

# MATH 324: Statistics

Julian Lore

Last updated: April 3, 2018

Notes from Masoud Asgharian's Winter 2018 lectures.

## Contents

<b>1</b>	<b>01/09/18</b>	<b>3</b>
1.1	Overview - What is Statistics? . . . . .	3
1.2	Point Estimation . . . . .	5
<b>2</b>	<b>01/11/18</b>	<b>6</b>
2.1	Chebyshev/Tchbycshev's Inequality . . . . .	7
<b>3</b>	<b>01/16/18</b>	<b>12</b>
<b>4</b>	<b>01/18/18</b>	<b>19</b>
<b>5</b>	<b>01/15/18</b>	<b>24</b>
5.1	Confidence Intervals . . . . .	24
5.2	Small Sample Size . . . . .	29
<b>6</b>	<b>01/30/18</b>	<b>31</b>
6.1	Sample Size Determination . . . . .	34
<b>7</b>	<b>02/01/18</b>	<b>37</b>
7.1	Relative Efficiency . . . . .	38
7.2	Consistency . . . . .	41
<b>8</b>	<b>02/06/18</b>	<b>44</b>
8.1	Consistency . . . . .	44
8.2	Kolmogarov's Theorem (Law of Large Numbers) . . . . .	47

8.3 Sufficiency . . . . .	49
<b>9 02/08/18</b>	<b>50</b>
9.1 Sufficiency . . . . .	50
9.2 Fisher-Neyman Theorem . . . . .	54
<b>10 02/13/18</b>	<b>55</b>
10.1 The Rao-Blackwell Theorem . . . . .	55
10.2 Methods of Estimation . . . . .	60
<b>11 02/15/18</b>	<b>60</b>
11.1 MLE . . . . .	60
11.2 Method of Moments . . . . .	64
11.3 Information . . . . .	65
<b>12 02/20/18</b>	<b>66</b>
12.1 Testing Statistical Hypotheses (TSH) . . . . .	68
12.2 Common Large-Sample Tests . . . . .	69
<b>13 02/22/18</b>	<b>70</b>
13.1 TSH (Large Sample Case) . . . . .	70
13.2 p-Value (Fisher's perspective) . . . . .	74
<b>14 02/27/18</b>	<b>75</b>
14.1 $p$ -Value . . . . .	75
14.2 TSH and CI . . . . .	76
14.3 Small-Sample TSH . . . . .	77
14.3.1 Small-Sample TSH for $\sigma^2$ . . . . .	80
<b>15 03/01/18</b>	<b>82</b>
15.1 Neyman-Pearson Lemma (TSH) . . . . .	84
<b>16 03/15/18</b>	<b>87</b>
16.1 Likelihood Ratio Tests . . . . .	87
16.2 Regression Analysis . . . . .	91
<b>17 03/20/18</b>	<b>92</b>
17.1 Linear Regression . . . . .	92

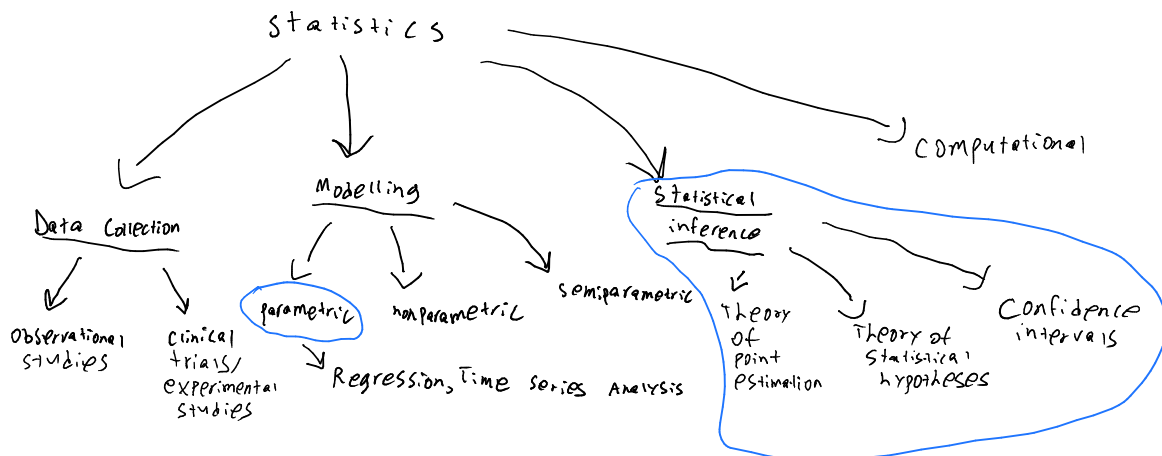
<b>18 03/22/18</b>	<b>93</b>
<b>19 03/27/18</b>	<b>98</b>
19.1 Analysis of Variance . . . . .	98
<b>20 03/29/18</b>	<b>100</b>
20.1 Analysis of Variance . . . . .	100
<b>21 04/03/18</b>	<b>103</b>
21.1 $\chi^2$ tests . . . . .	103
21.2 Bayesian Analysis . . . . .	104
21.3 Empirical Bayes . . . . .	106

## 1 01/09/18

**What we will cover this semester** Will essentially cover chapter 8,9,10. For chapter 11, he will give us his own notes. The first 6 sections of chapter 13 and a few sections from chapter 14. Occasionally we will go back to chapter 7 to revisit things like the t distribution. In 323, we made probabilistic models. Statistics is the breach in which we connect these models to real life. Otherwise, those are just models. A core part of data analysis and data sciences is statistics and computer science.

### 1.1 Overview - What is Statistics?

Inductive logic, we have a sample from the population we want to make inference about. With this data, we want to extend the results to the whole population. From small to large, sample to the population.



- Observational studies: we go to the population and make observations.
- Experimental studies: give test subjects something, i.e. give them cigarettes when trying to test for if cigarettes lead to cancer. Need to account for causation, other factors that can affect outcome. In order to do so we have to keep their diet and other factors controlled. We must also have some sort of randomization, we can't send all males to one group and all females to another, as males may have a tendency to smoke or something of the like. These are also called clinical trials.
- When we have data, the next step is modeling. May occasionally speak of this, but this is not part of the course. There are different approaches to modeling, can be split into 3 parts.
  - Parametric: the salary is distributed like a distribution (ex. Gamma), but we don't know the parameters. Take for example, we always know that the normal distribution is a bell curve, but we don't know where it's centered. Very useful, but we might have a miss-specification. How do we know our models are correct? Most of the time we will be talking about **parametric** models.
  - Semiparametric
  - Nonparametric: since we don't know if parametric models are correct, we make no assumption about the distribution. We just assume that  $X \sim F$ , all we assume about  $F$  is that it's continuous, nothing more. This is an infinite dimensional vector. Why? How do we know a function? We have a vector for  $F$ , like  $F(1), F(2), \dots$ . How do we approximate this?  $X_i \stackrel{iid}{\sim} F, i = 1, 2, \dots, n$ .  $n$  patients, with all the same distribution. So  $F(t) = P(X \leq t)$ . What does this

tell us? The proportion of time that  $x$  falls below  $t$ . So with  $n$  samples, how do we mimic this? We count the number of observations below  $t$ , i.e.  $\frac{\#X_i \leq t}{n}$ , which is an approximation of the above. This is an empirical observation. More mathematically:

$$\varepsilon(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

So we have  $\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^n \varepsilon(t - x_i)$ . This gives us a binomial distribution. But we are assuming they are all the same distribution.

Nonparametric approaches are good for functions of single variables, but not for multi variables, which is what semiparametric was made for.

- Bayesian inference: when you learn that  $X \sim N(\mu, \sigma^2)$ ,  $X$  is normally distributed and  $\mu$  is the average of the whole population. Bayes' approach says that these parameters are not constants, these are random variables themselves. Bayes did not look at probability as a frequentist approach, not the proportion of when something arrives (frequentist approach works when we have a huge sample). The other approach that Bayes had was an updating approach, that our parameters are unknown. This is good for when you have a stream of data (machine learning is a prime example). We have a lack of knowledge and then we update it using Bayesian's approach.  $\rightarrow X|\mu, \sigma^2 \sim N(\mu, \sigma^2)$ , i.e. the parameters are also normally distributed.

Most of the time we'll be at parametric modeling and statistical inference.

## 1.2 Point Estimation

What do we mean by point estimation? A scientific guess about the unknown parameter of the population. Consider the following situation:

$x_1, \dots, x_n \sim N(\mu, 1)$  (usually interested in the normal distribution, binomial and poisson). Suppose this is the IQ of high school graduates in Canada (the  $X_i$  are numbers). Why do we call this distribution normal? Because for a healthy population, most of the weight should be in the middle, just like the bell curve. The Normal distribution is especially important for modeling error. For insurance companies, we see at the tails that there aren't many large claims.

We want to find  $\mu$ . Recall that  $E(X_i) = \mu, i = 1, 2, \dots, n$  (if they all have the same observations, they have the same mean).

First, what is a point estimation? What properties should it have? If we know the value of  $\mu$ , we have the whole thing, can calculate everything. How do we estimate this? The whole

population is huge, so we take a sample part of the population, mimicking the real  $\mu$ , getting  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ .  $\bar{X}_n$  is useful, but  $\bar{X}_n - \mu$  isn't, as there's an unknown we have here.

**Statistic** A function of observations that does not depend on any unknown parameter.

**Ex**  $\bar{X}_n$  is a statistic.  $\bar{X}_n - \mu$  is not.

**Estimator** A statistic that aims at estimating an unknown parameter (we want to work with it). For example, if  $\mu$  moves from  $-\infty$  to  $\infty$ , we want to have an estimator that also has the same range, not one that is strictly positive. Example:  $\bar{X}_n$  is an estimator. However, consider:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

This is a statistic, but not an estimator, it always returns a positive value. Also, take for example in physics, where each measure has a unit of measurement. This statistic wouldn't even be the same unit, so it once again is a bad estimator.

When we take a mean and try to estimate it, the next step is to figure out how we quantify possible bias.

$$\varepsilon = |\bar{X}_n - \mu|$$

We can use Tchbycshev's inequality to put a bound on the error.

$$P(|X - \underbrace{E(X)}_{\mu_x}| > k \sqrt{\underbrace{Var(X)}_{\sigma_x^2}})$$

$$P(|X - \mu_x| > k\sigma_x) \leq \frac{1}{k^2}$$

Very useful, assume very little but get lots of information. One of the big hammers of probability and statistics. The only thing we assume here is the existence of the second moment.

Consider  $k = 3$ .

$$P(|X - \mu_x| > 3\sigma_x) \leq \frac{1}{9}$$

$$P(|X - \mu_x| \leq 3\sigma_x) \geq 1 - \frac{1}{9} \approx \%89$$

Without knowing anything else about the distribution, this tells us that about 89% of the population is within 3 times the variance of the mean.

## 2 01/11/18

Last lecture we learned about statistics, estimators and how we can measure deviation from the target and the estimation.

We had  $n$  random variables:  $x_1, \dots, x_n, \mu \rightarrow$  IQ in the population. We want to have a scientific guess of the average IQ (there are many more examples, like salary). Our  $n$  random variables are  $n$  random people chosen.

We then arrive at  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ , a scientific guess mimicking  $\mu$  but in the sample population.

How much deviation do we have?  $|\bar{X}_n - \mu|$

Since our observations are random, then  $\bar{X}_n$  will also be random, i.e.  $\bar{X}_n$  is a random variable itself. Each person gives us a different deviation, so we need a way to summarize all of this information, say, the expected value.

$$E[|\bar{X}_n - \mu|]$$

Another way to summarize it is with probability.

$$P(|\bar{X}_n - \mu| > \varepsilon)$$

What is the chance that what we produce is not within  $\varepsilon$  of the target? Often times we want to bound these things. What do you think might happen if instead of taking a sample of  $n = 50$ , we take  $n = 100$ ? As  $n$  increases, we should get closer to the target. But, the more samples I take, the more it'll cost me. So we want to have a balance. We want the distance from the target to be within some sort of value.

$$P(|\bar{X}_n - \mu| > \varepsilon) \leq \delta$$

Take for example a spam filter. Something is either spam or not spam. We start with messages and then start checking. We want to know how many messages we should check, i.e. how big our training set should be.

We will use Tchebyshev's Inequality for this!

## 2.1 Chebyshev/Tchbycshev's Inequality

Let  $X$  be a random variable. Suppose  $h(x)$  is a positive function (i.e. the range of this function consists of positive values). We can show that

$$P(h(x) \geq \lambda) \leq \frac{E[h(x)]}{\lambda}$$

for any  $\lambda > 0$ , if  $E[h(X)] \leq \infty$ , i.e. it exists. This is called **Markov's Inequality**

When we say that the expected value of a random variable exists, we mean  $E[|X|] < \infty$ .

When we talk about existence of a moment, we check the absolute value, but the actual value does not have an absolute value, it is just  $E[X]$ . Why? The trouble is when  $X$  can take positive and negative values and is not bound.

$$E[X] = \sum_{i=1}^{\infty} X_i P(X = x_i)$$

What if we have infinite values that we can take?

**Recall from Calculus**  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} < \infty$  is convergent, but not absolutely convergent because  $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \infty$ . Riemann has a result such that if a series is convergent but not absolutely convergent (like the example just mentioned), then it can converge to any real number (if we reorder the terms). Thus we don't like this and must check for absolute convergence for moments, or else the expected value will depend on the order we consider the numbers in.

Recall the theorem that says if we have a function of a random variable, we don't need its distribution, we can directly use the distribution of  $X$ . Note that integrals are another form of sums, we can use similar notation with  $x$  as a subscript to denote ranging over all  $x$ .

$$E[h(x)] = \int_x h(x) f_x(x) dx = \left( \int_{x:h(x) \geq \lambda} + \int_{x:h(x) < \lambda} h(x) f_x(x) dx \right)$$

(Note that the two integrals both apply on the right side)

$$\geq \int_{x:h(x) \geq \lambda} h(x) f_x(x) dx \geq \lambda \int_{x:h(x) \geq \lambda} f_x(x) dx = \lambda P(h(x) \geq \lambda)$$

So what did we get?

$$E[h(x)] \geq \lambda P(h(x) \geq \lambda)$$



$$P(h(x) \geq \lambda) \leq \frac{E[h(x)]}{\lambda}$$

Now consider:  $h(x) = (x - \mu)^2$ . Then what do we have?

$$P(|x - \mu| \geq \lambda) = P([x - \mu]^2 \geq \lambda^2) \stackrel{\text{By Markov's Inequality}}{\leq} \frac{E[(x - \mu)^2]}{\lambda^2}$$

$$P(|x - \mu| \geq \lambda) \leq \frac{\text{Var}(x)}{\lambda^2}$$

Replace  $\lambda$  by  $k\sigma_x$  where  $\sigma_x = \sqrt{\text{Var}(x)}$

$k$  is a constant, so we get:

$$P(|x - \mu| \geq k\sigma_x) \leq \frac{\text{Var}(x)}{k^2\sigma_x^2} = \frac{\text{Var}(x)}{k^2\text{Var}(x)} = \frac{1}{k^2}$$

This is **Tchbycshev's Inequality**. For  $k = 3$  we have:

$$P(|x - \mu| \geq 3\sigma_x) \leq \frac{1}{9}$$

$$P(|x - \mu| \leq 3\sigma_x) \geq \frac{8}{9} \approx 88\%$$

What does this say? For any random variable with 2 moments, 88% of the values fall within  $3\sigma_x$ s from the center of gravity (mean). This is a very crude lower bound that required almost no assumptions, all we need is that  $\mu = E(x)$  and  $\sigma_x^2 = \text{Var}(x)$  and the existence of the second moment.

Back to where we were before with  $|\bar{X}_n - \mu|$ :

$$P(|\bar{X}_n - \mu| > 3\sigma_{\bar{X}_n}) \leq \frac{1}{9}$$

Note that here, it must be true that  $\mu = E(\bar{X}_n)$ . Is this true?

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

$$E[\bar{X}_n] = E\left[\frac{1}{n} \sum_{i=1}^n X_i\right]$$

Recall:  $E[cY] = cE[Y]$ , think of expected values like integrals and sums, they have the same properties.

$$= \frac{1}{n} E \left[ \sum_{i=1}^n X_i \right] = \frac{1}{n} \sum_{i=1}^n E[X_i]$$

Remember that  $X_1, \dots, X_n \sim F$ , i.e. they all have the same distribution! So  $E[X_i] = \mu, i = 1, 2, \dots, n$

$$= \frac{1}{n} \sum_{i=1}^n \mu = \mu$$

Now how do we use this in practice? If everyone is sent to the population and asked to take a sample of size 10 (the same size for everyone) and everyone makes their own  $\bar{X}_n$ , their own sample average and then we take the average of all the sample averages and we obtain the actual average of the population, i.e. this is an average of averages (this is difficult though, as we need all the possible averages of size 10; in practice we only use one sample, more on this later).

**Example** Suppose we have the following 0-1 random variable representing what people will vote for

$$X_i = \begin{cases} 1 & \text{if NDP} \\ 0 & \text{otherwise} \end{cases}$$

We know that  $X_i \sim \text{Bernoulli}(p)$ ,  $p = P(X_i = 1)$

$$X_1, \dots, X_n \sim p = (P(X_i = 1), i = 1, 2, \dots, n)$$

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i = \hat{p}_n$$

**Side Note about Picking with Replacement**

$$P(X_2 = 1 | X_1 = 1) = \frac{M-1}{N-1}$$

(after taking 1 in favor of NDP, where  $N$  is total population size and  $M$  is total number of people in favor of NDP) Using Total Probability Theorem we get:

$$\begin{aligned} P(X_2 = 1) &= P(X_2 = 1|X_1 = 1)P(X_1 = 1) + P(X_2 = 1|X_1 = 0)P(X_1 = 0) \\ &= \frac{M-1}{N-1} \cdot \frac{M}{N} + \frac{M}{N-1} \left(1 - \frac{M}{N}\right) = \frac{M}{N} \\ P(X_2 = 1) &= \frac{M}{N} \\ P(X_2 = 1|X_1 = 1) &= \frac{M-1}{N-1} \end{aligned}$$

These are identically distributed, but not independent, this replacement is what differs hypergeometric from binomial. But when the sample size is very large we can just use binomial (also we don't ask someone who they're voting for twice).

Note we just showed that

$$P(|\bar{X}_n - \mu| > k\sigma_{\bar{X}_n}) \leq \frac{1}{k^2}$$

Recall that if  $X_i \sim \text{Bernoulli}(p)$ , then  $E[X_i] = p$

$$P(|\bar{X}_n - \mu| \geq k\sigma_{\bar{X}_n}) = P(|\hat{p}_n - p| > k\sigma_{\hat{p}_n})$$

Recall that:

$$\begin{aligned} \sigma_{\bar{X}_n}^2 &= \text{Var}(\bar{X}_n) \\ \text{Var}(cY) &= c^2 \text{Var}(Y) \\ \text{Var}(X \pm Y) &= \text{Var}(X) + \text{Var}(Y) \pm 2\text{Cov}(X, Y) \end{aligned}$$

If  $\text{Cov}(X, Y) = 0$ ,  $\text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y)$ . If  $X$  and  $Y$  are independent,  $\text{Cov}(X, Y) = 0$ .

$$\begin{aligned} \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\ X \perp\!\!\!\perp Y &\implies E[g(x)h(y)] = E[g(x)]E[h(y)] \\ X \perp\!\!\!\perp Y &\implies E(XY) = E(X)E(Y) \text{ therefore } \text{Cov}(X, Y) = 0 \end{aligned}$$

Thus,

$$\text{Var}(\bar{X}_n) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^n X_i\right)$$

So,

$$\frac{1}{n^2} \text{Var}\left(\sum_{i=1}^n X_i\right) \stackrel{\text{Assuming that } X_i \text{ s are ind}}{=} \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i)$$

Remember that  $X_i \sim F$

$$\stackrel{\text{Assuming that } X_i \text{ s have the same variance}}{=} \frac{1}{n^2} \cdot n \text{Var}(X_i) = \frac{\text{Var}(x)}{n}$$

Thus we have:

$$\text{Var}(\bar{X}_n) = \frac{\text{Var}(X)}{n}$$

i.e. variance gets smaller and smaller as the population size increases, so  $\bar{X}_n$  gets closer and closer to its center.

$$\text{Var}(\hat{p}_n) = \text{Var}(\bar{X}_n) = \frac{\text{Var}(X)}{n}$$

Where  $X \sim \text{Bernoulli}(p)$ , so  $\text{Var}(X) = p(1-p)$

$$\text{Var}(\hat{p}_n) = \frac{p(1-p)}{n}$$

$$P\left(|\hat{p}_n - p| > k \sqrt{\frac{p(1-p)}{n}}\right) \leq \frac{1}{k^2}$$

Now, back to the original problem:

$$P(|\hat{p}_n - p| > \varepsilon) \leq \delta$$

where  $\varepsilon, \delta$  are known (and small values). Next class we will see how to choose values to satisfy this.

### 3 01/16/18

**Recall** Last class we were talking about voting. We got to the point of estimating thousands of votes. We want to estimate the amount of Canadians voting NDP.

$$X_i = \begin{cases} 1 & \text{NDP} \\ 0 & \text{otherwise} \end{cases}$$

$i = 1, \dots, n$

$p(X_i = 1) = p, i = 1, \dots, n$ , we want to estimate this. Why can we make the assumption that this is  $p$ ? If we chose someone randomly, the probability that one of them is voting NDP is  $p$ , the same for each sample. We try not to get samples from the same family, so that the variables remain independent, as they might have the same views. We use the following notation to signify independence:

$$\perp\!\!\!\perp_{i=1}^n X_i$$

$$\hat{p}_n = \frac{1}{n} \underbrace{\sum_{i=1}^n x_i}_{\substack{\text{Number of people} \\ \text{in sample voting NDP}}} = \bar{X}_n$$

$$\varepsilon = |\hat{p}_n - p|$$

$$p(|\hat{p}_n - p| > \overbrace{\delta}^{\text{given}}) \leq \frac{E[|\hat{p}_n - p|^2]}{\delta^2}$$

$$E[|\hat{p}_n - p|^2]$$

$$\begin{aligned} E(\hat{p}_n) &= E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] \\ &= \frac{1}{n} E\left[\sum_{i=1}^n X_i\right] \\ &= \frac{1}{n} \sum_{i=1}^n E(X_i) \end{aligned}$$

$$\begin{aligned} E(X_i) &= 1 \cdot p(X_i = 1) + 0 \cdot p(X_i = 0) \\ &= 1 \cdot p + 0 \cdot (1 - p) \\ &= p(\text{Expected value of a Bernoulli variable}) \end{aligned}$$

$$\implies E(\hat{p}_n) = \frac{1}{n} \sum_{i=1}^n p = \frac{1}{n} np = p$$

Expectation of variable minus its expectation squared is variance:

$$\begin{aligned} E[|\hat{p}_n - p|^2] &= \text{Var}(\hat{p}_n) \\ \text{Var}(\hat{p}_n) &= \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \\ &= \left(\frac{1}{n}\right)^2 \text{Var}\left(\sum_{i=1}^n X_i\right) \\ \text{Var}\left(\sum_{i=1}^n X_i\right) &\stackrel{\text{Thm 5.12(b), p271}}{=} \sum_{i=1}^n \text{Var}(X_i) + 2 \sum \sum_{i \leq i < j \leq n} \text{Cov}(X_i, X_j) \end{aligned}$$

Recall that  $\perp \sum_{i=1}^n X_i \implies \text{Cov}(X_i, X_j) = 0, \forall i \neq j$ , so all the terms in the double sum will be 0.

$$\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i)$$

Now we must calculate the variance of  $X_i$ . We already have the first moment:

$$\begin{aligned} \text{Var}(X_i) &= E(X_i^2) - \underbrace{[E(X_i)]^2}_p \\ E(X_i^2) &= 1^2 \cdot p(X_i = 1) + 0^2 \cdot p(X_i = 0) \\ &= 1 \cdot p + 0 \cdot (1 - p) = p \end{aligned}$$

$$\begin{aligned} \text{Var}(X_i) &= p - [p]^2 = p - p^2 = p(1 - p) \\ \text{Var}\left(\sum_{i=1}^n X_i\right) &= \sum_{i=1}^n p(1 - p) = np(1 - p) \\ \text{Var}(\hat{p}_n) &= \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^n X_i\right) = \frac{1}{n^2} np(1 - p) = \frac{p(1 - p)}{n} \end{aligned}$$

Now, using Chebyshev's inequality:

$$\begin{aligned} P(|\hat{p}_n - p| > \delta) &\leq \frac{E[|\hat{p}_n - p|^2]}{\delta^2} = \frac{\text{Var}(\hat{p}_n)}{\delta^2} \\ P(|\hat{p}_n - p| > \delta) &\leq \frac{E[|\hat{p}_n - p|^2]}{\delta^2} = \frac{p(1 - p)}{n\delta^2} \end{aligned}$$

Is this useful? As  $n$  tends to  $\infty$ , the bound goes to 0, meaning, no matter what  $\delta$  you choose here, the chance gets smaller and smaller, i.e. the bigger  $n$  gets, the more were on track, heading to the right direction. But we don't know  $p$ , so how is this useful? But we know something else:

$$p(1-p) \leq \frac{1}{4}$$

Why?

$$\begin{aligned} f(x) &= x(1-x) \\ \frac{df}{dx} &= 1-2x = 0 \implies x = \frac{1}{2} \\ \frac{d^2f}{dx^2} &= -2 \end{aligned}$$

Thus we get a concave function with  $f\left(\frac{1}{2}\right)$  as the max.

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \frac{1}{2} \left(1 - \frac{1}{2}\right) = \frac{1}{4} \\ p(|\hat{p}_n - p| > \delta) &\leq \frac{1}{4n\delta^2} \end{aligned}$$

So now for any  $\delta$  we can determine how far off our estimate will be. Another application is sample size determination with  $\delta = 0.01$ , we want the deviation from the target to be less than 1% and we don't want it to fail (from being in the range) more than 5% of the time.

$$\begin{aligned} \frac{1}{4n\delta^2} &= 0.05 \\ n &= \frac{1}{4(0.05)(0.01)^2} \end{aligned}$$

This is conservative, as we've put a bound without knowing  $p$  and now we know how many samples we should obtain.

The bounds like Chebyshev's are very crude inequalities with no assumptions, when we have things like binary variables, we can make more assumptions and get better bounds, won't need to take as many samples. If we want to train a system (i.e. machine learning or spam filtering), if it's not expensive we don't care as much, but in other cases like sampling

patients, we might want to get better bounds so we can keep the cost lower.

$$p(|\bar{X}_n - \mu| > \delta)$$

We might want to minimize the distance between  $\bar{X}_n$  and  $\mu$ . In Euclidean space, we square the difference for the distance.

$$E[|\bar{X}_n - \mu|^2] = MSE(\bar{X}_n) \text{ (Mean Squared Error)}$$

Is it possible to decrease this error to 0? Aside from  $n$  being very large (i.e. everyone in the population; a census). It can be 0 if  $\mu = E(X)$ , meaning that  $Var(X) = 0$ , i.e. it is the same everywhere.

$$Var(X) = E[(X - \mu_x)^2]$$

Now let's look at MSE more.

$$MSE(\hat{\theta}_n) = E[(\hat{\theta}_n - \theta)^2]$$

Where MSE is the estimator and  $\theta$  is the estimand.

$$\begin{aligned} &= E[\{(\hat{\theta}_n - E(\hat{\theta}_n)) + E(\hat{\theta}_n - \theta)\}^2] \\ &= E[(\hat{\theta}_n)^2 + (E(\hat{\theta}_n - \theta))^2 + 2(\hat{\theta}_n - E(\hat{\theta}_n - E(\hat{\theta}_n)))(E(\hat{\theta}_n - \theta))] \\ &= E[(\hat{\theta}_n - E(\hat{\theta}_n))^2] + E[(E(\hat{\theta}_n) - \theta)^2] + 2E[(\hat{\theta}_n - E(\hat{\theta}_n))(\overbrace{E(\hat{\theta}_n)}^{\text{constant}} - \overbrace{\theta}^{\text{constant}})] \\ E(X - \mu_x) &= 0 \\ &= Var(\hat{\theta}_n) + \underbrace{[E(\hat{\theta}_n - \theta)]^2}_{Bias(\hat{\theta}_n)} \end{aligned}$$

Now what does Variance and Bias measure? Variance measures the reliability, the larger the variance, the less reliable our data is. If someone does the same type of experiment (in another country, with the same exact procedure) as me and comes up with another estimated value for the estimand, then this means we don't have much credibility. We want to minimize variance.

