ETH zürich

Lecture «Robot Dynamics»: Summary

151-0851-00 V

lecture:	HG F3	Tuesday 10:15 – 12:00, every week
exercise:	HG F3	Wednesday 8:15 – 10:00, according to schedule

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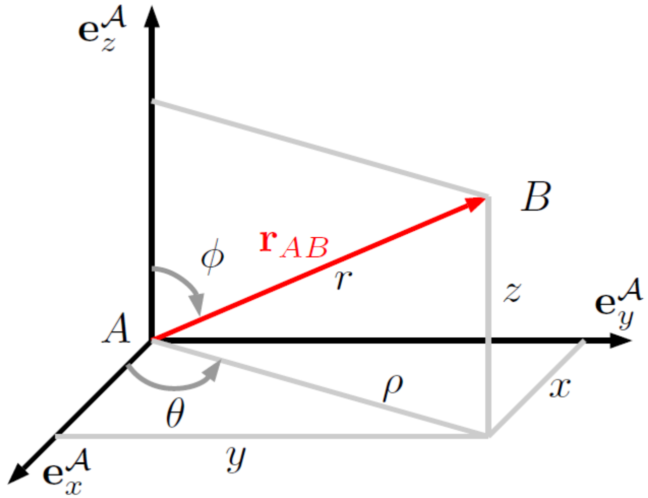
17.09.2019	Intro and Outline	Course Introduction; Recapitulation Position, Linear Velocity			
24.09.2019	Kinematics 1	Rotation and Angular Velocity; Rigid Body Formulation, Transformation	25.09.2019	Exercise 1a	Kinematics Modeling the ABB arm
01.10.2019	Kinematics 2	Kinematics of Systems of Bodies; Jacobians	02.10.2019	Exercise 1a	Differential Kinematics of the ABB arm
08.10.2019	Kinematics 3	Kinematic Control Methods: Inverse Differential Kinematics, Inverse Kinematics; Rotation Error; Multi-task Control	09.10.2019	Exercise 1b	Kinematic Control of the ABB Arm
15.10.2019	Dynamics L1	Multi-body Dynamics	16.10.2019	Midterm 1	Programming kinematics with matlab
22.10.2019	Dynamics L2	Floating Base Dynamics	23.10.2019	Exercise 2a	Dynamic Modeling of the ABB Arm
29.10.2019	Dynamics L3	Dynamic Model Based Control Methods	30.10.2019	Exercise 2b	Dynamic Control Methods Applied to the ABB arm
05.11.2019	Legged Robot	Dynamic Modeling of Legged Robots & Control	06.11.2019	Midterm 2	Programming dynamics with matlab
12.11.2019	Case Studies 1	Legged Robotics Case Study	13.11.2019	Exercise 3	Legged robot
19.11.2019	Rotorcraft	Dynamic Modeling of Rotorcraft & Control	20.11.2019		
26.11.2019	Case Studies 2	Rotor Craft Case Study	27.11.2019	Exercise 4	Modeling and Control of Multicopter
03.12.2019	Fixed-wing	Dynamic Modeling of Fixed-wing & Control	04.12.2019		
10.12.2019	Case Studies 3	Fixed-wing Case Study (Solar-powered UAVs - AtlantikSolar, Vertical Take-off and Landing UAVs – Wingtra)	11.12.2019	Exercise 5	Fixed-wing Control and Simulation
17.12.2019	Summery and Outlook	Summery; Wrap-up; Exam			

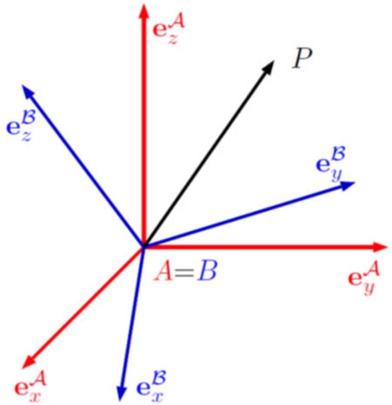
Position

- Position ${}_{\mathcal{A}}\mathbf{r}_{AB} \in \mathbb{R}^3$, reference frames $\mathcal{A}$ ${}_{\mathcal{A}}\mathbf{r}_{AP} = {}_{\mathcal{A}}\mathbf{r}_{AB} + {}_{\mathcal{A}}\mathbf{r}_{BP}$

- Different parameterizations, e.g.
  - Cartesian coordinates $\chi_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
    - Position vector ${}_{\mathcal{A}}\mathbf{r} = x\mathbf{e}_x^{\mathcal{A}} + y\mathbf{e}_y^{\mathcal{A}} + z\mathbf{e}_z^{\mathcal{A}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
  - Cylindrical coordinates $\chi_{Pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}$
    - Position vector ${}_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix}$



Rotation



- Rotation $\phi_{\mathcal{AB}} \in SO(3)$
- Rotation matrix ${}_{\mathcal{A}}\mathbf{r}_{AP} = \left[{}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} \quad {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} \quad {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}}\right] \cdot {}_{\mathcal{B}}\mathbf{r}_{AP} \;= \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP}$
- Different parameterizations, e.g.
  - Euler angles $\chi_{R,eulerZYZ} = \begin{pmatrix} z_1 \\ y \\ z_2 \end{pmatrix}$
  - Quaternions $\chi_{R,quat} = \boldsymbol{\xi} = \begin{pmatrix} \xi_0 \\ \check{\boldsymbol{\xi}} \end{pmatrix} \in \mathbb{H}$ $\quad \xi = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\mathbf{n} \end{pmatrix}$
  - …
- Relation to rotation matrix:

$$\mathbf{C}_{\mathcal{AB}} = \mathbb{I}_{3\times3} + 2\xi_0\left[\check{\boldsymbol{\xi}}\right]_\times + 2\left[\check{\boldsymbol{\xi}}\right]_\times^2 = \left(2\xi_0^2 - 1\right)\mathbb{I}_{3\times3} + 2\xi_0\left[\check{\boldsymbol{\xi}}\right]_\times + 2\check{\boldsymbol{\xi}}\check{\boldsymbol{\xi}}^T$$

$$= \begin{bmatrix} \xi_0^2+\xi_1^2-\xi_2^2-\xi_3^2 & 2\xi_1\xi_2-2\xi_0\xi_3 & 2\xi_0\xi_2+2\xi_1\xi_3 \\ 2\xi_0\xi_3+2\xi_1\xi_2 & \xi_0^2-\xi_1^2+\xi_2^2-\xi_3^2 & 2\xi_2\xi_3-2\xi_0\xi_1 \\ 2\xi_1\xi_3-2\xi_0\xi_2 & 2\xi_0\xi_1+2\xi_2\xi_3 & \xi_0^2-\xi_1^2-\xi_2^2+\xi_3^2 \end{bmatrix}.$$

Introduction to algebra with quaternions, e.g.

$$\boldsymbol{\xi}_{\mathcal{AB}} \otimes \boldsymbol{\xi}_{\mathcal{BC}} \longleftrightarrow \mathbf{C}_{\mathcal{AB}}\mathbf{C}_{\mathcal{BC}}$$

Velocity

- Linear velocity $\dot{\mathbf{r}}_{AB}$
  - Representation $\dot{\boldsymbol{\chi}}_P$

$$\dot{\mathbf{r}} = \mathbf{E}_P(\boldsymbol{\chi}_P)\,\dot{\boldsymbol{\chi}}_P$$
$$\dot{\boldsymbol{\chi}}_P = \mathbf{E}_P^{-1}(\boldsymbol{\chi}_P)\,\dot{\mathbf{r}}$$

e.g. cylindrical coordinates $\mathbf{E}_{P_z}(\boldsymbol{\chi}_{P_z}) = \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Angular velocity $[{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_\times = \dot{\mathbf{C}}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{AB}}^T$

$$[{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \qquad {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

  - Representation ${}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \mathbf{E}_R(\boldsymbol{\chi}_R) \cdot \dot{\boldsymbol{\chi}}_R$
  - e.g. quaternions

$$\mathbf{E}_{R,quat} = 2\mathbf{H}(\boldsymbol{\xi}),$$
$$\mathbf{E}_{R,quat}^{-1} = \frac{1}{2}\mathbf{H}(\boldsymbol{\xi})^T$$

$$\mathbf{H}(\boldsymbol{\xi}) = \begin{bmatrix} -\check{\boldsymbol{\xi}} & [\check{\boldsymbol{\xi}}]_\times + \xi_0 \mathbb{I}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times4}$$
$$= \begin{bmatrix} -\xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ -\xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ -\xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix}.$$

Kinematics of Systems of Bodies

- Generalized coordinates $\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$
- End-effector position and orientation $\mathbf{x}_e = \begin{pmatrix} \mathbf{r}_e \\ \phi_e \end{pmatrix} \in SE(3)$ paramterized by $\boldsymbol{\chi}_e = \begin{pmatrix} \boldsymbol{\chi}_{eP} \\ \boldsymbol{\chi}_{eR} \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_m \end{pmatrix} \in \mathbb{R}^m$
- Forward kinematics $\boldsymbol{\chi}_e = \boldsymbol{\chi}_e(\mathbf{q})$
- Forward differential kinematics
  - Analytic $\delta\boldsymbol{\chi}_e \approx \frac{\partial \boldsymbol{\chi}_e(\mathbf{q})}{\partial \mathbf{q}} \delta\mathbf{q} = \mathbf{J}_{eA}(\mathbf{q})\,\delta\mathbf{q}$ with $\mathbf{J}_{eA} = \frac{\partial \boldsymbol{\chi}_e}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial \chi_1}{\partial q_1} & \cdots & \frac{\partial \chi_1}{\partial q_{n_j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \chi_m}{\partial q_1} & \cdots & \frac{\partial \chi_m}{\partial q_{n_j}} \end{bmatrix}$

$$\boxed{\dot{\boldsymbol{\chi}}_e = \mathbf{J}_{eA}(\mathbf{q})\,\dot{\mathbf{q}} \quad \text{with } \mathbf{J}_{eA}(\mathbf{q}) \in \mathbb{R}^{m_e \times n_j}}$$

Depending on parameterization!!

