ETH zürich

Lecture «Robot Dynamics»: Kinematics 1

151-0851-00 V

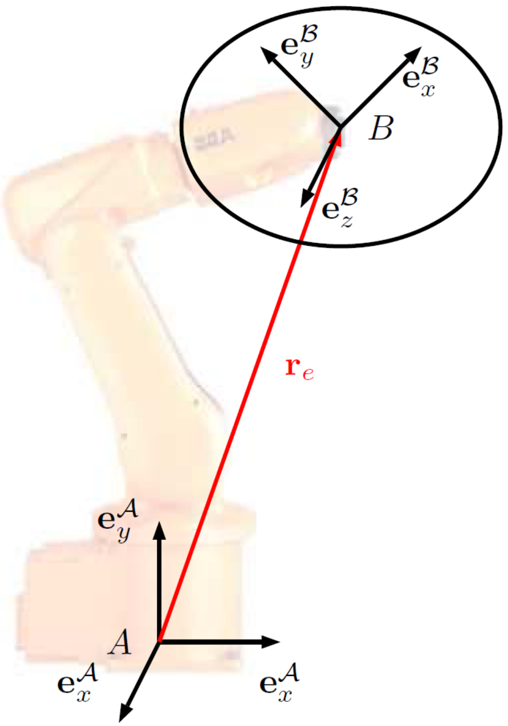
lecture:	HG F3	Tuesday 10:15 – 12:00, every week
exercise:	HG F3	Wednesday 8:15 – 10:00, according to schedule

Marco Hutter, Roland Siegwart, and Thomas Stastny

17.09.2019	Intro and Outline	Course Introduction; Recapitulation Position, Linear Velocity			
24.09.2019	Kinematics 1	Rotation and Angular Velocity; Rigid Body Formulation, Transformation	25.09.2019	Exercise 1a	Kinematics Modeling the ABB arm
01.10.2019	Kinematics 2	Kinematics of Systems of Bodies; Jacobians	02.10.2019	Exercise 1a	Differential Kinematics of the ABB arm
08.10.2019	Kinematics 3	Kinematic Control Methods: Inverse Differential Kinematics, Inverse Kinematics; Rotation Error; Multi-task Control	09.10.2019	Exercise 1b	Kinematic Control of the ABB Arm
15.10.2019	Dynamics L1	Multi-body Dynamics	16.10.2019	Midterm 1	Programming kinematics with matlab
22.10.2019	Dynamics L2	Floating Base Dynamics	23.10.2019	Exercise 2a	Dynamic Modeling of the ABB Arm
29.10.2019	Dynamics L3	Dynamic Model Based Control Methods	30.10.2019	Exercise 2b	Dynamic Control Methods Applied to the ABB arm
05.11.2019	Legged Robot	Dynamic Modeling of Legged Robots & Control	06.11.2019	Midterm 2	Programming dynamics with matlab
12.11.2019	Case Studies 1	Legged Robotics Case Study	13.11.2019	Exercise 3	Legged robot
19.11.2019	Rotorcraft	Dynamic Modeling of Rotorcraft & Control	20.11.2019		
26.11.2019	Case Studies 2	Rotor Craft Case Study	27.11.2019	Exercise 4	Modeling and Control of Multicopter
03.12.2019	Fixed-wing	Dynamic Modeling of Fixed-wing & Control	04.12.2019		
10.12.2019	Case Studies 3	Fixed-wing Case Study (Solar-powered UAVs - AtlantikSolar, Vertical Take-off and Landing UAVs – Wingtra)	11.12.2019	Exercise 5	Fixed-wing Control and Simulation
17.12.2019	Summery and Outlook	Summery; Wrap-up; Exam			

Last Time: Position Parameterization

- Position vector: $\mathbf{r}_e = \mathbf{r}_e(\chi) \in \mathbb{R}^3$
- Parameterization: $\chi_P \in \mathbb{R}^3$
  - Cartesian $\chi_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
  - Cylindrical coordinates $\chi_{Pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}$
  - Spherical coordinates $\chi_{Ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$



- Relation between linear velocity and parameter differentiation $\dot{\mathbf{r}}_e = \frac{\partial \mathbf{r}_e}{\partial \chi_P}\dot{\chi}_P = \mathbf{E}_P \dot{\chi}_P$

  … with the parameterization specific matrix $\mathbf{E}_P(\chi_P) \in \mathbb{R}^{3\times 3}$

Rotation Parameterization

- Rotation matrix:
  - 3x3 = 9 parameters
  - Orthonormality = 6 constraints

$$\mathbf{C}_{\mathcal{AB}} = \begin{bmatrix} {}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}} \end{bmatrix}$$

- Euler Angles
  - 3 parameters, singularity problem

$$\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$$

- Angle Axis
  - 4 parameters, unitary constraint, singularity problem

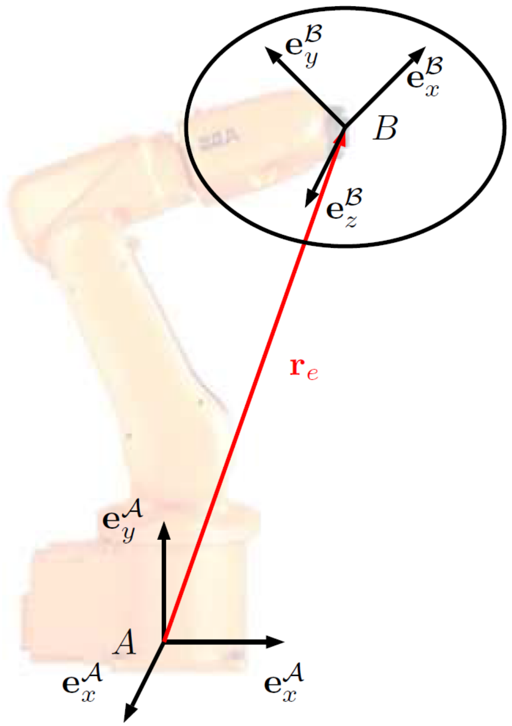
$$\chi_{R,AngleAxis} = \begin{pmatrix} \theta \\ \mathbf{n} \end{pmatrix}$$

- Rotation vector
  - 3 parameters, singularity problem

$$\chi_{R,rotvec} = \boldsymbol{\varphi} = \theta\mathbf{n}$$

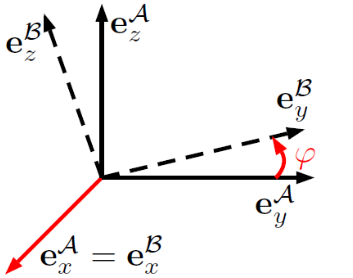
- Quaternions
  - 4 parameters
  - no singularity

$$\chi_{R,quat} = \boldsymbol{\xi} = \begin{pmatrix} \xi_0 \\ \check{\boldsymbol{\xi}} \end{pmatrix}$$

