Convergence of approximate Brownian distributions to the Wiener measure

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Outline

Definition (Random Process)

Let T be a set, $(\Omega, \mathcal{F}, P)$ a probability space, $(X, \mathcal{B})$ a measurable space. For every $t \in T$, we have an $X-$valued $\mathcal{F}-$measurable function $\xi_t = \xi_t(\omega)$, and ξ_t is a **random process** on T with values in X. For each ω, the function $\xi_t(\omega)$, viewed as a function of t, is called a **path** or a **trajectory of the process**.

Notation

Let $(X, \mathcal{B}(X))$ be a Polish space and T a set. The set of all $X-$valued functions on T is denoted by X^T. Denote by $x_\cdot$ the points in X^T, and by x_t the value of $x_\cdot$ at t.

Definitions (cont.)

In this section, we will consider real–valued random processes on $T = [0, 1]$. These processes are called **continuous** if all its trajectories are continuous functions on T.

For each ω, we have a continuous trajectory, i.e., we have an element in $C = C([0, 1])$, the space of continuous functions on $[0, 1]$, a Polish space equipped with the metric $\rho(x_., y_.) = \sup_{t\in[0,1]} |x_t - y_t|$.

Definition (Cylinder $\sigma-$field $\Sigma(C)$)

The **cylinder** $\sigma-$**field** $\Sigma(C)$ is the $\sigma-$algebra of C generated by all subsets of the form

$$\{x_. \in C : x_t \in \Gamma\}, t \in [0, 1], \Gamma \in \mathfrak{B}(\mathbb{R}).$$

Lemma 1

We will need the following theorem.

Theorem (Ch.1, §2: Theorem 1, p.6, [?])

Let X and Y be Polish spaces, and let $f : X \to Y$ be a continuous function. Then f is Borel.

Our first goal is to prove the following lemma.

Lemma (Ch.1, §4: Lemma, 1, p.16, [?])

$$\Sigma(C) = \mathfrak{B}(C)$$

Lemma 1 (cont.)

Proof: Fix t, and let π_t be defined as

$$\pi_t(x_.) = x_t.$$

First, we show that $\pi_t : C \to \mathbb{R}$ is a continuous function, i.e. that the preimage of an open set is open. WLOG, let $\pi_t(A)$ be the open ball

$$B_\epsilon(x_t) = (x_t - \epsilon, x_t + \epsilon) \subset \mathbb{R}$$

If $\rho(x_., y_.) = \sup_s |x_s - y_s| < \epsilon$, then $\pi_t y_. = y_t \in B_\epsilon(x_t)$.
Therefore A is open, so π_t is continuous.
A cylinder set (defined $\{x_. : \pi_t x_. \in \Gamma\}$, for $\Gamma \in \mathfrak{B}(\mathbb{R})$) is the preimage of a Borel set, so Borel (in C). Thus $\Sigma(C) \subset \mathfrak{B}(C)$.
For the other inclusion, we show that closed balls in C are generated by cylinder sets (thus generating the Borel algebra).

PROOF: (cont.) Claim:

$$\begin{aligned}\overline{B_\epsilon}(x^0_\cdot) &= \left\{x_\cdot \in C : \rho(x^0_\cdot, x_\cdot) \leq \epsilon\right\} \\ &= \bigcap_{r \in \mathbb{Q} \cap [0,1]} \left\{x_\cdot \in C : x_r \in \left[x^0_r - \epsilon, x^0_r + \epsilon\right]\right\}\end{aligned}$$

This follows from the continuity of $x_\cdot$ and density of $\mathbb{Q}$ in $\mathbb{R}$, which gives the left implication below:

$$\rho(x^0_\cdot, x_\cdot) = \sup_{t \in [0,1]} \left|x^0_t - x_t\right| \leq \epsilon \Leftrightarrow \left|x^0_r - x_r\right| \leq \epsilon, \forall r \in \mathbb{Q} \cap [0,1]$$

As the countable intersection of elements of a $\sigma-$algebra is in the $\sigma-$algebra, closed balls in C are elements of $\Sigma(C)$. $\square$

Theorem 2

Theorem (Ch.1, §4. Theorem 2, p.17, [?])

