\frac{1}{p}}(x)\right\|^p dx\right]^{\frac{p'}{p}}+(r^n-1)^{1+\frac{p'}{p}}\right\}\\
&\lesssim 1+\int_{B(\mathbf{0},1)}\left\|W_{\alpha,\beta}^{-\frac{1}{p}}(y)\right\|^{p'}dy+\left[\int_{B(\mathbf{0},1)}\left\|W_{\alpha,\beta}^{\frac{1}{p}}(x)\right\|^p dx\right]^{\frac{p'}{p}}\\
&\le 1+\int_{B(\mathbf{0},1)}\max\left\{|y|^{-\beta\frac{p'}{p}},1\right\}dy+\left[\int_{B(\mathbf{0},1)}\max\left\{|x|^{\beta},1\right\}dx\right]^{\frac{p'}{p}}\sim 1.
\end{aligned}$$

Thus, for any $r\in(0,\infty)$,

$$I_1(B(\mathbf{0},r))\lesssim 1. \tag{2.16}$$

Now, we show that $\sup_{\text{ball}\,B}I_1(B)<\infty$. To this end, we consider four cases for B.

Case (1) $|c_B|\in[2,\infty)$ and $r_B\in(0,\frac{1}{2}|c_B|]$. In this case, for any $x\in B$,

$$|x|\ge|c_B|-|x-c_B|>|c_B|-r\ge\frac{1}{2}|c_B|\ge 1$$

and hence $B\bigcap B(\mathbf{0},1)=\emptyset$. Applying this and (2.11), we obtain, for any $x,y\in B\setminus\{\mathbf{0}\}$,

$$\left\|W_{\alpha,\beta}^{\frac{1}{p}}(x)W_{\alpha,\beta}^{-\frac{1}{p}}(y)\right\|^p=1. \tag{2.17}$$

Therefore, $I_1(B)=1$.

Case (2) $|c_B|\in(\frac{2}{3},2)$ and $r_B\in(0,\frac{1}{2}|c_B|]$. In this case, for any $x\in B$,

$$\frac{1}{3}<\frac{1}{2}|c_B|\le|c_B|-r_B\le|c_B|-|c_B-x|\le|x|\le|c_B|+|c_B-x|<|c_B|+r_B\le\frac{3}{2}|c_B|<3$$

and hence $|x|\sim 1$. Using this, (2.5), and the definition of $W_{\alpha,\beta}$, we conclude that, for any $x,y\in B\setminus\{\mathbf{0}\}$,

$$\left\|W_{\alpha,\beta}^{\frac{1}{p}}(x)W_{\alpha,\beta}^{-\frac{1}{p}}(y)\right\|^p\le\left\|W_{\alpha,\beta}^{\frac{1}{p}}(x)\right\|^p\left\|W_{\alpha,\beta}^{-\frac{1}{p}}(y)\right\|^p\le\max\left\{|x|^{\beta},1\right\}\max\left\{|y|^{-\beta},1\right\}\sim 1. \tag{2.18}$$

Therefore, $I_1(B)\lesssim 1$.

Case (3) $|c_B|\in(0,\frac{2}{3}]$ and $r_B\in(0,\frac{1}{2}|c_B|]$. In this case, for any $x\in B$,

$$|x|\le|c_B|+|x-c_B|<|c_B|+r_B\le\frac{3}{2}|c_B|\le 1$$

and hence $B\subset B(\mathbf{0},1)$. From this and (2.14), we infer that, for any $y\in B\setminus\{\mathbf{0}\}$,

$$⨍_B\left\|W^{\frac{1}{p}}_{\alpha,\beta}(x)W^{-\frac{1}{p}}_{\alpha,\beta}(y)\right\|^p dx\lesssim\left[(|c_B|+r_B)^\beta+(|c_B|+r_B)^{\alpha p}\right]|y|^{-\beta}+1. \tag{2.19}$$

This, combined with (2.15), Lemma 2.2, and $\alpha\in[\frac{|\beta|}{p},\infty)$, further implies that

$$I_1(B)\lesssim⨍_B\left[(|c_B|+r_B)^{\beta\frac{p'}{p}}+(|c_B|+r_B)^{\alpha p'}\right]|y|^{-\beta\frac{p'}{p}}\,dy+1\sim 1+(|c_B|+r_B)^{(\alpha p-\beta)\frac{p'}{p}}\sim 1.$$

Case (4) $r_B\in(\frac{1}{2}|c_B|,\infty)$. In this case, for any $x\in B$, $|x|<|c_B|+r_B<3r_B$ and hence $B\subset B(\mathbf{0},3r_B)$. By this and (2.16), we have

$$I_1(B)\lesssim I_1(B(\mathbf{0},3r_B))\lesssim 1. \tag{2.20}$$

Combining all above four cases, we obtain $\sup_{\mathrm{ball}\,B}I_1(B)<\infty$, which, together with [138, Corollary 3.3(iv)], further implies that $W_{\alpha,\beta}\in\mathscr{A}_p$. This finishes the proof of the sufficiency of (i).

Next, we show the necessity of (i), that is, for any $p\in(1,\infty)$, if $W_{\alpha,\beta}\in\mathscr{A}_p$, then

$$\beta\in(-n,n(p-1))\ \text{ and }\ \alpha\in\left[\frac{|\beta|}{p},\infty\right).$$

Let $W_{\alpha,\beta}\in\mathscr{A}_p$. By (2.12), the definition of θ_α, $\alpha\in(0,\infty)$, and Lemma 2.2, we conclude that, for any $r\in(0,1]$ and $y\in B(\mathbf{0},r)\setminus\{\mathbf{0}\}$,

$$\begin{aligned}⨍_{B(\mathbf{0},r)}\left\|W^{\frac{1}{p}}_{\alpha,\beta}(x)W^{-\frac{1}{p}}_{\alpha,\beta}(y)\right\|^p dx&\gtrsim⨍_{B(\mathbf{0},r)}\left(|x|^\beta+|y|^{-\beta}\right)\left|\theta_\alpha(x)-\theta_\alpha(y)\right|^p dx\\&\gtrsim⨍_{B(\mathbf{0},r)\setminus(0,\infty)^n}\left(|x|^\beta+|y|^{-\beta}\right)(|x|^\alpha+|y|^\alpha)^p\,dx\\&\sim⨍_{B(\mathbf{0},r)}\left(|x|^{\alpha p+\beta}+|x|^\beta|y|^{\alpha p}+|y|^{-\beta}|x|^{\alpha p}+|y|^{\alpha p-\beta}\right)dx\\&\sim r^{\alpha p+\beta}+r^\beta|y|^{\alpha p}+|y|^{-\beta}r^{\alpha p}+|y|^{\alpha p-\beta}.\end{aligned}\tag{2.21}$$

Applying this and $W_{\alpha,\beta}\in\mathscr{A}_p$, we conclude that, for any $r\in(0,1]$,

$$\infty>I_1(B(\mathbf{0},r))\gtrsim⨍_{B(\mathbf{0},r)}\left[r^{(\alpha p+\beta)\frac{p'}{p}}+r^{\beta\frac{p'}{p}}|y|^{\alpha p'}+|y|^{-\beta\frac{p'}{p}}r^{\alpha p'}+|y|^{(\alpha p-\beta)\frac{p'}{p}}\right]dy \tag{2.22}$$

and hence $-\beta\frac{p'}{p}>-n$, which is equivalent to $\beta<n\frac{p}{p'}=n(p-1)$. From this, (2.22), $\alpha\in(0,\infty)$, and Lemma 2.2, it follows that, for any $r\in(0,1]$,

$$\infty>r^{(\alpha p+\beta)\frac{p'}{p}}+r^{(\alpha p-\beta)\frac{p'}{p}}.$$

Letting $r\to 0^+$, we obtain $-\alpha p\le\beta\le\alpha p$, which is equivalent to $\alpha\in[\frac{|\beta|}{p},\infty)$. This finishes the proof of the necessity of (i) and hence (i).

Now, we show the sufficiency of (ii), that is, for any $p\in(0,1)$, if

$$\beta\in(-n,0] \text{ and } \alpha\in\left[-\frac{\beta}{p},\infty\right), \tag{2.23}$$

then $W_{\alpha,\beta}\in\mathscr{A}_p$. For any ball $B\subset\mathbb{R}^n$, define

$$I_2(B):=\operatorname*{ess\,sup}_{y\in B}\fint_B\left\|W_{\alpha,\beta}^{\frac{1}{p}}(x)W_{\alpha,\beta}^{-\frac{1}{p}}(y)\right\|^p dx.$$

Using (2.14) and (2.23), we obtain, for any $r\in(0,1]$,

$$I_2(B(\mathbf{0},r))\lesssim\operatorname*{ess\,sup}_{y\in B(\mathbf{0},r)}\left(r^\beta|y|^{-\beta}+r^{\alpha p}|y|^{-\beta}\right)\lesssim 1.$$

Applying this, (2.17), (2.18), (2.19), and an argument used in the proof of (2.20), we find that, for any ball $B\subset\mathbb{R}^n$, $I_2(B)\lesssim 1$ and hence $W_{\alpha,\beta}\in\mathscr{A}_p$. This finishes the proof of the sufficiency of (ii).

Next, we prove the necessity of (ii), that is, for any $p\in(0,1)$, if $W_{\alpha,\beta}\in\mathscr{A}_p$ then

$$\beta\in(-n,0] \text{ and } \alpha\in\left[-\frac{\beta}{p},\infty\right).$$

Let $W_{\alpha,\beta}\in\mathscr{A}_p$. By this and (2.21), we have, for any $r\in(0,1]$,

$$\infty>I_2(B(\mathbf{0},r))\gtrsim\operatorname*{ess\,sup}_{y\in B(\mathbf{0},r)}\left(r^{\alpha p+\beta}+r^\beta|y|^{\alpha p}+|y|^{-\beta}r^{\alpha p}+|y|^{\alpha p-\beta}\right) \tag{2.24}$$

and hence $\beta\in(-n,0]$, which, together with (2.24) and $\alpha\in(0,\infty)$, further implies that

$$\infty>r^{\alpha p+\beta}+r^{\alpha p-\beta}.$$

Letting $r\to 0^+$, we obtain $\alpha\in[-\frac{\beta}{p},\infty)$. This finishes the proof of the necessity of (ii) and hence (ii).

Next, we show the sufficiency of (iii), that is, for any $p\in(0,\infty)$, if $\alpha\in[\frac{|\beta|}{p},\infty)$, then $W\in\mathscr{A}_{p,\infty}$. From [27, Proposition 4.1 and Lemma 2.9], we deduce that $W_{\alpha,\beta}\in\mathscr{A}_{p,\infty}$ if and only if there exists $u\in(0,\infty)$ such that

$$\sup_{\text{ball}\,B\subset\mathbb{R}^n}I_3(B):=\sup_{\text{ball}\,B\subset\mathbb{R}^n}\fint_B\left[\fint_B\left\|W_{\alpha,\beta}^{\frac{1}{p}}(x)W_{\alpha,\beta}^{-\frac{1}{p}}(y)\right\|^p dx\right]^u dy<\infty. \tag{2.25}$$

Let $u\in(0,\frac{n}{|\beta|+1})$. Then $-\beta u\in(-n,\infty)$. Using this, (2.14), $\alpha\in[\frac{|\beta|}{p},\infty)$, and Lemma 2.2, we conclude that, for any $r\in(0,1]$,

$$I_3(B(\mathbf{0},r))\lesssim \fint_{B(\mathbf{0},r)}\Big(r^{\beta u}|y|^{-\beta u}+r^{\alpha pu}|y|^{-\beta u}+1\Big)dy\sim 1+r^{(\alpha p-\beta)u}\lesssim 1.$$

Applying this, (2.17), (2.18), (2.19), and an argument used in the proof of (2.20), we obtain, for any ball $B\subset\mathbb{R}^n$, $I_3(B)\lesssim 1$ and hence $W_{\alpha,\beta}\in\mathscr{A}_{p,\infty}$. This finishes the proof of the sufficiency of (iii).

Now, we prove the necessity of (iii), that is, for any $p\in(0,\infty)$, if $W_{\alpha,\beta}\in\mathscr{A}_{p,\infty}$, then $\alpha\in[\frac{|\beta|}{p},\infty)$. Let $W_{\alpha,\beta}\in\mathscr{A}_{p,\infty}$. By this, (2.25), and (2.21), we find that, for any $r\in(0,1]$,

$$\begin{aligned}\infty>I_3(B(\mathbf{0},r))&\gtrsim \fint_{B(\mathbf{0},r)}\Big[r^{(\alpha p+\beta)u}+r^{\beta u}|y|^{\alpha pu}+|y|^{-\beta u}r^{\alpha pu}+|y|^{(\alpha p-\beta)u}\Big]dy\\ &\sim r^{(\alpha p+\beta)u}+r^{(\alpha p-\beta)u}\end{aligned}$$

for some $u\in(0,\infty)$. Letting $r\to 0^+$, we obtain $-\alpha p\le\beta\le\alpha p$, which is equivalent to $\alpha\in[\frac{|\beta|}{p},\infty)$. This finishes the proof of the necessity of (iii) and hence (iii), which completes the proof of Example 2.4. □

The following are further nontrivial examples of matrix weights constructed by Bickel et al. [12, Remark 3.6].

Example 2.5. Let $A:=(a_{ij})_{m\times m}\in M_m(\mathbb{C})$. For any $x\in\mathbb{R}^n$, define

$$W(x):=\begin{pmatrix} a_{11}|x|^{\gamma_{11}} & \dots & a_{1m}|x|^{\gamma_{1m}}\\ \vdots & \ddots & \dots\\ a_{m1}|x|^{\gamma_{m1}} & \dots & a_{mm}|x|^{\gamma_{mm}}\end{pmatrix}, \tag{2.26}$$

where, for any $i,j\in\{1,\dots,m\}$, $\gamma_{ij}\in\mathbb{R}$. Then $W\in\mathscr{A}_2$ if and only if A is positive definite and, for any $i,j\in\{1,\dots,m\}$,

$$\gamma_{ii}\in(-n,n)\ \text{ and }\ \gamma_{ij}=\frac{\gamma_{ii}+\gamma_{jj}}{2}.$$

Applying Example 2.5, Bickel et al. [12, Example 3.5] showed that

$$W(\cdot):=\begin{pmatrix} 5|\cdot|^{\frac12} & 3|\cdot|^{-\frac{1}{12}}\\ 3|\cdot|^{-\frac{1}{12}} & 2|\cdot|^{-\frac23}\end{pmatrix}\in\mathscr{A}_2.$$

Naturally, we pose the following questions.

Question I. *Let $p\in(0,\infty)$ and W be as in* (2.26).

(i) *What conditions on A and $\{\gamma_{ij}\}_{i,j=1}^m$ characterize $W\in\mathscr{A}_p$?*

(ii) *What conditions on A and $\{\gamma_{ij}\}_{i,j=1}^m$ characterize $W\in\mathscr{A}_{p,\infty}$?*

Finally, we present some methods for constructing new weights. The following proposition is a generalization of [28, Lemma 4.30], which considers the product of power weights.

Proposition 2.5. Let $\{w_i\}_{i=1}^n$ be a sequence scalar weights on $\mathbb{R}$. For any $x:=(x_1,\ldots,x_n)\in\mathbb{R}^n$, let

$$w_{\rm prod}(x):=\prod_{i=1}^n w_i(x_i).$$

Then the following assertions hold.

(i) If $p\in[1,\infty]$, then $w_{\rm prod}\in A_p$ if and only if, for any $i\in\{1,\ldots,n\}$, $w_i\in A_p(\mathbb{R})$.

(ii) If $q\in(1,\infty]$, then $w_{\rm prod}\in RH_q$ if and only if, for any $i\in\{1,\ldots,n\}$, $w_i\in RH_q(\mathbb{R})$.

(iii) If $w_{\rm prod}\in A_\infty$, then $r_s(w_{\rm prod})=\max_{i\in\{1,\ldots,n\}}r_s(w_i)$.

(iv) If $w_{\rm prod}\in A_\infty$, then $r_h(w_{\rm prod})=\min_{i\in\{1,\ldots,n\}}r_h(w_i)$.

Proof. The assertions (iii) and (iv) follow directly from (i) and (ii), respectively. Therefore, it suffices to prove (i) and (ii). By similarity, we only prove (i) under the case where $p\in(1,\infty)$.

If $w_{\rm prod}\in A_p$, then, by Hölder's inequality, we find that, for any $i\in\{1,\ldots,n\}$ and any interval $I\subset\mathbb{R}$,

$$[w_{\rm prod}]_{A_p}\geq\fint_{I^n}w_{\rm prod}(x)\,dx\left\{\fint_{I^n}[w_{\rm prod}(x)]^{-\frac{p'}{p}}\,dx\right\}^{\frac{p}{p'}}\geq\fint_I w_i(x_i)\,dx_i\left\{\fint_I[w_i(x_i)]^{-\frac{p'}{p}}\,dx_i\right\}^{\frac{p}{p'}}$$

and hence $w_i\in A_p(\mathbb{R})$. Conversely, if, for any $i\in\{1,\ldots,n\}$, $w_i\in A_p(\mathbb{R})$, then, from the definition of $[w_i]_{A_p(\mathbb{R})}$, we infer that, for any cube $Q:=I_1\times\cdots\times I_n\subset\mathbb{R}^n$,

$$\fint_Q w_{\rm prod}(x)\,dx\left\{\fint_Q[w_{\rm prod}(x)]^{-\frac{p'}{p}}\,dx\right\}^{\frac{p}{p'}}\leq\prod_{i=1}^n[w_i]_{A_p(\mathbb{R})},$$

which further implies that $w_{\rm prod}\in A_p$. This finishes the proof of (i) and hence Proposition 2.5. $\square$

Next, we present a theorem on the construction of A_1 weights (see, for instance, [69, Theorem 7.2.7]), which originates from the work of Coifman and Rochberg [41, Proposition 2 and Corollary 3(a)].

Theorem 2.1. (i) *If $f\in L^1_{\rm loc}$ satisfies that, for almost every $x\in\mathbb{R}^n$, $Mf(x)\in(0,\infty)$, then, for any $\delta\in[0,1)$, $(Mf)^\delta\in A_1$. Moreover, there exists a positive constant C, depending only on n, such that $[(Mf)^\delta]_{A_1}\leq C\frac{1}{1-\delta}$.*

(ii) *Conversely, if $w\in A_1$, then there exist $f\in L^1_{\rm loc}$, $\delta\in(-\infty,0)$, and $K\in L^\infty$, such that $w=K(Mf)^\delta$.*

The following theorem is the famous Jones factorization, which shows that any scalar Muckenhoupt weight can be constructed from A_1 weights (see [95, Theorem]).

Theorem 2.2. *Let $p\in(1,\infty)$ and w be a scalar weight. Then $w\in A_p$ if and only if there exist $w_1,w_2\in A_1$ such that $w=w_1w_2^{1-p}$.*

By some basic calculations, we obtain the following conclusion.

Proposition 2.6. Let $p\in[1,\infty)$ and $w_1,w_2\in A_p$. Then the following statements hold.

(i) For any $\theta\in[0,1]$, $w_1^\theta w_2^{1-\theta}\in A_p$ and $[w_1^\theta w_2^{1-\theta}]_{A_p}\le[w_1]_{A_p}^\theta[w_2]_{A_p}^{1-\theta}$.

(ii) $w_1+w_2\in A_p$ and $[w_1+w_2]_{A_p}\le[w_1]_{A_p}+[w_2]_{A_p}$.

(iii) $\max\{w_1,w_2\}\in A_p$ and $[\max\{w_1,w_2\}]_{A_p}\le[w_1]_{A_p}+[w_2]_{A_p}$.

(iv) $\min\{w_1,w_2\}\in A_p$ and

$$[\min\{w_1,w_2\}]_{A_p}\le\begin{cases}\max\{[w_1]_{A_1},[w_2]_{A_1}\} & \text{if } p=1,\\ \left([w_1]_{A_p}^{\frac{p'}{p}}+[w_2]_{A_p}^{\frac{p'}{p}}\right)^{\frac{p}{p'}} & \text{if } p\in(1,\infty).\end{cases}$$

We now turn to the setting of matrix weights. Recall that two matrix weights W_1 and W_2 are said to be *commuting* if, for any $x\in\mathbb{R}^n$, $W_1(x)W_2(x)=W_2(x)W_1(x)$. Recently, Cruz-Uribe and Bownik [20, Theorem 1.3] established the Jones factorization of matrix weights as follows.

Theorem 2.3. *Let $p\in(1,\infty)$ and W be a matrix weight on $\mathbb{R}^n$. Then $W\in\mathscr{A}_p$ if and only if there exist two commuting matrix weights $W_1,W_2\in\mathscr{A}_1$ such that $W=W_1W_2^{1-p}$.*

Question II. (i) *For scalar weights, the class A_1 is the minimal Muckenhoupt weight class; however, this does not hold for matrix weights. A natural question is whether Theorem 2.3 remains valid when $\mathscr{A}_1$ is replaced by $\mathscr{A}_q$ for some $q\in(0,1)$.*

(ii) *When $p\in(0,\infty)$ and $W\in\mathscr{A}_{p,\infty}$, it remains unknown whether an analogue of the Jones factorization exists.*

(iii) *To construct more examples of matrix weights, it is natural to ask whether analogues of Propositions 2.5 and 2.6 and Theorem 2.1 can be established.*

2.2 Properties of Scalar Muckenhoupt Weights

While some results concerning scalar weights are well known, their proofs are often difficult to trace in the literature. For the convenience of the reader, we provide the details here. Let us begin with some fundamental characterizations of A_p-scalar weights with $p\in[1,\infty)$. To this end, we first recall the definition of weighted Lebesgue spaces.

Definition 2.5. Let $p\in(0,\infty]$ and w be a weight on $\mathbb{R}^n$. The *weighted Lebesgue space* $L^p(w)$ is defined to be the set of all $f\in\mathscr{M}$ on $\mathbb{R}^n$ such that $\|f\|_{L^p(w)}<\infty$, where

$$\|f\|_{L^p(w)}:=\begin{cases}\left[\displaystyle\int_{\mathbb{R}^n}\left|f(x)\right|^p w(x)\,dx\right]^{\frac{1}{p}} & \text{if } p\in(0,\infty),\\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}|f(x)| & \text{if } p=\infty.\end{cases}$$

The following three fundamental characterizations of A_p-scalar weights can be found in [69, Proposition 7.1.5(4)], [118, p. 109], and [69, Proposition 7.1.5(8)], respectively.

Proposition 2.7. Let $p\in(1,\infty)$ and w be a scalar weight. Then $w\in A_p$ if and only if $w^{1-p'}\in A_{p'}$. Moreover, for any $w\in A_p$, $[w^{1-p'}]_{A_{p'}}=[w]_{A_p}^{p'-1}$.

Proposition 2.8. Let $p\in(1,\infty)$. Then $w\in A_p$ if and only if $w\in A_\infty$ and $w^{1-p'}\in A_\infty$.

Proposition 2.9. Let $p\in[1,\infty)$ and w be a scalar weight. Then $w\in A_p$ if and only if

$$\sup_{\text{cube}\,Q\subset\mathbb{R}^n}\ \sup_{\|f\mathbf{1}_Q\|_{L^p(w)}=1} w(Q)\left[\fint_Q|f(x)|\,dx\right]^p<\infty. \tag{2.27}$$

Moreover, for any $w\in A_p$, $[w]_{A_p}$ is equal to the left-hand side of (2.27).

Next, we recall some equivalent characterizations of A_∞-scalar weights. The equivalences of all the following characterizations of A_∞ weights were given in [57, p. 959], except for (ii), (vii), and (xi). The mutual equivalences among the assertions (i), (ii), and (vii) are precisely [69, Proposition 7.3.2(5) and Theorem 7.3.3]. The equivalence between the assertions (i) and (xi) is exactly [45, Theorem 2.1].

Proposition 2.10. Let w be a scalar weight. Then the following assertions are equivalent.

(i) $w\in A_\infty$.

(ii)
$$\sup_{\text{cube}\,Q\subset\mathbb{R}^n}\ \sup_{\substack{\log|f|\in L^1(Q)\\ \|f\mathbf{1}_Q\|_{L^1(w)}=1}} w(Q)\exp\left(\fint_Q\log|f(t)|\,dt\right)<\infty.$$

(iii) $w\in\bigcup_{p\in[1,\infty)}A_p$.

(iv) There exist $\delta,C\in(0,\infty)$ such that, for any cube $Q\subset\mathbb{R}^n$ and any measurable set $E\subset Q$,

$$\frac{w(E)}{w(Q)}\le C\left(\frac{|E|}{|Q|}\right)^\delta.$$

(v) There exist $\delta, C \in (0,\infty)$ such that, for any cube $Q \subset \mathbb{R}^n$ and any measurable set $E \subset Q$,

$$\frac{|E|}{|Q|} \leq C\left[\frac{w(E)}{w(Q)}\right]^{\delta}.$$

(vi) There exist $\alpha, \beta \in (0,1)$ such that, for any cube $Q \subset \mathbb{R}^n$ and any measurable set $E \subset Q$, if $|E| \leq \alpha |Q|$, then $w(E) \leq \beta w(Q)$.

(vii) There exist $\alpha', \beta' \in (0,1)$ such that, for any cube $Q \subset \mathbb{R}^n$ and any measurable set $E \subset Q$, if $w(E) < \alpha' w(Q)$, then $|E| < \beta' |Q|$.

(viii)

$$[w]_{A_\infty^*} := \sup_{\text{cube}\, Q \subset \mathbb{R}^n} \frac{1}{w(Q)} \int_Q M(w\mathbf{1}_Q)(x)\,dx < \infty.$$

(ix)

$$\sup_{\text{cube}\, Q \subset \mathbb{R}^n} \frac{1}{w(Q)} \int_Q w(x) \log_+\left(\frac{w(x)}{\fint_Q w(y)\,dy}\right) dx < \infty.$$

(x) There exist $\varepsilon, C \in (0,\infty)$ such that, for any cube $Q \subset \mathbb{R}^n$,

$$\left\{\fint_Q [w(x)]^{1+\varepsilon}\,dx\right\}^{\frac{1}{1+\varepsilon}} \leq C \fint_Q w(x)\,dx.$$

(xi) There exists $s \in (1,\infty)$ such that $w^{\frac{1}{s}} \in RH_s$.

(xii) There exist $t \in (0,1)$ and a positive constant C such that, for any cube $Q \subset \mathbb{R}^n$,

$$\fint_Q w(x)\,dx \leq C\left\{\fint_Q [w(x)]^t\,dx\right\}^{\frac{1}{t}}.$$

(xiii) There exist $\alpha, \beta \in (0,1)$ such that, for any cube $Q \subset \mathbb{R}^n$,

$$\left|\left\{x \in Q:\ w(x) \leq \alpha \fint_Q w(y)\,dy\right\}\right| \leq \beta |Q|.$$

(xiv) There exists a positive constant C such that, for any cube $Q \subset \mathbb{R}^n$,

$$\fint_Q w(x)\,dx \leq C \sup\left\{t \in (0,\infty):\ |\{x \in Q:\ w(x) < t\}| \leq \frac{|Q|}{2}\right\}.$$

(xv) There exist $C, \beta \in (0,\infty)$ such that, for any cube $Q \subset \mathbb{R}^n$ and any $\lambda \in (\fint_Q w(x)\,dx, \infty)$,

$$w(\{x \in Q:\ w(x) > \lambda\}) \leq C\lambda \left|\{x \in Q:\ w(x) > \beta\lambda\}\right|.$$

In addition to the equivalent characterizations of Muckenhoupt weights themselves, there are several important properties associated with them. Before discussing these properties, let us recall some related spaces. The first one is the well-known space BMO introduced by John and Nirenberg in [93].

Definition 2.6. The *space of functions with bounded mean oscillation*, denoted by BMO, is defined to be the set of all $f\in L^1_{\mathrm{loc}}$ such that

$$\|f\|_{\mathrm{BMO}}:=\sup_{\mathrm{cube}\,Q\subset\mathbb{R}^n}\fint_Q\left|f(x)-f_Q\right|dx<\infty.$$

Based on the space BMO, Sarason [140] introduced the space VMO.

Definition 2.7. The *space of functions with vanishing mean oscillation*, denoted by VMO, is defined to be the set of all $f\in\mathrm{BMO}$ such that

$$\lim_{a\to0^+}\sup_{|Q|=a}\fint_Q\left|f(x)-f_Q\right|dx=0.$$

Now, we present some properties of scalar Muckenhoupt weights in Proposition 2.11 below.

Proposition 2.11. Let $p\in[1,\infty]$ and w be a scalar weight. Then the following assertions hold.

(i) For any $\lambda\in(0,\infty)$, $[w(\lambda\cdot)]_{A_p}=[w]_{A_p}$.

(ii) For any $z\in\mathbb{R}^n$, $[w(\cdot-z)]_{A_p}=[w]_{A_p}$.

(iii) For any $\lambda\in(0,\infty)$, $[\lambda w]_{A_p}=[w]_{A_p}$.

(iv) $[w]_{A_p}\geq1$ and the equality holds if and only if w is a constant.

(v) If $q\in[p,\infty]$, then $[w]_{A_q}\leq[w]_{A_p}$.

(vi) If $p\in[1,\infty)$ and $q\in(p,\infty]$, then $A_p\subsetneqq A_q$.

(vii) If $p\in[1,\infty)$, then $A_p\subsetneqq\bigcap_{q\in(p,\infty)}A_q$.

(viii) If $p\in(1,\infty]$, then $A_p=\bigcup_{q\in[1,p)}A_q$.

(ix) If $w\in A_p$, then, for any $\alpha\in(0,1)$, $w^\alpha\in A_p$ and $[w^\alpha]_{A_p}\leq[w]_{A_p}^\alpha$.

(x) If $w\in A_p$, then there exists $\alpha\in(1,\infty)$ such that $w^\alpha\in A_p$.

(xi) $\mathrm{VMO}\subset\{\log w:\ w\in\bigcap_{q\in(1,\infty)}A_q\}$.

