STA 4003 Homework 5 杨煜同 12109044G

3.15. For AR(1): $x_t = \phi x_{t-1} + w_t$.

$$x_{t+1} = \phi x_t + w_{t+1}$$

$$x_{t+1}^{(t)} = E(x_{t+1} \mid x_1 \cdots x_t) = \phi x_t$$

$$x_{t+2}^{(t)} = E(\phi x_{t+1} + w_{t+2} \mid x_1 \cdots x_t)$$

$$= E(\phi^2 x_t + \phi w_{t+1} + w_{t+2} \mid x_1 \cdots x_t)$$

$$= \phi^2 x_t$$

$$\vdots$$

$$x_{t+m}^{(t)} = \phi^m x_t.$$

$$E(x_{t+m} - x_{t+m}^{(t)})^2 = E(\phi^m x_t + \phi^{m-1} w_{t+1} \cdots w_{t+m} - \phi^m x_t)^2$$

$$= E(\phi^{m-1} w_{t+1} + \phi^{m-2} w_{t+2} \cdots w_{t+m})^2$$

$$= \sum_{i=0}^{m-1} E(\phi^{2i} w_{t+m-i}^2) + 2 \sum_{l>k} \phi^{l+k} E(w_l w_k)$$

Since $E(w_l w_k) = 0$, $E(w_i^2) = \sigma_w^2$

$$= \sum_{i=0}^{m-1} \phi^{2i} \cdot \sigma_w^2 = \sigma_w^2 \cdot \frac{1-\phi^{2m}}{1-\phi^2}$$

3.21

$$x_t = 0.9\, x_{t-1} + w_t + 0.5\, w_{t-1} \qquad w_t \sim N(0,1)$$

Let $\beta = (\mu, \phi, \theta)$.

① Unconditional MLE:

$$L(\beta, \sigma_w^2) = f(x_1) \cdot \prod_{t=2}^{n} f(x_n \mid x_{1:n-1})$$

where $x_1 - \mu = \sum_{j=0}^{\infty} \psi_j w_{t-j}$,

$$w_t \overset{iid}{\sim} N(0, \sigma_w^2) \qquad x_1 \sim N(\mu, \sigma_w^2 \sum_{j=0}^{\infty} \psi_j^2)$$

$$E(x_n \mid x_{1:n-1}) = x_n^{n-1}$$

$$\begin{aligned} \mathrm{Var}(x_n \mid x_{1:n-1}) &= E(x_n - x_n^{n-1})^2 \\ &= r(0) \prod_{j=1}^{n-1} (1 - \phi_{jj}^2) \\ &= \sigma_w^2 \sum_{j=0}^{\infty} \psi_j^2 \left[\prod_{j=1}^{n-1} (1 - \phi_{jj}^2) \right] \end{aligned}$$

$$= \sigma_w^2 r_n \quad \text{as} \quad r_n \triangleq \sum_{j=0}^{\infty} \psi_j^2 \left[\prod_{j=1}^{n-1} (1 - \phi_{jj}^2) \right]$$

Let $x_t^0 = \mu$, $r_1 = \sum_{j=0}^{\infty} \psi_j^2$

$$\Rightarrow \quad x_t \sim N(\mu, \sigma_w^2 r_1)$$

The likelihood function is given by:

$$L(\beta, \sigma_w^2) = \prod_{t=1}^{n} \left[2\pi \sigma_w^2 r_t(\beta) \right]^{-\frac{1}{2}} \cdot \exp\left\{ - \frac{(x_t - x_t^{t-1}(\beta))^2}{2\sigma_w^2 r_t(\beta)} \right\}$$

Let $S(\beta) = \sum_{t=1}^{n} \frac{(x_t - x_t^{t-1}(\beta))^2}{r_t(\beta)}$

$$\Rightarrow L(\beta, \sigma_w^2) = (2\pi\sigma_w^2)^{-\frac{n}{2}} \left[r_1(\beta) \cdots r_n(\beta) \right]^{-\frac{1}{2}} \cdot \exp\left\{ - \frac{S(\beta)}{2\sigma_w^2} \right\}$$

log-likelihood: $l(\beta, \sigma_w^2) = \log L(\beta, \sigma_w^2)$

$$= \log\left[\frac{1}{n} S(\beta)\right] + \frac{1}{n}\sum_{t=1}^{n} \log r_t(\beta)$$

② Unconditional Least Square:

$$\text{minimize} \quad S(\beta) = \sum_{t=1}^{n} \frac{(x_t - \hat{x}_t^{t-1}(\beta))^2}{r_t(\beta)}$$

③ Conditional Least Square.

