

In this notebook, we calculate the critical sensitivity, χ_u^* , as a function of the diffusivity of H-type individuals D_v and the wavenumber m , as studied in Section 3.2.2 of the main text and Section B.3 of the appendix. These calculations are used to produce Figure 2 and Figure 9, which are generated using MATLAB. The results from this notebook are cross-verified with the MATLAB scripts “chi_u_Dv_di.m” and “combine_chi_u.m”, ensuring consistency between the outputs.

The first part analytically calculates the critical χ_u^* under the equation $\det A=0$, which corresponds to one of the stability conditions, $\det A>0$.

```
In[ ]:= ClearAll[c11, c12, c13, c22, c23, c32, c33, m, L, Du, Dv, Dn, p0, q0, chiu, chiv]
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```
A = {{c11 - (m Pi / L) ^ 2 (Du p0 + Dv (1 - p0)),
      c12 - (m Pi / L) ^ 2 (Du - Dv) q0, c13 + (m Pi / L) ^ 2 (chiu p0 q0 + chiv q0 (1 - p0))},
      {-q0 ^ (-1) (m Pi / L) ^ 2 (1 - p0) p0 (Du - Dv), c22 - (m Pi / L) ^ 2 (Du (1 - p0) + Dv p0),
      c23 + (m Pi / L) ^ 2 p0 (1 - p0) (chiu - chiv)}, {0, c32, - (m Pi / L) ^ 2 Dn + c33}};
```

```
detA = Det[A];
eqn = detA == 0;
solution = Solve[eqn, chiu];
solution
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Out[ ]:=
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$$\left\{ \left\{ \text{chiu} \rightarrow \left(\frac{c_{32} \text{chiv} (Du - Dv) m^4 (1 - p_0)^2 p_0 \pi^4}{L^4} + c_{23} c_{32} \left(c_{11} - \frac{m^2 (Dv (1 - p_0) + Du p_0) \pi^2}{L^2} \right) - \frac{c_{32} \text{chiv} m^2 (1 - p_0) p_0 \pi^2 \left(c_{11} - \frac{m^2 (Dv (1 - p_0) + Du p_0) \pi^2}{L^2} \right)}{L^2} + \frac{c_{13} c_{32} (Du - Dv) m^2 (1 - p_0) p_0 \pi^2}{L^2 q_0} - \left(c_{33} - \frac{Dn m^2 \pi^2}{L^2} \right) \left(\left(c_{11} - \frac{m^2 (Dv (1 - p_0) + Du p_0) \pi^2}{L^2} \right) \left(c_{22} - \frac{m^2 (Du (1 - p_0) + Dv p_0) \pi^2}{L^2} \right) + \frac{(Du - Dv) m^2 (1 - p_0) p_0 \pi^2 \left(c_{12} - \frac{(Du - Dv) m^2 \pi^2 q_0}{L^2} \right)}{L^2 q_0} \right) \right) \right) / \left(- \frac{c_{32} (Du - Dv) m^4 (1 - p_0) p_0^2 \pi^4}{L^4} - \frac{c_{32} m^2 (1 - p_0) p_0 \pi^2 \left(c_{11} - \frac{m^2 (Dv (1 - p_0) + Du p_0) \pi^2}{L^2} \right)}{L^2} \right) \right\} \right\}$$

In the second part, we use the results from the first part and combine them with actual parameters to plot the minimum χ_u^* as a function of diffusivity D_v under $\det A=0$. This result is consistent with Figure 2 and the MATLAB script “chi_u_Dv_di.m”. The algorithm involves looping over different values of D_v , and for each D_v , iterating through different m values to identify the minimum χ_u^* across the range of m .