  - Geometric

$$\boxed{\mathbf{w}_e = \begin{pmatrix} \mathbf{v}_e \\ \boldsymbol{\omega}_e \end{pmatrix} = \mathbf{J}_{e0}(\mathbf{q})\,\dot{\mathbf{q}} \quad \text{with } \mathbf{J}_{e0}(\mathbf{q}) \in \mathbb{R}^{6 \times n_j}}$$

Independent of parameterization

$$\mathbf{w}_e = \mathbf{E}_e(\boldsymbol{\chi}_e)\,\dot{\boldsymbol{\chi}}_e$$

$$\downarrow$$

$$\mathbf{J}_{e0}(\mathbf{q}) = \mathbf{E}_e(\boldsymbol{\chi})\,\mathbf{J}_{eA}(\mathbf{q})$$

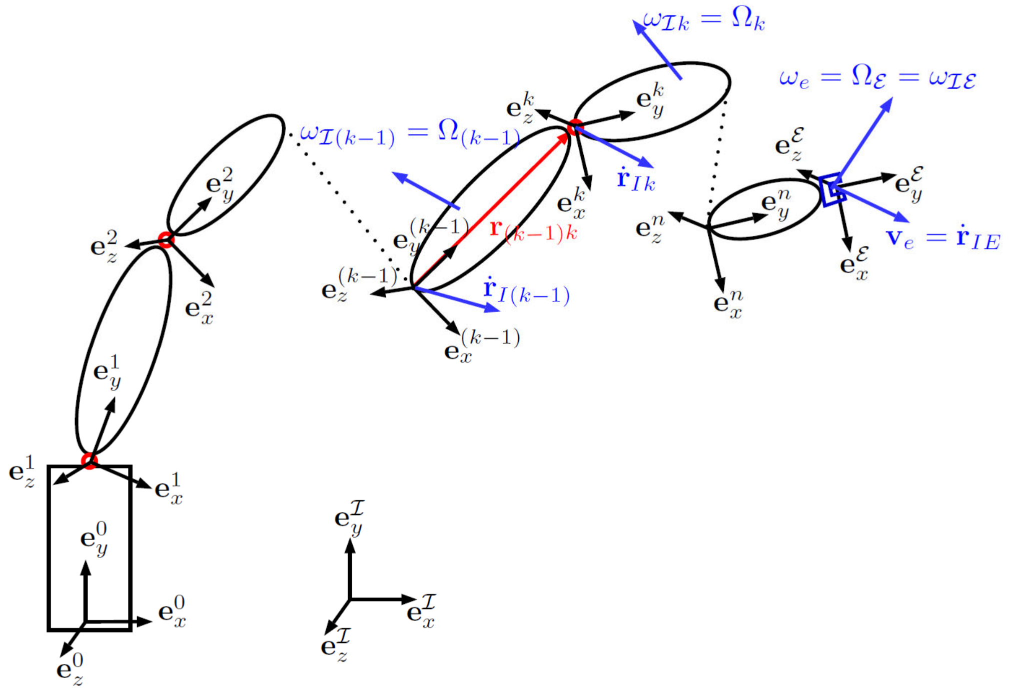
Geometric Jacobian Derivation

- Linear velocity

$$\dot{\mathbf{r}}_{IE} = \underbrace{\begin{bmatrix} \mathbf{n}_1 \times \mathbf{r}_{1(n+1)} & \mathbf{n}_2 \times \mathbf{r}_{2(n+1)} & \dots & \mathbf{n}_n \times \mathbf{r}_{n(n+1)} \end{bmatrix}}_{\mathbf{J}_{e0_P}} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

- Angular velocity

$$\omega_{\mathcal{IE}} = \sum_{i=1}^{n} \mathbf{n}_i \dot{q}_i = \underbrace{\begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \dots & \mathbf{n}_n \end{bmatrix}}_{\mathbf{J}_{e0_R}} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$



Analytical and Kinematic Jacobian

- Analytical Jacobian

$$\dot{\chi}_e = \mathbf{J}_{eA}(\mathbf{q})\,\dot{\mathbf{q}}$$

- Geometric (or basic) Jacobian

$$\mathbf{w}_e = \begin{pmatrix} \mathbf{v}_e \\ \boldsymbol{\omega}_e \end{pmatrix} = \mathbf{J}_{e0}(\mathbf{q})\,\dot{\mathbf{q}}$$

$$\mathbf{J}_{e0}(\mathbf{q}) = \mathbf{E}_e(\chi)\,\mathbf{J}_{eA}(\mathbf{q})$$

- Relates time-derivatives of config. parameters to generalized velocities
- Depending on selected parameterization (mainly rotation) in 3D $\Delta\chi \Leftrightarrow \Delta\mathbf{q}$
  Note: there exist no "rotation angle"
- Mainly used for numeric algorithms

- Relates end-effector velocity to generalized velocities
- Unique for every robot
- Used in most cases

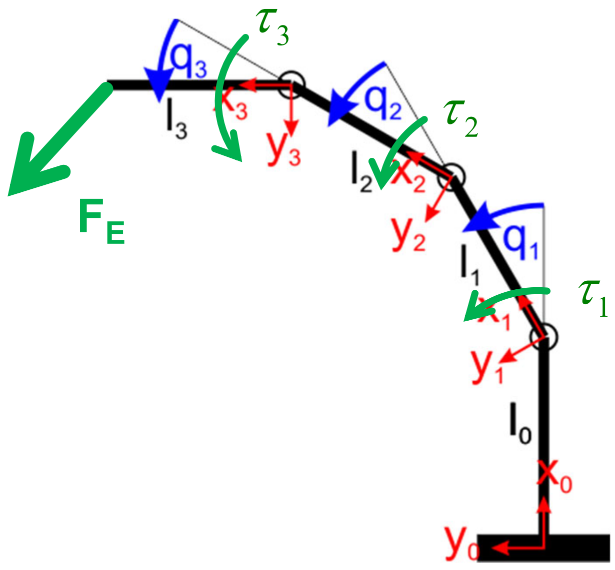
Importance of Jacobian

- Kinematics (mapping of changes from joint to task space)
  - Inverse kinematics control
  - Resolve redundancy problems
  - Express contact constraints
- Statics (and later also dynamics)
  - Principle of virtual work
    - Variations in work must cancel for all virtual displacement
    - Internal forces of ideal joint don't contribute

$$\delta W = \sum_i \mathbf{f}_i \mathbf{x}_i = \boldsymbol{\tau}^T \delta \mathbf{q} + \left(-\mathbf{F}_E\right)^T \delta \mathbf{x}_E$$
$$= \boldsymbol{\tau}^T \delta \mathbf{q} + \left(-\mathbf{F}_E\right)^T \mathbf{J} \delta \mathbf{q} = 0 \quad \forall \delta \mathbf{q}$$

➢ Dual problem from principle of virtual work

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{J}\dot{\mathbf{q}} \\ \boldsymbol{\tau} &= \mathbf{J}^T \mathbf{F} \end{aligned}}$$