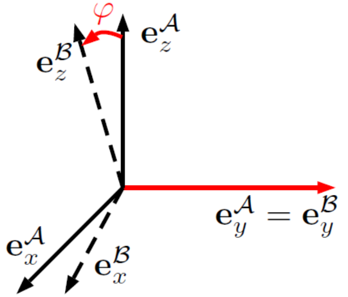


Euler Angles
Consecutive elementary rotations

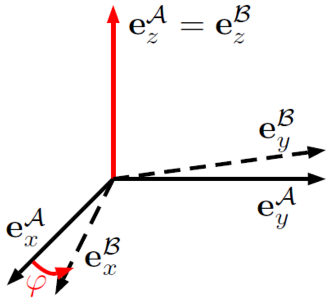
- Last time: Elementary rotation



$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$



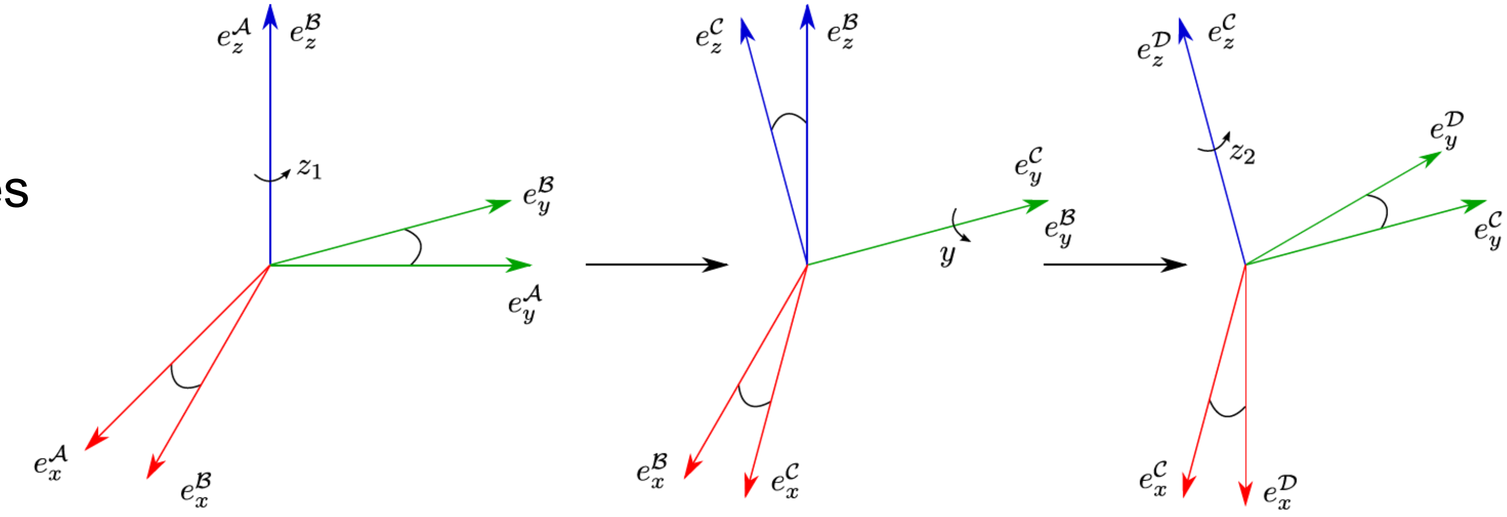
$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_y(\varphi) = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$



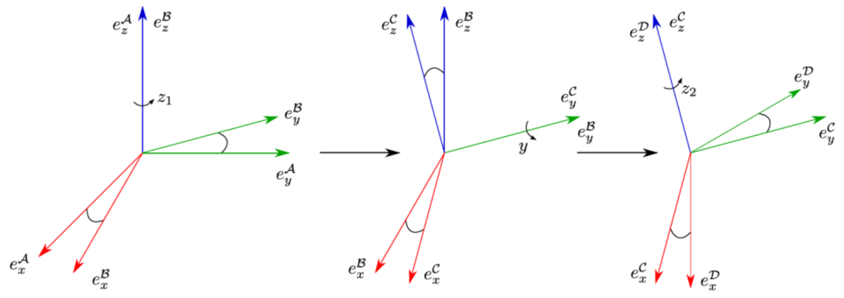
$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles
Consecutive elementary rotations

- Three elementary rotations
  - ZYZ and ZXZ: proper Euler angles
  - ZYX: Tait-Bryan angles
  - XYZ: Cardan angles

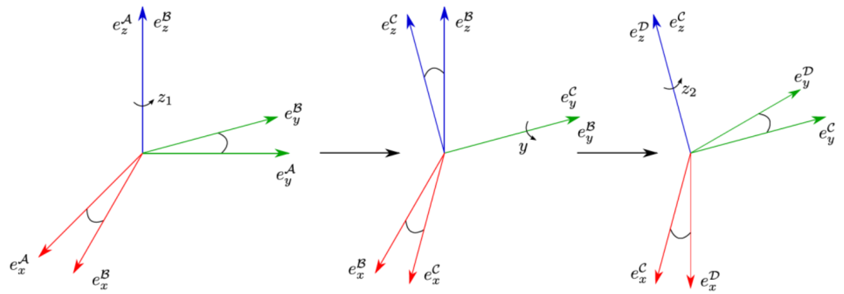


From Euler Angles to Rotation Matrix
ZYZ example



$$\mathbf{C}_{\mathcal{AD}} = \mathbf{C}_{\mathcal{AD}}\left(\chi_{R,eulerZYZ}\right) = \mathbf{C}_{\mathcal{AB}}\left(z_1\right)\mathbf{C}_{\mathcal{BC}}\left(y\right)\mathbf{C}_{\mathcal{CD}}\left(z_2\right)$$

From Euler Angles to Rotation Matrix
ZYZ example



$$\mathbf{C}_{\mathcal{AD}} = \mathbf{C}_{\mathcal{AD}}\left(\chi_{R,eulerZYZ}\right) = \mathbf{C}_{\mathcal{AB}}\left(z_1\right)\mathbf{C}_{\mathcal{BC}}\left(y\right)\mathbf{C}_{\mathcal{CD}}\left(z_2\right)$$

$$\mathbf{C}_{\mathcal{AD}} = \mathbf{C}_{\mathcal{AB}}(z_1)\mathbf{C}_{\mathcal{BC}}(y)\mathbf{C}_{\mathcal{CD}}(z_2) \quad \Rightarrow \quad {}_{\mathcal{A}}\mathbf{r} = \mathbf{C}_{\mathcal{AD}}\,{}_{\mathcal{D}}\mathbf{r}$$

$$= \begin{bmatrix} \cos z_1 & -\sin z_1 & 0 \\ \sin z_1 & \cos z_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix} \begin{bmatrix} \cos z_2 & -\sin z_2 & 0 \\ \sin z_2 & \cos z_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_y c_{z_1} c_{z_2} - s_{z_1} s_{z_2} & -c_{z_2} s_{z_1} - c_y c_{z_1} s_{z_2} & c_{z_1} s_y \\ c_{z_1} s_{z_2} + c_y c_{z_2} s_{z_1} & c_{z_1} c_{z_2} - c_y s_{z_1} s_{z_2} & s_y s_{z_1} \\ -c_{z_2} s_y & s_y s_{z_2} & c_y \end{bmatrix}.$$

From Rotation Matrix to Euler Angles
ZYZ example

- A rotation matrix has the following form

$$\mathbf{C}_{\mathcal{AD}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

- As a function of ZYZ Euler Angles, we found

$$\mathbf{C}_{\mathcal{AD}} = \begin{bmatrix} c_y c_{z_1} c_{z_2} - s_{z_1} s_{z_2} & -c_{z_2} s_{z_1} - c_y c_{z_1} s_{z_2} & c_{z_1} s_y \\ c_{z_1} s_{z_2} + c_y c_{z_2} s_{z_1} & c_{z_1} c_{z_2} - c_y s_{z_1} s_{z_2} & s_y s_{z_1} \\ -c_{z_2} s_y & s_y s_{z_2} & c_y \end{bmatrix}$$