Condition: x_1, w_1 are non-random

Let $w_t(\beta) = x_t - \phi x_{t-1} - \theta w_{t-1}(\beta)$

Since when $t-1 \geq 1$, x_t, w_t are non-random
$w_t(\beta)$ can be calculated iteratively.

Want to minimize:

$$S_{cm}(\beta) = \sum_{t=2}^{n} w_t^2(\beta)$$

STA4003_HW5_codes

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2023-12-06

```
library(tswge)
```

```
## Warning: 程辑包'tswge'是用R版本4.2.3 来建造的
```

```
## Registered S3 method overwritten by 'quantmod':
##   method            from
##   as.zoo.data.frame zoo
```

```
##
## 载入程辑包：'tswge'
```

```
## The following object is masked from 'package:datasets':
##
##     uspop
```

```
set.seed(123)

x <- list()

for(i in 1:10){
  x[[i]] <- gen.arima.wge(200, phi=0.9, theta=0.5, d=0,s=0,mu=0,vara=1,plot=F,sn=0)
  print(arima(x[[i]], order = c(1,0,1), method = "CSS-ML"))
  print(arima(x[[i]], order = c(1,0,1), method = "ML"))
  print(arima(x[[i]], order = c(1,0,1), method = "CSS"))
  cat("--------------------------------------------------------------------------------\n")
}
```

```
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS-ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.9144  -0.4367    -0.7202
## s.e.  0.0370   0.0818     0.4019
##
## sigma^2 estimated as 0.8196:  log likelihood = -264.39,  aic = 536.79
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.9143  -0.4366    -0.7202
## s.e.  0.0370   0.0818     0.4017
##
## sigma^2 estimated as 0.8196:  log likelihood = -264.39,  aic = 536.79
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.9124  -0.4444    -0.9037
## s.e.  0.0365   0.0811     0.4139
##
## sigma^2 estimated as 0.8158:  part log likelihood = -263.42
## ------------------------------------------------------------------------------
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS-ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8899  -0.4417    -0.2143
## s.e.  0.0424   0.0823     0.3589
##
## sigma^2 estimated as 1.074:  log likelihood = -291.32,  aic = 590.64
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8899  -0.4416    -0.2145
## s.e.  0.0424   0.0823     0.3588
##
## sigma^2 estimated as 1.074:  log likelihood = -291.32,  aic = 590.64
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS")
##
```

```
## Coefficients:
##          ar1      ma1  intercept
##       0.8784  -0.4156    -0.0992
## s.e.  0.0449   0.0812     0.3563
##
## sigma^2 estimated as 1.077:  part log likelihood = -291.18
## ------------------------------------------------------------------------------
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS-ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8503  -0.5200    -0.3946
## s.e.  0.0563   0.0878     0.2099
##
## sigma^2 estimated as 0.8961:  log likelihood = -273.03,  aic = 554.06
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8502  -0.5199    -0.3947
## s.e.  0.0564   0.0878     0.2099
##
## sigma^2 estimated as 0.8961:  log likelihood = -273.03,  aic = 554.06
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8544  -0.5226    -0.3866
## s.e.  0.0568   0.0883     0.2211
##
## sigma^2 estimated as 0.901:  part log likelihood = -273.36
## ------------------------------------------------------------------------------
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS-ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8614  -0.5485    -0.3148
## s.e.  0.0579   0.0944     0.2511
##
## sigma^2 estimated as 1.243:  log likelihood = -305.77,  aic = 619.54
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8613  -0.5485    -0.3150
## s.e.  0.0579   0.0944     0.2511
```

```
##
## sigma^2 estimated as 1.243:  log likelihood = -305.77,  aic = 619.54
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8651  -0.5531    -0.3527
## s.e.  0.0569   0.0933     0.2630
##
## sigma^2 estimated as 1.248:  part log likelihood = -305.93
## ------------------------------------------------------------------------------
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS-ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.9018  -0.5883     0.4698
## s.e.  0.0444   0.0853     0.2701
##
## sigma^2 estimated as 0.8932:  log likelihood = -272.79,  aic = 553.59
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.9018  -0.5882     0.4696
