43. Show that the two expressions for r_S in displays (8.63) and (8.64) are equivalent.

$$\hat{\rho} = \frac{\sum_{i=1}^{n}(R_i-\bar{R})(S_i-\bar{S})}{\sqrt{\sum_{i=1}^{n}(R_i-\bar{R})^2 \sum_{i=1}^{n}(S_i-\bar{S})^2}} \qquad \text{where } \bar{S}=\bar{R}=\frac{n+1}{2}$$

$$\sum_{i=1}^{n} R_i^2 - n(\bar{R})^2 = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{4} = \frac{n(n^2-1)}{12}$$

$$\sum_{i=1}^{n}(R_i-\bar{R})(S_i-\bar{S}) = \sum_{i=1}^{n} R_iS_i - \frac{n(n+1)^2}{4}$$

$$\Rightarrow \hat{\rho} = \frac{12}{n(n^2-1)}\sum_{i=1}^{n}(R_i-\bar{R})(S_i-\bar{S}) = \frac{12}{n(n^2-1)}\sum_{i=1}^{n} R_iS_i - \frac{3n+3}{n-1}$$

Let $d_i = R_i - S_i \Rightarrow \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} R_i^2 - \sum_{i=1}^{n} S_i^2 - 2\sum_{i=1}^{n} R_iS_i$

$$= \frac{n(n+1)(2n+1)}{3} - 2\sum_{i=1}^{n} R_iS_i$$

$$\Rightarrow \hat{\rho} = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)}$$

46. Give an example of a data set of $n \geq 10$ bivariate observations for which r_S has value 0.

S:	1	2	3	4	5	6	7	8	9	10	11
R:	11	2	3	7	6	5	4	8	9	10	1
d:	10	0	0	3	1	1	3	0	0	0	10

$$\sum_{i=1}^{11} d_i^2 = 2\times(10^2+3^2+1) = 220$$

$$\hat{\rho} = 1 - \frac{6\times 220}{11(121-1)} = 0$$

49. Let r_p be the Pearson product moment correlation coefficient defined in (8.78). Show that r_s (8.63) is simply this Pearson product moment correlation coefficient applied to the rank vectors $(R_1, \ldots, R_n)$ and $(S_1, \ldots, S_n)$ instead of the original $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ vectors.

$$r_p = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X})^2 \sum_{i=1}^{n}(Y_i - \bar{Y})}}$$

$(R_1 \cdots R_n)$ is rank associated with $(X_1 \cdots X_n$

$(S_1 \cdots S_n)$ is rank associated with $(Y_1 \cdots Y_n)$

$$r_s = \frac{\sum_{i=1}^{n}(R_i - \bar{R})(S_i - \bar{S})}{\sqrt{\sum_{i=1}^{n}(R_i - \bar{R})^2 \sum_{i=1}^{n}(S_i - \bar{S})^2}}$$ is r_p w.r.t. $(R_1 \cdots R_n)$ $(S_1 \cdots S_n)$

53. Let $(S_1, \ldots, S_n)$ be a vector of ranks that is uniformly distributed over the set of all $n!$ permutations of $(1, 2, \ldots, n)$. Show that the marginal probability distribution of each S_i, for $i = 1, \ldots, n$, is uniform over the set $\{1, 2, \ldots, n\}$. Use this fact to show that $E(S_i) = (n+1)/2$ and $\text{var}(S_i) = (n^2 - 1)/12$, for $i = 1, \ldots, n$.

For $\forall i \in \{1, 2, 3, \cdots n\}$ and $k \in \{1, 2, 3, \cdots n\}$

Let $P_j(\cdot)$ be a permutation function on

$(1, 2, 3, \cdots k-1, k+1, \cdots n) \longrightarrow (P_j(1), P_j(2), \cdots P_j(k-1), P_j(k+1) \cdots P_j(n))$

$\mathcal{P}$ be the family containing all such functions

$|\mathcal{P}| = (n-1)!$

$$\Rightarrow P\{S_i = k\} = \sum_{P_j \in \mathcal{P}} P\{S_i = k, (S_1, S_2 \cdots S_{i-1}, S_{i+1} \cdots S_n) = P_j(1, 2, 3, \cdots k-1, k+1, \cdots n)\}$$

$$= \frac{1}{n!} \cdot |\mathcal{P}| = \frac{1}{n!} \cdot (n-1)! = \frac{1}{n}.$$

S_i is uniformly distributed over $\{1, 2, \cdots n\}$

then $E(S_i) = \frac{n+1}{2}$

$$E(S_i^2) = \frac{1}{n} \sum_{k=1}^{n} k^2 = \frac{(n+1)(2n+1)}{6}$$

$$\Rightarrow Var(S_i) = E(S_i^2) - E^2(S_i)$$

$$= \frac{(n+1)^2(2n+1)^2}{36} - \frac{(n+1)^2}{4}$$

$$= \frac{(n+1)^2[(2n+1)^2-9]}{36}$$

$$= \frac{n^2-1}{12}$$

54. Let $(S_1, \ldots, S_n)$ be a vector of ranks that is uniformly distributed over the set of all $n!$ permutations of $(1, 2, \ldots, n)$. Show that the joint marginal probability distribution of (S_i, S_j), for $i \neq j = 1, \ldots, n$, is given by

$$P(S_i = s, S_j = t) = \begin{cases} \dfrac{1}{n(n-1)}, & s \neq t = 1, \ldots, n \\ 0, & \text{otherwise.} \end{cases}$$

Use this fact to show that $\text{cov}(S_i, S_j) = -(n+1)/12$, for $i \neq j = 1, \ldots, n$.

Similar to what we do in 53, define $P_k(\cdot)$ be the permutation function on $(1, 2, 3 \cdots s-1, s+1, \cdots\cdots t-1, t+1 \cdots n)$ ($s<t$ WLOG)

$\mathcal{P}$ be the family containing all such functions

$|\mathcal{P}| = (n-2)!$

$$\Rightarrow P\{S_i = s, S_j = t\} = \sum_{P_k \in \mathcal{P}} P\{S_i = s, S_j = t, (S_1, S_2 \cdots S_{i-1}, S_{i+1} \cdots S_{j-1}, S_{j+1} \cdots S_n) = P_k(1, 2, 3, \cdots s-1, s+1, \cdots t-1, t+1 \cdots n)\}$$

$$= \frac{1}{n!} \cdot |\mathcal{P}| = \frac{1}{n!} \cdot (n-2)! = \frac{1}{n(n-1)} \quad (s \neq t)$$

$P\{S_i = s, S_j = t\} = 0$ if $s = t$.

$$Cov(S_i, S_j) = E(S_i S_j) - E(S_i)E(S_j)$$

$$= \sum_{s \neq t}^{n} \frac{st}{n(n-1)} - \frac{(n+1)^2}{4}$$

$$= \frac{(n+1)(3n+2)}{12} - \frac{(n+1)^2}{4}$$

$$= -\frac{n+1}{12}$$

8. Describe a situation where it is natural to expect the underlying distribution to satisfy the IFRA property but not satisfy the IFR property (i.e., where F might be expected to be a member of the IFRA class but not a member of the IFR class).

Consider the faidure rate of a machine over time.

Clearly the failure rate increases as time goes on.

satisfying IFRA.

But if given periodic maintainence, the failure rate drops within a time interval, disobeying IFR.