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3.15. For AR(1): $x_t = \phi x_{t-1} + w_t$.

$$x_{t+1} = \phi x_t + w_{t+1}$$

$$x_{t+1}^{(t)} = \mathbb{E}(x_{t+1} \mid x_1 \cdots x_t) = \phi x_t$$

$$\begin{aligned} x_{t+2}^{(t)} &= \mathbb{E}(\phi x_{t+1} + w_{t+2} \mid x_1 \cdots x_t) \\ &= \mathbb{E}(\phi^2 x_t + \phi w_{t+1} + w_{t+2} \mid x_1 \cdots x_t) \\ &= \phi^2 x_t \end{aligned}$$

$\vdots$

$$x_{t+m}^{(t)} = \phi^m x_t.$$

$$\begin{aligned} \mathbb{E}(x_{t+m} - x_{t+m}^{(t)})^2 &= \mathbb{E}(\phi^m x_t + \phi^{m-1} w_{t+1} \cdots w_{t+m} - \phi^m x_t)^2 \\ &= \mathbb{E}(\phi^{m-1} w_{t+1} + \phi^{m-2} w_{t+2} \cdots w_{t+m})^2 \\ &= \sum_{i=0}^{m-1} \mathbb{E}(\phi^{2i} w_{t+m-i}^2) + 2\sum_{l>k} \phi^{l+k} \mathbb{E}(w_l w_k) \end{aligned}$$

Since $\mathbb{E}(w_l w_k) = 0$, $\mathbb{E}(w_i^2) = \sigma_w^2$

$$= \sum_{i=0}^{m-1} \phi^{2i} \cdot \sigma_w^2 = \sigma_w^2 \cdot \frac{1-\phi^{2m}}{1-\phi^2}$$

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$$x_t = 0.9 x_{t-1} + w_t + 0.5 w_{t-1} \qquad w_t \sim N(0,1)$$

Let $\beta = (\mu, \phi, \theta)$.

① Unconditional MLE:

$$L(\beta, \sigma_w^2) = f(x_1) \cdot \prod_{t=2}^{n} f(x_n | x_{1:n-1})$$

where $x_1 - \mu = \sum_{j=0}^{\infty} \psi_j w_{t-j}$,

$$w_t \overset{iid}{\sim} N(0, \sigma_w^2) \qquad x_1 \sim N(\mu, \sigma_w^2 \sum_{j=0}^{\infty} \psi_j^2$$

$E(x_n | x_{1:n-1}) = x_n^{n-1}$

$Var(x_n | x_{1:n-1}) = E(x_n - x_n^{n-1})^2$

$= \gamma(0) \prod_{j=1}^{n-1} (1 - \phi_{jj}^2)$

$= \sigma_w^2 \sum_{j=0}^{\infty} \psi_j^2 \left[\prod_{j=1}^{n-1} (1 - \phi_{jj}^2)\right]$

$= \sigma_w^2 r_n$ as $r_n \triangleq \sum_{j=0}^{\infty} \psi_j^2 \left[\prod_{j=1}^{n-1} (1 - \phi_{jj}^2)\right]$

Let $x_1^0 = \mu$, $r_1 = \sum_{j=0}^{\infty} \psi_j^2$

$\Rightarrow$ $x_1 \sim N(\mu, \sigma_w^2 r_1)$

The likelihood function is given by:

$$L(\beta, \sigma_w^2) = \prod_{t=1}^{n} \left[2\pi \sigma_w^2 r_t(\beta)\right]^{-\frac{1}{2}} \cdot \exp\left\{- \frac{(x_t - x_t^{t-1}(\beta))^2}{2\sigma_w^2 r_t(\beta)}\right\}$$

Let $S(\beta) = \sum_{t=1}^{n} \frac{(x_t - x_t^{t-1}(\beta))^2}{r_t(\beta)}$

$$\Rightarrow L(\beta, \sigma_w^2) = (2\pi\sigma_w^2)^{-\frac{n}{2}} \left[r_1(\beta) \cdots r_n(\beta)\right]^{-\frac{1}{2}} \cdot \exp\left\{- \frac{S(\beta)}{2\sigma_w^2}\right\}$$

log-likelihood: $l(\beta, \sigma_w^2) = \log L(\beta, \sigma_w^2)$

$$= \log\left[\frac{1}{n} S(\beta)\right] + \frac{1}{n}\sum_{t=1}^{n} \log r_t(\beta)$$

② Unconditional Least Square:

minimize
$$S(\beta) = \sum_{t=1}^{n} \frac{\left(x_t - x_t^{t-1}(\beta)\right)^2}{r_t(\beta)}$$

③ Conditional Least Square.

Condition: x_1, w_1 are non-random

Let $w_t(\beta) = x_t - \phi x_{t-1} - \theta w_{t-1}(\beta)$

Since when $t-1=1$, x_t, w_t are non-random
$w_t(\beta)$ can be calculated iteratively.

Want to minimize:
$$S_{cn}(\beta) = \sum_{t=2}^{n} w_t^2(\beta)$$