In this notebook, we calculate the critical sensitivity, χ_u^*, as a function of the diffusivity of H-type individuals D_v and the wavenumber m, as studied in Section 3.2.2 of the main text and Section B.3 of the appendix. These calculations are used to produce Figure 2 and Figure 9, which are generated using MATLAB. The results from this notebook are cross-verified with the MATLAB scripts "chi_u_Dv_di.m" and "combine_chi_u.m", ensuring consistency between the outputs.

The first part analytically calculates the critical χ_u^* under the equation detA=0, which corresponds to one of the stability conditions, detA>0.

In[∘]:=
```
ClearAll[c11, c12, c13, c22, c23, c32, c33, m, L, Du, Dv, Dn, p0, q0, chiu, chiv]

A = {{c11 - (m Pi / L) ^2 (Du p0 + Dv (1 - p0)),
    c12 - (m Pi / L) ^2 (Du - Dv) q0, c13 + (m Pi / L) ^2 (chiu p0 q0 + chiv q0 (1 - p0))},
   {-q0 ^ (-1) (m Pi / L) ^2 (1 - p0) p0 (Du - Dv), c22 - (m Pi / L) ^2 (Du (1 - p0) + Dv p0),
    c23 + (m Pi / L) ^2 p0 (1 - p0) (chiu - chiv)}, {0, c32, - (m Pi / L) ^2 Dn + c33}};

detA = Det[A];
eqn = detA == 0;
solution = Solve[eqn, chiu];
solution
```

Out[∘]=

$$\Big\{\Big\{\text{chiu}\to \Big(\frac{\text{c32 chiv}\,(\text{Du}-\text{Dv})\,m^4\,(1-\text{p0})^2\,\text{p0}\,\pi^4}{L^4}+\text{c23 c32}\left(\text{c11}-\frac{m^2\,(\text{Dv}\,(1-\text{p0})+\text{Du p0})\,\pi^2}{L^2}\right)-\frac{\text{c32 chiv}\,m^2\,(1-\text{p0})\,\text{p0}\,\pi^2\left(\text{c11}-\frac{m^2\,(\text{Dv}\,(1-\text{p0})+\text{Du p0})\,\pi^2}{L^2}\right)}{L^2}+\frac{\text{c13 c32}\,(\text{Du}-\text{Dv})\,m^2\,(1-\text{p0})\,\text{p0}\,\pi^2}{L^2\,\text{q0}}-\left(\text{c33}-\frac{\text{Dn}\,m^2\,\pi^2}{L^2}\right)\left(\left(\text{c11}-\frac{m^2\,(\text{Dv}\,(1-\text{p0})+\text{Du p0})\,\pi^2}{L^2}\right)\left(\text{c22}-\frac{m^2\,(\text{Du}\,(1-\text{p0})+\text{Dv p0})\,\pi^2}{L^2}\right)+\frac{(\text{Du}-\text{Dv})\,m^2\,(1-\text{p0})\,\text{p0}\,\pi^2\left(\text{c12}-\frac{(\text{Du}-\text{Dv})\,m^2\,\pi^2\,\text{q0}}{L^2}\right)}{L^2\,\text{q0}}\right)\Big)\Big/\left(-\frac{\text{c32}\,(\text{Du}-\text{Dv})\,m^4\,(1-\text{p0})\,\text{p0}^2\,\pi^4}{L^4}-\frac{\text{c32}\,m^2\,(1-\text{p0})\,\text{p0}\,\pi^2\left(\text{c11}-\frac{m^2\,(\text{Dv}\,(1-\text{p0})+\text{Du p0})\,\pi^2}{L^2}\right)}{L^2}\right)\Big\}\Big\}$$

In the second part, we use the results from the first part and combine them with actual parameters to plot the minimum χ_u^* as a function of diffusivity D_v under detA=0. This result is consistent with Figure 2 and the MATLAB script "chi_u_Dv_di.m". The algorithm involves looping over different values of D_v , and for each D_v , iterating through different m values to identify the minimum χ_u^* across the range of m.

