§1.1 Monotone Functions

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Problem 1. Let $\{a_n, n \geq 1\}$ be any given enumeration of the set of all rational numbers, and let $\{b_n, n \geq 1\}$ be a set of positive (≥ 0) numbers such that $\sum_{n=1}^{\infty} b_n < \infty$. For instance, we may take $b_n = 2^{-n}$. Consider now

$$f(x) = \sum_{n=1}^{\infty} b_n \delta_{a_n}(x), \tag{1}$$

where the point mass function δ_t at t is given by

$$\delta_t(x) = \begin{cases} 0 & \text{for } x < t, \\ 1 & \text{for } x \geq t. \end{cases}$$

Note that the series in (1) is absolutely and uniformly convergent, and f is increasing. Prove that

(i) $f(-\infty) = 0$,

(ii) $f(+\infty) = \sum_{n=1}^{\infty} b_n$

Solution. We first show that the limit of f at $-\infty$ is 0. Let $\epsilon > 0$ be given. By the convergence of the series $\sum_{n=1}^{\infty} b_n$, there exists a natural number N such that $\sum_{n=N+1}^{\infty} b_n < \epsilon$. Now let $x_0 = \inf_{n \leq N} a_n$, then $\forall x < x_0$, we have $\delta_{a_n}(x) = 0$ for all $n \leq N$. Hence

$$\begin{aligned} f(x) &= \sum_{n=1}^{N} b_n \delta_{a_n}(x) + \sum_{n=N+1}^{\infty} b_n \delta_{a_n}(x) \\ &\leq \sum_{n=1}^{N} b_n \cdot 0 + \sum_{n=N+1}^{\infty} b_n \cdot 1 \\ &< \epsilon, \end{aligned}$$

where the first inequality is due to the fact that $\delta_{a_n}(x) \leq 1$ for every n and x. Since f is obviously positive, we conclude that $f(-\infty) = 0$.

We proceed to show that the limit of f at $+\infty$ equals $\sum_{n=1}^{\infty} b_n$. Define $r(x) := \sum_{n=1}^{\infty} b_n - f(x) = \sum_{n=1}^{\infty} b_n(1 - \delta_{a_n}(x))$. Our aim is to show that $\lim_{x \to \infty} r(x) = 0$. Let $\epsilon > 0$ be given and choose a natural number $N \in \mathbb{N}$ satisfying $\sum_{n=N+1}^{\infty} b_n < \epsilon$. Let $y_0 = \sup_{n \le N} a_n$, then $\forall x > y_0$, we have $1 - \delta_{a_n}(x) = 0$ for all $n \le N$. Hence

$$\begin{aligned} r(x) &= \sum_{n=1}^{N} b_n(1 - \delta_{a_n}(x)) + \sum_{n=N+1}^{\infty} b_n(1 - \delta_{a_n}(x)) \\ &\le \sum_{n=1}^{N} b_n \cdot 0 + \sum_{n=N+1}^{\infty} b_n \cdot 1 \\ &< \epsilon, \end{aligned}$$

where the first inequality is due to the fact that $1 - \delta_{a_n}(x) \le 1$ for every n and x. Since r is positive, we conclude that $\lim_{x\to\infty} r(x) = 0$, which completes the proof. □

Problem 3. Suppose that f is increasing and that there exist real numbers A and B such that $\forall x : A \le f(x) \le B$. Show that for each $\epsilon > 0$, the number of jumps of size exceeding ϵ is at most $(B - A)/\epsilon$. Hence prove that the set of discontinuities of an increasing function is countable, first for bounded case and then in general.

Solution. Suppose the number of jumps of size exceeding ϵ is greater than $(B - A)/\epsilon$. Let $c_1 < \cdots < c_k$ be k of them arranged in increasing order, where $k = \lfloor (B - A)/\epsilon \rfloor + 1$. Then $f(c_i+) - f(c_i-) > \epsilon, i = 1, \ldots, k$. Since f is increasing, we also have $f(c_i+) \le f(c_{i+1}-), i = 1, \ldots, k-1$. But then

$$\begin{aligned} f(c_k+) - f(c_1-) &= \sum_{i=1}^{k}(f(c_i+) - f(c_i-)) + \sum_{i=1}^{k-1}(f(c_{i+1}-) - f(c_i+)) \\ &\ge \sum_{i=1}^{k} \epsilon + \sum_{i=1}^{k-1} 0 \\ &= k\epsilon > B - A, \end{aligned}$$

contradicting the fact that f is bounded between A and B. Therefore, the number of jumps of size exceeding ϵ is at most $(B - A)/\epsilon$.