$\xi_t(\omega)$ is a continuous process (all trajectories continuous) on $[0,1]$ if and only if $\xi_\cdot$ is a $C-$valued random variable.

Proof: ($\Rightarrow$) We want to show $\xi_\cdot^{-1}(B) := \{\omega : \xi_\cdot(\omega) \in B\}$ measurable for every $B \in \mathfrak{B}(C)$ (this is what it means to be a random variable). Note that $\{B \subset C : \xi_\cdot^{-1}(B) \text{ is measurable}\}$ form a sub-σ-algebra of $\mathfrak{B}(C)$. If B is a cylinder set, the measure of $\xi_\cdot^{-1}(B)$ is $\mathbb{P}(\xi_t \in \Gamma)$. In particular, this means that

$$\Sigma(C) \subset \{B \subset C : \xi_\cdot^{-1}(B) \text{ is measurable}\} \subset \mathfrak{B}(C)$$

By lemma 1, $\Sigma(C) = \mathfrak{B}(C)$.
($\Leftarrow$) $\pi_t \circ \xi_\cdot$ is measurable since π_t continuous. $\xi_\cdot(\omega) \in C$ for any ω, so $\xi_t(\omega) = \pi_t(\xi_\cdot(\omega))$ is a continuous function of t. $\square$

Lemma 4

Lemma (Ch. 1, §4, Lemma 4, p.18, [?])

Let x_t *be a real–valued function defined on* $[0,1]$*. Assume that there exist a constant* $a > 0$ *and an integer* $n \geq 0$ *such that*

$$\left| x_{(i+1)/2^m} - x_{i/2^m} \right| \leq \frac{1}{2^{ma}}$$

for all $m \geq n$ *and* $0 \leq i \leq 2^m - 1$*. Then for all binary numbers* $t, s \in [0,1]$*, i.e. numbers of the form* $\sum_{i=0}^{\infty} \epsilon 2^{-i}, \epsilon = 0$ *or* 1*, such that* $|t - s| \leq 2^{-n}$*, we have that*

$$|x_t - x_s| \leq N(a)|t - s|^a,$$

where $N(a) = 2^{2a+1}(2^a - 1)^{-1}$*.*

Theorem 6

Theorem (Ch.1, §4, Theorem 6, p.19 – 20, [?])

Let ξ_t *be a continuous process, and let* $\alpha > 0, \beta > 0, N \in (0, \infty)$ *be constants such that*

$$E|\xi_t - \xi_s|^\alpha \leq N|t-s|^{1+\beta}, \forall s, t \in [0,1]. \quad (1)$$

Then, for $0 < a < \frac{\beta}{\alpha}$ *and for every* $\epsilon > 0$*, there's an* $n \in \mathbb{N}$ *such that*

$$P\left\{\xi_. \in K_n(a)\right\} \geq 1 - \epsilon,$$

where
$K_n(a) = \{x_. : |x_0| \leq 2^n, |x_t - x_s| \leq N(a)|t-s|^a, \forall |t-s| \leq 2^{-n}\}$

Theorem 6 (cont.)

Proof: Let

$$A_n = \{\omega : |\xi_0| \geq 2^n\} \cup \left\{\omega : \sup_{m \geq n} \max_{i=0,\ldots,2^m-1} \left|\xi_{\frac{i+1}{2^m}} - \xi_{\frac{i}{2^m}}\right| \geq 2^{-ma}\right\}$$

and recall the definition of $K_n(a)$:

$$K_n(a) = \left\{x_\cdot : |x_0| \leq 2^n, |x_t - x_s| \leq N(a)|t-s|^a, \forall |t-s| \leq 2^{-n}\right\}$$