(xii) If $p\in(1,\infty]$, then $\mathrm{BMO}=\{\lambda\log w:\ \lambda\in\mathbb{R},\ w\in A_p\}$.

(xiii) $\mathrm{BLO}=\{\lambda\log w:\ \lambda\in[0,\infty),\ w\in A_1\}$.

(xiv) If $p\in[1,\infty)$, then, for any $\lambda\in(1,\infty)$ and any cube $Q\subset\mathbb{R}^n$, $w(\lambda Q)\le\lambda^{np}[w]_{A_p}w(Q)$.

(xv) For any $\lambda\in(1,\infty)$ and any cube $Q\subset\mathbb{R}^n$, $w(\lambda Q)\le 2^{\lambda^n}[w]_{A_\infty}^{\lambda^n}w(Q)$.

Proof. The assertions (i), (ii), and (iii) follow directly from the definition of $[w]_{A_p}$. Assertion (v) is precisely [69, Proposition 7.1.5(6)] for any $q\in[p,\infty)$ and [69, p. 525] for $q=\infty$. The assertions (viii) through (xi) are exactly [18, (3), (6), (7), and (11) on pp. 385–386]. Assertion (xii) is precisely [56, p. 151]. The assertions (xiii) and (xiv) are exactly [69, Propositions 7.15(9) and 7.3.2(6)]. It remains to prove the assertions (iv), (vi), and (vii).

We next prove (iv). When $p\in[1,\infty)$, (iv) is precisely [69, Propositions 7.1.5(5)]. When $p=\infty$, by Jensen's inequality, we have, for any cube $Q\subset\mathbb{R}^n$,

$$\fint_Q w(x)\,dx\,\exp\left(\fint_Q \log\left([w(x)]^{-1}\right)dx\right)\ge\fint_Q w(x)\,dx\left[\fint_Q w(x)\,dx\right]^{-1}=1,$$

where the equality holds if and only if w is a constant. Therefore, (iv) holds when $p=\infty$. This finishes the proof of (iv).

Now, we show (vi). By (v), we conclude that $A_p\subset A_q$. From (i) and (iii) of Example 2.2, it follows that, for any $r\in(p,q)$, $|x|^{n(r-1)}\in A_q\setminus A_p$ and hence $A_p\ne A_q$. This finishes the proof of (vi).

Finally, we prove (vii). By (v), we conclude that $A_p\subset\bigcap_{q\in(p,\infty)}A_q$. It remains to show $A_p\ne\bigcap_{q\in(p,\infty)}A_q$ by considering two cases for p. When $p\in(1,\infty)$, using Example 2.2(iii), we have $|x|^{n(p-1)}\notin A_p$ and $|x|^{n(p-1)}\in[\bigcap_{q\in(p,\infty)}A_q]\setminus A_p$. When $p=1$, from (ii) and (iii) of Example 2.2, it follows that $\log(2+|x|)\in[\bigcap_{q\in(1,\infty)}A_q]\setminus A_1$. Therefore, $A_p\ne\bigcap_{q\in(p,\infty)}A_q$. This finishes the proof of (vii) and hence Proposition 2.11. □

Remark 2.3. By Proposition 2.11(xiii), we find that

$$\mathrm{BLO}\cup-\mathrm{BLO}=\{\lambda\log w:\ \lambda\in\mathbb{R},\ w\in A_1\}.$$

It is well known that $\mathrm{BLO}\subsetneqq\mathrm{BMO}$. Indeed, $\mathrm{BLO}\cup-\mathrm{BLO}$ is still a proper subset of BMO. Notice that $|x|,|x-1|^{-\frac{n}{2}}\in A_\infty$. This, together with Proposition 2.11(xii), further implies that

$$f(x):=\log|x|+\log\frac{1}{|x-1|}=\log|x|+\frac{2}{n}\log\left(|x-1|^{-\frac{n}{2}}\right)\in\mathrm{BMO}.$$

However, $\operatorname{essinf}_{x\in B(\mathbf{0},2)}f(x)=-\infty=\operatorname{essinf}_{x\in B(\mathbf{0},2)}-f(x)$ and hence $\|f\|_{\mathrm{BLO}}=\infty=\|-f\|_{\mathrm{BLO}}$. Thus,

$$\mathrm{BLO}\cup-\mathrm{BLO}\subsetneqq\mathrm{BMO}.$$

Next, we recall an interesting result, which was used in [22, p. 699], but here we require the exact constant.

Lemma 2.5. *Let $p\in(1,\infty)$. Then, for any $\theta\in(0,\infty)$ and $a,b\in(0,\infty)$,*

$$(a+b)^p\le\frac{(\theta^{p'}+1)^{\frac{p}{p'}}}{\theta^p}a^p+\left(\theta^{p'}+1\right)^{\frac{p}{p'}}b^p.$$

Proof. By Hölder's inequality, we conclude that, for any $\theta\in(0,\infty)$ and $a,b\in(0,\infty)$,

$$a+b=\theta\frac{a}{\theta}+b\leq\left(\theta^{p'}+1\right)^{\frac{1}{p'}}\left[\left(\frac{a}{\theta}\right)^p+b^p\right]^{\frac{1}{p}}=\left[\frac{(\theta^{p'}+1)^{\frac{p}{p'}}}{\theta^p}a^p+\left(\theta^{p'}+1\right)^{\frac{p}{p'}}b^p\right]^{\frac{1}{p}}.$$

This finishes the proof of Lemma 2.5. □

Now, we recall the concept of the decreasing rearrangement (see, for instance, [69, Definition 1.4.1]). Let f be a complex-valued function defined on $\mathbb{R}^n$. The *decreasing rearrangement* f^* of f is defined by setting, for any $t\in[0,\infty)$,

$$f^*(t):=\inf\{s\geq 0:\ |\{x\in\mathbb{R}^n:\ |f(x)|>s\}|\leq t\}.$$

Let $E\subset\mathbb{R}^n$, the *restriction* of f to E is the function $f|_E:\ E\to\mathbb{C}$, $x\mapsto f(x)$. Repeating the argument in [163, p. 250], we obtain the following conclusion. For the convenience of the reader, we provide the details.

Proposition 2.12. Let $p\in(1,\infty)$ and $w\in A_p$. Then, for any cube $Q\subset\mathbb{R}^n$ and any $t\in[0,\infty)$,

$$(w|_Q)^*(t)\geq\frac{(|Q|-t)_+^{p-1}}{[w]_{A_p}}\frac{w(Q)}{|Q|^p}.\tag{2.28}$$

Proof. Let Q be a cube in $\mathbb{R}^n$. When $t\in[|Q|,\infty)$, (2.28) holds obviously. It remains to consider the case $t\in[0,|Q|)$.

Applying Proposition 2.9, we obtain, for any measurable set $E\subset Q$ with $|E|\neq 0$,

$$\begin{aligned}[w]_{A_p}\geq&\sup_{\|f\mathbf{1}_Q\|_{L^p(w)}=1}w(Q)\left[\fint_Q|f(x)|dx\right]^p\\ \geq& w(Q)\left[\fint_Q\frac{\mathbf{1}_E(x)}{\|\mathbf{1}_E\|_{L^p(w)}}dx\right]^p=\frac{w(Q)}{w(E)}\left(\frac{|E|}{|Q|}\right)^p.\end{aligned}\tag{2.29}$$

Let $E_t\subset Q$ be a measurable set satisfying

(i) for any $x\in E_t$ and $y\in Q\setminus E_t$, $w(x)\leq w(y)$;

(ii) $|E_t|=|Q|-t$.

These, combined with Cavalieri's principle (see, for example, [69, Proposition 1.1.4]) and the definition of the decreasing rearrangement, further imply that

$$\begin{aligned}w(E_t)&=\int_Q w(x)\mathbf{1}_{E_t}(x)dx=\int_0^\infty|\{x\in Q:\ w(x)\mathbf{1}_{E_t}(x)>s\}|ds\\ &=\int_0^\infty(|\{x\in Q:\ w(x)>s\}|-t)_+ds=\int_0^\infty(|\{r\in[0,|Q|]:\ (w|_Q)^*(r)>s\}|-t)_+ds\end{aligned}$$

$$=\int_0^\infty |\{r\in[t,|Q|]:\ (w|_Q)^*(r)>s\}|ds=\int_t^{|Q|}(w|_Q)^*(r)\,dr.$$

From this, (2.29), and (ii), it follows that

$$\int_t^{|Q|}(w|_Q)^*(r)\,dr=w(E_t)\geq\left(\frac{|Q|-t}{|Q|}\right)^p\frac{w(Q)}{[w]_{A_p}}.$$

This, together with the fact that $(w|_Q)^*$ is a decreasing function, further implies that

$$(w|_Q)^*(t)\geq\frac{1}{|Q|-t}\int_t^{|Q|}(w|_Q)^*(r)\,dr\geq\frac{(|Q|-t)^{p-1}}{[w]_{A_p}}\frac{w(Q)}{|Q|^p}.$$

This finishes the proof of Proposition 2.12. □

With the help of Proposition 2.12, we obtain the following conclusion.

Proposition 2.13. Let w be a scalar weight. Then the following statements hold.

(i) $\lim_{q\to1^+}[w]_{A_q}=[w]_{A_1}$.

(ii) $\lim_{q\to\infty}[w]_{A_q}=[w]_{A_\infty}$.

(iii) For any $p\in(1,\infty)$, $\lim_{q\to p}[w]_{A_q}=[w]_{A_p}$.

Proof. Statement (i) is exactly [69, Proposition 7.1.5(7)] and statement (ii) is precisely the open question posed by Johnson [94, Question 1], which was solved by Sbordone and Wik [145, Theorem 1].

It remains to prove (iii), whose proof borrows some ideas from the proofs of (i) and (ii). We first show

$$\lim_{q\to p^+}[w]_{A_q}=[w]_{A_p} \tag{2.30}$$

by consider two cases for w.

Case (1) $w\in A_p$. In this case, using Proposition 2.11(v), we obtain, for $q\in(p,\infty)$, $[w]_{A_q}\leq[w]_{A_p}<\infty$ and $[w]_{A_q}$ is decreasing in q. These further imply that $\lim_{q\to p^+}[w]_{A_q}\leq[w]_{A_p}$. On the other hand, let $\varepsilon\in(0,1)$. Applying the definition of $[w]_{A_p}$, we conclude that there exists cube $Q_\varepsilon\subset\mathbb{R}^n$ such that

$$[w]_{A_p}-\varepsilon<\fint_{Q_\varepsilon}w(x)\,dx\left\{\fint_{Q_\varepsilon}[w(x)]^{-\frac{p'}{p}}\,dx\right\}^{\frac{p}{p'}}. \tag{2.31}$$

By the monotone convergence theorem, we find that

$$\fint_{Q_\varepsilon}[w(x)]^{-\frac{p'}{p}}\,dx=\frac{1}{|Q_\varepsilon|}\int_{\{x\in Q_\varepsilon:\ w(x)\geq1\}}[w(x)]^{-\frac{p'}{p}}\,dx+\frac{1}{|Q_\varepsilon|}\int_{\{x\in Q_\varepsilon:\ w(x)<1\}}\cdots$$

$$
\begin{aligned}
&= \lim_{q\to p^+} \frac{1}{|Q_\varepsilon|}\int_{\{x\in Q_\varepsilon:\ w(x)\ge 1\}} [w(x)]^{-\frac{q'}{q}}\,dx + \lim_{q\to p^+} \frac{1}{|Q_\varepsilon|}\int_{\{x\in Q_\varepsilon:\ w(x)<1\}} \cdots \\
&= \lim_{q\to p^+} \fint_{Q_\varepsilon} [w(x)]^{-\frac{q'}{q}}\,dx. \qquad (2.32)
\end{aligned}
$$

This, combined with (2.31), further implies that

$$
[w]_{A_p}-\varepsilon < \fint_{Q_\varepsilon} w(x)\,dx \lim_{q\to p^+}\left\{\fint_{Q_\varepsilon} [w(x)]^{-\frac{q'}{q}}\,dx\right\}^{\frac{q}{q'}} \le \lim_{q\to p^+}[w]_{A_q}.
$$

Letting $\varepsilon\to 0^+$, we obtain $[w]_{A_p}\le \lim_{q\to p^+}[w]_{A_q}$. This finishes the proof of (2.30) in this case.

Case (2) $w\notin A_p$. In this case, (2.30) reduces to $\lim_{q\to p^+}[w]_{A_q}=\infty$. From $w\notin A_p$ and (2.32), it follows that, for any $L\in(0,\infty)$, there exists cube $Q_L\subset\mathbb{R}^n$ such that

$$
\begin{aligned}
L &< \fint_{Q_L} w(x)\,dx\left\{\fint_{Q_L}[w(x)]^{-\frac{p'}{p}}\,dx\right\}^{\frac{p}{p'}} \\
&= \fint_{Q_L} w(x)\,dx \lim_{q\to p^+}\left\{\fint_{Q_L}[w(x)]^{-\frac{q'}{q}}\,dx\right\}^{\frac{q}{q'}} \le \lim_{q\to p^+}[w]_{A_q}.
\end{aligned}
$$

This finishes the proof of (2.30) in this case and hence (2.30).

Next, we prove

$$
\lim_{q\to p^-}[w]_{A_q}=[w]_{A_p}. \qquad (2.33)
$$

If $w\notin A_p$, then, using Proposition 2.11(v), we obtain, for any $q\in(1,p)$, $w\notin A_q$ and hence

$$
\lim_{q\to p^-}[w]_{A_q}=\infty=[w]_{A_p}.
$$

Therefore, it suffices to consider the case where $w\in A_p$. In this case, from Proposition 2.11(v), it follows that there exists $r\in[1,p)$ such that $w\in A_r$. This, together with Proposition 2.11(v), further implies that, for any $q\in[r,p)$, $[w]_{A_p}\le[w]_{A_q}\le[w]_{A_r}<\infty$ and $[w]_{A_q}$ is decreasing in q. Therefore, $[w]_{A_p}\le\lim_{q\to p^-}[w]_{A_q}$.

Now, we show $\lim_{q\to p^-}[w]_{A_q}\le[w]_{A_p}$. Let Q be a cube in $\mathbb{R}^n$ and

$$
\widetilde{w}(\cdot):=\frac{w(\ell(Q)(\cdot+c_Q))}{\fint_Q w(x)\,dx}.
$$

Applying (i), (ii), and (iii) of Proposition 2.11, we find that, for any $q\in[r,p)$,

$$
[\widetilde{w}]_{A_q}=[w]_{A_q}<\infty. \qquad (2.34)
$$

Let Q_0 be the cube with center $\mathbf{0}$ and edge length 1. Observe that $\widetilde{w}(Q_0)=1$ and, for any $q\in[r,\infty)$,

$$
\fint_Q w(x)\,dx\left\{\fint_Q[w(x)]^{-\frac{q'}{q}}\,dx\right\}^{\frac{q'}{q}}=\left\{\int_{Q_0}[\widetilde{w}(x)]^{-\frac{q'}{q}}\,dx\right\}^{\frac{q'}{q}}\le[\widetilde{w}]_{A_q}=[w]_{A_q}<\infty. \qquad (2.35)
$$

From [69, Propositions 1.1.4 and 1.4.5(8)] and (2.35), it follows that, for any $q\in(1,\infty)$,

$$\begin{aligned}\int_{Q_0}[\widetilde{w}(x)]^{-\frac{q'}{q}}\,dx&=\frac{q'}{q}\int_0^\infty s^{\frac{q'}{q}-1}\left|\left\{x\in Q_0:\ [\widetilde{w}(x)]^{-1}>s\right\}\right|ds\\&=\frac{q'}{q}\int_0^\infty s^{\frac{q'}{q}-1}\left|\left\{t\in[0,1]:\ [(\widetilde{w}|_{Q_0})^*(t)]^{-1}>s\right\}\right|ds\\&=\int_0^1[(\widetilde{w}|_{Q_0})^*(t)]^{-\frac{q'}{q}}\,dt.\end{aligned}\tag{2.36}$$

By (2.34) with $q=r$, Proposition 2.12, and $\widetilde{w}(Q_0)=1$, we conclude that, for any $t\in[0,\infty)$,

$$(\widetilde{w}|_{Q_0})^*(t)\geq\frac{(1-t)_+^{r-1}}{[\widetilde{w}]_{A_r}}=\frac{(1-t)_+^{r-1}}{[w]_{A_r}}.\tag{2.37}$$

Let $\delta\in(0,1)$ and $\theta\in(0,\infty)$. For any $s\in(1,\infty)$, define

$$C_1(\theta,s):=\frac{(\theta^{s'}+1)^{\frac{s}{s'}}}{\theta^s}\quad\text{and}\quad C_2(\theta,s):=\left(\theta^{s'}+1\right)^{\frac{s}{s'}}.$$

Using (2.36) and Lemma 2.5, we find that, for any $q\in(r,p)$,

$$\begin{aligned}\left\{\int_{Q_0}[\widetilde{w}(x)]^{-\frac{q'}{q}}\,dx\right\}^{\frac{q}{q'}}&\leq C_1\left(\theta,\frac{q}{q'}\right)\left\{\int_0^\delta[(\widetilde{w}|_{Q_0})^*(t)]^{-\frac{q'}{q}}\,dt\right\}^{\frac{q}{q'}}\\&\quad+C_2\left(\theta,\frac{q}{q'}\right)\left\{\int_\delta^1[(\widetilde{w}|_{Q_0})^*(t)]^{-\frac{q'}{q}}\,dt\right\}^{\frac{q}{q'}}\\&=:I(\delta,\theta)+J(\delta,\theta).\end{aligned}\tag{2.38}$$

We first estimate $I(\delta,\theta)$. Notice that $\frac{p'}{p}-\frac{q'}{q}=p'-q'<0$. This, combined with (2.37), further implies that

$$\begin{aligned}I(\delta,\theta)&=C_1\left(\theta,\frac{q}{q'}\right)\left\{\int_0^\delta[(\widetilde{w}|_{Q_0})^*(t)]^{-\frac{p'}{p}}[(\widetilde{w}|_{Q_0})^*(t)]^{\frac{p'}{p}-\frac{q'}{q}}\,dt\right\}^{\frac{q}{q'}}\\&\leq C_1\left(\theta,\frac{q}{q'}\right)\left[\frac{(1-\delta)^{r-1}}{[w]_{A_r}}\right]^{(\frac{p'}{p}-\frac{q'}{q})\frac{q}{q'}}\left\{\int_0^\delta[(\widetilde{w}|_{Q_0})^*(t)]^{-\frac{p'}{p}}\,dt\right\}^{\frac{q}{q'}}.\end{aligned}\tag{2.39}$$

Applying both (2.36) and (2.35) with q replaced by p, we conclude that

$$\int_0^\delta[(\widetilde{w}|_{Q_0})^*(t)]^{-\frac{p'}{p}}\,dt\leq\int_0^1[(\widetilde{w}|_{Q_0})^*(t)]^{-\frac{p'}{p}}\,dt=\int_{Q_0}[\widetilde{w}(x)]^{-\frac{p'}{p}}\,dx\leq[w]_{A_p}^{\frac{p'}{p}}.$$

This, together with (2.39), further implies that

$$I(\delta,\theta)\leq C_1\left(\theta,\frac{q}{q'}\right)\left[\frac{(1-\delta)^{r-1}}{[w]_{A_r}}\right]^{(\frac{p'}{p}-\frac{q'}{q})\frac{q}{q'}}[w]_{A_p}^{\frac{p'}{p}\frac{q}{q'}}.\tag{2.40}$$

Next, we estimate $J(\delta,\theta)$. Observe that $(r-1)\frac{q'}{q}=\frac{r-1}{q-1}\in(0,1)$. This, combined with (2.37), further implies that

$$\begin{aligned} J(\delta,\theta) &\le C_2\left(\theta,\frac{q}{q'}\right)[w]_{A_r}\left[\int_\delta^1 (1-t)^{-(r-1)\frac{q'}{q}}\,dt\right]^{\frac{q}{q'}} \\ &\le C_2\left(\theta,\frac{q}{q'}\right)[w]_{A_r}\left[1-(r-1)\frac{q'}{q}\right]^{-\frac{q}{q'}}(1-\delta)^{\frac{q}{q'}-(r-1)}. \end{aligned} \tag{2.41}$$

By (2.35), (2.38), (2.40), and (2.41), we have, for any $q\in(r,p)$ and $\delta,\theta\in(0,1)$,

$$\begin{aligned} [w]_{A_q} \le{}& C_1\left(\theta,\frac{q}{q'}\right)\left[\frac{(1-\delta)^{r-1}}{[w]_{A_r}}\right]^{(\frac{p'}{p}-\frac{q'}{q})\frac{q}{q'}}[w]_{A_p}^{\frac{p'}{p}\frac{q}{q'}} \\ &+C_2\left(\theta,\frac{q}{q'}\right)[w]_{A_r}\left[1-(r-1)\frac{q'}{q}\right]^{-\frac{q}{q'}}(1-\delta)^{\frac{q}{q'}-(r-1)}. \end{aligned}$$

Letting $q\to p^-$, we obtain

$$\lim_{q\to p^-}[w]_{A_q}\le C_1\left(\theta,\frac{p}{p'}\right)[w]_{A_p}+C_2\left(\theta,\frac{p}{p'}\right)[w]_{A_r}\left[1-(r-1)\frac{p'}{p}\right]^{-\frac{p}{p'}}(1-\delta)^{\frac{p}{p'}-(r-1)}.$$

Notice that $\frac{p}{p'}-(r-1)=p-r>0$. Letting $\delta\to1^-$, we find that

$$\lim_{q\to p^-}[w]_{A_q}\le C_1\left(\theta,\frac{p}{p'}\right)[w]_{A_p}.$$

Finally, letting $\theta\to\infty$, we obtain $\lim_{q\to p^-}[w]_{A_q}\le[w]_{A_p}$. This finishes the proof of (2.33) and hence Proposition 2.13. □

The following conclusion is an immediate consequence of Proposition 2.13.

Corollary 2.1. *Let w be a scalar weight. Then the following assertions hold.*

(i) *If* $p\in(1,\infty]$*, then* $w\in A_p$ *if and only if* $\inf_{q\in(1,p)}[w]_{A_q}<\infty$.

(ii) *If* $p\in[1,\infty)$*, then* $w\in A_p$ *if and only if* $\sup_{q\in(p,\infty)}[w]_{A_q}<\infty$.

Remark 2.4. Corollary 2.1(i) also follows from Proposition 2.11(viii).

The reverse Hölder's inequality is a significant characterization of A_∞. Based on the A_∞ condition introduced by Wilson [163], that is, $[w]_{A_\infty^*}<\infty$ [see Proposition 2.10(viii)], Hytönen and Pérez [83, Theorem 2.3(a)] improved the classic reverse Hölder's inequality and obtained the sharp constant.

Theorem 2.4. *If $w\in A_\infty$, then, for any $r\in(0,1+\frac{1}{2^{11+n}[w]_{A_\infty^*}}]$ and any cube $Q\subset\mathbb{R}^n$,*

$$\left\{\fint_Q [w(x)]^r\,dx\right\}^{\frac{1}{r}}\le 2\fint_Q w(x)\,dx.$$

Remark 2.5. It is worth noting that the constant in the reverse Hölder's inequality stated in Theorem 2.4 is exactly 2, and it is independent of the characteristic constant of w. Using this sharp reverse Hölder's inequality, Hytönen and Pérez [83] improved several weighted inequalities. An extension of Theorem 2.4 to spaces of homogeneous type was later established by Anderson et al. [3]. More recently, Grafakos [70, Theorem 8.6.9] established a reverse Hölder's inequality for any A_p weight with $p\in(1,\infty)$.

2.3 Properties of Matrix Muckenhoupt Weights

In this subsection, corresponding to Subsection 2.2, we study the analogous properties of matrix Muckenhoupt weights.

Now, we recall the definition of reducing operators introduced by Volberg [155], which is a common tool in the related study of matrix weights.

Definition 2.8. Let $p\in(0,\infty)$, W be a matrix weight, and $E\subset\mathbb{R}^n$ a bounded measurable set satisfying $|E|\in(0,\infty)$. The matrix $A_E\in M_m(\mathbb{C})$ is called a *reducing operator* of order p for W if A_E is positive definite and there exists a positive constant C, depending only on m and p, such that, for any $\vec{z}\in\mathbb{C}^m$,

$$C^{-1}\left|A_E\vec{z}\right|\le\left[\fint_E\left|W^{\frac{1}{p}}(x)\vec{z}\right|^p dx\right]^{\frac{1}{p}}\le C|A_E\vec{z}|.$$

Remark 2.6. In Definition 2.8, the existence of A_E is guaranteed by [67, Proposition 1.2] for any $p\in(1,\infty)$ and [63, p. 1237] for any $p\in(0,1]$. Actually, when $p\in(1,\infty)$, by [67, Proposition 1.2], we find that there exists a positive definite matrix A_E such that

$$m^{-\frac{1}{2}}|A_E\vec{z}|\le\left[\fint_E\left|W^{\frac{1}{p}}(x)\vec{z}\right|^p dx\right]^{\frac{1}{p}}\le|A_E\vec{z}|.$$

The following proposition is precisely [23, Proposition 2.14].

Proposition 2.14. Let $p\in(0,\infty)$, W be a matrix weight, and $E\subset\mathbb{R}^n$ a bounded measurable set satisfying $|E|\in(0,\infty)$. Then A_E is a reducing operator of order p for W if and only if, for any matrix $M\in M_m(\mathbb{C})$,

$$\|A_E M\|\sim\left[\fint_E\left\|W^{\frac{1}{p}}(x)M\right\|^p dx\right]^{\frac{1}{p}},$$

where the positive equivalence constants depend only on m and p.

In correspondence with Proposition 2.7, we have the following conclusion, where the statements (i), (ii), and (iii) are exactly [138, Corollary 3.3], [24, Corollaries 2.15 and 2.17], and [27, Lemma 3.5].

Lemma 2.6. *Let W be a matrix weight and $\{A_Q\}_{\mathrm{cube}\,Q}$ be a family of reducing operators of order p for W. Then the following statements hold.*

(i) *If $p\in(1,\infty)$, then $W\in\mathscr{A}_p$ if and only if $W^{1-p'}\in\mathscr{A}_{p'}$. Moreover, for any $W\in\mathscr{A}_p$, $[W]_{\mathscr{A}_p}^{p'-1}\sim[W^{1-p'}]_{\mathscr{A}_{p'}}$, where the positive equivalence constants depend only on m and p.*

(ii) *If $p\in(1,\infty)$ and $W\in\mathscr{A}_p$, then, for any cube $Q\subset\mathbb{R}^n$ and any $M\in M_m(\mathbb{C})$,*

$$\left\|A_Q^{-1}M\right\|\sim\left[\fint_Q\left\|W^{-\frac{1}{p}}(x)M\right\|^{p'}dx\right]^{\frac{1}{p'}},$$

where the positive equivalence constants depend only on m, p, and $[W]_{\mathscr{A}_p}$.

(iii) *If $p\in(0,1]$ and $W\in\mathscr{A}_p$, then, for any cube $Q\subset\mathbb{R}^n$ and any $M\in M_m(\mathbb{C})$,*

$$\left\|A_Q^{-1}M\right\|\sim\operatorname*{ess\,sup}_{x\in Q}\left\|W^{-\frac{1}{p}}(x)M\right\|,$$

where the positive equivalence constants depend only on m, p, and $[W]_{\mathscr{A}_p}$.

(iv) *If $p\in(0,\infty)$ and $W\in\mathscr{A}_{p,\infty}$, then, for any cube $Q\subset\mathbb{R}^n$ and any $M\in M_m(\mathbb{C})$,*

$$\left\|A_Q^{-1}M\right\|\sim\exp\left(\fint_Q\log\left(\left\|W^{-\frac{1}{p}}(x)M\right\|\right)dx\right),$$

where the positive equivalence constants depend only on m, p, and $[W]_{\mathscr{A}_{p,\infty}}$.

Nazarov and Treil [118, Subsection 11.2] showed a matrix-weighted version of Proposition 2.8 in the case where $n=1$ by invoking the inverse volume of weights. Instead of this, using the equivalent definitions of $\mathscr{A}_p$-matrix weights and $\mathscr{A}_{p,\infty}$-matrix weights given, respectively, by Roudenko [138, Corollary 3.3] (see also Definition 2.3) and by Bu et al. [27, Definition 3.1] (see also Definition 2.4), we give another simpler proof for all $n\in\mathbb{N}$.

Proposition 2.15. Let $p\in(1,\infty)$ and W be a matrix weight. Then $W\in\mathscr{A}_p$ if and only if $W\in\mathscr{A}_{p,\infty}$ and $W^{-\frac{p'}{p}}\in\mathscr{A}_{p',\infty}$.

Proof. If $W\in\mathscr{A}_p$, then, by Lemma 2.6(i) and [27, Proposition 4.2(i)], we find that

$$W\in\mathscr{A}_p\subset\mathscr{A}_{p,\infty}\quad\text{and}\quad W^{-\frac{p'}{p}}\in\mathscr{A}_{p'}\subset\mathscr{A}_{p',\infty}.$$

On the other hand, assume that $W\in\mathscr{A}_{p,\infty}$ and $W^{-\frac{p'}{p}}\in\mathscr{A}_{p',\infty}$. From the invertibility of W, it follows that, for any cube $Q\subset\mathbb{R}^n$ and almost every $z\in Q$,

$$\begin{aligned}I(Q)&:=\fint_Q\left[\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{p'}dy\right]^{\frac{p}{p'}}dx\\&\le\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(z)\right\|^{p}dx\left[\fint_Q\left\|W^{\frac{1}{p}}(z)W^{-\frac{1}{p}}(y)\right\|^{p'}dy\right]^{\frac{p}{p'}}\end{aligned}$$

and hence

$$\begin{aligned}\log I(Q)&\le\fint_Q\log\left(\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(z)\right\|^{p}dx\right)dz\\&\quad+\frac{p}{p'}\fint_Q\log\left(\fint_Q\left\|W^{\frac{1}{p}}(z)W^{-\frac{1}{p}}(y)\right\|^{p'}dy\right)dz\\&\le\log[W]_{\mathscr{A}_{p,\infty}}+\frac{p}{p'}\log\left[W^{-\frac{p'}{p}}\right]_{\mathscr{A}_{p',\infty}},\end{aligned}$$

which further implies that

$$[W]_{A_p}=\sup_{\text{cube}\,Q}I(Q)\le[W]_{\mathscr{A}_{p,\infty}}\left[W^{-\frac{p'}{p}}\right]_{\mathscr{A}_{p',\infty}}^{\frac{p}{p'}}<\infty.$$

This finishes the proof of Proposition 2.15. □

Next, we provide a characterization of $\mathscr{A}_p$ for any $p\in(0,1]$ in terms of $\mathscr{A}_{p,\infty}$.