```
In[ ]:= p0 = 0.5;
        n0 = 0.5;
```

```

q0 = 4000;
L = 10;
Du = 0.01;
Dn = 0.01;
chiv = 0.02;
r = 1;
a = 0.5;
eL = 0.2;
eH = 0.5;
R0 = 0.35;
S0 = 0.3;
T0 = 0.5;
P0 = 0.2;
R1 = 0.6;
S1 = 0.35;
T1 = 0.6;
P1 = 0.3;
epsilon = 10;
kappa = 0.0001;

```

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fL = (1 - n0) (R0 p0 + S0 (1 - p0)) + n0 (R1 p0 + S1 (1 - p0));
fH = (1 - n0) (T0 p0 + P0 (1 - p0)) + n0 (T1 p0 + P1 (1 - p0));

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fLp = (1 - n0) (R0 - S0) + n0 (R1 - S1);
fHp = (1 - n0) (T0 - P0) + n0 (T1 - P1);
fLn = p0 (R1 - R0) + (1 - p0) (S1 - S0);
fHn = p0 (T1 - T0) + (1 - p0) (P1 - P0);
kp = fLp - fHp;
kn = fLn - fHn;

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c11 = ((1 - p0) fH + p0 fL - 2 kappa q0) / epsilon;
c12 = (q0 (fHp + p0 kp)) / epsilon;
c13 = (q0 (fHn + p0 kn)) / epsilon;
c22 = (p0 (1 - p0) kp) / epsilon;
c23 = (p0 (1 - p0) kn) / epsilon;
c32 = r - a (eL n0 + eH (1 - n0));
c33 = -(r - a (eL n0 + eH (1 - n0)));

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ChiU[m_, Dv_] := Module[{chiuNum, chiuDen, chiuStar},

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    chiuNum =

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$$\left(\frac{c_{32} \text{chiv} (Du - Dv) m^4 (1 - p_0)^2 p_0 \pi^4}{L^4} + c_{23} c_{32} \left(c_{11} - \frac{m^2 (Dv (1 - p_0) + Du p_0) \pi^2}{L^2} \right) - \right.$$

$$\left. \frac{c_{32} \text{chiv} m^2 (1 - p_0) p_0 \pi^2 \left(c_{11} - \frac{m^2 (Dv (1 - p_0) + Du p_0) \pi^2}{L^2} \right)}{L^2} + \right.$$

$$\frac{c_{13} c_{32} (Du - Dv) m^2 (1 - p\theta) p\theta \pi^2}{L^2 q\theta} - \left(c_{33} - \frac{Dn m^2 \pi^2}{L^2} \right) \left(\left(c_{11} - \frac{m^2 (Dv (1 - p\theta) + Du p\theta) \pi^2}{L^2} \right) \left(c_{22} - \frac{m^2 (Du (1 - p\theta) + Dv p\theta) \pi^2}{L^2} \right) + \frac{(Du - Dv) m^2 (1 - p\theta) p\theta \pi^2 \left(c_{12} - \frac{(Du - Dv) m^2 \pi^2 q\theta}{L^2} \right)}{L^2 q\theta} \right) \right);$$

chiuDen =

$$\left(-\frac{c_{32} (Du - Dv) m^4 (1 - p\theta) p\theta^2 \pi^4}{L^4} - \frac{c_{32} m^2 (1 - p\theta) p\theta \pi^2 \left(c_{11} - \frac{m^2 (Dv (1 - p\theta) + Du p\theta) \pi^2}{L^2} \right)}{L^2} \right);$$

chiuStar = If[chiuDen == 0 || Im[chiuNum / chiuDen] != 0, ∞, chiuNum / chiuDen];
chiuStar]

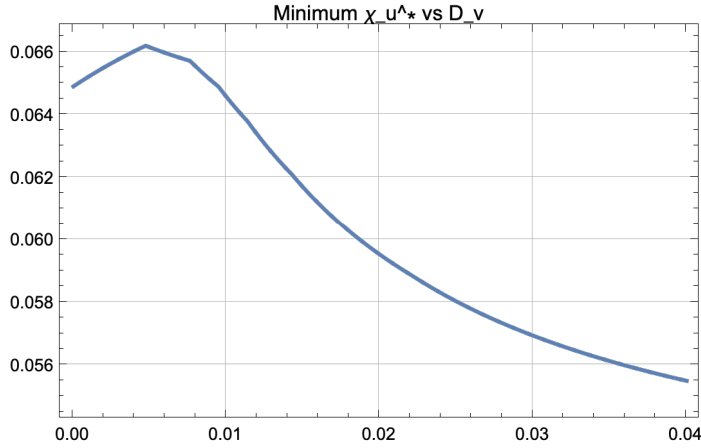
DvValues = Range[0, 0.04, 0.00001];

mValues = Range[0, 100, 1];

minChiUStar = Table[Min[Table[ChiU[m, Dv], {m, mValues}]], {Dv, DvValues}];

ListLinePlot[Transpose[{DvValues, minChiUStar}], PlotStyle → Thick,
AxesLabel → {"D_v", "Minimum χ_u^* "}, PlotLabel → "Minimum χ_u^* vs D_v",
GridLines → Automatic, Frame → True, PlotRange → All]

Out[]=



In the third part, we aim to analytically derive the expression for the critical χ_u^* under the equation $-1/2 (\text{tr}A^2 - \text{tr}A^2) \text{tr}A + \det A = 0$, which corresponds to the other stability condition, $-1/2 (\text{tr}A^2 - \text{tr}A^2) \text{tr}A + \det A > 0$.