Lemma 4 says $\left|x_{(i+1)/2^m} - x_{i/2^m}\right| \leq 2^{-ma}, \forall m \geq n, i < 2^m$
$\Longrightarrow |x_t - x_s| \leq N(a)|t-s|^a, \forall t, s$ binary s.t. $|t-s| \leq 2^{-n}$.
Note that $\xi_\cdot(\omega)$ satisfies the first requirement to be in $K_n(a)$ whenever ω is outside the first set comprising A_n. Lemma 4 does the same for the second, so $\omega \notin A_n \Longrightarrow \xi_\cdot(\omega) \in K_n(a)$. Thus, $P\{\xi_\cdot \notin K_n(a)\} \leq P(A_n)$

$$\leq P\{|\xi_0| \geq 2^n\} + P\left\{\sup_{m \geq n} \max_{i=0,\ldots,2^m-1} \left|\xi_{(i+1)/2^m} - \xi_{i/2^m}\right| \geq 2^{-ma}\right\}$$

Theorem 6 (cont.)

Proof: (cont.) By Chebyshev's inequality, $\forall \alpha > 0$ this is

$$\leq P\{|\xi_0| \geq 2^n\} + E \sup_{m \geq n} \max_{i=0,\ldots,2^m-1} \left|\xi_{(i+1)/2^m} - \xi_{i/2^m}\right|^{\alpha} 2^{ma\alpha}$$

$$\leq P\{|\xi_0| \geq 2^n\} + \sum_{m=n}^{\infty} \sum_{i=0}^{2^m-1} 2^{m\alpha a} E\left|\xi_{(i+1)/2^m} - \xi_{i/2^m}\right|^{\alpha}$$

$$\leq P\{|\xi_0| \geq 2^n\} + \sum_{m=n}^{\infty} \sum_{i=0}^{2^m-1} 2^{ma\alpha} N 2^{-m(1+\beta)} \ (\because \textbf{(??)})$$

$$\leq P\{|\xi_0| \geq 2^n\} + N \sum_{m=n}^{\infty} 2^{ma\alpha - m\beta} \xrightarrow{n \to \infty} 0$$

Since $P\{\xi_\cdot \notin K_n(a)\} \to 0$, we have $P\{\xi_\cdot \in K_n(a)\} \to 1$. $\square$

Theorem 7

Theorem (Ch.1, §4. Theorem 7, p.20, [?])

For $k = 1, 2, ...,$ let $\xi^k_\cdot$ be continuous processes on $[0,1]$ such that $\sup_k P(|\xi^k_0| \geq c) \xrightarrow[c\to\infty]{} 0$, and for some constants $\alpha, \beta, N > 0$,

$$E|\xi^k_t - \xi^k_s|^\alpha \leq N|t-s|^{1+\beta}, \forall s,t \in [0,1], k \geq 1$$

Then the sequence of distributions of $\xi^k_\cdot$ on C is relatively compact.

Proof: Since every $K_n(a)$ is compact, given any $\epsilon > 0$ the previous theorem allows us to find a compact set $K := K_n(c)$ such that $\mu_k(K^c) := P(\xi^k_\cdot \in K^c) = 1 - P(\xi^k_\cdot \in K) \leq \epsilon$. By hypothesis $P\{|\xi^k_0| \geq 2^n\} \xrightarrow[n\to\infty]{} 0$ for any k. Notice that this is exactly the term on the left in the above proof. Since N doesn't depend on k, the term on the right is the same $\forall \xi^k_\cdot$. We choose the same K for every μ_k and apply Prokhorov. $\square$

Lemma 2.1

Theorem (Ch.2, §1. Lemma 1, p.28, [?])