What about bias? If  $Bias(\hat{\theta}_n) = 0 \implies \hat{\theta}_n$  is an unbiased estimator. If the average of the estimator is equal to the target, then we say the estimator is an unbiased estimator. Unbiased estimators aren't a big topic but **bias is bad**. Say we want to know the salary of all Canadians. If we see that every time we take a sample and we get it over the target or



under the target, then we see that we are over/underestimating, so we need to make sure the bias is (if possible) 0. But sometimes we can allow for a bit of bias but the variance goes down and makes the MSE go down dramatically as a whole. Often we work with estimators that are biased, because if we don't allow for a bit of bias, then the variance remains very high and so does the MSE.

**Unbiased estimator**  $\hat{\theta}_n$  is said to be an unbiased estimator of  $\theta$  if  $E(\hat{\theta}_n) = \theta$ .

**Example**  $X_i \text{ iid } \sim N(\mu, \sigma^2)$ ,  $i = 1, \dots, n$  with  $\mu$  and  $\sigma^2$  both unknown.

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \implies E(\bar{X}_n) = \mu$$

$$\text{Var}(\bar{X}_n) = \frac{1}{n} \sum_{i=1}^n \text{Var}(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$\text{MSE}(\bar{X}_n) = \text{Var}(\bar{X}_n) + [\text{Bias}(\bar{X}_n)]^2 = \frac{\sigma^2}{n} + 0^2 = \frac{\sigma^2}{n}$$

i.e. for very large  $n$  our results are unbiased.

Can we extend this example?

---

Suppose  $X_1, X_2, \dots, X_n$  have the same mean  $\mu$ . Then

$$E(\bar{X}_n) = E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n \overbrace{E(X_i)}^{\mu} = \mu$$

We don't need normalized variables, just need the same distribution with the same mean, very little assumptions yet we can tell that it is unbiased.

Suppose further that  $\text{cov}(X_i, X_j) = 0, i \neq j$

Then

$$\begin{aligned} \text{Var}(\bar{X}_n) &= \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \\ &= \frac{1}{n^2} \left\{ \sum_{i=1}^n \text{Var}(X_i) + 2 \sum \sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j) \right\} \end{aligned}$$

$$= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i)$$

Assume that  $\text{Var}(X_i) = \sigma^2, i = 1, \dots, n$

$$\text{Then } \text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$$

$$\text{Then } \text{MSE}(\bar{X}_n) = \frac{\sigma^2}{n}$$

where  $\sigma^2 = \text{Var}(X_i), i = 1, \dots, n$ . i.e. if  $X_i$ 's have the same mean  $\mu$  & Variance  $\sigma^2$  and  $\text{Cov}(X_i, X_j) = 0, i \neq j$ . Very minimal assumptions, yet we know what  $\text{MSE}$  is.

This is called **Stein's paradox**.

$$X \sim N(\mu_X, 1), i = 1, \dots, n \rightarrow \bar{X}_n$$

$$Y \sim N(\mu_Y, 1), i = 1, \dots, n \rightarrow \bar{Y}_n$$

$$Z \sim N(\mu_Z, 1), i = 1, \dots, n \rightarrow \bar{Z}_n$$

**Admissibility** We say an estimator  $\hat{\theta}_n$  is admissible if there is no other estimator  $\tilde{\theta}$  such that (note that MSE is always a function of a parameter)

$$\text{MSE}(\tilde{\theta}) \leq \text{MSE}(\hat{\theta})$$

i.e. your estimator is always better than someone else's estimator. Now the paradox here is that each pairwise pairs of these 3 (vectors with 2 components, i.e.  $(\bar{X}_n, \bar{Y}_n)$ ) random variables have admissibility, but as soon as you introduce all 3 random variables, you no longer have admissibility.

So we have shown that the MSE of  $\bar{X}_n$  is unbiased with minimal assumptions. But we talked about several estimators. Are there any other estimators other than  $\bar{X}_n$ ? Actually, we have uncountably infinite estimators. Suppose  $X_i$  have the same mean  $\mu$ .

$$\tilde{X}_n = \sum_{i=1}^n c_i X_i \text{ where } \sum_{i=1}^n c_i = 1$$

$$E(\tilde{X}_n) = E\left[\sum_{i=1}^n c_i X_i\right] = \sum_{i=1}^n E[c_i X_i] = \sum_{i=1}^n c_i E(X_i) = \sum_{i=1}^n c_i \mu = \mu \sum_{i=1}^n c_i = \mu$$

$$MSE(\tilde{X}_n) = Var(\tilde{X}_n) + \underbrace{Bias^2(\tilde{X}_n)}_0$$

Next class we will show that we can minimize the variance if we set  $c_i = \frac{1}{n}$   
 Min  $MSE(\tilde{X}_n)$  with  $c = (c_1, \dots, c_n)$  such that  $\sum_{i=1}^n c_i = 1$

## 4 01/18/18

**Recall** We we're talking about number of estimators and showed that there can be uncountably infinite estimators. So the question was, how can we chose amongst uncountably infinite estimators? So we assumed that we wanted to confine ourselves to unbiased estimators. We then looked at sample average, which gives equal weight to each sample. A familiar example is GPA, which has weight based on the amount of credits the course is. We normalize the sample average by making the sum add up to 1. We noticed that if all our observations have the same average  $\mu$ , then so does our estimator. So how do we chose our estimator?

$$\tilde{X}_{n,\vec{c}} = \sum_{i=1}^n c_i x_i \text{ where } \vec{c} = (c_1, \dots, c_n)$$

$$\{\tilde{X}_{n,\vec{c}} : \vec{c} \in \mathbb{R}^n, \sum_{i=1}^n c_i = 1\}$$

One of the things we used to check how good an estimator was is MSE:

$$MSE(\vec{X}_{n,\vec{c}}) = Var(\tilde{X}_{n,\vec{c}}) + \underbrace{[Bias(\tilde{X}_{n,\vec{c}})]^2}_{E(\tilde{X}_{n,\vec{c}}) - \mu}$$

$$MSE(\tilde{X}_{n,\vec{c}}) = Var(\tilde{X}_{n,\vec{c}})$$

So we want:

$$\min_{\vec{c} \in \mathbb{R}} Var(\tilde{X}_{n,\vec{c}}) \tag{1}$$

subject to  $\sum_{i=1}^n c_i = 1$  (constraint)

This is a generalization of problem 8.6 on page 394.

Suppose  $X_i$ 's have the same mean  $\mu$  and variance  $\sigma^2$  and  $Cov(X_i, X_j) = 0, i \neq j$  (i.e. they are orthogonal). Then the solution to (1) is  $\bar{X}_n$ , i.e.  $c_i = \frac{1}{n}, i = 1, \dots, n$ . Now how do we solve this? We can use calculus to find a minimum. How many variables do we have here?  $n - 1$  variables, as the last one is specified by the constraint. The way we do this is with the

Lagrange Method.

So we note that problem (1) is equivalent to

$$\min_{\vec{c} \in \mathbb{R}} \{Var(\tilde{X}_{n,\vec{c}}) + \lambda \left( \sum_{i=1}^n c_i - 1 \right)\} \quad (2)$$

where  $\lambda$  is the lagrange multiplier. How do we show that these two mathematical programs are the same? We show that if we find a  $\vec{c}^*$  that minimizes (1), it must also minimize (2) and vice versa. How do we solve this?

$$\begin{aligned} Var(\tilde{X}_{n,\vec{c}}) &= Var\left(\sum_{i=1}^n c_i X_i\right) \stackrel{\text{Thm 5.12(b), p271}}{=} \sum_{i=1}^n Var(c_i X_i) + 2 \sum \sum_{1 \leq i \leq j \leq n} Cov(c_i X_i, c_j X_j) \\ &= \sum_{i=1}^n c_i^2 Var(X_i) + 2 \sum \sum_{1 \leq i \leq j \leq n} c_i c_j Cov(X_i, X_j) \\ &= \sum_{i=1}^n c_i^2 \overbrace{Var(X_i)}^{\sigma^2} = \sigma^2 \sum_{i=1}^n c_i^2 \end{aligned}$$

Now our problem (2) is equivalent to

$$\min_{\vec{c} \in \mathbb{R}} \underbrace{\left\{ \sigma^2 \sum_{i=1}^n c_i^2 + \lambda \left( \sum_{i=1}^n c_i - 1 \right) \right\}}_{\varphi(\vec{c})}$$

$$\begin{aligned} \frac{\partial}{\partial c_i} \varphi(\vec{c}) &= 2\sigma^2 c_i + \lambda, i = 1, 2, \dots, n \\ \frac{\partial}{\partial \lambda} \varphi(\vec{c}) &= \sum_{i=1}^n c_i - 1 \end{aligned}$$

So now we equate them to 0:

$$\begin{cases} \frac{\partial}{\partial c_i} \varphi(\vec{c}) = 0, i = 1, \dots, n \\ \frac{\partial}{\partial \lambda} \varphi(\vec{c}) = 0 \rightarrow \sum_{i=1}^n c_i = 1 \end{cases}$$

$$\begin{aligned} \frac{\partial}{\partial c_i} \varphi(\vec{c}) &= 2\sigma^2 c_i + \lambda = 0 \\ c_i &= -\frac{\lambda}{2\sigma^2}, i = 1, 2, \dots, n \end{aligned}$$

$$\frac{\partial \varphi_{\lambda}(\vec{c})}{\partial \lambda} = \sum_{i=1}^n c_i - 1 = 0 \implies \sum_{i=1}^n -\frac{\lambda}{2\sigma^2} = 1$$

$$\lambda = \frac{1}{-\sum_{i=1}^n \frac{1}{2\sigma^2}} = -\frac{2\sigma^2}{n}$$

$$\begin{cases} c_i = -\frac{\lambda}{2\sigma^2}, i = 1, \dots, n \\ \lambda = -\frac{2\sigma^2}{n} \end{cases}$$

$$\rightarrow c_i = \frac{1}{n}, i = 1, \dots, n$$

$$\vec{c} = (c_1, c_2, \dots, c_n)$$

How do we know this is the minimum? We could take the second derivative and look at the resulting matrix, but we won't be going into details for that in this class.

You don't need to check the matrix, but you should be able to take partial derivatives to minimize something (might be less variables, like 8.6), add the constraint to the objective function.

Essentially what we did was minimize  $MSE(\tilde{X}_{n,\vec{c}})$ . For any vector  $\vec{c}$  we have an estimator and this estimator has a distance, so we want to minimize the distance by choosing the components of  $\vec{c}$  subject to  $\sum_{i=1}^n c_i = 1$  and we found that the solution is the sample average where we give equal weight to each sample, i.e.

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

with  $c_i = \frac{1}{n}, i = 1, \dots, n$ .

So we have constrained ourselves to unbiased estimators that are linear combinations:  $\tilde{X}_{n,\vec{c}} = \sum_{i=1}^n c_i X_i$

How do we know that there isn't a better class of distributions that aren't linear combinations? We will see something in chapter 9.

So far we have looked at  $E(X) = \mu$  for population averages. But we can also look at other parameters that are important to us, like variance.

So we have a bunch of estimates and want to estimate the variation. The estimation of this variation is very important.  $Var(X) = \sigma_x^2$ , because sample size requires this and greatly affects the amount of money required to conduct an experiment. How can we solve this?

Suppose we have a sample from a population (no distribution assumption):

$$X_1, \dots, X_n$$

$$E(X_i) = \mu, i = 1, \dots, n$$

$$Var(X_i) = \sigma^2, i = 1, \dots, n$$

$$Cov(X_i, X_j) = 0, i \neq j$$

What would be a natural estimator for this variance? **Sample variance.**

$$Var(X) = E[(X - \mu)^2]$$

$$S_{n,*}^2 \text{ (sample variance)} = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

Do we have the following equality? This would imply unbiasedness.

$$\begin{aligned}
 E[S_{n,*}^2] &\stackrel{?}{=} \sigma^2 \\
 (X_i - \mu)^2 &= [(X_i - \bar{X}_n) + (\bar{X}_n - \mu)]^2 \\
 (X_i - \mu)^2 &= (X_i - \bar{X}_n)^2 + (\bar{X}_n - \mu)^2 + 2(X_i - \bar{X}_n)(\bar{X}_n - \mu), i = 1, \dots, n \\
 \sum_{i=1}^n (X_i - \mu)^2 &= \sum_{i=1}^n (X_i - \bar{X}_n)^2 + \sum_{i=1}^n (\bar{X}_n - \mu)^2 + 2 \sum_{i=1}^n (X_i - \bar{X}_n)(\bar{X}_n - \mu) \\
 &= \sum_{i=1}^n (X_i - \bar{X}_n)^2 + n(\bar{X}_n - \mu)^2 + 2(\bar{X}_n - \mu) \underbrace{\sum_{i=1}^n (X_i - \bar{X}_n)}_{\sum_x i - n\bar{X}_n = 0} \\
 \sum_{i=1}^n (X_i - \mu)^2 &= \sum_{i=1}^n (X_i - \bar{X}_n)^2 + n(\bar{X}_n - \mu)^2 \\
 E \left[ \sum_{i=1}^n (X_i - \mu)^2 \right] &= E \left[ \sum_{i=1}^n (X_i - \bar{X}_n)^2 \right] + E[n(\bar{X}_n - \mu)^2] \\
 \sum_{i=1}^n E(X_i - \mu)^2 &= E[nS_{n,*}^2] + E[n(\bar{X}_n - \mu)^2] \\
 \sum_{i=1}^n \underbrace{Var(X_i)}_{\sigma^2} &= nE(S_{n,*}^2) + nE[(\bar{X}_n - \mu)^2] \\
 n\sigma^2 &= nE(S_{n,*}^2) + n \underbrace{E[(\bar{X}_n - \mu)^2]}_{Var(\bar{X}_n)} \\
 \sigma^2 &= E(S_{n,*}^2) + Var(\bar{X}_n)
 \end{aligned}$$

$$\sigma^2 = E(S_{n,*}^2) + \frac{\sigma^2}{n}$$

$$E(S_{n,*}^2) = \sigma^2 \left(1 - \frac{1}{n}\right)$$

So is this unbiased? No! It's not equal to our target. When  $n$  gets really big, the other term will be negligible, but what if we want something unbiased without that condition? Multiply by the reciprocal!

$$\frac{n}{n-1} E(S_{n,*}^2) = \sigma^2$$

$$E\left(\frac{n}{n-1} S_{n,*}^2\right) = \sigma^2$$

$$S_n^2 = \frac{n}{n-1} \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

$$= \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

So now we have something unbiased. But why is sample variance happen to be biased? Why do we need  $n-1$ ? What is causing us trouble?  $\bar{X}_n$ . If we had  $\mu$  instead, we wouldn't have this problem.

Consider the following as vectors:

$$V = \text{Span}\{X_i - \bar{X}_n, i = 1, \dots, n\}, \sum_{i=1}^n (X_i - \bar{X}_n) = 0$$

Not all the vectors are linearly independent, so we lose 1 degree of freedom, the reason why we have  $n-1$ . But now, if we compare this with:

$$W = \text{Span}\{X_i - \mu, i = 1, \dots, n\}$$

$$\dim(V) = n-1, \dim(W) = n$$

The  $X_i$  do not depend linearly on  $\mu$ , if you sum up all the terms it won't be 0, because  $\mu$  is the population average, not the sample average, we are just shifting by a constant  $\mu$  and this will not affect covariance.

So far we have looked at the parameters:  $\mu, p, \sigma^2$ .

What happens if we have 2 populations, i.e. men  $(X_1, \dots, X_n)$  with mean  $\mu_M$  and women

$(Y_1, \dots, Y_n)$  with mean  $\mu_W$ . Say we want to compare the salaries of men and women. We may have the salaries of everyone in our university, but not everyone in Canada. So what's a natural estimator for  $\mu_M$  and  $\mu_w$ ?

$$\bar{X}_M = \frac{1}{m} \sum_{i=1}^m X_i$$

$$\bar{Y}_W = \frac{1}{n} \sum_{i=1}^n Y_i$$

**Exercise** Show that  $E(\bar{X}_M - \bar{Y}_W) = \mu_M - \mu_W$  (this should be immediately available to you from what we've done). Note that we assume they are all coming from the same population (i.e. they all have the same mean).

Furthermore, also show:

Suppose  $Cov(X_i, X_j) = 0, Cov(Y_i, Y_j) = 0, i \neq j$  and  $X$ s and  $Y$ s are independent (don't really need this condition, extra). Find  $Var(\bar{X}_m - \bar{Y}_n)$  (these are all immediately available to you, just work through them)

The formula that will be useful to you (Thm 5.12):

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) \pm 2ab Cov(X, Y)$$

The same idea applies if we want to see the proportion of a kind of population that votes for someone and more applications.

## 5 01/15/18

### 5.1 Confidence Intervals

Statistics is inductive logic, meaning that we have a sample of the whole population. Using this sample we want to make inference about the whole population. This is unlike theorems that are deductive, where we go to specifics from general things, we are going up, generalizing the sample to the whole population.

Suppose we have  $X_i \stackrel{iid}{\sim} N(\mu, 1), i = 1, \dots, n$  (IQ of high school graduates). We want to estimate the IQ for the whole population of Canada.

We've learned that  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$  is a reasonable point estimate. But we want to do an interval estimate.



**Pivotal quantity** A function of the observations  $X_1, \dots, X_n$  and some unknown parameters, ideally just the parameters of interest, such that the distribution of this function does **NOT** depend on any unknown parameter.

What do we mean? What is the distribution of  $\bar{X}_n$ ?  
 $\bar{X}_n \sim N(\mu, \frac{1}{n})$ . How do we get this? Remember that we had different approaches to getting distributions of a function of random variables.

- Transformation (essentially the change of variables theorem from Calculus, when you integrate a function and change the variable).
- Method of distribution (try to relate the distribution to an existing distribution)
- Method of mgf (most suitable when we have iid samples and want to find the distribution of the sum of random variables, exactly what we have here with  $\bar{X}_n$ )

How does the method of mgf work?

$$S = \sum_{i=1}^n X_i$$

$$m_S(t) = E[e^{tS}] = E\left[e^{t\sum_{i=1}^n X_i}\right] = E\left[\prod_{i=1}^n e^{tX_i}\right]$$

$$\text{Independence} \implies = \prod_{i=1}^n E[e^{tX_i}] = \prod_{i=1}^n m_{X_i}(t)$$

$$\text{Identically dist} \implies = \prod_{i=1}^n m(t) = [m(t)]^n$$

For example, if  $X_i \stackrel{iid}{\sim} N(\mu, \sigma^2)$  then

$$m_{X_i}(t) = \left[e^{\mu t + \frac{\sigma^2 t^2}{2}}\right]^n = e^{n\mu t + \frac{n\sigma^2 t^2}{2}}$$

Now if we call  $n\mu = \mu_*$  and  $n\sigma^2 = \sigma_*^2$

$$= e^{\mu_* t + \frac{\sigma_*^2 t^2}{2}}$$

Now this is the same mgf as a normal distribution with

$$S \sim N(\mu_*, \sigma_*^2) = N(n\mu, n\sigma^2)$$

$$\bar{X}_n = \frac{1}{n}S$$

$$\begin{aligned}
 m_{\bar{X}_n}(t) &= E[e^{t\bar{X}_n}] = E[e^{t\frac{1}{n}S}] \stackrel{t\frac{1}{n}=t_*}{=} E[e^{t_*S}] = m_S(t_*) = e^{n\mu t_* + \frac{n\sigma^2 t_*^2}{2}} \\
 &= e^{n\mu \frac{t}{n} + \frac{n\sigma^2 (\frac{t}{n})^2}{2}} = e^{\mu t + \frac{\sigma^2 t^2}{2n}}
 \end{aligned}$$

Now this tells us  $\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$  if  $X_i \stackrel{iid}{\sim} N(\mu, \sigma^2)$ .

In our example,  $\sigma^2 = 1$ , so  $\bar{X}_n \sim N(\mu, \frac{1}{n})$ . Standardizing we get:

$$Z = \underbrace{\frac{\bar{X}_n - \mu}{\sqrt{\frac{1}{n}}}}_{\text{Pivotal quantity}} \sim N(0, 1)$$

This distribution itself does not depend on any unknown parameter, it is just the standard normal distribution. All standardizations of normal distributions make them into pivotal quantities.

So we know that:

$$P(|Z| \leq 1.96) = 0.95$$

Now let's try and plug in for  $Z$ :

$$\begin{aligned}
 P\left(\left|\frac{\bar{X}_n - \mu}{\sqrt{\frac{1}{n}}}\right| \leq 1.96\right) &= 0.95 \\
 P\left(\bar{X}_n - 1.96\sqrt{\frac{1}{n}} \leq \mu \leq \bar{X}_n + 1.96\sqrt{\frac{1}{n}}\right) &= 0.95
 \end{aligned}$$

This gives me an interval estimate that  $\mu$  falls between these two limits with 95% confidence. Why were we able to do this? Because  $Z$  is a pivotal quantity so we could just use the table. If  $Z$  depended on unknown parameters we would need to find them. So we have found an upper and lower bound with a given confidence.

Now, we can generalize this.  $\left(\bar{X}_n \pm 1.96\sqrt{\frac{1}{n}}\right)$  is called a 95% confidence interval, for the unknown  $\mu$ . In real life application we have one sample and one average. So we plug in values and get the two bounds, say: (125, 135). Someone might come in and ask us how we can associate probability to this interval, either the average is in there or not, so what do we mean by 95% confidence? Notice that the probability statement we had above means

that, if we dispatch many students to test students and come up with confidence intervals just like we did and write them all on pieces of paper. Then we randomly select one of the papers from the bag, then we are 95% sure that the interval given **covers the truth**.

Let's see how we can generalize this. In general, we have large sample confidence intervals. When we started this example, we assumed that the observations are coming from a Normal distribution, but in practice, we don't know if something is normally distributed. This allowed us to normalize  $\bar{X}_n$ . So if the  $X_i$  weren't normally distributed, all of this would fail.

But we know that, from Central Limit Theorem (most celebrated result in all of probability & statistics, in baby form): If  $X_i$ 's are independent, have the same mean  $\mu$  and variance  $\sigma^2$ , then  $\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \stackrel{approx}{\sim} N(0, 1)$  for large  $n$ . Note, we **do not** need to know the distribution of  $X_i$ . This is a pivotal quantity. In many applications, we might only be interested in  $\mu$ , but we have another unknown parameter,  $\sigma$  sitting here. We could do the same calculation as above and get:

$$\left( \bar{X}_n \pm 1.96 \sqrt{\frac{\sigma^2}{n}} \right) \text{ is 95\% CI}$$

$$\left( \bar{X}_n \pm 1.96 \frac{\sigma}{\sqrt{n}} \right)$$

But how do we get rid of  $\sigma$ ? We can replace it by the sample  $\sigma$ , i.e.

$$\left( \bar{X}_n \pm 1.96 \frac{S}{\sqrt{n}} \right)$$

Note that we are still assuming very large  $n$  (how large is very large will not be addressed in this class).

What justifies replacing  $\sigma$  with  $S$ ? Since we have a large sample, then  $S$  is going to be close enough to  $\sigma$ . This is called **consistency** (to be discussed in the next chapter). But what justifies replacing a parameter by a consistent estimator?

Essentially, what we are saying is:

$$\frac{\bar{X}_n - \mu}{\frac{S}{\sqrt{n}}} \stackrel{approx}{\sim} N(0, 1)$$

But CLT only works with  $\sigma$ . Intuitively we say it works when  $S$  is sufficiently close to  $\sigma$ .

$$\frac{\bar{X}_n - \mu}{\frac{S}{\sqrt{n}}} = \frac{\sigma}{S} \cdot \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$$

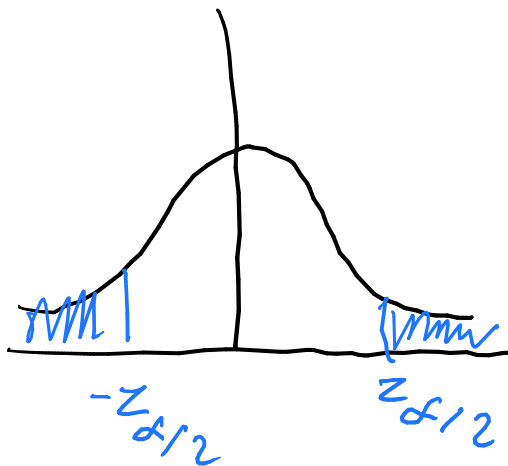
We have a theorem from Cramer, telling us that if we have a ratio of parameters that converge to 1 and a distribution converging to Normal, then the whole thing converges to Normal.

So we can use:

$$\left( \bar{X}_n \pm 1.96 \frac{S}{\sqrt{n}} \right) \text{ 95\% CI}$$

So if we need other confidence intervals:

$$\left( \bar{X}_n \pm Z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \right) 100(1 - \alpha)\% \text{ CI}$$



Note that this is just for  $\bar{X}_n$ . What if we have something different?

If  $n$  is large and  $\theta$  is a parameter of interest, then

$$\frac{\hat{\theta}_n - \theta}{\sqrt{\text{Var}(\hat{\theta}_n)}} \underset{\text{approx}}{\sim} N(0, 1)$$

where  $\hat{\theta}_n$  is an MLE (Maximum Likelihood Estimator, which we'll see in a future chapter)

For example, if  $X_i \sim \text{Ber}(p)$ , i.e.  $X_i = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$  and  $E(X_i) = p$ , then  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i = \hat{p}_n$ . So we want to estimate  $\hat{p}_n$ .

$$\frac{\hat{p}_n - p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0, 1)$$

$$\hat{p}_n \pm 1.96 \sqrt{\frac{p(1-p)}{n}}$$

But once again, we don't know what  $p$  is and it is not useful! So we can either:

- Replace  $p$  by  $\hat{p}_n$
- Or, take the conservative approach and replace  $p(1-p)$  by  $\frac{1}{4}$  (remember that this is the max value the it can take, as seen earlier)

$f(x) = x(1-x) \rightarrow f'(x) = 1-2x = 0 \implies x = \frac{1}{2}, f''(x) = -2, f(\frac{1}{2}) = \frac{1}{4}$ , a max. Since we are replacing by the max, we are accounting for the worse situation, so it gives us a large confidence interval (which is why it is conservative). These are the types of intervals that we hear about before elections. When they are talking about chance they are talking about  $\hat{p}_n$ , when talking about how accurate it is, they are referring to the  $\pm$  part and when saying it is right 19 out of 20 times, they are talking about the % of the CI.

Now suppose we are interested in estimating the variance  $\theta = p(1-p)$  (not interested in estimating  $p$ ). What would be a natural estimator?

$$\begin{aligned}\hat{\theta}_n &= \hat{p}_n(1 - \hat{p}_n) \\ \frac{\hat{\theta}_n - \theta}{\sqrt{Var(\hat{\theta}_n)}} &\sim N(0, 1)\end{aligned}$$

We then have the general recipe:

$$\hat{\theta}_n \pm Z_{\frac{\alpha}{2}} \sqrt{Var(\hat{\theta}_n)} \text{ is } 100(1 - \alpha)\% \text{ CI}$$

The only thing we need is a right estimator that gives us this asymptotic normality and then we are in business! As long as our  $\hat{\theta}_n$  can provide asymptotic normality for large  $n$ , then we can have this. Most of the estimators in the textbook follow this pattern.