Inverse Differential Kinematics

- Differential kinematics $\mathbf{w}_e = \mathbf{J}_{e0}\dot{\mathbf{q}}$
- Inverse differential kinematics $\dot{\mathbf{q}} = \mathbf{J}_{e0}^{+}\mathbf{w}_e^*.$
  - Singularity: minimizing $\left\|\mathbf{w}_e^* - \mathbf{J}_{e0}\dot{\mathbf{q}}\right\|^2$
  - Redundancy: $\dot{\mathbf{q}} = \mathbf{J}_{e0}^{+}\mathbf{w}_e^* + \mathbf{N}\dot{\mathbf{q}}_0$ null-space projection matrix $\mathbf{N} = \mathcal{N}\left(\mathbf{J}_{e0}\right)$
- Multi-task control: $task_i := \{\mathbf{J}_i, \mathbf{w}_i^*\}$
  - Equal priority

$$\dot{\mathbf{q}} = \underbrace{\begin{bmatrix} \mathbf{J}_1 \\ \vdots \\ \mathbf{J}_{n_t} \end{bmatrix}^{+}}_{\bar{\mathbf{J}}} \underbrace{\begin{pmatrix} \mathbf{w}_1^* \\ \vdots \\ \mathbf{w}_{n_t}^* \end{pmatrix}}_{\bar{\mathbf{w}}}$$

Multi-task with prioritization

$$\dot{\mathbf{q}} = \sum_{i=1}^{n_t} \mathbf{N}_i \dot{\mathbf{q}}_i, \quad \text{with} \quad \dot{\mathbf{q}}_i = (\mathbf{J}_i \mathbf{N}_i)^{+} \left(\mathbf{w}_i^* - \mathbf{J} \sum_{k=1}^{i-1} \mathbf{N}_k \dot{\mathbf{q}}_k\right)$$

Inverse Kinematics

- Numerical approach $\Delta\boldsymbol{\chi}_e = \mathbf{J}_{eA}\Delta\mathbf{q}$

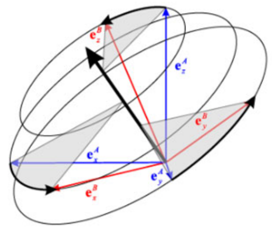
Algorithm 1 Numerical Inverse Kinematics

```
1: q ← q^0                                        ▷ Start configuration
2: while ||χ*_e − χ_e(q)|| > tol do               ▷ While the solution is not reached
3:     J_eA ← J_eA(q) = ∂χ_e/∂q (q)               ▷ Evaluate Jacobian for q
4:     J+_eA ← (J_eA)^+                           ▷ Calculate the pseudo inverse
5:     Δχ_e ← χ*_e − χ_e(q)                       ▷ Find the end-effector configuration error vector
6:     q ← q + J+_eA Δχ_e                         ▷ Update the generalized coordinates
7: end while
```

$$
\begin{array}{ll}
1{:}\ \mathbf{q} \leftarrow \mathbf{q}^0 & \triangleright \text{ Start configuration} \\
2{:}\ \textbf{while } \|\boldsymbol{\chi}_e^* - \boldsymbol{\chi}_e(\mathbf{q})\| > tol \textbf{ do} & \triangleright \text{ While the solution is not reached} \\
3{:}\quad \mathbf{J}_{eA} \leftarrow \mathbf{J}_{eA}(\mathbf{q}) = \frac{\partial \boldsymbol{\chi}_e}{\partial \mathbf{q}}(\mathbf{q}) & \triangleright \text{ Evaluate Jacobian for } \mathbf{q} \\
4{:}\quad \mathbf{J}_{eA}^+ \leftarrow (\mathbf{J}_{eA})^+ & \triangleright \text{ Calculate the pseudo inverse} \\
5{:}\quad \Delta\boldsymbol{\chi}_e \leftarrow \boldsymbol{\chi}_e^* - \boldsymbol{\chi}_e(\mathbf{q}) & \triangleright \text{ Find the end-effector configuration error vector} \\
6{:}\quad \mathbf{q} \leftarrow \mathbf{q} + \mathbf{J}_{eA}^+ \Delta\boldsymbol{\chi}_e & \triangleright \text{ Update the generalized coordinates} \\
7{:}\ \textbf{end while} &
\end{array}
$$

Position/Rotation Errors and Trajectory Control

- Position error $\Delta \mathbf{r}_e^t = \mathbf{r}_e^*(t) - \mathbf{r}_e(\mathbf{q}^t)$
  - Trajectory control with position-error feedback $\dot{\mathbf{q}} = \mathbf{J}_{e0_P}^+ (\dot{\mathbf{r}}^* + k_{PP} \Delta \mathbf{r}_e^t)$



- Rotation error $\Delta\varphi$ is not $\varphi^* - \varphi^t$ ➡ $\mathbf{C}_{\mathcal{GS}}(\Delta\varphi) = \mathbf{C}_{\mathcal{GI}}(\varphi^*)\mathbf{C}_{\mathcal{SI}}^T(\varphi^t)$
  - Trajectory control with rotation-error feedback $\dot{\mathbf{q}} = \mathbf{J}_{e0_R}^+ (\omega(t)_e^* + k_{PR} \Delta\varphi)$

Floating Base Kinematics

- Describe system by base and joint coordinates $$\mathbf{q} = \begin{pmatrix} \mathbf{q}_b \\ \mathbf{q}_j \end{pmatrix}$$
- Base coordinates: rotation and position of base $$\mathbf{q}_b = \begin{pmatrix} \mathbf{q}_{b_P} \\ \mathbf{q}_{b_R} \end{pmatrix} \in \mathbb{R}^3 \times SO(3)$$
- Contact constraints: $${}_{\mathcal{I}}\mathbf{r}_{IC_i} = const, \quad {}_{\mathcal{I}}\dot{\mathbf{r}}_{IC_i} = {}_{\mathcal{I}}\ddot{\mathbf{r}}_{IC_i} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Multi-body Dynamics

- We learned how to get the equation of motion in joint space
  - Newton-Euler
  - Projected Newton-Euler
  - Lagrange II

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) + \mathbf{J}_c^T\mathbf{F}_c = \boldsymbol{\tau}$$

$\ddot{\mathbf{q}}$	Generalized accelerations
$\mathbf{M}(\mathbf{q})$	Mass matrix
$\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})$	Centrifugal and Coriolis forces
$\mathbf{g}(\mathbf{q})$	Gravity forces
$\boldsymbol{\tau}$	Generalized forces
$\mathbf{F}_c$	External forces
$\mathbf{J}_c$	Contact Jacobian

- Started from the principle for virtual work

$$\delta W = \int_{\mathcal{B}} \delta\mathbf{r}^T \cdot (\ddot{\mathbf{r}}\mathrm{d}m - \mathrm{d}\mathbf{F}_{ext}) = 0$$

$\mathrm{d}\mathbf{F}_{ext}$	external forces acting on element *i*
$\ddot{\mathbf{r}}$	acceleration of element *i*
dm	mass of element *i*
$\delta\mathbf{r}$	virtual displacement of element *i*

Impulse and angular momentum

- Use the following definitions

$$\begin{aligned} \mathbf{p}_S &= m\mathbf{v}_S && \text{linear momentum} \\ \mathbf{N}_S &= \mathbf{\Theta}_S\mathbf{\Omega}_S && \text{angular momentum} \\ \dot{\mathbf{p}}_S &= m\mathbf{a}_S && \text{change in linear momentum} \\ \dot{\mathbf{N}}_S &= \mathbf{\Theta}_S\mathbf{\Psi} + \mathbf{\Omega} \times \mathbf{\Theta}_S\mathbf{\Omega} && \text{change in angular momentum} \end{aligned}$$

- Conservation of impulse and angular momentum

Newton

$$0 = \delta W = \begin{pmatrix} \delta\mathbf{r}_s \\ \delta\mathbf{\Phi} \end{pmatrix}^T \left(\begin{pmatrix} \dot{\mathbf{p}}_S \\ \dot{\mathbf{N}}_S \end{pmatrix} - \begin{pmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext} \end{pmatrix} \right) \quad \forall \begin{pmatrix} \delta\mathbf{r}_s \\ \delta\mathbf{\Phi} \end{pmatrix}$$

A free body can move
In all directions

$$\boxed{\begin{aligned} \dot{\mathbf{p}}_S &= \mathbf{F}_{ext} \\ \dot{\mathbf{N}}_S &= \mathbf{T}_{ext} \end{aligned}}$$

Euler

External forces and moments

Change in impulse and angular momentum

Projected Newton Euler

- Consider only directions the system can move (c.f. generalized coordinates)

$$0 = \delta W = \delta \mathbf{q}^T \sum_{i=1}^{n_b} \underbrace{\begin{pmatrix} \mathbf{J}_{S_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \begin{pmatrix} m\mathbf{J}_{S_i} \\ \mathbf{\Theta}_{S_i}\mathbf{J}_{R_i} \end{pmatrix}}_{\mathbf{M}(\mathbf{q})} \ddot{\mathbf{q}} + \begin{pmatrix} \mathbf{J}_{S_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \underbrace{\begin{pmatrix} m\dot{\mathbf{J}}_{S_i}\dot{\mathbf{q}} \\ \mathbf{\Theta}_{S_i}\dot{\mathbf{J}}_R\dot{\mathbf{q}} + \mathbf{J}_{R_i}\dot{\mathbf{q}} \times \mathbf{\Theta}_{S_i}\mathbf{J}_{R_i}\dot{\mathbf{q}} \end{pmatrix}}_{\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})} - \underbrace{\begin{pmatrix} \mathbf{J}_{P_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \begin{pmatrix} \mathbf{F}_{ext,i} \\ \mathbf{T}_{ext,i} \end{pmatrix}}_{\mathbf{g}(\mathbf{q})} \qquad \boxed{\forall \delta \mathbf{q}}$$