```

In[*]:= ClearAll[c11, c12, c13, c22, c23, c32, c33, m, L, Du, Dv, Dn, p0, q0, chiu, chiv]

A = {{c11 - (m Pi / L) ^ 2 (Du p0 + Dv (1 - p0)),
      c12 - (m Pi / L) ^ 2 (Du - Dv) q0, c13 + (m Pi / L) ^ 2 (chiu p0 q0 + chiv q0 (1 - p0))},
     {-q0 ^ (-1) (m Pi / L) ^ 2 (1 - p0) p0 (Du - Dv), c22 - (m Pi / L) ^ 2 (Du (1 - p0) + Dv p0),
      c23 + (m Pi / L) ^ 2 p0 (1 - p0) (chiu - chiv)}, {0, c32, - (m Pi / L) ^ 2 Dn + c33}};

detA = Det[A];
trA = Tr[A];

A2 = A.A;
trA2 = Tr[A2];

eqn = -1/2 (trA2 - trA^2) trA + detA == 0;
solution = Solve[eqn, chiu];
FullSimplify[solution]

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Out[*]:=
{ {chiu -> (c11 c12 (Du - Dv) L^4 m^2 (-1 + p0) p0 pi^2 - c13 c32 (Du - Dv) L^4 m^2 (-1 + p0) p0 pi^2 +
c12 (Du - Dv) L^2 m^2 (-1 + p0) p0 pi^2 ((c22 + 2 c33) L^2 - (2 Dn + Du + Dv) m^2 pi^2) +
c11^2 L^4 (-((c22 + c33) L^2) + m^2 (Dn + Du - Du p0 + Dv p0) pi^2) q0 -
c11 L^2 ((c22^2 - 2 c23 c32 + 4 c22 c33 + c33^2) L^4 - 2 L^2 m^2 (c22 (2 Dn + Du + Dv) +
c32 chiv (-1 + p0) p0 + c33 (Dn + 2 Du + Dv - Du p0 + Dv p0)) pi^2 +
m^4 (Dn^2 + 4 Dn Du + Du^2 + 2 Dn Dv + 2 Du Dv - (Du - Dv) (2 Dn + Du + Dv) p0) pi^4)
q0 + (-((c22 + c33) (-c23 c32 + c22 c33) L^6) +
L^4 m^2 (c33^2 (Du + Dv) + c22 c32 chiv (-1 + p0) p0 + c32 c33 chiv (-1 + p0) p0 +
c22^2 (Dn + Dv + Du p0 - Dv p0) - c23 c32 (Dn + Du + 2 Dv + Du p0 - Dv p0) +
2 c22 c33 (Dn + Du + 2 Dv + Du p0 - Dv p0)) pi^2 -
L^2 m^4 (c33 (Du^2 + 4 Du Dv + Dv^2 + 2 Dn (Du + Dv)) +
c32 chiv (Dn + 2 Du + Dv) (-1 + p0) p0 + c22
(Dn^2 + 2 Dn Du + 4 Dn Dv + 2 Du Dv + Dv^2 + (Du - Dv) (2 Dn + Du + Dv) p0)) pi^4 +
(Dn^2 (Du + Dv) + Du Dv (Du + Dv) + Dn (Du^2 + 4 Du Dv + Dv^2)) m^6 pi^6) q0) /
(c32 L^2 m^2 (-1 + p0) p0 pi^2 ((2 c11 + c22 + c33) L^2 - (Dn + Du + 2 Dv) m^2 pi^2) q0) } }

```

In the fourth part, we use the results derived in the third part and combine them with actual parameters to plot the minimum χ_u^* as a function of diffusivity D_v under the equation $-1/2 (\text{trA}2 - \text{trA}^2) \text{trA} + \text{detA} = 0$. This result aligns with part of Figure 9 and the MATLAB script “combine_chi_u.m”. The algorithm follows the same approach as in the second part, with the only difference being the use of the new expression for critical χ_u^* .

