Proving the Existence, Uniqueness, and Main Property of the Circumcenter for Triangles

Honors Geometry

January 11, 2021

Abstract

The purpose of this document is to present what we covered in the first week of the second semester of Honors Geometry as a fully fleshed-out mathematical proof. This is not something you should expect to be able to do before working out all the details just as we did in class. Keep in mind that, unlike a proof, mathematics is often not a linear process from hypothesis to conclusion; instead, we require time and effort to discover the best means of reaching the conclusion, at which point we can begin to draft up a clear chain of reasoning which can be developed into a formal proof.

1 Introduction

We begin by defining two terms. We say a circle *circumscribes* a polygon if every vertex of the polygon lies on the circle. The *circumcenter* of a polygon is the center of its circumscribing circle.

In the following sections, we will prove that all triangles have a circumcenter and it is located at the intersection of the perpendicular bisectors. This proof will proceed as follows: First we will show that for a given circle, the perpendicular bisector of any chord must pass through the center of the circle. Since the sides of an inscribed polygon are chords of the circle, this means that the perpendicular bisector of any side must pass through the circumcenter. This is the main property of the circumcenter. Next, we will show that every triangle is circumscribed by one specific circle. This is done by finding the equation for this circle. This guarantees the existence of a unique circumcenter for all triangles.

2 The perpendicular bisector of any chord passes through the center of the circle

Given a chord, label the two endpoints A and B. By translational symmetry, we choose the center of the circle as our origin $(0, 0)$. By scale symmetry, we set our unit (the distance between each tick mark on the graph) to the radius of the circle. Therefore, our circle is described by the equation

$$x^2 + y^2 = 1 \tag{1}$$

By rotational symmetry, we choose the orientation of our axes so that A and B are horizontal from one another. A and B have the same y-coordinate, so

$$A = (x_A, y_A) \tag{2}$$

$$B = (x_B, y_B) = (x_B, y_A) \tag{3}$$

Since A and B are on the circle, they must satisfy the equation for our circle. Plugging them in, we find

$$x_A^2 + y_A^2 = 1 = x_B^2 + y_A^2 \tag{4}$$

$$x_A^2 + y_A^2 = x_B^2 + y_A^2 \tag{5}$$

$$x_A^2 = x_B^2 \tag{6}$$

$$|x_A| = |x_B| \tag{7}$$

Since A and B are equal distance from the y-axis, their midpoint must lie on the y-axis. Because the chord is horizontal, it is perpendicular to the (vertical) y-axis. Therefore, the y-axis is the perpendicular bisector of our chord. The y-axis passes through the origin, which is the center of our circle.

3 Every triangle is circumscribed by a unique (meaning exactly one) circle

Label the vertices of our triangle A, B, C. By translational symmetry, we set our origin at A. By rotational symmetry, we orient our axes so that B

lies along the x-axis. By scale symmetry, we set our unit so that B is at a distance 2 from the origin. Summarizing, we have that

$$\begin{aligned} A &= (0,0) \\ B &= (0,2) \\ C &= (x_C, y_C) \end{aligned} \tag{8}$$

Note that A and B both lie on the x-axis, so are horizontal from each other (they have an x-coordinate of 0). Using our result from the previous section, we know that any circle that contains A and B has a center that lies directly above or below their midpoint. In this case, their midpoint is $(1,0)$, so the center of any circle passing through all three points has x-coordinate 1. Label the y-coordinate of the center of our circle y_0.

Now, we may plug the coordinates of A into the equation for a circle with center $(1, y_0)$:

$$R^2 = (x-1)^2 + (y-y_0)^2 \tag{9}$$

$$R^2 = (0-1)^2 + (0-y_0)^2 \tag{10}$$

$$R^2 = 1 + y_0^2 \tag{11}$$

$$R^2 - y_0^2 - 1 = 0 \tag{12}$$

Next, we plug the coordinates of C into the equation for our circle:

$$R^2 = (x_C - 1)^2 + (y_C - y_0)^2 \tag{13}$$

$$R^2 = x_C^2 - 2x_C + 1 + y_C^2 + 2y_C y_0 + y_0^2 \tag{14}$$

$$R^2 - y_0^2 - 1 = x_C^2 - 2x_C + y_C^2 + 2y_C y_0 \tag{15}$$

Looking at equation (12) above, we see that the left side of equation (15) is actually equal to 0!

$$0 = x_C^2 - 2x_C + y_C^2 + 2y_C y_0 \tag{16}$$

Remember that we are solving for the equation of the circle. In equation (16), we only need to solve for y_0 in terms of everything else, since we assume that the coordinates of C are given to us. Rearranging, we get that

$$2y_C y_0 = 2x_C - x_C^2 - y_C^2 \tag{17}$$

Which allows us to conclude that

$$y_0 = \frac{2x_C - x_C^2 - y_C^2}{2y_C} = \frac{x_C}{y_C}\left(1 - \frac{x_C}{2}\right) - \frac{y_C}{2} \tag{18}$$

We are almost done! Don't worry about making sense of this expression for y_0, the important thing is that we have an explicit value. By that, I mean that if we are given numbers for the coordinates of $C = (x_C, y_C)$, we could plug them into equation (18) and get a number for y_0.

Finally, notice that equation (11) from before allows us to substitute an expression in terms of only y_0 for R^2.

$$R^2 = 1 + y_0^2 \tag{11}$$

Equation (9) is the equation for our circle, which we can now use (11) to rewrite in terms of only y_0.

$$R^2 = (x-1)^2 + (y-y_0)^2 \tag{9}$$

$$1 + y_0^2 = (x-1)^2 + (y-y_0)^2 \tag{19}$$

Since we have solved for y_0 in terms of the known coordinates of C, equation (19) is our final equation for the circle passing through the points A, B, C. These are the vertices of our triangle, so by definition the circle circumscribes our triangle. Only one circle is described by this equation, so we say that it is the unique circle which circumscribes our triangle.

4 Conclusion

We have shown two things. The first is that the perpendicular bisector of a chord passes through the center of the circle. By definition, if a polygon is circumscribed by a circle, its vertices are on the circle. The side of a polygon is the line segment between its vertices, and a chord is the line segment between two points on a circle. Since its vertices are all on the circle, each side of the polygon is a chord. Our first result therefore implies that the circumcenter of a polygon (such as a triangle) is passed through by the perpendicular bisector of any side. Since this is true of all the sides, the circumcenter is at the point where all the perpendicular bisectors intersect. As stated previously, this is the main property of the circumcenter.

The second thing we have shown is that there is a unique circle that circumscribes every triangle. Since this circle has exactly one center, this means every triangle has a unique circumcenter. This establishes both the existence and uniqueness of the circumcenter for every triangle.