## s.e.  0.0444   0.0854     0.2700
##
## sigma^2 estimated as 0.8932:  log likelihood = -272.79,  aic = 553.59
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8953  -0.5677     0.5234
## s.e.  0.0481   0.0885     0.2801
##
## sigma^2 estimated as 0.9011:  part log likelihood = -273.38
## ------------------------------------------------------------------------------
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS-ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.9323  -0.6371     0.2527
## s.e.  0.0357   0.0815     0.3534
##
## sigma^2 estimated as 0.9589:  log likelihood = -279.97,  aic = 567.93
##
## Call:
```

```
## arima(x = x[[i]], order = c(1, 0, 1), method = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.9323  -0.6372     0.2531
## s.e.  0.0357   0.0815     0.3535
##
## sigma^2 estimated as 0.9589:  log likelihood = -279.97,  aic = 567.93
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.9285  -0.6178     0.1801
## s.e.  0.0389   0.0862     0.3751
##
## sigma^2 estimated as 0.9723:  part log likelihood = -280.98
## ------------------------------------------------------------------------------
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS-ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.9000  -0.5727    -0.6953
## s.e.  0.0437   0.0815     0.2711
##
## sigma^2 estimated as 0.8602:  log likelihood = -269.04,  aic = 546.07
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.9001  -0.5728    -0.6951
## s.e.  0.0437   0.0815     0.2712
##
## sigma^2 estimated as 0.8602:  log likelihood = -269.04,  aic = 546.07
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.9009  -0.5732    -0.6326
## s.e.  0.0439   0.0810     0.2844
##
## sigma^2 estimated as 0.8627:  part log likelihood = -269.02
## ------------------------------------------------------------------------------
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS-ML")
##
## Coefficients:
##          ar1      ma1  intercept
```

```
##       0.8090  -0.3083    -0.1806
## s.e.  0.0585   0.0919     0.2427
##
## sigma^2 estimated as 0.9316:  log likelihood = -276.99,  aic = 561.99
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8090  -0.3082    -0.1805
## s.e.  0.0585   0.0919     0.2426
##
## sigma^2 estimated as 0.9316:  log likelihood = -276.99,  aic = 561.99
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8121  -0.3111    -0.1475
## s.e.  0.0585   0.0921     0.2513
##
## sigma^2 estimated as 0.9345:  part log likelihood = -277.01
## ------------------------------------------------------------------------------
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS-ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8695  -0.5313    -0.5668
## s.e.  0.0480   0.0755     0.2429
##
## sigma^2 estimated as 0.9653:  log likelihood = -280.51,  aic = 569.01
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8694  -0.5313    -0.5667
## s.e.  0.0480   0.0755     0.2429
##
## sigma^2 estimated as 0.9653:  log likelihood = -280.51,  aic = 569.01
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8636  -0.5209    -0.5021
## s.e.  0.0495   0.0755     0.2460
##
## sigma^2 estimated as 0.9669:  part log likelihood = -280.43
## ------------------------------------------------------------------------------
```

```
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS-ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8820  -0.4946     0.2819
## s.e.  0.0459   0.0828     0.2890
##
## sigma^2 estimated as 0.9647:  log likelihood = -280.51,  aic = 569.02
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "ML")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8820  -0.4946     0.2816
## s.e.  0.0459   0.0828     0.2890
##
## sigma^2 estimated as 0.9647:  log likelihood = -280.51,  aic = 569.02
##
## Call:
## arima(x = x[[i]], order = c(1, 0, 1), method = "CSS")
##
## Coefficients:
##          ar1      ma1  intercept
##       0.8842  -0.4977     0.2208
## s.e.  0.0456   0.0821     0.3029
##
## sigma^2 estimated as 0.9674:  part log likelihood = -280.47
## ------------------------------------------------------------------------------
```