```

In[*]:= p0 = 0.5;
n0 = 0.5;
q0 = 4000;
L = 10;
Du = 0.01;
Dn = 0.01;

```

```

chiv = 0.02;
r = 1;
a = 0.5;
eL = 0.2;
eH = 0.5;
R0 = 0.35;
S0 = 0.3;
T0 = 0.5;
P0 = 0.2;
R1 = 0.6;
S1 = 0.35;
T1 = 0.6;
P1 = 0.3;
epsilon = 10;
kappa = 0.0001;

fL = (1 - n0) (R0 p0 + S0 (1 - p0)) + n0 (R1 p0 + S1 (1 - p0));
fH = (1 - n0) (T0 p0 + P0 (1 - p0)) + n0 (T1 p0 + P1 (1 - p0));

fLp = (1 - n0) (R0 - S0) + n0 (R1 - S1);
fHp = (1 - n0) (T0 - P0) + n0 (T1 - P1);
fLn = p0 (R1 - R0) + (1 - p0) (S1 - S0);
fHn = p0 (T1 - T0) + (1 - p0) (P1 - P0);
kp = fLp - fHp;
kn = fLn - fHn;

c11 = ((1 - p0) fH + p0 fL - 2 kappa q0) / epsilon;
c12 = (q0 (fHp + p0 kp)) / epsilon;
c13 = (q0 (fHn + p0 kn)) / epsilon;
c22 = (p0 (1 - p0) kp) / epsilon;
c23 = (p0 (1 - p0) kn) / epsilon;
c32 = r - a (eL n0 + eH (1 - n0));
c33 = -(r - a (eL n0 + eH (1 - n0)));

ChiU[m_, Dv_] := Module[{chiumNum, chiuDen, chiuStar},
  chiumNum =
    (-2 c13 c32 (Du - Dv) L^4 m^2 (-1 + p0) p0 pi^2 + 2 c32 chiv (Du - Dv) L^2 m^4 (-1 + p0)^2 p0 pi^4
      q0 + 2 c23 c32 L^4 (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 pi^2 - Du m^2 pi^2 - Dv m^2 pi^2) q0 + 2 c32
      chiv L^2 m^2 (-1 + p0) p0 pi^2 (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 pi^2 - Du m^2 pi^2 - Dv m^2 pi^2) q0 +
      (c33 L^2 - Dn m^2 pi^2)^2 (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 pi^2 - Du m^2 pi^2 - Dv m^2 pi^2) q0 -
      (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 pi^2 - Du m^2 pi^2 - Dv m^2 pi^2)^3 q0 +
      2 c23 c32 L^4 (c11 L^2 + m^2 (Dv (-1 + p0) - Du p0) pi^2) q0 -
      2 c32 chiv L^2 m^2 (1 - p0) p0 pi^2 (c11 L^2 + m^2 (Dv (-1 + p0) - Du p0) pi^2) q0 +
      (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 pi^2 - Du m^2 pi^2 - Dv m^2 pi^2)
      (c11 L^2 + m^2 (Dv (-1 + p0) - Du p0) pi^2)^2 q0 +
      (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 pi^2 - Du m^2 pi^2 - Dv m^2 pi^2)

```

$$\begin{aligned}
& \left(c_{22} L^2 + m^2 (Du (-1 + p_0) - Dv p_0) \pi^2 \right)^2 q_0 - \\
& 2 (Du - Dv) m^2 (1 - p_0) p_0 \pi^2 (c_{11} L^2 + c_{22} L^2 + c_{33} L^2 - Dn m^2 \pi^2 - Du m^2 \pi^2 - Dv m^2 \pi^2) \\
& (c_{12} L^2 + (-Du + Dv) m^2 \pi^2 q_0) + 2 (c_{33} L^2 - Dn m^2 \pi^2) \\
& (c_{12} (Du - Dv) L^2 m^2 (-1 + p_0) p_0 \pi^2 + (-c_{11} c_{22} L^4 + c_{22} L^2 m^2 (Dv + Du p_0 - Dv p_0) \\
& \pi^2 + c_{11} L^2 m^2 (Du - Du p_0 + Dv p_0) \pi^2 - Du Dv m^4 \pi^4) q_0));
\end{aligned}$$

```

chiuDen = (2 c32 L^2 m^2 (-1 + p0) p0 π^2
  (2 c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 π^2 - Du m^2 π^2 - 2 Dv m^2 π^2) q0);
chiuStar = If[chiuDen == 0 || Im[chiuNum / chiuDen] ≠ 0, ∞, chiuNum / chiuDen];
chiuStar]

```

```
DvValues = Range[0, 0.04, 0.00001];
```

```
mValues = Range[0, 100, 1];
```

```
minChiUStar = Table[Min[Table[ChiU[m, Dv], {m, mValues}]], {Dv, DvValues}];
```

```

ListLinePlot[Transpose[{DvValues, minChiUStar}], PlotStyle → Thick,
  AxesLabel → {"D_v", "Minimum χ_u^*"}, PlotLabel → "Minimum χ_u^* vs D_v",
  GridLines → Automatic, Frame → True, PlotRange → All]

```

Out[]=