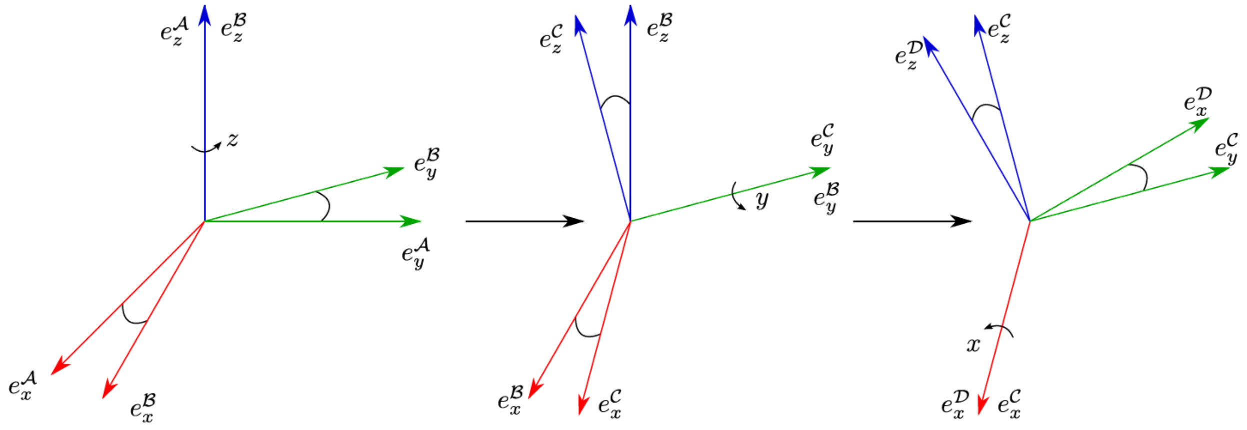
$$\boxed{\begin{aligned} \chi_{R,eulerZYZ} &= \begin{pmatrix} z_1 \\ y \\ z_2 \end{pmatrix} : \\ &= \begin{pmatrix} atan2\left(c_{23}, c_{13}\right) \\ atan2\left(\sqrt{c_{13}^2 + c_{23}^2}, c_{33}\right) \\ atan2\left(c_{32}, -c_{31}\right) \end{pmatrix} \end{aligned}}$$

Atan2 function:
uses sign of both arguments to determine the correct quadrant

Euler Angles ⇔ Rotation Matrix
ZYX example

- Rotation parameters

$$\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$$



- Rotation matrix from Euler Angles

$$\mathbf{C}_{\mathcal{AD}} = \mathbf{C}_{\mathcal{AB}}(z)\mathbf{C}_{\mathcal{BC}}(y)\mathbf{C}_{\mathcal{CD}}(x) \quad \Rightarrow \quad {}_{\mathcal{A}}\mathbf{r} = \mathbf{C}_{\mathcal{AD}\,\mathcal{D}}\mathbf{r}$$

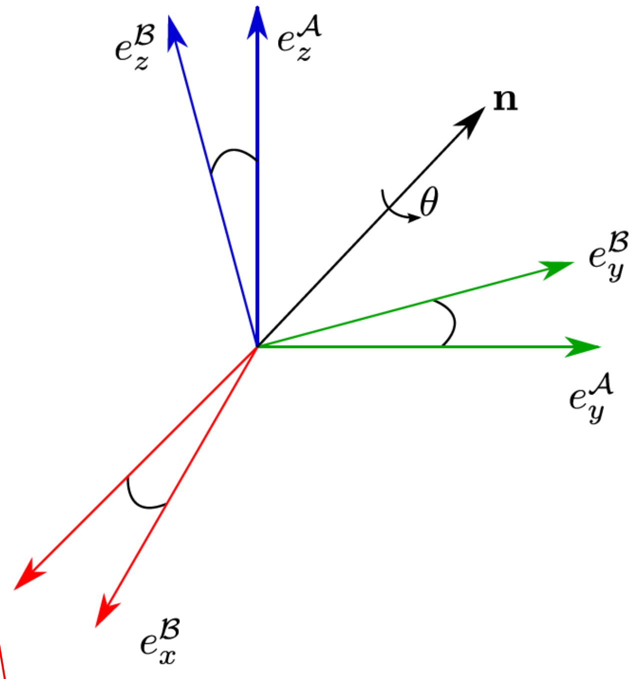
$$= \begin{bmatrix} \cos z & -\sin z & 0 \\ \sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{bmatrix}$$

$$= \begin{bmatrix} c_y c_z & c_z s_x s_y - c_x s_z & s_x s_z + c_x c_z s_y \\ c_y s_z & c_x c_z + s_x s_y s_z & c_x s_y s_z - c_z s_x \\ -s_y & c_y s_x & c_x c_y \end{bmatrix}.$$

- Euler Angles from Rotation matrix

$$\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} atan2\left(c_{21}, c_{11}\right) \\ atan2\left(-c_{31}, \sqrt{c_{32}^2 + c_{33}^2}\right) \\ atan2\left(c_{32}, c_{33}\right) \end{pmatrix}$$

Angle Axis and Rotation Vector



- Angle axis parameterize the rotation by:

$$\chi_{R,AngleAxis} = \begin{pmatrix} \theta \\ \mathbf{n} \end{pmatrix}$$

Rotation angle θ
Rotation axis $\mathbf{n} \in \mathbb{R}^3$

- Rotation vector (aka Euler vectors)

$$\varphi = \theta \cdot \mathbf{n} \in \mathbb{R}^3$$

Idea for the proof:
- Use unit-rotations to align z axis with n
- Rotate with angle θ around **n**
- Use unit-rotations to rotate back

[Robotics – Modelling, Planning and Control (Siciliano), p.53]

- Rotation matrix is given by:

$$\mathbf{C}_{\mathcal{AB}}(\theta,\mathbf{n}) = \cos(\theta)\mathbf{I}_{3\times3} - \sin(\theta)[\mathbf{n}]_\times + (1-\cos(\theta))\mathbf{n}\mathbf{n}^T$$

$$\mathbf{C}_{\mathcal{AB}} = \begin{bmatrix} n_x^2(1-c_\theta)+c_\theta & n_xn_y(1-c_\theta)-n_zs_\theta & n_xn_z(1-c_\theta)+n_ys_\theta \\ n_xn_y(1-c_\theta)+n_zs_\theta & n_y^2(1-c_\theta)+c_\theta & n_yn_z(1-c_\theta)-n_xs_\theta \\ n_xn_z(1-c_\theta)-n_ys_\theta & n_yn_z(1-c_\theta)+n_xs_\theta & n_z^2(1-c_\theta)+c_\theta \end{bmatrix}$$

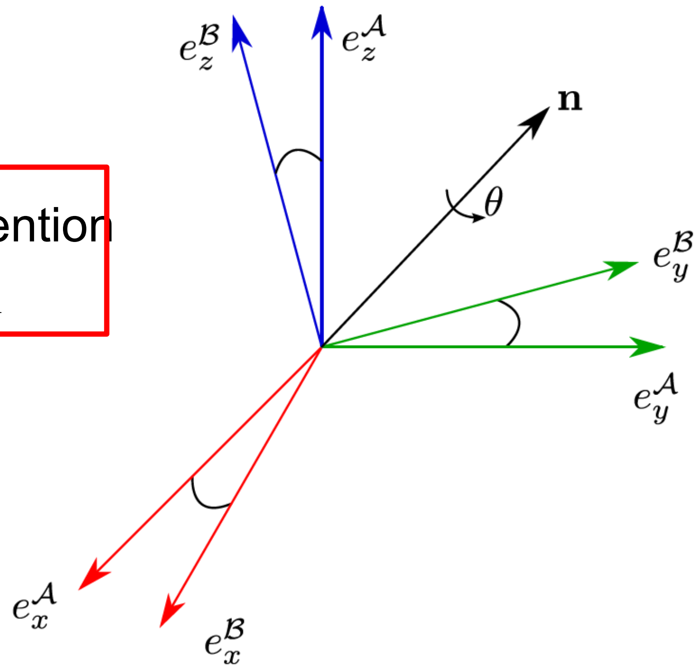
- Parameters from rotation matrix

$$\theta = cos^{-1}\left(\frac{c_{11}+c_{22}+c_{33}-1}{2}\right)$$

$$\mathbf{n} = \frac{1}{2\boxed{sin(\theta)}}\begin{pmatrix} c_{32}-c_{23} \\ c_{13}-c_{31} \\ c_{21}-c_{12} \end{pmatrix}$$