Proposition 2.16. Let $p\in(0,1]$ and W be a matrix weight. Then $W\in\mathscr{A}_p$ if and only if $W\in\mathscr{A}_{p,\infty}$ and

$$\mathrm{J}:=\sup_{\text{cube}\,Q}\exp\left(\fint_Q\log\left(\operatorname*{ess\,sup}_{y\in Q}\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{p}\right)dx\right)<\infty.$$

Proof. Assume that $W\in\mathscr{A}_p$. Applying [27, Proposition 4.2(i)], we conclude that $W\in\mathscr{A}_{p,\infty}$. By Jensen's inequality and Proposition 2.4, we have

$$\mathrm{J}\le\sup_{\text{cube}\,Q}\fint_Q\operatorname*{ess\,sup}_{y\in Q}\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{p}dx\sim[W]_{A_p}.$$

On the other hand, assume $W\in\mathscr{A}_{p,\infty}$ and $\mathrm{J}<\infty$. Using the invertibility of W, we find that, for any cube $Q\subset\mathbb{R}^n$ and almost every $z\in Q$,

$$\begin{aligned}I(Q)&:=\operatorname*{ess\,sup}_{y\in Q}\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{p}dx\\&\le\left[\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(z)\right\|^{p}dx\right]\operatorname*{ess\,sup}_{y\in Q}\left\|W^{\frac{1}{p}}(z)W^{-\frac{1}{p}}(y)\right\|^{p}\end{aligned}$$

and hence

$$\begin{aligned}\log I(Q) &\le \fint_Q \log\left(\fint_Q \left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(z)\right\|^p dx\right)dz\\ &\quad+\fint_Q \log\left(\operatorname*{ess\,sup}_{y\in Q}\left\|W^{\frac{1}{p}}(z)W^{-\frac{1}{p}}(y)\right\|^p\right)dz\\ &\le \log[W]_{\mathscr{A}_{p,\infty}}+\log \mathrm{J},\end{aligned}$$

which further implies that $[W]_{\mathscr{A}_p}=\sup_{\mathrm{cube}\,Q} I(Q)\le [W]_{\mathscr{A}_{p,\infty}}\mathrm{J}<\infty$. This finishes the proof of Proposition 2.16. □

When $m=1$, Proposition 2.16 reduces to the following conclusion, which can be formally regarded as the duality of A_1 weights.

Proposition 2.17. Let w be a scalar weight. Then $w\in A_1$ if and only if $w\in A_\infty$ and

$$\sup_{\mathrm{cube}\,Q}\left\|w^{-1}\right\|_{L^\infty(Q)}\exp\left(\fint_Q \log(w(x))\,dx\right)<\infty. \tag{2.42}$$

Remark 2.7. Let w be a scalar weight. Observe that

$$\begin{aligned} w\in A_1 &\Longleftrightarrow \text{ for any cube } Q\subset\mathbb{R}^n,\ \fint_Q w(x)\,dx\sim\left\|w^{-1}\right\|_{L^\infty(Q)}^{-1},\\ w\in A_\infty &\Longleftrightarrow \text{ for any cube } Q\subset\mathbb{R}^n,\ \fint_Q w(x)\,dx\sim\exp\left(\fint_Q \log w(x)\,dx\right),\end{aligned}$$

and

$$(2.42)\Longleftrightarrow \text{ for any cube } Q\subset\mathbb{R}^n,\ \left\|w^{-1}\right\|_{L^\infty(Q)}^{-1}\sim\exp\left(\fint_Q \log w(x)\,dx\right).$$

Therefore, Proposition 2.17 is quite natural.

Now, we extend Proposition 2.9 to the matrix-weighted setting.

Proposition 2.18. Let $p\in(0,1]$ and W be a matrix weight. Then $W\in\mathscr{A}_p$ if and only if

$$[W]_{\mathscr{A}_p}^{\#}:=\sup_{\mathrm{cube}\,Q}\sup_{H\in\mathcal{H}_Q}\fint_Q\fint_Q\left\|W^{\frac{1}{p}}(x)H(y)\right\|^p dx\,dy<\infty,$$

where

$$\mathcal{H}_Q:=\left\{H:\ \mathbb{R}^n\to M_m(\mathbb{C}) \text{ measurable}:\ \fint_Q\left\|W^{\frac{1}{p}}(y)H(y)\right\|^p dy=1\right\}. \tag{2.43}$$

Moreover, for any $W\in\mathscr{A}_p$, $[W]_{\mathscr{A}_p}=[W]_{\mathscr{A}_p}^{\#}$.

Proof. By the definition of $[W]_{\mathscr{A}_p}$, we conclude that, for any cube $Q\subset\mathbb{R}^n$ and any $H\in\mathcal{H}_Q$,

$$\begin{aligned}\fint_Q\fint_Q\left\|W^{\frac{1}{p}}(x)H(y)\right\|^p dxdy&\le\fint_Q\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\left\|W^{\frac{1}{p}}(y)H(y)\right\|^p dy\\&\le\fint_Q[W]_{\mathscr{A}_p}\left\|W^{\frac{1}{p}}(y)H(y)\right\|^p dy=[W]_{\mathscr{A}_p}\end{aligned}$$

and hence $[W]^{\#}_{\mathscr{A}_p}\le[W]_{\mathscr{A}_p}$.

On the other hand, let $[W]^{\#}_{\mathscr{A}_p}<\infty$. Then, for any cubes $Q,R\subset\mathbb{R}^n$ with $R\subset Q$,

$$\begin{aligned}&\fint_R\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dxdy\\&\quad=\fint_Q\fint_Q\left\|W^{\frac{1}{p}}(x)\left[\left(\frac{|Q|}{|R|}\right)^{\frac{1}{p}}W^{-\frac{1}{p}}(y)\mathbf{1}_R(y)\right]\right\|^p dxdy\le[W]^{\#}_{\mathscr{A}_p}<\infty,\end{aligned}\tag{2.44}$$

which further implies that, for any cube $Q\subset\mathbb{R}^n$, $\fint_Q\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(\cdot)\|^p dx\in L^1_{\rm loc}$. This, together with Lebesgue's differentiation theorem and (2.44), further implies that, for almost every $z\in Q$, there exists a sequence of cubes $\{R_k\}_{k=1}^\infty$ contained in Q, with $R_1\supset R_2\supset\cdots$, $\lim_{k\to\infty}\ell(R_k)=0$, and $z\in\bigcap_{k=1}^\infty R_k$, such that

$$\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(z)\right\|^p dx=\lim_{k\to\infty}\fint_{R_k}\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dxdy\le[W]^{\#}_{\mathscr{A}_p}$$

and hence $[W]_{\mathscr{A}_p}\le[W]^{\#}_{\mathscr{A}_p}$. This finishes the proof of Proposition 2.18. $\square$

To establish an analogue of Proposition 2.18 for $\mathscr{A}_p$ with $p\in(1,\infty)$, we need the following technical lemma.

Lemma 2.7. *Let $p\in(1,\infty)$ and W be a matrix weight. Then, for any cube $Q\subset\mathbb{R}^n$ and any measurable matrix-valued function $M\colon\mathbb{R}^n\to M_m(\mathbb{C})$,*

$$\sup_{H\in\mathcal{H}_Q}\fint_Q\|M(x)H(x)\|dx=\left[\fint_Q\left\|M(x)W^{-\frac{1}{p}}(x)\right\|^{p'}dx\right]^{\frac{1}{p'}},$$

where $\mathcal{H}_Q$ is as in (2.43).

Proof. Using Hölder's inequality, we obtain, for any $H\in\mathcal{H}_Q$,

$$\begin{aligned}\fint_Q\|M(x)H(x)\|dx&\le\fint_Q\left\|M(x)W^{-\frac{1}{p}}(x)\right\|\left\|W^{\frac{1}{p}}(x)H(x)\right\|dx\\&\le\left[\fint_Q\left\|M(x)W^{-\frac{1}{p}}(x)\right\|^{p'}dx\right]^{\frac{1}{p'}}=:I(M).\end{aligned}$$

Next, we prove

$$\sup_{H\in\mathcal{H}_Q}\fint_Q \|M(x)H(x)\|dx \ge I(M) \tag{2.45}$$

by consider three cases for $I(M)$.

Case (1) $I(M)=0$. In this case, (2.45) holds obviously.

Case (2) $I(M)\in(0,\infty)$. In this case, let

$$H:=W^{-\frac{1}{p}}\left\|MW^{-\frac{1}{p}}\right\|^{p'-1}\left[\fint_Q\left\|M(y)W^{-\frac{1}{p}}(y)\right\|^{p'}dy\right]^{-\frac{1}{p}}.$$

Then $H\in\mathcal{H}_Q$ and

$$\fint_Q\|M(x)H(x)\|dx=\left[\fint_Q\left\|M(x)W^{-\frac{1}{p}}(x)\right\|^{p'}dx\right]^{1-\frac{1}{p}}=I(M).$$

This finishes the proof of (2.45) in this case.

Case (3) $I(M)=\infty$. In this case, for any $k\in\mathbb{N}$ and $x\in\mathbb{R}^n$,

$$M_k(x):=\begin{cases} M(x) & \text{if } \|M(x)W^{-\frac{1}{p}}(x)\|<k,\\ 0 & \text{otherwise.}\end{cases}$$

Applying the monotone convergence theorem and Case (2), we find that

$$\begin{aligned}\sup_{H\in\mathcal{H}_Q}\fint_Q\|M(x)H(x)\|dx&=\sup_{H\in\mathcal{H}_Q}\sup_{k\in\mathbb{N}}\fint_Q\|M_k(x)H(x)\|dx\\&=\sup_{k\in\mathbb{N}}\sup_{H\in\mathcal{H}_Q}\fint_Q\|M_k(x)H(x)\|dx\ge\sup_{k\in\mathbb{N}}I(M_k)=I(M).\end{aligned}$$

This finishes the proof of (2.45) in this case and hence Lemma 2.7. □

Remark 2.8. Let all the symbols be as in Lemma 2.7. By repeating the proof of Lemma 2.7 with some slight modifications, we obtain

$$\sup_{H\in\mathcal{H}}\int_{\mathbb{R}^n}\|M(x)H(x)\|dx=\left[\int_{\mathbb{R}^n}\left\|M(x)W^{-\frac{1}{p}}(x)\right\|^{p'}dx\right]^{\frac{1}{p'}},$$

where

$$\mathcal{H}:=\left\{H:\,\mathbb{R}^n\to M_m(\mathbb{C})\text{ measurable}:\,\int_{\mathbb{R}^n}\left\|W^{\frac{1}{p}}(y)H(y)\right\|^p dy=1\right\}.$$

Proposition 2.19. Let $p\in(1,\infty)$ and W be a matrix weight. Then $W\in\mathscr{A}_p$ if and only if

$$[W]^{\sharp}_{\mathscr{A}_p}:=\sup_{\mathrm{cube}\,Q}\sup_{H\in\mathcal{H}_Q}\left\{⨍_Q\left[⨍_Q\left\|W^{\frac{1}{p}}(x)H(y)\right\|^p dx\right]^{\frac{1}{p}}dy\right\}^p<\infty,$$

where $\mathcal{H}_Q$ is as in (2.43). Moreover, for any $W\in\mathscr{A}_p$, $[W]_{\mathscr{A}_p}\sim[W]^{\sharp}_{\mathscr{A}_p}$, where the positive equivalence constants depend only on m and p.

Proof. Let $\{A_Q\}_{\mathrm{cube}\,Q}$ be a family of reducing operators of order p for W. From Proposition 2.14 and Lemma 2.7, we infer that, for any cube $Q\subset\mathbb{R}^n$,

$$\begin{aligned}
&\sup_{H\in\mathcal{H}_Q}\left\{⨍_Q\left[⨍_Q\left\|W^{\frac{1}{p}}(x)H(y)\right\|^p dx\right]^{\frac{1}{p}}dy\right\}^p\\
&\sim\sup_{H\in\mathcal{H}_Q}\left\{⨍_Q\left\|A_QH(y)\right\|dy\right\}^p=\left\{⨍_Q\left\|A_QW^{-\frac{1}{p}}(y)\right\|^{p'}dy\right\}^{\frac{p}{p'}}\\
&\sim\left\{⨍_Q\left[⨍_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\right]^{\frac{p'}{p}}dy\right\}^{\frac{p}{p'}},
\end{aligned}$$

which, combined with Lemma 2.6, further implies that

$$[W]^{\sharp}_{\mathscr{A}_p}\sim\left[W^{1-p'}\right]^{\frac{1}{p'-1}}_{\mathscr{A}_{p'}}\sim[W]_{\mathscr{A}_p}.$$

This finishes the proof of Proposition 2.19. □

Remark 2.9. The combination of Propositions 2.18 and 2.19 reduces to Proposition 2.9 in the case where $m=1$.

A similar equivalent characterization also holds for $\mathscr{A}_{p,\infty}$ (see [27, Proposition 3.8]), which reduces to Proposition 2.10(ii) in the case where $m=1$.

Proposition 2.20. Let $p\in(0,\infty)$ and W be a matrix weight. Then $W\in\mathscr{A}_{p,\infty}$ if and only if

$$[W]^{\#}_{\mathscr{A}_{p,\infty}}:=\sup_{\mathrm{cube}\,Q}\sup_{H\in\mathcal{F}_Q}\exp\left(⨍_Q\log\left(⨍_Q\left\|W^{\frac{1}{p}}(x)H(y)\right\|^p dx\right)dy\right)<\infty,$$

where

$$\begin{aligned}
\mathcal{F}_Q:=\Bigg\{&H:\ \mathbb{R}^n\to M_m(\mathbb{C})\text{ measurable: }⨍_Q\left\|W^{\frac{1}{p}}(y)H(y)\right\|^p dy=1,\\
&\log_+\left(⨍_Q\left\|W^{\frac{1}{p}}(x)H(\cdot)\right\|^p dx\right)\in L^1(Q)\Bigg\}.
\end{aligned}$$

Moreover, for any $W\in\mathscr{A}_{p,\infty}$, $[W]_{\mathscr{A}_{p,\infty}}=[W]^{\#}_{\mathscr{A}_{p,\infty}}$.

The following characterizations of $\mathscr{A}_{p,\infty}$ are precisely [27, Proposition 3.7]; in particular, Proposition 2.21(ii) in the case where $n=1$ and $p\in(1,\infty)$ is the original definition of $\mathscr{A}_{p,\infty}$-matrix weights given by Volberg [155, (2.2)].

Proposition 2.21. Let $p\in(0,\infty)$ and W be a matrix weight. For any cube $Q\subset\mathbb{R}^n$, let A_Q be the reducing operator of order p for W as in Definition 2.8. Then there exists a positive constant C such that the following assertions are equivalent.

(i) $W\in\mathscr{A}_{p,\infty}$.

(ii) For any cube $Q\subset\mathbb{R}^n$ and any $\vec{z}\in\mathbb{C}^m$,

$$\exp\left(\fint_Q \log\left|W^{-\frac{1}{p}}(x)\vec{z}\right|dx\right)\le C\sup_{\vec{u}\in\mathbb{C}^m\setminus\{\vec{\mathbf{0}}\}}|(\vec{z},\vec{u})|\left[\fint_Q\left|W^{\frac{1}{p}}(x)\vec{u}\right|^p dx\right]^{-\frac{1}{p}}.$$

(iii) For any cube $Q\subset\mathbb{R}^n$ and any $\vec{z}\in\mathbb{C}^m$,

$$\exp\left(\fint_Q \log\left|W^{-\frac{1}{p}}(x)\vec{z}\right|dx\right)\le C\left|A_Q^{-1}\vec{z}\right|.$$

(iv) For any $\vec{v}\in\mathbb{C}^m$ with $|\vec{v}|=1$,

$$\sup_{\text{cube}\,Q}\exp\left(\fint_Q \log_+\left|W^{-\frac{1}{p}}(x)A_Q\vec{v}\right|dx\right)\le C.$$

(v) For any $U\in M_m(\mathbb{C})$ with $\|U\|=1$,

$$\sup_{\text{cube}\,Q}\exp\left(\fint_Q \log_+\left\|W^{-\frac{1}{p}}(x)A_QU\right\|dx\right)\le C.$$

(vi)

$$\sup_{\text{cube}\,Q}\exp\left(\fint_Q \log_+\left\|W^{-\frac{1}{p}}(x)A_Q\right\|dx\right)\le C.$$

(vii)

$$\sup_{\text{cube}\,Q}\exp\left(\fint_Q \log_+\left(\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|dx\right)dy\right)\le C.$$

(viii) For any cube $Q\subset\mathbb{R}^n$ and any $M\in M_m(\mathbb{C})$,

$$\exp\left(\fint_Q \log\left\|W^{-\frac{1}{p}}(x)M\right\|dx\right)\le C\left\|A_Q^{-1}M\right\|.$$

When $m=1$, Proposition 2.21 reduces to the following conclusion.

Proposition 2.22. Let w be a scalar weight. Then the following statements are equivalent.

(i) $w\in A_\infty$.

(ii)
$$\sup_{\mathrm{cube}\,Q\subset\mathbb{R}^n}\exp\left(\fint_Q\log\left(\fint_Q w(x)\,dx[w(y)]^{-1}\right)dy\right)<\infty.$$

(iii)
$$\sup_{\mathrm{cube}\,Q\subset\mathbb{R}^n}\exp\left(\fint_Q\log_+\left(\fint_Q w(x)\,dx[w(y)]^{-1}\right)dy\right)<\infty.$$

Similarly to A_∞ weights, $\mathscr{A}_{p,\infty}$ weights also exhibit a self-improvement property (see [27, Proposition 4.1]).

Proposition 2.23. Let $p\in(0,\infty)$ and W be a matrix weight. Then $W\in\mathscr{A}_{p,\infty}$ if and only if there exists $u\in(0,\infty)$ such that

$$\sup_{\mathrm{cube}\,Q\subset\mathbb{R}^n}\fint_Q\left\|A_QW^{-\frac{1}{p}}(x)\right\|^u dx<\infty,$$

where A_Q is the reducing operator of order p for W.

Remark 2.10. When $m=1$, Proposition 2.23 reduces to $A_\infty=\bigcup_{p\in[1,\infty)}A_p$.

Let $p\in(0,\infty)$ and $W\in\mathscr{A}_{p,\infty}$. By [27, Lemma 5.3], we find that, for any nonzero matrix $M\in M_m(\mathbb{C})$, $w_M:=\|W^{\frac{1}{p}}M\|^p\in A_\infty$. Based on this, we define

$$[W]^{\mathrm{sc}}_{\mathscr{A}_{p,\infty}}:=\sup_{M\in M_m(\mathbb{C})\setminus\{O_m\}}[w_M]^*_{A_\infty},$$

where $[\cdot]^*_{A_\infty}$ is as in Proposition 2.10(viii).

Proposition 2.24. Let $p\in(0,\infty)$, $W\in\mathscr{A}_{p,\infty}$, and $\{A_Q\}_{\mathrm{cube}\,Q}$ be a family of reducing operators of order p for W. Then the following assertions hold.

(i) For any $M\in M_m(\mathbb{C})$, any $r\in[1,1+\frac{1}{2^{n+1}[W]^{\mathrm{sc}}_{\mathscr{A}_{p,\infty}}-1}]$, and any cube $Q\subset\mathbb{R}^n$,

$$\left[\fint_Q\left\|W^{\frac{1}{p}}(x)M\right\|^{pr}dx\right]^{\frac{1}{r}}\le 2^{\frac{1}{r}}\fint_Q\left\|W^{\frac{1}{p}}(x)M\right\|^p dx.$$

(ii) There exist $\delta\in(0,1)$ and $C\in(0,\infty)$ such that, for any cube $Q\subset\mathbb{R}^n$ and any measurable set $E\subset Q$,

$$\frac{1}{|Q|}\int_E\left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p dx\le C\left(\frac{|E|}{|Q|}\right)^\delta.$$

(iii) There exist $\delta\in(0,1)$ and $C\in(0,\infty)$ such that, for any cube $Q\subset\mathbb{R}^n$ and any measurable set $E\subset Q$,

$$\frac{|E|}{|Q|}\le C\left[\frac{1}{|Q|}\int_E\left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p dx\right]^{\delta}.$$

(iv) For any $\beta\in(0,1)$, there exists $\alpha\in(0,1)$ such that, for any cube $Q\subset\mathbb{R}^n$ and any measurable set $E\subset Q$, if $|E|<\alpha|Q|$, then

$$\frac{1}{|Q|}\int_E\left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p dx<\beta.$$

(v) For any $\beta\in(0,1)$, there exists $\alpha\in(0,1)$ such that, for any cube $Q\subset\mathbb{R}^n$ and any measurable set $E\subset Q$, if

$$\frac{1}{|Q|}\int_E\left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p dx<\alpha,$$

then $|E|<\beta|Q|$.

(vi) There exist $\delta,C\in(0,\infty)$ such that, for any cube $Q\subset\mathbb{R}^n$ and any $\alpha\in(0,\infty)$,

$$\left|\left\{x\in Q:\ \left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p\le\alpha\right\}\right|\le C\alpha^{\delta}|Q|.$$

(vii) There exist $\delta,C\in(0,\infty)$ such that, for any cube $Q\subset\mathbb{R}^n$ and any $\alpha\in(0,\infty)$,

$$\left|\left\{x\in Q:\ \left\|A_QW^{-\frac{1}{p}}(x)\right\|^p\ge\alpha\right\}\right|\le C\alpha^{-\delta}|Q|.$$

(viii) There exists a positive constant C, depending only on m and p, such that, for any cube $Q\subset\mathbb{R}^n$ and any $M\in(0,\infty)$,

$$\left|\left\{x\in Q:\ \left\|A_QW^{-\frac{1}{p}}(x)\right\|^p\ge e^M\right\}\right|\le\frac{\log(C[W]_{\mathscr{A}_{p,\infty}})}{M}|Q|.$$

(ix) For any cube $Q\subset\mathbb{R}^n$,

$$m(W;Q):=\sup\left\{t\in(0,\infty):\ \left|\left\{x\in Q:\ \left\|A_QW^{-\frac{1}{p}}(x)\right\|^p>\frac{1}{t}\right\}\right|\le\frac{|Q|}{2}\right\}\sim 1,$$

where the positive equivalence constants are independent of Q.

(x) For any cube $Q\subset\mathbb{R}^n$,

$$\widetilde{m}(W;Q):=\sup\left\{t\in(0,\infty):\ \left|\left\{x\in Q:\ \left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p<t\right\}\right|\le\frac{|Q|}{2}\right\}\sim 1,$$

where the positive equivalence constants are independent of Q.

Proof. Assertion (i) is exactly [27, Proposition 5.6].

Now, we prove (ii). Applying Hölder's inequality and (i), we conclude that, for any cube Q and any measurable set $E\subset Q$, there exists $r\in(1,\infty)$ such that

$$\begin{aligned}\frac{1}{|Q|}\int_E\left\|W^{\frac1p}(x)A_Q^{-1}\right\|^p dx&\le\left(\frac{|E|}{|Q|}\right)\left[\fint_E\left\|W^{\frac1p}(x)A_Q^{-1}\right\|^{pr}dx\right]^{\frac1r}\\&\le\left(\frac{|E|}{|Q|}\right)^{1-\frac1r}\left[\fint_Q\left\|W^{\frac1p}(x)A_Q^{-1}\right\|^{pr}dx\right]^{\frac1r}\\&\le\left(\frac{|E|}{|Q|}\right)^{1-\frac1r}\fint_Q\left\|W^{\frac1p}(x)A_Q^{-1}\right\|^{p}dx\sim\left(\frac{|E|}{|Q|}\right)^{1-\frac1r}.\end{aligned}$$

This finishes the proof of (ii).

Next, we show (iii). Let u be as in Proposition 2.23 and $s:=\frac{pu}{p+u}$. Then

$$\frac{p}{s}\in(1,\infty)\ \text{ and }\ s\left(\frac{p}{s}\right)'=\frac{ps}{p-s}=u.$$

By these, Hölder's inequality, and Proposition 2.23, we obtain, for any cube $Q\subset\mathbb{R}^n$ and any measurable set $E\subset Q$,

$$\begin{aligned}\frac{|E|}{|Q|}&=\fint_Q\mathbf{1}_E(y)\,dy\le\fint_Q\left\|A_QW^{-\frac1p}(y)\right\|^s\left\|W^{\frac1p}(y)A_Q^{-1}\right\|^s\mathbf{1}_E(y)\,dy\\&\le\left[\fint_Q\left\|A_QW^{-\frac1p}(y)\right\|^{s(\frac{p}{s})'}dy\right]^{1/(\frac{p}{s})'}\left[\fint_Q\left\|W^{\frac1p}(y)A_Q^{-1}\right\|^p\mathbf{1}_E(y)\,dy\right]^{\frac{s}{p}}\\&\lesssim\left[\frac{1}{|Q|}\int_E\left\|W^{\frac1p}(x)A_Q^{-1}\right\|^p dx\right]^{\frac{s}{p}},\end{aligned}$$

where $\frac{s}{p}=\frac{u}{p+u}\in(0,1)$. This finishes the proof of (iii).

The assertions (iv) and (v) follow directly from (ii) and (iii), respectively.

Now, we prove (vi). Let u be as in Proposition 2.23. Let, for any cube $Q\subset\mathbb{R}^n$ and any $\alpha\in(0,\infty)$,

$$E:=\left\{x\in Q:\ \left\|W^{\frac1p}(x)A_Q^{-1}\right\|^p\le\alpha\right\}=\left\{x\in Q:\ \left\|W^{\frac1p}(x)A_Q^{-1}\right\|^u\le\alpha^{\frac{u}{p}}\right\},$$

which, together with Proposition 2.23, further implies that

$$\begin{aligned}|E|&\le\int_E\alpha^{\frac{u}{p}}\left\|W^{\frac1p}(x)A_Q^{-1}\right\|^{-u}dx\le\alpha^{\frac{u}{p}}\int_E\left\|A_QW^{-\frac1p}(x)\right\|^u dx\\&\le\alpha^{\frac{u}{p}}|Q|\fint_Q\left\|A_QW^{-\frac1p}(x)\right\|^u dx\lesssim\alpha^{\frac{u}{p}}|Q|.\end{aligned}$$

This finishes the proof of (vi).

Next, we prove (vii). Let u be as in Proposition 2.23. Let, for any cube $Q\subset\mathbb{R}^n$ and any $\alpha\in(0,\infty)$,

$$F:=\left\{x\in Q:\ \left\|A_QW^{-\frac{1}{p}}(x)\right\|^p\geq\alpha\right\}=\left\{x\in Q:\ \left\|A_QW^{-\frac{1}{p}}(x)\right\|^u\geq\alpha^{\frac{u}{p}}\right\},$$

which, combined with Proposition 2.23, further implies that

$$|F|\leq\int_F\alpha^{-\frac{u}{p}}\left\|A_QW^{-\frac{1}{p}}(x)\right\|^u dx\leq\alpha^{-\frac{u}{p}}|Q|\fint_Q\left\|A_QW^{-\frac{1}{p}}(x)\right\|^u dx\lesssim\alpha^{-\frac{u}{p}}|Q|.$$

This finishes the proof of (vii).

Assertion (viii) is precisely [27, Corollary 3.9].

Now, we prove (ix). Let u be as in Proposition 2.23. We first prove $m(W;Q)\gtrsim1$. For any cube $Q\subset\mathbb{R}^n$, let

$$\begin{aligned}E:=&\left\{y\in Q:\ \left\|A_QW^{-\frac{1}{p}}(y)\right\|^p>\frac{1}{2m(W;Q)}\right\}\\=&\left\{y\in Q:\ [2m(W;Q)]^{\frac{u}{p}}\left\|A_QW^{-\frac{1}{p}}(y)\right\|^u>1\right\}.\end{aligned}$$

Then, using the definitions of $m(W;Q)$ and E, we find that

$$\begin{aligned}\frac{|Q|}{2}<|E|<&\int_E[2m(W;Q)]^{\frac{u}{p}}\left\|A_QW^{-\frac{1}{p}}(y)\right\|^u dy\\\leq&[2m(W;Q)]^{\frac{u}{p}}|Q|\fint_Q\left\|A_QW^{-\frac{1}{p}}(y)\right\|^u dy\lesssim[m(W;Q)]^{\frac{u}{p}}|Q|\end{aligned}$$

and hence $m(W;Q)\gtrsim1$.

Next, we show $m(W;Q)\lesssim1$. For any cube $Q\subset\mathbb{R}^n$, let

$$F:=\left\{x\in Q:\ \left\|A_QW^{-\frac{1}{p}}(x)\right\|^p>\frac{2}{m(W;Q)}\right\}.$$

Then, from the definition of $m(W;Q)$, we infer that $|F|\leq\frac{|Q|}{2}$. Therefore,

$$Q\setminus F=\left\{x\in Q:\ m(W;Q)\left\|A_QW^{-\frac{1}{p}}(x)\right\|^p\leq2\right\}$$

and $|Q\setminus F|=|Q|-|F|\geq\frac{|Q|}{2}$. These further imply that

$$\begin{aligned}m(W;Q)\leq&\fint_{Q\setminus F}2\left\|A_QW^{-\frac{1}{p}}(x)\right\|^{-p}dx\leq4\fint_Q\left\|A_QW^{-\frac{1}{p}}(x)\right\|^{-p}dx\\\leq&4\fint_Q\left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p dx\sim1,\end{aligned}$$

which completes the proof of (ix).