- Resulting in

$$\mathbf{M} = \sum_{i=1}^{n_b} \left({}_{\mathcal{A}}\mathbf{J}_{S_i}^T \cdot m \cdot {}_{\mathcal{A}}\mathbf{J}_{S_i} + {}_{\mathcal{B}}\mathbf{J}_{R_i}^T \cdot {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\mathbf{J}_{R_i} \right)$$

$$\mathbf{b} = \sum_{i=1}^{n_b} \left({}_{\mathcal{A}}\mathbf{J}_{S_i}^T \cdot m \cdot {}_{\mathcal{A}}\dot{\mathbf{J}}_{S_i} \cdot \dot{\mathbf{q}} + {}_{\mathcal{B}}\mathbf{J}_{R_i}^T \cdot \left({}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\dot{\mathbf{J}}_{R_i} \cdot \dot{\mathbf{q}} + {}_{\mathcal{B}}\mathbf{\Omega}_{S_i} \times {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\mathbf{\Omega}_{S_i} \right) \right)$$

$$\mathbf{g} = \sum_{i=1}^{n_b} \left(-{}_{\mathcal{A}}\mathbf{J}_{S_i}^T {}_{\mathcal{A}}\mathbf{F}_{g,i} \right)$$

Lagrange II

- Get equation of motion from $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)-\frac{\partial T}{\partial \mathbf{q}}+\frac{\partial U}{\partial \dot{\mathbf{q}}}=\boldsymbol{\tau}$$

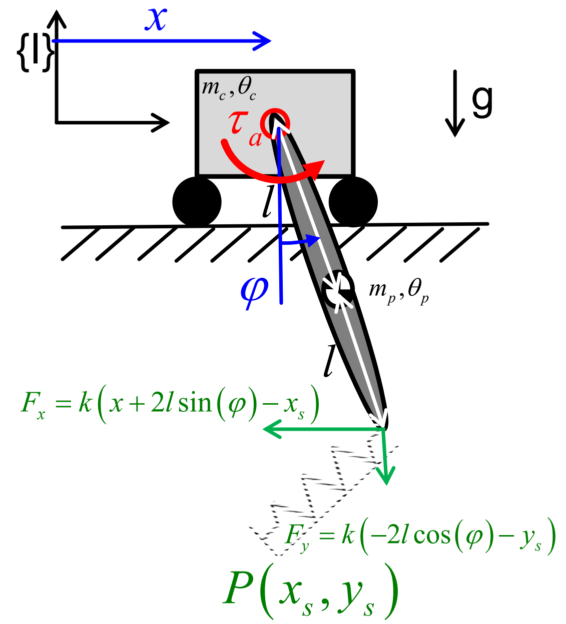
  - Kinetic energy $$\mathcal{T}=\sum_{i=1}^{n_b}\left(\frac{1}{2}m_i\,{}_{\mathcal{A}}\dot{\mathbf{r}}_{S_i}^T\,{}_{\mathcal{A}}\dot{\mathbf{r}}_{S_i}+\frac{1}{2}\,{}_{\mathcal{B}}\mathbf{\Omega}_{S_i}^T\cdot{}_{\mathcal{B}}\mathbf{\Theta}_{S_i}\cdot{}_{\mathcal{B}}\mathbf{\Omega}_{S_i}\right)$$

  - Potential energy $$\mathbf{F}_{g_i}=m_i\,g\,{}_I\mathbf{e}_g$$ $$\mathcal{U}_g=-\sum_{i=1}^{n_b}\mathbf{r}_{S_i}^T\,\mathbf{F}_{g_i}$$ $$\mathcal{U}_{E_j}=\frac{1}{2}k_j\left(d(\mathbf{q})-d_0\right)^2$$

External Forces
Cart pendulum example

- Equation of motion without actuation

$$\underbrace{\begin{bmatrix} m_c + m_p & l m_p \cos(\varphi) \\ l m_p \cos(\varphi) & m_p l^2 + \theta_p \end{bmatrix}}_{\mathbf{M}} \ddot{\mathbf{q}} + \underbrace{\begin{pmatrix} -\dot{\varphi}^2 l m_p \sin(\varphi) \\ 0 \end{pmatrix}}_{\mathbf{b}} + \underbrace{\begin{pmatrix} 0 \\ m_p g l \sin(\varphi) \end{pmatrix}}_{\mathbf{g}} = \mathbf{0}$$



- Add actuator for the pendulum τ_a
  - Action on pendulum $\quad T_p = \tau_a \quad \mathbf{J}_{Rp} = [0 \;\; 1]$
  - Reaction on cart $\quad T_c = -\tau_a \quad \mathbf{J}_{Rc} = [0 \;\; 0]$

$$\boldsymbol{\tau} = \sum \mathbf{J}_{R,i}^T \mathbf{T}_i = \mathbf{J}_{Rc}^T \mathbf{T}_c + \mathbf{J}_{Rp}^T \mathbf{T}_p = \begin{pmatrix} 0 \\ \tau_a \end{pmatrix}$$

- Add spring to the pendulum
  - (world attachment point P, zero length 0, stiffness k)
  - Action on pendulum

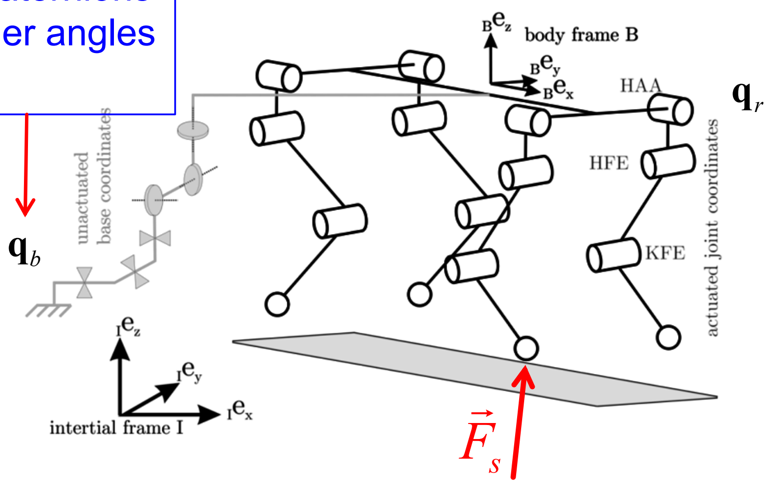
$$\mathbf{r} = \begin{pmatrix} x + 2l\sin(\varphi) \\ -2l\cos(\varphi) \end{pmatrix}$$

$$\mathbf{F}_s = \begin{pmatrix} -F_x \\ -F_y \end{pmatrix} \qquad \mathbf{J}_s = \frac{\partial \mathbf{r}_s}{\partial \mathbf{q}} = \begin{bmatrix} 1 & 2l\cos(\varphi) \\ 0 & 2l\sin(\varphi) \end{bmatrix}$$

$$\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}_s = \begin{pmatrix} -F_x \\ -2l\left(F_x \cos(\varphi) + F_y \sin(\varphi)\right) \end{pmatrix}$$

Dynamics of Floating Base Systems

- Quaternions
- Euler angles
- …



$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_b \\ \mathbf{q}_j \end{pmatrix} \begin{matrix} \text{Un-actuated base} \\ \text{Actuated joints} \end{matrix}$$

- EoM from last time

$$\mathbf{M}\ddot{\mathbf{q}}_j + \mathbf{b} + \mathbf{g} = \boldsymbol{\tau}$$

- Not all joint are actuated

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g} = \mathbf{S}^T \boldsymbol{\tau}$$

  - Selection matrix of actuated joints

$$\mathbf{S} = \begin{bmatrix} \mathbf{0}_{n\times 6} & \mathbf{I}_{n\times n} \end{bmatrix} \qquad \mathbf{q}_j = \mathbf{S}\mathbf{q}$$

- Contact force acting on system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g} = \mathbf{S}^T \boldsymbol{\tau} + \mathbf{J}_s^T \mathbf{F}_{s,\text{ acting on system}}$$

two external force

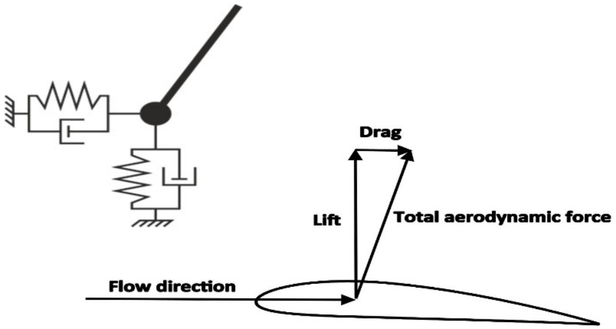
$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_s^T \mathbf{F}_{s,\text{ exerted by robot}} = \mathbf{S}^T \boldsymbol{\tau}$$