In[∘]:=
```
p0 = 0.5;
n0 = 0.5;
```

```
q0 = 4000;
L = 10;
Du = 0.01;
Dn = 0.01;
chiv = 0.02;
r = 1;
a = 0.5;
eL = 0.2;
eH = 0.5;
R0 = 0.35;
S0 = 0.3;
T0 = 0.5;
P0 = 0.2;
R1 = 0.6;
S1 = 0.35;
T1 = 0.6;
P1 = 0.3;
epsilon = 10;
kappa = 0.0001;

fL = (1 - n0) (R0 p0 + S0 (1 - p0)) + n0 (R1 p0 + S1 (1 - p0));
fH = (1 - n0) (T0 p0 + P0 (1 - p0)) + n0 (T1 p0 + P1 (1 - p0));

fLp = (1 - n0) (R0 - S0) + n0 (R1 - S1);
fHp = (1 - n0) (T0 - P0) + n0 (T1 - P1);
fLn = p0 (R1 - R0) + (1 - p0) (S1 - S0);
fHn = p0 (T1 - T0) + (1 - p0) (P1 - P0);
kp = fLp - fHp;
kn = fLn - fHn;

c11 = ((1 - p0) fH + p0 fL - 2 kappa q0) / epsilon;
c12 = (q0 (fHp + p0 kp)) / epsilon;
c13 = (q0 (fHn + p0 kn)) / epsilon;
c22 = (p0 (1 - p0) kp) / epsilon;
c23 = (p0 (1 - p0) kn) / epsilon;
c32 = r - a (eL n0 + eH (1 - n0));
c33 = - (r - a (eL n0 + eH (1 - n0)));

ChiU[m_, Dv_] := Module[{chiuNum, chiuDen, chiuStar},
  chiuNum =
```

$$\left(\frac{\text{c32 chiv }(\text{Du}-\text{Dv})\, m^4\,(1-\text{p0})^2\,\text{p0}\,\pi^4}{L^4}+\text{c23 c32}\left(\text{c11}-\frac{m^2\,(\text{Dv}\,(1-\text{p0})+\text{Du p0})\,\pi^2}{L^2}\right)-\right.$$

$$\frac{\text{c32 chiv}\, m^2\,(1-\text{p0})\,\text{p0}\,\pi^2\left(\text{c11}-\frac{m^2\,(\text{Dv}\,(1-\text{p0})+\text{Du p0})\,\pi^2}{L^2}\right)}{L^2}+$$

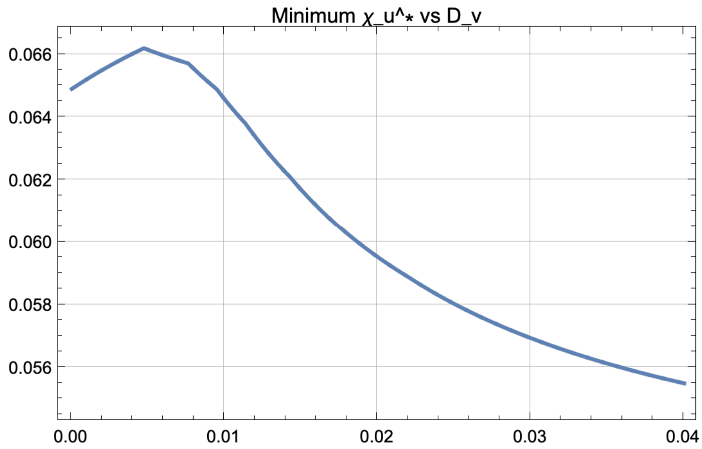
```
    c13 c32 (Du - Dv) m^2 (1 - p0) p0 π^2 / (L^2 q0) - (c33 - Dn m^2 π^2 / L^2)
     ((c11 - m^2 (Dv (1 - p0) + Du p0) π^2 / L^2) (c22 - m^2 (Du (1 - p0) + Dv p0) π^2 / L^2) +
       (Du - Dv) m^2 (1 - p0) p0 π^2 (c12 - (Du-Dv) m^2 π^2 q0 / L^2) / (L^2 q0)));
  chiuDen =
   (- c32 (Du - Dv) m^4 (1 - p0) p0^2 π^4 / L^4 - c32 m^2 (1 - p0) p0 π^2 (c11 - m^2 (Dv (1-p0) + Du p0) π^2 / L^2) / L^2);
  chiuStar = If[chiuDen == 0 || Im[chiuNum / chiuDen] ≠ 0, ∞, chiuNum / chiuDen];
  chiuStar]

DvValues = Range[0, 0.04, 0.00001];
mValues = Range[0, 100, 1];

minChiUStar = Table[Min[Table[ChiU[m, Dv], {m, mValues}]], {Dv, DvValues}];

ListLinePlot[Transpose[{DvValues, minChiUStar}], PlotStyle → Thick,
 AxesLabel → {"D_v", "Minimum χ_u^*"}, PlotLabel → "Minimum χ_u^* vs D_v",
 GridLines → Automatic, Frame → True, PlotRange → All]
```