Now, we prove (x). Let u be as in Proposition 2.23. We first prove $\widetilde{m}(W;Q)\gtrsim 1$. For any cube $Q\subset\mathbb{R}^n$, let

$$E:=\left\{y\in Q:\ \left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p<2\widetilde{m}(W;Q)\right\}=\left\{y\in Q:\ \left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^u<[2\widetilde{m}(W;Q)]^{\frac{u}{p}}\right\}.$$

Then, by the definitions of $\widetilde{m}(W;Q)$ and E, we find that

$$\begin{aligned}\frac{|Q|}{2}<|E|&<\int_E[2\widetilde{m}(W;Q)]^{\frac{u}{p}}\left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^{-u}dx\\&\le[2\widetilde{m}(W;Q)]^{\frac{u}{p}}|Q|\fint_Q\left\|A_QW^{-\frac{1}{p}}(x)\right\|^u dx\lesssim[\widetilde{m}(W;Q)]^{\frac{u}{p}}|Q|\end{aligned}$$

and hence $\widetilde{m}(W;Q)\gtrsim 1$.

Next, we show $\widetilde{m}(W;Q)\lesssim 1$. For any cube $Q\subset\mathbb{R}^n$, let

$$F:=\left\{x\in Q:\ \left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p<\frac{\widetilde{m}(W;Q)}{2}\right\}.$$

Then, using the definition of $\widetilde{m}(W;Q)$, we conclude that $|F|\le\frac{|Q|}{2}$. Therefore,

$$Q\setminus F=\left\{y\in Q:\ 2\left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p\ge\widetilde{m}(W;Q)\right\}$$

and $|Q\setminus F|=|Q|-|F|\ge\frac{|Q|}{2}$. These further imply that

$$\widetilde{m}(W;Q)\le\fint_{Q\setminus F}2\left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p dy\le 4\fint_Q\left\|W^{\frac{1}{p}}(x)A_Q^{-1}\right\|^p dy\sim 1,$$

which completes the proof of (x) and hence Proposition 2.24. □

Remark 2.11. (i) When $m=1$, Proposition 2.24 reduces to the assertions (iv) through (vii), (x), (xiii), and (xiv) of Proposition 2.10, all of which are equivalent to $w\in A_\infty$. This naturally leads to the question of whether all the statements in Proposition 2.24 are likewise equivalent to $W\in\mathscr{A}_{p,\infty}$.

(ii) Note that Proposition 2.24(i) is equivalent to the assertion that, for any $M\in M_m(\mathbb{C})\setminus\{O_m\}$, $\|W^{\frac{1}{p}}M\|^p\in A_\infty$. However, this may not be equivalent to $W\in\mathscr{A}_{p,\infty}$ because it transforms the matrix weight into the scalar weight and thus loses information.

(iii) Up to this point, most of the assertions in Proposition 2.10 have matrix-weighted variants, except (ix) through (xii) and (xv). Whether corresponding matrix-weighted analogues of these assertions exist remains unclear.

Next, we recall some lemmas about matrix weights. The following conclusion is exactly [18, Lemma 2.1].

Lemma 2.8. *Let $A\in M_m(\mathbb{C})$ be a positive definite matrix. Then, for any $\alpha\in(0,1)$ and $\vec{z}\in\mathbb{C}^m$ with $|\vec{z}|=1$.*

$$|A^\alpha\vec{z}|\le|A\vec{z}|^\alpha.$$

The following lemma originates from the proof of [91, Lemma 2]. However, since we need the exact constant, we provide the details.

Lemma 2.9. *Let $A,B\in M_m(\mathbb{C})$ be positive definite matrices. Then, for any $p,q\in(0,\infty)$ with $p<q$,*

$$\left\|A^{\frac{1}{q}}B^{-\frac{1}{q}}\right\|^q\le m^q\left\|A^{\frac{1}{p}}B^{-\frac{1}{p}}\right\|^p.$$

Proof. Let $\{\lambda_i\}_{i=1}^m$ be the eigenvalues of B and $\{\vec{e}_i\}_{i=1}^m$ the corresponding eigenvectors forming an orthonormal basis of $\mathbb{C}^m$. Applying [29, Lemma A.1] and Lemma 2.8 with α replaced by $\frac{p}{q}$, we find that, for any $p,q\in(0,\infty)$ with $p<q$,

$$\begin{aligned}\left\|A^{\frac{1}{q}}B^{-\frac{1}{q}}\right\|&\le\sum_{i=1}^m\left|A^{\frac{1}{q}}B^{-\frac{1}{q}}\vec{e}_i\right|=\sum_{i=1}^m\lambda_i^{-\frac{1}{q}}\left|A^{\frac{1}{q}}\vec{e}_i\right|\le\sum_{i=1}^m\lambda_i^{-\frac{1}{q}}\left|A^{\frac{1}{p}}\vec{e}_i\right|^{\frac{p}{q}}\\&=\sum_{i=1}^m\left|A^{\frac{1}{p}}B^{-\frac{1}{p}}\vec{e}_i\right|^{\frac{p}{q}}\le m\left\|A^{\frac{1}{p}}B^{-\frac{1}{p}}\right\|^{\frac{p}{q}}.\end{aligned}$$

This finishes the proof of Lemma 2.9. □

Now, we present some properties of $\mathscr{A}_p$.

Proposition 2.25. Let $p\in(0,\infty)$ and $W\in\mathscr{A}_p$. Then the following assertions hold.

(i) For any $\lambda\in(0,\infty)$, $[W(\lambda\cdot)]_{\mathscr{A}_p}=[W]_{\mathscr{A}_p}$.

(ii) For any $z\in\mathbb{R}^n$, $[W(\cdot-z)]_{\mathscr{A}_p}=[W]_{\mathscr{A}_p}$.

(iii) For any $\lambda\in(0,\infty)$, $[\lambda W]_{\mathscr{A}_p}=[W]_{\mathscr{A}_p}$.

(iv) $[W]_{\mathscr{A}_p}\in[1,\infty)$.

(v) Let $q\in(p,\infty)$. Then $\mathscr{A}_p\subsetneqq\mathscr{A}_q$. Moreover, there exists a positive constant C, depending only on m, p, and q, such that $[W]_{\mathscr{A}_q}\le C[W]_{\mathscr{A}_p}$.

(vi) $\mathscr{A}_p\subset\bigcap_{q\in(p,\infty)}\mathscr{A}_q$. Moreover, if $p\in[1,\infty)$, then $\mathscr{A}_p\subsetneqq\bigcap_{q\in(p,\infty)}\mathscr{A}_q$.

(vii) If $m\ge2$, then $\bigcup_{q\in(0,p)}\mathscr{A}_q\subsetneqq\mathscr{A}_p$.

(viii) For any $\alpha\in(0,1)$, $W^\alpha\in\mathscr{A}_p$. Moreover, for any $\alpha\in(0,1)$, $[W^\alpha]_{\mathscr{A}_p}\le m^p[W]_{\mathscr{A}_p}^\alpha$.

(ix) There exists $r\in(1,\infty)$ such that, for any $\alpha\in[1,r]$, $W^\alpha\in\mathscr{A}_{p\alpha}$.

Proof. The assertions (i) through (iii) follow immediately from the definition of $[W]_{\mathscr{A}_p}$. Assertion (iv) is precisely [27, Proposition 5.5(ii)].

Next, we prove (v). By [24, Proposition 2.12], we conclude that $\mathscr{A}_p\subset\mathscr{A}_q$ and $[W]_{\mathscr{A}_q}\lesssim[W]_{\mathscr{A}_p}$. Let $\alpha\in(0,\frac{n}{q})$ and $\beta=-\alpha q$. Then $\beta\in(-n,0)$ and $\alpha=-\frac{\beta}{q}<-\frac{\beta}{p}$. Applying these and Example 2.4, we obtain $W_{\alpha,\beta}\notin\mathscr{A}_p$ but $W_{\alpha,\beta}\in\mathscr{A}_q$, where $W_{\alpha,\beta}$ is as in Example 2.4. This finishes the proof of (v).

Now, we prove (vi). Using (v), we find that $\mathscr{A}_p\subset\bigcap_{q\in(p,\infty)}\mathscr{A}_q$. From Proposition 2.9(vii), it follows that there exists $w\in(\bigcap_{q\in(p,\infty)}A_q)\setminus A_p$. This, together with Lemma 2.3(i), further implies that $wI_m\in(\bigcap_{q\in(p,\infty)}\mathscr{A}_q)\setminus\mathscr{A}_p$, which completes the proof of (vi).

Next, we prove (vii). By (v), we have $\bigcup_{q\in(0,p)}\mathscr{A}_q\subset\mathscr{A}_p$. Let $\alpha\in(0,\frac{n}{p})$ and $\beta=-\alpha p$. Then $\beta\in(-n,0)$ and $\alpha=-\frac{\beta}{p}<-\frac{\beta}{q}$. Applying these and Example 2.4, we find that $W_{\alpha,\beta}\notin\bigcup_{q\in(0,p)}\mathscr{A}_q$ but $W_{\alpha,\beta}\in\mathscr{A}_p$, where $W_{\alpha,\beta}$ is as in Example 2.4. This finishes the proof of (vii).

Now, we prove (viii) by considering two cases for p.

Case (1) $p\in(0,1]$. In this case, using Lemma 2.9 and Hölder's inequality, we conclude that, for any cube $Q\subset\mathbb{R}^n$ and almost every $y\in Q$,

$$\begin{aligned}\fint_Q\left\|W^{\frac{\alpha}{p}}(x)W^{-\frac{\alpha}{p}}(y)\right\|^p dx&\le m^p\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{\alpha p}dx\\&\le m^p\left[\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\right]^\alpha\end{aligned}$$

and hence $[W^\alpha]_{\mathscr{A}_p}\le m^p[W]_{\mathscr{A}_p}^\alpha$. This finishes the proof of (viii) in this case.

Case (2) $p\in(1,\infty)$. In this case, from Lemma 2.9 and Hölder's inequality, it follows that, for any cube $Q\subset\mathbb{R}^n$,

$$\begin{aligned}\fint_Q\left[\fint_Q\left\|W^{\frac{\alpha}{p}}(x)W^{-\frac{\alpha}{p}}(y)\right\|^{p'}dy\right]^{\frac{p}{p'}}dx&\le\fint_Q\left[m^{p'}\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{\alpha p'}dy\right]^{\frac{p}{p'}}dx\\&\le m^p\fint_Q\left[\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{p'}dy\right]^{\alpha\frac{p}{p'}}dx\\&\le m^p\left\{\fint_Q\left[\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{p'}dy\right]^{\frac{p}{p'}}dx\right\}^\alpha,\end{aligned}$$

which further implies that $[W^\alpha]_{\mathscr{A}_p}\le m^p[W]_{\mathscr{A}_p}^\alpha$. This finishes the proof of (viii) in this case and hence (viii).

Finally, let $r:=1+\frac{1}{2^{n+1}[W]^{\mathrm{sc}}_{\mathscr{A}_{p,\infty}}-1}$ and $\alpha\in[1,r]$. We prove (ix) by considering three cases for p.

Case (1) $p\alpha\in(0,1]$. In this case, $p\in(0,1]$. Using Proposition 2.24(i), we find that, for any cube $Q\subset\mathbb{R}^n$ and almost every $y\in Q$,

$$\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{p\alpha}dx\le2\left[\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\right]^\alpha\le2[W]_{\mathscr{A}_p}^\alpha \tag{2.46}$$

and hence $[W^\alpha]_{\mathscr{A}_{p\alpha}}\le2[W]_{\mathscr{A}_p}^\alpha$. This finishes the proof of (ix) in this case.

Case (2) $p\in(0,1]$ and $p\alpha\in(1,\infty)$. In this case, applying Lemma 2.6(i) and (2.46), we conclude that, for any cube $Q\subset\mathbb{R}^n$ and almost every $y\in Q$,

$$\begin{aligned}[W^\alpha]_{\mathscr{A}_{p\alpha}}^{(p\alpha)'-1} &\sim \left[W^{\alpha[1-(p\alpha)']}\right]_{\mathscr{A}_{(p\alpha)'}}\\ &=\sup_{\mathrm{cube}\,Q\subset\mathbb{R}^n}\fint_Q\left[\fint_Q\left\|W^{\frac1p}(x)W^{-\frac1p}(y)\right\|^{p\alpha}dx\right]^{\frac{(p\alpha)'}{p\alpha}}dy\lesssim[W]_{\mathscr{A}_p}^{\frac{(p\alpha)'}{p}}.\end{aligned}$$

This finishes the proof of (ix) in this case.

Case (3) $p\in(1,\infty)$. In this case, $p\alpha\in(1,\infty)$. From Hölder's inequality, Lemma 2.6(ii), and Propositions 2.24(i) and 2.14, we deduce that, for any cube $Q\subset\mathbb{R}^n$,

$$\begin{aligned}&\fint_Q\left[\fint_Q\left\|W^{\frac1p}(x)W^{-\frac1p}(y)\right\|^{(p\alpha)'}dy\right]^{\frac{p\alpha}{(p\alpha)'}}dx\\ &\quad\le\fint_Q\left[\fint_Q\left\|W^{\frac1p}(x)W^{-\frac1p}(y)\right\|^{p'}dy\right]^{\frac{p\alpha}{p'}}dx\sim\fint_Q\left\|W^{\frac1p}(x)A_Q^{-1}\right\|^{p\alpha}dx\\ &\quad\le2\left[\fint_Q\left\|W^{\frac1p}(x)A_Q^{-1}\right\|^{p}dx\right]^{\alpha}\sim1\end{aligned}$$

and hence $[W^\alpha]_{\mathscr{A}_{p\alpha}}<\infty$. This finishes the proof of (ix) in this case and hence (ix), which completes the proof of Proposition 2.25. □

Remark 2.12. (i) For Proposition 2.25(vi), we conjecture that $\mathscr{A}_p\subsetneqq\bigcap_{q\in(p,\infty)}\mathscr{A}_q$ still holds for any $p\in(0,1)$, which is still not clear now.

(ii) The assertions (vii), (viii), and (ix) of Proposition 2.25 reduce to [18, Remark 5.4 and Corollaries 2.6 and 3.5] in the case where $n=1$ and $p\in(1,\infty)$.

(iii) The assertions (xi) and (xii) of Proposition 2.11 cannot be extended to the matrix setting in the sense that all entries belong to VMO or BMO (see [18, p. 405]).

(iv) By introducing the concept of the dimension of weights, Bu et al. [24] improved (xiii) and (xiv) of Proposition 2.11 (see Proposition 2.35).

Next, we present some properties of $\mathscr{A}_{p,\infty}$.

Proposition 2.26. Let $p\in(0,\infty)$ and $W\in\mathscr{A}_{p,\infty}$. Then the following assertions hold.

(i) For any $\lambda\in(0,\infty)$, $[W(\lambda\cdot)]_{\mathscr{A}_{p,\infty}}=[W]_{\mathscr{A}_{p,\infty}}$.

(ii) For any $z\in\mathbb{R}^n$, $[W(\cdot-z)]_{\mathscr{A}_{p,\infty}}=[W]_{\mathscr{A}_{p,\infty}}$.

(iii) For any $\lambda\in(0,\infty)$, $[\lambda W]_{\mathscr{A}_{p,\infty}}=[W]_{\mathscr{A}_{p,\infty}}$.

(iv) $[W]_{\mathscr{A}_{p,\infty}}\in[1,\infty)$.

(v) Let $q\in(p,\infty)$. Then $\mathscr{A}_{p,\infty}\subset\mathscr{A}_{q,\infty}$ and $[W]_{\mathscr{A}_{q,\infty}}\le m^q[W]_{\mathscr{A}_{p,\infty}}$. Specially, if $m\ge2$, then $\mathscr{A}_{p,\infty}\subsetneqq\mathscr{A}_{q,\infty}$.

(vi) $\mathscr{A}_p\subsetneqq\mathscr{A}_{p,\infty}$. Moreover, there exists a positive constant C, depending only on m and p, such that $[W]_{\mathscr{A}_{p,\infty}}\le C[W]_{\mathscr{A}_p}$, where $C=1$ when $p\in(0,1]$.

(vii) $\mathscr{A}_{p,\infty}\subset\bigcup_{q\in(0,\infty)}\mathscr{A}_q$. Specially, if $m\ge2$, then $\mathscr{A}_{p,\infty}\subsetneqq\bigcup_{q\in(0,\infty)}\mathscr{A}_q$.

(viii) For any $\alpha\in(0,1)$, $W^\alpha\in\mathscr{A}_{p,\infty}$. Moreover, for any $\alpha\in(0,1)$, $[W^\alpha]_{\mathscr{A}_{p,\infty}}\le m^p[W]^\alpha_{\mathscr{A}_{p,\infty}}$.

(ix) There exists $r\in(1,\infty)$ such that, for any $\alpha\in[1,r]$, $W^\alpha\in\mathscr{A}_{p\alpha,\infty}$. Moreover, for any $\alpha\in[1,r]$, $[W^\alpha]_{\mathscr{A}_{p\alpha,\infty}}\le2[W]^\alpha_{\mathscr{A}_{p,\infty}}$.

Proof. The assertions (i) through (iii) follow immediately from the definition of $[W]_{\mathscr{A}_p}$. Assertion (iv) is exactly [27, Proposition 5.5(i)].

Now, we prove (v). Applying Lemma 2.9 and a similar argument to that used in the proof of [27, Proposition 4.2(i)], we conclude that $\mathscr{A}_{p,\infty}\subset\mathscr{A}_{q,\infty}$ and $[W]_{\mathscr{A}_{q,\infty}}\le m^q[W]_{\mathscr{A}_{p,\infty}}$. Let $\alpha\in(0,\frac{n}{q})$ and $\beta=-\alpha q$. Then $\beta\in(-n,0)$ and $\alpha=-\frac{\beta}{q}<-\frac{\beta}{p}$. From these and Example 2.4(iii), we infer that $W_{\alpha,\beta}\notin\mathscr{A}_{p,\infty}$ but $W_{\alpha,\beta}\in\mathscr{A}_{q,\infty}$ and hence $\mathscr{A}_{p,\infty}\subsetneqq\mathscr{A}_{q,\infty}$. This finishes the proof of (v).

Assertion (vi) is a part of [27, Proposition 4.2(ii)].

Next, we show (vii). Applying [27, Proposition 4.2(ii)], we obtain $\mathscr{A}_{p,\infty}\subset\bigcup_{q\in(0,\infty)}\mathscr{A}_q$. Let $r\in(2p+1,\infty)$. Then $2p<n(r-1)$ and $1>\frac{2p}{r}$. From these and Example 2.4, it follows that

$$W_{1,2p}\notin\mathscr{A}_{p,\infty}\ \text{ but }\ W_{1,2p}\in A_r\subset\bigcup_{q\in(0,\infty)}\mathscr{A}_q$$

and hence $\mathscr{A}_{p,\infty}\subsetneqq\bigcup_{q\in(0,\infty)}\mathscr{A}_q$. This finishes the proof of (vi).

Now, we prove (viii). By Lemma 2.9 and Hölder's inequality, we have, for any cube $Q\subset\mathbb{R}^n$,

$$\begin{aligned}
&\exp\left(\fint_Q\log\left(\fint_Q\left\|W^{\frac{\alpha}{p}}(x)W^{-\frac{\alpha}{p}}(y)\right\|^p dx\right)dy\right)\\
&\quad\le\exp\left(\fint_Q\log\left(m^p\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{p\alpha} dx\right)dy\right)\\
&\quad\le m^p\ \exp\left(\fint_Q\log\left(\left[\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\right]^\alpha\right)dy\right)\\
&\quad=m^p\left[\exp\left(\fint_Q\log\left(\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\right)dy\right)\right]^\alpha
\end{aligned}$$

and hence $[W^\alpha]_{\mathscr{A}_{p,\infty}}\le m^p[W]^\alpha_{\mathscr{A}_{p,\infty}}$. This finishes the proof of (viii).

Next, we prove (ix). Let $r:=1+\frac{1}{2^{n+1}[W]^{\mathrm{sc}}_{\mathscr{A}_{p,\infty}}-1}$ and $\alpha\in[1,r]$. Using Proposition 2.24(i), we conclude that, for any cube $Q\subset\mathbb{R}^n$,

$$\exp\left(\fint_Q\log\left(\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{p\alpha} dx\right)dy\right)$$

$$
\begin{aligned}
&\le \exp\left(\fint_Q \log\left(2\left[\fint_Q \left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\right]^\alpha\right)dy\right)\\
&=2\left[\exp\left(\fint_Q \log\left(\fint_Q \left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\right)dy\right)\right]^\alpha
\end{aligned}
$$

and hence $[W^\alpha]_{\mathscr{A}_{p\alpha,\infty}} \le 2[W]^\alpha_{\mathscr{A}_{p,\infty}}$. This finishes the proof of (viii) and hence Proposition 2.26. $\square$

Although matrix weights do not have the self-improvement property, Bownik [18, Theorem 3.6] identifies conditions ensuring that a matrix weight can have the self-improvement property in the case where $n=1$ and $p\in(1,\infty)$. The following proposition is a general version.

Proposition 2.27. Let $p\in(0,\infty)$ and $W\in\mathscr{A}_p$. Then the following statements are equivalent.

(i) There exists $q\in(0,p)$ such that $W\in\mathscr{A}_q$.

(ii) There exists $r\in(1,\infty)$ such that, for any $\alpha\in(1,r]$, $W^\alpha\in\mathscr{A}_p$.

(iii) There exists $\alpha\in(1,\infty)$ such that $W^\alpha\in\mathscr{A}_p$.

Proof. We first show that (i) implies (ii). By $W\in\mathscr{A}_q$ and Proposition 2.25(ix), we find that there exists $r\in(1,\infty)$ such that, for any $\alpha\in[1,r]$, $W^\alpha\in\mathscr{A}_{q\alpha}$. This, combined with $q\in(0,p)$ and Proposition 2.25(v), further implies that, for any $\alpha\in(1,\min\{r,\frac{p}{q}\}]$, $W^\alpha\in\mathscr{A}_{q\alpha}\subset\mathscr{A}_p$.

That statement (ii) implies statement (iii) is obvious.

Now, we prove that (iii) implies (i) by consider two cases for p.

Case (1) $p\in(0,1]$. In this case, let $q\in(\frac{p}{\alpha},p)$. Applying Lemma 2.9 and Hölder's inequality, we find that, for any cube $Q\subset\mathbb{R}^n$ and almost every $y\in Q$,

$$
\begin{aligned}
\fint_Q \left\|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\right\|^q dx &\le m^q \fint_Q \left\|W^{\frac{\alpha}{p}}(x)W^{-\frac{\alpha}{p}}(y)\right\|^{\frac{p}{\alpha}} dx\\
&\le m^q\left[\fint_Q \left\|W^{\frac{\alpha}{p}}(x)W^{-\frac{\alpha}{p}}(y)\right\|^p dx\right]^{\frac{1}{\alpha}}
\end{aligned}
$$

and hence $[W]_{\mathscr{A}_q}\le m^q[W^\alpha]^{\frac{1}{\alpha}}_{\mathscr{A}_p}$.

Case (2) $p\in(1,\infty)$. In this case, let $q\in(\frac{p-1}{\alpha}+1,p)$. Then

$$
q>\frac{p}{\alpha} \quad\text{and}\quad \frac{p}{p'}\frac{q'}{q}\frac{1}{\alpha}=\frac{p-1}{q-1}\frac{1}{\alpha}<1.
$$

From these, Lemma 2.9, and Hölder's inequality, it follows that, for any cube $Q\subset\mathbb{R}^n$,

$$
\begin{aligned}
&\fint_Q\left[\fint_Q \left\|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\right\|^{q'} dy\right]^{\frac{q}{q'}} dx\\
&\quad\le m^q\fint_Q\left[\fint_Q \left\|W^{\frac{\alpha}{p}}(x)W^{-\frac{\alpha}{p}}(y)\right\|^{p'\frac{p}{p'}\frac{q'}{q}\frac{1}{\alpha}} dy\right]^{\frac{q}{q'}} dx
\end{aligned}
$$

$$\leq m^q \fint_Q \left[\fint_Q \left\| W^{\frac{\alpha}{p}}(x) W^{-\frac{\alpha}{p}}(y) \right\|^{p'} dy \right]^{\frac{p}{p'}\frac{1}{\alpha}} dx$$

$$\leq m^q \left\{ \fint_Q \left[\fint_Q \left\| W^{\frac{\alpha}{p}}(x) W^{-\frac{\alpha}{p}}(y) \right\|^{p'} dy \right]^{\frac{p}{p'}} dx \right\}^{\frac{1}{\alpha}}$$

and hence $[W]_{\mathscr{A}_q} \leq m^q [W^\alpha]_{\mathscr{A}_p}$. This finishes the proof of Proposition 2.27. □

Some similar conclusions to Proposition 2.27 also hold for $\mathscr{A}_{q,\infty}$.

Proposition 2.28. Let $p \in (0,\infty)$ and $W \in \mathscr{A}_{p,\infty}$. Then the following assertions are equivalent.

(i) There exists $q \in (0,p)$ such that $W \in \mathscr{A}_{q,\infty}$.

(ii) There exists $r \in (1,\infty)$ such that, for any $\alpha \in (1,r]$, $W^\alpha \in \mathscr{A}_{p,\infty}$.

(iii) There exists $\alpha \in (1,\infty)$ such that $W^\alpha \in \mathscr{A}_{p,\infty}$.

Proof. We first show that (i) implies (ii). By $W \in \mathscr{A}_{q,\infty}$ and Proposition 2.26(ix), we find that there exists $r \in (1,\infty)$ such that, for any $\alpha \in [1,r]$, $W^\alpha \in \mathscr{A}_{q\alpha,\infty}$. This, together with $q \in (0,p)$ and Proposition 2.25(v), further implies that, for any $\alpha \in (1, \min\{r, \frac{p}{q}\}]$, $W^\alpha \in \mathscr{A}_{q\alpha,\infty} \subset \mathscr{A}_{p,\infty}$.

Assertion (ii) implies that assertion (iii) is obviously.

Next, we prove that (iii) implies (i). Let $q \in (\frac{p}{\alpha}, p)$. Using Lemma 2.9 and Hölder's inequality, we obtain, for any cube $Q \subset \mathbb{R}^n$ and almost every $y \in Q$,

$$\fint_Q \left\| W^{\frac{1}{q}}(x) W^{-\frac{1}{q}}(y) \right\|^q dx \leq m^q \fint_Q \left\| W^{\frac{\alpha}{p}}(x) W^{-\frac{\alpha}{p}}(y) \right\|^{\frac{p}{\alpha}} dx$$

$$\leq m^q \left[\fint_Q \left\| W^{\frac{\alpha}{p}}(x) W^{-\frac{\alpha}{p}}(y) \right\|^p dx \right]^{\frac{1}{\alpha}}$$

and hence $[W]_{\mathscr{A}_{q,\infty}} \leq m^q [W^\alpha]_{\mathscr{A}_{p,\infty}}^{\frac{1}{\alpha}}$. This finishes the proof of Proposition 2.28. □

Corresponding to Proposition 2.13, we have the following question.

Question III. *If $p \in (0,\infty)$ and W is a matrix weight, then*

(i) $\lim_{q \to p^+} [W]_{\mathscr{A}_q} = [W]_{\mathscr{A}_p}$?

(ii) $\lim_{q \to p^+} [W]_{\mathscr{A}_{q,\infty}} = [W]_{\mathscr{A}_{p,\infty}}$?

Remark 2.13. Since matrix weights do not have the self-improvement property, the limit $q \to p^-$ is not meaningful. Applying a similar argument to that used in the proof of Proposition 2.13, we obtain

$$\liminf_{q \to p^+} [W]_{\mathscr{A}_q} \geq [W]_{\mathscr{A}_p} \quad \text{and} \quad \liminf_{q \to p^+} [W]_{\mathscr{A}_{q,\infty}} \geq [W]_{\mathscr{A}_{p,\infty}}.$$

However, it remains unknown whether

$$\limsup_{q \to p^+} [W]_{\mathscr{A}_q} \leq [W]_{\mathscr{A}_p} \quad \text{and} \quad \limsup_{q \to p^+} [W]_{\mathscr{A}_{q,\infty}} \leq [W]_{\mathscr{A}_{p,\infty}}.$$

So far, the corresponding matrix-weighted versions of the properties of scalar weights mentioned in Subsection 2.2 have been fully discussed.

2.4 Dimensions of Weights

Let W be a matrix weight and $\{A_Q\}_{\mathrm{cube}\,Q}$ a family of reducing operators of order p for W. For any cubes $Q,R\subset\mathbb{R}^n$, the estimates of $\|A_QA_R^{-1}\|$ play an important role in the study of matrix-weighted spaces; see [24,25,65]. Such estimates establish a connection between the matrix weight and the geometric relationship of the cubes. To obtain the sharp estimates of $\|A_QA_R^{-1}\|$, Bu et al. [24,27] introduced concept of dimensions of matrix weights. The following definition is precisely [24, Definition 2.22].

Definition 2.9. Let $p\in(0,\infty)$, $d\in\mathbb{R}$, and W be a matrix weight. Then W is said to have the $\mathscr{A}_p$-*dimension* d, denoted by $W\in\mathbb{D}_{p,d}(\mathbb{R}^n,\mathbb{C}^m)$, if there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$, when $p\in(0,1]$,

$$\operatorname*{ess\,sup}_{y\in\lambda Q}\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\le C\lambda^d$$

or, when $p\in(1,\infty)$,

$$\fint_Q\left[\fint_{\lambda Q}\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{p'} dy\right]^{\frac{p}{p'}} dx\le C\lambda^d.$$

For simplicity, we denote $\mathbb{D}_{p,d}(\mathbb{R}^n,\mathbb{C}^m)$ by $\mathbb{D}_{p,d}$. Analogous to Proposition 2.4, we obtain the following equivalent characterization of $\mathbb{D}_{p,d}$.