Unit Quaternions
Rotation parameterization w/o singularity problem

Hamiltonian convention
$i^2 = j^2 = k^2 = ijk = -1$



- Complex numbers in 4D $\xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$
- As vector $\chi_{R,quat} = \boldsymbol{\xi} = \begin{pmatrix} \xi_0 \\ \check{\boldsymbol{\xi}} \end{pmatrix} \in \mathbb{H}$
- Real part $\xi_0 = \cos\left(\frac{\|\boldsymbol{\varphi}\|}{2}\right) = \cos\left(\frac{\theta}{2}\right)$
- Imaginary part $\check{\boldsymbol{\xi}} = \sin\left(\frac{\|\boldsymbol{\varphi}\|}{2}\right)\frac{\boldsymbol{\varphi}}{\|\boldsymbol{\varphi}\|} = \sin\left(\frac{\theta}{2}\right)\mathbf{n} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$
- Unitary constraint $\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$
- Inverse $\boldsymbol{\xi} = \begin{pmatrix} \xi \\ \check{\boldsymbol{\xi}} \end{pmatrix} \xrightarrow{inverse} \boldsymbol{\xi}^{-1} = \begin{pmatrix} \xi \\ -\check{\boldsymbol{\xi}} \end{pmatrix}$
- Identity $\xi = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T$

Unit Quaternions ⇔ Rotation matrix

- Rotation matrix from unit quaternion

$$\mathbf{C}_{\mathcal{AD}} = \mathbb{I}_{3\times3} + 2\xi_0 \left[\check{\boldsymbol{\xi}}\right]_\times + 2\left[\check{\boldsymbol{\xi}}\right]_\times^2 = \left(2\xi_0^2 - 1\right)\mathbb{I}_{3\times3} + 2\xi_0\left[\check{\boldsymbol{\xi}}\right]_\times + 2\check{\boldsymbol{\xi}}\check{\boldsymbol{\xi}}^T$$
$$= \begin{bmatrix} \xi_0^2+\xi_1^2-\xi_2^2-\xi_3^2 & 2\xi_1\xi_2-2\xi_0\xi_3 & 2\xi_0\xi_2+2\xi_1\xi_3 \\ 2\xi_0\xi_3+2\xi_1\xi_2 & \xi_0^2-\xi_1^2+\xi_2^2-\xi_3^2 & 2\xi_2\xi_3-2\xi_0\xi_1 \\ 2\xi_1\xi_3-2\xi_0\xi_2 & 2\xi_0\xi_1+2\xi_2\xi_3 & \xi_0^2-\xi_1^2-\xi_2^2+\xi_3^2 \end{bmatrix}.$$

- Unit quaternions from rotation matrix

$$\chi_{R,quat} = \boldsymbol{\xi}_{\mathcal{AD}} = \frac{1}{2}\begin{pmatrix} \sqrt{c_{11}+c_{22}+c_{33}+1} \\ sgn(c_{32}-c_{23})\sqrt{c_{11}-c_{22}-c_{33}+1} \\ sgn(c_{13}-c_{31})\sqrt{c_{22}-c_{33}-c_{11}+1} \\ sgn(c_{21}-c_{12})\sqrt{c_{33}-c_{11}-c_{22}+1} \end{pmatrix}$$

Quiz

3:00

60

Quizz
3min
1min

- Rotation matrix $\mathbf{C}_{\mathcal{AB}}$

- EulerZYX $\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} atan2\left(c_{21}, c_{11}\right) \\ atan2\left(-c_{31}, \sqrt{c_{32}^2 + c_{33}^2}\right) \\ atan2\left(c_{32}, c_{33}\right) \end{pmatrix}$

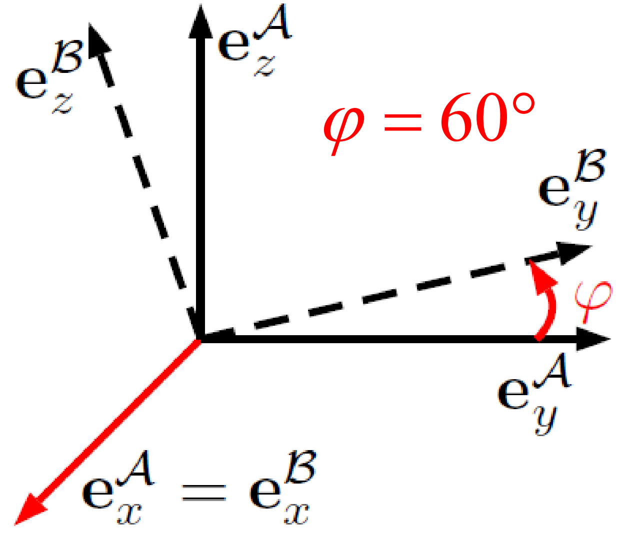
- Angle Axis

$$\theta = cos^{-1}\left(\frac{c_{11} + c_{22} + c_{33} - 1}{2}\right)$$

$$\mathbf{n} = \frac{1}{2sin(\theta)}\begin{pmatrix} c_{32} - c_{23} \\ c_{13} - c_{31} \\ c_{21} - c_{12} \end{pmatrix}$$

- Quaternions

$$\chi_{R,quat} = \frac{1}{2}\begin{pmatrix} \sqrt{c_{11} + c_{22} + c_{33} + 1} \\ sgn(c_{32} - c_{23})\sqrt{c_{11} - c_{22} - c_{33} + 1} \\ sgn(c_{13} - c_{31})\sqrt{c_{22} - c_{33} - c_{11} + 1} \\ sgn(c_{21} - c_{12})\sqrt{c_{33} - c_{11} - c_{22} + 1} \end{pmatrix}$$



