§2.1 Classes of Sets

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Problem 1. The union of a countable collection of B.F.'s $\{\mathcal{F}_j\}$ such that $\mathcal{F}_j \subset \mathcal{F}_{j+1}$ need not be a B.F., but there is a minimal B.F. containing all of them, denoted by $\bigvee_j \mathcal{F}_j$. In general $\bigvee_{\alpha\in A} \mathcal{F}_\alpha$ denotes the minimal B.F. containing all $\mathcal{F}_\alpha, \alpha \in A$. [HINT: $\Omega =$ the set of positive integers; $\mathcal{F}_j =$ the B.F. generated by those up to j.]

Solution. Following the hint, we consider $\Omega =$ the set of positive integers and $\mathcal{F}_j =$ the B.F. generated by those up to j. Note that for every positive interger n, the singleton $\{n\}$ belongs to $\cup_{j=1}^{\infty}\mathcal{F}_j$, since it belongs to $\mathcal{F}_n$. However, the union of these singletons equals the whole set of positive integers, which is not included in the union of $\mathcal{F}_j$'s, since every single one of them only contains finite subsets. We conclude that $\cup_{j=1}^{\infty}\mathcal{F}_j$ is not a B.F. $\square$

Problem 8. Let $\mathcal{F}$ be a B.F. generated by an arbitrary collection of sets $\{E_\alpha, \alpha \in A\}$. Prove that for each $E \in \mathcal{F}$, there exists a countable subcollection $\{E_{\alpha_j}, j \geq 1\}$ (depending on E) such that E belongs already to the B.F. generated by this subcollection. [HINT: Consider the class of all sets with the asserted property and show that it is a B.F. containing each E_α.]

Solution. Following the hint, we consider the class of all sets with the asserted property and denote it by $\mathcal{C}$. We proceed to show that $\mathcal{C}$ is a B.F.

Let $E \in \mathcal{C}$ be given. By definition of $\mathcal{C}$, E belongs to the B.F. generated by some subcollection from the sets $\{E_\alpha, \alpha \in A\}$. Since $E^{\complement}$ belongs to the same B.F., E also has the asserted property and hence belongs to $\mathcal{C}$.

Now let $\{E_i, i \geq 1\}$ be any sequence of sets from $\mathcal{C}$. For each E_i in the sequence, there exists a countable collection $\mathcal{S}_i$ of sets $\{E_{\alpha_j^i}, j \geq 1\}$ such that E_i belongs to the B.F. generated by $\mathcal{S}_i$. Let $\mathcal{S} = \cup_{i=1}^{\infty}\mathcal{S}_i$ then S is countable as a countable union of countable collections, and each E_i belongs to the B.F. generated by $\mathcal{S}$. The property of a B.F. guarantees that $\cup_{i=1}^{\infty}E_i$ belongs to the B.F generated by $\mathcal{S}$ and hence is a member of $\mathcal{C}$. We conclude that $\mathcal{C}$ is a B.F.

Note that each E_α belongs to $\mathcal{C}$ as it is contained in the B.F. generated by itself. But the whole $\mathcal{F}$ is generated by $\{E_\alpha, \alpha \in A\}$. We have $\mathcal{F} \subset \mathcal{C}$ by the minimality of a generating B.F. In other words, every set in $\mathcal{F}$ has the asserted property and we are done. $\square$

Problem 10. Let $\mathcal{D}$ be a class of subsets of Ω having the closure property (iii); let $\mathcal{A}$ be a class of sets containing Ω as well as $\mathcal{D}$, and having the closure properties (vi) and (x).

Then $\mathcal{A}$ contains the B.F. generated by $\mathcal{D}$. (This is Dynkin's form of a monotone class theorem which is expedient for certain applications. The proof proceeds as in Theorem 2.1.2 by replacing $\mathcal{F}_0$ and $\mathcal{G}$ with $\mathcal{D}$ and $\mathcal{A}$ respectively.)

Solution. Let $\mathcal{A}_0$ be the minimal class of sets containing Ω as well as $\mathcal{D}$, and having the same closure properties (vi) and (x). Clearly $\mathcal{A}_0 \subset \mathcal{A}$. Note that $\mathcal{A}_0$ is closed under complementation by having property (x) and containing Ω. We shall show that $\mathcal{A}_0$ is also closed under intersection and hence is a field. Define two classes of subsets of $\mathcal{A}_0$ as follows:

$$\mathcal{C}_1 = \{E \in \mathcal{A}_0 : E \cap F \in \mathcal{A}_0 \text{ for all } F \in \mathcal{D}\},$$

$$\mathcal{C}_2 = \{E \in \mathcal{A}_0 : E \cap F \in \mathcal{A}_0 \text{ for all } F \in \mathcal{A}_0\}.$$

The identities

$$F \cap (\cup_{j=1}^{\infty} E_f) = \cup_{j=1}^{\infty} (F \cap E_j)$$

$$F \cap (E_2 \setminus E_1) = (F \cap E_2) \setminus (F \cap E_1)$$

show that both $\mathcal{C}_1$ and $\mathcal{C}_2$ have the closure properties (vi) and (x). It is also clear that they both contain Ω. Since $\mathcal{D}$ is closed under intersection and contained in $\mathcal{A}_0$, we have $\mathcal{D} \subset \mathcal{C}_1$. Hence $\mathcal{A}_0 \subset \mathcal{C}_1$ by the minimality of $\mathcal{C}_1$ and so $\mathcal{A}_0 = \mathcal{C}_1$. This means for any $F \in \mathcal{D}$ and $E \in \mathcal{A}_0$ we have $E \cap E \in \mathcal{A}_0$, which in turn means $\mathcal{D} \subset \mathcal{C}_2$. Hence $\mathcal{A}_0 = \mathcal{C}_2$ and this means $\mathcal{A}_0$ is closed under intersection.

