1 Review of big-Oh notation

Definition: $f(n)$ is $O(g)$ iff $\exists n_0 \in \mathbf{N}, \mathbf{c} \in \mathbf{R} : \mathbf{f(n) \leq c \cdot g(n) \forall n \geq n_0}$

Intuition: $f(n)$ is $O(g(n))$ if $f(n)$ grows at most as fast as some constant times $g(n)$, for large n.

IMPORTANT: The running time of selection sort on an array of n elements was $1 + 5n + 15n^2$, which is $O(n^2)$. But it is also $O(n^3)$, and $O(n^4)$, and $O($any function that grows at least as fast as $n^2)$. However, we usually try to give the snuggest big-Oh description possible.

2 Hierarchy of big-Oh classes

$O(g(n))$ can be seen as the set of all functions $f(n)$ that are $O(g(n))$: $O(g(n)) = \{f(n) : \exists c, n_0, \forall n \geq n_0, f(n) \leq c \cdot g(n)\}$.

Then we can write $n^2 + 10n + 2 \in O(n^2)$.

We have the following (incomplete) hierarchy of big-Oh classes:

$$O(1) \subset O(\log n) \subset O(\sqrt{n}) \subset O(n) \subset O(n^k) \subset O(2^n)$$

- $O(1)$: functions bounded above by a constant. $f(n) = 100 \in O(1)$, $10 + \sin(n) \in O(1)$. All primitive operations can be executed in time $O(1)$.
- $O(\log n)$: logarithmic functions. Running time of binarySearch is $O(\log n)$.
- $O(\sqrt{n})$: square root of n. $2 + \sqrt{n + 10} \in O(\sqrt{n})$
- $O(n)$: linear functions. Running time of findMin is $O(n)$
- $O(n^k)$, for some integer $k > 1$: polynomials. Running time of selectionSort is $O(n^2)$.
- $O(a^n)$, for some $a > 1$: exponential functions. Running time of Fibonnaci recursive algorithm is $O(2^n)$.

3 Shortcuts

Always relying on $O()$ definition to prove statements is tiring... Instead, we can use the following rules:

1. **Sum rule.** If $f_1(n) \in O(g(n))$ and $f_2(n) \in O(g(n))$ then $f_1(n) + f_2(n) \in O(g(n))$.

   Proof:

Prove that $\underbrace{log(n)}_{f_1(n)} + \underbrace{5 \cdot n}_{f_2(n)}$ is $O \underbrace{(n)}_{g(n)}$

$f_1(n)$ is $O(log(n)) \subset O(n)$ and $f_2(n)$ is $O(n) \implies f_1(n) + f_2(n)$ is $O(n)$

if $f_1(n) \in O(g(n))$ and $f_2(n) \in O(g(n))$, there must exist constants c_1, n_1, c_2, n_2 such that $f_1(n) \leq c_1 g(n)$ $\forall n \geq n_1$ and $f_2(n) \leq c_2 g(n)$ $\forall n \geq n_2$. Thus, if we pick $c_3 = c_1 + c_2$ and $n_3 = \max(n_1, n_2)$, we have that if $n \geq n_3$, then $f(n) + g(n) \leq c_1 g(n) + c_2 g(n) = (c_1 + c_2) g(n) = c_3 g(n)$. Thus $f_1(n) + f_2(n)$ is $O(g(n))$.

2. **Constant factors rule**. if $f_1(n) \in O(g(n))$ then $k f_1(n) \in O(g(n))$ for any *constant* k.

   Example: $n^3 + 10n^2 + \log(n) \in O(n^3)$ because
   $10n^2 \in O(n^2) \subset O(n^3)$ and $\log(n) \in O(\log(n)) \subset O(n^3)$.
   By rule (1), $n^3 + 10n^2 + \log(n) \in O(n^3)$
   Example: for any polynomial $p(n) = a_k n^k + a_{k-1} n^{k-1} + ... + a_1 n^1 + a_0 n^0$, we have $p(n) \in O(n^k)$.

3. **Product rule.** if $d(n) \in O(f(n))$ and $e(n) \in O(g(n))$, then $d(n) \cdot e(n) \in O(f(n) \cdot g(n))$.
   Example: $(1 + 10n) \cdot (2 \log(n) + 3) \in O(n \cdot \log(n))$, because...

4. $n^x \in O(a^n)$ for any fixed $x > 0$ and $a > 1$. However, $a^n \notin O(n^x)$ for any fixed $x > 0$ and $a > 1$.
   Example: $n^{1000} \in O(1.0001^n)$. However, the constants c and n_0 for which $n^{1000} \leq c \cdot 1.0001^n$ $\forall n \geq n_0$ are very large.

5. $\log(n^x) \in O(log(n))$ for any fixed $x > 0$.
   Proof: $\log(n^x) = x \log(n) \in O(log(n))$ (by rule 2).

6. $\log_a(n) \in O(log_b(n))$ for any fixed $a > 1, b > 1$.
   Proof: $\log_a(n) = log_b(n)/log_b(a) \in O(log_b(n))$ (by rule 2).

4 More shortcuts

Theorem: Let $f(n)$ and $g(n)$ be two non-negative functions. Then

1. if $\lim_{n \to +\infty} f(n)/g(n) = 0$, then $f(n) \in O(g(n))$ and $g(n) \notin O(f(n))$.
2. if $\lim_{n \to +\infty} f(n)/g(n) = x \neq 0$, then $f(n) \in O(g(n))$ and $g(n) \in O(f(n))$.
3. if $\lim_{n \to +\infty} f(n)/g(n) = +\infty$, then $g(n) \in O(f(n))$ and $f(n) \notin O(g(n))$.
4. if $\lim_{n \to +\infty} f(n)/g(n)$ does not exist, then we can't say anything

Reminder: l'Hopital rule:
$\lim_{n \to +\infty} f(n)/g(n) = \lim_{n \to +\infty} \frac{df(n)/dn}{dg(n)/dn}$
Example: Prove that $\log(n) \in O(\sqrt{n})$.