Manipulator:	interaction forces at end-effector
Legged robot:	ground contact forces
UAV:	lift force

Note: for simplicity we don't use here ***u*** *but only time derivatives of* ***q***

External Forces
Some notes

- External forces from force elements or actuator
  - E.g. soft contact $\mathbf{F}_s = k_p(\mathbf{r}_{des} - \mathbf{r}) + k_d(-\dot{\mathbf{r}})$
  - Aerodynamics $F_s = \frac{1}{2}\rho c_v A c_L$
- External forces from constraints…



Support Consistent Dynamics

delete the contact force

- Equation of motion $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_s^T \mathbf{F}_s = \mathbf{S}^T \boldsymbol{\tau}$ (1)
  - Cannot directly be used for control due to the occurrence of contact forces
- Contact constraint $\ddot{\mathbf{r}}_s = \mathbf{J}_s \ddot{\mathbf{q}} + \dot{\mathbf{J}}_s \dot{\mathbf{q}} = \mathbf{0}$
- Contact force $\mathbf{F}_s = \left(\mathbf{J}_s \mathbf{M}^{-1} \mathbf{J}_s^T\right)^{-1} \left(\mathbf{J}_s \mathbf{M}^{-1} \left(\mathbf{S}^T \boldsymbol{\tau} - (\mathbf{b} + \mathbf{g})\right) + \dot{\mathbf{J}}_s \dot{\mathbf{q}}\right)$
  - Back-substitute in (1), replace $\dot{\mathbf{J}}_s \dot{\mathbf{q}} = -\mathbf{J}_s \ddot{\mathbf{q}}$ and use support null-space projection

$$\mathbf{N}_s = \mathbf{I} - \mathbf{M}^{-1} \mathbf{J}_s^T \left(\mathbf{J}_s \mathbf{M}^{-1} \mathbf{J}_s^T\right)^{-1} \mathbf{J}_s \qquad \boxed{\mathbf{J}_s \mathbf{N}_s = \mathbf{0}}$$

- Support consistent dynamics $\boxed{\mathbf{N}_s^T \mathbf{M}\ddot{\mathbf{q}} + \mathbf{N}_s^T (\mathbf{b} + \mathbf{g}) = \mathbf{N}_s^T \mathbf{S}^T \boldsymbol{\tau}}$

$$\boldsymbol{\tau}^* = \mathrm{P}_c \left(\mathrm{M}\ddot{\mathrm{q}}^* + \mathrm{b} + \mathrm{g}\right) + \mathrm{N}_{\mathrm{P}_c} \boldsymbol{\tau}_0^*$$

$$\mathrm{N}_c = \mathbb{I} - \mathrm{M}^{-1} {}_I\mathrm{J}_c \left({}_I\mathrm{J}_c \mathrm{M}^{-1} {}_I\mathrm{J}_c^\top\right)^{-1} {}_I\mathrm{J}_c$$

$$\mathrm{P}_c = \left(\mathrm{N}_c \mathrm{S}^\top\right)^+ \mathrm{N}_c$$

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Dynamic Control Methods

- Joint impedance control (w/o and w/ gravity compensation)
- Inverse dynamics control
- Generalized motion and force control

Joint Impedance Control

$$\boxed{\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}}$$

- Torque as function of position and velocity error $\boldsymbol{\tau}^* = \mathbf{k}_p(\mathbf{q}^* - \mathbf{q}) + \mathbf{k}_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}})$

- Closed loop behavior

$$\cancel{\mathbf{M}(\mathbf{q})}\ddot{\mathbf{q}} + \cancel{\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} = \mathbf{k}_p(\mathbf{q}^* - \mathbf{q}) + \mathbf{k}_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}})$$

  - Static offset due to gravity

- Impedance control and gravity compensation

$$\boldsymbol{\tau}^* = \mathbf{k}_p(\mathbf{q}^* - \mathbf{q}) + \mathbf{k}_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}}) + \hat{\mathbf{g}}(\mathbf{q})$$

Estimated gravity term

Simple setup…
but configuration dependent load

Inverse Dynamics Control

$$\boxed{\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}}$$

- Compensate for system dynamics $\boldsymbol{\tau} = \hat{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{q}}^* + \hat{\mathbf{b}}(\mathbf{q},\dot{\mathbf{q}}) + \hat{\mathbf{g}}(\mathbf{q})$
- In case of no modeling errors,
  - the desired dynamics can be perfectly prescribed $\mathbb{I}\ddot{\mathbf{q}} = \ddot{\mathbf{q}}^*$
- PD-control law $\mathbb{I}\ddot{\mathbf{q}} = \ddot{\mathbf{q}}^* = \mathbf{k}_p(\mathbf{q}^* - \mathbf{q}) + \mathbf{k}_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}})$

Can achieve great performance…
but requires accurate modeling

Operational Space Control
Generalized framework to control motion and force

$$\boldsymbol{\tau} = \mathbf{J}_e^T \mathbf{F}_e$$

$$\boldsymbol{\Lambda} = \left(\mathbf{J}_e \mathbf{M}^{-1} \mathbf{J}_e^T\right)^{-1}$$
$$\boldsymbol{\mu} = \boldsymbol{\Lambda} \mathbf{J}_e \mathbf{M}^{-1} \mathbf{b} - \boldsymbol{\Lambda} \dot{\mathbf{J}}_e \dot{\mathbf{q}}$$
$$\mathbf{p} = \boldsymbol{\Lambda} \mathbf{J}_e \mathbf{M}^{-1} \mathbf{g}$$

- Joint-space dynamics

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$$

End-effector dynamics

$$\boldsymbol{\Lambda}\dot{\mathbf{w}}_e + \boldsymbol{\mu} + \mathbf{p} = \mathbf{F}_e$$

  - Determine the corresponding joint torque

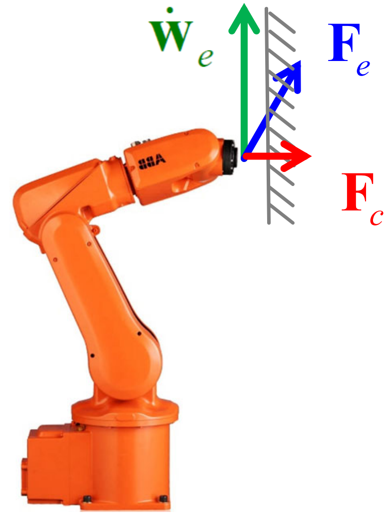
$$\boldsymbol{\tau}^* = \hat{\mathbf{J}}^T \left(\hat{\boldsymbol{\Lambda}}_e \dot{\mathbf{w}}_e^* + \hat{\boldsymbol{\mu}} + \hat{\mathbf{p}}\right)$$

- Extend end-effector dynamics in contact with contact force

$$\mathbf{F}_c + \boldsymbol{\Lambda}\boxed{\dot{\mathbf{w}}_e} + \boldsymbol{\mu} + \mathbf{p} = \boxed{\mathbf{F}_e}$$

- Introduce selection matrices to separate motion force directions

$$\boldsymbol{\tau}^* = \hat{\mathbf{J}}^T \left(\underline{\hat{\boldsymbol{\Lambda}}\mathbf{S}}_M \dot{\mathbf{w}}_e + \mathbf{S}_F \underline{\mathbf{F}_c} + \hat{\tilde{\boldsymbol{\mu}}} + \hat{\mathbf{p}}\right)$$



Inverse Dynamics of Floating Base Systems

- Equation of motion of floating base systems

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{u}} + \mathbf{b}(\mathbf{q},\mathbf{u}) + \mathbf{g}(\mathbf{q}) + \mathbf{J}_c^T \mathbf{F}_c = \mathbf{S}^T \boldsymbol{\tau}$$

- Support-consistent $\mathbf{N}_c^T(\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g}) = \mathbf{N}_c^T \mathbf{S}^T \boldsymbol{\tau}$
- Inverse-dynamics $\boldsymbol{\tau}^* = (\mathbf{N}_c^T \mathbf{S}^T)^+ \mathbf{N}_c^T (\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g})$
- Multiple solutions $\boldsymbol{\tau}^* = (\mathbf{N}_c^T \mathbf{S}^T)^+ \mathbf{N}_c^T (\mathbf{M}\ddot{\mathbf{q}}^* + \mathbf{b} + \mathbf{g}) + \mathcal{N}(\mathbf{N}_c^T \mathbf{S}^T)\boldsymbol{\tau}_0^*$