## 5.2 Small Sample Size

There are some cases we can solve and some that we cannot. The case of interest that we can solve, is the normal case. We are focusing a lot on normal distributions because we will see later that many things relate to the normal distribution, like comparing control groups

on placebo and not on placebo.  $X_i \stackrel{iid}{\sim} N(\mu, \sigma^2), i = 1, \dots, n$  where  $n$  is “small”. Here  $\mu$  is of interest but  $\sigma^2$  is a nuisance.

We learned that  $\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \stackrel{\text{Exact}}{\sim} N(0, 1)$ . So we can use this as a pivotal quantity to come up with a confidence interval for  $\mu$ . But we still have the  $\sigma$  here that is a nuisance. Before we replaced  $\sigma$  by  $S$ , but that was well justified because our sample size was large. But that’s the best thing we have, so we’ll use it. But can we say it is normally distributed? We were only able to say that earlier because we had a large distribution. So what might happen?

$$\frac{\bar{X}_n - \mu}{\frac{S}{\sqrt{n}}} \sim T_{n-1}$$

This will be similar to the  $X_i$ , i.e. normally distributed and it will still have a bell curve. But what about the tails? The uncertainty of  $S$  means that the tails will die out more slowly.

If  $X_i \stackrel{iid}{\sim} N(\mu, \sigma^2), i = 1, \dots, n$ , then

$$\frac{\bar{X}_n - \mu}{\frac{S}{\sqrt{n}}} \sim T_{n-1}$$

where  $S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$

What is the  $T$  distribution?  $\frac{N(0,1)}{\sqrt{\frac{\chi_\nu^2}{\nu}}}$  where  $N(0,1) \perp \chi_\nu^2$  and  $\nu$  denotes the degrees of freedom. How does this relate to what we have?

$$\frac{(\bar{X}_n - \mu)}{\frac{\sigma}{\sqrt{n}}} / \left( \left( \frac{\sigma}{\sqrt{n}} \right)^{-1} \sqrt{\frac{S^2}{n}} \right)$$

Where the numerator is

$$\sim N(0, 1)$$

$$= \frac{\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{S^2}{\sigma^2}}}$$

$$\sqrt{\frac{S^2}{\sigma^2}} = \sqrt{\frac{\sum_{i=1}^n \left( \frac{X_i - \bar{X}_n}{\sigma} \right)^2}{n-1}}$$

$$\sum_{i=1}^n \left( \frac{X_i - \bar{X}_n}{\sigma} \right)^2 \sim \chi_{n-1}^2$$

## 6 01/30/18

**Recall** When it comes to confidence intervals, the main tool is **pivotal quantities**. So whatever we did using large or small samples, was to come up with a pivotal quantity. In large samples, we used Central Limit Theorem. For small samples, the case that we studied assumed that observations came from a Normal distribution and so we used normality assumption in order to come up with a pivotal quantity. What happens when we go beyond Normal distribution? There is a general approach to get a pivotal quantity that covers all continuous random variables, but in practice it is only useful if we can find the distribution function of the variable of interest, but it is only useful if its easy to acquire. We will present the recipe today, which is sometimes successful and sometimes unsuccessful. Pivotal quantities are case by case, vary by distribution. This is already in the notes on myCourses:

Say we have  $X_1, \dots, X_n \sim F(\text{cdf}) \rightarrow f(\text{pdf})$ . Suppose this is information from Revenue Canada and we made a histogram of the range of salaries and we fit a distribution over several years (this is our pdf). Now for this year, we'll be taking random people ( $x_i$ ) from Canada who have filled out their tax forms.

Probability Integral Transform is a simple result that is very useful, especially in simulation. What it says is that if  $X(\text{continuous}) \sim F \implies \underbrace{F(X)}_Y \sim \text{Unif}(0, 1)$ . I.e. apply  $F$  on a continuous distribution and get another distribution, a uniform distribution on  $(0, 1)$ . So how can we use this? If we can generate observations on a uniform distribution, then we can generate results for any continuous distribution by applying  $F^{-1}$ , (since  $F$  is a monotone decreasing function it has an inverse) and then we get  $X$ s that are distributed like  $X$ , although uniform randomness is very difficult to achieve.

$$X \sim N(0, 1)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < +\infty$$

$$F_X(x) = \int_{-\infty}^x f_X(t) dt = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

This is a monotone function, but its inverse is very hard to find. What we do is generate uniform random numbers to approximate this and then get normal distribution. Try this at home:

$$X \sim FY = F(x)$$

You will see that the distribution of  $Y$  is uniform. Use the method of transformation here (look in Chapter 6). Now our result is:

$$\begin{aligned}
 X \stackrel{F}{\Rightarrow} \underbrace{Y = F(X) \sim Unif(0, 1)}_{PIT} &\Rightarrow -2 \log Y \sim \chi_2^2 \\
 X_i &\stackrel{iid}{\sim} F, i = 1, \dots, n \\
 Y_i = F(X_i) &\stackrel{iid}{\sim} Unif(0, 1), i = 1, 2, \dots, n \\
 V_i = -2 \log Y_i &\stackrel{iid}{\sim} \chi_2^2, i = 1, \dots, n
 \end{aligned}$$

$$\sum_{i=1}^n V_i \sim \chi_{2n}^2 \quad (3)$$

This can easily be gotten to via the method of mgf. If  $F$  has any unknown parameters, it will still be present in  $V_i$ . So  $\sum_{i=1}^n V_i$  will depend on  $X_i$ s and parameters of  $F$ , but its distribution doesn't, so this is a pivotal quantity.

**Ex**  $X_i \stackrel{iid}{\sim} Exp(\lambda), i = 1, 2, \dots, n$

$$\begin{aligned}
 f_X(x) &= \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases} \quad \lambda > 0 \\
 F_X(x) &= \int_{-\infty}^x f_X(t) dt = \int_0^x \lambda e^{-\lambda t} dt \\
 &= -e^{-\lambda t} \Big|_0^x = 1 - e^{-\lambda x}
 \end{aligned}$$

The dual to (3):

$$\sum_{i=1}^n W_i \sim \chi_{2n}^2$$

where  $W_i = -2 \log(1 - Y_i)$ . This is because, if  $U \sim Unif(0, 1)$  then  $1 - U \sim Unif(0, 1)$ . This is useful because we often want  $1 - F$ , because the probability of  $F$  is that something is less than something, so  $1 - F$  means that the value is past a certain point, i.e. this is good



for the survivor function, checking the chances that someone survives past something.

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-\lambda x} & \text{if } x > 0 \end{cases}$$

$$S_X(x) = 1 - F_X(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ e^{-\lambda x} & \text{if } x > 0 \end{cases}$$

$$\begin{aligned} \sum_{i=1}^n W_i &= \sum_{i=1}^n -2 \log[1 - F(X_i)] \\ &= \sum_{i=1}^n -2 \log e^{-\lambda X_i} \\ &= 2\lambda \sum_{i=1}^n X_i = 2\lambda n \bar{X}_n \end{aligned}$$

$$2\lambda n \bar{X}_n = \sum_{i=1}^n W_i \sim \chi_{2n}^2$$

This is a pivotal quantity. So we can use this for a confidence interval of lambda.

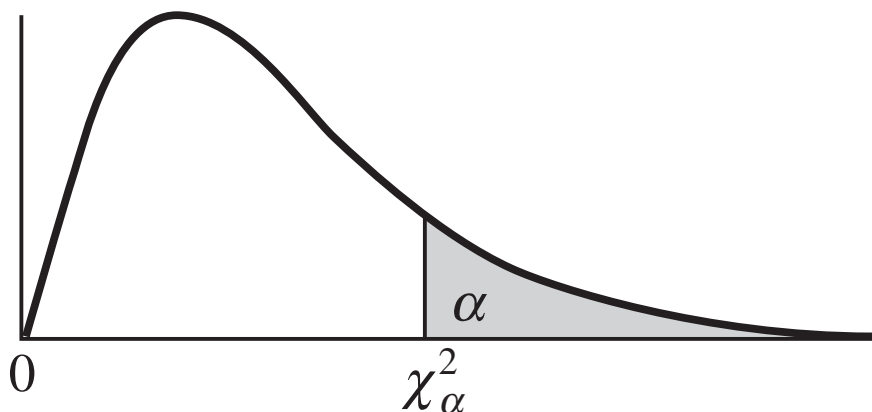
Using the  $\chi^2$  table (Appendix 3, p850-851), we can find:

$$\chi_{2n,0.025}^2 \& \chi_{2n,0.975}^2$$

$$P(\chi_{2n,0.975}^2 < 2\lambda n \bar{X}_n < \chi_{2n,0.025}^2) = 95\% = 0.95$$

$$P\left(\frac{\chi_{2n,0.975}^2}{2n\bar{X}_n} < \lambda < \frac{\chi_{2n,0.025}^2}{2n\bar{X}_n}\right) = 95\% = 0.95$$

So:  $\left(\frac{\chi_{2n,0.975}^2}{2n\bar{X}_n}, \frac{\chi_{2n,0.025}^2}{2n\bar{X}_n}\right)$  is a 95% C.I. for  $\lambda$ .



## 6.1 Sample Size Determination

Say we want to predict the results of an election. We want to be, say 99% sure of the results (why not 100%? because then we'd need infinite samples).

$$X_i = \begin{cases} 1 & \text{NDP} \\ 0 & \text{otherwise} \end{cases}, i = 1, \dots, n$$

$$p(X_i = 1) = p$$

$$X_i \stackrel{iid}{\sim} \text{Bernoulli}(p), i = 1, \dots, n$$

$$\hat{p}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

We want  $|\hat{p}_n - p| \leq 0.01$  with 95%. We can use Chebyshev's for this, but since Chebyshev's makes very little assumptions, the bound is very crude. We want a tighter bound. So we can use CLT:

$$\hat{p}_n \pm \underbrace{1.96 \sqrt{\frac{p(1-p)}{n}}}_B$$

This is called a symmetric confidence interval, since it is centered around a point. Normal distribution is symmetric,  $\chi$  is not. If it is symmetric, we can talk about length or half length, if it isn't we talk about the whole length.

$$(\text{e.g. } 0.01) \rightarrow B = 1.96 \sqrt{\frac{p(1-p)}{n}}$$

Say we want it in terms of  $100(1 - \alpha)\%$ :

$$B = Z_{\frac{\alpha}{2}} \sqrt{\frac{p(1-p)}{n}}$$

We obtain:

$$n = \frac{p(1-p)(Z_{\frac{\alpha}{2}})^2}{B^2}$$

We have the following 2 options:

- Replace  $p$  by  $\hat{p}_n$  (a prior estimate, although often we don't have an estimate).
- Replace  $p(1-p)$  by  $\frac{1}{4}$  (conservative approach).

So then we get:

$$n = \frac{Z_{\frac{\alpha}{2}}^2}{4B^2}$$

(we round up  $n$  if it is not an integer) This gives us a sample size! This is a proportion estimate for the sample size for Bernoulli.

**Ex**  $B = 0.01, \alpha = 0.05$

$$n = \frac{(1.96)^2}{4(0.01)^2} = 9604 \approx 10,000$$

Now, generalizing:

C.I. for  $\mu$ :

$$\begin{aligned} \bar{X}_n \pm \underbrace{Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}}_B \\ B = Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \\ n = \frac{Z_{\frac{\alpha}{2}}^2 \sigma^2}{B^2} \text{ for } \mu \end{aligned}$$

Here we can't just maximize  $\sigma$ , so we do pilot studies, where we try to justify the need for a study, so we receive a little bit of money to make a small study, to get an idea of some of the parameters, like  $\sigma$  in this case and then we use this parameter to get a bit of information on other parameters so that we can apply for the real study. Now determining the sample size goes back to confidence intervals.

Other variables:

$$\frac{\hat{\theta}_n - \theta}{\sqrt{\text{Var}(\hat{\theta}_n)}} \stackrel{app}{\sim} N(0, 1) \text{ for large } n \quad (4)$$

Here  $\hat{\theta}_n$  is the MLE (will see in the future) and  $\theta$  is the estimand.

$$\hat{\theta}_n \pm Z_{\frac{\alpha}{2}} \sqrt{\text{Var}(\hat{\theta}_n)}$$

$100(1 - \alpha)\%$  C.I. for  $\theta$

$$B = Z_{\frac{\alpha}{2}} \sqrt{\text{Var}(\hat{\theta}_n)}$$

So we get:

- Bernoulli  $\text{Var}(\hat{\theta}_n) = \frac{p(1-p)}{n}$
- $\mu$   $\text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$

These are based on a large sample. What if we have a small sample size? As long as we have the confidence interval, if it is symmetric we can bound by half length, if it isn't we can bound by the whole length.

Normal case:

$$\bar{X}_n \pm t_{\frac{\alpha}{2}, (n-1)} \frac{\sigma}{\sqrt{n}}$$

$$B = t_{\frac{\alpha}{2}, (n-1)} \frac{s}{\sqrt{n}}$$

---

So far we've talked about one population. So what happens if we have two populations? Say we have men and women and want to estimate average salary.

$$\frac{\hat{\theta}_n - \theta}{\sqrt{\text{Var}(\hat{\theta}_n)}} \sim N(0, 1)$$

How do we use this if we have more than one sample, say  $X_1, \dots, X_m$  (men,  $\bar{X}_m$ ) and  $Y_1, \dots, Y_n$  (women,  $\bar{Y}_n$ ). Say we want to estimate  $\theta = \mu_M - \mu_W$ . Then  $\hat{\theta} = \bar{X}_m - \bar{Y}_n$ . Now, using the formula above, we get:

$$\frac{(\bar{X}_m - \bar{Y}_n) - (\mu_M - \mu_W)}{\sqrt{\text{Var}(\bar{X}_m - \bar{Y}_n)}}$$

So the general recipe is:

Pivotal quantity -  $i$  confidence interval -  $i$  margin of error -  $i$  sample size

$$\text{Var}(\bar{X}_m - \bar{Y}_n) = \text{Var}(\bar{X}_m) + \text{Var}(\bar{Y}_n) - 2\text{Cov}(\bar{X}_m, \bar{Y}_n)$$

Assuming that  $X$ s &  $Y$ s are independent (note that we don't need independence, just orthogonality such that  $Cov$  is 0, independence is much stronger). Then  $Cov(\bar{X}_m, \bar{Y}_n) = 0$

$$Var(\bar{X}_m - \bar{Y}_n) = Var(\bar{X}_n) + Var(\bar{Y}_n) = \frac{\sigma_M^2}{m} + \frac{\sigma_W^2}{n}$$

$$\frac{(\bar{X}_m - \bar{Y}_n) - (\mu_M - \mu_W)}{\sqrt{\frac{\sigma_M^2}{m} + \frac{\sigma_W^2}{n}}} \stackrel{approx}{\sim} N(0, 1) \text{ for large } m \& n$$

$$(\bar{X}_m - \bar{Y}_n) \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_M^2}{m} + \frac{\sigma_W^2}{n}}$$

is a  $100(1 - \alpha)\%$  C.I. for  $\mu_M - \mu_w$

$$B = Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_M^2}{m} + \frac{\sigma_W^2}{n}}$$

$$\left( \frac{B}{Z_{\frac{\alpha}{2}}} \right)^2 = \frac{\sigma_M^2}{m} + \frac{\sigma_W^2}{n}$$

So we have one equation with 2 unknowns. What should we do? Set them equal! What is the rational for setting them equal? Cost. If  $u = km$ , we can solve:

$$\left( \frac{B}{Z_{\frac{\alpha}{2}}} \right)^2 = \frac{1}{m} \left( \sigma_M^2 + \frac{\sigma_W^2}{k} \right)$$

$$m = \frac{\sigma_M^2 + \frac{\sigma_W^2}{k}}{\left( \frac{B}{Z_{\frac{\alpha}{2}}} \right)^2}$$

With proportions  $\sigma_M^2 = p_M(1 - p_M)$  can replace with  $\frac{1}{u}$ , much simpler:

$$\rightarrow \frac{1 + \frac{1}{k}}{\left( \frac{B}{Z_{\frac{\alpha}{2}}} \right)^2}$$

## 7 02/01/18

Today we are starting **Chapter 9**.

## 7.1 Relative Efficiency

We learned that we can have many point estimates and we discussed estimator errors with Chebyshev's inequality as well as MSE, etc. We noticed that MSE could be split into 2 parts, bias and variance. So perhaps a reasonable property for an estimator would be that we want it to be unbiased. But we also saw that unbiased estimators are not unique, so which do we chose? We came to the point with two unbiased estimators and decided that the one with the **smaller variance** is better. Now we want to quantify this deviation by looking at the difference in variances of unbiased estimators, or the ratio. For us, ratios and differences are the same (if you take logarithm of a ratio you get differences). We do prefer to work with ratios though.

**Def** The *relative efficiency* of two **unbiased estimators**,  $\hat{\theta}_1$  and  $\hat{\theta}_2$ , is defined to be

$$eff(\hat{\theta}_1, \hat{\theta}_2) = \frac{Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)}$$

Note that we can also extend this to bias estimators, looking at MSEs instead. But there is a reason that we are using unbiased estimators. For a fixed sample size  $n$ , we may have biased estimators, but for large  $n$ , as  $n$  increases, we require unbiasedness.

The reason we look at the ratio:

Suppose we have an unfair coin, with  $P(head) = 0.6$ ,  $P(tail) = 0.4$ . If you toss this over and over again, you'll see that the ratio of heads and tails should be bounded, converge to some number. But for difference, it won't converge, because the difference increases more and more to infinity.

Recall determining sample size. We looked at the margin of error, the part we'd get in the confidence interval. If we want to compare two different procedures/estimators.

Recall that we had  $Var(\bar{X}_n) = \frac{\sigma^2}{n}$ . If we have a symmetric distribution we can replace  $\bar{X}_n$  by  $Var(median_n) = \frac{\sigma^2}{\dots 1.6n}$ . So using a procedure that uses the median requires say, 1.6 times the sample size to get the same information, much more costly. So the ratio tells us about the ratio between two estimators and which one requires a larger sample size for the same information.

**Ex. 9.1, p446**  $Y_i \stackrel{iid}{\sim} Unif(0, \theta), i = 1, 2, \dots, n$ , where  $\theta$  is what we want to estimate. Now consider the following 2 estimators:

$$\hat{\theta}_1 = 2\bar{Y}_n$$

$$\hat{\theta}_2 = \left( \frac{n+1}{n} \right) Y_{(n)}$$

where  $Y_{(n)} = \max\{Y_1, \dots, Y_n\}$ . Which one intuitively seems better? Try to look at the extreme cases, like when  $n = 1$  or  $n$  is very large. When  $n$  is very large,  $\hat{\theta}_2$  gets larger and larger and since  $\theta$  is the upper bound, it will get closer and closer to  $\theta$ , but it still remains slightly under, which is why we multiply by  $\frac{n+1}{n}$  to increase it slightly.

$$\begin{aligned} \hat{\theta}_1 : E(\hat{\theta}_1) &= E(2\bar{Y}_n) = 2E(\bar{Y}_n) = 2E\left(\frac{1}{n} \sum_{i=1}^n Y_i\right) \\ &= \frac{2}{n} E\left(\sum_{i=1}^n Y_i\right) = \frac{2}{n} \sum_{i=1}^n E(Y_i) \end{aligned}$$

Because  $Y$  is a symmetric distribution, the mean is the median:

$$\begin{aligned} &= \frac{2}{n} \cdot \sum_{i=1}^n \frac{\theta}{2} = \theta \\ \text{Var}(\hat{\theta}_1) &= \text{Var}(2\bar{Y}_n) = \frac{4\text{Var}(Y)}{n} \\ \text{Var}(Y) &= E(Y^2) - \underbrace{[E(Y)]^2}_{\frac{\theta^2}{4}} \\ E(Y^2) &= \int_{-\infty}^{\infty} y^2 f_Y(y) dy = \int_0^{\theta} y^2 \frac{1}{\theta} dy = \frac{1}{\theta} \int_0^{\theta} y^2 dy \\ &= \frac{1}{\theta} \cdot \frac{1}{3} y^3 \Big|_0^{\theta} = \frac{\theta^3}{3\theta} = \frac{\theta^2}{3} \\ \text{Var}(Y) &= \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12} \end{aligned}$$

Plugging in:

$$\text{Var}(\hat{\theta}_1) = \frac{4 \cdot \frac{\theta^2}{12}}{n} = \frac{\theta^2}{3n}$$

$$\begin{aligned} \hat{\theta}_2 : F_{Y_{(n)}} &= P(Y_{(n)} \leq t) = P(Y_1 \leq t, Y_2 \leq t, \dots, Y_n \leq t) \\ &= \prod_{i=1}^n P(Y_i \leq t) = \prod_{i=1}^n F_{Y_i}(t) = [F_Y(t)]^n \\ F_{Y_{(n)}}(t) &= [F_Y(t)]^n \end{aligned}$$

$$\begin{aligned}
\delta_{Y_{(n)}}(t) &= \frac{d}{dt} F_{Y_{(n)}}(t) = \frac{d}{dt} [F_Y(t)]^n \\
&= n f_Y(t) (F_Y(t))^{n-1} \\
f_{Y_{(n)}}(t) &= \begin{cases} n \frac{1}{\theta} \left(\frac{t}{\theta}\right)^{n-1} & \text{if } 0 < t < \theta \\ 0 & \text{otherwise} \end{cases} \\
E(\hat{\theta}_2) &= E \left[ \left( \frac{n+1}{n} \right) Y_{(n)} \right] = \left( \frac{n+1}{n} \right) E[Y_{(n)}] \\
&= \left( \frac{n+1}{n} \right) \int_0^\theta y \underbrace{n \cdot \frac{1}{\theta} \left(\frac{y}{\theta}\right)^{n-1}}_{f_{Y_{(n)}}(y)} dy \\
&= \left( \frac{n+1}{n} \right) \cdot \frac{n}{\theta^n} \int_0^\theta y^n dy \\
&= \left( \frac{n+1}{n} \right) \cdot \frac{n}{\theta^n} \left[ \frac{1}{n+1} y^{n+1} \right]_0^\theta \\
&= \left( \frac{n+1}{n} \right) \cdot \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \theta \\
E(\hat{\theta}_2) &= \theta
\end{aligned}$$

$\hat{\theta}_2$  is an unbiased estimator of  $\theta$ .

$$\begin{aligned}
Var(\hat{\theta}_2) &= E(\hat{\theta}_2^2) - \underbrace{[E(\hat{\theta}_1)]^2}_{\theta^2} \\
E(\hat{\theta}_2^2) &= \int_0^\theta \left( \frac{n+1}{n} \right)^2 y^2 \delta_{Y_{(n)}} dy \\
&= \left( \frac{n+1}{n} \right)^2 \int_0^\theta y^2 \cdot n \cdot \frac{1}{\theta} \left(\frac{y}{\theta}\right)^{n-1} dy \\
&= \left( \frac{n+1}{n} \right)^2 \frac{n}{\theta^n} \int_0^\theta y^{n+1} dy \\
&= \left( \frac{n+1}{n} \right)^2 \cdot \frac{n}{\theta^n} \left[ \frac{1}{n+2} y^{n+2} \right]_0^\theta \\
&= \left( \frac{n+1}{n} \right)^2 \frac{n}{\theta^n} \frac{\theta^{n+2}}{n+2} \\
&= \frac{(n+1)^2}{n(n+2)} \cdot \theta^2 \\
Var(\hat{\theta}_2^2) &= \frac{(n+1)^2}{n(n+1)} \cdot \theta^2 - \theta^2
\end{aligned}$$



$$\begin{aligned}
&= \theta^2 \left[ \frac{(n+1)^2 - n(n+2)}{n(n+1)} \right] \\
&= \frac{\theta^2}{n(n+2)}
\end{aligned}$$

So what do we have?

$$\begin{aligned}
\text{Var}(\hat{\theta}_1) &= \frac{\theta^2}{3n} \\
\text{Var}(\hat{\theta}_2) &= \frac{\theta^2}{n(n+2)} \\
\text{eff}(\hat{\theta}_1, \hat{\theta}_2) &= \frac{\text{Var}(\hat{\theta}_2)}{\text{Var}(\hat{\theta}_1)} \\
&= \frac{\frac{\theta^2}{n(n+2)}}{\frac{\theta^2}{3n}} = \frac{3}{n+2} \\
\frac{3}{n+2} &\leq 1 \text{ if } n > 1, \frac{3}{n+2} < 1 \\
n \rightarrow \infty \frac{3}{n+2} &\rightarrow 0
\end{aligned}$$

So the meaning of this ratio is that as  $n$  increases, the efficiency of procedure 1 decreases compared to the efficiency of procedure 2. It's definitely better to use the second procedure to estimate  $\theta$ .

## 7.2 Consistency

Minimum requirement an estimator should have. More rigorously:

**Def** We say  $\hat{\theta}_n$  (subscript meaning it was based on  $n$  observations) is a *consistent estimator* of  $\theta$  if  $\hat{\theta}_n \xrightarrow{P} \theta$  as  $n \rightarrow \infty$ .

Recall that converges with probability ( $\xrightarrow{P}$ ) means:

$$\lim_{n \rightarrow \infty} P(|\hat{\theta}_n - \theta| > \varepsilon) = 0, \forall \varepsilon > 0$$

i.e. the more and more elements we have, the more our estimates will converge to the actual

value.

Recall:

$$\lim_{n \rightarrow \infty} a_n = a$$

$$\forall \varepsilon > 0, \exists N(\varepsilon) \implies |a_n - a| < \varepsilon \text{ if } n \geq N(\varepsilon)$$

What does this say? It means we can get as close as  $a$  as we'd like by increasing  $n$ .

But this isn't the same with random variables, because they have **random fluctuations**! For example, if we had a fair coin, we'd try flipping many many times in order to try to induct that the probability is 0.5, but after many many trials, say 10000, we may only have 4820 heads or something of the like. i.e. we increased the sample size but we noticed an even bigger deviation and this happens in practice because of randomness. So what is the complement of converges with probability?

$$\lim_{n \rightarrow \infty} P(|\hat{\theta}_n - \theta| < \varepsilon) = 1, \forall \varepsilon > 0$$

So this may not happen with certainty (because of random fluctuations, but the complement will converge to 1).

**Ex**  $X_i \sim \text{Ber}(p)$

$$X_i = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}, i = 1, \dots, n.$$

What justifies us in using  $\hat{p}_n = \frac{1}{n} \sum_{i=1}^n X_i$ ? We said we were mimicking that proportion in the population. Now we want to formalize this. To establish that  $\hat{p}_n$  is on the right track:

$$\lim_{n \rightarrow \infty} P(|\hat{p}_n - p| > \varepsilon) = 0, \forall \varepsilon > 0$$

What's our big hammer here? Chebychev's inequality. Can we use that here? Recall:

$$P(|X - E(x)| > k\sigma) \leq \frac{1}{k^2}$$

Our random variable  $X$ , is  $\hat{p}_n$ . The mean of  $\hat{p}_n$  is  $p$ . Now  $\varepsilon$  plays the role of our  $k\sigma$ , i.e.

$$E(\hat{p}_n) = p$$

$$\varepsilon = k\sqrt{\text{var}(\hat{p}_n)}$$

$$\begin{aligned} \text{Var}(\hat{p}_n) &= \frac{p(1-p)}{n} \\ k &= \frac{\varepsilon}{\sqrt{\text{Var}(\hat{p}_n)}} \end{aligned}$$

So if we want to use Chebychev's:

$$\begin{aligned} P(|\hat{p}_n - p| > \varepsilon) &\leq \frac{1}{\left[ \frac{\varepsilon}{\sqrt{\text{Var}(\hat{p}_n)}} \right]^2} \\ P(|\hat{p}_n - p| > \varepsilon) &\leq \frac{\left( \frac{p(1-p)}{n} \right)}{\varepsilon^2} \\ p(1-p) \leq \frac{1}{4} &\implies = \frac{p(1-p)}{n\varepsilon^2} \\ &\leq \frac{1}{4n\varepsilon^2} \rightarrow 0, \text{ as } n \rightarrow \infty, \forall \varepsilon > 0 \end{aligned}$$

So we have obtained:

$$P(|\hat{p}_n - p| > \varepsilon) \leq \frac{1}{4n\varepsilon^2}$$

So what if we want to decrease  $\varepsilon$  as  $n$  increases?

$$\begin{aligned} \varepsilon_n &= \frac{\log n}{\sqrt{n}} \\ P\left(|\hat{p}_n - p| > \frac{\log n}{\sqrt{n}}\right) &\leq \frac{1}{4n\left(\frac{\log n}{\sqrt{n}}\right)^2} \leq \frac{1}{4\log n} \rightarrow 0, \text{ as } n \rightarrow \infty \end{aligned}$$

So this tells us the rate at which we can decrease  $\varepsilon$ . This brings us closer to the definition of limits from Calculus from numbers.