Unit Quaternions
Algebra

- Product of quaternions
  - Given two quaternions **q** and **p**, the product is defined as

$$
\begin{aligned}
\boldsymbol{q} \otimes \boldsymbol{p} &= (q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k})(p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}) \\
&= q_0p_0 + q_0p_1\boldsymbol{i} + q_0p_2\boldsymbol{j} + q_0p_3\boldsymbol{k} \\
&\quad + q_1p_0\boldsymbol{i} + q_1p_1\boldsymbol{ii} + q_1p_2\boldsymbol{ij} + q_1p_3\boldsymbol{ik} \\
&\quad + q_2p_0\boldsymbol{j} + q_2p_1\boldsymbol{ji} + q_2p_2\boldsymbol{jj} + q_2p_3\boldsymbol{jk} \\
&\quad + q_3p_0\boldsymbol{k} + q_3p_1\boldsymbol{ki} + q_3p_2\boldsymbol{kj} + q_3p_3\boldsymbol{kk} \\
&= q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\
&\quad + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\boldsymbol{i} \\
&\quad + (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)\boldsymbol{j} \\
&\quad + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\boldsymbol{k} \\
&= \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \underbrace{\begin{bmatrix} q_0 & -\check{\boldsymbol{q}}^T \\ \check{\boldsymbol{q}} & q_0\boldsymbol{I} + [\check{\boldsymbol{q}}]_\times \end{bmatrix}}_{=:\boldsymbol{M}_l(\boldsymbol{q})} \boldsymbol{p} = \boldsymbol{M}_l(\boldsymbol{q})\boldsymbol{p} \\
&= \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{bmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \underbrace{\begin{bmatrix} p_0 & -\check{\boldsymbol{p}}^T \\ \check{\boldsymbol{p}} & p_0\boldsymbol{I} - [\check{\boldsymbol{p}}]_\times \end{bmatrix}}_{=:\boldsymbol{M}_r(\boldsymbol{p})} \boldsymbol{q} = \boldsymbol{M}_r(\boldsymbol{p})\boldsymbol{q}
\end{aligned}
$$

Hamiltonian convention

$$\xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$$

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = -ji = -ijk^2 = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

Unit Quaternions
Rotating a vector

- The pure (imaginary) quaternion of a coordinate vector ${}_I\boldsymbol{r}$ is given by

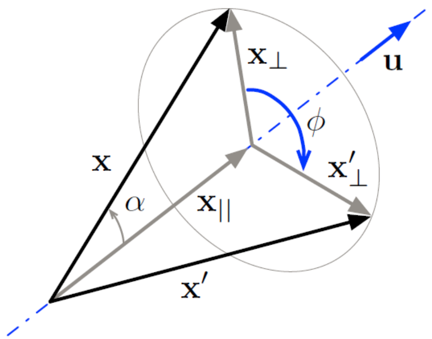
$$\boldsymbol{p}({}_I\boldsymbol{r}) = \begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$$

- Given the unit quaternion $\boldsymbol{\zeta}_{BI}$:

$$\boldsymbol{p}({}_B\boldsymbol{r}) = \boldsymbol{\zeta}_{BI} \otimes \boldsymbol{p}({}_I\boldsymbol{r}) \otimes \boldsymbol{\zeta}_{BI}^T \quad \longleftrightarrow \quad {}_{\mathcal{B}}\mathbf{r} = \mathbf{C}_{\mathcal{BI}} \cdot {}_{\mathcal{I}}\mathbf{r}$$

- Proof (see Quaternion Kinematics by Joan Solà)
  - Decompose vector in parallel and orthogonal part to get vector rotation formula

$$\boxed{\mathbf{x}' = \mathbf{x}_{||} + \mathbf{x}_{\perp}\cos\phi + (\mathbf{u} \times \mathbf{x})\sin\phi}$$



$$\mathbf{x} = \mathbf{x}_{||} + \mathbf{x}_{\perp}$$
$$\mathbf{x}_{||} = \mathbf{u}\,\mathbf{u}^\top\mathbf{x}$$
$$\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{u}\,\mathbf{u}^\top\mathbf{x}$$

  - Show that equation above does exactly the same

Unit Quaternion
Derivation of rotation matrix

- Derivation of rotation matrix ($\boldsymbol{\zeta} = \boldsymbol{\zeta}_{BI}$):
  - $\boldsymbol{p}({}_B\boldsymbol{r}) = \boldsymbol{\zeta} \otimes \boldsymbol{p}({}_I\boldsymbol{r}) \otimes \boldsymbol{\zeta}^T = \boldsymbol{M_l}(\boldsymbol{\zeta})\boldsymbol{M_r}(\boldsymbol{\zeta}^T)\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$
  - $\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} \zeta_0 & -\check{\boldsymbol{\zeta}}^T \\ \check{\boldsymbol{\zeta}} & \zeta_0\boldsymbol{I} + [\check{\boldsymbol{\zeta}}]_\times \end{bmatrix}\begin{bmatrix} \zeta_0 & \check{\boldsymbol{\zeta}}^T \\ -\check{\boldsymbol{\zeta}} & \zeta_0\boldsymbol{I} + [\check{\boldsymbol{\zeta}}]_\times \end{bmatrix}\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$
  - $\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} \zeta_0^2 + |\check{\boldsymbol{\zeta}}|^2 & \zeta_0\check{\boldsymbol{\zeta}}^T - \zeta_0\check{\boldsymbol{\zeta}}^T - \check{\boldsymbol{\zeta}}^T[\check{\boldsymbol{\zeta}}]_\times \\ \zeta_0\check{\boldsymbol{\zeta}} - \zeta_0\check{\boldsymbol{\zeta}} - [\check{\boldsymbol{\zeta}}]_\times\check{\boldsymbol{\zeta}} & \check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T + \zeta_0^2\boldsymbol{I} + 2\zeta_0[\check{\boldsymbol{\zeta}}]_\times + \underline{[\check{\boldsymbol{\zeta}}]_\times[\check{\boldsymbol{\zeta}}]_\times} \end{bmatrix}\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$
  - $\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \left(\zeta_0^2 - |\check{\boldsymbol{\zeta}}|^2\right)\boldsymbol{I} + 2\zeta_0[\check{\boldsymbol{\zeta}}]_\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T \end{bmatrix}\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$
  - ${}_B\boldsymbol{r} = \left(\left(\zeta_0^2 - |\check{\boldsymbol{\zeta}}|^2\right)\boldsymbol{I} + 2\zeta_0[\check{\boldsymbol{\zeta}}]_\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T\right)\ {}_I\boldsymbol{r}$
  - $\boldsymbol{C}(\boldsymbol{\zeta}) = \left(2\zeta_0^2 - 1\right)\boldsymbol{I} + 2\zeta_0[\check{\boldsymbol{\zeta}}]_\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T$

$\boxed{[\check{\zeta}]_\times[\check{\zeta}]_\times = \check{\zeta}\check{\zeta}^T - |\check{\zeta}|^2\mathbf{I}}$

$$\boxed{\begin{aligned} \boldsymbol{M_r}(\boldsymbol{\zeta}) &= \begin{bmatrix} \zeta_0 & -\check{\boldsymbol{\zeta}}^T \\ \check{\boldsymbol{\zeta}} & \zeta_0\boldsymbol{I} - [\check{\boldsymbol{\zeta}}]_\times \end{bmatrix} \\ [\check{\boldsymbol{\zeta}}^T]_\times &= -[\check{\boldsymbol{\zeta}}]_\times \\ \boldsymbol{\zeta}^{-1} &= \boldsymbol{\zeta}^T = \begin{pmatrix} \zeta_0 \\ -\check{\boldsymbol{\zeta}} \end{pmatrix} \end{aligned}}$$

2:00

60

Quizz

2min

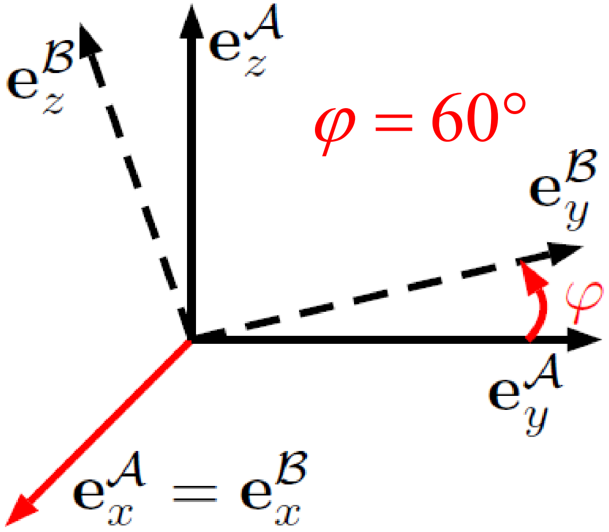
1min

Quiz 2

- Given a vector in $\mathcal{A}$ frame $\quad {}_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

