Math Modeling Project 2

Ruoxi Wang, Yvan N'guettia, Qin Yang

October 2025

1 Background Story

We model five groups of runners with varying speeds. They are transported from the starting line to the finish line, and they diffuse because speeds within each cohort differ slightly, causing the group to flatten out along the course. A hill in the middle slows runners as they climb. Finally, a decay term captures the possibility that some runners quit the race.

2 Terms

$$\frac{\partial u_k}{\partial t} \;+\; \frac{\partial}{\partial x}\big(v_k(x)\,u_k(x,t)\big) \;=\; D\,\frac{\partial^2 u_k}{\partial x^2} \;-\; \lambda\,u_k(x,t).$$

Term	**Meaning**	**Description**
$u_k(x,t)$	Runner density	Runners from cohort k per meter at position x and time t.
$\partial_t u_k$	Time change	How the density changes over time.
$\partial_x\big(v_k u_k\big)$	Advection	Movement along the course at local speed $v_k(x)$.
$v_k(x)$	Local speed	Speed of cohort k at position x.
$v_k(x) = s_k\,\phi(x)$	Speed model	Base speed s_k modulated by terrain slowdown $\phi(x)$.
$\phi(x)$	Terrain slowdown	Smaller near the hill; closer to 1 away from it.
Hill profile (unnormalized): $\phi(x) = 3 - \exp\Big(-2\big(\frac{x-h}{w}\big)^2\Big)$. *Optional normalized:* $\phi_{\text{norm}}(x) = \frac{1}{3}\phi(x) = 1 - \frac{1}{3}\exp\Big(-2\big(\frac{x-h}{w}\big)^2\Big)$.		
$D\,\partial_{xx} u_k$	Diffusion	Smooths density; captures small speed differences within a cohort.
$-\lambda\,u_k$	Decay	Loss of runners with rate $\lambda \geq 0$.

3 Model

Boundary	Mathematical Condition
Left Boundary (x=0)	$v_k(0)u_k(0,t) = 0$
Right Boundary (x=L)	Outflowflux $F_{finish}(t) = \sum_k v_k(L)u_k(L,t)$
left and right (diffusion)	$u_{xx}(0,t) = u_{xx}(L,t) = 0$