So we can use Chebyshev's inequality to establish consistency for many estimators.

Suppose  $X_1, \dots, X_n$  have the same mean  $\mu$  and variance  $\sigma^2$ . Suppose further that  $\text{Cov}(X_i, X_j) = 0, i \neq j$ . Then

$$\overline{X}_n \xrightarrow{P} \mu$$

which we can easily show using Chebyshev's inequality as well.

$$P(|\bar{X}_n - \mu| > \varepsilon)$$

Recall that Chebyshev's:

$$P(|X - E(X)| > k\sqrt{\text{Var}(x)}) \leq \frac{1}{k^2}$$

So  $\bar{X}_n \rightarrow X, \mu \rightarrow E(x), \varepsilon \rightarrow k\sqrt{\text{Var}(x)}$

$$\begin{aligned}\varepsilon &= k\sqrt{\text{Var}(\bar{X}_n)} \\ k &= \frac{\varepsilon}{\sqrt{\text{Var}(\bar{X}_n)}} \\ \text{Var}(\bar{X}_n) &= \frac{\sigma^2}{n} \\ P(|\bar{X}_n - \mu| > \varepsilon) &\leq \frac{1}{\left(\frac{\varepsilon}{\sqrt{\frac{\sigma^2}{n}}}\right)^2} \\ &= \frac{\sigma^2}{n\varepsilon^2} \rightarrow 0, \text{ as } n \rightarrow \infty, \text{ as long as } \sigma^2 \text{ finite}\end{aligned}$$

But can we also say:

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2 \xrightarrow{P} \sigma^2$$

No, there is much more difficulty with this, which we'll see next class.

## 8 02/06/18

### 8.1 Consistency

Last class we discussed consistency and went over a few examples, played a bit with  $\varepsilon$  and convergent probability. We then extended the consistency result for sample proportions to sample averages for estimating the population average. Then we wanted to start discussing the consistency of population variance, although we noted that the usual big hammer (Chebyshev's/Markov's Inequality) would be harder to use.

$$\hat{p}_n \xrightarrow{P} p \rightarrow p = P(X = 1)$$

This essentially justifies using relative frequency as an estimate to  $p$ . As we repeat our experiments more and more, the relative frequency  $\hat{p}_n$  gets closer and closer to  $p$ . The whole

theorem was trying to find an upper bound of the difference between relative frequency and  $p$ , using Chebyshev's Inequality (a special case of Markov's).

$$P(|\hat{p}_n - p| > \varepsilon)$$

$$\begin{aligned} P(|\hat{p}_n - p| > \varepsilon) &= P(|\hat{p}_n - p|^2 > \varepsilon^2) \\ &\stackrel{\text{Markov's}}{\leq} \frac{E[|\hat{p}_n - p|^2]}{\varepsilon^2} \end{aligned} \quad (5)$$

Recall: If  $g$  is a non-negative function and  $X$  a r.v., then

$$P(g(X) \geq \lambda) \leq \frac{E[g(X)]}{\lambda}, \lambda > 0$$

To apply Markov's inequality to obtain (5),  $g(x) = (x - E(x))^2$ . Now replace  $X$  by  $\hat{p}_n$ . Note that  $E(\hat{p}_n) = p$ . Why did we square before applying Markov's? Because it is easier to compute variance with.

$$\begin{aligned} E[(\hat{p}_n - p)^2] &= \text{Var}(\hat{p}_n) = \frac{p(1-p)}{n} \\ p(|\hat{p}_n - p| > \varepsilon) &= p(|\hat{p}_n - p|^2 > \varepsilon^2) \\ &\leq \frac{E[|\hat{p}_n - p|^2]}{\varepsilon^2} = \frac{p(1-p)}{n\varepsilon^2} \rightarrow 0, \text{ as } n \rightarrow \infty, \forall \varepsilon > 0 \end{aligned}$$

This means the variance approaches 0, so the estimator gets closer and closer to its center, what its trying to estimate.

**Theorem** (Thm 9.1, p450)

Suppose  $\hat{\theta}_n$  is an estimate for  $\theta$  and  $MSE = (\hat{\theta}) \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\hat{\theta}_n \xrightarrow{P} \theta$  meaning that  $\hat{\theta}_n$  is a consistent estimator of  $e$ .

**Proof**

$$\begin{aligned} P(|\hat{\theta}_n - \theta| > \varepsilon) &= P(|\hat{\theta}_n - \theta|^2 > \varepsilon^2) \\ &\stackrel{\text{Markov's}}{\leq} \frac{E[|\hat{\theta}_n - \theta|^2]}{\varepsilon^2} \\ &= \frac{MSE(\hat{\theta}_n)}{\varepsilon^2} \rightarrow 0 \end{aligned}$$

So  $\hat{\theta}_n \xrightarrow{P} \theta$ .

**Corollary** Suppose  $\hat{\theta}_n$  is an unbiased estimator of  $\theta$  such that  $Var(\hat{\theta}_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\hat{\theta}_n \xrightarrow{P} \theta$ , i.e.  $\hat{\theta}_n$  is a consistent estimator of  $\theta$ .

**Recall** Proportion:

$$Var(\hat{p}_n) = \frac{p(1-p)}{n} \rightarrow 0, \text{ as } n \rightarrow \infty$$

Sample average:

$$Var(\bar{X}_n) = \frac{\sigma_X^2}{n} \rightarrow 0, \text{ as } n \rightarrow \infty$$

$$E(\hat{p}_n) = p$$

$$E(\bar{X}_n) = \mu_X$$

$$MSE(\hat{p}_n) = Var(\hat{p}_n)$$

$$MSE(\bar{X}_n) = Var(\bar{X}_n)$$

**Recall**

$$MSE(\hat{\theta}_n) = Var(\hat{\theta}_n) + \underbrace{Bias^2(\hat{\theta}_n)}_{(E(\hat{\theta}_n) - \theta)^2} \quad (6)$$

**Corollary 2** Suppose  $\hat{\theta}_n$  is asymptotically unbiased for  $\theta$ , i.e.  $\lim_{n \rightarrow \infty} E(\hat{\theta}_n) = \theta$ , and  $Var(\hat{\theta}_n) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\hat{\theta}_n \xrightarrow{P} \theta$ , i.e.  $\hat{\theta}_n$  is a consistent estimator of  $\theta$ .

Using (6):

$$\lim_{n \rightarrow \infty} MSE(\hat{\theta}_n) = \lim_{n \rightarrow \infty} Var(\hat{\theta}_n) + \underbrace{\lim_{n \rightarrow \infty} Bias^2(\hat{\theta}_n)}_0 = \lim_{n \rightarrow \infty} Var(\hat{\theta}_n) = 0$$

---

Suppose  $X_1, \dots, X_n$  for a random sample (i.e. independent and identically distributed) from a population with mean  $\mu$  and variance  $\sigma^2$ . We want to estimate  $\sigma^2$ . We learned that perhaps a reasonable estimator is:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

$$E(S^2) = \sigma^2$$

But we need to divide by  $n - 1$ .

$$S_{n,*}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 = \left(1 - \frac{1}{n}\right) S_n^2$$

$$E(S_{n,*}^2) = \left(1 - \frac{1}{n}\right) E(S_n^2) = \left(1 - \frac{1}{n}\right) \sigma^2$$

i.e.  $S_{n,*}^2$  is biased, but it will be our maximum likelihood estimator later.

Can we show:

$$S_n^2 \xrightarrow{P} \sigma^2?$$

$$\begin{aligned} P(|S_n^2 - \sigma^2| > \varepsilon) &= P(|S_n^2 - \sigma^2| > \varepsilon) \\ &\leq \frac{E[(S_n^2 - \sigma^2)]}{\varepsilon^2} \\ &= \frac{\text{Var}(S_n^2)}{\varepsilon^2} \end{aligned}$$

Note that all of these depend heavily on the second moment and give us difficulty. They require variance, but Kolmogorov's Theorem doesn't and makes it important/hard to prove.

## 8.2 Kolmogorov's Theorem (Law of Large Numbers)

Suppose  $X_1, \dots, X_n$  is a random sample with population mean  $\mu$ . Then  $\bar{X}_n \xrightarrow{P} \mu = E(X)$ .

$$\begin{aligned} S_{n,*}^2 &= \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 \\ &= \frac{1}{n} \sum_{i=1}^n X_i^2 - \bar{X}_n^2 \\ &\quad \underbrace{\hspace{1.5cm}}_{E(X^2)} \end{aligned}$$

Now  $X_1, \dots, X_n$  iid and  $X_1^k, \dots, X_n^k$  iid  $\rightarrow E(X^k)$

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \underbrace{X_i^k}_{Y_i} &\rightarrow E(X^k) \\ \frac{1}{n} \sum_{i=1}^n X_i^k &= \frac{1}{n} \sum_{i=1}^n Y_i = \bar{Y}_n \end{aligned}$$

Using Kolmogorov's theorem:

$$\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow E(X^2) \quad (7)$$

$$\bar{X}_n \xrightarrow{P} E(X) \quad (8)$$

$$(\bar{X}_n)^2 \rightarrow [E(X)]^2 \quad (9)$$

Thm 9.7, page 451

If  $\hat{\theta}_n \xrightarrow{P} \theta$ , then  $\hat{\phi}_n \xrightarrow{P} \phi$ . Then

a)  $\hat{\theta}_n \hat{\phi}_n \rightarrow \theta \phi$

b)  $\hat{\theta}_n \pm \hat{\phi}_n \rightarrow \theta \pm \phi$

c)  $\frac{\hat{\theta}_n}{\hat{\phi}_n} \rightarrow \frac{\theta}{\phi}$ , provided that  $\hat{\phi}_n \neq 0$  and  $\phi \neq 0$ .

d)  $g(\hat{\theta}_n) \rightarrow g(\theta)$  if  $g$  is a continuous function.

This is called the continuous mapping theorem.

Then (9) follows from 9.2(d), the continuous mapping theorem.

$$\hat{\theta}_n \xrightarrow{P} \theta : P(|\hat{\theta}_n - \theta| > \varepsilon) \rightarrow 0, \text{ as } n \rightarrow \infty, \forall \varepsilon > 0$$

Step (8):  $\bar{X}_n^2 \rightarrow [E(X)]^2$

Step (9): Using 9.2(b)

$$\frac{1}{n} \sum_{i=1}^n X_i^2 - \bar{X}_n^2 \rightarrow \underbrace{E(X^2) - [E(X)]^2}_{\text{Var}(X) = \sigma^2}$$

So  $S_{n,*}^2 \xrightarrow{P} \sigma^2$

Question:

$$S_{n,*}^2 = \frac{1}{n} \sum_{i=1}^n \underbrace{(X_i - \bar{X}_n)^2}_{Z_i} = \frac{1}{n} \sum_{i=1}^n Z_i$$

Why can't we use Kolmogorov's theorem directly here? They are not independent.

$$Z_i = (X_i - \bar{X}_n)^2$$

$$\sum_{i=1}^n (X_i - \bar{X}_n) = 0$$



Consistency means being right-headed, if you have an estimator that is not consistent, throw it out.

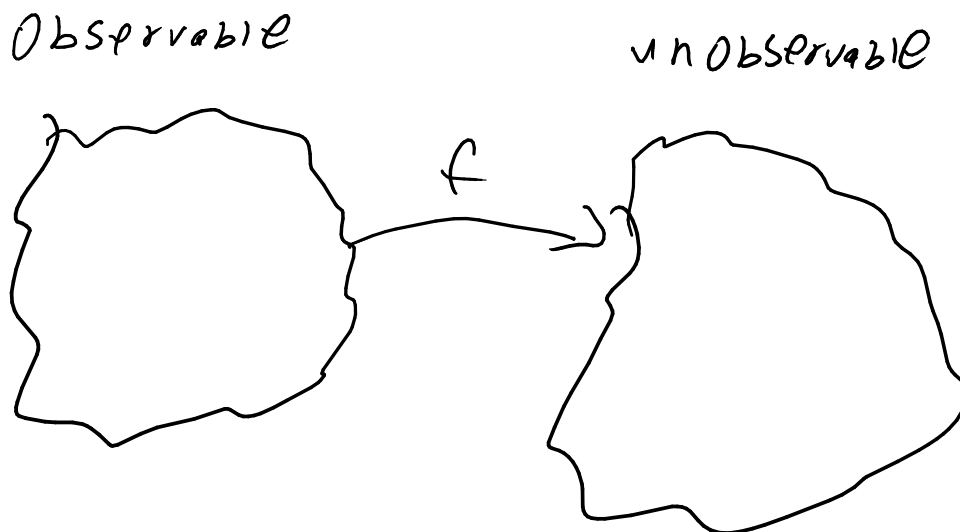
Now suppose that you get the whole population (magically somehow), can you hit the target parameter? This is the minimum requirement, consistency.

Recall that before our goal was to get the “best” estimator, the one that is closest to the target. Sufficiency paves the way to that.

### 8.3 Sufficiency

Sufficiency was first introduced by Fisher, at first for compression.

#### Likelihood Inference



For example:  $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ , where  $x$  is observable and  $\mu$  and  $\sigma$  are unobservable.

Lets say that  $\sigma$  is known and  $\mu$  is unknown. How do we quantify this information? What will compressing the data do? If we compress, we also want to be able to decompress to get the original data. It's a bit too much to expect to get the exact original data after decompressing, so maybe we can lower our expectations.

$$\vec{x} = (X_1, \dots, X_n)$$

$$\vec{y} = (Y_1, \dots, Y_n)$$

Now suppose  $T$  is the compressed value as follows:

$$T = T(X_1, \dots, X_n)$$

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

We want  $m < n$ , or else we aren't compressing anything. We want to work with a smaller dimension, the smaller the  $m$  the better.

We say that  $\vec{X}$  &  $\vec{Y}$  are T-similar if

$$P_{\vec{X}|T}(\vec{u}|t_1\theta) = P_{\vec{Y}|T}(\vec{u}|t_1\theta), \forall \vec{u}$$

Given compressed values, we want to come back to something similar (T-similar) if not the same. They are similar, we cannot tell the difference, we come up with something equally likely.

A realization of  $\vec{X}$  and a realization of  $\vec{Y}$  are T-similar if

a)  $\vec{X}$  &  $\vec{Y}$  are T-Similar

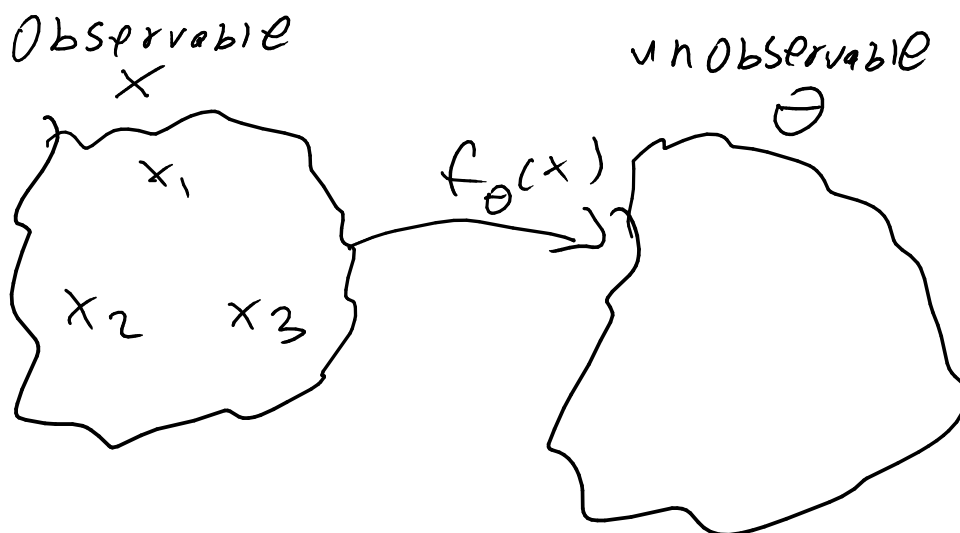
b)  $T(\vec{X}) = T(\vec{Y})$

## 9 02/08/18

### 9.1 Sufficiency

We were discussing sufficiency last class. We said how sufficiency is essentially just compression.

Say we have a vector:  $\vec{X} = (X_1, \dots, X_n)$ . We have observable and unobservable things.



We want to use the observable things to find the true values of the observable parameters, i.e.  $\theta$ , given that we have  $f_\theta = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}}$ ,  $X \sim N(\theta, 1)$

We want to compress this data and be able to decompress it to get the original data, but since these are random variables we want to get something similar, since getting the exact same thing back is asking for a bit too much. So we defined:

$\vec{X}, \vec{Y}$  (random variables) are T-similar if

$$P_{\vec{X}|T=t}(\vec{u}|T=t) = P_{\vec{Y}|T=t}(\vec{u}|T=t), \forall t$$

$\vec{x}, \vec{y}$  are realizations of  $\vec{X}, \vec{Y}$ ,  $T(\vec{x}) = T(\vec{y})$ .

**Def** A statistic  $T_n = T(X_1, \dots, X_N)$  is called a sufficient statistic for a parameter  $\theta$  if

- $T_n$  can preserve all “information” in  $\vec{X} = (X_1, \dots, X_n)$  pertinent to  $\theta$ .
- For any given value of  $T_n$ , say  $t$ , we can generate a  $T_n$ -similar sample of  $x$ ’s. (this regeneration is decompression)

Say we have  $P(X_1 = x_1, \dots, X_n = x_n | T_n = t)$  and we want to generate the original values, say  $x_1^*, \dots, x_n^*$ . What do I need for this? We need the distribution to not depend on any unknown parameters, say  $\theta$ . The information from  $f_\theta(x)$  has been absorbed enough by  $T$  such that we don’t need  $\theta$  to work with anymore, we can just use  $T$  instead.

**Ex**  $X_i \stackrel{iid}{\sim} \text{Bernoulli}(p), i = 1, 2, \dots, n$

$$P(X_i = 1) = p, P(X_i = 0) = 1 - p$$

$$p(x) = P(X = x) = p^x(1 - p)^{1-x}, x = 0, 1$$

$$T_n = T(X_1, \dots, X_n) = \sum_{i=1}^n X_i$$

$$\begin{aligned} P(\overbrace{X_1 = x_1, \dots, X_n = x_n}^A | \overbrace{T_n = t}^B) &= \begin{cases} \frac{P(X_1=x_1, \dots, x_n=x_n, T_n=t)}{P(T_n=t)} & \text{if } t = \sum_{i=1}^n X_i \\ 0 & \text{if } t \neq \sum_{i=1}^n X_i \end{cases} \\ &= \begin{cases} \frac{P(X_i=x_1, \dots, X_n=x_n)}{P(T_n=t)} & \text{if } t = \sum_{i=1}^n X_i \\ 0 & \text{o.w.} \end{cases} \\ T_n = \sum_{i=1}^n X_i &\sim \text{Bin}(n, p) \\ P(T_n = t) &= \binom{n}{t} p^t (1-p)^{n-t}, t = 0, \dots, n \\ &= \begin{cases} \frac{\prod_{i=1}^n P(X_i=x_i)}{\binom{n}{t} p^t (1-p)^{n-t}} & \text{if } t = \sum_{i=1}^n X_i \\ 0 & \text{o.w.} \end{cases} \\ &= \begin{cases} \frac{\prod_{i=1}^n p^{x_i} (1-p)^{1-x_i}}{\binom{n}{t} p^t (1-p)^{n-t}} & \text{if } t = \sum_{i=1}^n X_i \\ 0 & \text{o.w.} \end{cases} \\ &= \begin{cases} \frac{p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i}}{\binom{n}{t} p^t (1-p)^{n-t}} & \text{if } t = \sum_{i=1}^n X_i \\ 0 & \text{o.w.} \end{cases} \\ &= \begin{cases} \frac{1}{\binom{n}{t}} & \text{if } t = \sum_{i=1}^n x_i \\ 0 & \text{o.w.} \end{cases} \end{aligned}$$

The parameter  $p$  that was unknown has been eliminated and thus we see that  $t$  is sufficient.

$\mathfrak{A}_t = \{(X_1, \dots, X_n) : \sum_{i=1}^n x_i = t\}$ , now this contains  $n$ -tuples that are  $T$ -similar and sum up to the same value as  $t$ . Then we can choose one of these at random to decide which to use.

**Ex**  $X_i \stackrel{iid}{\sim} Po(\lambda), i = 1, \dots, b, P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$$\begin{aligned}
 T_n &= \sum_{i=1}^n X_i \sim Po(n\lambda) \\
 P(X_1 = x_1, \dots, X_n = x_n | T_n = t) &= \begin{cases} \frac{P(X_1=x_1, \dots, X_n=x_n)}{P(T_n=t)} & \text{if } t = \sum_{i=1}^n X_i \\ 0 & \text{o.w} \end{cases} \\
 &= \begin{cases} \frac{\prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}}{e^{-n\lambda} (n\lambda)^t / t!} & \text{if } t = \sum_{i=1}^n X_i \\ 0 & \text{o.w} \end{cases} \\
 &= \begin{cases} \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i} / (\prod_{i=1}^n x_i!)}{e^{-n\lambda} n^t \lambda^t / t!} & \text{if } t = \sum_{i=1}^n x_i \\ 0 & \text{o.w} \end{cases} \\
 &= \begin{cases} \frac{t!}{\prod_{i=1}^n x_i!} \left(\frac{1}{n}\right)^t & \text{if } t = \sum_{i=1}^n X_i \\ 0 & \text{o.w} \end{cases} \\
 &= \begin{cases} \binom{t}{x_1, \dots, x_n} \prod_{i=1}^n \left(\frac{1}{n}\right)^{x_i} & \text{if } t = \sum_{i=1}^n X_i \\ 0 & \text{o.w} \end{cases}
 \end{aligned}$$

As you can see, there is no  $\lambda$  anymore in here. So if we want to mimic the population we can use this and generate a sample via a multinomial distribution, so this is once again a sufficient statistic.

---

Note that this is the multinomial distribution! Recall:

$$\begin{aligned}
 (X_1, \dots, X_k) &\sim \text{Multinomial}(n, p_1, \dots, p_k) \\
 \text{if } P(X_1 = x_1, \dots, X_k = x_k) &= \binom{n}{x_1, \dots, x_k} \prod_{i=1}^k p_i^{x_i}
 \end{aligned}$$

We have  $\sum_{i=1}^n X_i = n \sum_{k=1}^n p_i = 1$

$$\begin{aligned}
 \binom{n}{x_1, \dots, x_k} &= \frac{n!}{x_1! x_2! \dots x_k!} \\
 \binom{n}{x, n-x} &= \binom{n}{\lambda} = \frac{n!}{x!(n-x)!}
 \end{aligned}$$


---

**Def 9.4, p460, Likelihood** suppose  $y_1, \dots, y_n$  are realizations of  $Y_1, \dots, Y_n$  whose distribution depends on an unknown parameter  $\theta$ . Thus the likelihood of  $y_1, \dots, y_n$  is defined to be

$$\mathfrak{L}(\theta : y_1, \dots, y_n) = P_\theta(Y_1 = y_1, \dots, Y_n = y_n)$$

**Ex**  $X_i \stackrel{iid}{\sim} \text{Bernoulli}(p)$

$$\begin{aligned} \mathfrak{L}(p, x_1, \dots, x_n) &= P(X_1 = x_1, \dots, X_n = x_n) \\ \text{iid} &= \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i} = p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i} \end{aligned}$$

**Ex**  $X_i \stackrel{iid}{\sim} N(\mu, \sigma^2)$

$$\begin{aligned} \mathfrak{L}(\mu, \sigma^2, x_1, \dots, x_n) &= f_{\mu, \sigma^2}(x_1, \dots, x_n) = \prod_{i=1}^n f_{\mu, \sigma^2}(x_i) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} = \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^n e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2} \end{aligned}$$

**Ex.**  $X_i \stackrel{iid}{\sim} \text{Po}(\lambda), i = 1, \dots, n$

$$\begin{aligned} \mathfrak{L}(\lambda, x_1, \dots, x_n) &= P_\lambda(X_1 = x_1, \dots, X_n = x_n) \\ \text{iid} &= \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \\ &= \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} \end{aligned}$$

## 9.2 Fisher-Neyman Theorem

Thm 9.4, p 461

### Factorization Theorem

**Theorem** A statistic  $T = T(Y_1, \dots, Y_n)$  is sufficient for the parameters of the distribution of  $Y_1, \dots, Y_n$  if and only if

$$\mathfrak{L}(\theta, y_1, \dots, y_n) = g(t, \theta) h(y_1, \dots, y_n)$$

for any realization  $(y_1, \dots, y_n)$  &  $t$ , where  $t = T(y_1, \dots, y_n)$

**Ex.**  $X_i \stackrel{iid}{\sim} \text{Bernoulli}(p)$

$$\mathfrak{L}(p, x_1, \dots, x_n) = p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i} = \left( \frac{p}{1-p} \right)^{\sum_{i=1}^n x_i} (1-p)^n$$

Where  $p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i}$  can be  $g(t, p)$ , where  $t = \sum_{i=1}^n x_i$ ,  $h(x_1, \dots, x_n) \equiv 1$ .

**Ex.**  $X_i \stackrel{iid}{\sim} \text{Po}(\lambda)$

$$\mathfrak{L}(\lambda, x_1, \dots, x_n) = \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} = \underbrace{\left( e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i} \right)}_{g(t, \lambda)} \underbrace{\left( \frac{1}{\prod_{i=1}^n x_i!} \right)}_{h(x_1, \dots, x_n)}$$

where again  $t = \sum_{i=1}^n x_i$ . Note that the kernel is important, the part we cannot factorize unknown parameters from observations.  $h$  is just a function of the random observations.

**Ex.**  $X_i \stackrel{iid}{\sim} N(\mu, \sigma^2)$

$$\mathfrak{L}(\mu, \sigma^2, x_1, \dots, x_n) = \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^n e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2}$$

$$\begin{aligned} \sum_{i=1}^n (x_i - \mu)^2 &= \sum_{i=1}^n x_i^2 - 2\mu \sum_{i=1}^n x_i + n\mu^2 \\ \implies \mathfrak{L}(\mu, \sigma^2, x_1, \dots, x_n) &= \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^n \underbrace{e^{-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}} e^{\frac{2\mu \sum_{i=1}^n x_i}{2\sigma^2}}}_{\text{kernel}} \underbrace{e^{-\frac{n\mu^2}{2\sigma^2}}}_{\text{just a function of params}} \\ T &= \left( \sum_{i=1}^n x_i, \sum_{i=1}^n x_i^2 \right) \end{aligned}$$

## 10 02/13/18

Why did we introduce sufficiency? How do we use this?

### 10.1 The Rao-Blackwell Theorem

(Thm 9.5, p464)

Let  $\hat{\theta}$  be an unbiased estimator of an unknown parameter  $\theta$  such that  $\text{Var}(\hat{\theta}) < \infty$  (i.e. it exists). If  $T$  is a sufficient statistic for  $\theta$ , then

a)  $\hat{\theta}^* = E(\hat{\theta}|T)$  is also an unbiased estimator for  $\theta$ .

b)  $\text{var}(\hat{\theta}^*) \leq \text{Var}(\hat{\theta})$ .

Why is this so remarkable? Remember at the beginning of chapter 8 we wanted to come up with an estimator for  $\theta$  and we measured the goodness via bias and MSE. With an unbiased estimator, bias disappears so MSE is just variance and the estimator with smaller variance is closer to the target. So if we can find the unbiased estimator with the smallest variance, then we have the best among the class of estimators. This theorem is important because it says if we condition on a sufficient statistic, then we can find another estimator that is closer to the target.