- Rotate this to $\mathcal{B}$ frame using quaternions

$$\mathbf{p}({}_{\mathcal{B}}\mathbf{r}) = \begin{pmatrix} 0 \\ {}_{\mathcal{B}}\mathbf{r} \end{pmatrix} = \xi_{\mathcal{BA}} \otimes \mathbf{p}({}_{\mathcal{A}}\mathbf{r}) \otimes \xi_{\mathcal{BA}}^T = \mathbf{M}_l(\xi_{\mathcal{BA}})\mathbf{M}_r(\xi_{\mathcal{BA}}^T)\begin{pmatrix} 0 \\ {}_{\mathcal{A}}\mathbf{r} \end{pmatrix}$$



2:00

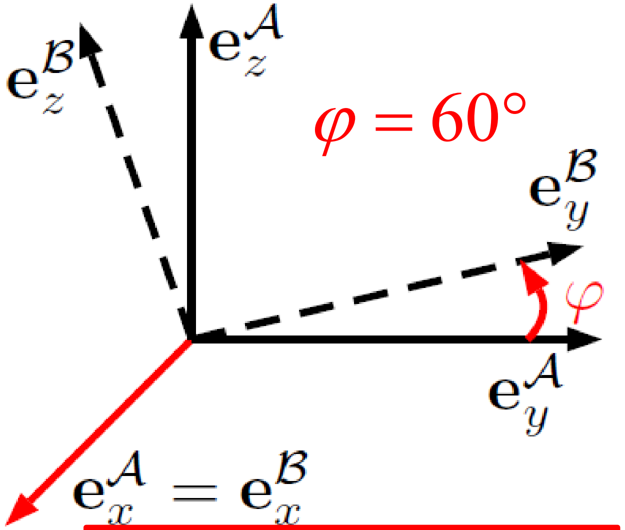
60

Quizz
2min
1min

Quiz 2b

- Given a vector in $\mathcal{A}$ frame $\quad {}_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

- Rotate this to $\mathcal{B}$ frame using directly the complex numbers
  - $\boldsymbol{q} \otimes \boldsymbol{p} = (q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k})(p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k})$



Hamiltonian convention

$$\xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$$

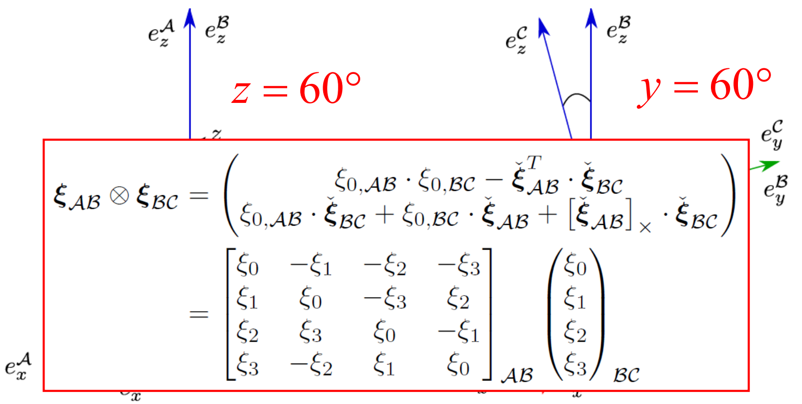
$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = -ji = -ijk^2 = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

Quiz 3



$$\xi_{\mathcal{AB}} \otimes \xi_{\mathcal{BC}} = \begin{pmatrix} \xi_{0,\mathcal{AB}} \cdot \xi_{0,\mathcal{BC}} - \check{\xi}_{\mathcal{AB}}^T \cdot \check{\xi}_{\mathcal{BC}} \\ \xi_{0,\mathcal{AB}} \cdot \check{\xi}_{\mathcal{BC}} + \xi_{0,\mathcal{BC}} \cdot \check{\xi}_{\mathcal{AB}} + \left[\check{\xi}_{\mathcal{AB}}\right]_\times \cdot \check{\xi}_{\mathcal{BC}} \end{pmatrix} = \begin{bmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix}_{\mathcal{AB}} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}_{\mathcal{BC}}$$

- Rotation matrix $\mathbf{C}_{\mathcal{AC}} = ?$ $\quad \mathbf{C}_{\mathcal{AB}} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \mathbf{C}_{\mathcal{BC}} = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix}$

- Quaternion $\xi_{\mathcal{AB}} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad \xi_{\mathcal{BC}} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$\xi_{\mathcal{AC}} = \xi_{\mathcal{AB}} \otimes \xi_{\mathcal{BC}} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \\ 1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\frac{1}{2}\begin{bmatrix} \sqrt{3} & 0 & 0 & -1 \\ 0 & \sqrt{3} & -1 & 0 \\ 0 & 1 & \sqrt{3} & 0 \\ 1 & 0 & 0 & \sqrt{3} \end{bmatrix}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\frac{1}{2}\begin{pmatrix} 3 \\ -1 \\ \sqrt{3} \\ \sqrt{3} \end{pmatrix}$$

$$\begin{aligned} \mathbf{C} &= \mathbb{I}_{3\times 3} + 2\xi_0 \left[\check{\xi}\right]_\times + 2\left[\check{\xi}\right]_\times^2 = \left(2\xi_0^2 - 1\right)\mathbb{I}_{3\times 3} + 2\xi_0\left[\check{\xi}\right]_\times + 2\check{\xi}\check{\xi}^T \\ &= \begin{bmatrix} \xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 & 2\xi_1\xi_2 - 2\xi_0\xi_3 & 2\xi_0\xi_2 + 2\xi_1\xi_3 \\ 2\xi_0\xi_3 + 2\xi_1\xi_2 & \xi_0^2 - \xi_1^2 + \xi_2^2 - \xi_3^2 & 2\xi_2\xi_3 - 2\xi_0\xi_1 \\ 2\xi_1\xi_3 - 2\xi_0\xi_2 & 2\xi_0\xi_1 + 2\xi_2\xi_3 & \xi_0^2 - \xi_1^2 - \xi_2^2 + \xi_3^2 \end{bmatrix}. \end{aligned}$$