It is clear that $\mathcal{A}_0$ is also a M.C. So it is a B.F. by Theorem 2.1.1. We conclude that the B.F. generated by $\mathcal{D}$ is contained in $\mathcal{A}_0$ and hence in $\mathcal{A}$. □

Problem 11. Take $\Omega = \mathbb{R}^n$ or a separable metric space in Exercise 10 and let $\mathcal{D}$ be the class of all open sets. Let $\mathcal{H}$ be a class of real-valued functions on Ω satisfying the following conditions.

(a) $1 \in \mathcal{H}$ and $1_D \in \mathcal{H}$ for each $D \in \mathcal{D}$;

(b) $\mathcal{H}$ is a vector space, namely: if $f_1 \in \mathcal{H}$, $f_2 \in \mathcal{H}$ and c_1, c_2 are any two real constants, then $c_1 f_1 + c_2 f_2 \in \mathcal{H}$;

(c) $\mathcal{H}$ is closed with respect to increasing limits of positive functions, namely: if $f_n \in \mathcal{H}$, $0 \le f_n \le f_{n+1}$ for all n, and $f = \lim_n \uparrow f_n < \infty$, then $f \in \mathcal{H}$.

Then $\mathcal{H}$ contains all Borel measurable functions on Ω, namely all finite-valued functions measurable with respect to the topological Borel field (= the minimal B.F. containing all open sets of Ω). [HINT: let $\mathcal{C} = \{E \subset \Omega : 1_E \in \mathcal{H}\}$; apply Exercise 10 to show that $\mathcal{C}$ contains the B.F. just defined. Each positive Borel measurable function is the limit of an increasing sequence of simple (finitely-valued) functions.]

Solution. Following the hint, we let $\mathcal{C} = \{E \subset \Omega : 1_E \in \mathcal{H}\}$. Since $\mathcal{H}$ satisfies the condition (a), $\mathcal{H}$ contains Ω as well as $\mathcal{D}$. Let $E_j \in \mathcal{C}, E_j \subset E_{j+1}, 1 \le j < \infty$, then $1_{\cup E_j} = \lim_j \uparrow 1_{E_j} \in \mathcal{H}$ by condition (c). It follows that $\cup E_j \in \mathcal{C}$ and so $\mathcal{C}$ has the closure property (vi). Let $E_1 \in \mathcal{C}, E_2 \in \mathcal{C}, E_1 \subset E_2$, then $1_{E_2 \setminus E_1} = 1_{E_2} - 1_{E_1} \in \mathcal{H}$ by condition (c). It follows that $E_2 \setminus E_1 \in \mathcal{C}$

and so $\mathcal{C}$ has the closure property (x). Now Exercise 10 gives that $\mathcal{C}$ contains the B.F. generated by $\mathcal{D}$, or the topological Borel field. Note that every finitely-valued simple function is a linear combination of indicator functions 1_{F_k} where $F_k \in \mathcal{C}$, and therefore belongs to $\mathcal{H}$. By condition (c), we conclude that $\mathcal{H}$ contains every Borel measurable functions as the limit of an increasing sequence of simple (finitely-valued) functions. □

Problem 12. Let $\mathcal{C}$ be a M.C. of subsets of $\mathbb{R}^n$ (or a separable metric space) containing all the open sets and closed sets. Prove that $\mathcal{C} \supset \mathcal{B}^n$ (the topological Borel field defined in Exercise 11). [HINT: Show that the minimal such class is a field.]

Solution. (TO BE COMPLETED) Denote the collection of all open and closed sets by $\mathcal{F}_0$ and the minimal M.C. containing $\mathcal{F}_0$ by $\mathcal{C}_0$. It is sufficient to show that $\mathcal{C}_0$ is a field, because then it will also be a Borel field and hence contain $\mathcal{B}^n$. We show show that it is closed under intersection and complementation. Define a class of sets in $\mathcal{C}_0$ as follows:

$$\mathcal{C}_3 = \{E \in \mathcal{C}_0 : E^{\complement} \in \mathcal{C}_0\}$$

The (DeMorgan) identities

$$(\cup_{j=1}^{\infty} E_j)^{\complement} = \cap_{j=1}^{\infty} E_j^{\complement}$$

$$(\cap_{j=1}^{\infty} E_j)^{\complement} = \cup_{j=1}^{\infty} E_j^{\complement}$$

show that $\mathcal{C}_3$ is a M.C. Since $\mathcal{C}_3 \supset \mathcal{F}_0$, we have $\mathcal{C}_0 \subset \mathcal{C}_3$ by the minimality of $\mathcal{C}_0$. Hence $\mathcal{C}_0 = \mathcal{C}_3$, which means $\mathcal{C}_0$ is closed under complementation. It remains to be shown that it is also closed under intersection. □