We now prove that the set of discontinuities of f is countable. Since the only type of discontinuities of an increasing function are jumps, we can write the set of discontinuities of f as a countable union $\cup_{n=1}^{\infty} J_{1/n}$ where J_α denotes the set of jumps of size exceeding α for $\alpha > 0$. But each such set is finite by what we have just proved. Therefore, the set of discontinuities of f is countable as a countable union of countable sets.

For a general increasing function g that is possibly unbounded, consider the restriction of g on the interval $[-k, k], k = 1, 2, \ldots$. Let D_k be the set of discontinuities of g restricted on $[-k, k]$. The set of discontinuities of g on the whole real line is can be written as $\cup_{k=1}^{\infty} D_k$. Now each restriction is bounded by the function values of g at the interval endpoints. Hence D_k is countable for all positive integer k. We conclude that the set of discontinuities of a general increasing function is countable as a countable union of countable sets. □

Problem 4. Let f be an arbitrary function on $(-\infty, +\infty)$ and L be the set of x where f is right continuous but not left continuous. Prove that L is a countable set. [HINT: Consider $L \cap M_n$, where $M_n = \{x \mid O(f; x) > 1/n\}$ and $O(f; x)$ is the oscillation of f at x.]

Solution. Since continuity of a function at a point is equivalent to a non-zero oscillation at this point, every point of discontinuity belongs to some M_n for sufficiently large n. Hence $L = \cup_{n=1}^{\infty}(L \cap M_n)$. It suffices to show that $L \cap M_n$ is countable for all positive integer n.

Let $n \in \mathbb{N}^*$ be given. For any $x \in L \cap M_n$, since x is right continuous, there exists $\delta_x > 0$ such that if $y > x$ and $y - x < \delta_x$, then $|f(y) - f(x)| < 1/2n$. It follows that $\sup_{y\in(x,x+\delta_x)} f(y) \le f(x) + 1/2n$, $\inf_{y\in(x,x+\delta_x)} f(y) \ge f(x) - 1/2n$, and so

$$\sup_{y\in(x,x+\delta_x)} f(y) - \inf_{y\in(x,x+\delta_x)} f(y) \le 1/n.$$

The above inequality implies that the interval $(x, x + \delta_x)$ contains no element of $L \cap M_n$, because every point in $(x, x+\delta_x)$ has an oscillation no greater than $1/n$. Now associate each $x \in L \cap M_n$ with the interval $(x, x + \delta_x)$ as defined above, then these intervals are disjoint. But such a collection of pairwise disjoint open intervals is necessarily a countable one. We conclude that $L \cap M_n$ is countable, which completes the proof. □

Problem 5. Let D be dense in $(-\infty, +\infty)$ and let f be an increasing function on D. Define $\tilde{f}(x)$ on $(-\infty, +\infty)$ as follows:

$$\forall x : \tilde{f}(x) = \inf_{x<t\in D} f(t).$$

Show that the continuity of f on D does not imply that of $\tilde{f}$ on $(-\infty, +\infty)$, but uniform continuity does imply uniform continuity.

Solution. For the first part of the statement, we provide a counter example. Let $D = \mathbb{R}\setminus\{0\}$. Define $f(x)$ on D as the restriction of the point mass function $\delta_0(x)$ at D. Then f is obviously continuous on D. However, the corresponding $\tilde{f}$ is the original δ_0 defined on the whole of $\mathbb{R}$, which has a jump at 0.

Now if f is uniformly continuous on D, we show that $\tilde{f}$ is then necessarily continuous on $(-\infty, +\infty)$. Let $\epsilon > 0$ be given. The uniform continuity of f guarantees that there exists $\delta > 0$ such that if $x', y' \in D$ and $|x'-y'| < \delta$, then $|f(x')-f(y')| < \epsilon/2$. Let $x, y \in (-\infty, +\infty)$. We claim that if $|x - y| < \delta/2$, then $|\tilde{f}(x) - \tilde{f}(y)| < \epsilon$. By definition of $\tilde{f}$, there exist $x', y' \in D, x < x' < x + \delta/4, y < y' < y + \delta/4$, such that $|\tilde{f}(x) - f(x')| < \epsilon/4, |\tilde{f}(y) - f(y')| < \epsilon/4$. Since $|x'-y'| \le |x'-x|+|x-y|+|y-y'| < \delta/4+\delta/2+\delta/4 = \delta$, we have $|f(x')-f(y')| < \epsilon/2$. Hence $|\tilde{f}(x) - \tilde{f}(y)| \le |\tilde{f}(x) - f(x')| + |f(x') - f(y')| + |f(y') - \tilde{f}(y)| < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon$. We conclude that $\tilde{f}$ is uniformly continuous. □