Operational Space Control as Quadratic Program

A general problem

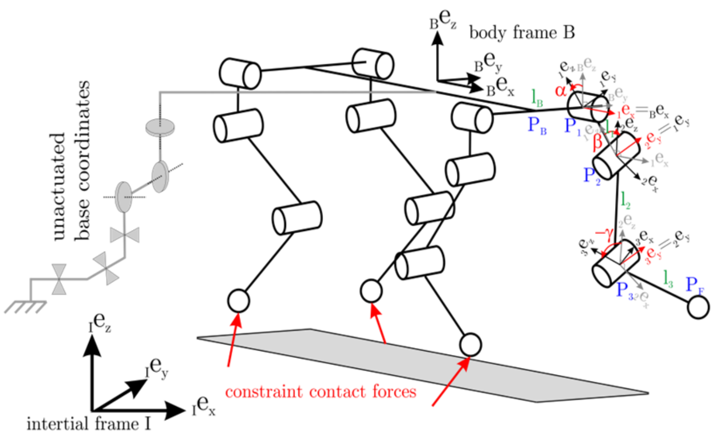
$$\boxed{\min_{\mathbf{x}} \quad \|\mathbf{A}_i\mathbf{x} - \mathbf{b}_i\|_2 \qquad \mathbf{x} = \begin{pmatrix} \dot{\mathbf{u}} \\ \mathbf{F}_c \\ \boldsymbol{\tau} \end{pmatrix}}$$

- We search for a solution that fulfills the equation of motion

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{u}} + \mathbf{b}(\mathbf{q},\mathbf{u}) + \mathbf{g}(\mathbf{q}) + \mathbf{J}_c^T\mathbf{F}_c = \mathbf{S}^T\boldsymbol{\tau} \Rightarrow \mathbf{A} = \begin{bmatrix}\hat{\mathbf{M}} & \hat{\mathbf{J}}_c^T & -\mathbf{S}^T\end{bmatrix} \qquad \mathbf{b} = -\hat{\mathbf{b}} - \hat{\mathbf{g}}$$

- Motion tasks: $\mathbf{J}\dot{\mathbf{u}} + \dot{\mathbf{J}}\mathbf{u} = \dot{\mathbf{w}}^*$ $\Rightarrow \mathbf{A} = \begin{bmatrix}\hat{\mathbf{J}}_i & 0 & 0\end{bmatrix} \qquad \mathbf{b} = \dot{\mathbf{w}}^* - \hat{\dot{\mathbf{J}}}_i\mathbf{u}$
- Force tasks: $\mathbf{F}_i = \mathbf{F}_i^*$ $\Rightarrow \mathbf{A} = \begin{bmatrix}0 & \hat{\mathbf{J}}_i^T & 0\end{bmatrix} \qquad \mathbf{b} = \mathbf{F}_i^*$
- Torque min: $\min\|\boldsymbol{\tau}\|_2$ $\Rightarrow \mathbf{A} = \begin{bmatrix}0 & 0 & \mathbb{I}\end{bmatrix} \qquad \mathbf{b} = 0$

Kinematics of Floating Base / Mobile Systems
example

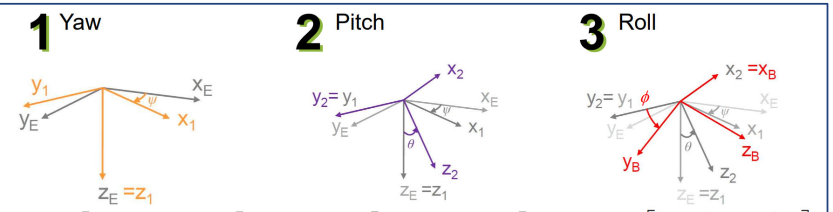


- Quadrupedal robot
- Static walking
- 3 legs in stance [NR 1,2,3]
- 1 in swing [NR 4]

1. How many generalized coordinates?
2. How many base coordinates?
3. How many actuated joint coordinates?
4. How many contact constraints?
5. Write down the contact constraint
6. How many DoFs remain adjustable?
7. Which DoFs remain adjustable?
8. Given a desired swing velocity, what is the generalized velocity?
9. Is it unique?
10. Is it possible to follow the desired swing trajectory without moving the joints of leg 4? How?

Modeling of Rotorcrafts

- Representing Altitude
- Rotational Velocity
- Gimbal lock



$$C_{E1}(z,\psi)=\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}\quad C_{12}(y,\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}\quad C_{2B}(x,\phi)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi & \cos\phi\end{bmatrix}$$

$\phi)-s(\psi)c(\phi)\quad c(\psi)s(\theta)c(\phi)+s(\psi)s(\phi)$
$\phi)+c(\psi)c(\phi)\quad s(\psi)s(\theta)c(\phi)-c(\psi)s(\phi)$
$)s(\phi)\quad c(\theta)c(\phi)$

Roll ($-\pi<\phi<\pi$)
Pitch ($-\pi/2<\theta<\pi/2$)
Yaw ($-\pi<\psi<\pi$)

- Split angular velocity into the three basic rotations
  - $_B\boldsymbol{\omega} = {_B\boldsymbol{\omega}_{roll}} + {_B\boldsymbol{\omega}_{pitch}} + {_B\boldsymbol{\omega}_{yaw}}$
  - Angular velocity due to change in roll angle
    - $_B\boldsymbol{\omega}_{roll} = (\dot{\phi},0,0)^T$

$$_B\omega_{roll}=\begin{bmatrix}\dot{\phi}\\0\\0\end{bmatrix}$$

  - Angular velocity due to change in pitch angle
    - $_B\boldsymbol{\omega}_{pitch} = \mathbf{C}^T_{2B}(x,\phi)\ (0,\ \dot{\theta},0)^T$

$$_B\omega_{pitch}=\mathbf{C}^T_{2B}(\mathbf{x},\phi)\cdot\begin{bmatrix}0\\\dot{\theta}\\0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{bmatrix}\cdot\begin{bmatrix}0\\\dot{\theta}\\0\end{bmatrix}=\begin{bmatrix}0\\\dot{\theta}\cdot\cos\phi\\\dot{\theta}\cdot(-\sin\phi)\end{bmatrix}$$

  - Angular velocity due to change in yaw angle
    - $_B\boldsymbol{\omega}_{yaw} = [\mathbf{C}_{12}(y,\theta)\ \mathbf{C}_{2B}(x,\ \phi)]^T\ (0,0,\dot{\psi})^T$

$$_B\omega_{yaw}=[\mathbf{C}_{12}(y,\theta)\cdot\mathbf{C}_{2B}(x$$

$$_B\omega_{yaw}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{bmatrix}\cdot\begin{bmatrix}\cos\theta\\ 0\\ \sin\theta\end{bmatrix}$$

Autonomous Systems Lab

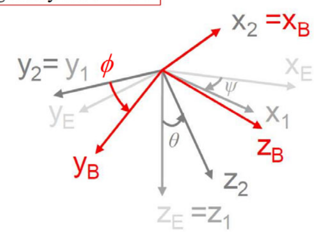
- Relation between Tait-Bryan angles χ and angular velocities

$$E_r\dot{\chi}_r = {_B\boldsymbol{\omega}}\qquad E_r=\begin{bmatrix}1 & 0 & -\sin\theta\\ 0 & \cos\phi & \sin\phi\cos\theta\\ 0 & -\sin\phi & \cos\phi\cos\theta\end{bmatrix};\quad \dot{\chi}_r=\begin{bmatrix}\dot{\phi}\\\dot{\theta}\\\dot{\psi}\end{bmatrix}$$

$\Rightarrow$ Singularity for $\theta=\pm 90^o$

- Linearized relation at hover

$$E_r\big|_{\phi=0,\theta=0}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}\dot{\phi}\\\dot{\theta}\\\dot{\psi}\end{bmatrix}={_B\boldsymbol{\omega}}$$



E*body frame = inertia frame

Modeling of Rotorcrafts
Body dynamics

$$\mathbf{M}(\vec{\varphi})\ddot{\vec{\varphi}} + \vec{b}(\vec{\varphi},\dot{\vec{\varphi}}) + \vec{g}(\vec{\varphi}) + \mathbf{J}_{ex}^T \vec{F}_{ex} = \mathbf{S}^T \vec{\tau}_{act}$$

- Change of momentum and spin in the body frame

$$\begin{bmatrix} mE_{3x3} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} {}_B\dot{\boldsymbol{v}} \\ {}_B\dot{\boldsymbol{\omega}} \end{bmatrix} + \begin{bmatrix} {}_B\boldsymbol{\omega} \times m\ {}_B\boldsymbol{v} \\ {}_B\boldsymbol{\omega} \times \boldsymbol{I}\ {}_B\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} {}_B\boldsymbol{F} \\ {}_B\boldsymbol{M} \end{bmatrix}$$

E_{3x3}: Identity matrix

Torque!!

