COMP250: Loop invariants

Jérôme Waldispühl
School of Computer Science
McGill University

Based on (CRLS, 2009) & slides from (Sora,2015)

Algorithm specification

- An algorithm is described by:
  - Input data
  - Output data
  - ***Preconditions***: specifies restrictions on input data
  - ***Postconditions***: specifies what is the result
- Example: Binary Search
  - Input data: `a:array of integer; x:integer;`
  - Output data: `index:integer;`
  - Precondition: `a` is sorted in ascending order
  - Postcondition: `index` of `x` if `x` is in `a`, and -1 otherwise.

Correctness of an algorithm

An algorithm is correct if:

- for any correct input data:
  - it stops and
  - it produces correct output.
- Correct input data: satisfies precondition
- Correct output data: satisfies postcondition

Problem: Proving the correctness of an algorithm may be complicated when the latter is repetitive or contains loop instructions.

Loop invariant

A **loop invariant** is a loop property that hold before and after each iteration of a loop.

Proof using loop invariants

We must show:

1. **Initialization:** It is true prior to the first iteration of the loop.
2. **Maintenance:** If it is true before an iteration of the loop, it remains true before the next iteration.
3. **Termination:** When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct.

Analogy to induction proofs

Using loop invariants is like mathematical induction.

- You prove a **base case** and an **inductive step**.
- Showing that the invariant holds before the first iteration is like the base case.
- Showing that the invariant holds from iteration to iteration is like the inductive step.
- The **termination** part differs from classical mathematical induction. Here, we stop the ``induction'' when the loop terminates instead of using it infinitely.

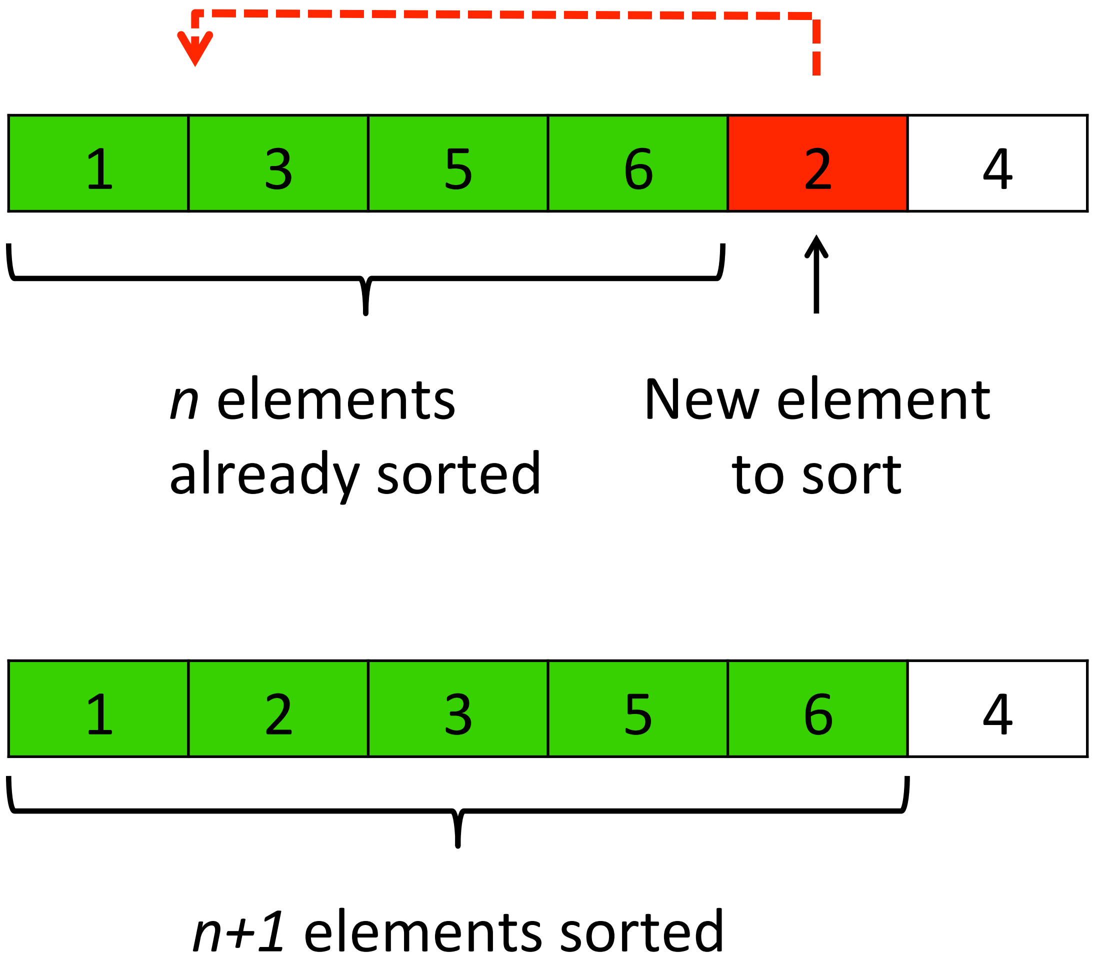
We can show the three parts in any order.

