Writing proofs better than your teacher

Honors Geometry

1 Perpendicular Bisectors

First, the definition. The **perpendicular bisector** of a line segment is the line (or a line segment) that is perpendicular to and bisects it. Bisects just means cut in half. For example, let's say we have two points, A and B, with the line segment AB between them and midpoint M. The perpendicular bisector would go through M and make a right angle with AB.

1.1 Proving every point on the perpendicular bisector of a line segment is equidistant from the endpoints

We will do this with a two column proof, but first we must draw a picture and figure out our givens. If you are reading this, please go through the exercise of drawing the picture as described. For the rest of the proofs I will omit most of this sort of exposition, so make sure you're solid on drawing.

Let's begin by drawing a line segment AB. We already talked about how the perpendicular bisector bisects AB at a right angle, so make sure your picture reflects this. Label the point where they intersect M, and pick and arbitrary point C on the perpendicular bisector. Now that we have two points on it, we can refer to the perpendicular bisector as MC.

We already talked about how M is the midpoint of AB, so we know that AM and BM are congruent. Last semester, this would be one of the givens! In our proof, we will say $AM = BM$ by the definition of the perpendicular bisector. This is because M is the point where the perpendicular bisector crosses AB. By the definition of the perpendicular bisector, M is the midpoint of AB. This is the same as saying $AM = BM$. Our second given would have been $AB \perp MC$. This is also part of the definition!

The last thing to include in your picture is two line segments connecting C with A and B. Our goal is to prove that these line segments are congruent, or that $AC = BC$. Before we actually begin the proof, it's a good idea to think about how we could actually get to this. Looking at your picture, what result would allow us to conclude that AC and BC are congruent?

Below is a proof that leads to the desired conclusion. Could you do the same thing using the Pythagorean Theorem?

Premise: MC is the perpendicular bisector of AB, with M lying on AB.

$AM = BM$	Premise, definition of perpendicular bisector	(1)
$AB \perp MC$	Premise, definition of perpendicular bisector	(2)
$MC = MC$	Reflexivity	(3)
$\triangle AMC \cong \triangle BMC$	1, 2, 3: SAS (Side-Angle-Side) $\triangle$ Congruence	(4)
$AC = BC$	5: Corresponding parts of congruent $\triangle$s	(5)

Conclusion: C is equidistant from A and B. Since C was arbitrary, any point on the perpendicular bisector of AB is equidistant from the endpoints of AB.

1.2 Proving the converse: any point equidistant from two points is on the perpendicular bisector of the line segment connecting them

Let AB be the line segment in question. We start out by picking an arbitrary point C that is equidistant from the endpoints of AB. So, what is our premise? Well, it's what we start out with: C is equidistant from A and B. Next, we want to draw a line segment from C to AB that is perpendicular to AB. Let's pick the point M so that MC is perpendicular to AB.

Premise: C equidistant from A and B.

Construction: Choose M on AB so that $MC \perp AB$.

$AC = AB$	Premise	(1)
$MC = MC$	Reflexivity	(2)
$\triangle AMC \cong \triangle BMC$	Construction, 1, 2: HL $\triangle$ Congruence	(3)
$AM = BM$	3: Corresponding parts of congruent $\triangle$s	(4)
MC is $\perp$ bisector of AB	Construction, 4: definition of $\perp$ bisector	(5)

Conclusion: C is on the $\perp$ bisector of AB. Since C was arbitrary, any point equidistant from the endpoints of AB is on the perpendicular bisector of AB.

1.3 Proving the circumcenter of a triangle is at the intersection of the perpendicular bisectors

Wait, what's the **circumcenter** again? It's the center of the circle that all the vertices are on. We're just going to prove that the center of this circle is on the perpendicular bisector of each of the sides!

First of all, what do we know about circles? Well, any point on a circle is equidistant from the center. Now, you might be thinking this is starting to sound like the premise of our last proof. To recap, if we pick two vertices of a triangle, they are equidistant from the circumcenter. By what we proved in **1.2**, this means the circumcenter is on the perpendicular bisector of the side between those two vertices! Since that's true for all the sides, all the perpendicular bisectors must intersect at the circumcenter.