Out[ ]=



In the third part, we aim to analytically derive the expression for the critical χ_u^* under the equation -1/2 (trA2-trA^2) trA+detA=0, which corresponds to the other stability condition, -1/2 (trA2-trA^2) trA+detA>0.

In[ ]:=
```
ClearAll[c11, c12, c13, c22, c23, c32, c33, m, L, Du, Dv, Dn, p0, q0, chiu, chiv]

A = {{c11 - (m Pi / L) ^ 2 (Du p0 + Dv (1 - p0)),
    c12 - (m Pi / L) ^ 2 (Du - Dv) q0, c13 + (m Pi / L) ^ 2 (chiu p0 q0 + chiv q0 (1 - p0))},
   {-q0 ^ (-1) (m Pi / L) ^ 2 (1 - p0) p0 (Du - Dv), c22 - (m Pi / L) ^ 2 (Du (1 - p0) + Dv p0),
    c23 + (m Pi / L) ^ 2 p0 (1 - p0) (chiu - chiv)}, {0, c32, - (m Pi / L) ^ 2 Dn + c33}};

detA = Det[A];
trA = Tr[A];

A2 = A.A;
trA2 = Tr[A2];

eqn = -1 / 2 (trA2 - trA ^ 2) trA + detA == 0;
solution = Solve[eqn, chiu];
FullSimplify[solution]
```