Proposition 2.29. Let $p\in(0,1]$, $d\in\mathbb{R}$, and W be a matrix weight. Then $W\in\mathbb{D}_{p,d}$ if and only if there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$,

$$\fint_Q\operatorname*{ess\,sup}_{y\in\lambda Q}\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\le C\lambda^d.$$

Proof. For any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$, let

$$I_\lambda(Q):=\operatorname*{ess\,sup}_{y\in\lambda Q}\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx$$

and

$$J_\lambda(Q):=\fint_Q\operatorname*{ess\,sup}_{y\in\lambda Q}\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx.$$

Repeating the proof of [24, Proposition 2.11] with $[W]_{\mathscr{A}_p}$ and $[W]_{\mathscr{A}_p}^*$ therein replaced, respectively, by $I_\lambda(Q)$ and $J_\lambda(Q)$, we find that $I_\lambda(Q)\sim J_\lambda(Q)$ with the positive equivalence constants independent of Q. This finishes the proof of Proposition 2.29. $\square$

Next, we recall some fundamental properties of the above dimensions. The following proposition is exactly [24, Propositions 2.23 and 2.43].

Proposition 2.30. Let $p\in(0,\infty)$. Then the following statements hold.

(i) For any $d\in(-\infty,0)$, $\mathbb{D}_{p,d}=\emptyset$.

(ii) For any $d\in[0,n)$, $\mathbb{D}_{p,d}\subsetneqq\mathscr{A}_p$.

(iii) $\bigcup_{d\in[0,n)}\mathbb{D}_{p,d}=\mathscr{A}_p$.

(iv) For any $d\in[n,\infty)$, $\mathbb{D}_{p,d}=\mathscr{A}_p$.

(v) If $0\le d_1<d_2\le n$, then $\mathbb{D}_{p,d_1}\subsetneqq\mathbb{D}_{p,d_2}$.

(vi) For any $q\in(p,\infty)$ and $d\in[0,\infty)$, $\mathbb{D}_{p,d}\subset\mathbb{D}_{q,d}$.

Remark 2.14. Proposition 2.30(iii) shows that the family $\{\mathbb{D}_{p,d}\}_{d\in[0,n)}$ forms a classification of $\mathscr{A}_p$. Therefore, the essential and useful range of $\mathscr{A}_p$-dimensions is $[0,n)$.

Now, we recall the characterization of $\mathbb{D}_{p,d}$ in terms of reducing operators (see [24, Proposition 2.24]).

Proposition 2.31. Let $p\in(0,\infty)$, $W\in\mathscr{A}_p$, and $\{A_Q\}_{\mathrm{cube}\,Q}$ be a family of reducing operators of order p for W. Then, for any $d\in[0,\infty)$, $W\in\mathbb{D}_{p,d}$ if and only if there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$, $\|A_QA_{\lambda Q}^{-1}\|^p\le C\lambda^d$.

Based on Remark 2.14, for any $W\in\mathscr{A}_p$, let

$$d_p(W):=\inf\{d\in[0,n):\ W\in\mathbb{D}_{p,d}\},$$

see [24, Remark 2.34]. It is easy to show that $d_p(W)$ is the *critical point* of A_p-dimension, that is, for any $d\in(-\infty,d_p(W))$, $W\notin\mathbb{D}_{p,d}$ and, for any $d\in(d_p(W),\infty)$, $W\in\mathbb{D}_{p,d}$. However, the critical value $d_p(W)$ is more subtle (see [24, Proposition 2.42]).

Proposition 2.32. Let $p\in(0,\infty)$ and $d\in[0,n)$. Then the following statements hold.

(i) There exists $W\in A_p$ having the A_p-dimension $d_p(W)=d$.

(ii) There exists $W\in A_p$ such that $d_p(W)=d$ but $d_p(W)$ is not the A_p-dimension of W.

Bu et al. [24, Proposition 2.33] provided a method for computing $d_p(W)$. Using [24, Proposition 2.33] and [24, Lemma 2.10 and Corollaries 2.15 and 2.17], we directly obtain an alternative method for computing $d_p(W)$.

Proposition 2.33. Let $p\in(0,\infty)$, $W\in A_p$, and $\{A_Q\}_{\mathrm{cube}\,Q}$ be a family of reducing operators of order p for W. Then

$$d_p(W)=\limsup_{i\to\infty}\frac{1}{i}\log_2\left(\left\|A_QA_{2^iQ}^{-1}\right\|^p\right).$$

In Definition 2.9, what happens if we exchange the positions of Q and λQ? The following proposition, originally due to Bu et al. [24, Proposition 2.27], shows that this situation is trivial when $p\in(0,1]$ and can be described in terms of the dimension of matrix weights when $p\in(1,\infty)$. Therefore, it does not require introducing a new concept.

Proposition 2.34. Let $p\in(0,\infty)$ and $W\in\mathscr{A}_p$. Then the following assertions hold.

(i) If $p\in(0,1]$, then there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$,

$$\operatorname*{ess\,sup}_{y\in Q}\fint_{\lambda Q}\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\le C.$$

(ii) If $p\in(1,\infty)$ and $d\in\mathbb{R}$, then there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$,

$$\fint_{\lambda Q}\left[\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^{p'}dy\right]^{\frac{p}{p'}}dx\le C\lambda^d$$

if and only if $W^{1-p'}$ (which belongs to $\mathscr{A}_{p'}$) has the $\mathscr{A}_{p'}$-dimension $\frac{d}{p-1}$.

The following sharp estimate of $\|A_QA_R^{-1}\|$ is precisely [24, Lemma 2.28]. For the sharpness, see [24, Lemmas 2.44 and 2.46].

Proposition 2.35. Let $p\in(0,\infty)$, $W\in\mathscr{A}_p$ have the $\mathscr{A}_p$-dimension $d\in[0,n)$, and $\{A_Q\}_{\mathrm{cube}\,Q}$ be a family of reducing operators of order p for W. Then the following statements hold.

(i) If $p\in(0,1]$, then there exists a positive constant C such that, for any cubes $Q,R\subset\mathbb{R}^n$,

$$\left\|A_QA_R^{-1}\right\|\le C\max\left\{\left[\frac{\ell(R)}{\ell(Q)}\right]^{\frac{d}{p}},1\right\}\left[1+\frac{|c_Q-c_R|}{\ell(Q)\vee\ell(R)}\right]^{\frac{d}{p}}.$$

(ii) If $p\in(1,\infty)$ and $W^{1-p'}$ (which belongs to $\mathscr{A}_{p'}$) have the $\mathscr{A}_{p'}$-dimension $\widetilde{d}\in[0,n)$, then there exists a positive constant C such that, for any cubes $Q,R\subset\mathbb{R}^n$,

$$\left\|A_QA_R^{-1}\right\|\le C\max\left\{\left[\frac{\ell(R)}{\ell(Q)}\right]^{\frac{d}{p}},\left[\frac{\ell(Q)}{\ell(R)}\right]^{\frac{\widetilde{d}}{p'}}\right\}\left[1+\frac{|c_Q-c_R|}{\ell(Q)\vee\ell(R)}\right]^{\frac{d}{p}+\frac{\widetilde{d}}{p'}}.$$

Next, we recall the concepts of the upper and the lower dimensions of $\mathscr{A}_{p,\infty}$ (see [27, Difinition 6.2]).

Definition 2.10. Let $p\in(0,\infty)$ and $d\in\mathbb{R}$. A matrix weight W is said to have $\mathscr{A}_{p,\infty}$*-lower dimension* d, denoted by $W\in\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}(\mathbb{R}^n,\mathbb{C}^m)$, if there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$,

$$\exp\left(\fint_{\lambda Q}\log\left(\fint_Q\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\right)dy\right)\le C\lambda^d.$$

A matrix weight W is said to have $\mathscr{A}_{p,\infty}$*-upper dimension* d, denoted by $W\in\mathbb{D}^{\mathrm{upper}}_{p,\infty,d}(\mathbb{R}^n,\mathbb{C}^m)$, if there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$,

$$\exp\left(\fint_{Q}\log\left(\fint_{\lambda Q}\left\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\right\|^p dx\right)dy\right)\le C\lambda^d.$$

For simplicity, we denote $\mathbb{D}^{\mathrm{upper}}_{p,\infty,d}(\mathbb{R}^n,\mathbb{C}^m)$ and $\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}(\mathbb{R}^n,\mathbb{C}^m)$, respectively, by $\mathbb{D}^{\mathrm{upper}}_{p,\infty,d}$ and $\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}$. Now, we recall some fundamental properties of the lower and the upper dimensions of $\mathscr{A}_{p,\infty}$ (see [27, Propositions 6.4 and 6.5]). Specially, assertion (iii) is new and follows from (iv) and (vi).

Proposition 2.36. Let $p\in(0,\infty)$. Then the following assertions hold.

(i) For any $d\in(-\infty,0)$, $\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}=\emptyset=\mathbb{D}^{\mathrm{upper}}_{p,\infty,d}$.

(ii) For any $d\in[0,n)$, $\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}\subsetneqq\mathscr{A}_{p,\infty}$; for any $d\in[n,\infty)$, $\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}=\mathscr{A}_{p,\infty}$.

(iii) For any $d\in[0,\infty)$, $\mathbb{D}^{\mathrm{upper}}_{p,\infty,d}\subsetneqq\mathscr{A}_{p,\infty}$.

(iv) $\bigcup_{d\in[0,n)}\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}=\mathscr{A}_{p,\infty}=\bigcup_{d\in[0,\infty)}\mathbb{D}^{\mathrm{upper}}_{p,\infty,d}$.

(v) If $0\le d_1<d_2\le n$, then $\mathbb{D}^{\mathrm{lower}}_{p,\infty,d_1}\subsetneqq\mathbb{D}^{\mathrm{lower}}_{p,\infty,d_2}$.

(vi) If $0\le d_1<d_2<\infty$, then $\mathbb{D}^{\mathrm{upper}}_{p,\infty,d_1}\subsetneqq\mathbb{D}^{\mathrm{upper}}_{p,\infty,d_2}$.

(vii) If $p\in(0,1]$, then $\mathscr{A}_p=\mathbb{D}^{\mathrm{upper}}_{p,\infty,0}$.

Remark 2.15. Proposition 2.36(iv) shows that the families $\{\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}\}_{d\in[0,n)}$ and $\{\mathbb{D}^{\mathrm{upper}}_{p,\infty,d}\}_{d\in[0,\infty)}$ form two different classifications of $\mathscr{A}_{p,\infty}$. Therefore, the essential and useful ranges of $\mathscr{A}_{p,\infty}$-lower dimension and $\mathscr{A}_{p,\infty}$-upper dimension are, respectively, $[0,n)$ and $[0,\infty)$.

By [27, Proposition 6.1] and the definitions of $\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}$ and $\mathbb{D}^{\mathrm{upper}}_{p,\infty,d}$, we obtain the following conclusion.

Proposition 2.37. Let $p\in(0,\infty)$, $W\in\mathscr{A}_{p,\infty}$, and $\{A_Q\}_{\mathrm{cube}\,Q}$ be a family of reducing operators of order p for W. Then the following statements hold.

(i) For any $d\in[0,\infty)$, $W\in\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}$ if and only if there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$, $\|A_QA_{\lambda Q}^{-1}\|^p\le C\lambda^d$.

(ii) For any $d\in[0,\infty)$, $W\in\mathbb{D}^{\mathrm{upper}}_{p,\infty,d}$ if and only if there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$, $\|A_{\lambda Q}A_Q^{-1}\|^p\le C\lambda^d$.

Next, we recall the critical point of $\mathscr{A}_{p,\infty}$-dimensions (see [27, p. 27]). Based on Remark 2.15, for any $W\in\mathscr{A}_{p,\infty}$, let

$$d^{\mathrm{lower}}_{p,\infty}(W):=\inf\left\{d\in[0,n):\ W\in\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}\right\}$$

and

$$d^{\mathrm{upper}}_{p,\infty}(W):=\inf\left\{d\in[0,\infty):\ W\in\mathbb{D}^{\mathrm{upper}}_{p,\infty,d}\right\}.$$

Then, for any $d\in(-\infty,d^{\mathrm{lower}}_{p,\infty}(W))$, $W\notin\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}$ and, for any $d\in(d^{\mathrm{lower}}_{p,\infty}(W),\infty)$, $W\in\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}$. An analogous conclusion holds for the upper dimension. As in Proposition 2.32, the critical values $d^{\mathrm{lower}}_{p,\infty}(W)$ and $d^{\mathrm{upper}}_{p,\infty}(W)$ of dimensions are more subtle. The following proposition follows from the proof of [27, Lemma 7.10].

Proposition 2.38. Let $p\in(0,\infty)$, $d_1\in[0,n)$, and $d_2\in[0,\infty)$. Then the following assertions hold.

(i) There exists $W\in A_{p,\infty}$ such that

$$d^{\mathrm{lower}}_{p,\infty}(W)=d_1,\ \ d^{\mathrm{upper}}_{p,\infty}(W)=d_2,\tag{2.47}$$

$W\in\mathbb{D}^{\mathrm{lower}}_{p,\infty,d_1}$, and $W\in\mathbb{D}^{\mathrm{upper}}_{p,\infty,d_2}$.

(ii) There exists $W\in A_{p,\infty}$ such that (2.47) holds, $W\notin\mathbb{D}^{\mathrm{lower}}_{p,\infty,d_1}$, and $W\in\mathbb{D}^{\mathrm{upper}}_{p,\infty,d_2}$.

(iii) There exists $W\in A_{p,\infty}$ such that (2.47) holds, $W\in\mathbb{D}^{\mathrm{lower}}_{p,\infty,d_1}$, and $W\notin\mathbb{D}^{\mathrm{upper}}_{p,\infty,d_2}$.

(iv) There exists $W\in A_{p,\infty}$ such that (2.47) holds, $W\notin\mathbb{D}^{\mathrm{lower}}_{p,\infty,d_1}$, and $W\notin\mathbb{D}^{\mathrm{upper}}_{p,\infty,d_2}$.

Applying Proposition 2.37 and an argument similar to that used in the proof of [24, Proposition 2.33], we have the following conclusion, which is an analogue of Proposition 2.33.

Proposition 2.39. Let $p\in(0,\infty)$, $W\in\mathscr{A}_{p,\infty}$, and $\{A_Q\}_{\mathrm{cube}\,Q}$ be a family of reducing operators of order p for W. Then

$$d^{\mathrm{lower}}_{p,\infty}(W)=\limsup_{i\to\infty}\frac{1}{i}\log_2\left(\sup_{\mathrm{cube}\,Q\subset\mathbb{R}^n}\left\|A_QA_{2^iQ}^{-1}\right\|^p\right)$$

and

$$d^{\mathrm{upper}}_{p,\infty}(W)=\limsup_{i\to\infty}\frac{1}{i}\log_2\left(\sup_{\mathrm{cube}\,Q\subset\mathbb{R}^n}\left\|A_{2^iQ}A_Q^{-1}\right\|^p\right).$$

Before recall the sharp estimate of $\|A_Q A_R^{-1}\|$, we need the following symbols (see [27, p. 27]). Let

$$[[d_{p,\infty}^{\mathrm{lower}}(W),n):=\begin{cases}[d_{p,\infty}^{\mathrm{lower}}(W),n) & \text{if } d_{p,\infty}^{\mathrm{lower}}(W) \text{ is } \mathscr{A}_{p,\infty}\text{-lower dimension of } W,\\ (d_{p,\infty}^{\mathrm{lower}}(W),n) & \text{otherwise}\end{cases}$$

and

$$[[d_{p,\infty}^{\mathrm{upper}}(W),\infty):=\begin{cases}[d_{p,\infty}^{\mathrm{upper}}(W),\infty) & \text{if } d_{p,\infty}^{\mathrm{upper}}(W) \text{ is } \mathscr{A}_{p,\infty}\text{-upper dimension of } W,\\ (d_{p,\infty}^{\mathrm{upper}}(W),\infty) & \text{otherwise.}\end{cases}$$

The following sharp estimate of $\|A_Q A_R^{-1}\|$ is exactly [27, Lemma 6.5]. For the sharpness, see [27, Lemma 7.9].

Proposition 2.40. Let $p\in(0,\infty)$, $W\in\mathscr{A}_{p,\infty}$, and $\{A_Q\}_{\mathrm{cube}\,Q}$ be a family of reducing operators of order p for W. Assume that $d_1\in[[d_{p,\infty}^{\mathrm{lower}}(W),\infty)$ and $d_2\in[[d_{p,\infty}^{\mathrm{upper}}(W),\infty)$. Then there exists a positive constant C such that, for any cubes $Q,R\subset\mathbb{R}^n$,

$$\left\|A_Q A_R^{-1}\right\|\le C\max\left\{\left[\frac{\ell(R)}{\ell(Q)}\right]^{\frac{d_1}{p}},\left[\frac{\ell(Q)}{\ell(R)}\right]^{\frac{d_2}{p}}\right\}\left[1+\frac{|c_Q-c_R|}{\ell(Q)\vee\ell(R)}\right]^{\frac{d_1+d_2}{p}}.$$

Now, we establish the relationship between $\mathscr{A}_{p,\infty}$ and $\mathscr{A}_p$.

Proposition 2.41. Let $p\in(0,\infty)$, $W\in\mathscr{A}_p$, and $d\in[0,\infty)$. Then the following statements hold.

(i) $W\in\mathbb{D}_{p,\infty,d}^{\mathrm{lower}}$ if and only if $W\in\mathbb{D}_{p,d}$.

(ii) If $p\in(0,1]$, then $\mathbb{D}_{p,\infty,0}^{\mathrm{upper}}=\mathscr{A}_p$.

(iii) If $p\in(1,\infty)$, then $W\in\mathbb{D}_{p,\infty,d}^{\mathrm{upper}}$ if and only if $W^{1-p'}\in\mathbb{D}_{p',\frac{d}{p-1}}$.

(iv) If $p\in(1,\infty)$, then $W^{1-p'}\in\mathbb{D}_{p',\infty,\frac{d}{p-1}}^{\mathrm{upper}}$ if and only if $W\in\mathbb{D}_{p,d}$.

(v) $d_{p,\infty}^{\mathrm{lower}}(W)=d_p(W)$.

(vi) If $p\in(0,1]$, then $d_{p,\infty}^{\mathrm{upper}}(W)=0$.

(vii) If $p\in(1,\infty)$, then $d_{p,\infty}^{\mathrm{upper}}(W)=(p-1)d_p(W^{1-p'})$.

Proof. From Propositions 2.31 and 2.37(i), we infer that (i) holds. Statement (ii) is precisely [27, Proposition 6.5(iv)]. By Propositions 2.34(ii) and 2.14 and Lemma 2.6(ii), we find that $W^{1-p'}\in\mathbb{D}_{p',\frac{d}{p-1}}$ if and only if, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$, $\|A_{\lambda Q}A_Q^{-1}\|^p\lesssim\lambda^d$. This, combined with Proposition 2.37(i), further implies (iii). Applying Lemma 2.6(i), we obtain $W^{1-p'}\in A_{p'}$. Using (iii) with W, p, and d replaced, respectively, by $W^{1-p'}$, p', and $(p-1)d$, we obtain (iv). From (i) through (iii), we infer that (v) through (vii) hold. This finishes the proof of Proposition 2.41. $\square$

Next, we show what the above results characterize in the case where $m=1$. Let us begin with introducing some concepts. To avoid confusion, we retain the underlying space $\mathbb{R}^n$ in the symbol $\mathbb{D}_{p,d}(\mathbb{R}^n):=\mathbb{D}_{p,d}(\mathbb{R}^n,\mathbb{C})$ for any $p\in[1,\infty)$. In this case, the upper dimension and the lower dimension of $\mathscr{A}_{p,\infty}$ reduce to, respectively, the upper dimension and the lower dimension of A_∞ (see [27, Definition 7.1]).

Definition 2.11. Let $d\in\mathbb{R}$. A scalar weight w is said to have A_∞*-lower dimension* d, denoted by $w\in\mathbb{D}^{\mathrm{lower}}_{\infty,d}$, if there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$,

$$\fint_Q w(x)\,dx\exp\left(\fint_{\lambda Q}\log\left([w(x)]^{-1}\right)dx\right)\le C\lambda^d.$$

A matrix weight w is said to have A_∞*-upper dimension* d, denoted by $w\in\mathbb{D}^{\mathrm{upper}}_{\infty,d}$, if there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$,

$$\fint_{\lambda Q} w(x)\,dx\exp\left(\fint_{Q}\log\left([w(x)]^{-1}\right)dx\right)\le C\lambda^d.$$

Next, we recall some fundamental properties of the above dimensions.

Proposition 2.42. Let $p\in[1,\infty)$. Then the following assertions hold.

(i) For any $d\in(-\infty,0)$, $\mathbb{D}_{p,d}(\mathbb{R}^n)=\emptyset$.

(ii) For any $d\in[0,n)$, $\mathbb{D}_{p,d}(\mathbb{R}^n)\subsetneqq A_p$.

(iii) $\bigcup_{d\in[0,n)}\mathbb{D}_{p,d}(\mathbb{R}^n)=A_p$.

(iv) For any $d\in[n,\infty)$, $\mathbb{D}_{p,d}(\mathbb{R}^n)=A_p$.

(v) If $0\le d_1<d_2\le n$, then $\mathbb{D}_{p,d_1}(\mathbb{R}^n)\subsetneqq\mathbb{D}_{p,d_2}(\mathbb{R}^n)$.

(vi) For any $q\in(p,\infty)$ and $d\in[0,\infty)$, $\mathbb{D}_{p,d}(\mathbb{R}^n)\subset\mathbb{D}_{q,d}(\mathbb{R}^n)$.

Proposition 2.43. The following statements hold.

(i) For any $d\in(-\infty,0)$, $\mathbb{D}^{\mathrm{lower}}_{\infty,d}=\emptyset=\mathbb{D}^{\mathrm{upper}}_{\infty,d}$.

(ii) For any $d\in[0,n)$, $\mathbb{D}^{\mathrm{lower}}_{\infty,d}\subsetneqq A_\infty$; for any $d\in[n,\infty)$, $\mathbb{D}^{\mathrm{lower}}_{\infty,d}=A_\infty$.

(iii) For any $d\in[0,\infty)$, $\mathbb{D}^{\mathrm{upper}}_{\infty,d}\subsetneqq A_\infty$.

(iv) $\bigcup_{d\in[0,n)}\mathbb{D}^{\mathrm{lower}}_{\infty,d}=A_\infty=\bigcup_{d\in[0,\infty)}\mathbb{D}^{\mathrm{upper}}_{\infty,d}$.

(v) If $0\le d_1<d_2\le n$, then $\mathbb{D}^{\mathrm{lower}}_{\infty,d_1}\subsetneqq\mathbb{D}^{\mathrm{lower}}_{\infty,d_2}$.

(vi) If $0\le d_1<d_2<\infty$, then $\mathbb{D}^{\mathrm{upper}}_{\infty,d_1}\subsetneqq\mathbb{D}^{\mathrm{upper}}_{\infty,d_2}$.

(vii) $A_1 = \mathbb{D}^{\rm upper}_{\infty,0}$.

Proposition 2.44. Let $w\in A_\infty$. Then the following assertions hold.

(i) For any $d\in[0,\infty)$, $w\in\mathbb{D}^{\rm lower}_{\infty,d}$ if and only if there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$,

$$w(Q)\le C\lambda^{d-n}w(\lambda Q). \tag{2.48}$$

(ii) For any $d\in[0,\infty)$, $w\in\mathbb{D}^{\rm upper}_{\infty,d}$ if and only if there exists a positive constant C such that, for any $\lambda\in[1,\infty)$ and any cube $Q\subset\mathbb{R}^n$,

$$w(\lambda Q)\le C\lambda^{d+n}w(Q). \tag{2.49}$$

Note that (2.48) and (2.49) are precisely the *reverse doubling condition* and the *doubling condition* of weight w under consideration.

Proposition 2.45. Let $w\in A_\infty$. Then

$$d^{\rm lower}_\infty(w)=\limsup_{i\to\infty}\frac{1}{i}\log_2\left(\sup_{{\rm cube}\,Q\subset\mathbb{R}^n}\frac{w(Q)}{w(2^iQ)}\right)+n$$

and

$$d^{\rm upper}_\infty(w)=\limsup_{i\to\infty}\frac{1}{i}\log_2\left(\sup_{{\rm cube}\,Q\subset\mathbb{R}^n}\frac{w(2^iQ)}{w(Q)}\right)-n.$$

Proposition 2.46. Let $p\in[1,\infty)$ and $w\in A_p$ have the A_p-dimension $d\in[0,n)$. Then the following statements hold.

(i) If $p=1$, then there exists a positive constant C such that, for any cubes $Q,R\subset\mathbb{R}^n$,

$$\frac{w(Q)}{w(R)}\le C\max\left\{\left[\frac{\ell(R)}{\ell(Q)}\right]^{d-n},\left[\frac{\ell(Q)}{\ell(R)}\right]^{n}\right\}\left[1+\frac{|c_Q-c_R|}{\ell(Q)\vee\ell(R)}\right]^{d}.$$

(ii) If $p\in(1,\infty)$, letting further $w^{1-p'}$ (which belongs to $A_{p'}$) have the $A_{p'}$-dimension $\widetilde{d}\in[0,n)$, then there exists a positive constant C such that, for any cubes $Q,R\subset\mathbb{R}^n$,

$$\frac{w(Q)}{w(R)}\le C\max\left\{\left[\frac{\ell(R)}{\ell(Q)}\right]^{d-n},\left[\frac{\ell(Q)}{\ell(R)}\right]^{(p-1)\widetilde{d}+n}\right\}\left[1+\frac{|c_Q-c_R|}{\ell(Q)\vee\ell(R)}\right]^{d+(p-1)\widetilde{d}}.$$

For any $w\in A_\infty$, let

$$d^{\rm lower}_\infty(w):=\inf\left\{d\in[0,n):\ w\in\mathbb{D}^{\rm lower}_{\infty,d}\right\}$$

and

$$d_\infty^{\rm upper}(w):=\inf\left\{d\in[0,\infty):\ w\in\mathbb{D}_{\infty,d}^{\rm upper}\right\}.$$

Let

$$[[d_\infty^{\rm lower}(w),n):=\begin{cases}[d_\infty^{\rm lower}(w),n) & \text{if } d_\infty^{\rm lower}(w) \text{ is } A_\infty\text{-lower dimension of } w,\\ (d_\infty^{\rm lower}(w),n) & \text{otherwise}\end{cases}$$

and

$$[[d_\infty^{\rm upper}(w),\infty):=\begin{cases}[d_\infty^{\rm upper}(w),\infty) & \text{if } d_\infty^{\rm upper}(w) \text{ is } A_\infty\text{-upper dimension of } w,\\ (d_\infty^{\rm upper}(w),\infty) & \text{otherwise.}\end{cases}$$

Proposition 2.47. Let $w\in A_\infty$. Assume that $d_1\in[[d_\infty^{\rm lower}(w),\infty)$ and $d_2\in[[d_\infty^{\rm upper}(w),\infty)$. Then there exists a positive constant C such that, for any cubes $Q,R\subset\mathbb{R}^n$,

$$\frac{w(Q)}{w(R)}\le C\max\left\{\left[\frac{\ell(R)}{\ell(Q)}\right]^{d_1-n},\left[\frac{\ell(Q)}{\ell(R)}\right]^{d_2+n}\right\}\left[1+\frac{|c_Q-c_R|}{\ell(Q)\vee\ell(R)}\right]^{d_1+d_2}.$$

Proposition 2.48. Let $p\in[1,\infty)$, $w\in A_p$, and $d\in[0,\infty)$. Then the following assertions hold.

(i) $w\in\mathbb{D}_{\infty,d}^{\rm lower}$ if and only if $w\in\mathbb{D}_{p,d}(\mathbb{R}^n)$.

(ii) $\mathbb{D}_{\infty,0}^{\rm upper}=A_1$.

(iii) If $p\in(1,\infty)$, then $w\in\mathbb{D}_{p,\infty,d}^{\rm upper}$ if and only if $w^{1-p'}\in\mathbb{D}_{p',\frac{d}{p-1}}$.

(iv) If $p\in(1,\infty)$, then $w^{1-p'}\in\mathbb{D}_{\infty,\frac{d}{p-1}}^{\rm upper}$ if and only if $w\in\mathbb{D}_{p,d}$.

(v) $d_\infty^{\rm lower}(w)=d_p(w)$.

(vi) If $p=1$, then $d_\infty^{\rm upper}(w)=0$.

(vii) If $p\in(1,\infty)$, then $d_\infty^{\rm upper}(w)=(p-1)d_p(w^{1-p'})$.

The following lemma serves as a bridge connecting the dimensions of scalar weights and those of matrix weights; see [24, Lemma 2.38(ii)] and [27, Lemma 7.2]. These statements follow directly from the definitions of dimensions; we omit the details.

Lemma 2.10. *Let $p\in(0,\infty)$ and $d\in[0,\infty)$. Let w be a scalar weight and $W:=wI_m$, where I_m is the identity matrix. Then the following statements hold.*

(i) *$W\in\mathbb{D}_{p,d}$ if and only if $w\in\mathbb{D}_{\max\{1,p\},d}$.*

(ii) *$W\in\mathbb{D}_{p,\infty,d}^{\rm upper}$ if and only if $w\in\mathbb{D}_{\infty,d}^{\rm upper}$.*

(iii) $W\in\mathbb{D}^{\mathrm{lower}}_{p,\infty,d}$ *if and only if* $w\in\mathbb{D}^{\mathrm{lower}}_{\infty,d}$.

Now, we compute the dimensions of two scalar weights discussed in Subsection 2.1.

Example 2.6. Let $a\in(-n,\infty)$ and $b\in\mathbb{R}$. Assume that $w_{a,b}$ and $\widetilde{w}_{a,b}$ are as in Lemma 2.2. Then the following assertions hold.

(i) $d^{\mathrm{lower}}_{\infty}(w_{a,b})=d^{\mathrm{lower}}_{\infty}(\widetilde{w}_{a,b})=a_-$.

(ii) $d^{\mathrm{upper}}_{\infty}(w_{a,b})=d^{\mathrm{upper}}_{\infty}(\widetilde{w}_{a,b})=a_+$.