That's literally the whole proof, wow. No two columns about it.

2 Angle Bisectors

An **angle bisector** splits an angle in half. If you'd like, go ahead and try to prove the results listed below for angle bisectors. We've gone through analogous proofs in class. To check, I'll just refer you to a textbook chapter: `https://www.ck12.org/book/ck-12-geometry-concepts/section/5.3/`

2.1 Prove that all points on an angle bisector are equidistant from the two lines making the bisected angle

2.2 Prove the converse: any point equidistant from two lines is on their angle bisector

2.3 Proof that all three angle bisectors of a triangle intersect at the incenter

The **incenter** is the center of the largest circle that's completely inside the triangle. If a circle touches each side exactly once, it couldn't be any bigger right? We say it's **inscribed** in the triangle. Here's a proof I think is really clever (from Art of Problem Solving).

Label our triangle $\triangle ABC$ and start out by considering the intersection of two of the angle bisectors. If we label their point of intersection O, we have bisectors AO and BO (haha). We showed in 2.1 that every point of the angle bisector is equidistant from the two sides making the angle.

Premise: The angle bisectors of A and B intersect at O.

O is on $\angle$ bisector of A	Premise	(1)
O is equidistant from AB and AC	1: **2.1**	(2)
O is on $\angle$ bisector of B	Premise	(3)
O is equidistant from AB and BC	3: **2.1**	(4)
O is equidistant from BC and AC	2, 4: Transitivity	(5)
O is on $\angle$ bisector of C	5: **2.2**	(6)

Conclusion: All angle bisectors of $\triangle ABC$ intersect at O. Also, O is equidistant from all the sides, so there's a circle centered there that touches each side in exactly one spot. This makes it an inscribed circle, so O is the incenter!

3 Altitudes

An **altitude** is the line segment perpendicular to a side that goes to the opposite vertex. If your triangle was a mountain, with one side being sea level, then the length of the altitude from the bottom side to the peak would be how you'd measure the height.

I posted a picture of the whiteboard from class when we proved that any altitude next to an obtuse angle has to be outside the triangle. This was our first example of an indirect proof, or a proof by contradiction. We will discuss these in more detail fairly soon. The following class we did the same thing to prove that any altitude next to two acute angles must be inside the triangle. I won't be duplicating those proofs here since it uses a technique we haven't formalized yet.

However we did show that the two legs of a right triangle are both altitudes using a very simple direct proof.

3.1 Proof that both legs of a right triangle are altitudes

Premise: $\triangle ABC$ is a right triangle, with $AC \perp BC$.

The vertex A is on AC	Premise, duh	(1)
AC is an altitude	Premise, 1: definition of altitude	(2)
The vertex B is on BC	Premise, also duh	(3)
BC is an altitude	Premise, 3: definition of altitude	(4)

Conclusion: AC and BC, the legs of right triangle $\triangle ABC$, are altitudes.

Notice that I made sure to cite two things in lines 2 and 4. Why did I do this? Well, an altitude has to satisfy two requirements! What are they?

3.2 Proof that the orthocenter actually exists

Congratulations on making it this far, I know it hasn't been easy. Take a second to pat yourself on the back. In this section, I'm not gonna go through all the formalities of doing this proof since we did it in class, but I'll give you a picture to fill in with congruencies. Feel free to write out a standard two column proof for this, just be warned that it'll get quite lengthy.

As preamble, the basic idea here is that once we have a triangle $\triangle ABC$, we can construct another triangle around it composed of parallel line segments. Then, the perpendicular bisectors of this bigger triangle, let's call it $\triangle A'B'C'$, will actually be altitudes of $\triangle ABC$. Since the perpendicular bisectors intersect at the circumcenter of $\triangle A'B'C'$, the altitudes of $\triangle ABC$ will intersect at this same point, which we'll call the **orthocenter** of $\triangle ABC$.