Out[ ]=

$$\{\{\text{chiu} \to (\text{c11 c12 (Du} - \text{Dv}) L^4 m^2 (-1 + \text{p0}) \text{p0} \pi^2 - \text{c13 c32 (Du} - \text{Dv}) L^4 m^2 (-1 + \text{p0}) \text{p0} \pi^2 + \text{c12 (Du} - \text{Dv}) L^2 m^2 (-1 + \text{p0}) \text{p0} \pi^2 \left((\text{c22} + 2\,\text{c33}) L^2 - (2\,\text{Dn} + \text{Du} + \text{Dv}) m^2 \pi^2\right) + \text{c11}^2 L^4 \left(-\left((\text{c22} + \text{c33}) L^2\right) + m^2 (\text{Dn} + \text{Du} - \text{Du p0} + \text{Dv p0}) \pi^2\right) \text{q0} - \text{c11} L^2 \left(\left(\text{c22}^2 - 2\,\text{c23 c32} + 4\,\text{c22 c33} + \text{c33}^2\right) L^4 - 2 L^2 m^2 (\text{c22} (2\,\text{Dn} + \text{Du} + \text{Dv}) + \text{c32 chiv} (-1 + \text{p0}) \text{p0} + \text{c33} (\text{Dn} + 2\,\text{Du} + \text{Dv} - \text{Du p0} + \text{Dv p0})) \pi^2 + m^4 \left(\text{Dn}^2 + 4\,\text{Dn Du} + \text{Du}^2 + 2\,\text{Dn Dv} + 2\,\text{Du Dv} - (\text{Du} - \text{Dv}) (2\,\text{Dn} + \text{Du} + \text{Dv}) \text{p0}\right) \pi^4\right) \text{q0} + \left(-\left((\text{c22} + \text{c33}) (-\text{c23 c32} + \text{c22 c33}) L^6\right) + L^4 m^2 \left(\text{c33}^2 (\text{Du} + \text{Dv}) + \text{c22 c32 chiv} (-1 + \text{p0}) \text{p0} + \text{c32 c33 chiv} (-1 + \text{p0}) \text{p0} + \text{c22}^2 (\text{Dn} + \text{Dv} + \text{Du p0} - \text{Dv p0}) - \text{c23 c32} (\text{Dn} + \text{Du} + 2\,\text{Dv} + \text{Du p0} - \text{Dv p0}) + 2\,\text{c22 c33} (\text{Dn} + \text{Du} + 2\,\text{Dv} + \text{Du p0} - \text{Dv p0})\right) \pi^2 - L^2 m^4 \left(\text{c33} \left(\text{Du}^2 + 4\,\text{Du Dv} + \text{Dv}^2 + 2\,\text{Dn} (\text{Du} + \text{Dv})\right) + \text{c32 chiv} (\text{Dn} + 2\,\text{Du} + \text{Dv}) (-1 + \text{p0}) \text{p0} + \text{c22} \left(\text{Dn}^2 + 2\,\text{Dn Du} + 4\,\text{Dn Dv} + 2\,\text{Du Dv} + \text{Dv}^2 + (\text{Du} - \text{Dv}) (2\,\text{Dn} + \text{Du} + \text{Dv}) \text{p0}\right)\right) \pi^4 + \left(\text{Dn}^2 (\text{Du} + \text{Dv}) + \text{Du Dv} (\text{Du} + \text{Dv}) + \text{Dn} \left(\text{Du}^2 + 4\,\text{Du Dv} + \text{Dv}^2\right)\right) m^6 \pi^6\right) \text{q0}) \Big/ \left(\text{c32} L^2 m^2 (-1 + \text{p0}) \text{p0} \pi^2 \left((2\,\text{c11} + \text{c22} + \text{c33}) L^2 - (\text{Dn} + \text{Du} + 2\,\text{Dv}) m^2 \pi^2\right) \text{q0}\right)\}\}$$

In the fourth part, we use the results derived in the third part and combine them with actual parameters to plot the minimum χ_u^* as a function of diffusivity D_v under the equation -1/2 (trA2-trA^2) trA+detA=0. This result aligns with part of Figure 9 and the MATLAB script “combine_chi_u.m”. The algorithm follows the same approach as in the second part, with the only difference being the use of the new expression for critical χ_u^*.