Insertion sort

```
for i ← 1 to length(A) - 1
    j ← i
    while j > 0 and A[j-1] > A[j]
        swap A[j] and A[j-1]
        j ← j - 1
    end while
end for
```

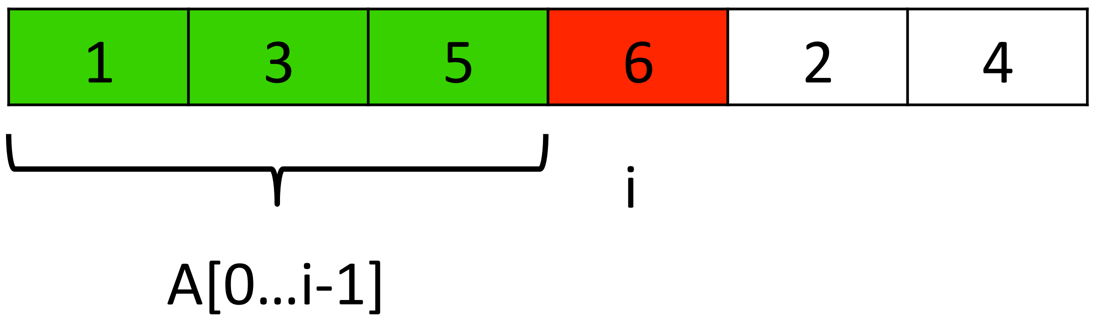
(Seen in Lecture 7)

Insertion sort



Loop invariant

The array A[0…i-1] is fully sorted.