Let $\xi_t^n = \frac{1}{\sqrt{n}}\left(S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)\eta_{\lfloor nt \rfloor}\right)$, *where* $S_k = \eta_1 + ... + \eta_k$, $\xi_0^n = 0$ *for all* n, *and the* η_j *are i.i.d. with mean 0 and variance 1. The sequence of distributions of* $\xi_{\cdot}^n$ *on* C *is relatively compact.*

Proof: For simplicity, assume the η_k have finite 4th moment which we denote $m_4 := E\eta_k^4$ (see Billingsley [?] for general case). Since $\xi_0^n = 0$, it's sufficient to prove that $\xi_{\cdot}^n$ satisfies the inequality of the previous theorem. Note that η_{k+1} is independent of S_k, so

$$
\begin{aligned}
&ES_{k+1}^4 = E(S_k + \eta_{k+1})^4 \\
&= ES_k^4 + 4ES_k^3\eta_{k+1} + 6ES_k^2\eta_{k+1}^2 + 4ES_k\eta_{k+1}^3 + E\eta_k^4 \\
&= ES_k^4 + 4(ES_k^3)(E\eta_{k+1}) + 6(ES_k^2)(E\eta_{k+1}^2) + 4(ES_k)(E\eta_{k+1}^3) + m_4 \\
&= ES_k^4 + 6ES_k^2 + 4(ES_k)(E\eta_{k+1}^3) + m_4 = ES_k^4 + 6k + m_4
\end{aligned}
$$

Lemma 2.1 (cont.)

Proof: (cont.) $ES_k^4 = ES_{k-1}^4 + 6k + m_4$ and $ES_1^4 = E\eta_1^4 = m_4$, so $ES_k^4 = \sum_{j=1}^k \{6(j-1) + m_4\} = 3k(k-1) + km_4 \le 3k^2 + km_4$
When s, t are endpoints of intervals (i.e. ns, nt integers) we have

$$E|\xi_t^n - \xi_s^n|^4 = \frac{1}{n^2} E|S_{tn} - S_{sn}|^4 = \frac{1}{n^2} ES_{n|t-s|}^4 \le 3|t-s|^2 + \frac{m_4}{n}|t-s|$$

Since s, t are not inside the same interval $|t-s| \ge 1/n$, so

$$E|\xi_t^n - \xi_s^n|^4 \le (3 + m_4)|t-s|^2 \tag{2}$$

This will also prove useful when s, t are in different intervals.
If s, t are both in the interval $[k/n, (k+1)/n]$, we have that

$$E|\xi_t^n - \xi_s^n|^4 = E\left(\sqrt{n}|\eta_{k+1}||t-s|\right)^4 = n^2 m_4 |t-s|^4 \le m_4|t-s|^2$$

For the remaining case, we split up $[s,t] = [s,s'] \cup [s',t'] \cup [t',t]$ where s', t' are the closest endpoints between s, t. Note also that

$$(a+b+c)^4 \le 81(a^4 + b^4 + c^4)$$

so $E|\xi_t^n - \xi_s^n|^4 \le E\left(|\xi_t^n - \xi_{t'}^n| + |\xi_{t'}^n - \xi_{s'}^n| + |\xi_{s'}^n - \xi_s^n|\right)^4$

Lemma 2.1 (cont.

Proof: (cont.) $\leq 81 \left(E|\xi_t^n - \xi_{t'}^n|^4 + E|\xi_{t'}^n - \xi_{s'}^n|^4 + E|\xi_{s'}^n - \xi_s^n|^4 \right)$
We apply our previous result to s, s' and t', t and (2) to s', t':

$$E|\xi_t^n - \xi_s^n|^4 \leq 81 \left(m_4|t - t'|^2 + (3 + m_4)|t' - s'|^2 + m_4|s' - s|^2 \right)$$

Since each of $[s, s'], [s', t'], [t', t]$ are subsets of $[s, t]$, this is

$$\leq 81 \left(m_4|t - s|^2 + (3 + m_4)|t - s|^2 + m_4|t - s|^2 \right) = 81(3+3m_4)|t-s|^2$$

This is precisely the condition for $\alpha = 4, \beta = 1, N = 243(1 + m_4)$, which are all independent of n, s, t. Thus $\forall n, s, t$ we have that

$$E|\xi_t^n - \xi_s^n|^4 \leq N|t - s|^2 := 243(1 + m_4)|t - s|^2$$

Therefore the sequence of distributions on is relatively compact. $\square$

Bibliography

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