Proposition 2.49. Let $\{w_i\}_{i=1}^n$ be a sequence of scalar weights on $\mathbb{R}$, and let w_{prod} be as in Proposition 2.5. Then the following statements hold.

(i) If $w_{\mathrm{prod}}\in A_\infty$, then $d^{\mathrm{lower}}_{\infty}(w_{\mathrm{prod}})\le\sum_{i=1}^n d^{\mathrm{lower}}_{\infty}(w_i)$.

(ii) If $w_{\mathrm{prod}}\in A_\infty$, then $d^{\mathrm{upper}}_{\infty}(w_{\mathrm{prod}})\le\sum_{i=1}^n d^{\mathrm{upper}}_{\infty}(w_i)$.

Proof. By similarity, we only prove (i). From Proposition 2.45, we infer that

$$\begin{aligned}
d^{\mathrm{lower}}_{\infty}(w_{\mathrm{prod}})-n&=\limsup_{j\to\infty}\frac{1}{j}\log_2\left(\sup_{Q:=I_1\times\cdots\times I_n}\frac{w(Q)}{w(2^jQ)}\right)\\
&\le\limsup_{i\to\infty}\frac{1}{j}\log_2\left(\prod_{i=1}^n\sup_{I\subset\mathbb{R}}\frac{w_i(I)}{w_i(2^jI)}\right)\\
&\le\sum_{i=1}^n\limsup_{j\to\infty}\frac{1}{j}\log_2\left(\sup_{I\subset\mathbb{R}}\frac{w_i(I)}{w_i(2^jI)}\right)=\sum_{i=1}^n\left[d^{\mathrm{upper}}_{\infty}(w_i)-1\right].
\end{aligned}$$

This finishes the proof of (i) and hence Proposition 2.49. □

Specifically, if w_{prod} is constructed by power weights, then we obtain the following example.

Example 2.7. Let $\vec{a}:=\{a_i\}_{i=1}^n\subset\mathbb{R}$ and, for any $x:=(x_1,\dots,x_n)\in\mathbb{R}^n$, $w_{\vec{a}}(x):=\prod_{i=1}^n|x_i|^{a_i}$. Then the following statements hold.

(i) $w_{\vec{a}}\in A_\infty$ if and only if $a_i\in(-1,\infty)$ for every $i\in\{1,\dots,n\}$.

(ii) $w_{\vec{a}}\in A_1$ if and only if $a_i\in(-1,0]$ for every $i\in\{1,\dots,n\}$.

(iii) If $p\in(1,\infty)$, then $w_{\vec{a}}\in A_p$ if and only if $a_i\in(-1,p-1)$ for every $i\in\{1,\dots,n\}$.

(iv) If $w_{\vec{a}}\in A_\infty$, then $r_s(w_{\vec{a}})=1+\max_{i\in\{1,\dots,n\}}(a_i)_+$.

(v) If $w_{\vec{a}}\in A_\infty$, then $d^{\mathrm{upper}}_{\infty}(w_{\vec{a}})=\sum_{i=1}^n(a_i)_+$ is an A_∞-upper dimension of $w_{\vec{a}}$.

(vi) If $w_{\vec{a}}\in A_\infty$, then $d^{\mathrm{lower}}_{\infty}(w_{\vec{a}})=\sum_{i=1}^n(a_i)_-$ is an A_∞-lower dimension of $w_{\vec{a}}$.

(vii) $w_{\vec{a}}\in RH_\infty$ if and only if $a_i\in[0,\infty)$ for every $i\in\{1,\dots,n\}$.

(viii) If $q\in(1,\infty)$, then $w_{\vec{a}}\in RH_q$ if and only if $a_i\in(-\frac{1}{q},\infty)$ for every $i\in\{1,\dots,n\}$.

(ix) If $w_{\vec{a}}\in A_\infty$, then $r_h(w_{\vec{a}})=\min_{i\in\{1,\dots,n\}}\frac{1}{(a_i)_-}$, where $\frac{1}{0}:=\infty$.

Proof. Note that the assertions (i) through (v) are precisely [28, Lemma 4.30]. Repeating the argument used in the proof of [28, Lemma 4.30(v)], we obtain (vi). By Proposition 2.5 and Example 2.3, we conclude that (vii) through (ix) hold. This finishes the proof of Example 2.7. $\square$

Finally, we studied the relationship between dimensions of weights and the critical index of weights. To this end, we first present the following technical lemma.

Lemma 2.11. *If $q\in(1,\infty]$ and $w\in RH_q$, then $w\in\mathbb{D}^{\mathrm{lower}}_{\infty,\frac{n}{q}}$.*

Proof. Using Hölder's inequality and $w\in RH_q$, we find that, for any cube $Q\subset\mathbb{R}^n$,

$$\fint_Q w(x)\,dx\le\left\{\fint_Q[w(x)]^q\,dx\right\}^{\frac{1}{q}}\le\lambda^{\frac{n}{q}}\left\{\fint_{\lambda Q}[w(x)]^q\,dx\right\}^{\frac{1}{q}}\lesssim\lambda^{\frac{n}{q}}\fint_{\lambda Q}w(x)\,dx$$

and hence $w(Q)\lesssim\lambda^{\frac{n}{q}-n}w(\lambda Q)$. This, together with Proposition 2.44(i), further implies that $w\in\mathbb{D}^{\mathrm{lower}}_{\infty,\frac{n}{q}}$, which completes the proof of Lemma 2.11. $\square$

The following theorem illustrates that, in some situations, the dimensions of weights offer a more finer property than the critical indices of weights; see [28, Section 4.2] for more details.

Theorem 2.5. *The following statements hold.*

(i) *If $w\in A_\infty$, then $d^{\mathrm{upper}}_\infty(w)\le n[r_s(w)-1]$.*

(ii) *For any $d\in[0,\infty)$, there exists $w\in A_\infty$ such that $d^{\mathrm{upper}}_\infty(w)=n[r_s(w)-1]=d$.*

(iii) *If $n\ge2$, then there exists $w\in A_\infty$ such that $d^{\mathrm{upper}}_\infty(w)<n[r_s(w)-1]$.*

(iv) *If $w\in A_\infty$, then $d^{\mathrm{lower}}_\infty(w)\le\frac{n}{r_h(w)}$.*

(v) *For any $d\in[0,n)$, then there exists $w\in A_\infty$ such that $d^{\mathrm{lower}}_\infty(w)=\frac{n}{r_h(w)}=d$.*

(vi) *If $n\ge2$, then there exists $w\in A_\infty$ such that $d^{\mathrm{lower}}_\infty(w)<\frac{n}{r_h(w)}$.*

Proof. The statements (i) through (iii) are exactly [28, Theorem 4.28].

Next, we prove (iv). From the definition of $r_h(w)$, we infer that, for any $q\in(0,r_h(w))$, $w\in RH_q$. Applying this and Lemma 2.11, we conclude that $w\in\mathbb{D}^{\mathrm{lower}}_{\infty,\frac{n}{q}}$, which further implies that $d^{\mathrm{lower}}_\infty(w)\le\frac{n}{q}$. Letting $q\to r_h(w)$, we obtain $d^{\mathrm{lower}}_\infty(w)\le\frac{n}{r_h(w)}$. This finishes the proof of (iv).

Now, we prove (v). Let $d\in[0,n)$ and $\vec{a}:=(-\frac{d}{n},\dots,-\frac{d}{n})\in\mathbb{R}^n$. Using (vi) and (ix) of Example 2.7, we have $d^{\mathrm{lower}}_\infty(w_{\vec{a}})=\frac{n}{r_h(w_{\vec{a}})}=d$, where $w_{\vec{a}}$ is as in Example 2.7. This finishes the proof of (v).

Finally, we prove (vi). Let $\vec{a}:=(-\frac{1}{2},\ldots,-\frac{1}{n+1})$. Then, by (vi) and (ix) of Example 2.7 again, we conclude that

$$d_\infty^{\mathrm{lower}}(w_{\vec{a}})=\sum_{i=1}^n\frac{1}{i+1}<\frac{n}{2}=\frac{n}{r_h(w_{\vec{a}})}.$$

This finishes the proof of (vi) and hence Theorem 2.5. □

Question IV. *It is interesting to know whether there exists* $w\in A_\infty(\mathbb{R})$ *such that* $d_\infty^{\mathrm{upper}}(w)<r_s(w)-1$ *or* $d_\infty^{\mathrm{lower}}(w)<\frac{1}{r_h(w)}$.

3 Weighted Inequalities

The subject, which is the most closely related to weights, is the weighted norm inequalities of the Hardy–Littlewood maximal operator and the Calderón–Zygmund operator.

In particular, the theory of Muckenhoupt weights provides a powerful framework for understanding how certain integral operators behave under weighted norms. A central object in this theory is the Hardy–Littlewood maximal operator, whose boundedness properties on weighted L^p spaces are deeply intertwined with the structure of these weights. The interplay between weights and maximal operators forms the basis for many results in modern analysis, including the development of weighted inequalities and the characterization of function spaces.

The boundedness of the Calderón–Zygmund operator is crucial in solving partial differential equations, studying the regularity of solutions, and analyzing complex structures in harmonic analysis and related fields. We first recall the definition of Calderón–Zygmund operators (see, for instance, [56, p. 99]). Let $\Delta:=\{(x,x):\ x\in\mathbb{R}^n\}$. Then we say that $K:\ \mathbb{R}^n\times\mathbb{R}^n\setminus\Delta\to\mathbb{C}$ is a *standard kernel* if there exist $\delta,C\in(0,\infty)$ such that

(i) for any $x,y\in\mathbb{R}^n$ with $x\neq y$,

$$|K(x,y)|\le C\frac{1}{|x-y|^n},$$

(ii) for any $x,y,z\in\mathbb{R}^n$ with $|x-y|>2|y-z|$,

$$|K(x,y)-K(x,z)|\le C\frac{|y-z|^\delta}{|x-y|^{n+\delta}},$$

(iii) for any $x,y,w\in\mathbb{R}^n$ with $|x-y|>2|x-w|$,

$$|K(x,y)-K(w,y)|\le C\frac{|x-w|^\delta}{|x-y|^{n+\delta}}.$$

Definition 3.1. An operator T is called a *Calderón–Zygmund operator* if

(i) T is bounded on L^2;

(ii) there exists a standard kernel K such that, for any $f\in L^2$ with compact support and for any $x\in\mathbb{R}^n\setminus\operatorname{supp} f$,

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)\,dy.$$

Let $\Omega\subset\mathbb{R}^n$ be a *domain* which means a non-empty connected open subset in $\mathbb{R}^n$. In what follows, for any $k\in\mathbb{N}$, we use $C^k(\Omega)$ to denote the set of all k times continuously differentiable functions on Ω and $C^\infty(\Omega)$ to denote the set of all infinitely differentiable functions on Ω. Now, we recall the concept of convolution Calderón–Zygmund operators (see, for instance, [146, p. 204]). We say that $K\colon\,\mathbb{R}^n\setminus\{\mathbf{0}\}\to\mathbb{C}$ satisfies the *gradient condition* if $K\in C^1(\mathbb{R}^n)$ and, for any $x\in\mathbb{R}^n\setminus\{\mathbf{0}\}$ and $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|\le 1$,

$$\left|\partial^\alpha K(x)\right|\le C|x|^{-n-\alpha}.$$

For a function K satisfying the gradient condition, we say that $T(f):=f*K$ is a *convolution Calderón–Zygmund operator* if there exists a positive constant C such that, for any $f\in C^\infty(\mathbb{R}^n)$ with compact support,

$$\|Tf\|_{L^2}\le C\|f\|_{L^2}.$$

Letting $\widetilde{K}(x,y):=K(x-y)$, we find that convolution Calderón–Zygmund operators satisfy all conditions of Definition 3.1. Thus, they are special cases of Calderón–Zygmund operators.

Furthermore, we say that a convolution Calderón–Zygmund operator satisfies the *nondegeneracy hypothesis* (see, for instance, [146, p. 210]) if there exists a unit vector $u\in\mathbb{R}^n$ such that $|K(ru)|\ge a|r|^{-n}$ for all $r\in\mathbb{R}\setminus\{\mathbf{0}\}$. Notably, the Hilbert transform and the Riesz transform both satisfy the nondegeneracy hypothesis.

Before proceeding to the matrix-weighted setting, we first recall some related function spaces. Let $p\in(0,\infty)$. The *vector-valued Lebesgue space* $\mathcal{L}^p$ is defined to be the set of all measurable vector-valued functions $\vec{f}\colon\,\mathbb{R}^n\to\mathbb{C}^m$ such that

$$\left\|\vec{f}\right\|_{\mathcal{L}^p}:=\left[\int_{\mathbb{R}^n}\left|\vec{f}(x)\right|^p dx\right]^{\frac{1}{p}}<\infty.$$

For Calderón–Zygmund operators, compared with the scalar case, the boundedness properties in the matrix-weighted setting are considerably less understood and substantially more difficult to establish. This increased complexity arises from the intricate interaction between the operators and matrix weights. Notably, certain operators that behave well under scalar weights may exhibit markedly different behavior when matrix weights are involved. Due to these challenges, matrix-weighted norm inequalities have only recently obtained some systematic study. References [148, 155] discuss these difficulties in detail and offer valuable approaches to overcoming them.

If a scalar weight w satisfies the Muckenhoupt condition, the boundedness of any Calderón–Zygmund operator can be deduced from the boundedness of the Hardy–Littlewood maximal operator M. A similar strategy is applicable in the matrix-weighted setting: one may investigate the boundedness of Calderón–Zygmund operators by first analyzing the corresponding matrix-weighted maximal operator. In this direction, Goldberg [67] studied the boundedness of both the Hardy–Littlewood maximal operator and the Calderón–Zygmund operator in the matrix setting.

The remainder of this section is organized as follows. In Subsection 3.1, we review the quantitative weighted inequalities for the Hardy–Littlewood maximal operator. In Subsection 3.2, we address the corresponding results for the Calderón–Zygmund operator. In Subsection 3.3, we discuss the inequalities related to some other important operators.

3.1 The Hardy–Littlewood Maximal Operator

In what follows, for any quasi-Banach spaces $\mathscr{X},\mathscr{Y}$ and for any linear or sublinear operator $T:\mathscr{X}\to\mathscr{Y}$, let

$$\|T\|_{\mathscr{X}\to\mathscr{Y}}:=\sup_{\|f\|_{\mathscr{X}}=1}\|Tf\|_{\mathscr{Y}}.$$

We first recall the well-known strong-type weighted inequality on the Hardy–Littlewood maximal operator in the scalar case, which is precisely Buckley [31, Theorem 2.5].

Theorem 3.1. *Let $p\in(1,\infty)$ and w be a scalar weight. Then the following assertions hold.*

(i) *If $w\in A_p$, then there exists a positive constant C, depending only on n and p, such that, for any $f\in L^p(w)$,*

$$\|Mf\|_{L^p(w)}\le C[w]_{A_p}^{\frac{1}{p-1}}\|f\|_{L^p(w)},$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[w]_{A_p}\le t}\|M\|_{L^p(w)\to L^p(w)}\ge\widetilde{C}t^{\frac{1}{p-1}}$.

(ii) *Conversely, if M is bounded on $L^p(w)$, then $w\in A_p$.*

Although the strong-type inequality fails when $p=1$, the corresponding weak-type inequality remains valid.

Definition 3.2. Let $p\in(0,\infty)$ and w be a weight on $\mathbb{R}^n$. The *weighted weak Lebesgue space* $L^{p,\infty}(w)$ is defined to be the set of all $f\in\mathscr{M}$ such that

$$\|f\|_{L^{p,\infty}(w)}:=\sup_{t\in(0,\infty)}t\left[w(\{x\in\mathbb{R}^n:\ f(x)>t\})\right]^{\frac{1}{p}}<\infty.$$

The *unweighted weak Lebesgue space* $L^{p,\infty}:=L^{p,\infty}(1)$. The respective weak-type weighted inequality is due to Muckenhoupt [112, Theorem 1].

Theorem 3.2. *Let $p\in[1,\infty)$ and w be a scalar weight. Then the following statements hold.*

(i) *If $w\in A_p$, then there exists a positive constant C, depending only on n and p, such that, for any $f\in L^p(w)$,*

$$\|Mf\|_{L^{p,\infty}(w)}\le C[w]_{A_p}^{\frac{1}{p}}\|f\|_{L^p(w)},$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[w]_{A_p}\le t}\|M\|_{L^p(w)\to L^{p,\infty}(w)}\ge\widetilde{C}t^{\frac{1}{p}}$.

(ii) *Conversely, if M is bounded from $L^p(w)$ to $L^{p,\infty}(w)$, then $w\in A_p$.*

Another weak-type inequality was established by Muckenhoupt and Wheeden [115, Theorem 1]. Recall that, for any $p\in(0,\infty)$ and any scalar weight w, the *Christ–Goldberg maximal operator* $M_{w,p}$ is defined by setting, for any $f\in\mathscr{M}$ and $x\in\mathbb{R}^n$,

$$M_{w,p}f(x):=\sup_{\text{ball}\,B\ni x}\fint_B\left|[w(x)]^{\frac{1}{p}}[w(y)]^{-\frac{1}{p}}f(y)\right|dy.$$

Recently, Cruz-Uribe et al. [49, Theorem 1.5] [for $p=1$] and [43, Corollary 1.5] [for any $p\in(1,\infty)$] obtained the following quantitative version of the aforementioned inequality in [115, Theorem 1]: for any $p\in[1,\infty)$ and $w\in A_p$, there exists a positive constant C, depending only on n and p, such that, for any $f\in L^p(w)$,

$$\left\|M_{w,p}f\right\|_{L^{p,\infty}}\le C[w]_{A_p}^{1+\frac{1}{p}}\|f\|_{L^p}.$$

Subsequently, Lerner et al. [102] proved the sharpness of this bound in the case where $p=1$ (see [102, Theorem 1]) and improved this bound into the (almost) sharp one in the case where $p\in(1,\infty)$ (see [103, (1.4) and Theorems 1.1 and 1.3]), which is stated as follows.

Theorem 3.3. *Let $p\in[1,\infty)$ and $w\in A_p$. Then the following assertions hold.*

(i) *If $p\in[1,2)$, then there exists a positive constant C, depending only on n and p, such that, for any $f\in L^p$,*

$$\left\|M_{w,p}f\right\|_{L^{p,\infty}}\le C[w]_{A_p}^{\frac{2}{p}}\|f\|_{L^p},$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[w]_{A_p}\le t}\|M_{w,p}\|_{L^p\to L^{p,\infty}}\ge\widetilde{C}t^{\frac{2}{p}}$.

(ii) *If $p\in[2,\infty)$, then there exists a positive constant C, depending only on n and p, such that, for any $f\in L^p$,*

$$\left\|M_{w,p}f\right\|_{L^{p,\infty}}\le C[w]_{A_p}^{\frac{1}{p-1}}\|f\|_{L^p}, \tag{3.1}$$

where the bound is almost sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[w]_{A_p}\le t}\|M_{w,p}\|_{L^p\to L^{p,\infty}}\ge\widetilde{C}t^{\frac{1}{p-1}}[\log(t+e)]^{-\frac{1}{p}}$.

Remark 3.1. Theorem 3.3(ii) shows that the exponent $\frac{1}{p-1}$ in (3.1) cannot be decreased. It remains open whether $\sup_{[w]_{A_p}\le t}\|M_{w,p}\|_{L^p\to L^{p,\infty}}\ge Ct^{\frac{1}{p-1}}$ holds for any $t\in[1,\infty)$ and some positive constant C independent of t.

It is worth mentioning that Muckenhoupt and Wheeden [115, Theorem 6] showed that, if w is a scalar weight, $p\in(1,\infty)$, and $M_{w,p}$ is bounded from L^p to $L^{p,\infty}$, then

$$[w]_{A_p^*}:=\sup_{\text{cube}\,Q\subset\mathbb{R}^n}\frac{1}{|Q|}\|w\mathbf{1}_Q\|_{L^{1,\infty}}\left\{\fint_Q[w(x)]^{-\frac{p'}{p}}dx\right\}^{\frac{p}{p'}}<\infty.$$

Notice that $A_p \subset A_p^*$ and $|\cdot|^{-n} \in A_p^* \backslash A_p$, which further imply $A_p \subsetneq A_p^*$. This fact was observed by Muckenhoupt and Wheeden [115, Section 4]. A natural question is whether the condition $w \in A_p^*$ is sufficient for this weak-type inequality. This question was resolved by Sweeting [147, Theorem 4.1] as follows.

Theorem 3.4. *Let $p \in (1,\infty)$ and w be a scalar weight. Then $w \in A_p^*$ if and only if $M_{w,p}$ is bounded from L^p to $L^{p,\infty}$.*

Question V. *It remains open whether one can introduce a scalar weight class A_1^* such that Theorem 3.4 holds in the case where $p=1$. The other natural question is whether a quantitative version of this weak-type inequality can be established.*

We now recall the corresponding results in the matrix setting. The strong-type weighted inequality associated with matrix-weighted Hardy–Littlewood maximal operators was established by Christ and Goldberg [38] for $p=2$. Later, Goldberg [67] extended this result to the general case where $p \in (1,\infty)$.

For any $p \in (0,\infty)$ and any matrix weight W, the *matrix-weighted maximal operator* $M_{W,p}$ is defined by setting, for any $\vec{f} \in (\mathscr{M})^m$ and $x \in \mathbb{R}^n$,

$$M_{W,p}\vec{f}(x) := \sup_{\text{ball}\, B \ni x} \fint_B \left| W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y) \vec{f}(y) \right| dy.$$

A quantitative version associated with the above matrix-weighted maximal operators is a special case of Isralowitz and Moen [90, Theorem 1.3], which is on matrix-weighted fractional maximal operators.

Theorem 3.5. *Let $p \in (1,\infty)$ and W be a matrix weight. Then the following statements hold.*

(i) *If $W \in \mathscr{A}_p$, then there exists a positive constant C, depending only on n, m, and p, such that, for any $\vec{f} \in \mathscr{L}^p$,*

$$\left\| M_{W,p}\vec{f} \right\|_{L^p} \le C [W]_{\mathscr{A}_p}^{\frac{1}{p-1}} \left\| \vec{f} \right\|_{\mathscr{L}^p},$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t \in [1,\infty)$, $\sup_{[W]_{\mathscr{A}_p} \le t} \|M_{W,p}\|_{\mathscr{L}^p \to L^p} \ge \widetilde{C} t^{\frac{1}{p-1}}$.

(ii) *Conversely, if $M_{W,p}$ is bounded from $\mathscr{L}^p$ to L^p, then $W \in \mathscr{A}_p$.*

The matrix-weighted weak-type inequality counterpart to Theorem 3.2 remains unknown. However, Cruz-Uribe et al. [43, 49] established a counterpart to Theorem 3.3 (see [43, Theorem 1.3] for $p=1$ and [49, Theorem 1.6] for any $p \in (1,\infty)$). Subsequently, Lerner et al. [102, 103] proved the sharpness of their bound in the case where $p=1$ (see [102, Theorem 1]) and improved the bound in the case where $p \in (1,\infty)$ (see [103, (1.4) and Theorems 1.1 and 1.3]), which read as follows.

Theorem 3.6. *Let $p\in[1,\infty)$ and $W\in\mathscr{A}_p$. Then the following assertions hold.*

(i) *If $p\in[1,2)$, then there exists a positive constant C, depending only on n, m, and p, such that, for any $\vec{f}\in\mathcal{L}^p$,*

$$\left\|M_{W,p}\vec{f}\right\|_{L^{p,\infty}}\le C[W]_{\mathscr{A}_p}^{\frac{2}{p}}\left\|\vec{f}\right\|_{\mathcal{L}^p}.$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[W]_{\mathscr{A}_p}\le t}\|M_{W,p}\|_{\mathcal{L}^p\to L^{p,\infty}}\ge\widetilde{C}t^{\frac{2}{p}}$.

(ii) *If $p\in[2,\infty)$, then there exists a positive constant C, depending only on n, m, and p, such that, for any $\vec{f}\in\mathcal{L}^p$,*

$$\left\|M_{W,p}\vec{f}\right\|_{L^{p,\infty}}\le C[W]_{\mathscr{A}_p}^{\frac{1}{p-1}}\left\|\vec{f}\right\|_{\mathcal{L}^p},$$

where the bound is almost sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[W]_{\mathscr{A}_p}\le t}\|M_{W,p}\|_{\mathcal{L}^p\to L^{p,\infty}}\ge\widetilde{C}t^{\frac{1}{p-1}}[\log(t+e)]^{-\frac{1}{p}}$.

Question VI. *Let $p\in(0,\infty)$. Motivated by Theorem 3.4, it is natural to consider introducing a suitable matrix weight class $\mathscr{A}_p^*$ such that $W\in\mathscr{A}_p^*$ if and only if $M_{W,p}$ is bounded from $\mathcal{L}^p$ to $L^{p,\infty}$.*

3.2 Calderón–Zygmund Operators and A_2 Conjecture

The strong-type weighted inequality for Calderón–Zygmund operators has been well known since the seminal work of Coifman and Fefferman [40, Theorem III]. Later, Buckley [31, Theorem 2.9] established a quantitative version of this inequality. Let $p\in(1,\infty)$ and $w\in A_p$. Then there exists a positive constant C, depending only on n and p, such that, for any $f\in L^p(w)$,

$$\|Tf\|_{L^p(w)}\le C[w]_{A_p}^{1+\frac{1}{p-1}}\|f\|_{L^p(w)},$$

where T is a convolution Calderón–Zygmund operator. Moreover, the power of $[w]_{A_p}$ in this inequality is at least $\max\{1,\frac{1}{p-1}\}$. A natural question is whether the sharp power is precisely $\max\{1,\frac{1}{p-1}\}$. This question has attracted considerable attention, especially after Astala et al. [4, Proposition 22] observed that it has applications to the study of the sharp regularity estimate for solutions to the Beltrami equation. Subsequently, this question was solved for several special Calderón–Zygmund operators, including the Hilbert transform [131], the Riesz transform [132], and the Beurling–Ahlfors operator [133]. Eventually, the problem was completely solved by Hytönen [80, Theorem 1.3] for general Calderón–Zygmund operators.

Theorem 3.7. *Let $p\in(1,\infty)$, $w\in A_p$, and T be a Calderón–Zygmund operator. Then there exists a positive constant C, depending only on n and p, such that, for any $f\in L^p(w)$,*

$$\|Tf\|_{L^p(w)}\le C[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}, \tag{3.2}$$

where the bound is sharp in the sense that there exist a Calderón–Zygmund operator $\widetilde{T}$ and a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[w]_{A_p}\le t}\|\widetilde{T}\|_{L^p(w)\to L^p(w)}\ge\widetilde{C}t^{\max\{1,\frac{1}{p-1}\}}$.

Remark 3.2. Let $p\in(1,\infty)$ and T be a non-degenerate convolution Calderón–Zygmund operator (for example, the Hilbert transform or the Riesz transform). Stein [146, p. 210] showed that, if T is bounded on $L^p(w)$, then $w\in A_p$.

This question is referred to as the A_2 *conjecture* because, if (3.2) holds for $p=2$, then, from this and the Rubio de Francia extrapolation theorem (see, for instance, [69, Theorem 7.5.3]), we infer that (3.2) holds for any $p\in(1,\infty)$.

Theorem 3.8. *Let $p_0\in[1,\infty)$ and T be an operator defined on $\bigcup_{q\in[1,\infty)}\bigcup_{w\in A_q}L^q(w)$. Assume that there exists an increasing function $K\colon\ [1,\infty)\to(0,\infty)$ such that, for any $w_0\in A_{p_0}$ and $f\in L^{p_0}(w_0)$,*

$$\|Tf\|_{L^{p_0}(w_0)}\le K\Big([w_0]_{A_{p_0}}\Big)\|f\|_{L^{p_0}(w_0)}.$$

Then, for any $p\in(1,\infty)$ and $w\in A_p$, there exist both a positive constant C_{p,p_0}, depending only on p and p_0, and a positive constant C_{n,p,p_0}, depending only on n, p, and p_0, such that, for any $f\in L^p(w)$,

$$\|Tf\|_{L^p(w)}\le C_{p,p_0}K\left(C_{n,p,p_0}[w]_{A_p}^{\max\{1,\frac{p_0-1}{p-1}\}}\right)\|f\|_{L^p(w)}.$$

It is worth mentioning that Lerner [101] provided a simpler proof of the A_2 conjecture by introducing the technique of the sparse domination, which has since been widely used in the study of weighted inequalities.

Next, we recall the history of weak-type weighted inequalities. In [105, Theorem 1.1], Lerner et al. established the quantitative weak-type weighted inequality for $p=1$. In [82, Theorem 1.2], Hytönen et al. obtained the corresponding result for any $p\in(1,\infty)$. The sharpness of these results was due to Lerner et al. [104, Theorem 1.1] for $p=1$ and Pérez and Rivera-Ríos [130, Theorem 2] for any $p\in(1,\infty)$.

Theorem 3.9. *Let $p\in[1,\infty)$, $w\in A_p$, and T be a Calderón–Zygmund operator. Then the following statements hold.*

(i) *If $p=1$, then there exists a positive constant C, depending only on n and p, such that, for any $f\in L^1(w)$,*

$$\|Tf\|_{L^{1,\infty}(w)}\le C[w]_{A_1}(1+\log[w]_{A_1})\|f\|_{L^1(w)},$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[w]_{A_1(\mathbb{R})}\le t}\|H\|_{L^1(w,\mathbb{R})\to L^{1,\infty}(w,\mathbb{R})}\ge\widetilde{C}t(1+\log t)$, where H is the Hilbert transform.

(ii) *If $p\in(1,\infty)$, then there exists a positive constant C, depending only on n and p, such that, for any $f\in L^p(w)$,*

$$\|Tf\|_{L^{p,\infty}(w)}\le C[w]_{A_p}\|f\|_{L^p(w)},$$

where the bound is sharp in the sense that the power of $[w]_{A_p}$ in this inequality is at least 1.