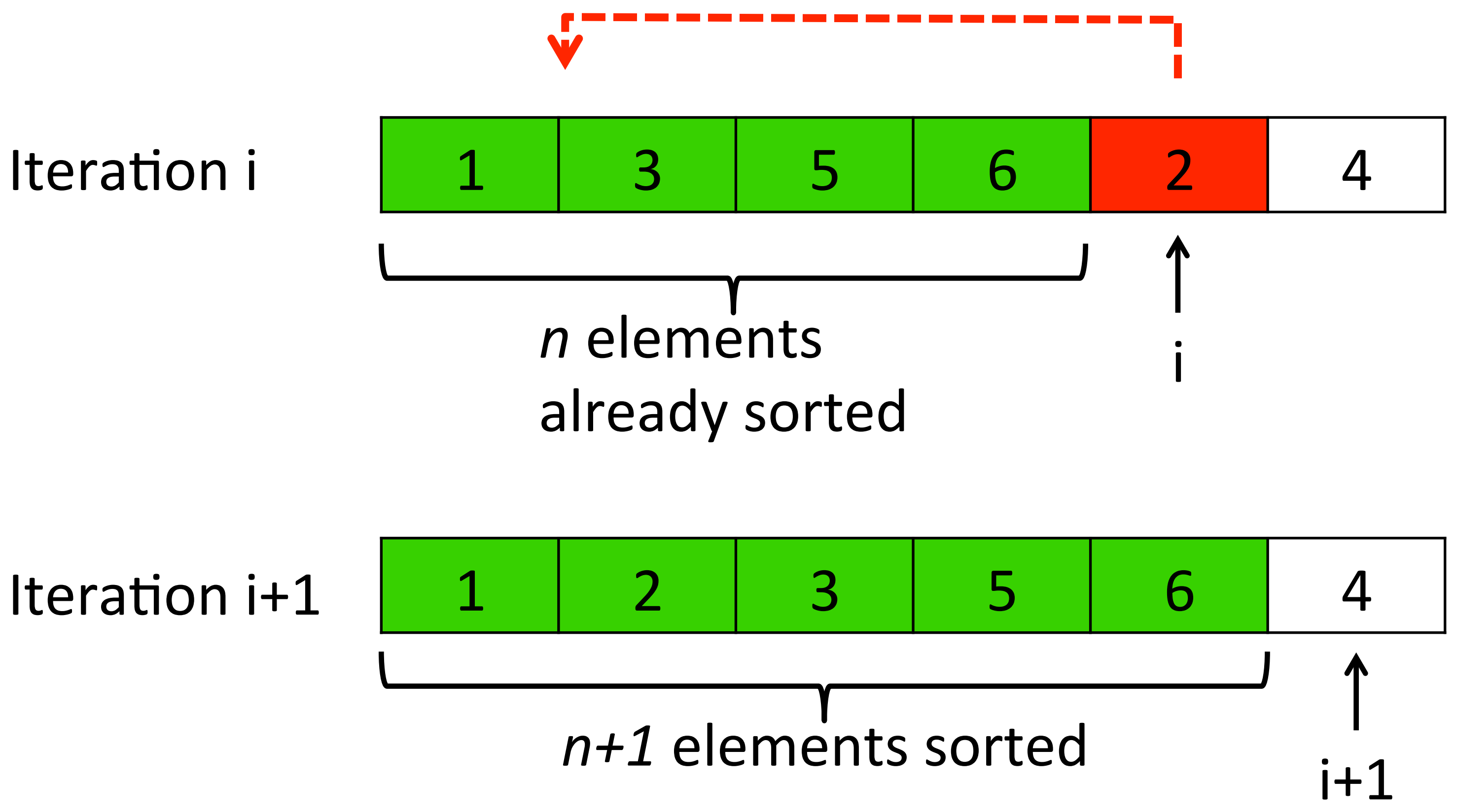
Initialization

Just before the first iteration (i = 1), the sub-array *A*[0 … i−1] is the single element *A*[0], which is the element originally in *A*[0], and it is trivially sorted.

1	3	5	6	2	4

i=1

Maintenance



Note: To be precise, we would need to state and prove a loop invariant for the ``inner'' **while** loop.

Termination

The outer **for** loop ends when $i \geq length(A)$ and increment by 1 at each iteration starting from 1.

Therefore, $i = length(A)$.

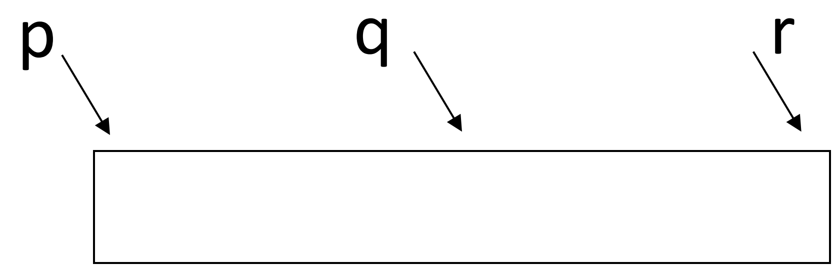
Plugging *length(A)* in for $i-1$ in the loop invariant, the subarray $A[0 \ldots length(A)-1]$ consists of the elements originally in $A[0 \ldots length(A)-1]$ but in sorted order.

$A[0 \ldots length(A)-1]$ contains *length(A) elements (i.e. all initial elements!) and no element is duplicated/deleted.*

In other words, the entire array is sorted!

Merge Sort

```
MERGE-SORT(A,p,r)
   if p < r
        q= (p+r)/2
        MERGE-SORT(A,p,q)
        MERGE-SORT(A,q+1,r)
        MERGE(A,p,q,r)
```



Precondition:

Array A has at least 1 element between indexes p and r (p<r)

Postcondition:

The elements between indexes p and r are sorted

Merge Sort (combine)

- MERGE-SORT calls a function MERGE(A,p,q,r) to merge the sorted subarrays of A into a single sorted one
- The proof of MERGE can be done separately, using loop invariants
- We assume here that MERGE has been proved to fulfill its postconditions (Exercise!)

MERGE (A,p,q,r)

Precondition: A is an array and p, q, and r are indices into the array such that p <= q < r. The subarrays A[p.. q] and A[q +1.. r] are sorted

Postcondition: The subarray A[p..r] is sorted

Correctness proof for Merge-Sort

- Recursive property: Elements in A[p,r] are be sorted.
- **Base Case**: n = 1
  - A contains a single element (which is trivially "sorted")
- **Inductive Hypothesis:**
  - Assume that MergeSort correctly sorts n=1, 2, ..., **k** elements
- **Inductive Step:**
  - Show that MergeSort correctly sorts n = **k + 1** elements.
- **Termination Step:**
  - MergeSort terminate and all elements are sorted.

Maintenance

- **Inductive Hypothesis:**
  - Assume MergeSort correctly sorts n=1, ... , **k** elements
- **Inductive Step:**
  - Show that MergeSort correctly sorts n = **k + 1** elements.
- **Proof:**
  - First recursive call n_1=q-p+1=(k+1)/2 ≤ k

    => subarray A[p .. q] is sorted
  - Second recursive call n_2=r-q=(k+1)/2 ≤ k

    => subarray A[q+1 .. r] is sorted
  - A, p q, r fulfill now the precondition of Merge
  - The post-condition of Merge guarantees that the array A[p .. r] is sorted => post-condition of MergeSort satisfied.

Termination

We have to find a quantity that decreases with every recursive call: the length of the subarray of A to be sorted MergeSort.

At each recursive call of MergeSort, the length of the subarray is strictly decreasing.

When MergeSort is called on a array of size ≤ 1 (i.e. the base case), the algorithm terminates without making additional recursive calls.

Calling MergeSort(A,0,n) returns a fully sorted array.