- Position in the inertial frame and the attitude

$${}_E\dot{\boldsymbol{x}} = \boldsymbol{C}_{EB}\ {}_B\boldsymbol{v} \qquad E_r \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = {}_B\boldsymbol{\omega}$$

- Forces and moments
  - Aerodynamics and gravity

$${}_B\boldsymbol{F} = {}_B\boldsymbol{F}_G + {}_B\boldsymbol{F}_{Aero} \qquad {}_B\boldsymbol{M} = {}_B\boldsymbol{M}_{Aero} \qquad {}_B\boldsymbol{F}_G = \boldsymbol{C}_{EB}^T\ {}_E\begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}$$

- Hover forces
  - Thrust forces in the shaft direction

$${}_B F_{Aero} = \sum_{i=1}^{4} {}_B\begin{bmatrix} 0 \\ 0 \\ -T_i \end{bmatrix} \qquad T_i = b_i \omega_{p,i}^2$$

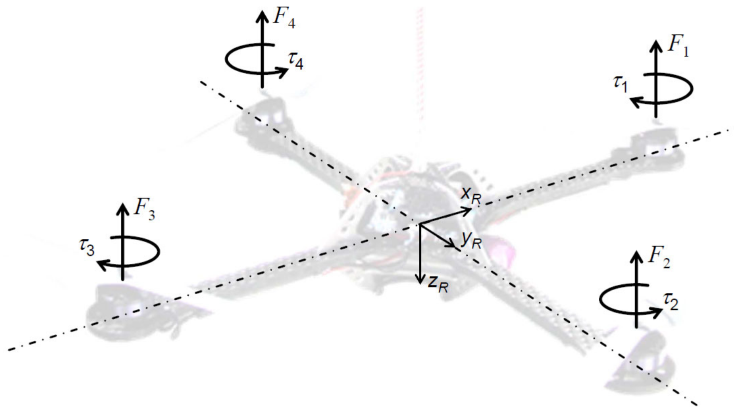
- Hover moments
  - Thrust induced moment
  - Drag torques

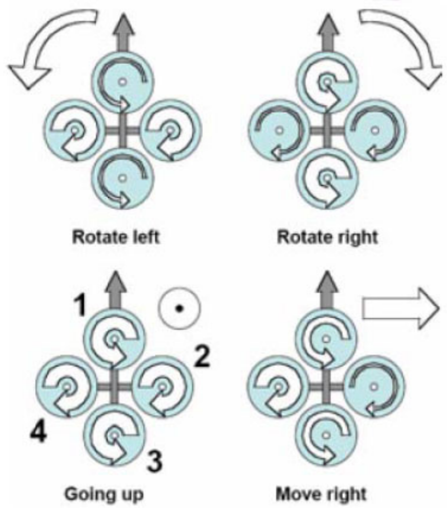
$${}_B M_{Aero} = {}_B M_T + {}_B Q = {}_B\begin{bmatrix} l(T_4 - T_2) \\ l(T_1 - T_3) \\ 0 \end{bmatrix} + {}_B\begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{4} Q_i (-1)^{(i-1)} \end{bmatrix}$$

$$T_i = b_i \omega_{p,i}^2 \qquad Q_i = d_i \omega_{p,i}^2$$

Modeling of a Rotorcraft

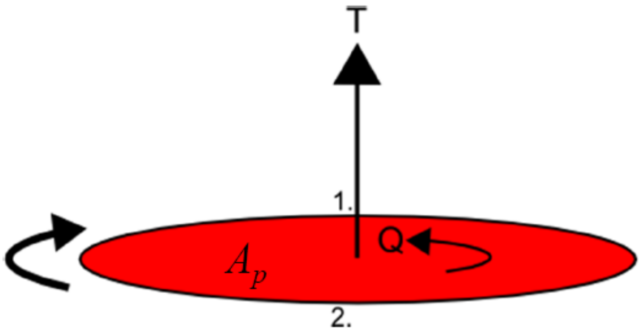
- Singularity issues (Tait-bryan angles)
- Analysis of Dynamics
  - Linearization of dynamics at hover
- Control
  - Virtual control input for decoupled attitude control





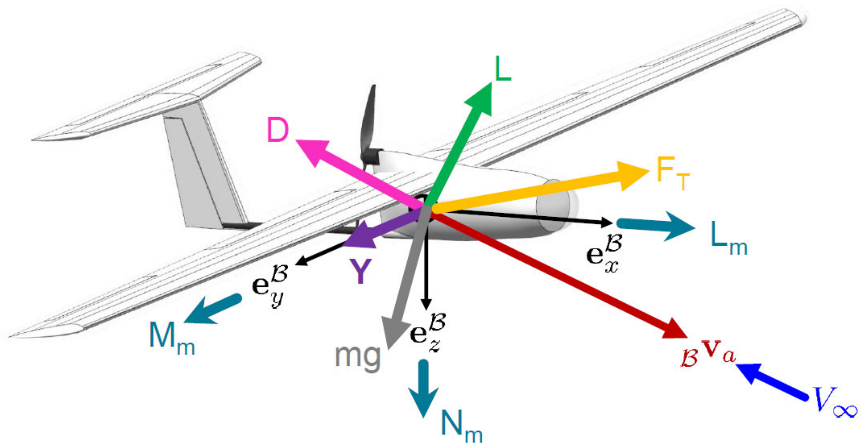
Introduction to propeller aerodynamics

- Thrust force
- Drag torque
- Momentum theory
  - Conservation of fluid mass, fluid momentum and energy
- BEMT (blade element momentum theory)



Fixed wing aerodynamics

- Lift force and drag force
  - Polars of airfoils
- Kinematics and dynamics of an aircraft
  - Forces and moments acting on airplane
    - Lift $L = \frac{1}{2}\rho V^2 S c_L$
    - Drag $D = \frac{1}{2}\rho V^2 S c_D$
    - Thrust
    - Gravity
- Cascaded control of FWs
  - Attitude controller, body rate controller



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Lecture «Robot Dynamics»: Exam

151-0851-00 V

lecture: CAB G11 Tuesday 10:15 – 12:00, every week

exercise: HG E1.2 Wednesday 8:15 – 10:00, according to schedule

Marco Hutter, Roland Siegwart

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5

Quiz 3min

Multiple Choice

A. Multiple Choice 24 pts

Decide whether the following statements are true or false. Cross the checkbox on the corresponding answer. You will be credited 1 point for a correct answer, while 1 pt will be subtracted from the total, if your answer is wrong.

(1) Parameterizing the orientation of a free floating body with three parameters always results in a singularity problem.	◯ True	◯ False	[1 pt]
(2) There exists a parameterization $\chi_{\mathcal{AB}}$ for rotations for which it holds that $\dot{\chi}_{\mathcal{AB}} = {}_{\mathcal{A}}\omega_{\mathcal{AB}}$.	◯ True	◯ False	[1 pt]
(3) The quaternion corresponding to the elementary rotation of 60° around the z axis is $\xi = \frac{1}{2}\begin{pmatrix}\sqrt{3} & 0 & 0 & 1\end{pmatrix}^T$.	◯ True	◯ False	[1 pt]
(4) A six-legged walking robot with three joints per leg has a total of 18 generalized coordinates. 24	◯ True	◯ False	[1 pt]
(5) The definition of the generalized coordinates for a six-legged walking robot is unique.	◯ True	◯ False	[1 pt]
(6) A point contact imposes three constraints on the generalized velocities of a legged robot.	◯ True	◯ False	[1 pt]
(7) The generalized velocity of a free floating base can be uniquely defined with 6 parameters.	◯ True	◯ False	[1 pt]
(8) Given the linar velocity at all point feet of a six-legged walking robot, it is possible to determine the generalized velocity.	◯ True	◯ False	[1 pt]
(9) The geometric Jacobian is independent of the choice of generalized coordinates.	◯ True	◯ False	[1 pt]
(10) The analytical Jacobian has always different dimension than the geometric Jacobian.	◯ True	◯ False	[1 pt]
(11) Gain-scheduled linear controllers can be used as a computationally efficient means of controlling nonlinear effects of aircraft when further away from the nominal operating point.	◯ True	◯ False	[1 pt]
(12) Static aerodynamic reaction torques acting on an aircraft may be measured in a wind tunnel.	◯ True	◯ False	[1 pt]

B. Kinematics of an excavator 13 pts

The excavator depicted in Fig. 1 has a hydraulic arm with a bucket at the end-effector. The arm has a total of four degrees of freedom (DOF), namely three rotational DOFs ($\varphi_1, \varphi_2, \varphi_3$) actuated through parallel linkages as well as a linear DOF (x).