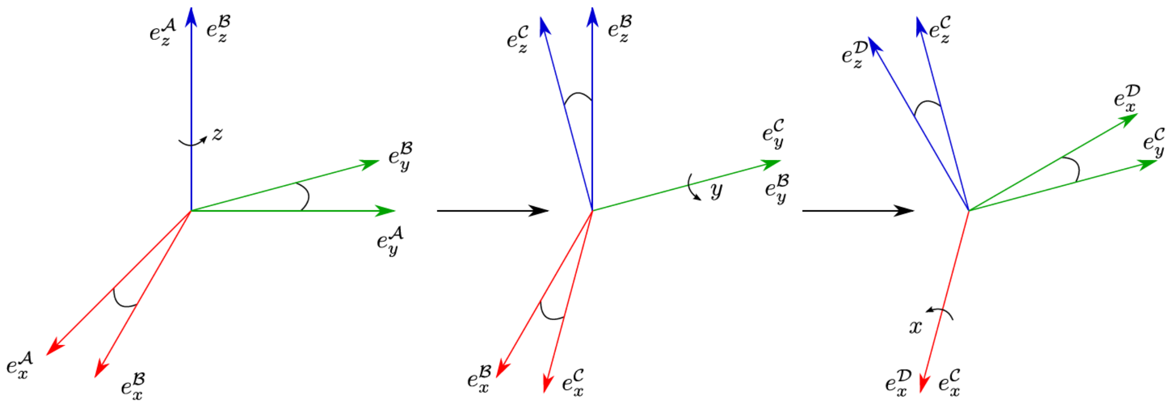
$\Rightarrow$

$$\mathbf{C}_{\mathcal{AC}} = \left(\frac{1}{2}\frac{1}{2}\right)^2 \begin{bmatrix} 9+1-3-3 & -2\sqrt{3}-6\sqrt{3} & 6\sqrt{3}-2\sqrt{3} \\ 6\sqrt{3}-2\sqrt{3} & 9-1+3-3 & 2\sqrt{3}\sqrt{3}+6 \\ -2\sqrt{3}-6\sqrt{3} & -6+2\sqrt{3}\sqrt{3} & 9-1-3+3 \end{bmatrix}$$

Time Derivatives and Rotational Velocity

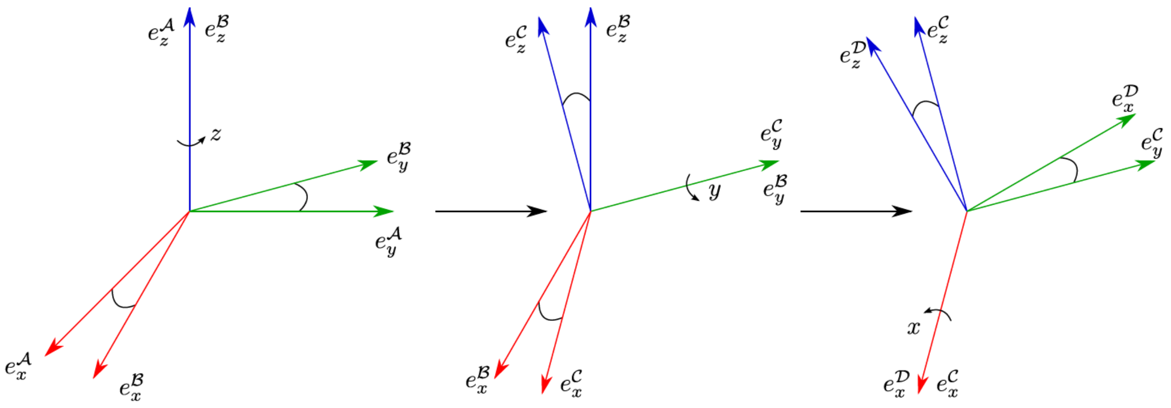
- What is the relation $\boldsymbol{\omega}_{\mathcal{AD}} \Leftrightarrow \dot{\boldsymbol{\chi}}_{\mathcal{AD}}$
- Analog to linear velocity: Find $\mathbf{E}_R(\boldsymbol{\chi}_R)$, s.t. $_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \mathbf{E}_R(\boldsymbol{\chi}_R) \cdot \dot{\boldsymbol{\chi}}_R$

Time Derivatives and Rotational Velocity
ZYX example



Time Derivatives and Rotational Velocity

ZYX example



$$
\begin{aligned}
{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AD}} &= {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} + {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{BC}} + {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{CD}} \\
&= {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} + \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{BC}} + \mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\boldsymbol{\omega}_{\mathcal{CD}} \\
&= {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{A}} \cdot \dot{z} + \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{e}_y^{\mathcal{B}} \cdot \dot{y} + \mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\mathbf{e}_x^{\mathcal{C}} \cdot \dot{x} \\
&= \begin{bmatrix} {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{A}} & \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{e}_y^{\mathcal{B}} & \mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\mathbf{e}_x^{\mathcal{C}} \end{bmatrix} \begin{pmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{pmatrix}
\end{aligned}
$$

$$
\mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{e}_y^{\mathcal{B}} = \begin{bmatrix} c_z & -s_z & 0 \\ s_z & c_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -s_z \\ c_z \\ 0 \end{pmatrix}
$$

$$
\mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\mathbf{e}_x^{\mathcal{C}} = \begin{bmatrix} c_z & -s_z & 0 \\ s_z & c_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_y & 0 & s_y \\ 0 & 1 & 0 \\ -s_y & 0 & c_y \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_y c_z \\ c_y s_z \\ -s_y \end{pmatrix}
$$

$$
\boxed{\boldsymbol{\omega} = \begin{bmatrix} 0 & -s_z & c_y c_z \\ 0 & c_z & c_y s_z \\ 1 & 0 & -s_y \end{bmatrix} \dot{\boldsymbol{\chi}}}
\quad
\boxed{\det\left(\mathbf{E}_{R,eulerZYX}\right) = -\cos(y)}
$$

$$
\boxed{\mathbf{E}_{R,eulerZYX}^{-1} = \begin{bmatrix} \frac{\cos(z)\sin(y)}{\cos(y)} & \frac{\sin(y)\sin(z)}{\cos(y)} & 1 \\ -\sin(z) & \cos(z) & 0 \\ \frac{\cos(z)}{\cos(y)} & \frac{\sin(z)}{\cos(y)} & 0 \end{bmatrix}}
$$

Derivative Angle Axis, Rotation Vector, Quaternions ⇔ Angular Velocity

$$\boxed{{}_{\mathcal{A}}\omega_{\mathcal{AB}} = \mathbf{E}_R(\boldsymbol{\chi}_R) \cdot \dot{\boldsymbol{\chi}}_R}$$

- Angle Axis

$$\mathbf{E}_{R,angleaxis} = \begin{bmatrix} \mathbf{n} & \sin\theta \mathbb{I}_{3\times 3} + (1-\cos\theta)[\mathbf{n}]_\times \end{bmatrix}$$

$$\mathbf{E}^{-1}_{R,angleaxis} = \begin{bmatrix} \mathbf{n}^T \\ -\frac{1}{2}\frac{\sin\theta}{1-cos\theta}[\mathbf{n}]^2_\times - \frac{1}{2}[\mathbf{n}]_\times \end{bmatrix}$$

- Rotation Vector

$$\mathbf{E}_{R,rotationvector} = \left[\mathbb{I}_{3\times 3} + [\boldsymbol{\varphi}]_\times \left(\frac{1-\cos\|\boldsymbol{\varphi}\|}{\|\boldsymbol{\varphi}\|^2}\right) + [\boldsymbol{\varphi}]^2_\times \left(\frac{\|\boldsymbol{\varphi}\| - \sin\|\boldsymbol{\varphi}\|}{\|\boldsymbol{\varphi}\|^3}\right)\right]$$

$$\mathbf{E}^{-1}_{R,rotationvector} = \left[\mathbb{I}_{3\times 3} - \frac{1}{2}[\boldsymbol{\varphi}]_\times + [\boldsymbol{\varphi}]^2_\times \frac{1}{\|\boldsymbol{\varphi}\|^2}\left(1 - \frac{\|\boldsymbol{\varphi}\|}{2}\frac{\sin\|\boldsymbol{\varphi}\|}{1-\cos\|\boldsymbol{\varphi}\|}\right)\right]$$

