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STA 4003 Assignment 3.

2.6.

$$x_t = \beta_0 + \beta_1 t + w_t \quad E(w_t) = 0 \quad \text{and} \quad Var(w_t) = \sigma_w^2$$

(a) $E(x_t) = \beta_0 + \beta_1 t$ → not a constant with respect to t,

x_t is not stationary.

(b)

$\nabla x_t = x_t - x_{t-1} = \beta_1 + w_t - w_{t-1}$

$E(\nabla x_t) = \beta_1 \Rightarrow$ constant.

$\gamma(h) = Cov(\nabla x_t, \nabla x_{t-h}) = Cov(w_t - w_{t-1}, w_{t-h} - w_{t-h-1})$

① $h=0$ $\quad \gamma(h) = Var(\nabla x_t) = 2\sigma_w^2$

② $h=1$ $\quad \gamma(h) = Cov(w_t - w_{t-1}, w_{t-1} - w_{t-2})$
$= -\sigma_w^2$

③ for $h \geq 2$ $\quad \gamma(h) = 0$.

$\gamma(h)$ only depend on h. $\Rightarrow \nabla x_t$ is stationary.

(c)

$\nabla x_t = x_t - x_{t-1} = \beta_1 + y_t - y_{t-1}$.

$E(\nabla x_t) = \beta_1 \Rightarrow$ constant

$\gamma(h) = Cov(y_t - y_{t-1}, y_{t-h} - y_{t-h-1})$
$= 2\gamma_y(h) - \gamma_y(h-1) - \gamma_y(h+1) \Rightarrow$ only depend on h

∇x_t is stationary.

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3.4.

(a) $x_t = 0.8x_{t-1} - 0.15x_{t-2} + w_t - 0.3w_{t-1}$

$$x_t - 0.8x_{t-1} + 0.15x_{t-2} = w_t - 0.3w_{t-1}$$

$$(1-0.8B+0.15B^2)x_t = (1-\overset{0.3B}{\cancel{B}})w_t$$

$$(1-0.3B)(1-0.5B)x_t = (1-0.3B)w_t \Leftrightarrow (1-0.5B)x_t = (\cancel{1-0.5B})w_t$$

$$\phi(B)x_t = \theta(B)w_t$$

AR(1) model

The root for $\phi(z)=0$ is $z_1 = \frac{10}{3} > 1$, $z_2 = 2 > 1$ $\Rightarrow$ causal

The root for $\theta(z)=0$ is $z_1 = \frac{10}{3} > 1$ $\Rightarrow$ invertible

(b) $x_t = x_{t-1} - 0.5x_{t-2} + w_t - w_{t-1}$

$$(1-B+0.5B^2)x_t = (1-B)w_t \Leftrightarrow \text{ARMA}(2,1)$$

$$\phi(B)x_t = \theta(B)w_t$$

$\phi(z)=0 \Leftrightarrow z_1 = 1+i$, $z_2 = 1-i$

$|z_1|>1$, $|z_2|>1$ $\Rightarrow$ not causal

$\theta(z)=0 \Rightarrow z_1 = 1 \Rightarrow$ not invertible

3. Let $\hat{\theta}$ be an estimate of a parameter θ. Suppose

$$\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0,V) \quad \text{for some } V, \text{ as } n\to\infty$$

Show that $\hat{\theta}-\theta = O_p(n^{-\frac{1}{2}})$

Proof: Let $\Phi_V(x)$ be the c.d.f of $N(0,V)$,
$F_{\hat{\theta}}(x)$ be the c.d.f of $\sqrt{n}(\hat{\theta}-\theta)$

$$\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0,V) \text{ as } n\to\infty$$

$\Leftrightarrow$ $\forall \varepsilon>0$, $\exists n_0$, s.t. when $n\geq n_0$,

$$|F_{\hat{\theta}}(x) - \Phi_V(x)| < \varepsilon.$$

For random variable $X \sim N(0,V)$, by Chebyshev's Inequality,

$$\mathbb{P}\{|X|>\delta\} < \frac{V}{\delta^2}$$

Let $\varepsilon = \frac{V}{\delta^2}$, $\Rightarrow$ $\delta = \sqrt{\frac{V}{\varepsilon}}$.

For $\forall\varepsilon>0$, $\exists \delta(\varepsilon)$, s.t.

$$\mathbb{P}\{|X|>\delta(\varepsilon)\} < \varepsilon$$

$$\Rightarrow 1-\Phi_V(\delta)+\Phi_V(-\delta) < \varepsilon$$

For $\forall\varepsilon$, $\exists \delta(\varepsilon) = \sqrt{\frac{V}{\varepsilon}}$

$$\mathbb{P}\left\{\frac{|\hat{\theta}-\theta|}{n^{-\frac{1}{2}}} > \delta(\varepsilon)\right\} = 1-F_{\hat{\theta}}(\delta)+F_{\hat{\theta}}(-\delta)$$

$$\mathbb{P}\left\{\frac{\hat{\theta}-\theta}{n^{-\frac{1}{2}}} > \delta\right\} - \mathbb{P}\{|X|>\delta(\varepsilon)\}$$

$$= (F_{\hat{\theta}}(\delta)-\Phi(\delta)) + (\Phi(-\delta)-F_{\hat{\theta}}(\delta)) < 2\varepsilon.$$

Since $\mathbb{P}\{|X|>\delta(\varepsilon)\} < \varepsilon$

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$$\Rightarrow \mathbb{P}\left\{ \frac{|\hat{\theta}-\theta|}{n^{-\frac{1}{2}}} > \delta(\varepsilon) \right\} < 3\varepsilon$$

$$\Rightarrow \sqrt{n}(\hat{\theta}-\theta) = O_p(1)$$

$$\hat{\theta}-\theta = O_p(n^{-\frac{1}{2}})$$

Campus