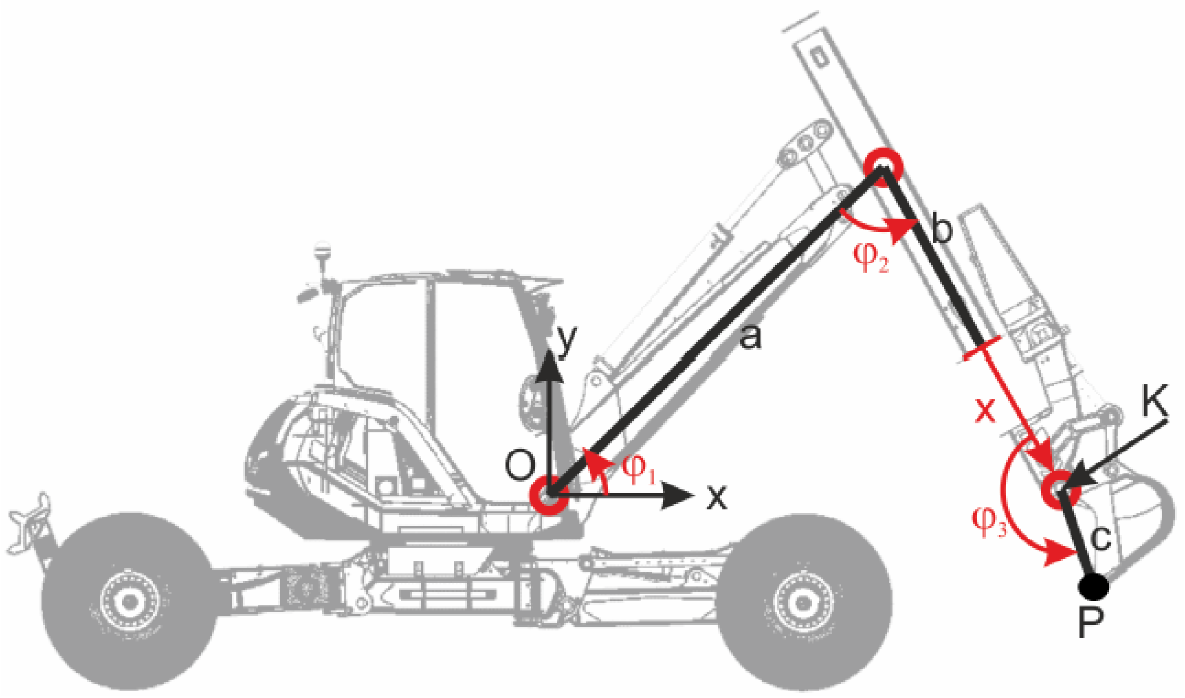
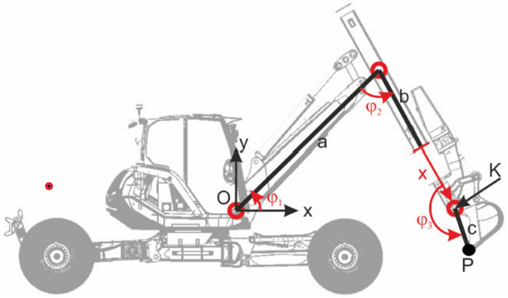


Figure 1: Excavator with four DOF arm and bucket.

5

Quiz
3min

(1) What is the generalized coordinate vector of the excavator arm? [1 pt]



$\mathbf{q} =$

(2) Please express the bucket position $\mathbf{r}_P = \mathbf{r}_{OP} \in \mathbb{R}^2$ (point P) and orientation $\varphi_e \in \mathbb{R}$ w.r.t the cabin frame as a function of the generalized coordinates. [3 pts]

$$\mathbf{r}_P = \begin{pmatrix} \\ \\ \end{pmatrix}$$

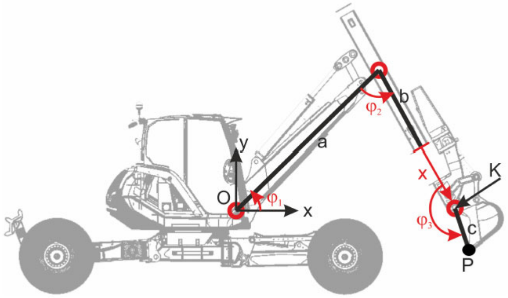
$\varphi_e =$

(3) What is the rotation Jacobian of the end-effector? 1*4 [1 pt]

5

Quiz 3min

(4) Assume that we provide you with the position Jacobian $\mathbf{J}_K = \frac{\partial \mathbf{r}(\mathbf{q})}{\partial \mathbf{q}}$ of point K (mounting point of bucket). What is the position Jacobian $\mathbf{J}_P$ of point P?

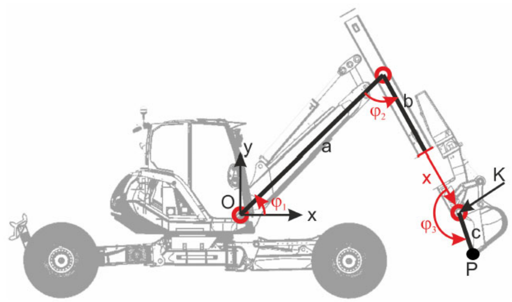


$$\mathbf{J}_P = \mathbf{J}_K + \begin{bmatrix} \quad \end{bmatrix}$$

(5) Assume you want to move the bucket (point P) along a horizontal trajectory with the velocity $\dot{\mathbf{r}}^* = (-v_x, 0)^T$. What least squares minimal generalized velocity produces such motion? (you can use all previously introduced variables)

$\dot{\mathbf{q}} =$

(6) What is the generalized velocity if you want to move horizontally and keep the bucket orientation constant? Please provide a solution where keeping the orientation has higher priority than moving horizontally. (you can use all previously introduced variables)



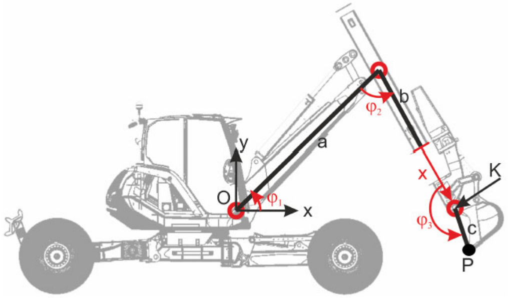
$\dot{q} = ...$

(7) Is the previously determined generalized velocity the unique answer to Question 6?

3dof 3 constraints

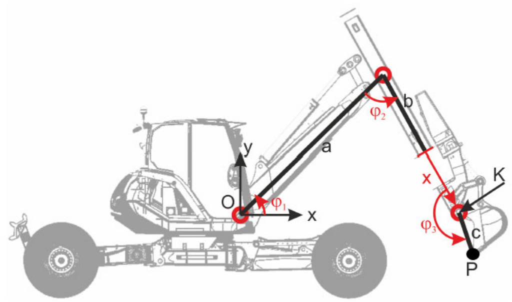
(8) What is the formula to determine the generalized velocity for problem 6 if keeping the orientation has the same priority as moving horizontally? (you can use all previously introduced variables)

$\dot{\mathrm{q}} = \ldots$



C. Statics and Dynamics 7 pts

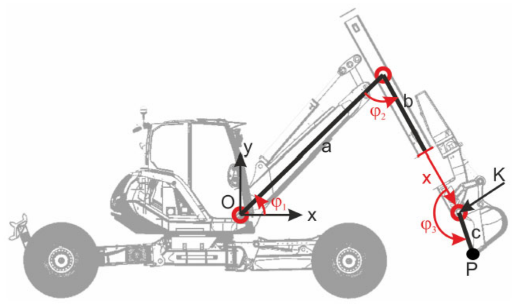
In the following exercise, the excavator bucket is in contact with the ground at point P. In vertical direction, the bucket does not move and produces a contact force F_y. Moving in horizontal direction through earth imposes a drag force F_{drag}. Furthermore, assume that we have already determined the equations of motion $\mathrm{M}\ddot{\mathrm{q}} + \mathrm{b} + \mathrm{g} = \boldsymbol{\tau}$ for the excavator arm in free motion.



(1) Given all previously introduced variables , please express the equations of motion in end-effector coordinates $\chi_E = \begin{pmatrix}\mathrm{r}_P^T & \varphi_e\end{pmatrix}^T \in \mathbb{R}^3$ for the arm in contact with the ground and as a function of a generalized end-effector force $\mathbf{F}_E \in \mathbb{R}^3$. Hint: remember that the generalized end-effector force $\mathbf{F}_E$ is directly related to the joint torque τ.

$\ddot{\chi}_E = \ldots$

(2) You now want to develop a controller that calculates the necessary generalized force $\boldsymbol{\tau}$ that must be applied in order to move the arm with constant velocity $\dot{\mathbf{r}}^* = \begin{pmatrix} -v_x & 0 \end{pmatrix}^T$. Thereby, the bucket orientation should not be changed (i.e. $\dot{\varphi}_e^* = 0$). The contact force is defined as in C.1. *Hint:* Use the equation derived in the previous question and introduce control gains where necessary.



(3) Assume that the excavator has filled the bucket with 150 kg of dirt which must be lifted. What is the necessary joint torque for a given arm configuration $\mathbf{q}_t$ if the excavator moves very slowly $\ddot{\mathrm{q}} \approx \dot{\mathrm{q}} \approx 0$.