Now, we recall another weak-type weighted inequality. Let $p\in[1,\infty)$, $w\in A_p$, and T be a Calderón–Zygmund operator. For any measurable function f and $x\in\mathbb{R}^n$,

$$T_{w,p}(f)(x):=[w(x)]^{\frac{1}{p}}T\left(w^{-\frac{1}{p}}f\right)(x).$$

In [43, Corollary 1.5], Cruz-Uribe et al. established a weak-type weighted inequality associated with $T_{w,1}$, which was later shown to be sharp by Lerner et al. [102, Theorem 1]. In [49, Theorem 1.5], Cruz-Uribe and Sweeting extended this inequality to any $p\in(1,\infty)$. Their result was subsequently improved by Lerner et al. [103, Theorems 1.2 and 1.4].

Theorem 3.10. *Let $p\in[1,\infty)$, $w\in A_p$, and T be a Calderón–Zygmund operator. Then the following assertions hold.*

(i) *If $p=1$, then there exists a positive constant C, depending only on n, such that, for any $f\in L^1$,*

$$\left\|T_{w,p}f\right\|_{L^{1,\infty}}\le C[w]_{A_1}^2\|f\|_{L^1}.$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[w]_{A_1(\mathbb{R})}\le t}\|H_{w,1}\|_{L^1(\mathbb{R})\to L^{1,\infty}(\mathbb{R})}\ge\widetilde{C}t^2$, where H is the Hilbert transform.

(ii) *If $p\in(1,2)$, then there exists a positive constant C, depending only on n and p, such that, for any $f\in L^p$,*

$$\left\|T_{w,p}f\right\|_{L^{p,\infty}}\le C[w]_{A_p}^{1+\frac{1}{p^2}}\left[\log([w]_{A_p}+e)\right]^{\frac{1}{p}}\|f\|_{L^p}. \tag{3.3}$$

(iii) *If $p\in[2,\infty)$, then there exists a positive constant C, depending only on n and p, such that, for any $f\in L^p$,*

$$\left\|T_{w,p}f\right\|_{L^{p,\infty}}\le C[w]_{A_p}\|f\|_{L^p},$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[w]_{A_p(\mathbb{R})}\le t}\|H_{w,p}\|_{L^p(\mathbb{R})\to L^{p,\infty}(\mathbb{R})}\ge\widetilde{C}t$, where H is the Hilbert transform.

Based on Theorem 3.10, we ask the following questions.

Question VII. (i) *What is the sharp bound in inequality* (3.3)*?*

(ii) *Does Theorem 3.10 still hold for any $p\in(1,\infty)$ and $w\in A_p^*$?*

Now, we recall strong-type matrix-weighted inequalities. Let $p\in(0,\infty)$ and W be a matrix weight. The *matrix-weighted Lebesgue space* $\mathcal{L}^p(W)$ is defined to be the set of all measurable vector-valued functions $\vec{f}$: $\mathbb{R}^n\to\mathbb{C}^m$ such that

$$\left\|\vec{f}\right\|_{\mathcal{L}^p(W)}:=\left\|\left|W^{\frac{1}{p}}\vec{f}\right|\right\|_{L^p}<\infty.$$

In 1990s, Treil and Volberg [148, 149] proved that $W\in\mathscr{A}_2(\mathbb{R},\mathbb{C}^m)$ if and only if the Hilbert transform H is bounded on $\mathcal{L}^2(W,\mathbb{R})$. This result was later extended to the case where $p\in(1,\infty)$

by Nazarov and Treil [118] and independently by Volberg [155]. Subsequently, Goldberg [67] showed that, if $p\in(1,\infty)$, $W\in\mathscr{A}_2$, and T is a convolution Calderón–Zygmund operator, then, for any $\vec{f}\in\mathcal{L}^p(W)$,

$$\left\|T\vec{f}\right\|_{\mathcal{L}^p(W)}\lesssim\left\|\vec{f}\right\|_{\mathcal{L}^p(W)}$$

with the implicit constant independent of $\vec{f}$. In particular, if T is additionally assumed to be non-degenerate, then $W\in\mathscr{A}_p$ if and only if T is bounded on $\mathcal{L}^p(W)$.

Later, a lot of attention has been paid to establish quantitative versions of strong-type matrix-weighted inequalities; see, for example, [13, 15, 135]. A remarkable breakthrough was made independently by Nazarov et al. [117] and Culiuc et al. [50]. By introducing the technique of the convex body sparse domination, they showed that, if $W\in\mathscr{A}_2$, then, for any $\vec{f}\in\mathcal{L}^2(W)$,

$$\left\|T\vec{f}\right\|_{\mathcal{L}^2(W)}\le C[W]_{\mathscr{A}_2}^{\frac{3}{2}}\left\|\vec{f}\right\|_{\mathcal{L}^2(W)},\tag{3.4}$$

where T is a Calderón–Zygmund operator and C is a positive constant depending only on n and m. This convex body sparse domination technique has since been widely used in the study of matrix-weighted inequalities; see, for example, [53, 54, 58, 81, 84, 91, 116, 161]. Specially, Cruz-Uribe et al. [42, Corollary 1.16] extended (3.4) to the case where $p\in(1,\infty)$.

Theorem 3.11. *Let $p\in(1,\infty)$, $W\in\mathscr{A}_p$, and T be a Calderón–Zygmund operator. Then there exists a positive constant C, depending only on n, m, and p, such that, for any $\vec{f}\in\mathcal{L}^p(W)$,*

$$\left\|T\vec{f}\right\|_{\mathcal{L}^p(W)}\le C[W]_{\mathscr{A}_p}^{1+\frac{1}{p(p-1)}}\left\|\vec{f}\right\|_{\mathcal{L}^p(W)}.\tag{3.5}$$

More recently, a breakthrough was made by Bownik and Cruz-Uribe [20, Theorem 1.4], who established a matrix-valued extrapolation theorem by further developing the theory of convex-set valued functions.

Theorem 3.12. *Let $p_0\in[1,\infty)$ and T be an operator defined on $\bigcup_{q\in[1,\infty)}\bigcup_{W\in\mathscr{A}_q}\mathcal{L}^q(W)$. Assume that there exists a increasing function $K:\ [1,\infty)\to(0,\infty)$ such that, for any $W_0\in\mathscr{A}_{p_0}$ and $\vec{f}\in\mathcal{L}^{p_0}(W_0)$,*

$$\left\|T\vec{f}\right\|_{\mathcal{L}^{p_0}(W_0)}\le K\left([W_0]_{\mathscr{A}_{p_0}}\right)\left\|\vec{f}\right\|_{\mathcal{L}^{p_0}(W_0)}.$$

Then, for any $p\in(1,\infty)$, there exist both a positive constant C_{p,p_0}, depending only on p and p_0, and a positive constant C_{n,m,p,p_0}, depending only on n, m, p, and p_0, such that, for any $W\in\mathscr{A}_p$ and $\vec{f}\in\mathcal{L}^p(W)$,

$$\left\|T\vec{f}\right\|_{\mathcal{L}^p(W)}\le C_{p,p_0}K\left(C_{n,m,p,p_0}[W]_{\mathscr{A}_p}^{\max\{1,\frac{p_0-1}{p-1}\}}\right)\left\|\vec{f}\right\|_{\mathcal{L}^p(W)}.$$

Based on Theorem 3.7, one naturally conjectures that, for any $W\in\mathscr{A}_2$ and $\vec{f}\in\mathcal{L}^2(W)$,

$$\left\|T\vec{f}\right\|_{\mathcal{L}^2(W)}\lesssim[W]_{\mathscr{A}_2}\left\|\vec{f}\right\|_{\mathcal{L}^2(W)}$$

with the implicit positive constant depending only on n and m. If this conjecture holds, then, from this and Theorem 3.12, it follows that, for any $p\in(1,\infty)$, $W\in\mathscr{A}_p$, and $\vec{f}\in\mathcal{L}^p(W)$,

$$\left\|T\vec{f}\right\|_{\mathcal{L}^p(W)}\lesssim[W]_{\mathscr{A}_p}^{\max\{1,\frac{p_0-1}{p-1}\}}\left\|\vec{f}\right\|_{\mathcal{L}^p(W)} \tag{3.6}$$

with the implicit positive constant depending only on n, m, and p. This would completely solve the question of the sharp bound in (3.6). Surprisingly, Domelevo et al. [55, Theorem 1.1] showed that this conjecture is false, while the sharp power of $[W]_{\mathscr{A}_2(\mathbb{R},\mathbb{C}^m)}$ is $\frac{3}{2}$, not 1. Indeed, their result was proved for the case where $m=2$, but it can be easily extended to the general case where $m\geq3$ as follows.

Theorem 3.13. *Let $m\geq2$, $W\in\mathscr{A}_2(\mathbb{R},\mathbb{C}^m)$, and H be the Hilbert transform. Then there exists a positive constant C, depending only on n and m, such that, for any $t\in[1,\infty)$,*

$$\sup_{[W]_{\mathscr{A}_2(\mathbb{R},\mathbb{R}^m)}\leq t}\|H\|_{\mathcal{L}^2(W,\mathbb{R})\to\mathcal{L}^2(W,\mathbb{R})}\geq Ct^{\frac{3}{2}}.$$

Proof. From [55, Theorem 1.1], it follows that there exists a positive constant C such that, for any $t\in[1,\infty)$,

$$\sup_{[W_0]_{\mathscr{A}_2(\mathbb{R},\mathbb{R}^2)}\leq t}\|H\|_{\mathcal{L}^2(W_0,\mathbb{R})\to\mathcal{L}^2(W_0,\mathbb{R})}\geq Ct^{\frac{3}{2}}.$$

Let $t\in[1,\infty)$ be fixed. Then there exist a matrix weight $W_0\in\mathscr{A}_2(\mathbb{R},\mathbb{R}^2)$ with $[W_0]_{\mathscr{A}_2(\mathbb{R},\mathbb{R}^2)}\leq t$ and a nonzero function $\vec{g}\in L^2(W_0,\mathbb{R})$ such that

$$\left\|H\vec{g}\right\|_{\mathcal{L}^2(W_0,\mathbb{R})}\geq\frac{C}{2}t^{\frac{3}{2}}\left\|\vec{g}\right\|_{\mathcal{L}^2(W_0,\mathbb{R})}. \tag{3.7}$$

Let

$$W:=\begin{pmatrix}W_0 & O_{2,m-2}\\ O_{m-2,2} & I_{m-2}\end{pmatrix}\text{ and }\vec{f}:=\begin{pmatrix}\vec{g}\\ \vec{\mathbf{0}}_{m-2}\end{pmatrix}.$$

Then $\vec{f}$ is a nonzero function. Using (2.9), we obtain

$$\begin{aligned}[W]_{\mathscr{A}_2(\mathbb{R},\mathbb{R}^m)}&\sim\sup_{\text{interval}\,I\subset\mathbb{R}}\fint_I\fint_I\left\|W_0^{\frac{1}{2}}(x)W_0^{-\frac{1}{2}}(y)\right\|^2+(m-2)\,dy\,dx\\&=[W_0]_{\mathscr{A}_2(\mathbb{R},\mathbb{R}^2)}+m+2\leq t+m+2\leq(m+3)t\end{aligned}$$

and hence there exists $K\in(1,\infty)$ such that $[W]_{\mathscr{A}_2(\mathbb{R},\mathbb{R}^m)}\le s:=Kt$. By (3.7), we find that

$$\left\|H\vec{f}\right\|_{\mathcal{L}^2(W,\mathbb{R})}=\|H\vec{g}\|_{\mathcal{L}^2(W_0,\mathbb{R})}\ge\frac{C}{2}t^{\frac{3}{2}}\|\vec{g}\|_{\mathcal{L}^2(W_0,\mathbb{R})}=\frac{C}{2}K^{-\frac{3}{2}}s^{\frac{3}{2}}\left\|\vec{f}\right\|_{\mathcal{L}^2(W,\mathbb{R})}.$$

These further imply that, for any $s\in[K,\infty)$,

$$\sup_{[W]_{\mathscr{A}_2(\mathbb{R},\mathbb{R}^m)}\le s}\|H\|_{\mathcal{L}^2(W,\mathbb{R})\to\mathcal{L}^2(W,\mathbb{R})}\ge\frac{C}{2}K^{-\frac{3}{2}}s^{\frac{3}{2}}.$$

On the other hand, for any $s\in[1,K)$,

$$\sup_{[W]_{\mathscr{A}_2(\mathbb{R},\mathbb{R}^m)}\le s}\|H\|_{\mathcal{L}^2(W,\mathbb{R})\to\mathcal{L}^2(W,\mathbb{R})}\ge\|H\|_{\mathcal{L}^2(I_m,\mathbb{R})\to\mathcal{L}^2(I_m,\mathbb{R})}K^{-\frac{3}{2}}s^{\frac{3}{2}}.$$

This finishes the proof of Theorem 3.13. □

Question VIII. *When $p\in(1,2)\cup(2,\infty)$, the sharp bound in inequality* (3.5) *remains unknown.*

Let $p\in(1,\infty)$. If we apply the matrix-weighted extrapolation theorem (Theorem 3.12) and the sharp inequality

$$\left\|T\vec{f}\right\|_{\mathcal{L}^2(W)}\lesssim[W]_{\mathscr{A}_2}^{\frac{3}{2}}\left\|\vec{f}\right\|_{\mathcal{L}^2(W)},$$

we obtain

$$\left\|T\vec{f}\right\|_{\mathcal{L}^p(W)}\lesssim[W]_{\mathscr{A}_p}^{\frac{3}{2}\max\{1,\frac{1}{p-1}\}}\left\|\vec{f}\right\|_{\mathcal{L}^p(W)}$$

with the implicit positive constant depending only on n, m, and p, which is even worse than the estimate in Theorem 3.11. Although the matrix-weighted extrapolation theorem cannot be used to solve Question VIII, it allows one to reduce the problem of establishing the boundedness of an operator on $\mathcal{L}^p(W)$ to the case of $\mathcal{L}^2(W)$. This reduction is valuable, as the space $\mathcal{L}^2(W)$ and the class $\mathscr{A}_2$ possess richer properties compared to $\mathcal{L}^p(W)$ and $\mathscr{A}_p$.

Now, we turn our attention to weak-type matrix-weighted inequalities. Quantitative bounds were established by Cruz-Uribe et al. [43,49]; see [43, Theorem 1.4] for $p=1$ and [49, Theorem 1.6] for any $p\in(1,\infty)$. Subsequently, Lerner et al. [102, Theorem 1] proved the sharpness of the bound in the case where $p=1$. Moreover, using Theorem 3.11, we can further improve the bound in the case where $p\in(2,\infty)$.

Theorem 3.14. *Let $p\in[1,\infty)$ and $W\in\mathscr{A}_p$. Then the following statements hold.*

(i) *There exists a positive constant C, depending only on n and m, such that, for any $\vec{f}\in\mathcal{L}^1$,*

$$\left\|T_{W,1}\vec{f}\right\|_{L^{1,\infty}}\le C[W]_{\mathscr{A}_1}^2\left\|\vec{f}\right\|_{\mathcal{L}^1},$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[W]_{A_1(\mathbb{R},\mathbb{C}^m)}\le t}\|H_{W,1}\|_{\mathcal{L}^1(\mathbb{R})\to L^{1,\infty}(\mathbb{R})}\ge\widetilde{C}t^2$, where H is the Hilbert transform.

(ii) *If $p\in(1,2]$, then there exists a positive constant C, depending only on n, m, and p, such that, for any $\vec{f}\in\mathcal{L}^p$,*

$$\left\|T_{W,p}\vec{f}\right\|_{L^{p,\infty}}\le C[W]_{\mathscr{A}_p}^{1+\frac{1}{p}}\left\|\vec{f}\right\|_{\mathcal{L}^p}. \tag{3.8}$$

(ii) *If $p\in(2,\infty)$, then there exists a positive constant C, depending only on n, m, and p, such that, for any $\vec{f}\in\mathcal{L}^p$,*

$$\left\|T_{W,p}\vec{f}\right\|_{L^{p,\infty}}\le C[W]_{\mathscr{A}_p}^{1+\frac{1}{p(p-1)}}\left\|\vec{f}\right\|_{\mathcal{L}^p}. \tag{3.9}$$

Question IX. *What are the sharp bounds in the inequalities* (3.8) *and* (3.9).

3.3 Further Remarks

Apart from the Hardy–Littlewood maximal operators and Calderón–Zygmund operators, there are several other important operators in harmonic analysis. The study of fractional maximal operators and fractional integral operators is motivated by their applications in partial differential equations and potential theory. These operators involve a fractional scaling, which enables more refined control over the behavior of functions, especially those with limited regularity (see, for example, [1, 98, 114, 143]).

Now, we recall the definition of the class $A_{p,q}$.

Definition 3.3. Let $p,q\in(1,\infty)$. A scalar weight w is called an *$A_{p,q}$-scalar weight* if w satisfies that

$$[w]_{A_{p,q}}:=\sup_{\text{cube}\,Q\subset\mathbb{R}^n}\fint_Q w(x)\,dx\left\{\fint_Q[w(x)]^{-\frac{p'}{q}}\right\}^{\frac{q}{p'}}dx<\infty.$$

Let $\alpha\in[0,n)$. The *fractional maximal operator M_α* is defined by setting, for any $f\in L^1_{\mathrm{loc}}$ and $x\in\mathbb{R}^n$,

$$M_\alpha f(x)=\sup_{\text{ball}\,B\ni x}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B|f(y)|\,dy.$$

The sharp boundedness of M_α was investigated by Lacey et al. [98, Theorem 2.9].

Theorem 3.15. *Let $\alpha\in[0,n)$, $p\in(1,\frac{n}{\alpha})$, and $q:=(\frac{1}{p}-\frac{\alpha}{n})^{-1}$. If $w\in A_{p,q}$, then there exists a positive constant C, depending only on n, α, and p, such that, for any $f\in L^p(w^{\frac{p}{q}})$,*

$$\|M_\alpha f\|_{L^q(w)}\le C[w]_{A_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{n})}\|f\|_{L^p(w^{\frac{p}{q}})},$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[w]_{A_{p,q}}\le t}\|M_\alpha\|_{L^p(w^{\frac{p}{q}})\to L^q(w)}\ge\widetilde{C}t^{\frac{p'}{q}(1-\frac{\alpha}{n})}$.

Let $\alpha\in(0,n)$. The *fractional integral operator* I_α is defined by setting, for any suitable f and any $x\in\mathbb{R}^n$,

$$I_\alpha f(x):=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

The sharp bound of I_α was established by Lacey et al. [98, Theorem 2.6].

Theorem 3.16. *Let $\alpha\in(0,n)$, $p\in(1,\frac{n}{\alpha})$, and $q:=(\frac{1}{p}-\frac{\alpha}{n})^{-1}$. If $w\in A_{p,q}$, then there exists a positive constant C, depending only on n, α, and p, such that, for any $f\in L^p(w^{\frac{p}{q}})$,*

$$\|I_\alpha f\|_{L^q(w)}\le C[w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max\{1,\frac{p'}{q}\}}\|f\|_{L^p(w^{\frac{p}{q}})},$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[w]_{A_{p,q}}\le t}\|I_\alpha\|_{L^p(w^{\frac{p}{q}})\to L^q(w)}\ge\widetilde{C}t^{(1-\frac{\alpha}{n})\max\{1,\frac{p'}{q}\}}$.

The corresponding matrix-weighted inequalities were established by Isralowitz and Moen [90]. Now, we recall the definition of the class $\mathscr{A}_{p,q}$.

Definition 3.4. Let $p,q\in(1,\infty)$. A matrix weight W is called an *$\mathscr{A}_{p,q}$-matrix weight* if W satisfies that

$$[W]_{\mathscr{A}_{p,q}(\mathbb{R}^n,\mathbb{C}^m)}:=\sup_{\mathrm{cube}\,Q\subset\mathbb{R}^n}\fint_Q\left[\fint_Q\left\|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\right\|^{p'}dy\right]^{\frac{q}{p'}}dx<\infty.$$

It is easy to show $\mathscr{A}_{p,q}(\mathbb{R}^n,\mathbb{C})=A_{p,q}$. For simplicity of presentation, we denote $\mathscr{A}_{p,q}(\mathbb{R}^n,\mathbb{C}^m)$ simply by $\mathscr{A}_{p,q}$.

For any $\alpha\in[0,n)$ and any matrix weight W, the *matrix-weighted fractional maximal operator* $M_{W,\alpha}$ is defined by setting, for any $\vec{f}\in(L^1_{\mathrm{loc}})^m$ and $x\in\mathbb{R}^n$,

$$\mathcal{M}_{W,\alpha}\vec{f}(x):=\sup_{\mathrm{ball}\,B\ni x}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B\left|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\vec{f}(y)\right|dy.$$

The following theorem is exactly [90, Theorem 1.3].

Theorem 3.17. *Let $\alpha\in[0,n)$, $p\in(1,\frac{n}{\alpha})$, and $q:=(\frac{1}{p}-\frac{\alpha}{n})^{-1}$. If $W\in\mathscr{A}_{p,q}$, then there exists a positive constant C, depending only on n, m, α, and p, such that, for any $\vec{f}\in\mathcal{L}^p$,*

$$\left\|\mathcal{M}_{W,\alpha}\vec{f}\right\|_{L^q}\le C[W]_{\mathscr{A}_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{n})}\left\|\vec{f}\right\|_{\mathcal{L}^p},$$

where the bound is sharp in the sense that there exists a positive constant $\widetilde{C}$ such that, for any $t\in[1,\infty)$, $\sup_{[W]_{\mathscr{A}_{p,q}}\le t}\|\mathcal{M}_{W,\alpha}\|_{\mathcal{L}^p\to L^q}\ge\widetilde{C}t^{\frac{p'}{q}(1-\frac{\alpha}{n})}$.

The following theorem is precisely [90, Theorem 1.4].

Theorem 3.18. *Let* $\alpha\in(0,n)$*,* $p\in(1,\frac{n}{\alpha})$ *and* $q:=(\frac{1}{p}-\frac{\alpha}{n})^{-1}$*. If* $W\in\mathscr{A}_{p,q}$*, then there exists a positive constant C, depending only on n, m,* α*, and p, such that, for any* $\vec{f}\in\mathscr{L}^p(W^{\frac{p}{q}})$*,*

$$\left\|I_\alpha\vec{f}\right\|_{\mathscr{L}^q(W)}\le C[W]_{\mathscr{A}_{p,q}}^{(1-\frac{\alpha}{n})\frac{p'}{q}+\frac{1}{q'}}\left\|\vec{f}\right\|_{\mathscr{L}^p(W^{\frac{p}{q}})}. \tag{3.10}$$

It is worth noting that, when $\alpha=0$ and $p=q=2$, the estimate (3.10) reduces to

$$\left\|I_0\vec{f}\right\|_{\mathscr{L}^2(W)}\lesssim[W]_{\mathscr{A}_2}^{\frac{3}{2}}\left\|\vec{f}\right\|_{\mathscr{L}^2(W)}$$

with the implicit positive constant depending only on n and m, which coincides with the known sharp bound for Calderón–Zygmund operators. This naturally leads to the following question.

Question X. *Is the bound in* (3.10) *sharp?*

On the other hand, Cruz-Uribe et al. [42] extended Theorems 3.17 and 3.18 to the two weight bump condition on $\mathbb{R}^n$.

There are also several other results concerning the boundedness of various operators on matrix-weighted Lebesgue spaces. Bickel et al. [11] studied the boundedness of well-localized operators. Vuorinen [157] investigated the boundedness of the strong matrix-weighted maximal operator. Kakaroumpas and Nieraeth [97] also extend matrix weights to multilinear scale. Nieraeth and Penrod [125] proved the boundedness of Calderón–Zygmund operators and the matrix-weighted maximal operator on matrix-weighted variable exponent Lebesgue spaces. Di Plinio et al. [53] developed quantitative sparse bounds for maximal rough singular integrals. Bownik and Cruz-Uribe [20] introduced the convex set-valued maximal operator and studied its boundedness on convex-set valued Lebesgue spaces. This result was later extended by Nieraeth [124] to the setting of directional Banach function spaces. Furthermore, Cruz-Uribe and Penrod [47] investigated the boundedness of averaging operators in variable exponent matrix-weighted spaces. See [14, 37, 48, 51, 123, 137] for further studies on matrix-weighted inequalities.

Furthermore, in the setting of spaces of homogeneous type, Bu et al. [29, Lemma A.22] studied the boundedness of matrix-weighted maximal operators. Based on this, Zhang and Zhou [173] extended Theorems 3.17 and 3.18 to the setting of spaces of homogeneous type. However, they did not consider the quantitative bounds.

4 Matrix-Weighted Function Spaces

In this section, we recall recent developments in matrix-weighted Sobolev spaces in Section 4.1, matrix-weighted BMO spaces in Section 4.2, matrix-weighted Besov–Triebel–Lizorkin-type spaces in Section 4.3, and other matrix-weighted function spaces in Section 4.4.

4.1 Matrix-Weighted Sobolev Spaces

Before presenting the definition of matrix-weighted Sobolev spaces, we first recall some symbols. Let Ω be a domain. The *local Sobolev space* $\mathscr{W}^{1,1}_{\rm loc}(\Omega)$ is defined to be the set of all $f\in L^1_{\rm loc}(\Omega)$ such that, for any $i\in\{1,\ldots,n\}$, its weak derivatives $\partial_i f$ exists and $\partial_i f\in L^1_{\rm loc}(\Omega)$. Cruz-Uribe et al. [44, p. 2810] introduced the degenerate matrix-weighted Sobolev spaces.

Definition 4.1. Let $p\in(0,\infty)$ and W be a matrix weight. Then the *degenerate matrix-weighted Sobolev space* $\widetilde{\mathscr{W}}^{1,p}(W,\Omega)$ is defined to be the set of all $f\in\mathscr{W}^{1,1}_{\rm loc}(\Omega)$ such that

$$\|f\|_{\widetilde{\mathscr{W}}^{1,p}(W,\Omega)}:=\|f\|_{L^p(\|W\|)}+\|\nabla f\|_{\mathcal{L}^p(W)}<\infty,$$

where $\nabla f:=(\partial_1 f,\ldots,\partial_n f)$ in the sense of weak derivatives.

Cruz-Uribe et al. [44, Theorem 5.3] also extended the Meyers–Serrin density theorem to the degenerate matrix-weighted Sobolev space. Subsequently, Cruz-Uribe and Penrod [46, Theorem 1.4] further extended this to the variable exponent degenerate matrix-weighted Sobolev space.

Degenerate Sobolev spaces have gained extensive attention and it is useful in the study of degenerate elliptic equations. For details and further applications, we refer to, for example, [39, 73, 90, 144, 154]. Moreover, Liu et al. [108, Corollary 3.3] established Kolmogorov–Riesz's compactness theorem on $\widetilde{\mathscr{W}}^{1,p}(W,\Omega)$. Later, Isralowitz and Moen [90, p. 1360] introduced another matrix-weighted Sobolev space.

Definition 4.2. Let $p\in(0,\infty)$ and W be a matrix weight. The *matrix-weighted Sobolev space* $\mathscr{W}^{1,p}(W,\Omega)$ is defined to be the set of the completion of $\{\vec f\in[C^\infty(\Omega)]^m\colon |W^{\frac1p}\vec f|,\|W^{\frac1p}D\vec f\|\in L^p(\Omega)\}$ with respect to the norm

$$\left\|\vec f\right\|_{\mathscr{W}^{1,p}(W,\Omega)}:=\left[\int_\Omega\left|W^{\frac1p}(x)\vec f(x)\right|^p dx+\int_\Omega\left\|W^{\frac1p}(x)D\vec f(x)\right\|^p dx\right]^{\frac1p},$$

where $D\vec f$ is the standard Jacobian matrix of $\vec f$.

Isralowitz and Moen [90, p. 1361] pointed out that, by repeating the arguments used in the proof of [44, Theorem 5.3], one can obtain the following conclusion.

Proposition 4.1. Let $p\in[1,\infty)$ and $W\in\mathscr{A}_p$. Then

$$\mathscr{W}^{1,p}(W,\Omega)=\left\{\vec f\in\left[\mathscr{W}^{1,1}_{\rm loc}(\Omega)\right]^m\colon \left|W^{\frac1p}\vec f\right|,\left\|W^{\frac1p}D\vec f\right\|\in L^p(\Omega)\right\},$$

where $D\vec f$ is the Jacobian matrix in the sense of weak derivatives.

Indeed, $[\widetilde{\mathscr{W}}^{1,p}(W,\Omega)]^m$ is continuously embedded into $\mathscr{W}^{1,p}(W,\Omega)$.

Proposition 4.2. Let $p\in[1,\infty)$ and $W\in\mathscr{A}_p$. Then $[\widetilde{\mathscr{W}}^{1,p}(W,\Omega)]^m\subset\mathscr{W}^{1,p}(W,\Omega)$. Moreover, there exists a positive constant C such that, for any $\vec f:=(f_1,\ldots,f_n)\in[\widetilde{\mathscr{W}}^{1,p}(W,\Omega)]^m$,

$$\left\|\vec f\right\|_{\mathscr{W}^{1,p}(W,\Omega)}\le C\left\|\vec f\right\|_{[\widetilde{\mathscr{W}}^{1,p}(W,\Omega)]^m}:=\sum_{i=1}^m\|f_i\|_{\widetilde{\mathscr{W}}^{1,p}(W,\Omega)}.$$

Proof. By (2.5), Lemma 2.4, and the definitions of both $\|\cdot\|_{[\widetilde{\mathscr{W}}^{1,p}(W,\Omega)]^m}$ and $\|\cdot\|_{\mathscr{W}^{1,p}(W,\Omega)}$, we obtain, for any $\vec{f}\in[\widetilde{\mathscr{W}}^{1,p}(W,\Omega)]^m$,

$$\left\|\vec{f}\right\|_{\mathscr{W}^{1,p}(W,\Omega)}\le\left[\int_\Omega\|W(x)\|\left|\vec{f}(x)\right|^p dx+\int_\Omega\left\|W^{\frac{1}{p}}(x)D\vec{f}(x)\right\|^p dx\right]^{\frac{1}{p}}\sim\left\|\vec{f}\right\|_{[\widetilde{\mathscr{W}}^{1,p}(W,\Omega)]^m}.$$

This finished the proof of Proposition 4.2. □

When $m=1$, both $\widetilde{\mathscr{W}}^{1,p}(W,\Omega)$ and $\mathscr{W}^{1,p}(W,\Omega)$ reduce to scalar-weighted Sobolev spaces (see, for example, [68]).

