Assignment

1. Consider a 3 state Markov chain observed in iid zero mean Gaussian noise.
   (a) Illustrate via simulation the performance of the HMM filter. Compare the mean square error of the estimate with the optimal smoother for a batch of $N = 1000$ points.
   Illustrate via simulation how the mean squared error of the estimate for both filter and smoother changes with noise variance.
   (b) Let θ^o denote the model parameter values that you used to generate the observations above. Compute via simulation the likelihood of the model θ^o given $N = 1000$ data points. Choose another model value $\bar{\theta}$ and compute the likelihood. Do this for several values $\bar{\theta}$ and evaluate the likelihood in each case.
   (c) Derive and simulate the EM algorithm to estimate the parameters of the HMM.
2. Derive the Kalman filter equations for the scalar case starting with the Bayesian recursion.
3. Consider a ship moving in 2 dimensional space with position and velocity represented by the state $z_k = (p_k^{(1)}, \dot{p}_k^{(1)}, p_k^{(2)}, \dot{p}_k^{(2)})'$. Then

$$z_{k+1} = A\, z_k + f\, r_{k+1} + w_k, \quad w_k \sim \mathbf{N}(0, Q), \quad k = 0, 1, \ldots,$$

$$A = \begin{bmatrix} 1 & \Delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad f = \begin{bmatrix} \Delta^2/2 & 0 \\ \Delta & 0 \\ 0 & \Delta^2/2 \\ 0 & \Delta \end{bmatrix}, \quad r_{k+1} = \begin{bmatrix} q_k^{(1)} \\ q_k^{(2)} \end{bmatrix}.$$

   Here r_{k+1} denotes the acceleration (maneuvers) during the k-th sampling period. Assume the acceleration is a known constant and does not evolve with time.
   Suppose that the position of the target is observed in noise at each time as

$$y_k = C z_k + v_k, \quad v_k \sim \mathbf{N}(0, R).$$

   Here $y_k \in \mathbb{R}^2$, and C is a 2×4 matrix with $C_{1,1} = C_{2,3} = 1$ and the remaining elements are zero.
   Using Matlab simulation, illustrate the use of the Kalman filter for estimating that target's state (position and velocity) for various state and observation noise variances.
   Also illustrate the performance of the optimal predictor for the target's state.
4. Consider the scalar stochastic model

$$\begin{aligned} x_{k+1} &= a x_k + w_k \\ y_k &= \tan^{-1}(x_k) + v_k \end{aligned} \tag{1}$$

where w_k and v_k are iid zero mean scalar Gaussian processes and $|a| < 1$.
Simulate a particle filter for estimating the state x_k given observations $y_1, \ldots, y_k$ for $k = 1, 2, \ldots$. Compare the particle filter performance to a grid quantization of the posterior filtering equation update. (The comparison should be in terms of the mean square error of the state estimate.)