Survey of problems on graphs

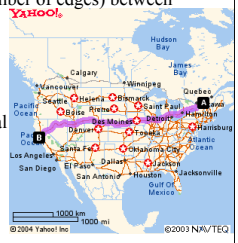
Lecture 31

Outline

- Graphs model many real-world applications
- Graphs lend themselves to nice computer science problems:
  - Shortest path
  - Cycles: Eulerian, Hamiltonian
  - Cliques and independent sets
  - Coloring
  - Matching
- We will only consider undirected graphs

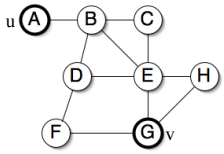
Shortest path problem

- Unweighted Graph Shortest Path:
  - Given an unweighted graph and two vertices u and v,
  - Find the shortest path (minimum number of edges) between u and v
- Weighted Graph Shorted Path:
  - Given an weighted graph and two vertices u and v,
  - Find the shortest path (minimum total edges weight) between u and v
- Applications:
  - Driving from one city to another
  - Routing packets through the internet
  - Solving the Rubik's cube using the least number of moves



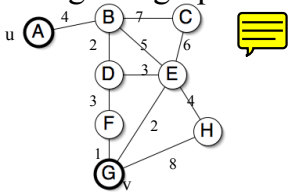
Algo. for unweighted graph shortest path

- Algorithm for unweighted graph:
  - Do a breadth-first search starting at u, until v is reached
  - For each vertex visited, remember from which vertex it was reached



  - Works because vertices are visited in increasing order of distance from u

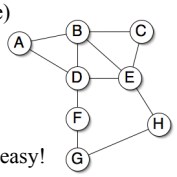
Algo. for weighted graph shortest path



- Idea:
  - Visit vertices in increasing order of distance from u
  - The first time you get to v, you came to it via the shortest path.
  - This can be done efficiently using a priority queue (see HW5)

Eulerian cycles

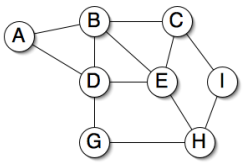
- Recall: A cycle is a path that returns to its starting vertex
- An **eulerian cycle** visits each **edge** exactly once (but vertices can be visited more than once)
- Problem:
  - Given a undirected graph
  - Find an eulerian cycle (if one exists)



- Algorithm: Sounds hard, but actually easy!
  - Start at any vertex u and follow any unvisited edge, as long as this does not result in a graph whose unvisited edges are unreachable
  - No need to plan ahead, so algorithm is fast

Hamiltonian cycles

- A **Hamiltonian cycle** visits each **vertex** *exactly* once
- Problem:
  - Given a undirected graph
  - Find an Hamiltonian cycle (if one exists)



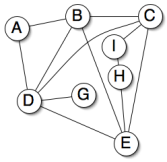
- Algorithm: Very hard!
  - Nobody knows how to do much better than trying all (n-1)! possible vertex orderings
  - **Be famous: find an algorithm that runs in polynomial time**

Graph coloring

- Problem: Given an undirected graph
  - Find the minimum number of colors needed to paint the vertices so that no pair of adjacent vertices have the same color
- Application: Coloring maps
  - Color countries so that neighbors always have different colors
  - Draw "contact graph"
    - One vertex per contry
    - Edges between touching contries
- **Be famous: Find a poly. time algo. for graph coloring**

Cliques

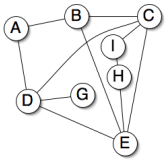
- Given an undirected graph, a **clique** is a subset of vertices where all vertices are adjacent
- Problem:
  - Given an undirected graph
  - Find the largest clique it contains



- **Be famous: Find a poly. time algo for finding maximal cliques**

Matching

- Example:
  - n people want to get married (vertices)
  - Some pairs of people are compatible (good horoscope, shown by edges), others are incompatible (no edge)
  - Question:
    Can we match everybody?



- NB:The graph contains triangles: what does that mean?
- Efficient algorithms are known but quite complicated