§1.3 Absolutely Continuous and Singular Distributions

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Problem 1. A d.f. F is singular if and only if $F = F_s$; it is absolutely continuous if and only if $F = F_{ac}$

Solution. If F is singular, then $F' = 0$ a.e. For every x, we have $F_{ac}(x) = \int_{-\infty}^{x} F'(t)dt = 0$. Hence $F_s(x) = F(x) - F_{ac}(x) = F(x)$. Conversely, if $F = F_s$, then $F' = 0$ a.e. and hence F is singular.

If F is absolutely continuous, then there exists a function $f \in L^1$ such that we have for every x, $F(x) = \int_{-\infty}^{x} f(t)dt$. Since $F' = f$ a.e., we have $F_{ac}(x) = \int_{-\infty}^{x} F'(t)dt = \int_{-\infty}^{x} f(t)dt = F(x)$. Conversely, if $F = F_{ac}$, we have $F(x) = \int_{-\infty}^{x} F'(t)dt$. We conclude that F is absolutely continuous by definition. $\square$

Problem 2. Every d.f. F can be written as the convex combination of a discrete, a singular continuous, and an absolutely continuous d.f. Such a decomposition is unique.

Solution. Theorem 1.3.1 from the textbook provides a decomposition of F into an absolutely continuous part F_ac and a singular part. We first prove that such a decomposition is unique. Let G_{ac} and G_s be another decomposition. Since $G_s' = 0$ a.e., we have $G_{ac}' = F' - G_s' = F' = F_{ac}'$ a.e. It follows from the absolute continuity of G_{ac} and F_{ac} that $G_{ac} = F_{ac}$, which gives the uniqueness.

Suppose $F_{ac} \not\equiv 0$, $F_s \not\equiv 0$, then we may set $\alpha = F_{ac}(\infty)$ so that $0 < \alpha < 1$,

$$F_1 = \frac{1}{\alpha} F_{ac}, \quad F_2 = \frac{1}{1-\alpha} F_s,$$

and write

$$F = \alpha F_1 + (1-\alpha) F_2 \tag{1}$$

Now, F_1 is an absolutely continuous d.f., F_2 is a singular d.f., and F is exhibited as a convex combination of them. If $F_s \equiv 0$, then F is absolutely continuous and we set $\alpha = 1$, $F_1 \equiv F$, $F_2 \equiv 0$; if $F_{ac} \equiv 0$, then F is singular and we set $\alpha = 0$, $F_1 \equiv 0$, $F_2 \equiv F$; in either extreme case (1) remains valid. According to Theorem 1.2.3 from the textbook, F_2 can be further uniquely written as a convex combination of a discrete d.f. F_3 and a continuous d.f. F_4, i.e.

$$F_2 = \beta F_3 + (1-\beta) F_4, \tag{2}$$

where $0 \leq \beta \leq 1$. Since both F_2 and F_3 are singular, F_4 is also singular. Substitute (2) into (1), and we obtain

$$F = \alpha F_1 + (1-\alpha)\beta F_3 + (1-\alpha)(1-\beta)F_4.$$

It is clear that the right-hand side of the above equation is a convex combination of F_1, F_3 and F_4. Now suppose we have another decomposition

$$F = \lambda_1 G_1 + \lambda_2 G_3 + \lambda_3 G_4,$$

where G_1 is an absolutely continuous d.f., G_3 is a discrete d.f., G_4 is a singular continuous d.f., $0 \leq \lambda_i \leq 1$, $\sum_i \lambda_i = 1$. Note that both $(1-\alpha)\beta F_3 + (1-\alpha)(1-\beta)F_4$ and $\lambda_2 G_3 + \lambda_3 G_4$ are singular. We conclude that $\alpha F_1 = \lambda_1 G_1$ and $(1-\alpha)\beta F_3 + (1-\alpha)(1-\beta)F_4 = \lambda_2 G_3 + \lambda_3 G_4$ since the decomposition of F into an absolutely continuous part and a singular part is unique. But the decomposition of the singular part into a discrete part and a singular continuous part is again unique by Theorem 1.2.3. We conclude that every d.f. F can be uniquely written as the convex combination of a discrete, a singular continuous, and an absolutely continuous d.f. □

Problem 3. If the support of a d.f. is of measure zero, then F is singular. The converse if false.