In[ ]:=
```
p0 = 0.5;
n0 = 0.5;
q0 = 4000;
L = 10;
Du = 0.01;
Dn = 0.01;
```

```
chiv = 0.02;
r = 1;
a = 0.5;
eL = 0.2;
eH = 0.5;
R0 = 0.35;
S0 = 0.3;
T0 = 0.5;
P0 = 0.2;
R1 = 0.6;
S1 = 0.35;
T1 = 0.6;
P1 = 0.3;
epsilon = 10;
kappa = 0.0001;

fL = (1 - n0) (R0 p0 + S0 (1 - p0)) + n0 (R1 p0 + S1 (1 - p0));
fH = (1 - n0) (T0 p0 + P0 (1 - p0)) + n0 (T1 p0 + P1 (1 - p0));

fLp = (1 - n0) (R0 - S0) + n0 (R1 - S1);
fHp = (1 - n0) (T0 - P0) + n0 (T1 - P1);
fLn = p0 (R1 - R0) + (1 - p0) (S1 - S0);
fHn = p0 (T1 - T0) + (1 - p0) (P1 - P0);
kp = fLp - fHp;
kn = fLn - fHn;

c11 = ((1 - p0) fH + p0 fL - 2 kappa q0) / epsilon;
c12 = (q0 (fHp + p0 kp)) / epsilon;
c13 = (q0 (fHn + p0 kn)) / epsilon;
c22 = (p0 (1 - p0) kp) / epsilon;
c23 = (p0 (1 - p0) kn) / epsilon;
c32 = r - a (eL n0 + eH (1 - n0));
c33 = -(r - a (eL n0 + eH (1 - n0)));

ChiU[m_, Dv_] := Module[{chiuNum, chiuDen, chiuStar},
  chiuNum =
   (-2 c13 c32 (Du - Dv) L^4 m^2 (-1 + p0) p0 π^2 + 2 c32 chiv (Du - Dv) L^2 m^4 (-1 + p0)^2 p0 π^4
       q0 + 2 c23 c32 L^4 (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 π^2 - Du m^2 π^2 - Dv m^2 π^2) q0 + 2 c32
       chiv L^2 m^2 (-1 + p0) p0 π^2 (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 π^2 - Du m^2 π^2 - Dv m^2 π^2) q0 +
     (c33 L^2 - Dn m^2 π^2)^2 (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 π^2 - Du m^2 π^2 - Dv m^2 π^2) q0 -
     (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 π^2 - Du m^2 π^2 - Dv m^2 π^2)^3 q0 +
     2 c23 c32 L^4 (c11 L^2 + m^2 (Dv (-1 + p0) - Du p0) π^2) q0 -
     2 c32 chiv L^2 m^2 (1 - p0) p0 π^2 (c11 L^2 + m^2 (Dv (-1 + p0) - Du p0) π^2) q0 +
     (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 π^2 - Du m^2 π^2 - Dv m^2 π^2)
      (c11 L^2 + m^2 (Dv (-1 + p0) - Du p0) π^2)^2 q0 +
     (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 π^2 - Du m^2 π^2 - Dv m^2 π^2)
```

```
    (c22 L^2 + m^2 (Du (-1 + p0) - Dv p0) π^2)^2 q0 -
   2 (Du - Dv) m^2 (1 - p0) p0 π^2 (c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 π^2 - Du m^2 π^2 - Dv m^2 π^2)
    (c12 L^2 + (-Du + Dv) m^2 π^2 q0) + 2 (c33 L^2 - Dn m^2 π^2)
    (c12 (Du - Dv) L^2 m^2 (-1 + p0) p0 π^2 + (-c11 c22 L^4 + c22 L^2 m^2 (Dv + Du p0 - Dv p0)
          π^2 + c11 L^2 m^2 (Du - Du p0 + Dv p0) π^2 - Du Dv m^4 π^4) q0));

  chiuDen = (2 c32 L^2 m^2 (-1 + p0) p0 π^2
      (2 c11 L^2 + c22 L^2 + c33 L^2 - Dn m^2 π^2 - Du m^2 π^2 - 2 Dv m^2 π^2) q0);
  chiuStar = If[chiuDen == 0 || Im[chiuNum / chiuDen] ≠ 0, ∞, chiuNum / chiuDen];
  chiuStar]

DvValues = Range[0, 0.04, 0.00001];
mValues = Range[0, 100, 1];

minChiUStar = Table[Min[Table[ChiU[m, Dv], {m, mValues}]], {Dv, DvValues}];

ListLinePlot[Transpose[{DvValues, minChiUStar}], PlotStyle → Thick,
 AxesLabel → {"D_v", "Minimum χ_u^*"}, PlotLabel → "Minimum χ_u^* vs D_v",
 GridLines → Automatic, Frame → True, PlotRange → All]
```

Out[ ]=