**Proof** (Simply an application of Thm 5.14 (p286) and Thm 5.15 (p287))

Thm 5.14:

$$E(X) = E\{E(X|Y)\} \quad (10)$$

Note that:

$$E(X|Y = y) = \int_x x f_{X|Y}(x|y) dx$$

gives you a function in terms of  $y$ , so we can take the expectation of this as well.

$$\begin{aligned} E\{E(X|Y)\} &= \int_y \underbrace{\int_x x f_{X|Y}(x|y) dx}_{E(X|Y=y)} f(y) dy \\ &= \int_x \int_y x \underbrace{f_{X|Y}(x|y) f(y)}_{f_{X,Y}(x,y)} dy dx \\ &= \int_x \int_y x f_{X,Y}(x, y) dy dx \\ &= \int_x x \left( \underbrace{\int_y f_{X,Y}(x, y) dy}_{f_X(x)} \right) dx \\ &= \int_x x f_X(x) dx = E(X) \end{aligned}$$

Do you know why conditioning is important in the first place? First of all, if we have a big area and we're wondering about the truth in the big area, it's much harder to find what we want than if we're looking at smaller areas and then we can glue them all together, using the law of total probability.

So, using 10 for part a):



a)

$$E(\hat{\theta}^*) = \underbrace{E\{E(\hat{\theta}|T)\}}_{E(\hat{\theta})} = E(\hat{\theta})$$

 $\hat{\theta}$  is unbiased  $\implies$ 

$$E(\hat{\theta}) = \theta \text{ for any } \theta$$

$$E(\hat{\theta}^*) = \theta \text{ for any } \theta$$

b) Thm 5.15

$$V(X) = V\{E(X|Y)\} + E\{V(X|Y)\} \quad (11)$$

$$\hat{\theta} = E(\hat{\theta}|T)$$

$$V(\hat{\theta}^*) = V\{E(\hat{\theta}|T)\}$$

Using 11

$$\begin{aligned} &= V(\hat{\theta}) - \underbrace{E\{V(\hat{\theta}|T)\}}_{\substack{\geq 0 \\ \geq 0}} \\ \implies V(\hat{\theta}^*) &= V(\hat{\theta}) \\ E(\hat{\theta}_n|T) &= \int \hat{\theta} f_{\hat{\theta}|T} \end{aligned}$$

**Completeness (Lehmann & Schemé)** Suppose  $X$  is a r.v. with pdf  $f_\theta$ . The r.v.  $X$  is called complete if

$$\begin{aligned} E_\theta[g(x)] &= \int_x g(x) f_\theta(x) dx = 0, \forall \theta \\ \implies g(x) &\equiv 0 \quad \forall x \\ Ax &= 0 \end{aligned}$$

**To find MVUE (minimum variance and unbiased estimator):**

1. Find an unbiased estimator of your estimand.

2. Find a sufficient statistic using Fisher-Newmann Theorem.

3. Find conditional expectation,  $\hat{\theta}^* = E(\hat{\theta}|T)$

$\hat{\theta}^*$  is the MVUE.

**Ex 1**  $X_i \stackrel{iid}{\sim} \text{Bernoulli}(p), i = 1, \dots, n$

Example 9.6, p466

$$\begin{aligned}\mathfrak{L}(p; x_1, \dots, x_n) &= P_p(X_1 = x_1, \dots, X_n = x_n) \\ &= \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i} = p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i} \\ T &= \sum_{i=1}^n X_i \text{ is sufficient} \\ g(t; p) &= p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i} \\ h(x_1, \dots, x_n) &= 1 \\ E(T) &= E\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \underbrace{E(X_i)}_p = np \\ E\left(\frac{T}{n}\right) &= p\end{aligned}$$

So  $\frac{T}{n}$  is the MVUE.

**Ex 2**  $X_i \stackrel{iid}{\sim} N(\mu, \sigma^2), i = 1, 2, \dots, n$

Ex. 9.8, p467

$$\begin{aligned}\mathfrak{L}(\mu, \sigma^2; x_1, \dots, x_n) &= f_{\mu, \sigma^2}(x_1, \dots, x_n) = \prod_{i=1}^n f_{\mu, \sigma^2}(x_i) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \\ &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2} \left[\sum_{i=1}^n x_i^2 - 2\mu \sum_{i=1}^n x_i + n\mu^2\right]\right\} \\ T &= \left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\right) \\ E(X_i) &= \sum_{i=1}^n E(X_i) = n\mu \\ \frac{\sum_{i=1}^n X_i}{n} &= \bar{X}_n \text{ is the MVUE of } \mu\end{aligned}$$

$$S^2 = \frac{1}{n-1} \left[ \sum_{i=1}^n X_i^2 - n\bar{X}_n \right]$$

$$E(S^2) = \sigma^2$$

$S^2$  is a function of  $T = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$

Then  $S^2$  is the MVUE of  $\sigma^2$

**Ex 3.**  $Y_i \stackrel{iid}{\sim} Weibull(m=2, \theta), i=1, \dots, n$

Ex. 9.7 p466-7

$$f_{\theta}(y) = \begin{cases} \left( \frac{2y}{\theta} e^{-\frac{y^2}{\theta}} \right) & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

MVUE of  $\theta$ ?

$$\mathcal{L}(\theta; y_1, \dots, y_n) = \prod_{i=1}^n \left( \frac{2y_i}{\theta} \right) e^{-\frac{y_i^2}{\theta}} = \left( \frac{2}{\theta} \right)^n \prod_{i=1}^n y_i e^{-\frac{1}{\theta} \sum_{i=1}^n y_i^2}$$

$$g(t; \theta) = \left( \frac{2}{\theta} \right)^n e^{\frac{1}{\theta} t}$$

$$h(y_1, \dots, y_n) = \prod_{i=1}^n Y_i$$

$$t = \sum_{i=1}^n y_i^2, T = \underbrace{\sum_{i=1}^n Y_i^2}_{\text{Sufficient}}$$

$$W_i = Y_i^2, i=1, \dots, n \implies Y = \sqrt{w}$$

$$f_W(w) = f_Y(\sqrt{w}) \left| \frac{d\sqrt{w}}{dw} \right| \quad (\text{recall, derivative is } original/new, \text{ i.e. } \mathbf{ON})$$

$$= \begin{cases} \left( \frac{2\sqrt{w}}{\theta} \right) e^{-\frac{(\sqrt{w})^2}{\theta}} \frac{1}{2\sqrt{w}} & w > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{\theta} e^{-\frac{w}{\theta}} & w > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$W \sim Exp(\theta)$$

$$\begin{aligned}
E(W) &= \theta \\
T &= \sum_{i=1}^n Y_i^2, E(T) = E\left(\sum_{i=1}^n Y_i^2\right) = E\left(\sum_{i=1}^n W_i\right) = \sum_{i=1}^n E(W_i) = n\theta \\
E\left(\frac{T}{n}\right) &= \theta
\end{aligned}$$

So  $\frac{T}{n}$  is MVUE for  $\theta$ .

**Ex. 9.10, p468** (Read at home)

## 10.2 Methods of Estimation

1. Method of moments (best method for most of the textbook problems)
2. Maximum likelihood approach (common method of approximation we use every day, MLE)

$$\begin{aligned}
\hat{\theta}_n &\xrightarrow{P} \theta \\
\sqrt{n}(\hat{\theta}_n - \theta) &\xrightarrow{D} N(0, I^{-1}(\theta))
\end{aligned}$$

Where  $D$  means it converges into distribution.

$$\frac{\hat{\theta}_n - \theta}{S_{\hat{\theta}_n}} \stackrel{app}{\sim} N(0, 1)$$

## 11 02/15/18

### 11.1 MLE

Method of Maximum Likelihood (p477)

$$\hat{\theta}_{ML} = \operatorname{argmax} \mathfrak{L}(\theta; y_1, \dots, y_n)$$

Step 1:  $Y_1 = y_1, \dots, Y_m = y_m$   
 $\mathfrak{L}(\theta; y_1, \dots, y_m)$

Step 2: Find the maximizer of  $\mathfrak{L}(\theta; y_1, \dots, y_n)$

Why maximize the likelihood? What does likelihood do for us? Likelihood is the only link we have between unobservables (like the average salary of Canadians) and

observables. So why did Fisher want to maximize likelihood, not minimize? So what we have is a bunch of observations. These should be typical observations, observations that anyone else could also observe. So these observations should happen most frequently. If we look at the observations as a function of the unknown parameters (likelihood), we want to see what makes this a common observation. In short, what we have observed is what we should have expected to observe.

**Ex**  $X_i \stackrel{iid}{\sim} \text{Bernoulli}(p), i = 1, 2, \dots, n$

$$P(X = x) = p^x(1-p)^{1-x}$$

$$\mathcal{L}(p; x_1, \dots, x_n) = p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i}$$

The little  $x_i$ 's are values we have observed, so this is a function of  $p$ .

$$\ell(p; x_1, \dots, x_n) = \log \mathcal{L}(p; x_1, \dots, x_n)$$

Note that log is a monotone increasing function, so it has the same maximizer as  $\mathcal{L}$

$$\begin{aligned} &= \left( \sum_{i=1}^n x_i \right) \log p + \left( n - \sum_{i=1}^n x_i \right) \log(1-p) \\ \frac{d\ell}{dp} &= \frac{\sum_{i=1}^n x_i}{p} - \frac{n - \sum_{i=1}^n x_i}{1-p}, \quad t = \sum_{i=1}^n x_i \\ \frac{t}{\hat{p}} - \frac{n-t}{1-\hat{p}} &= 0 \\ \frac{t}{\hat{p}} &= \frac{n-t}{1-\hat{p}} \implies \frac{t}{n-t} = \frac{\hat{p}}{1-\hat{p}} \\ \frac{t/n}{1-t/n} &= \frac{\hat{p}}{1-\hat{p}} \implies \hat{p} = \frac{t}{n} = \frac{\sum_{i=1}^n x_i}{n} \\ \hat{p} &= \frac{\sum_{i=1}^n x_i}{n} \\ \frac{d^2\ell}{dp^2} &= -\frac{\sum_{i=1}^n x_i}{p^2} - \frac{n - \sum_{i=1}^n x_i}{(1-p)^2} = -\frac{t}{p^2} - \frac{n-t}{(1-p)^2} < 0 \end{aligned}$$

$\hat{p}$  is a maximizer. (might be expected to find a maximizer ourselves, but only for single variable unknowns).

**Ex**  $X_i \stackrel{iid}{\sim} N(\mu, \sigma^2), i = 1, \dots, n$

$$\begin{aligned}\mathfrak{L}(\mu, \sigma^2; x_1, \dots, x_n) &= \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^n \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right\} \\ \ell(\mu, \sigma^2; x_1, \dots, x_n) &= \log \mathfrak{L}(\mu, \sigma^2; x_1, \dots, x_n) \\ &= -n \log \sqrt{2\pi} - n \frac{1}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \\ \frac{\partial \ell(\mu, \sigma^2)}{\partial \mu} &= -\frac{1}{2\sigma^2} \sum_{i=1}^n -2(x_i - \mu) \\ \frac{\partial \ell(\mu, \sigma^2)}{\partial \mu} &= -\frac{n}{2\sigma^2} - \frac{1}{2\sigma^4} \sum_{i=1}^n (x_i - \mu)^2 \\ &\begin{cases} -\frac{1}{2\sigma^2} \sum_{i=1}^n -2(x_i - \hat{\mu}) = 0 \\ -\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4} \sum_{i=1}^n (x_i - \hat{\mu})^2 = 0 \end{cases}\end{aligned}$$

From eq 1

$$\begin{aligned}\sum_{i=1}^n (x_i - \hat{\mu}) = 0 &\implies \sum_{i=1}^n x_i - n\hat{\mu} = 0 \\ \implies \hat{\mu} &= \frac{1}{n} \sum_{i=1}^n x_i = \bar{X}_n\end{aligned}$$

Plug in 2<sup>nd</sup> eq

$$\begin{aligned}-\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4} \sum_{i=1}^n (x_i - \bar{x})^2 &= 0 \\ \frac{1}{2\hat{\sigma}^4} \sum_{i=1}^n (x_i - \hat{\mu})^2 &= \frac{n}{2\hat{\sigma}^2} \\ \hat{\sigma}^2 &= \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu})^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\end{aligned}$$

So we have:

$$\begin{aligned}\hat{\mu} &= \bar{x} \\ \hat{\sigma}^2 &= \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\end{aligned}$$

**Ex**  $X_i \stackrel{iid}{\sim} Unif(0, \theta), i = 1, 2, \dots, n$

$$\begin{aligned}\delta_x(x) &= \begin{cases} \frac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases} \\ \mathfrak{L}(\theta; x_1, \dots, x_n) &= \prod_{i=1}^n \delta(x_i) = \prod_{i=1}^n \frac{1}{\theta} = \frac{1}{\theta^n} \\ f_X(x) &= \frac{1}{\theta} I_{(0, \theta)}(x) \\ I_{(0, \theta)}(x) &= \begin{cases} 1 & \text{if } x \in (0, \theta) \\ 0 & \text{otherwise} \end{cases} \\ \mathfrak{L}(\theta; x_1, \dots, x_n) &= \prod_{i=1}^n \frac{1}{\theta} I_{(0, \theta)}(x_i) \\ &= \frac{1}{\theta^n} \prod_{i=1}^n I_{(0, \theta)}(x_i) \\ &0 < x_i < \theta, i = 1, \dots, n \\ &0 < \max_{1 \leq i \leq n} x_i < \theta \\ \prod_{i=1}^n I_{(0, \theta)}(x_i) &= I_{(0, \theta)}(\max_{1 \leq i \leq n} x_i) \\ \mathfrak{L}(\theta; x_1, \dots, x_n) &= \frac{1}{\theta^n} I_{(0, \theta)}(\max_{1 \leq i \leq n} x_i)\end{aligned}$$

$\max_{1 \leq i \leq n} x_i$  is a sufficient statistic for  $\theta$ .

$$\hat{\theta} = \max_{1 \leq i \leq n} x_i \text{ since } \max_{1 \leq i \leq n} x_i < \theta$$

So this one was a bit tricky, don't always have to take the derivatives, sometimes have to do other things. Note that, with the capture recapture problem where we estimate the population with the hypergeometric distribution, we cannot take derivatives because the population is not continuous, it's a discrete distribution.

So we can show:

- Consistency:  $\hat{\theta}_{ML}$  is a consistent estimator.
- Asymptotic normality:  $\sqrt{n}(\hat{\theta}_{ML} - \theta) \stackrel{app}{\rightsquigarrow} N(0, \Sigma)$ , where  $\Sigma$  is the inverse of the information matrix.
- Asymptotic efficiency (around  $\theta$ ,  $\hat{\theta}_{ML}$  has the smallest variance)

- Invariance: If  $\hat{\theta}_{ML}$  is the Maximum Likelihood Estimator of  $\theta$  then  $g(\hat{\theta}_{ML})$  is the MLE of  $g(\theta)$ . This is great because if we find MLE of  $\theta$  then we know MLE for any function of  $\theta$ .

Recall, in  $Bernoulli(p)$ ,  $\hat{p}_n$  UMVUE for  $p$ . So MLE for  $p(1-p)$  is simply  $\hat{p}_n(1-\hat{p}_n)$ .

---

## 11.2 Method of Moments

(The other method that was used more in the past)

Say we have a bunch of observations coming from some distribution  $f : X_1, \dots, X_n \sim f$

$$\mu_k = E(X^k) < \infty, k = 1, 2, \dots, r$$

If you have  $r$  parameters, you use the following equations to find your parameters:

$$m_k = \mu_k, k = 1, 2, \dots, r$$

where  $m_k = \frac{1}{n} \sum_{i=1}^n x_i^k$ .

**Ex**  $X_i \stackrel{iid}{\sim} \Gamma(\alpha, \beta), i = 1, \dots, n$

$$\begin{aligned} \mu_1 &= \alpha\beta \\ \mu_2 &= E(X^2) = Var(X) + [E(X)]^2 \\ &= \alpha\beta^2 + [\alpha\beta]^2 \\ &= \alpha\beta^2 + \alpha^2\beta^2 \\ \begin{cases} \alpha\beta = \frac{1}{n} \sum_{i=1}^n x_i & \implies \alpha = \frac{\bar{X}_n}{\beta} \\ \alpha\beta^2 = \alpha^2\beta^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 & \implies \alpha\beta(\beta + \alpha\beta) \end{cases} \end{aligned}$$

So we have 2 equations and 2 unknowns.

$$\begin{aligned} \beta^2(\alpha + \alpha^2) &= \frac{1}{n} \sum_{i=1}^n x_i^2 \\ \beta &= \frac{\frac{1}{n} \sum_{i=1}^n x_i^2}{\bar{X}_n} - \bar{X}_n \\ \alpha &= \frac{\bar{X}_n}{\beta} = \frac{\bar{X}_n}{\frac{\frac{1}{n} \sum_{i=1}^n x_i^2}{\bar{X}_n} - \bar{X}_n} \end{aligned}$$



Could also have worked out this example with:

$$\begin{aligned} &\begin{cases} \mu = \bar{X} = \alpha\beta \\ \text{Var}(X) = S^2 = \alpha\beta^2 \end{cases} \\ &\beta = \frac{\alpha\beta^2}{\alpha\beta} = \frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2}{\bar{X}} \\ &\alpha\beta = \bar{X} \implies \alpha = \frac{\bar{X}}{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2} \end{aligned}$$

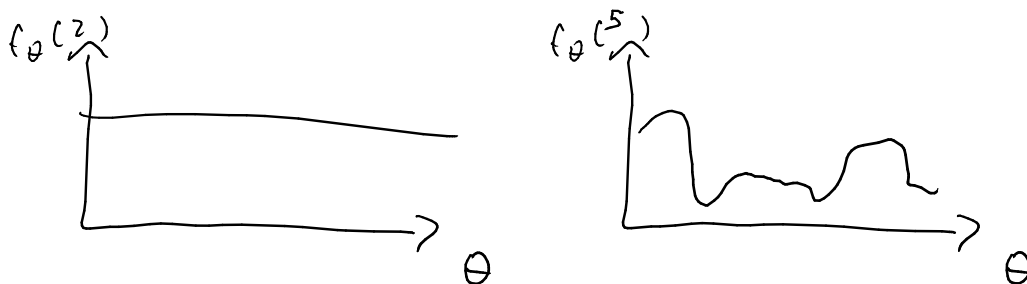
In general, we always use MLE (method of moments was used earlier when we didn't have the computing power to use MLE).

---

### 11.3 Information

$$f_{\theta}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}}$$

Say we have two observations,  $f_{\theta}(2)$  and  $f_{\theta}(5)$ , then the only unknown will be  $\theta$ :



Which of these can have more information about the unknown parameter  $\theta$ ? The second one, as the first one is not sensitive to changes in  $\theta$ . The more bumpy the function is the more it shows that the changes in  $\theta$  are reflected. So we can guess that this has lots to do with derivatives. Now, we use the expected value to summarize the information:

$$-E \left[ \frac{\partial^2 \log f_{\theta}(x)}{\partial \theta^2} \right] = E \left[ \left\{ \frac{\partial \log f_{\theta}(x)}{\partial \theta} \right\}^2 \right]$$

## 12 02/20/18

Section 9.8, optional in the textbook, but we will be going over it as it is slightly important.

$$\begin{aligned}\hat{\theta}_{ML} &= \hat{\theta}(X_1, \dots, X_n) \\ \sqrt{n}(\hat{\theta}_{ML} - \theta) &\overset{app}{\approx} N(0, I^{-1}(\theta)) \\ I(\theta) &= E \left\{ \left[ \frac{\partial}{\partial \theta} \ln f_{\theta}(X) \right]^2 \right\} \\ E \left\{ \left[ \frac{\partial}{\partial \theta} \ln f_{\theta}(X) \right]^2 \right\} &= E \left\{ -\frac{\partial^2}{\partial \theta^2} \ln f_{\theta}(X) \right\}\end{aligned}$$

$$\begin{aligned}\hat{\theta}_{ML} &\text{ is the MLE for } \theta \\ \implies \tau(\hat{\theta}_{ML}) &\text{ is the MLE for } \tau(\theta)\end{aligned}$$

This is called invariance. <sup>1</sup>

$$\begin{aligned}f_{\mu, \sigma^2}(x) &= \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R} \\ f_{\theta}(x) &= \begin{cases} \frac{1}{\theta} & \text{if } 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases} \\ f_{\sigma}(x) &= \frac{1}{\gamma(\alpha)\beta^2} x^{\alpha-1} e^{-\frac{x}{\beta}}, x \geq 0\end{aligned}$$

So we see that only the uniform distribution's support depends on  $\theta$ .

$$\begin{aligned}\sqrt{n}(\tau(\hat{\theta}_{ML}) - \tau(\theta)) &\overset{app}{\approx} N \left( 0, \left( \frac{\partial \tau(\theta)}{\partial \theta} \right)^2 I^{-1}(\theta) \right) \\ \frac{\sqrt{n}(\tau(\hat{\theta}_{ML}) - \tau(\theta))}{\sqrt{\left[ \frac{\partial \tau(\theta)}{\partial \theta} \right]^2 I^{-1}(\theta)}} &\approx N(0, 1) \\ I(\theta) &= E \left\{ \left[ \frac{\partial}{\partial \theta} \ln f_{\theta}(x) \right]^2 \right\} = E \left\{ -\frac{\partial^2}{\partial \theta^2} \ln f_{\theta}(X) \right\} \\ \frac{\sqrt{n}(\tau(\hat{\theta}_{ML}) - \tau(\theta))}{\sqrt{\left[ \frac{\partial \tau(\theta)}{\partial \theta} \right]^2 I^{-1}(\theta)}} &\overset{app}{\approx} N(0, 1)\end{aligned}$$

<sup>1</sup>The support (all possible values that can carry some probability mass) of  $f_{\theta}(x)$  should not depend on  $\theta$ .

$$\tau(\hat{\theta}_{ML}) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\left[\frac{\alpha\tau(\theta)}{\partial\theta}\right]^2 I^{-1}(\theta)}{n}}$$

where  $z_{\frac{\alpha}{2}}, P(Z > z_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$

$$Z \sim N(0, 1)$$

$$\tau(\hat{\theta}_{ML}) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\left[\frac{\alpha\tau(\theta)}{\partial\theta}\right]^2 I^{-1}(\hat{\theta}_{ML})}{n}}$$

**Ex.** #9.18, p484

$$Y_i \stackrel{iid}{\sim} Ber(p), i = 1, 2, \dots, n$$

$$\tau(p) = p(1-p) = p - p^2$$

$$\frac{\partial\tau(p)}{\partial p} = 1 - 2p$$

$$P_p(Y = y) = p^y(1-p)^{1-y}, y = 0, 1$$

$$\ln P_p(Y = y) = y \ln p + (1-y) \ln(1-p)$$

$$\frac{\partial}{\partial p} \ln P_p(Y = y) = \frac{y}{p} - \frac{1-y}{1-p}$$

$$\frac{\partial^2}{\partial p^2} \ln P_p(Y = y) = -\frac{y}{p^2} - \frac{1-y}{(1-p)^2}$$

$$E \left\{ \frac{\partial^2}{\partial p^2} \ln P_p(Y = y) \right\} = I(p)$$

$$I(p) = E \left\{ \frac{Y}{p^2} + \frac{1-Y}{(1-p)^2} \right\} = \frac{1}{p^2} E(Y) + \frac{1}{(1-p)^2} E[1-Y]$$

$$= \frac{1}{p^2} p + \frac{1}{(1-p)^2} (1-p) = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$$

$$I(p) = \frac{1}{p(1-p)}, I^{-1}(p) = p(1-p)$$

Inverse of information is variance, makes perfect sense as they go in opposite ways, the more information you have the less variance you have and vice versa.

$$C.I.\hat{p}_{ML} = \hat{p}_n = \frac{1}{n} \sum_{i=1}^n Y_i$$

$$\tau(\hat{\theta}_{ML}) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\left[\frac{\partial\tau(\theta)}{\partial\theta}\right]^2 I^{-1}(\theta)}{n}}$$

$$\tau(\hat{p}_n) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{(1 - 2\hat{p}_{ML})^2 \hat{p}_{ML}(1 - \hat{p}_{ML})}{n}}$$

$$\frac{\partial \tau(p)}{\partial p} = 1 - 2p$$

## 12.1 Testing Statistical Hypotheses (TSH)

1.  $H_0$ : null hypothesis.
2.  $H_a$ : alternative hypothesis.
3. Test statistic.
4. Rejection Region.

Data used for exploration or validation. But do not use the same data set for exploration (forming a hypothesis) and validation. If you use the same data for both, you will end up being circular and never rejecting your hypothesis.

For a null hypothesis  $H_0$  we have:

	True	False
Accept	✓	Type II error
Reject	Type I error	✓

In statistics, usually the statement that goes under the null hypothesis is the status quo, we really don't want type I error to happen, i.e. in court of law we don't want to send guilty people to jail or death. But the bigger one of the type of errors becomes the smaller the other one becomes, we can't minimize both. So if we want to control one of them by bounding it tightly, we also want to optimize it. i.e. given that we are controlling type I error, what is the lowest that type II error can get?

$$\alpha = P(\text{Type I error}) = P(\text{Rejecting } H_0 \text{ when } H_0 \text{ is true})$$

$$\beta = P(\text{Type II error}) = P(\text{Accepting } H_0 \text{ when } H_0 \text{ is false})$$

Notice that here we have a binary decision to make, either accept null hypothesis and reject alternative hypothesis or reject null hypothesis and accept alternative hypothesis.

**Ex.** 10.1, p491

We have a bunch of voters,  $x_i \stackrel{iid}{\sim} \text{Ber}(p), i = 1, 2, \dots, 15 = n$ .

$$Y = \sum_{i=1}^{15} X_i$$

$$H_0 : p = 0.5$$

$$H_a : p < 0.5$$

$$RR = \{Y \leq 2\}$$

$$\alpha = P(\text{Type I error}) = P(\text{Rejecting } H_0 \text{ when } H_0 \text{ is true})$$

$$= P(Y \leq 2 \text{ when } H_0 \text{ is true}) = P(Y \leq 2, p = 0.5)$$

$$Y = \sum_{i=1}^{15} X_i \sim \text{Bin}(n = 15, \overbrace{p = 0.5}^{\text{if } H_0 \text{ is true}})$$

$$\alpha = P_{p=0.5}(Y \leq 2) = \sum_{y=0}^2 \binom{15}{y} \left(\frac{1}{2}\right)^y \left(1 - \frac{1}{2}\right)^{15-y} \approx 0.004$$

**Ex.** 10.2, p492

$$\beta = P(\text{Type II error}) = P(\text{Accepting } H_0 \text{ when } H_0 \text{ is false})$$

$$= P(Y > 2, \underbrace{H_a \text{ is true}}_{p=0.3})$$

$$= P(Y > 2, p = 0.3) = \sum_{y=3}^{15} \binom{15}{y} (0.3)^y (1 - 0.3)^{15-y}$$

Table 1 in Appendix 3

$$\Rightarrow = 1 - \sum_{y=0}^2 \binom{15}{y} (0.3)^y (1 - 0.3)^{15-y}$$

$$\beta = 0.873$$

If you try  $p = 0.1, \beta = 0.184$

## 12.2 Common Large-Sample Tests

**Ex** 10.6, p498

- $H_0 : p = 0.1$

- $H_A : p > 0.1$

$$X_i \stackrel{iid}{\sim} \text{Ber}(p), i = 1, \dots, 15$$

$$X_i = \begin{cases} 1 & \text{if defective} \\ 0 & \text{otherwise} \end{cases}$$

Now how do we come up with test statistic? We usually try to come up with something close to the test parameter, like MLE.

$$\begin{aligned} \text{Test statistic} &\rightarrow |\hat{p}_{ML} - p| \\ z &= \frac{\hat{p}_{ML} - p}{\sqrt{\text{Var}(\hat{p}_{ML})}} \stackrel{app}{\sim} \text{ for large sample sizes} \\ \text{Var}(\hat{p}_{ML}) &= \text{Var}\left(\frac{1}{n} \sum_{i=1}^n x_i\right) = \frac{p(1-p)}{n} \end{aligned}$$

Suppose we want  $\alpha = P(\text{Type I error} \leq 0.05)$

$$P_{p=0.1}(Z > k) \leq 0.05$$

$$P_{p=0.1}(Z > k) = 0.05$$

Using the table for Normal dist

$$\begin{aligned} k &= 1.65 \\ P_{p=0.1}(Z > 1.65) &\approx 0.05 \\ Z &= \frac{\hat{p}_{ML} - p}{\sqrt{\frac{p(1-p)}{n}}} \\ \frac{\hat{p}_{ML} - 0.1}{\sqrt{0.1(1-0.1)/n}} &= \frac{\hat{p}_{ML} - 0.1}{\sqrt{\frac{(0.1)(0.9)}{15}}} \\ \frac{\hat{p}_{ML} - 0.1}{\sqrt{0.1(0.9)/15}} &> 1.65 \end{aligned}$$

## 13 02/22/18

### 13.1 TSH (Large Sample Case)

We arrived at the fundamentals of testing statistical hypotheses:

- $H_0$
- $H_A$
- Test statistic
- Rejection Region

The two types of errors:

- Type I (rejecting  $H_0$  when it is true),  $\alpha = P(\text{Type I error}) = P(\text{Rejecting } H_0 \text{ when } H_0 \text{ true})$
- Type II (accepting  $H_0$  when it is false),  $\beta = P(\text{Type II error}) = P(\text{Accepting } H_0 \text{ when } H_0 \text{ false})$

$$H_0 : \theta = \theta_0$$

$$\begin{cases} H_A : \theta > \theta_0 \\ H_A : \theta < \theta_0 \\ H_A : \theta \neq \theta_0 \end{cases}$$

Where the first two are one sided and the last one is two sided.