Solution. Any point outside of the support has a neighborhood on which F takes constant value. It follows that the derivative of F on this point is zero. But the support of F has zero measure, so F' vanishes a.e. We conclude that F is singular. Note that a discrete distribution is singular, and in Problem 6 from section §1.2 we have provided an example of a discrete d.f. whose support is the whole line. Hence we see that the converse is no true. □

Problem 4. Suppose that F is a d.f. and there exists a continuous function f in L^1 such that

$$\forall x : F(x) = \int_{-\infty}^{x} f(t)dt.$$

Then $F' = f \geq 0$ everywhere.

Solution. Fix $x \in (-\infty, \infty)$. By definition, $F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt$. Let $\epsilon > 0$ be given. Since f is continuous at x, there exists $\delta > 0$ such that whenever $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$. Hence if $0 < |h| < \delta$, we have $|f(t) - f(x)| < \epsilon$ for all t that falls between x and $x + h$. It follows that

$$\begin{aligned}
\left|\frac{1}{h}\int_x^{x+h} f(t)dt - f(x)\right| &= \left|\frac{1}{h}\int_x^{x+h} f(t)dt - \frac{1}{h}\int_x^{x+h} f(x)dt\right| \\
&= \left|\frac{1}{h}\int_x^{x+h} (f(t) - f(x))dt\right| \\
&\leq \frac{1}{h}\int_x^{x+h} |f(t) - f(x)|dt \\
&< \frac{1}{h}\int_x^{x+h} \epsilon dt = \epsilon.
\end{aligned}$$

Therefore, $F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt = f(x)$. Since F is increasing, we conclude that $F' = f \geq 0$ everywhere. □

Problem 5. Under the conditions in the preceding exercise, the support of F is the closure of the set $\{t \mid f(t) > 0\}$; the complement of the support is the interior of the set $\{t \mid f(t) = 0\}$

Solution. Since the derivative of F vanishes outside of its support, every point where F has nonzero derivative is a point of support. But the support of F is closed, so it necessarily contains the closure of the set $\{t \mid f(t) > 0\}$. Now the complement of the closure is the interior of the complement. It follows that the complement of the support is contained in the interior of the set $\{t \mid f(t) > 0\}^{\complement} = \{t \mid f(t) = 0\}$. Let x be an interior point of the set $\{t \mid f(t) = 0\}$, then there exists $\epsilon > 0$ such that $f(t) = 0$ for $t \in (x - \epsilon, x + \epsilon)$, and so $F(x+\epsilon) - F(x-\epsilon) = \int_{x-\epsilon}^{x+\epsilon} f(t)dt = \int_{x-\epsilon}^{x+\epsilon} 0dt = 0$. In other words, x lies outside of the support of F. We conclude that the complement of the support is exactly the interior of the set $\{t \mid f(t) = 0\}$, and that the support of F is exactly the closure of the set $\{t \mid f(t) > 0\}$. □

Problem 6. Prove that a discrete distribution is singular.

Solution. Let $F = \sum_j b_j \delta_{a_j}$ be a discrete d.f., where $\{a_j\}$ are its points of jump and b_j the corresponding jump sizes. Note that for all j, δ_{a_j} is increasing and has zero derivative except at the point a_j. By Fubini's theorem on differentiation, we have $F'(x) = \sum_j b_j \delta'_{a_j}(x) = \sum_j b_j \cdot 0 = 0$ a.e. We conclude that F is singular. □

Problem 7. Prove that a singular function as defined here is (Lebesgue) measurable but need not be of bounded variation even locally. [HINT: Such a function is continuous except on a set of Lebesgue measure zero; use the completeness of the Lebesgue measure.]

Solution. Let F be a singular function, then there exists a set D of Lebesgue measure zero such that $F' = 0$ on $D^{\complement}$. Since differentiability implies continuity, F is necessarily continuous on $D^{\complement}$. Fix a real number c and consider the set $F^{-1}((c,\infty))$. We have

$$F^{-1}((c,\infty)) = [F^{-1}((c,\infty)) \cap D^{\complement}] \cup [F^{-1}((c,\infty)) \cap D]. \tag{3}$$

Because F is continuous on $D^{\complement}$, the first set on the right of (3) is open in $D^{\complement}$. Thus it equals $D^{\complement} \cap U$ for some U open in $\mathbb{R}$. Because $D^{\complement}$, U are both measurable, so is $F^{-1}((c,\infty)) \cap D^{\complement}$. By the completeness of the Lebesgue measure, any subset of the zero-measure set D is measurable. Therefore, $F^{-1}((c,\infty))$ in (3) is the union of two measurable sets, and hence is measurable. We conclude that F is Lebesgue measurable.

TODO: provide an example of a singular function that is not of locally bounded variation. □