3.10 《机器学习基础》第1次作业

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T_1

1) 证：考虑

$$Var[aX] = E[(aX - E[aX])^2]$$
$$= E[(aX - aE[X])^2]$$
$$= E[a^2 (X - E[X])^2]$$
$$= a^2 E[(X - E[X])^2]$$
$$= a^2 Var[X];$$

$$Var[X] = E[(X - E[X])^2]$$
$$= E[X^2 - 2XE[X] + E[X]^2]$$
$$= E[X^2] - 2E[X] \cdot E[X] + E[X]^2$$
$$= E[X^2] - E[X]^2$$

2) 证：

$$Var[X+Y] = E[(X+Y - E(X+Y))^2]$$
$$= E[((X - E[X]) + (Y - E[Y]))^2]$$
$$= E[(X - E[X])^2] + E[(Y - E[Y])^2] + 2E[(X - E[X])(Y - E[Y])]$$
$$= Var[X] + Var[Y] + 2Cov(X,Y)$$

考虑 $Cov(X,Y) = E[XY - XE[Y] - YE[X] + E[X]E[Y]]$

$$= E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y]$$
$$= E[XY] - E[X]E[Y]$$

X、Y独立时，有 $E[XY] = E[X]E[Y]$

故 $Cov(X,Y) = 0$

故 $Var[X+Y] = Var[X] + Var[Y]$

3) 证：取 $\forall \lambda \in \mathbb{R}$，考虑

$$Var[Y - \lambda X] = E[(Y - \lambda X - E[Y - \lambda X])^2]$$
$$= E[((Y - E[Y]) - \lambda(X - E[X]))^2]$$
$$= E[(Y - E[Y])^2] + \lambda^2 E[(X - E[X])^2] - 2\lambda E[(Y - E[Y])(X - E[X])]$$
$$= Var[Y] + \lambda^2 Var[X] - 2\lambda Cov(X,Y)$$

取 $\lambda = \frac{Cov(X,Y)}{Var[X]}$，即得

$$Var[Y - \lambda X] = Var[Y] + \frac{Cov^2(X,Y)}{Var[X]} - 2\frac{Cov^2(X,Y)}{Var[X]}$$
$$= Var[Y] - \frac{Cov^2(X,Y)}{Var[X]}$$
$$\geqslant 0$$

故 $Cov^2(X,Y) \leqslant Var[X] Var[Y]$

即 $Cov(X,Y) \leqslant \sqrt{Var[X] Var[Y]}$ 证毕！

T_2

证：考虑 Chebyshev 不等式，取 $t = \frac{\varepsilon}{\sqrt{Var[\bar{X}_n]}}$

由

$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$，$X_i$ iid $\sim (\mu, \sigma^2)$

知 $E[\bar{X}_n] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n} \cdot n\mu = \mu$

又 $Var[\bar{X}_n] = \frac{1}{n^2}\sum_{i=1}^{n} Var[X_i]$

$$= \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}$$

故 $t^2 = \frac{n\varepsilon^2}{\sigma^2}$

代回 Chebyshev 不等式中，即有

$$Pr[|\bar{X}_n - \mu| \geqslant \varepsilon] \leqslant \frac{\sigma^2}{n\varepsilon^2}$$

取 $n \to \infty$，即得

$$\lim_{n\to\infty} Pr[|\bar{X}_n - \mu| \geqslant \varepsilon] \leqslant \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0$$ 证毕！