Let's say  $\theta$  is the population mean  $\mu$ .

- A good estimator for  $\theta$  based on the observations:  $\theta = \mu \rightarrow \hat{\theta} = \bar{X}_n$
- $\bar{X}_n - \mu \rightarrow \mu_0$  under  $H_0$  (so we look at what we can estimate with the data and what our hypothesis was and how close they are, whether or not the hypothesis was reasonable or not)
- Of course, it is easier for us if we standardize this, i.e.  $\frac{\bar{X}_n - \mu}{\sigma_{\bar{X}_n}}$ . We say it is worse for the null hypothesis when these two are far from each other, but what do we mean by far? This fraction will be our test statistic, we divide to get rid of the scale in variance, it is **scale invariant**, the units cancel out.

$$\bullet \text{ Rejection Region } \begin{cases} \{Z : Z > K\} & \text{if } H_A : \theta > \theta_0 \\ \{Z : Z < K\} & \text{if } H_A : \theta < \theta_0 \\ \{Z : |Z| > K\} & \text{if } H_A : \theta \neq \theta_0 \end{cases}$$

So we have:

$$\frac{\text{good estimator} - \text{Value of the parameter under } H_0}{\text{Std dev of the estimator } H \text{ under } H_0}$$

**Ex. 10.5, p497**  $n = 36, \bar{X}_{36} = 17, S_{36-1}^2 = 9, \alpha = 0.05, H_0 = \mu = 15, H_A : \mu > 15, \alpha = 0.05$ .

Test statistic:

$$Z = \frac{\bar{X}_n - 15}{\sigma_{\bar{X}_n}} = \frac{\bar{X}_n - 15}{\sigma \frac{\sigma}{\sqrt{n}}} = \frac{\bar{X}_n - 15}{\sigma \frac{3}{\sqrt{36}}} = \frac{\bar{X}_n - 15}{\frac{1}{2}}$$

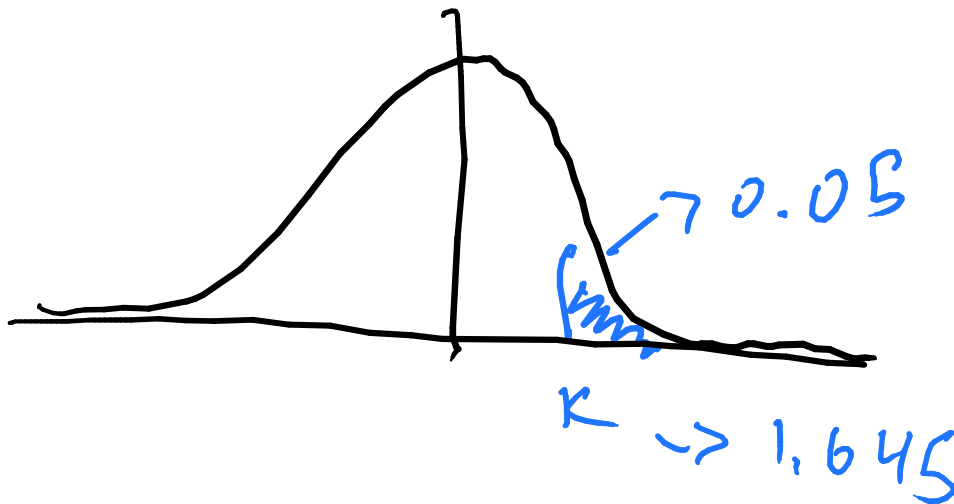
$$= 2(\bar{X}_n - 15)$$

$$RR = \{Z : Z > K\}$$

$$\alpha = P_{H_0}(RR)$$

$$= P(\text{Rejecting } H_0 \text{ when } H_0 \text{ is true})$$

$$P\{Z > K\} = 0.05$$



$$P_{H_0}(Z > 1.645) = 0.01$$

$$\bar{X}_{36} = \frac{\bar{X}_n - 15}{\frac{1}{2}}$$

$$\frac{17 - 15}{\frac{1}{2}} = 4$$

$H_0$  is rejected since  $4 > 1.645$ .

**Ex. 10.6, p498**  $\alpha = 0.01, n = 100, \#$  defectives 15,  $n = 100$ .  $H_o : p = 0.1, H_A : p > 0.1$ .

$$\hat{p}_{ML} = \hat{p}_n = \frac{1}{n} \sum_{i=1}^n X_i = \frac{1}{100} \sum_{i=1}^{100} X_i = \frac{1}{100} \times 15 = 0.15$$

$$X_i = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ item is defective} \\ 0 & \text{otherwise} \end{cases}$$



$$Z = \frac{\hat{p}_{ML} - 0.1}{\sigma_{\hat{p}_{ML}}} = \frac{\hat{p}_{ML} - 0.1}{\sqrt{\frac{p(1-p)}{n}}}$$

$p$  should be replaced by 0.1 ( $H_0 : p_0 = 1$ )

$$Z = \frac{\hat{p}_{ML} - 0.1}{\sqrt{\frac{0.1(1-0.1)}{100}}}$$

$$RR = \{Z : Z > K\}$$

$$P_{H_0}\{Z > K\} = 0.01$$

Table 4, App 3 gives  $K = 2.33$

Note, using CLT,  $Z \sim N(0, 1)$

The observed  $\hat{p}_{ML} = 0.15$

$$Z_{obs} = \frac{0.15 - 0.1}{\sqrt{\frac{0.1 \cdot 0.9}{100}}} = \frac{5}{3} = 1.667$$

$$1.667 < 2.33 \implies \text{cannot reject } H_0$$

**Ex. 10.8, p508**  $H_0 : \mu = 15, H_A : \mu = 16$  Find  $\beta$ ,  $\bar{X}_{36} = 17, S^2 = 9, \alpha = 0.05$ .

$$RR = \{Z : Z > 1.645\}$$

$$\beta = P(\text{Type II error}) = P(\text{Accept } H_0 \text{ when } H_0 \text{ is false})$$

$$Z = \frac{\bar{X}_n - \mu_A}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{X}_n - 16}{\frac{3}{\sqrt{36}}}$$

$$\frac{\bar{X}_n - 15}{\frac{3}{\sqrt{36}}} > 1.645$$

$$RR : \bar{X}_n > 1.645 \cdot \frac{3}{\sqrt{36}} + 15 \approx 15.823$$

Accepts  $H_0 \implies P_{H_A}(\bar{X}_n \leq 15.823)$

$$\beta = P_{H_A}(\bar{X}_n \leq 15.823) = P_{H_A}\left(\frac{\bar{X}_n - 16}{\frac{3}{\sqrt{36}}} < \frac{15.823 - 16}{\frac{3}{\sqrt{36}}}\right)$$

$$= P_{H_A}(Z \leq -0.36) \approx 0.36$$

Type I error was controlled to be 5%, but we see that type II error is quite high. The question is, can we choose sample size such that we can also control type II error?

**Choosing the sample size such that we can control Type II error at a given level**

$$H_0 : \mu = \mu_0, H_A : \mu = \mu_A$$

$$\begin{aligned} \alpha &= P_{\mu_0}(\bar{X}_n > K) = P_{\mu_0}\left(\frac{\bar{X}_n - \mu_0}{\frac{\sigma}{\sqrt{n}}} > \frac{K - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right) \\ &= P_{\mu_0}(Z > Z_\alpha) \\ \beta &= P_{\mu_A}(\bar{X}_n \leq K) = P_{\mu_A}\left(\frac{\bar{X}_n - \mu_A}{\frac{\sigma}{\sqrt{n}}} \leq \frac{K - \mu_A}{\frac{\sigma}{\sqrt{n}}}\right) \\ &= P_{\mu_A}(Z < -Z_\beta) \end{aligned}$$

$$\left. \begin{aligned} \frac{K - \mu_0}{\frac{\sigma}{\sqrt{n}}} &= Z_\alpha \\ \frac{K - \mu_A}{\frac{\sigma}{\sqrt{n}}} &= -Z_\beta \end{aligned} \right\} \implies K = \mu_0 + Z_\alpha \frac{\sigma}{\sqrt{n}} = \mu_A - Z_\beta \frac{\sigma}{\sqrt{n}}$$

$$\begin{aligned} \mu_A - \mu_0 &= \frac{\sigma}{\sqrt{n}}(Z_\alpha + Z_\beta) \\ n &= \frac{(Z_\alpha + Z_\beta)^2 \sigma^2}{(\mu_A - \mu_0)^2} \end{aligned}$$

**Ex. 10.9**  $H_0 : \mu = 15, H_A : \mu = 16$ . Want  $\alpha = \beta = 0.05$

$$\begin{aligned} Z_\alpha &= 1.645 = Z_\beta, \sigma^2 \rightarrow 9 \\ n &= \frac{(1.645 + 1.645)^2 (9)}{(16 - 15)^2} = 97.4 \approx 98 \end{aligned}$$

## 13.2 p-Value (Fisher's perspective)

The view that was presented was the Neyman-Pearson approach.

You have a claim and some data, want to know how likely it is to have the data under that claim.

Table 1:

$\theta$	$\mu$	$p$	$\sigma^2$	$\mu_1 - \mu_2$	$\frac{\sigma_1^2}{\sigma_2^2}$
$\hat{\theta}$	$\bar{X}_n$	$\hat{p}_n$	$S^2$	$\bar{X}_1 - \bar{X}_2$	$\frac{S_1^2}{S_2^2}$

Table 2:

$H_A$	$\theta > \theta_0$	$\theta < \theta_0$	$\theta \neq \theta_0$
RR	$\hat{\theta} > K$	$\hat{\theta} < K$	$ \hat{\theta}  > K$

**Ex. 10.5 again**  $H_0 : \mu = 15, \bar{X}_{36} = 17$

Instead of calculating  $P_{H_0}(\bar{X}_n = 17)$ , get  $P_{H_0}(\bar{X}_n > 17)$ , probability to observe something as bad as we observed with our null hypothesis.

$$P_{H_0} \left( \frac{X_n - 15}{\frac{3}{\sqrt{36}}} > \frac{17 - 15}{\frac{3}{\sqrt{36}}} \right)$$

$$\underbrace{P_{H_0}(Z > 4) \approx 0}_{p\text{-value}}$$

If this probability is small, what does it mean? A small  $p$ -value means the chance of observing something similar to what we observed against the null is small but we reject it anyway, why? Because we have observed something against the null, so either the hypothesis is wrong or the data is and so we let go of the hypothesis.

$$p\text{-value} = P \left( \underbrace{\bar{X}_n}_{\text{test statistic}} \overset{\text{aligned with } H_A}{>} \underbrace{17}_{\text{observed value of test statistic}} \right)$$

## 14 02/27/18

### 14.1 $p$ -Value

**Ex.** # 10.7 (p500) & 10.11 (p515)

$\alpha = 0.05, n_1 = 50, \bar{Y}_1 = 3.65, n_2 = 50, \bar{Y}_2 = 3.85, S_1^2 = 0.13, S_2^2 = 0.14, H_0 : \mu_1 - \mu_2 = 0, H_A : \mu_1 - \mu_2 \neq 0$

- Using Table 1, the test statistic,  $\bar{Y}_1 - \bar{Y}_2$ .
- Using Table 2, the RR =  $\{|\bar{Y}_1 - \bar{Y}_2| > k\}$

$$0.05\alpha = P_{H_0}(|\bar{Y}_1 - \bar{Y}_2| > K) = P_{H_0:\mu_1-\mu_2=0} \left( \left| \frac{\bar{Y}_1 - \bar{Y}_2 - 0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \right| > \frac{k - 0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \right)$$

$$\text{CLT} \approx P \left( |Z| > \frac{k}{\sqrt{\frac{0.18}{50} + \frac{0.19}{50}}} \right)$$

Using Table 4, App 3, p848

$$\frac{k}{\sqrt{\frac{0.18}{50} + \frac{0.19}{50}}} = 1.96 \implies k \approx 0.1568$$

$$\begin{aligned} p\text{-Value} &= P_{H_0:\mu_1-\mu_2=0} (|\bar{Y}_1 - \bar{Y}_2| > |\bar{y}_1 - \bar{y}_2|) \\ &= P_{H_0:\mu_1-\mu_2=0} (|\bar{Y}_1 - \bar{Y}_2| > |3.6 - 3.8|) \\ &= P_{H_0:\mu_1-\mu_2=0} (|\bar{Y}_1 - \bar{Y}_2| > 0.2) \\ &= P_{H_0:\mu_1-\mu_2=0} \left( \left| \frac{(\bar{Y}_1 - \bar{Y}_2) - 0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \right| > \frac{0.2 - 0}{\sqrt{\frac{0.18}{50} + \frac{0.19}{50}}} \right) \\ \text{CLT} &\approx P_{H_0:\mu_1-\mu_2} (|Z| > 2.5) \end{aligned}$$

Using Table 4, App 3

$$= 2P(Z > 2.5) = 2(0.0062) = 0.0124$$

## 14.2 TSH and CI

We note that there's a strong relation between testing statistical hypotheses and confidence intervals, about whether we want to accept something that falls in the interval or not. So, using confidence intervals, we can produce acceptance regions and complements of acceptance regions, i.e. rejection regions.

$$100(1 - \alpha)\% \text{ CI, } \hat{\theta}_{ML} \pm Z_{\frac{\alpha}{2}} \sigma_{\hat{\theta}_{ML}}$$

$$\overline{RR}_\alpha = \{\hat{\theta}_{ML} : \hat{\theta}_{ML} - z_{\frac{\alpha}{2}} \sigma_{\hat{\theta}_{ML}} \leq \theta_0 \leq \hat{\theta}_{ML} + z_{\frac{\alpha}{2}} \sigma_{\hat{\theta}_{ML}}\}$$

a  $100(1 - \alpha)\%$  acceptance region.

$$RR_\alpha = \left\{ \hat{\theta}_{ML} : \left| \frac{\hat{\theta}_{ML} - \theta_0}{\sigma_{\hat{\theta}_{ML}}} \right| > z_{\frac{\alpha}{2}} \right\}$$

Most people prefer using confidence intervals instead of tsh, but you can transform CIs into TSH.

### 14.3 Small-Sample TSH

- For  $\mu$ :

Assumptions:  $Y_1, \dots, Y_n$  are independent and identically distributed (iid) samples from  $N(\mu, \sigma^2)$ ,  $Y_i \stackrel{iid}{\sim} N(\mu, \sigma^2), i = 1, \dots, n$

$$H_0 : \mu = \mu_0$$

$$H_A : \begin{cases} \mu > \mu_0 & \text{(upper tail)} \\ \mu < \mu_0 & \text{(lower tail)} \\ \mu \neq \mu_0 & \text{(Two-tailed)} \end{cases}$$

Test statistic  $\bar{Y}$

$$RR = \begin{cases} \{\bar{Y} > k\} & \text{for } \mu > \mu_0 \\ \{\bar{Y} < k\} & \text{for } \mu < \mu_0 \\ \{|\bar{Y} - \mu_0| > k\} & \text{for } \mu \neq \mu_0 \end{cases}$$

$$k : P_{H_0}(RR) = \alpha$$

$$\begin{aligned} \alpha &= P_{H_0}(|\bar{Y} - \mu_0| > k) \\ &= P_{H_0} \left( \left| \frac{\bar{Y} - \mu_0}{\frac{s}{\sqrt{n}}} \right| > \frac{k}{\frac{s}{\sqrt{n}}} \right) \end{aligned}$$

Note that

$$T = P_{H_0}(|T| > k_1)$$

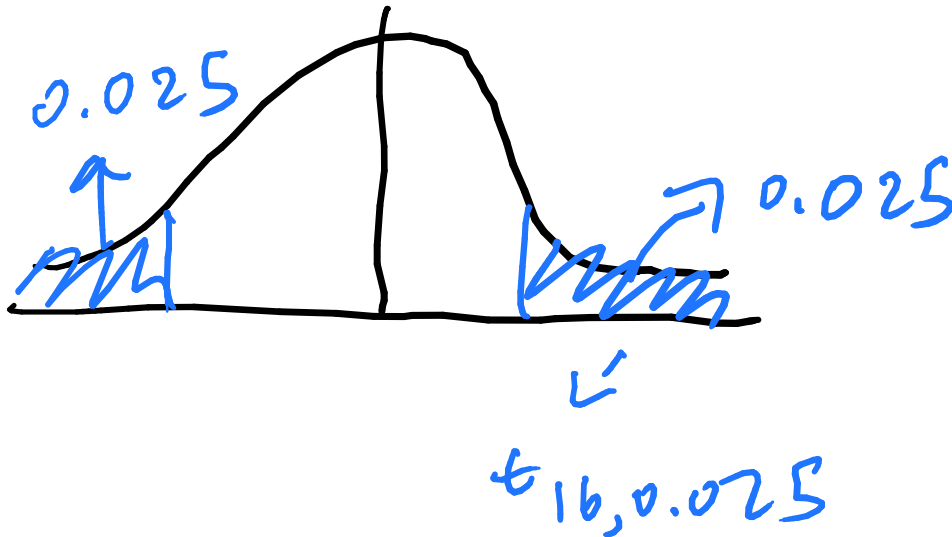
Where

$$T = \frac{\bar{Y} - \mu_0}{\frac{s}{\sqrt{n}}} \sim T_{(n-1)}$$

$$\text{and } s^2 = \frac{1}{n-1} \sum_{i=1}^k (Y_i - \bar{Y})^2 \text{ Table 5, App 3, p845}$$

$$k_1 \text{ s.t. } P_{H_0}(|T| > k_1) = \alpha$$

For example, if  $\alpha = 0.05, n = 17, df = n - 1 = 17 - 1 = 16$



$$t_{16,0.025} = 2.12$$

$$RR = \{|T| > 2.12\}$$

You calculate the observed  $t$

$$t_{obs} = \frac{\bar{y} - \mu_0}{\frac{s}{\sqrt{n}}}$$

- For the two-sample case, for  $\mu_1 - \mu_2$

Assumptions:

- $X_1, \dots, X_{n_1}$  and  $Y_1, \dots, Y_{n_2}$  are random samples
- $X_i \sim N(\mu_1, \sigma_1^2), Y_j \sim N(\mu_2, \sigma_2^2)$

- $\sigma_1^2 = \sigma_2^2$  (this is the strange one to keep in mind that isn't always there)
- $X_s \perp Y_s$

$$H_0 : \mu_1 - \mu_2 = D_0$$

$$H_A : \left\{ \mu_1 - \mu_2 > D_0 \text{ or } \mu_1 - \mu_2 < D_0 \right\}$$

Test

$$T = \frac{(\bar{X} - \bar{Y}) - D_0}{s_P \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$\text{where } s_p^2 = \frac{\underbrace{\sum_{i=1}^{n_1} (X_i - \bar{X})^2}_{(n_1-1)s_1^2} + \underbrace{\sum_{i=1}^{n_2} (Y_i - \bar{Y})^2}_{(n_2-1)s_2^2}}{n_1 + n_2 - 2}$$

under  $H_0 : T \sim T_{(n_1+n_2-2)}$

$$RR : \begin{cases} t > t_{\alpha, \nu} & (\text{upper}) \\ t < t_{\alpha, \nu} & (\text{lower}) \\ |t| > t_{\frac{\alpha}{2}, \nu} & (\text{two-sided}) \end{cases}$$

$$\nu = n_1 + n_2 - 2 = (n_1 - 1) + (n_2 - 1)$$

**Ex.** #10.14, p524

$$\alpha = 0.05, n_1 = 9, \bar{y}_1 = 35.22s, n_2 = 9, \bar{y}_2 = 31.56s$$

$$\sum_{i=1}^9 (y_{1i} - \bar{y}_1) = 195.56$$

$$H_0 : \mu_1 - \mu_2 = 0$$

$$\sum_{i=1}^9 (y_{2i} - \bar{y}_2) = 160.22$$

$$H_A : \mu_1 - \mu_2 \neq 0$$

$$T = \frac{(\bar{Y}_1 - \bar{Y}_2) - D_0}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$s_p^2 = \frac{\sum_{i=1}^9 (y_{1i} - \bar{y}_1)^2 + \sum_{i=1}^9 (y_{2i} - \bar{y}_2)^2}{n_1 + n_2 - 2}$$

$$= \frac{195.56 - 160.22}{9 + 9 - 2} = 22.24$$

$$s_p = 4.716$$

$$\text{The observed } t = \frac{(\bar{y}_1 - \bar{y}_2) - 0}{4.716 \sqrt{\frac{1}{9} + \frac{1}{9}}} = \frac{35.22 - 31.56}{4.716 \sqrt{\frac{2}{9}}} = 1.56.$$

The observed  $t$  value is 1.56

$$t_{obs} = 1.56$$

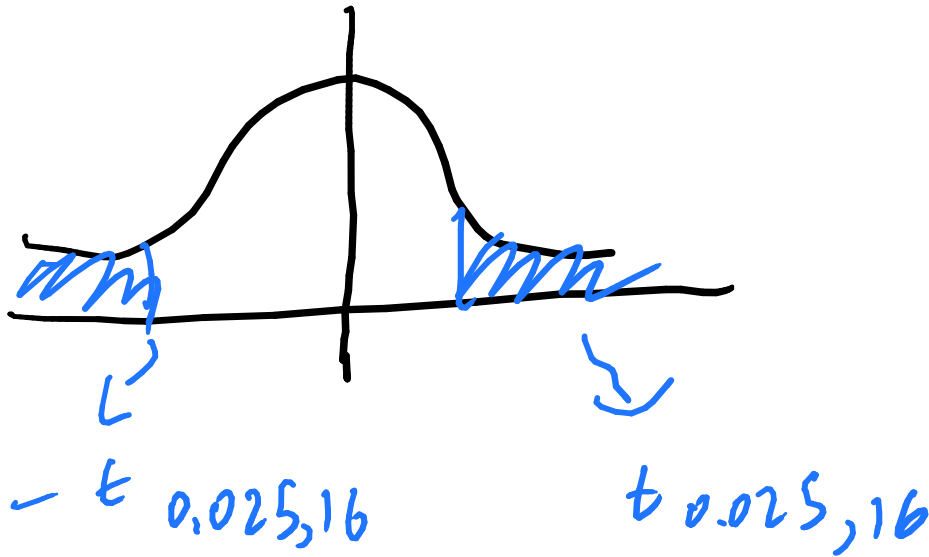


Table 5, App 3, p849

$$t_{0.025, 16} = 2.12$$

$$-2.12 < 1.56 < 2.12$$

We do **NOT** reject  $H_0$ .

### 14.3.1 Small-Sample TSH for $\sigma^2$

Assumption:  $Y_i \stackrel{iid}{\sim} N(\mu, \sigma^2)$

$$H_0 : \sigma^2 = \sigma_0^2$$

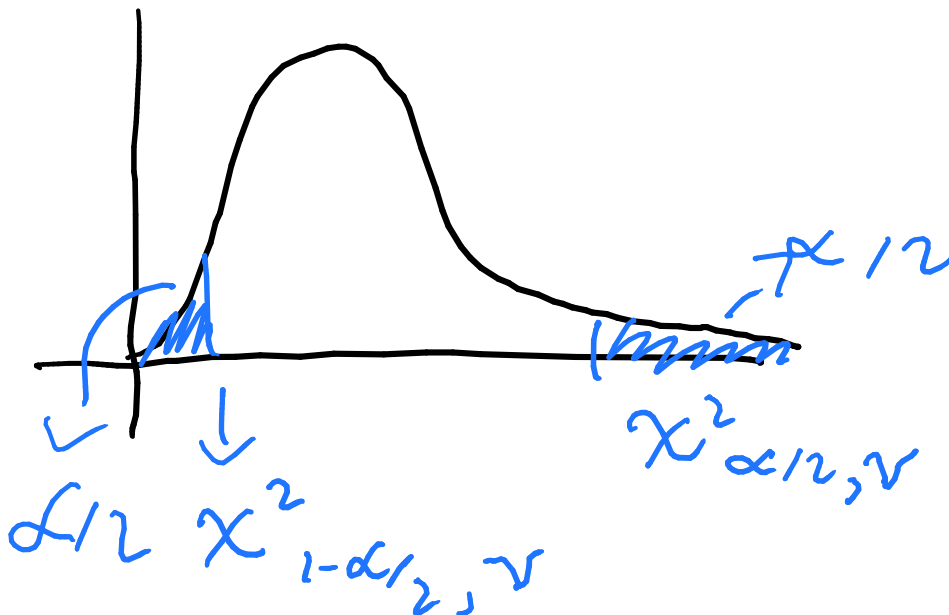
$$H_A : \begin{cases} \sigma^2 > \sigma_0^2 \\ \sigma^2 < \sigma_0^2 \\ \sigma^2 \neq \sigma_0^2 \end{cases}$$

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi_{(n-1)}^2 \text{ under } H_0 : \sigma^2 = \sigma_0^2$$



$$RR : \begin{cases} \chi^2 > \chi_{\alpha, \nu}^2 \\ \chi^2 < \chi_{1-\alpha, \nu}^2 \\ \chi^2 > \chi_{\frac{\alpha}{2}, \nu}^2 \text{ or } \chi^2 < \chi_{1-\frac{\alpha}{2}, \nu}^2 \text{ (not symmetric, no absolute val)} \end{cases} \quad \nu = n - 1$$

Table 6, App 3, p850:



**Ex.** 10.16, p532

$$n = 10, \alpha = 0.05, s^2 = 0.0003$$

$$H_0 : \sigma^2 = 0.0002, H_A : \sigma^2 > 0.0002$$

$$\chi_{obs}^2 = \frac{(10 - 1)(0.0003)}{0.0002} = 13.5$$

$$\chi^2 = \frac{(n - 1)S^2}{\sigma_0^2} \sim \chi_9^2$$

$$\chi_{0.05, 9}^2 = 16.919$$

$$13.5 < 16.919$$

$H_0$  is **NOT** rejected.