Isralowitz and Moen [90, Theorem 1.2] established matrix-weighted Poincaré's inequality.

Theorem 4.1. *Let $p\in(1,\infty)$ and $W\in\mathscr{A}_p$. Then there exist two positive constants δ and C such that, for any $\varepsilon\in(0,\delta[W]_{\mathscr{A}_p}^{-\max\{1,\frac{p'}{p}\}}]$, any cube $Q\subset\mathbb{R}^n$, and any $\vec{f}\in C^1(Q)$,*

$$\begin{aligned}&\left\{\fint_Q\left|W^{\frac{1}{p}}(x)\left[\vec{f}(x)-\fint_Q\vec{f}(y)\,dy\right]\right|^{p+\varepsilon}dx\right\}^{\frac{1}{p+\varepsilon}}\\&\quad\le C[W]_{\mathscr{A}_p}^{\max\{1+2\frac{p'}{p},2+\frac{p'}{p}\}}|Q|^{\frac{1}{n}}\left[\fint_Q\left\|W^{\frac{1}{p}}(x)D\vec{f}(x)\right\|^{p-\varepsilon}dx\right]^{\frac{1}{p-\varepsilon}}.\end{aligned}$$

By Theorem 4.1, Isralowitz and Moen [90, Theorem1.5] also established a reverse Meyers–Hölder inequality. Based on those, Isralowitz and Moen [90, Theorem 1.7] further proved some regularity results for the weak solution to the degenerate elliptic equation.

We refer to [111, 171] for more results about matrix-weighted Sobolev spaces.

4.2 Matrix-Weighted BMO Spaces

The study of matrix-valued BMO functions serves as a natural extension of the scalar BMO theory and provides essential tools for addressing noncommutative structures in harmonic analysis. It plays a significant role in the theory of weighted inequalities, noncommutative harmonic analysis, and the regularity theory of partial differential equations, particularly in capturing the interplay between matrix weights and singular integral operators. Moreover, the investigation of matrix-valued BMO contributes to the development of operator space theory, revealing deeper structural phenomena.

Let $p\in(1,\infty)$, $w\in A_p$, and T be one of the Riesz transform. It is well known that the commutator $[b,T]$ is bounded on $L^p(w)$ if and only if $b\in$BMO (see, for example, [16,17,75]). Later, Isralowitz, et al. [89] further extended this conclusion to the matrix setting. We first recall the definition of matrix-weighted BMO spaces (see, for example, [36, p. 4]).

Definition 4.3. For $p,q\in(1,\infty)$ and $U,V\in\mathscr{A}_{p,q}$, the *two matrix-weighted* BMO *space* $\mathrm{BMO}_{U,V}^{p,q}$ is defined to be the set of all locally integrable functions $B\colon\mathbb{R}^n\to M_m(\mathbb{C})$ such that

$$\|B\|_{\mathrm{BMO}_{U,V}^{p,q}}:=\sup_{\mathrm{cube}\,Q\subset\mathbb{R}^n}\fint_Q\left\{\fint_Q\left\|U^{\frac{1}{q}}(x)\,[B(x)-B(y)]\,V^{\frac{1}{q}}(y)\right\|^{p'}dy\right\}^{\frac{q}{p'}}dx<\infty.$$

Particularly, when $p=q$, we denote it by $\mathrm{BMO}_{U,V}^{p}$. Furthermore, when $p=q$ and $U=V=W$, we denote it by BMO_{W}^{p}. When $m=1$, $p=q$, and $U=V=1$,

$$\|B\|_{\mathrm{BMO}_{U,V}^{p,q}}=\sup_{\mathrm{cube}\,Q\subset\mathbb{R}^n}\fint_Q\fint_Q\left|B(x)-B(y)\right|^{p'}dydx.$$

For any vector-valued function $\vec{f}$ and any matrix-valued function B, let

$$[B,T]\vec{f}:=BT\vec{f}-TB\vec{f}$$

be the commutator in the matrix settings. Isralowitz [85, 86] studied the boundedness of commutators and H^1-BMO duality in the two matrix-weighted setting; additionally, Isralowitz identified this BMO space for $p=2$ as the dual in the scale of two matrix-weighted H^1 space. By these results, one can establish the converse to Bloom's $\mathscr{A}_2$ matrix theorem, which, as a particular case, verifies Buckley's summation condition for $\mathscr{A}_2$ matrix weights. Moreover, Isralowitz established a matrix-weighted John–Nirenberg inequality. It is worth mentioning that Bu et al. [23] studied the vector-valued matrix-weighted Hardy spaces and vector-valued matrix-weighted BMO spaces as well were also considered.

More recently, Isralowitz et al. [89] investigated matrix weighted norm inequalities for commutators and dyadic paraproducts with matrix-valued symbols. Isralowitz et al. [92] significantly advanced the matrix-weighted theory of commutators initiated in [89] by extending the framework from one-weight to two-weight settings and also from qualitative to quantitative estimates. They introduced a more flexible space $\mathrm{BMO}_{U,V}^{p}$ adapted to arbitrary pairs of matrix weights $\{u,v\}$ and provided nearly complete characterizations of the two-weight boundedness for Calderón–Zygmund commutators.

Moreover, Cardenas and Isralowitz [36] extended the two-weight theory of commutators to the setting of fractional integral operators with matrix symbols and matrix weights. Based on earlier results on Calderón–Zygmund operators, Cruz-Uribe and Isralowitz developed new matrix-weighted norm inequalities and introduced matrix-valued BMO-type spaces adapted to fractional integrals. Their results apply to arbitrary pairs of $\mathscr{A}_{p,q}$-matrix weights and remain novel even in scalar settings, providing both upper and lower bounds as well as Orlicz bump conditions for the boundedness. The following theorem is exactly [36, Theorem 1.1].

Theorem 4.2. *Let $p,q\in(1,\infty)$ and T be any linear operator, defined on scalar valued functions, which satisfies that, for any $W\in\mathscr{A}_p$, T is bounded from $\mathscr{L}^p(W^{\frac{p}{q}})$ to $\mathscr{L}^q(W)$ with bound depending on T, n, m, p, and $[W]_{\mathscr{A}_{p,q}}$. Then, for any $U,V\in\mathscr{A}_{p,q}$ and any locally integrable matrix-valued function B, there exists positive constant C, depending only on T, n, m, p, $[U]_{\mathscr{A}_{p,q}}$, and $[V]_{\mathscr{A}_{p,q}}$, such that*

$$\left\|[B,T]\right\|_{\mathscr{L}^p(V^{\frac{p}{q}})\to\mathscr{L}^q(U)}\le C\|B\|_{\mathrm{BMO}_{U,V}^{p,q}}.$$

Conversely, if, for any $W\in\mathscr{A}_{p,q}$,

$$[W]_{\mathscr{A}_{p,q}}^{\frac{1}{q}}\lesssim\|T\|_{\mathscr{L}^p(W^{\frac{p}{q}})\to\mathscr{L}^q(W)}<\infty$$

with the bound independent of W, then

$$\|B\|_{\mathrm{BMO}_{U,V}^{p,q}} \lesssim \|[B,T]\|_{\mathcal{L}^p(V^{\frac{p}{q}})\to\mathcal{L}^q(U)}.$$

4.3 Matrix-Weighted Besov–Triebel–Lizorkin-Type Spaces

Based on the works of Bernšteĭn [8] and Zygmund [175], Nikol'skiĭ [126] introduced Besov spaces $B^s_{p,\infty}$. Later, Besov [9, 10] extended this framework by introducing the additional index q. Around 1970, Lizorkin [109, 110] and Triebel [150] introduced Triebel–Lizorkin spaces. Subsequently, Peetre [127–129] extended the range of parameters in these spaces. For further studies on Besov–Triebel–Lizorkin spaces, we refer to [19, 32–34, 141, 142, 151–153]. Around 2010, in order to clarify the relationships among Besov–Triebel–Lizorkin spaces and Q spaces, Yang et al. [165, 166, 169] introduced Besov–Triebel–Lizorkin-type spaces. Various properties and characterizations of Besov–Triebel–Lizorkin-type spaces were subsequently studied in [71, 168, 170, 172].

Matrix-weighted Besov spaces were first introduced by Frazier and Roudenko [63, 138] and further developed by the same authors in [64, 139]. Special matrix-weighted Triebel–Lizorkin spaces $\dot F^0_{p,2}(W)$ with $p\in(1,\infty)$ were studied in [118, 155], where Nazarov, Triel, and Volberg showed the fact that $\dot F^0_{p,2}(W)=\mathcal{L}^p(W)$. A simpler proof was later given by Isralowitz [87]. Recently, Frazier and Roudenko [65] introduced matrix-weighted Triebel–Lizorkin spaces with full scale. Shortly thereafter, Wang et al. [160] studied the Littlewood–Paley characterization of such spaces. Based on these, Bai and Xu [5, 6] established the boundedness of pseudo-differential operators on matrix-weighted Besov–Triebel–Lizorkin spaces. Meanwhile, matrix-weighted Besov–Triebel–Lizorkin spaces with logarithmic smoothness were studied by Li et al. [106]. For further studies on matrix-weighted Besov–Triebel–Lizorkin spaces, see [29, 106, 107]. Very recently, Bu et al. introduced and systematically studied matrix-weighted Besov–Triebel–Lizorkin-type spaces in a series of five successive articles [24–28]. In the remainder of this subsection, we present two key theorems from their work.

To this end, we first recall some concepts. For any $j\in\mathbb{Z}$ and $k\in\mathbb{Z}^n$, let

$$Q_{j,k}:=2^{-j}([0,1)^n+k),\ \ \mathscr{Q}:=\{Q_{j,k}:\ j\in\mathbb{Z},\ k\in\mathbb{Z}^n\},$$

and $\mathscr{Q}_j:=\{Q_{j,k}:\ k\in\mathbb{Z}^n\}$. For any $Q:=Q_{j,k}\in\mathscr{Q}$, $j_Q:=j$ denotes the *generation* of Q and $x_Q:=2^{-j}k$ denotes the *lower left corner* of Q. Let $p,q\in(0,\infty]$. For any sequence $\{f_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ and any $P\in\mathscr{Q}$, define

$$\|\{f_j\}_{j\in\mathbb{Z}}\|_{L\dot B_{pq}(P)}:=\left[\sum_{j=j_P}^{\infty}\|f_j\|^q_{L^p(P)}\right]^{\frac1q}\ \text{ and }\ \|\{f_j\}_{j\in\mathbb{Z}}\|_{L\dot F_{pq}(P)}:=\left\|\left(\sum_{j=j_P}^{\infty}|f_j|^q\right)^{\frac1q}\right\|_{L^p(P)}.$$

We use $L\dot A_{pq}(P)\in\{L\dot B_{pq}(P),L\dot F_{pq}(P)\}$ as a generic symbol in statements that apply to both types of spaces.

Let $\mathcal{S}$ be the space of all Schwartz functions on $\mathbb{R}^n$, equipped with the well-known topology determined by a countable family of norms, and let $\mathcal{S}'$ be the set of all continuous linear functionals on $\mathcal{S}$, equipped with the weak-$*$ topology. For any $f\in L^1$ and $\xi\in\mathbb{R}^n$, let

$$\widehat{f}(\xi):=\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx$$

denote its *Fourier transform* and $\check{f}(\xi):=(2\pi)^{-n}\widehat{f}(-\xi)$ its *inverse Fourier transform*. In this way, we still have $(\widehat{f})^\vee=f$. For any complex-valued function g on $\mathbb{R}^n$, let $\operatorname{supp} g:=\overline{\{x\in\mathbb{R}^n:\ g(x)\neq 0\}}$.

Let $\varphi\in\mathcal{S}$ satisfy

$$\begin{cases}\operatorname{supp}\widehat{\varphi}\subset\left\{\xi\in\mathbb{R}^n:\ \dfrac{1}{2}\le|\xi|\le 2\right\},\\ \left|\widehat{\varphi}(\xi)\right|\ge C>0 \text{ if } \xi\in\mathbb{R}^n \text{ with } \dfrac{3}{5}\le|\xi|\le\dfrac{5}{3},\end{cases} \tag{4.1}$$

where C is a positive constant independent of ξ. It is easy to prove that

$$\varphi\in\mathcal{S}_\infty:=\left\{\varphi\in\mathcal{S}:\ \int_{\mathbb{R}^n} x^\gamma\varphi(x)\,dx=0 \text{ for any } \gamma\in\mathbb{Z}_+^n\right\}$$

and it is well known that there exists $\psi\in\mathcal{S}$ satisfying (4.1) such that, for any $\xi\in\mathbb{R}^n\setminus\{\mathbf{0}\}$,

$$\sum_{j\in\mathbb{Z}}\overline{\widehat{\varphi}(2^{-j}\xi)}\widehat{\psi}\left(2^{-j}\xi\right)=1. \tag{4.2}$$

Let φ be a complex-valued function on $\mathbb{R}^n$. For any $j\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, let $\varphi_j(x):=2^{jn}\varphi(2^jx)$. For any $Q:=Q_{j,k}\in\mathscr{Q}$ and $x\in\mathbb{R}^n$, let $\varphi_Q(x):=|Q|^{-\frac{1}{2}}\varphi(2^jx-k)=|Q|^{\frac{1}{2}}\varphi_j(x-x_Q)$. Recall that, for any $Q\in\mathscr{Q}$, let $\widetilde{\mathbf{1}}_Q:=|Q|^{-\frac{1}{2}}\mathbf{1}_Q$.

Next, we recall the matrix-weighted Besov–Triebel–Lizorkin-type spaces, which is precisely [28, Definitions 3.21 and 3.24].

Definition 4.4. Let $s\in\mathbb{R}$, $\tau\in[0,\infty)$, $p\in(0,\infty)$, $q\in(0,\infty]$, and $W\in A_{p,\infty}$. Assume that $\varphi\in\mathcal{S}$ satisfy (4.1). The *homogeneous matrix-weighted Besov-type space* $\dot{B}^{s,\tau}_{p,q}(W,\varphi)$ and the *homogeneous matrix-weighted Triebel–Lizorkin-type space* $\dot{F}^{s,\tau}_{p,q}(W,\varphi)$ are defined by setting

$$\dot{A}^{s,\tau}_{p,q}(W,\varphi):=\left\{\vec{f}\in(\mathcal{S}'_\infty)^m:\ \left\|\vec{f}\right\|_{\dot{A}^{s,\tau}_{p,q}(W,\varphi)}<\infty\right\},$$

where, for any $\vec{f}\in(\mathcal{S}'_\infty)^m$,

$$\left\|\vec{f}\right\|_{\dot{A}^{s,\tau}_{p,q}(W,\varphi)}:=\sup_{P\in\mathscr{Q}}\frac{1}{|P|^\tau}\left\|\left\{2^{js}\left|W^{\frac{1}{p}}\left(\varphi_j*\vec{f}\right)\right|\right\}_{j\in\mathbb{Z}}\right\|_{L\dot{A}_{pq}(P)}.$$

Remark 4.1. By [28, Proposition 3.31], we find that the space $\dot{A}^{s,\tau}_{p,q}(W,\varphi)$ defined in Definition 4.4 is independent of the choice of φ. Based on this, we denote $\dot{A}^{s,\tau}_{p,q}(W,\varphi)$ simply by $\dot{A}^{s,\tau}_{p,q}(W)$.

Definition 4.5. Let $s\in\mathbb{R}$, $\tau\in[0,\infty)$, $p\in(0,\infty)$, $q\in(0,\infty]$, and $W\in A_{p,\infty}$. The *homogeneous matrix-weighted Besov-type sequence space* $\dot{b}^{s,\tau}_{p,q}(W)$ and the *homogeneous matrix-weighted Triebel–Lizorkin-type sequence space* $\dot{f}^{s,\tau}_{p,q}(W)$ are defined to be the sets of all sequences $\vec{t}:=\{\vec{t}_Q\}_{Q\in\mathscr{Q}}$ in $\mathbb{C}^m$ such that

$$\left\|\vec{t}\right\|_{\dot{a}^{s,\tau}_{p,q}(W)}:=\sup_{P\in\mathscr{Q}}\frac{1}{|P|^\tau}\left\|\left\{2^{js}\left|W^{\frac{1}{p}}\sum_{Q\in\mathscr{Q}_j}\vec{t}_Q\mathbf{1}_Q\right|\right\}_{j\in\mathbb{Z}}\right\|_{L\dot{A}_{p,q}(P)}<\infty.$$

Recall that the *φ-transform* is defined to be the map taking each $\vec{f}\in(\mathcal{S}'_\infty)^m$ to the sequence $S_\varphi\vec{f}:=\{(S_\varphi\vec{f})_Q\}_{Q\in\mathscr{Q}}$, where $(S_\varphi\vec{f})_Q:=\langle\vec{f},\varphi_Q\rangle$ and the *inverse φ-transform* is defined to be the map taking a sequence $\vec{t}:=\{\vec{t}_Q\}_{Q\in\mathscr{Q}}\subset\mathbb{C}^m$ to $T_\psi\vec{t}:=\sum_{Q\in\mathscr{Q}}\vec{t}_Q\psi_Q$ in $(\mathcal{S}'_\infty)^m$ (see, for instance, [61,62]). The following theorem is the φ-transform characterization of $\dot{A}^{s,\tau}_{p,q}(W)$, which is exactly [28, Theorem 3.30].

Theorem 4.3. *Let $s\in\mathbb{R}$, $\tau\in[0,\infty)$, $p\in(0,\infty)$, $q\in(0,\infty]$, and $W\in\mathscr{A}_{p,\infty}$. Assume that $\varphi,\psi\in\mathcal{S}$ satisfy* (4.1). *Then the operators $S_\varphi:\ \dot{A}^{s,\tau}_{p,q}(W)\to\dot{a}^{s,\tau}_{p,q}(W)$ and $T_\psi:\ \dot{a}^{s,\tau}_{p,q}(W)\to\dot{A}^{s,\tau}_{p,q}(W)$ are bounded. Furthermore, if φ and ψ satisfy* (4.2)*, then $T_\psi\circ S_\varphi$ is the identity on $\dot{A}^{s,\tau}_{p,q}(W)$.*

Now, we recall the definition of almost diagonal operators. Let $B:=\{b_{Q,P}\}_{Q,P\in\mathscr{Q}}$ be a sequence in $\mathbb{C}$. For any sequence $\vec{t}:=\{\vec{t}_R\}_{R\in\mathscr{Q}}\subset\mathbb{C}^m$, we define $B\vec{t}:=\{(B\vec{t})_Q\}_{Q\in\mathscr{Q}}$ by setting, for any $Q\in\mathscr{Q}$,

$$\left(B\vec{t}\right)_Q:=\sum_{R\in\mathscr{Q}}b_{Q,R}\vec{t}_R$$

if the above summation is absolutely convergent. The following definition is precisely [28, Definition 4.1] (see also [25, Definition 4.1]).

Definition 4.6. Let $D,E,F\in\mathbb{R}$. An infinite matrix $B:=\{b_{Q,R}\}_{Q,R\in\mathscr{Q}}\subset\mathbb{C}$ is said to be (D,E,F)-*almost diagonal* if there exists a positive constant C such that, for any $Q,R\in\mathscr{Q}$,

$$|b_{Q,R}|\le C\left[1+\frac{|x_Q-x_R|}{\ell(Q)\vee\ell(R)}\right]^{-D}\begin{cases}\left[\dfrac{\ell(Q)}{\ell(R)}\right]^E & \text{if } \ell(Q)\le\ell(R),\\ \left[\dfrac{\ell(R)}{\ell(Q)}\right]^F & \text{if } \ell(R)<\ell(Q).\end{cases}$$

In what follows, the *symbol J* always carries the following meaning:

$$J:=\begin{cases}\dfrac{n}{\min\{1,p\}} & \text{if a Besov-type space is dealt,}\\ \dfrac{n}{\min\{1,p,q\}} & \text{if a Triebel–Lizorkin-type space is dealt.}\end{cases}$$

The following theorem is exactly [28, Theorem 4.5].

Theorem 4.4. *Let $s\in\mathbb{R}$, $\tau\in[0,\infty)$, $p\in(0,\infty)$, $q\in(0,\infty]$, and $W\in A_{p,\infty}$. Let*

$$J_\tau:=\begin{cases} n & \text{if } \tau>\frac{1}{p} \text{ or } (\tau,q)=(\frac{1}{p},\infty), \\ \dfrac{n}{\min\{1,q\}} & \text{if } \tau=\frac{1}{p} \text{ and } q<\infty \text{ and } \dot{a}=\dot{f}, \\ J & \text{if } \tau<\frac{1}{p}, \text{ or if } \tau=\frac{1}{p} \text{ and } q<\infty \text{ and } \dot{a}=\dot{b}, \end{cases}$$

$$\widehat{\tau}:=\left[\tau-\frac{1}{p}+\frac{d_{p,\infty}^{\mathrm{lower}}(W)}{np}\right]_+,$$

$$\widetilde{J}:=J_\tau+\left[(n\widehat{\tau})\wedge\frac{d_{p,\infty}^{\mathrm{lower}}(W)}{p}\right]+\frac{d_{p,\infty}^{\mathrm{upper}}(W)}{p},$$

and $\widetilde{s}:=s+n\widehat{\tau}$. If B is (D,E,F)-almost diagonal, then B is bounded on $\dot{a}_{p,q}^{s,\tau}(W)$ whenever

$$D>\widetilde{J},\ \ E>\frac{n}{2}+\widetilde{s},\ \ \text{and}\ \ F>\widetilde{J}-\frac{n}{2}-\widetilde{s}.$$

Remark 4.2. If we assume that $W\in\mathscr{A}_p$, then Theorem 4.4 holds with the term $d_{p,\infty}^{\mathrm{upper}}(W)/p$ removed. This result is sharp in most cases (see [25, Remark 13.2(i)]). Moreover, the additional term $d_{p,\infty}^{\mathrm{upper}}(W)/p$ cannot be removed in general in the case $W\in\mathscr{A}_{p,\infty}$ (see [28, Lemmas 4.13 and 4.18]).

Specially, Bu et al. [25] obtained two more interesting results. The first is precisely [25, Theorem 7.1].

Theorem 4.5. *Let $s\in\mathbb{R}$, $p,q\in(0,\infty]$, and $D,E,F\in\mathbb{R}$. Then every (D,E,F)-almost diagonal matrix B is bounded on $\dot{b}_{p,q}^{s}$ if and only if $D>J$, $E>\frac{n}{2}+s$, and $F>J-\frac{n}{2}-s$.*

The second is exactly [25, Theorem 8.6 and Lemma 9.1].

Theorem 4.6. *Let $s\in\mathbb{R}$, $p\in(0,\infty)$, $q\in[\min\{1,p\},\infty]$, and $D,E,F\in\mathbb{R}$. Then every (D,E,F)-almost diagonal matrix B is bounded on $\dot{f}_{p,q}^{s}$ if and only if $D>J$, $E>\frac{n}{2}+s$, and $F>J-\frac{n}{2}-s$, where $J:=\frac{n}{\min\{1,p\}}$.*

Question XI. *We conjecture that Theorem 4.6 holds also when $q\in(0,\min\{1,p\})$, which is still unknown.*

Using the φ-transform characterization of $\dot{A}_{p,q}^{s,\tau}(W)$, the boundedness of some operators on $\dot{A}_{p,q}^{s,\tau}(W)$ can be reduced to the boundedness of almost diagonal operators on $\dot{a}_{p,q}^{s,\tau}(W)$. Applying Theorems 4.3 and 4.4, Bu et al. [26] further characterized $\dot{A}_{p,q}^{s,\tau}(W)$ via molecules and wavelets, and established the boundedness of pseudo-differential operators, trace operators, pointwise multipliers, and Calderón–Zygmund operators on $\dot{A}_{p,q}^{s,\tau}(W)$. The corresponding inhomogeneous results also hold; see [28].

4.4 Further Remarks

Nielsen [119–121] investigated periodic matrix weights and their applications in function spaces. Let $\mathbb{T}$ denotes the unit circle identified with interval $[0,1]$ and $\mathbb{T}^n$ the torus $[0,1]^n$. Nielsen [119] focused on 2π-periodic matrix weights W for which the vector-valued trigonometric system forms a Schauder basis in the matrix-weighted space $\mathcal{L}^p(W,\mathbb{T})$. Nilesen examined quasi-greedy bases, where the thresholding approximation algorithm converges in $\mathcal{L}^p(W)$ norm and proved that the basis only suits when $p=2$, in which case it is also a Riesz basis. Nielsen [120] also studied periodic matrix weights W defined on $\mathbb{R}^d$, taking values in $N\times N$ positive-definite matrices. Nielsen established transference results between multiplier operators on $\mathcal{L}^p(W)$ and $\mathcal{L}^p(W,\mathbb{T}^n)$ for any $p\in(1,\infty)$, with a specific application to the transference for homogeneous multipliers of degree zero. Moreover, Nielsen [121] has also gotten some results about bandlimited Fourier multipliers on matrix-weighted $\mathcal{L}^p$ spaces.

Very recently, Xu et al. [7, 158, 159] worked on the theory of matrix-weighted function spaces, such as matrix-weighted modulation spaces and matrix-weighted Bourgain–Morrey spaces, and obtained some advances in pseudo-differential operators, precompactness, and embeddings.

Precompact sets in a metric space are sets where every sequence has a Cauchy subsequence. This property provides insight into the geometry of the space and how close the elements are to forming compact structures. It is well known that, in a complete metric space, a set is a precompact set if and only if it is totally bounded. Bai and Xu [7] also studied the precompactness of weighted Bourgain–Morrey spaces in [7, Defintion 2.10]. Observe that, when $m=1$ and $W=1$, the matrix-weighted Bourgain–Morrey space reduces to the classic Bourgain–Morrey space in [72, Definition 1.1]. Moreover, Bai and Xu [7] obtained got a sufficient condition for a totally bounded set in matrix-weighted Bourgain–Morrey spaces $M_p^{t,r}(W)$ in [7, Theorem 3.2]. Besides, as an application, Bai and Xu [7] established a criterion for a totally bounded set in degenerate matrix-weighted Bourgain–Morrey spaces in [7, Corollary 3.6]. Additionally, Bai and Xu [7] obtained the Kolmogorov–Riesz compactness theorem in matrix-weighted Bourgain–Morrey spaces and the boundedness of average and translation operators in [7, Theorem 4.1].

Also, Wang et al. [158] investigated the embedding and the duality of matrix-weighted modulation spaces in [158, Theorem 6.4]. The modulation space is extension of Besov spaces. Wang et al. gave the connection between averaging matrix-weighted modulation spaces and matrix-weighted modulation spaces and studied the embeddings and the duality of the matrix-weighted modulation spaces. For more about matrix-weighted modulation spaces, we refer to [122].

Subsequently, Wang et al. [159] established a sufficient condition for the precompactness in matrix-weighted Lebesgue spaces with variable exponents using the translation operator in [159, Theorem 3.1] and then presented a criterion for precompactness in these spaces based on the average operator, followed by another criterion involving approximate identities in [159, Theorem 4.1]. Finally, Wang et al. extended their analysis to consider the precompactness in matrix-weighted Sobolev spaces with variable exponents in [159, Theorems 6.1, 6.2 and 6.3].

Another intriguing extension in the study of matrix-weighted function spaces is to replace the base space $\mathbb{R}^n$ with an infinite-dimensional Banach space. Let $p\in(0,\infty)$, $\mathscr{X}$ be a Banach space, and a operator weight

$$W\colon \mathbb{R}^n \to \mathscr{L}(\mathscr{X}),$$

where $\mathscr{L}(\mathscr{X})$ is defined as the set of all continuous linear operators on $\mathscr{X}$. The *Banach-valued operator-weighted Lebesgue space* $\mathcal{L}^p(W,\mathbb{R}^n,\mathscr{X})$ is defined to be the set of all measurable Banach-valued functions $\vec{f}\colon \mathbb{R}^n \to \mathscr{X}$ such that

$$\left\|\vec{f}\right\|_{\mathcal{L}^p(W,\mathbb{R}^n,\mathscr{X})} := \left[\int_{\mathbb{R}^n} \left\|W^{\frac{1}{p}}(x)\vec{f}(x)\right\|_{\mathscr{X}}^p dx\right]^{\frac{1}{p}} < \infty.$$

Indeed, Gillespie et al. [66] indicated that the operator norm of the Hilbert transform exhibits $(\log n)^{1/2}$ growth, which means that the Hilbert transform is unbound on $\mathcal{L}^2(W,\mathbb{R}^n,\mathscr{X})$ if

$$\sup_{\mathrm{cube}\, Q\subset\mathbb{R}^n} \fint_Q \fint_Q \left\|W^{\frac{1}{2}}(x)W^{-\frac{1}{2}}(y)\right\|_{\mathscr{L}(\mathscr{X})}^2 dydx < \infty.$$

Later, Lauzon [99] provided a sufficient condition of W for the boundedness of the Hilbert transform and more general Calderón–Zygmund operators in $\mathcal{L}^p(W,\mathbb{R}^n,\mathscr{X})$. Aleman and Constantin [2] gave the sufficient and necessary condition for boundedness of the Bergman projection. For more about the boundedness of the Bergman projection, we refer to [79].

The main difficulty in studying operator-weighted function spaces is that the functions take values in infinite-dimensional Banach spaces. The biggest challenge is that these operator weights do not commute, so many standard tools from harmonic analysis cannot be used. Therefore, it is essential to skillfully combine tools from functional analysis, such as operator functional calculus and Banach space geometry, with the techniques of harmonic analysis. The problem under investigation is subject to non-commutative harmonic analysis. Currently, the research in function space theory under this framework is nearly unexplored. The results of this study will significantly enrich the theory of vector-valued function spaces and provide new frameworks, conceptual methods, and research tools for analysis and probability.

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