**Two-Sample case** Assumptions:

- $X_1, \dots, X_n$  and  $Y_1, \dots, Y_m$  are random samples.
- $X_i \sim N(\mu_1, \sigma_1^2)$ ,  $Y_j \sim N(\mu_2, \sigma_2^2)$
- $X_i \perp\!\!\!\perp Y_j$

$$H_0 : \sigma_1^2 = \sigma_2^2$$

$$H_A : \begin{cases} \sigma_1^2 > \sigma_2^2 \\ \sigma_1^2 < \sigma_2^2 \\ \sigma_1^2 \neq \sigma_2^2 \end{cases}$$

$$\frac{\sigma_1^2}{\sigma_2^2} \rightarrow \text{Test dist: } \frac{S_X^2}{S_Y^2} \rightarrow \text{F-distribution}$$

## 15 03/01/18

Continuation from last class, with small sample TSH and two samples.

$$S_X^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (x_i - \bar{x})^2$$

$$S_Y^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (y_i - \bar{y})^2$$

Independence of  $X_i$  is required for:

$$\frac{(n-1)S_X^2}{\sigma_1^2} \sim \chi_{(n-1)}^2$$

Similarly, for  $Y_i$ :

$$\frac{(n-1)S_Y^2}{\sigma_2^2} \sim \chi_{(n-1)}^2$$

Independence of  $X_i$ s and  $Y_i$ s needed for:

$$\frac{\chi_{n_1-1}^2/(n_1-1)}{\chi_{n_2-1}^2/(n_2-1)} \sim F_{(n_1-1), (n_2-1)}$$

$$\frac{S_X^2}{S_Y^2} = \frac{(n_1-1)S_X^2/(n_1-1)}{(n_2-1)S_Y^2/(n_2-1)} = \frac{\left[ \frac{(n_1-1)S_X^2}{\sigma^2} \right] / (n_1-1)}{\left[ \frac{(n_2-1)S_Y^2}{\sigma^2} \right] / (n_2-1)}$$

Where  $\sigma^2 = \sigma_1^2 = \sigma_2^2$  under  $H_0$ .

$$RR : \begin{cases} F > F_{\alpha,(n_1-1),(n_2-1)} \\ F < F_{1-\alpha,(n_1-1),(n_2-1)} \\ F > F_{\frac{\alpha}{2},(n_1-1),(n_2-1)} \\ \text{or } F < F_{1-\frac{\alpha}{2},(n_1-1),(n_2-1)} \end{cases}$$

		Women	Men
<b>Ex.</b> #10.21, p536	$n$	10	14
	$\bar{y}$	14.9	16.2
	$S^2$	26.4	12.7

$$H_0 : \sigma_W^2 = \sigma_M^2$$

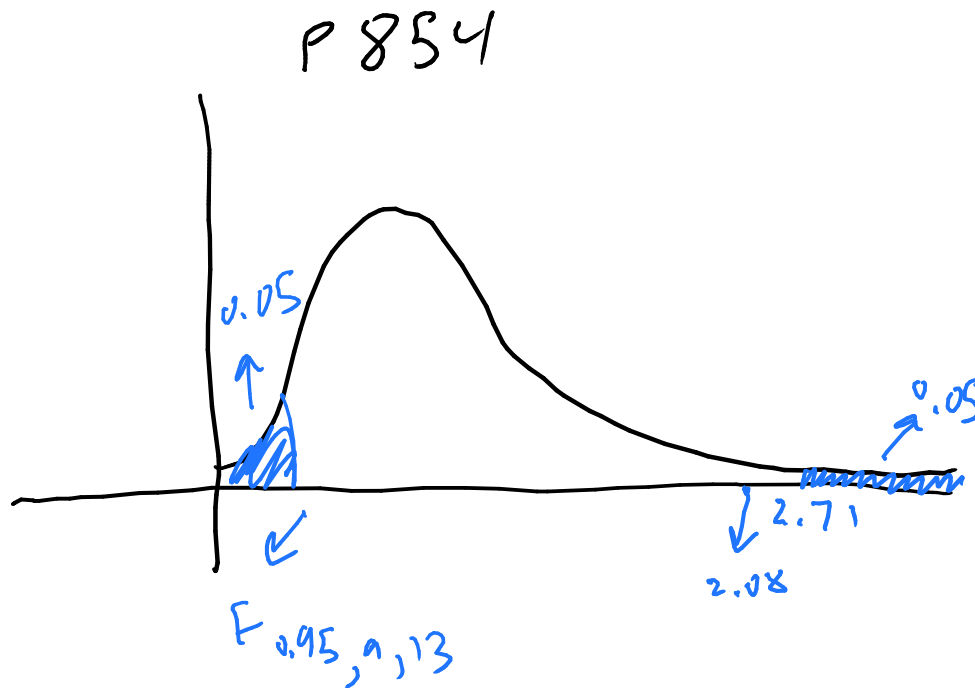
$$H_A : \sigma_W^2 \neq \sigma_M^2$$

$$\alpha = 0.1$$

$$F = \frac{\overbrace{26.4}^{\nu=10-1=9}}{\underbrace{12.7}_{\nu=14-1=13}} \approx 2.08$$

$$F_{0.05,9,13} = 2.71$$

$$\underbrace{F_{1-0.05,9,13}}_{0.95}$$



We do not reject  $H_0$ .

If  $F \sim \text{Fisher}_{\nu_1, \nu_2}$ :

$$F^{-1} \sim \text{Fisher}_{\nu_2, \nu_1}$$

$$F_{0.95, 9, 13} = F_{0.05, 13, 9}$$

$$P\text{-value} = 2P(F > F_{obs})$$

$$F_{obs} = \frac{\text{Larger of the two } S^2}{\text{Smaller of the two}}$$

$$P\text{-value} = 2P(F_{0,13} > \overbrace{2.08}^{\approx 2.16}) \approx 0.2$$

## 15.1 Neyman-Pearson Lemma (TSH)

$$H_0 : \theta = \theta_0, H_A : \theta = \theta_1$$

$$X_1, \dots, X_n \stackrel{iid}{\sim} f_\theta(x), \theta = \theta_1, \theta_2$$

$$\mathfrak{L}(\theta; x_1, \dots, x_n) = \prod_{i=1}^n f_\theta(x_i)$$

If you treat both likelihoods equally, we can see which one we should prefer going with:

$$\frac{\mathfrak{L}(\theta_0; x_1, \dots, x_n)}{\mathfrak{L}(\theta_1; x_1, \dots, x_n)} > 1$$

But what if we don't treat the two hypotheses equally? For example, considering giving people a high credit limit given information about them. It's more costly to give a conman who will disappear a high credit limit, versus not giving a high enough credit limit to someone who deserves it.

$$\frac{\mathfrak{L}(\theta_0; x_1, \dots, x_n)}{\mathfrak{L}(\theta_1; x_1, \dots, x_n)} < K$$

$$\begin{cases} \text{Reject } H_0 = 1 & \frac{\mathfrak{L}(\theta_0; x_1, \dots, x_n)}{\mathfrak{L}(\theta_1; x_1, \dots, x_n)} < K \\ \gamma & \frac{\mathfrak{L}(\theta_0; x_1, \dots, x_n)}{\mathfrak{L}(\theta_1; x_1, \dots, x_n)} = K \\ \text{Accept } H_0 = 0 & \frac{\mathfrak{L}(\theta_0; x_1, \dots, x_n)}{\mathfrak{L}(\theta_1; x_1, \dots, x_n)} > K \end{cases}$$

If  $X$  is a continuous r.v. “=” happens with probability zero.

---

$K$  is chosen such that

$$\begin{aligned} P_{H_0}(\text{Rejecting } H_0) &= \alpha \\ P\left(\frac{\mathfrak{L}(\theta_0; x_1, \dots, x_n)}{\mathfrak{L}(\theta_1; x_1, \dots, x_n)} < K\right) &= \alpha \end{aligned}$$

For the discrete case we have:

$$P\left(\frac{\mathfrak{L}(\theta_0; x_1, \dots, x_n)}{\mathfrak{L}(\theta_1; x_1, \dots, x_n)} < K\right) + \gamma P_{H_0}P\left(\frac{\mathfrak{L}(\theta_0; x_1, \dots, x_n)}{\mathfrak{L}(\theta_1; x_1, \dots, x_n)} = K\right) = \alpha$$

**Ex**  $X_i \stackrel{iid}{\sim} \text{Ber}(p), i = 1, \dots, n, H_0 : p = p_0, H_A : p = p_A, p_A > p_0.$

Step 1:

$$\mathfrak{L}(p; x_1, \dots, x_n) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i} = p^{\sum_{i=1}^n x_i} (1-p)^{1-\sum_{i=1}^n x_i}$$

Step 2:

$$\frac{\mathfrak{L}(p_0; x_1, \dots, x_n)}{\mathfrak{L}(p_A; x_1, \dots, x_n)} = \frac{p_0^{\sum_{i=1}^n x_i} (1-p_0)^{1-\sum_{i=1}^n x_i}}{p_A^{\sum_{i=1}^n x_i} (1-p_A)^{1-\sum_{i=1}^n x_i}}$$

Want to find  $K$  such that 2 holds for a given  $\alpha$ .

$$\begin{aligned}\alpha &= P_{H_0:p=p_0} \left( \left( \frac{p_0}{p_A} \right)^{\sum_{i=1}^n x_i} \left[ \frac{1-p_0}{1-p_A} \right]^{n-\sum_{i=1}^n x_i} < K \right) \\ &\quad + P_{H_0:p=p_0} \left( \left( \frac{p_0}{p_A} \right)^{\sum_{i=1}^n x_i} \left[ \frac{1-p_0}{1-p_A} \right]^{n-\sum_{i=1}^n x_i} = K \right) \\ RR &= \left\{ (x_1, \dots, x_n) : \left( \frac{p_0}{p_A} \right)^{\sum_{i=1}^n x_i} \left( \frac{1-p_0}{1-p_A} \right)^{n-\sum_{i=1}^n x_i} < K \right\} \\ &\cup \left\{ (x_1, \dots, x_n) : \left( \frac{p_0}{p_A} \right)^{\sum_{i=1}^n x_i} \left( \frac{1-p_0}{1-p_A} \right)^{n-\sum_{i=1}^n x_i} = K \right\}\end{aligned}$$

Let  $T = \sum_{i=1}^n X_i$ .

$$\begin{aligned}\phi(t) &= \left( \frac{p_0}{p_A} \right)^t \left( \frac{1-p_0}{1-p_A} \right)^{n-t} \text{ is a monotone function when } p_0 > p_A \\ \log \phi(t) &= t \log \frac{p_0}{p_A} + (n-t) \log \frac{1-p_0}{1-p_A} \\ &= t \log \left[ \frac{p_0/(1-p_0)}{p_A/(1-p_A)} \right] + n \log \frac{1-p_0}{1-p_A} = at + b\end{aligned}$$

$$\text{Now } \overbrace{p_0 > p_A}^{(1)} \iff (1-p_0) < (1-p_A) \iff \overbrace{\frac{1}{1-p_0} > \frac{1}{1-p_A}}^{(2)}$$

Using (1) & (2) we have:

$$\frac{p_0}{1-p_0} > \frac{p_A}{1-p_A} \iff \frac{p_0/(1-p_0)}{p_A/(1-p_A)} > 1$$

and hence  $\log \phi(t)$  is a monotone increasing ( $\phi \uparrow$ ) function of  $t$  if  $p_0 > p_A$

$$\frac{\mathfrak{L}(p_0; x_1, \dots, x_n)}{\mathfrak{L}(p_A; x_1, \dots, x_n)} < K \iff T < K_1$$

$$\begin{cases} 1 & \text{if } T < K_1 \\ \gamma & \text{if } T = K_1 \\ \gamma & \text{if } T > K_1 \end{cases}$$

“1” is rejection, “0” is acceptance.  $T = \sum_{i=1}^n X_i \stackrel{\text{under } H_0}{\sim} \text{Bin}(n, p_0)$ .

Choose  $K_1$ :

$$P_{H_0:p=p_0}(T < K_1) + \gamma P_{H_0}(T = K_1) = \alpha$$

Find  $K_1$  such that

$$P_{H_0:p=p_0}(T < K_1) < \alpha < P_{H_0:p=p_0}(T \leq K_1)$$

and

$$\gamma = \frac{\alpha - P_{H_0}(T < K_1)}{P_{H_0}(T = K_1)}$$

## 16 03/15/18

### 16.1 Likelihood Ratio Tests

We have a bunch of observations:  $\lambda(x_1, \dots, x_n) = \frac{\max_{\vec{\theta} \in \Theta_0} \mathcal{L}(\theta; x_1, \dots, x_n)}{\max_{\vec{\theta} \in \Theta} \mathcal{L}(\theta; x_1, \dots, x_n)} < K$

$$\mathcal{L}(\theta; X_1, \dots, X_n) = \prod_{i=1}^n f(x_i, \theta)$$

If we have a random sample, i.e.  $X_i \stackrel{iid}{\sim} f$

$$X_i \stackrel{iid}{\sim} N(\overbrace{\mu, \sigma^2}^{\text{unknown}}, \vec{\theta} = (\mu, \sigma^2))$$

$H_0 : \mu = \mu_0$  (Composite hypothesis)

$$\Theta = \left\{ \vec{\theta} = (\mu, \sigma^2) : \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2} dx = 1 \right\}$$

$$= \left\{ \vec{\theta} = (\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 > 0 \right\}$$

$$\Theta_0 = \left\{ \vec{\theta} = (\mu_0, \sigma^2) : \sigma^2 > 0 \right\}$$

What values of  $\lambda$  support the null and which support the alternative?  $\Theta_0$  is a subset of  $\Theta$ . If  $\mu_0$  is really the true value, then the numerator will be close to the denominator, i.e. the ratio  $\lambda$  will be close to 1. The further away  $\mu_0$  is from the true  $\mu$ , the further the ratio will be from 1. This is because if the maximum is contained in a subset, then maximizing over the subset will be the same as maximizing over the whole set.

The denominator here is based off the maximum likelihood method. When  $n$  is very large, the maximum likelihood is close to the true value of  $\theta$ . The fact that this is right headed and more concentrated around the truth for large  $n$ , then the denominator is the

best proxy for this.

**Ex.**  $X_i \stackrel{iid}{\sim} N(\mu, \sigma^2), i = 1, 2, \dots, n$

The example that motivated Fisher, fertilizing plants with perfect weather and conditions.

$$H_0 : \mu = \mu_0$$

$$X_{ij} = \mu + \alpha_i + \overbrace{\varepsilon_{ij}}^{\text{Fluctuation}}, i = 1, 2 \text{ (2 different fertilizers)}, j = 1, \dots, n_i$$

$$\varepsilon_{ik} \sim N(0, \tau^2)$$

$$H_0 : \alpha_1 = \alpha_2 \text{ (if we accept this then we can say the fertilizers are the same)}$$

$$X_i \stackrel{iid}{\sim} N(\mu, \sigma^2), i = 1, \dots, n$$

$$H_0 : \mu = \mu_0 \rightarrow \text{known value}$$

$$\lambda(x_1, \dots, x_n) = \frac{\max_{\theta \in \Theta_0} \mathcal{L}(\theta; x_1, \dots, x_n)}{\max_{\theta \in \Theta} \mathcal{L}(\theta; x_1, \dots, x_n)}$$

Step 1:

$$\max_{\theta \in \Theta_0} = \max_{\mu_0, \sigma^2} \prod_{i=1}^n \frac{1}{\sigma \sqrt{2\pi}} e^{-\left(\frac{1}{2\sigma^2}\right)(x_i - \mu)^2} = \max_{\sigma^2} \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^n e^{-\sum_{i=1}^n \frac{(x_i - \mu_0)^2}{2\sigma^2}}$$

Taking logarithm:

$$\max_{\sigma^2} \left[ \frac{-n}{2} \log 2\pi - \frac{n}{2} \log \sigma^2 - \sum_{i=1}^n \frac{(x_i - \mu_0)^2}{2\sigma^2} \right]$$

Taking  $\frac{d}{d\sigma^2}$ :

$$-\frac{n}{2} \cdot \frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (x_i - \mu_0)^2 = 0$$

$$\sigma_0^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu_0)^2$$

$$\begin{aligned} \max_{\theta \in \Theta_0} \mathcal{L}(\vec{\theta}; x_1, \dots, x_n) &= \mathcal{L}(\mu_0, \hat{\sigma}_0^2; x_1, \dots, x_n) \\ &= \left( \frac{1}{\sqrt{2\pi}\hat{\sigma}_0} \right)^n e^{-\frac{\sum_{i=1}^n (x_i - \mu_0)^2}{2\hat{\sigma}_0^2}} \end{aligned}$$



$$= \left( \frac{1}{\sqrt{2\pi\hat{\sigma}_0}} \right)^n e^{-\frac{n}{2}}$$

Step 2:

$$\begin{aligned} & \max_{\theta \in \Theta} \mathcal{L}(\vec{\theta}; x_1, \dots, x_n) \\ \hat{\mu}_{ML} &= \bar{x}, \hat{\sigma}_{ML}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \\ \max_{\theta \in \Theta} \mathcal{L}(\vec{\theta}; x_1, \dots, x_n) &= \left( \frac{1}{\sqrt{2\pi\hat{\sigma}_{ML}}} \right)^n e^{-\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{2\hat{\sigma}_{ML}^2}} \\ & \quad \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\hat{\sigma}_{ML}^2} = n \\ \max_{\theta \in \Theta} \mathcal{L}(\vec{\theta}; x_1, \dots, x_n) &= \left( \frac{1}{\sqrt{2\pi\hat{\mu}_{ML}}} \right)^n e^{-\frac{n}{2}} \\ \lambda(x_1, \dots, x_n) &= \frac{\left( \frac{1}{\sqrt{2\pi\hat{\sigma}_0}} \right)^n e^{-\frac{n}{2}}}{\left( \frac{1}{\sqrt{2\pi\hat{\sigma}_{ML}}} \right)^n e^{-\frac{n}{2}}} \\ &= \left( \frac{\hat{\sigma}_{ML}^2}{\hat{\sigma}_0^2} \right)^{\frac{n}{2}} \\ \hat{\sigma}_0^2 &= \frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2, \hat{\sigma}_{ML}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}_n)^2 \\ \sum_{i=1}^n (x_i - \mu_0)^2 &= \sum_{i=1}^n [(x_i - \bar{x}_n) + (\bar{x}_n - \mu_0)]^2 \\ &= \sum_{i=1}^n (x_i - \bar{x}_n)^2 + n(\bar{x}_n - \mu_0)^2 + 2(\bar{x}_n - \mu_0) \underbrace{\sum_{i=1}^n (x_i - \bar{x}_n)}_0 \\ &= \sum_{i=1}^n (x_i - \bar{x}_n)^2 + n(\bar{x}_n - \mu_0)^2 \\ \hat{\sigma}_0^2 &= \hat{\sigma}_{ML}^2 + (\bar{x}_n - \mu_0)^2 \\ \lambda(x_1, \dots, x_n) &= \left( \frac{\hat{\sigma}_{ML}^2}{\hat{\sigma}_0^2} \right)^{\frac{n}{2}} = \left( \frac{\hat{\sigma}_{ML}^2}{\hat{\sigma}_{ML}^2 + (\bar{x}_n - \mu_0)^2} \right)^{\frac{n}{2}} \\ &= \left( \frac{1}{1 + \frac{(\bar{x}_n - \mu_0)^2}{\hat{\sigma}_{ML}^2}} \right)^{\frac{n}{2}} \end{aligned}$$

$$P_{H_0:\mu=\mu_0}(\lambda(X_1, \dots, X_n) < K) = \alpha$$

$$\{\lambda(X_1, \dots, X_n) < K\} \text{ if and only if } \{U > K'\}$$

$$P_{H_0:\mu=\mu_0}(\lambda(X_1, \dots, X_n) < K) = P_{H_0:\mu=\mu_0}(U > K')$$

I need to find  $K'$  such that:

$$P_{H_0:\mu=\mu_0}(U > K') = \alpha \rightarrow \text{prespecified}$$

$$U = \frac{n(\bar{X}_n - \mu_0)^2}{\sum_{i=1}^n (X_i - \bar{X}_n)^2}$$

Assume  $\sigma_0^2$  is the true value of the variance, unknown to us

$$U = \frac{n(\bar{x}_n - \mu_0)^2/\sigma_0^2}{\sum_{i=1}^n (x_i - \bar{x}_n)^2/\sigma_0^2} = \frac{n(\bar{x}_n - \mu_0)^2/\sigma_0^2}{\underbrace{(n-1)S_n^2}_{\sim \chi_{(n-1)}^2}}$$

$$\begin{aligned} \frac{\bar{X}_n - \mu_0}{\frac{\sigma_0}{\sqrt{n}}} &\sim N(0, 1) \\ \implies \left[ \frac{\bar{X}_n - \mu_0}{\frac{\sigma_0}{\sqrt{n}}} \right]^2 &\sim \chi_{(1)}^2 \\ Z \sim N(0, 1) &\implies Z^2 \sim \chi_{(1)}^2 \end{aligned}$$

$\bar{X}_n$  is independent of  $S_n^2$  if  $X_i \stackrel{iid}{\sim} N(0, 1)$ .  $X_i = \mu + \varepsilon_i, \varepsilon_i \stackrel{iid}{\sim} N(0, \sigma^2)$

$$X_i = \underbrace{\mu}_{\bar{X}} + \underbrace{(X_i - \bar{X})}_{\varepsilon_i}, \bar{X} \perp (X_1 - \bar{X}, \dots, X_n - \bar{X}_n)$$

Using these three facts ( $\chi^2$  distributed and independence):

$$(n-1)U = \frac{\chi_{(1)}^2/1}{\chi_{(n-1)}^2/n-1} \implies (n-1)U \sim F_{1,n-1}$$

**Theorem:** Thm 10.2

$$-2 \ln \lambda(\vec{X}) \xrightarrow{D} \chi_{d.f.}^2 \text{ for large } n$$

where  $d.f. = \dim \Theta - \dim \Theta_0$

In our case:

$$\Theta(\mu, \sigma^2) \implies \dim \Theta = 2$$

$$\Theta_0(\mu_0, \sigma^2) \implies \dim \Theta_0 = 1$$

$$H_0 : \mu \in (a, b) \rightarrow \text{Problem!}$$

## 16.2 Regression Analysis

(Not following the book, look at typed notes uploaded on myCourses)

$$(X, Y)$$

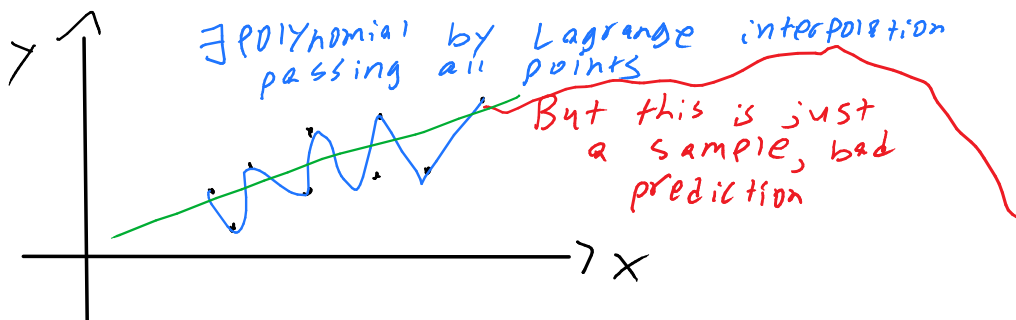
Get  $Y = f(X)$ . In this course we'll do linear regression.

$$(X_i, Y_i), i = 1, 2, \dots, n$$

Say height and weight.

$$Y_i = \alpha + \beta X_i + \varepsilon_i, \varepsilon_i \stackrel{iid}{\sim} F, E(\varepsilon_i) = 0$$

$$E[Y_i | X_i = x_i] = \alpha + \beta x_i$$



The challenge in statistics is fit vs prediction.

## 17 03/20/18

### 17.1 Linear Regression

$$E(Y) = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k$$

$x_i$  independent variables. For example, let's say:

$$E(\underbrace{Y}_{\text{Height}}) = \underbrace{\beta_0}_{\text{intercept}} + \underbrace{\beta_1}_{\text{slope}} \underbrace{x_1}_{\text{Age}}$$

Fitted model:  $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

Deterministic model (no error, ex.  $F = ma$ ) vs probabilistic model ( $y = \beta_0 + \beta_1 x + \varepsilon$ ,  $E(\varepsilon) = 0$ )

We are looking at  $y$ s that are linear in  $\beta_i$ .

Simple linear model:  $y = \beta_0 + \beta_1 x + \varepsilon$ . We want to minimize the distance from the line generated to the points, i.e.

$$\begin{aligned} Q = \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 x_i)^2 &\implies \begin{cases} \frac{\partial Q}{\partial \beta_0} = -2 \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) = 0 \implies \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \\ \frac{\partial Q}{\partial \beta_1} = -2 \sum_{i=1}^n x_i (y_i - \beta_0 - \beta_1 x_i) = 0 \end{cases} \\ &\implies \hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i - n \bar{x} \bar{y}}{\sum_{i=1}^n x_i^2 - n \bar{x}^2} = \frac{S_{XY}}{S_{XX}} \end{aligned}$$

$$\begin{aligned} \text{Var}(\hat{\beta}_1) &= \text{Var}(\bar{y} - \hat{\beta}_1 \bar{x}) = \text{Var}(\bar{y}) + \text{Var}(\hat{\beta}_1 \bar{x}) - 2\text{Cov}(\bar{y}, \hat{\beta}_1 \bar{x}) = \frac{\sigma^2}{n} \\ \hat{\beta}_1 &= \frac{S_{XY}}{S_{XX}} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{\sum_{i=1}^n (x_i - \bar{x})y_i - \bar{y} \sum_{i=1}^n (x_i - \bar{x})}{S_{XX}} \\ &= \frac{\sum_{i=1}^n (x_i - \bar{x})y_i}{S_{XX}} = \sum_{i=1}^n c_i y_i \text{ s.t. } c_i = \frac{x_i - \bar{x}}{S_{XX}}, \sum_{i=1}^n c_i = 0 \\ \implies \text{Var}(\hat{\beta}_1) &= \text{Var}\left(\sum_{i=1}^n c_i y_i\right) = \sum_{i=1}^n c_i^2 \text{Var}(y_i) = \sigma^2 \sum_{i=1}^n c_i^2 = \frac{\sigma^2}{S_{XX}} \\ \text{Cov}(\bar{y}, \hat{\beta}_1) &= \text{Cov}\left(\sum_{i=1}^n y_i/n, \sum_{i=1}^n c_i y_i\right) = \sum_{i=1}^n \frac{c_i}{n} \text{Cov}(y_i, y_i) = \frac{\sigma^2}{n} \sum_{i=1}^n c_i = 0 \end{aligned}$$

The least squares method is useful since we don't need to know the distribution of the population. In fact, if the population is normally distributed, the least squares method uses

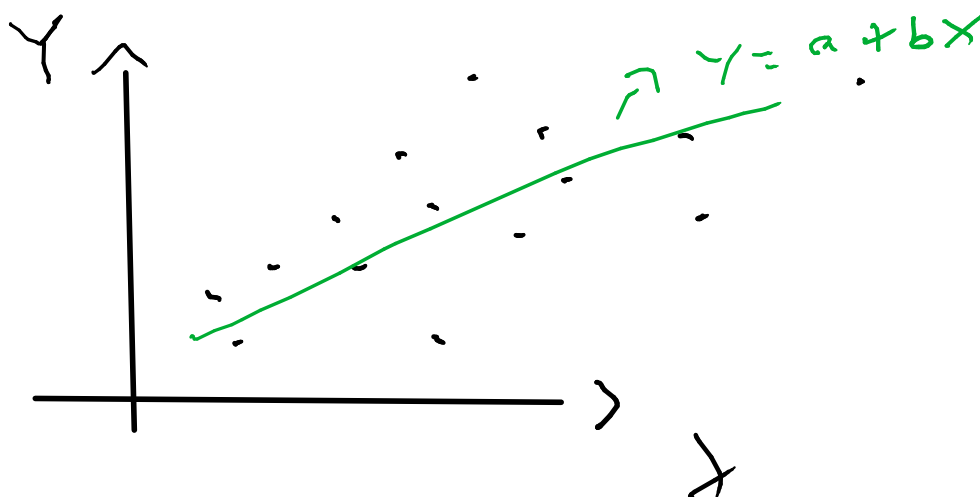
the same estimators as the MLE.

Why does  $E(\hat{\beta}_0) = \beta_0$  and  $E(\hat{\beta}_1) = \beta_1$ , i.e. they are unbiased?

$$\begin{aligned}\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\ E(\bar{y} - \hat{\beta}_1 \bar{x}) &= E(\bar{y}) - \bar{x} E(\hat{\beta}_1) = \beta_0 \\ E(\hat{\beta}_1) &= E\left(\sum_{i=1}^n c_i y_i\right) = \sum_{i=1}^n c_i E(\beta_0 + \beta_1 x_i) = \beta_1 \sum_{i=1}^n c_i x_i \\ Cov(\hat{\beta}_0, \hat{\beta}_1) &= Cov(\bar{y}, \hat{\beta}_1) - \bar{x} Cov(\hat{\beta}_1, \hat{\beta}_1) = 0 - \bar{x} Var(\hat{\beta}_1) = -\bar{x} \frac{\sigma^2}{S_{XX}}\end{aligned}$$

18 03/22/18

**Recap:** We have  $\underbrace{(X_i)}_h, \underbrace{(Y_i)}_w, i = 1, 2, \dots, n$

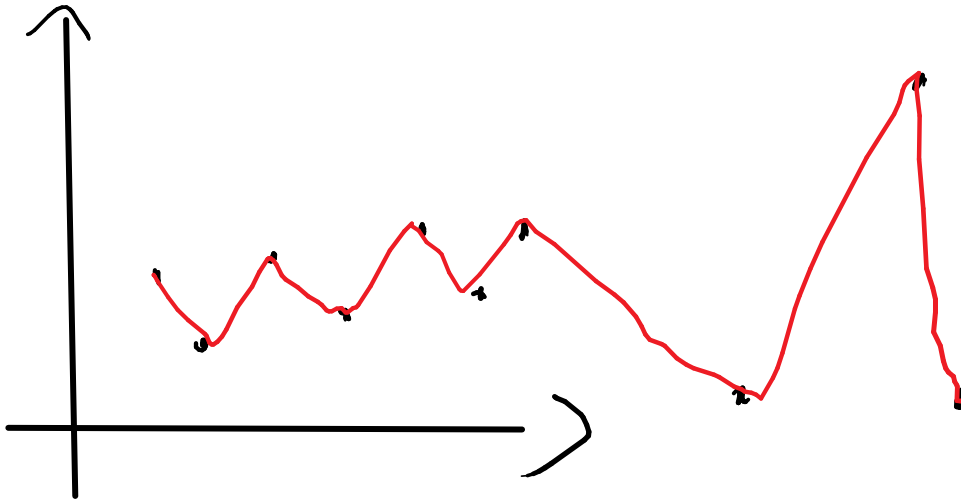


There exists a polynomial that passes through all these points (Lagrange interpolation), but statisticians use a simpler function for prediction, we do linear regression. Now, we want to minimize the distance squared between the line and the points given:

$$\begin{aligned}\hat{y}_i &= a + bx_i \\ \overbrace{Q(a, b)}^{\text{residual}} &= \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n (y_i - (a + bx_i))^2 \\ \min_{(a, b)} Q(a, b)\end{aligned}$$

Assuming  $E[Y_i|X = x_i] = \alpha + \beta x_i$ ,  $Y_i = \underbrace{\alpha + \beta x_i}_{a + bx_i} + \overbrace{\varepsilon_i}^{\text{error}}$

With Lagrange, we may have a few outliers and Lagrange interpolation will fluctuate a lot to catch up with these points. Since the function fluctuates a lot and is unstable, it does not have as much predictive power, but it provides an exact fit. So the trade-off in statistics is fit versus predictive power.

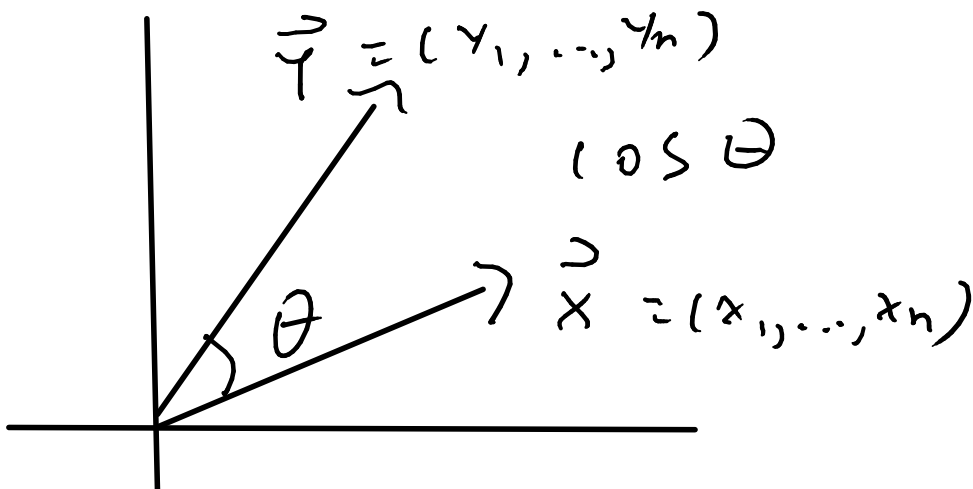


$$\min_{(a,b)} Q(a,b) \implies \begin{cases} \frac{\partial Q(a,b)}{\partial a} = 0 \implies a = \bar{y} - b\bar{x} \\ \frac{\partial Q(a,b)}{\partial b} = 0 \implies b = \frac{S_{xy}}{S_{xx}} \end{cases}$$

Where  $S_{xy} = \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$ ,  $S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2$ . What do these represent?

$$E[(X - \mu_X)(Y - \mu_Y)] = \text{Cov}(X, Y) \approx \frac{1}{n} S_{xy}$$

Recall, what is covariance? If we want to see how aligned two vectors are, we usually use  $\cos \theta$ .



$$\cos \theta = \frac{\vec{X} \cdot \vec{Y}}{\|\vec{X}\| \|\vec{Y}\|}$$

So we first wanted to use  $\vec{X} \cdot \vec{Y} = \sum_{i=1}^n x_i y_i$ , but this is not location invariant, if you shift all values by something this will change. So they subtract the mean.

$$\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

This is  $E[(X - \mu_x)(Y - \mu_Y)]$  and we have the norm as  $\left(\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\right)^{\frac{1}{2}} = \sqrt{\text{Var}(X)}$ ,  $\left(\frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2\right)^{\frac{1}{2}} = \sqrt{\text{Var}(Y)}$ .

$$\text{Cov}(X, Y) = \frac{E[(X - \mu_x)(Y - \mu_Y)]}{\sigma_x \sigma_Y} = \rho(x, y)$$

The sample counterpart:

$$\frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2}} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}$$

When we divide by norm we stay scale invariant, so we expect the same thing to happen

here, shouldn't matter what unit of measurement we use.

$$\rho(X, Y) = \cos(X, Y), |\rho| \leq 1$$

---

Back to where we were before:

$$a = \bar{y} - b\bar{x}$$

$$b = \frac{S_{xy}}{S_{xx}} \sqrt{\frac{S_{yy}}{S_{yy}}} = \sqrt{\frac{S_{yy}}{S_{xx}}} \underbrace{\frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}}_{\text{estimate of correlation}} = \sqrt{\frac{S_{yy}}{S_{xx}}} \hat{\rho}(x, y) = \sqrt{\frac{\hat{\sigma}_Y}{\hat{\sigma}_X}} \hat{\rho}(X, Y)$$

Which means we have  $b \propto \hat{\rho}$ .  $\hat{\rho}$  tells us how linearly correlated  $x$  and  $y$  are. If the deviation of  $Y$  is high and  $X$  is low, we get a very high ratio, so we get a big  $b$  that makes it look like  $x$  can really predict  $y$ , but that is not the case. So it's better to standardize the data first (subtract mean and divide by standard deviation).

---

- $a$  is an unbiased estimator of  $\alpha$
- $b$  is an unbiased estimator of  $\beta$
- $a$  &  $b$  are both consistent-estimators

$$\sqrt{n}(b - \beta) \xrightarrow{D} N(0, \frac{\sigma^2}{k^2})$$

where  $k^2 = \lim_{n \rightarrow \infty} \frac{S_{xx}}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$ ,  $Y_i = \alpha + \beta x_i + \varepsilon$ ,  $\sigma^2 = \text{Var}(\varepsilon) = \text{Var}(Y|X = x)$

$$b \overset{app}{\approx} N\left(\beta, \frac{\sigma^2}{\kappa}\right)$$

when  $n$  is large.

A C.I. for  $\beta$

$$b \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{S_{xx}}}$$

a  $100(1 - \alpha)\%$  C.I. for  $\beta$ .

Intercept:

$$\sqrt{n}(a - \alpha) \xrightarrow{D} N(0, \sigma^2 \left[1 + \frac{\bar{x}^2}{\kappa^2}\right])$$



$$a \stackrel{app}{\sim}, N\left(\alpha, \sigma^2 \left(1 + \frac{\bar{x}^2}{\kappa^2}\right)\right)$$

a  $100(1 - \alpha)\%$  C.I. for  $\alpha$ .

$$a \pm Z_{\alpha/2} \sigma \sqrt{\left[ \frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right]}$$

$$P(Z > Z_{\alpha/2}) = \alpha/2, Z \sim N(0, 1)$$

We want to estimate  $\sigma$ .

$$\varepsilon \rightarrow y_i - \hat{y}_i$$

$$S^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2 / (n - 2), \text{ estimator of } \sigma^2$$

$$\sum_{i=1}^n (y_i - \bar{y})^2 / (n - 1)$$

$$A = \{Y_i - \bar{Y} : i = 1, \dots, n\}, \sum_{i=1}^n (Y_i - \bar{Y}) = 0$$

$$\dim(\text{Span } A)$$

$$\{y_i - \hat{y}_i : i = 1, \dots, n\}, \hat{y}_i = a + bx_i$$

$$1. \sum_{i=1}^n (y_i - \hat{y}_i) = 0$$

$$2. \sum_{i=1}^n (y_i - \hat{y}_i)x_i = 0$$

$$3. \dim(\text{Span } B) = n - 2$$

Suppose we have  $(x_i, y_i), i = 1, \dots, n \rightarrow a, b$  Then we have  $\hat{Y}_1 = a + bx_i$  and  $\alpha + \beta x_i = E[Y_i | X = x_i]$  where  $Y_i = \alpha + \beta x_i + \varepsilon_i$ . But we also want a estimate for a specific value of  $x$  as well, say  $\hat{Y}_{\text{Masoud}} = a + bx^*$

$$\hat{y}(x^*) \pm Z_{\alpha/2} S \sqrt{\frac{1}{n} + \frac{(x^* - \bar{x})^2}{S_{xx}}}$$

$100(1 - \alpha)\%$  C.I. for  $E[Y | X = x^*]$ . Is this right headed?  $\frac{(x^* - \bar{x})^2}{S_{xx}}$  increases the further we get from the center, so there's more uncertainty when we get further from our data points, which

is what we'd expect. What if we want to give a confidence interval for Masoud himself? Then we'd have even more uncertainty.

$$\hat{y}(x^*) \pm Z_{\alpha/2} S \sqrt{1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{S_{xx}}}$$

100(1 -  $\alpha$ )% C.I. for  $Y(x^*) = \alpha + \beta x^* + \varepsilon$ , where we see that there's an extra term, 1 in the square root. This is to account for  $\varepsilon$ .

If you have small  $n$ , we need normality assumption.

$$Y_i = \alpha + \beta x_i + \varepsilon_i \rightarrow \varepsilon_i \stackrel{iid}{\sim} N(0, \sigma^2)$$

We have several assumptions here:

1.  $E[Y|X = x] = \alpha + \beta x$  (Linearity assumption)
2.  $\varepsilon$  is Normally distributed. (Normality assumption)
3. Observations are independent. (Independence assumption)
4.  $Var(\varepsilon)$  does not depend on  $x$ . (Homogeneity of variance)

## 19 03/27/18

**Recall** (Regression notes are online on myCourses)

Last class we looked at estimating the intersection  $a$  and looked at the asymptotic distribution of  $a$ . We also got a confidence interval for  $a$ . The problem here is  $\sigma$ , which we also need to estimate and we ended up using  $s^2$ . The denominator is  $n - 2$  and we have a loss of 2 degrees of freedom because of the linear dependence of the sums  $\sum_{i=1}^n (y_i - \hat{y}_i) = 0$  and  $\sum_{i=1}^n (y_i - \hat{y}_i)x_i = 0$ . So with all these tests we can accept or reject if  $\beta = \beta_0$ . Note that we use the mean response confidence interval if looking to estimate the average and the predicted value confidence interval if looking at one specific individual. For smaller sample sizes, we replace  $z$  by  $t$ .

### 19.1 Analysis of Variance

We want to see how well a linear relationship can describe the relationship between  $x$  and  $y$ .

$$y_i - \bar{y} = (\hat{y}_i - \bar{y}) + (y_i - \hat{y}_i)$$

$$\implies y_i = \underbrace{\bar{y}}_{\alpha} + \underbrace{(\hat{y}_i - \bar{y})}_{\beta x_i} + \underbrace{(y_i - \hat{y}_i)}_{\varepsilon_i}$$

Where we obtained the idea from:

$$y_i = \underbrace{\hat{y}_i}_{\alpha + \beta x_i} + \underbrace{(y_i - \hat{y}_i)}_{\varepsilon_i}$$

From  $(y_i - \bar{y}) = (\hat{y}_i - \bar{y}) + (y_i - \hat{y}_i)$  and  $\sum_{i=1}^n (\hat{y}_i - \bar{y})(y_i - \hat{y}_i) = 0$ , then we get:

$$\sum_{i=1}^n (y_i - \bar{y})^2 = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 + \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

$$SS_{Total} = SS_{Reg} + SS_{Res}$$

(Total variability in response) = (Variability explained by the model) + (Variability unexplained)

We can arrive at the ratio:

$$F = \frac{SS_{Reg}/1}{SS_{Res}/(n-2)} \sim F_{1,(n-2)}$$

Table 3: Analysis of Variance Table

Source	Sum of Squares (SS)	df	Mean Square (MS)	F
Regression	$SS_{Reg}$	1	$SS_{Reg}/1$	$F = MS_{Reg}/s^2$
Residual	$SS_{Res}$	$n - 2$	$s^2$	
Total	$SS_{Total}$	$n - 1$		

What portion of the total variability can be explained by the regression line? This is where  $R^2$  comes in.

$$R^2 = \frac{SS_{Reg}}{SS_{Total}} = \frac{SS_{Reg}}{SS_{Reg} + SS_{Res}}$$

**ANOVA** Analysis of Variance

$T_1 : x_{11}, \dots, x_{1m}, T_2 : y_{21}, \dots, y_{2n}$ , two treatments.

---

$H_0 : \mu_1 = \mu_2$ , so we look at  $\bar{x}_m$  vs  $\bar{y}_n$ , which have a small fluctuation, say  $x_{ij} = \mu_i + \varepsilon_i, i = 1, 2, \dots, k$  (groups),  $j = 1, \dots, n_i$ . We don't have any control over the fluctuation  $\varepsilon_i$ , but we

can decompose the average:  $x_{ij} = \mu + \alpha_i + \varepsilon_i$ . How do we estimate these things?

$$x_{ij} = \overbrace{\bar{x}_{..}}^{\text{Average of all } x_{ij}} + (\bar{x}_{i.} - \bar{x}_{..}) + \underbrace{(x_{ij} - \bar{x}_{i.})}_{\varepsilon_i}$$

Where  $\bar{x}_{i.} = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}$ ,  $\bar{x}_{..} = \frac{1}{\sum_{i=1}^k n_i} \sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij}$ .

$$x_{ij} - \bar{x}_{..} = (\bar{x}_{i.} - \bar{x}_{..}) + (x_{ij} - \bar{x}_{i.})$$

## 20 03/29/18

### 20.1 Analysis of Variance

$x_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \varepsilon_{ij} \stackrel{iid}{\sim} N(0, \sigma^2), i = 1, \dots, k$  (treatments)  $j = 1, \dots, n_i$  (subjects under treatment  $i$ )

$H_0 : \alpha_1 = \alpha_2 = \dots = \alpha_k$

$$x_{ij} = \bar{x}_{..} + (\bar{x}_{i.} - \bar{x}_{..}) + (x_{ij} - \bar{x}_{i.})$$

$$\bar{x}_{i.} = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}$$

$$\bar{x}_{..} = \frac{1}{N} \sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij}$$

where  $N = \sum_{i=1}^k n_i$ .

$$\begin{aligned} (x_{ij} - \bar{x}_{..}) &= (\bar{x}_{i.} - \bar{x}_{..}) + (x_{ij} - \bar{x}_{i.}) \\ \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_{..})^2 &= \sum_{i=1}^k \sum_{j=1}^{n_i} (\bar{x}_{i.} - \bar{x}_{..})^2 + \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_{i.})^2 \\ &= \sum_{i=1}^k \sum_{j=1}^{n_i} (\bar{x}_{i.} - \bar{x}_{..})(x_{ij} - \bar{x}_{i.}) \\ &= \sum_{i=1}^k (\bar{x}_{i.} - \bar{x}_{..}) \left[ \underbrace{\sum_{j=1}^{n_i} (x_{ij} - \bar{x}_{i.})}_0 \right] = 0 \end{aligned}$$

$$SSTotal = SSTr + SSE$$

So we look at the ratio to get the  $F$  distribution:

$$F = \frac{SSTr}{SSE}$$

$$\mathcal{A} = \{x_{ij} - \bar{x}_{..} : i = 1, \dots, k, j = 1, \dots, n\}$$

$$\mathcal{B} = \{\bar{x}_{i.} - \bar{x}_{..}, i = 1 \dots k\}$$

$$\mathcal{C} = \{x_{ij} - \bar{x}_{i.} : i = 1, \dots, k, j = 1, \dots, n_i\}$$

$$\dim(\text{Span} \mathcal{B}) = K - 1, \sum_{i=1}^k (\bar{x}_{i.} - \bar{x}_{..}) = 0$$

$$\dim(\text{Span}(\mathcal{A})) = N - 1, \sum_{i=1}^k \sum_{j=1}^{n_i} j = 1^{n_i} (x_{ij} - \bar{x}_{..})$$

$$x_1 \perp\!\!\!\perp x_2 \iff M_{x_1, x_2}(t_1, t_2) = \mu_{x_1}(t_1) \mu_{x_2}(t_2)$$

$$F_{K-1, N-K} = \frac{SST/K - 1}{SSE/N - K}, N = \sum_{i=1}^K n_i$$

$X \backslash Y$	1	2	...	$c$	
1	$n_{11}$	$n_{12}$	...	$n_{1c}$	$n_{1.}$
2	$n_{21}$	$n_{22}$	...	$n_{2c}$	$n_{2.}$
...	...	...	...	...	...
$r$	$n_{r1}$	$n_{r2}$	...	$n_{rc}$	$n_{r.}$
	$n_{.1}$	$n_{.2}$	...	$n_{.c}$	

So we want to see if  $X$  and  $Y$  are independent, i.e.

$$P(X = x, Y = y) = P(X = x)P(Y = y), \forall x, y$$

$$N = \sum_{i=1}^r \sum_{j=1}^c n_{ij}$$

$$p_{ij} = P(X = i, Y = j), N = \sum_{i=1}^r \sum_{j=1}^c n_{ij}$$

$$\hat{p}_{11} = \frac{n_{11}}{N}, \hat{p}_{ij} = \frac{n_{ij}}{N}$$

$$n_{.j} = \sum_{i=1}^r n_{ij}, n_{i.} = \sum_{j=1}^c n_{ij}$$

$$\begin{aligned}
\hat{p}_{i.} &= P(\widehat{X} = i) = \frac{n_{i.}}{N}, \hat{p}_{.j} = P(\widehat{Y} = j) = \frac{n_{.j}}{N} \\
\hat{p}_{ij} &= \frac{n_{ij}}{N}, \hat{p}_{i.} = \frac{n_{i.}}{N}, \hat{p}_{.j} = \frac{n_{.j}}{N} \\
\hat{p}_{ij} &\approx \hat{p}_{i.}\hat{p}_{.j}, i = 1, \dots, r, j = 1, \dots, c \\
\sum_{i=1}^r \sum_{j=1}^c [\hat{p}_{ij} - \hat{p}_{i.}\hat{p}_{.j}]^2 / (\hat{p}_{i.}\hat{p}_{.j})
\end{aligned}$$

In general we have:

$$\sum_{i=1}^r \sum_{j=1}^c \frac{[o_{ij} - e_{ij}]^2}{e_{ij}} \sim \chi^2$$

Where  $o$  is observed and  $e$  is expected.

$$\sum_{i=1}^r \sum_{j=1}^c \frac{[n_{ij} - \widehat{E}(n_{ij})]^2}{\widehat{E}(n_{ij})} = \sum_{i=1}^r \sum_{j=1}^c \left[ \frac{n_{ij} - \widehat{E}(n_{ij})}{\sqrt{\widehat{E}(n_{ij})}} \right]^2 \sim \chi^2$$

Now this is similar to the  $\chi^2$  formula if this were Poisson distributed, since the expectation is the same as the variance.

Why do we have  $(r-1)(c-1)$  df?

$$\begin{aligned}
&\dim \text{span} \left\{ (n_{ij} - \widehat{E}(n_{ij})) : i = 1, \dots, r; j = 1, \dots, c \right\} \\
&rc - r - c + 1 \text{ (Like inclusion exclusion)} = (r-1)(c-1) \\
&\sum_{j=1}^c n_{ij} = n_{i.}, i = 1, \dots, r \\
&\sum_{i=1}^r n_{ij} = n_{.j}, j = 1, \dots, c \\
&\sum_{i=1}^r \sum_{j=1}^c n_{ij} = N \\
&\sum_{i=1}^r \sum_{j=1}^c \frac{[n_{ij} - \widehat{E}(n_{ij})]^2}{\widehat{E}(n_{ij})} \sim \chi_{(r-1)(c-1)}^2
\end{aligned}$$

$$E(n_{ij}) = Np_{ij} \stackrel{H_0}{=} Np_{i.p.j} \approx N\hat{p}_{i.}\hat{p}_{.j} = N\frac{n_{i.}}{N} \cdot \frac{n_{.j}}{N} = \frac{n_{i.} \cdot n_{.j}}{N}$$

## 21 04/03/18

Chapters 8,9 and 10 have been covered, which we have Masoud's notes for. For linear regression, we have his typed up notes. Final will have 2 questions from chapter 8, 2 from 9, 2 from 10, 1 from linear, 1 from anova, 1 from contingency. Questions will be aligned with questions from the assignments. Material from now on will not be examined.

---

### 21.1 $\chi^2$ tests

Applications:

1. To test independence between two categorical variables.

	Republicans	Democrats	
2. pop 1 (African Americans)	$n_{11}$	$n_{12}$	$n_{1\bullet}$
pop 2 (Others)	$n_{21}$	$n_{22}$	$n_{2\bullet}$
			$N = n_{1\bullet} + n_{2\bullet}$

$H_0 : p_1 = p_2 = p$ , where  $p_1$  is the proportion of people in the first population and  $p_2$  is the proportion of people in the second population who are pro republican.

$$\sum \left( \frac{o_{ij} - E_{ij}}{\sqrt{E_{ij}}} \right)^2 \sim \chi_{(r-1)(c-1)}^2$$

Where  $r = \#$  rows,  $c = \#$  columns.

3. Goodness-of-fit tests

$$Y_i = \alpha + \beta x_i + \varepsilon_i, \varepsilon_i \stackrel{iid}{\sim} N(0, \sigma^2), i = 1, \dots, n$$

---

$X_1, \dots, X_n \sim F$ , where  $F$  is an unknown distribution.

$H_0 : F(x) = \Phi(x), \Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ , where  $\Phi(x)$  is the cdf of  $N(0, 1)$ . So we make a table and count the number of  $X_i$  falling in each interval:

	$-\infty$	$t_1$	$t_2$	$\dots$	$t_k$	$+\infty$
Observed	$\#X_i$	$\#X_i$	$\#X_i$		$\#X_i$	$\#X_i$
Expected						

Recall the multinomial distribution:

$$P(O_1 = o_1, \dots, O_k = o_k) = \binom{n}{o_1, \dots, o_k} \prod_{j=1}^k p_j^{o_j}$$

where

$$\binom{n}{o_1, \dots, o_k} = \frac{n!}{o_1! o_2! \dots o_k!}$$

$$E(O_j) = np_j$$

---


$$\sum_{j=1}^k \left( \frac{o_j - E_j}{\sqrt{E_j}} \right)^2 \sim \chi_{k-1}^2$$


---

## 21.2 Bayesian Analysis

$$A_1, \dots, A_k, A_i \cup A_j = \emptyset, \bigcup_{i=1}^k A_i = S$$

**Law of total probability**

$$P(B) = \sum_{j=1}^k P(B \cap A_j) = \sum_{j=1}^k P(B|A_j)P(A_j)$$

$$\underbrace{P(A_j|B)}_{\text{Posterior}} = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i) \overbrace{P(A_i)}^{\text{Prior}}}{\sum_{j=1}^k P(B|A_j)P(A_j)}$$

$A_1$  = Existence of order,  $G$  = Existence of God

$A_2$  = None existence of order,  $NG$  = None existence of God

$$P(A_1|G) = 1, P(G) = P(NG) = \frac{1}{2}$$

$$P(A_1|NG) = \varepsilon$$



$$P(G|A_1) = \frac{P(A_1|G)P(G)}{P(A_1|G)P(G) + P(A_1|NG)(1 - P(G))} = \frac{1/2}{1/2 + 1/2\varepsilon} = \frac{1}{1 + \varepsilon} \approx 1$$

$$X_i \stackrel{iid}{\sim} \text{Ber}(p)$$

$$P(X_i = x) = p^x(1 - p)^{1-x}, x = 0, 1$$

When  $p$  is unknown, some people would model  $p$  as a random variable:

$$P(X_i = x|p) = p^x(1 - p)^{1-x}, x = 0, 1$$

$$p \sim \text{Beta}(a, b)$$

$$f(p) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} p^{a-1}(1-p)^{b-1}$$

If  $a = 1$  and  $b = 1$ , then we get 1 everywhere, uniform on  $[0, 1]$ , noninformative prior.

Now we want to make inferences about  $p$ :

$$P(X_i = x|p), f(p)$$

$$f(p|X = x) = \frac{P(X = x|p)f(p)}{\int_0^1 P(X = x|p)f(p) dp} = Cp^{x+a-1}(1-p)^{1-x+b-1}$$

$$P(X = x|p)f(p) = p^x(1-p)^x \kappa p^{a-1}(1-p)^{b-1} = \kappa p^{x+a-1}(1-p)^{b-1+x} = \kappa p^{x+a-1}(1-p)^{b-x}$$

$$\implies f(p|x) \propto p^{x+a-1}(1-p)^{b-x} \text{ refined prior}$$

**Dirichlet Distribution** (Generalization of this)

$$\frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\prod_{i=1}^k \Gamma(\alpha_i)} \prod_{i=1}^k p_i^{\alpha_i-1}, \sum_{i=1}^k p_i = 1, \alpha_i > 0, i = 1, \dots, n$$

$$P(X_i = x_i, i = 1, \dots, k | p_1, \dots, p_k) = \binom{n}{x_1, \dots, x_k} \prod_{i=1}^k p_i^{x_i}$$

$$\overbrace{(p_1, \dots, p_k)}^{\text{Prior}} \sim \text{Dirichlet Dist}$$

$$\overbrace{(p_1, \dots, p_k)}^{\text{Posterior}} \bigg|_{\vec{x}=(x_1, \dots, x_k)} \propto \prod_{i=1}^k p_i^{\alpha_i+x_i-1}$$

## 21.3 Empirical Bayes