- Quaternion

$$\mathbf{E}_{R,quat} = 2\mathbf{H}(\boldsymbol{\xi}),$$

$$\mathbf{E}^{-1}_{R,quat} = \frac{1}{2}\mathbf{H}(\boldsymbol{\xi})^T$$

with

$$\mathbf{H}(\boldsymbol{\xi}) = \begin{bmatrix} -\check{\boldsymbol{\xi}} & [\check{\boldsymbol{\xi}}]_\times + \xi_0 \mathbb{I}_{3\times 3} \end{bmatrix} \in \mathbb{R}^{3\times 4}$$

$$= \begin{bmatrix} -\xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ -\xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ -\xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix}.$$

Position and Orientation of a Single Body

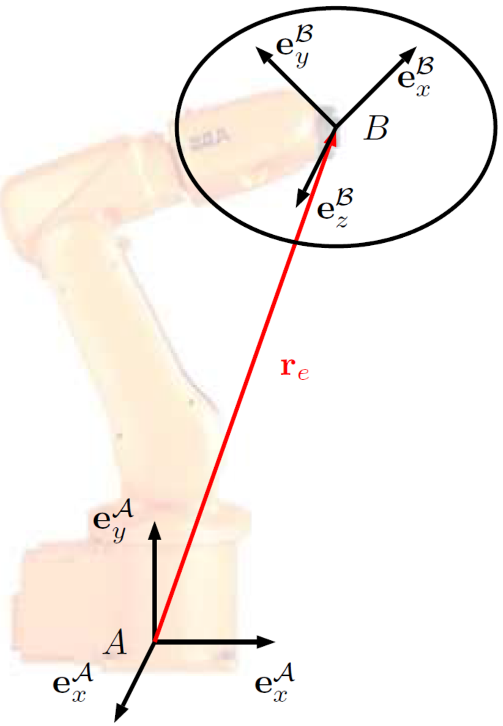
- Position vector: $\mathbf{r}_e = \mathbf{r}_e(\chi) \in \mathbb{R}^3$
  - Cartesian $\chi_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
  - Cylindrical coordinates $\chi_{Pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}$
  - Spherical coordinates $\chi_{Ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$
- Rotation $\phi_e = \phi_e(\chi_R) \in SO(3)$

$$\boxed{\mathbf{x}_e = \begin{pmatrix} \mathbf{r}_e \\ \boldsymbol{\phi}_e \end{pmatrix} \in SE(3)}$$

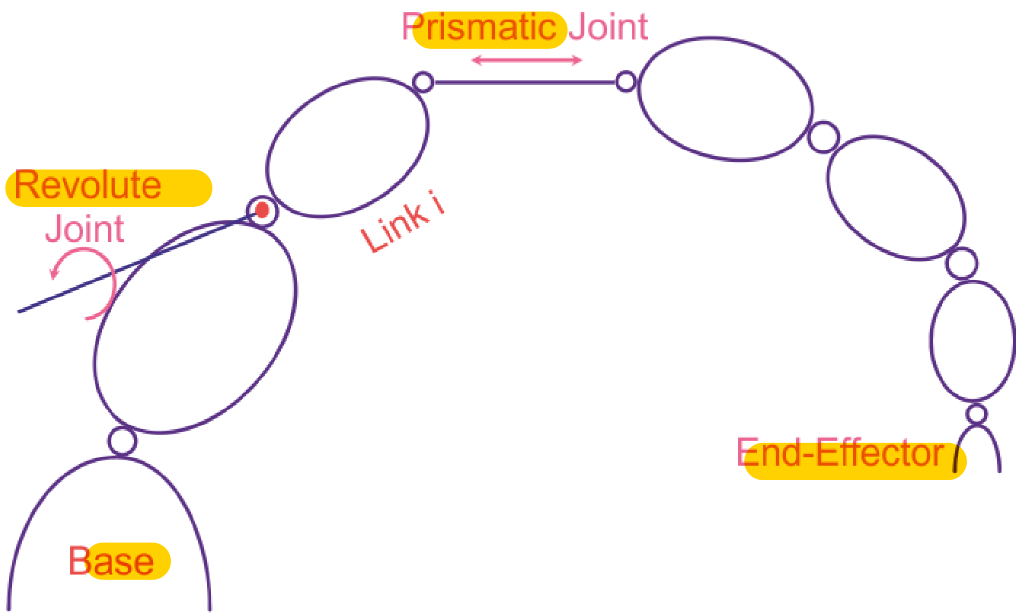
Rotation matrix: $\mathbf{C}_{\mathcal{AB}} = \begin{bmatrix} {}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}} \end{bmatrix}$

Euler Angles: $\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$

Quaternions: $\chi_{R,quat} = \boldsymbol{\xi} = \begin{pmatrix} \xi_0 \\ \check{\boldsymbol{\xi}} \end{pmatrix}$

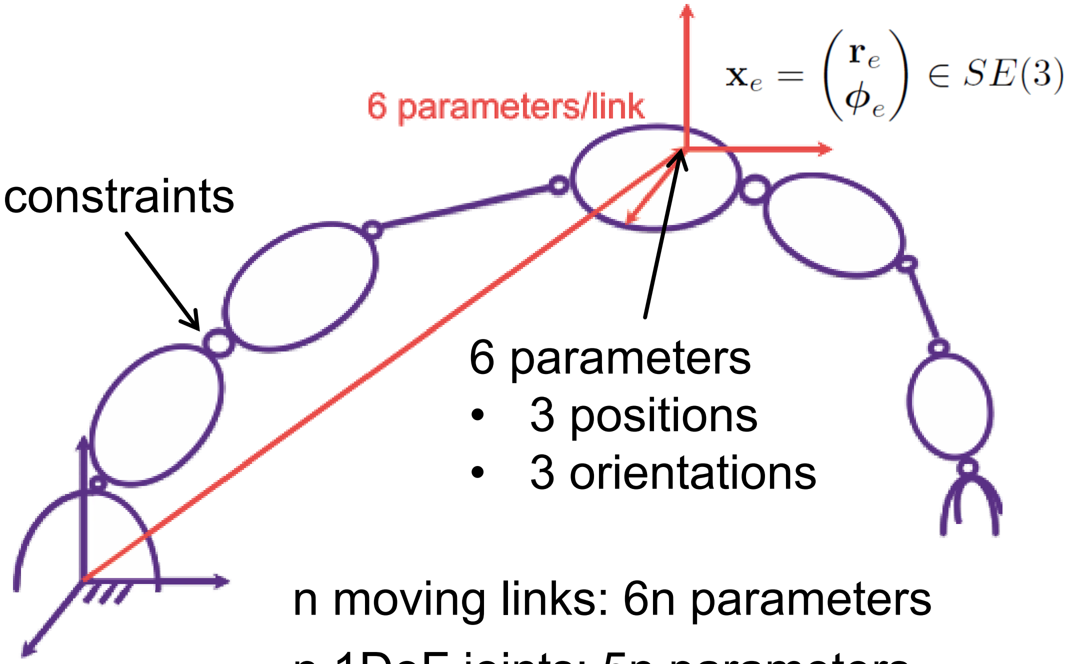


Classical Serial Kinematic Linkages
Generalized robot arm



- n_j joints
  - revolute (1DOF)
  - prismatic (1DOF)
- $n_l = n_j + 1$ links
  - n_j moving links
  - 1 fixed link

Configuration Parameters
Generalized coordinates



n moving links: 6n parameters

n 1DoF joints: 5n parameters

6n – 5n = n DoFs

Generalized coordinates

A set of scalar parameters **q** that describe the robot's configuration

- Must be **complete**
- (Must be **independent**)
  => minimal coordinates
- Is **not unique**

$$\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_{n_j} \end{pmatrix} \in \mathbb{R}^{n_j}$$

number of Gene.
Coor. is unique

Degrees of Freedom

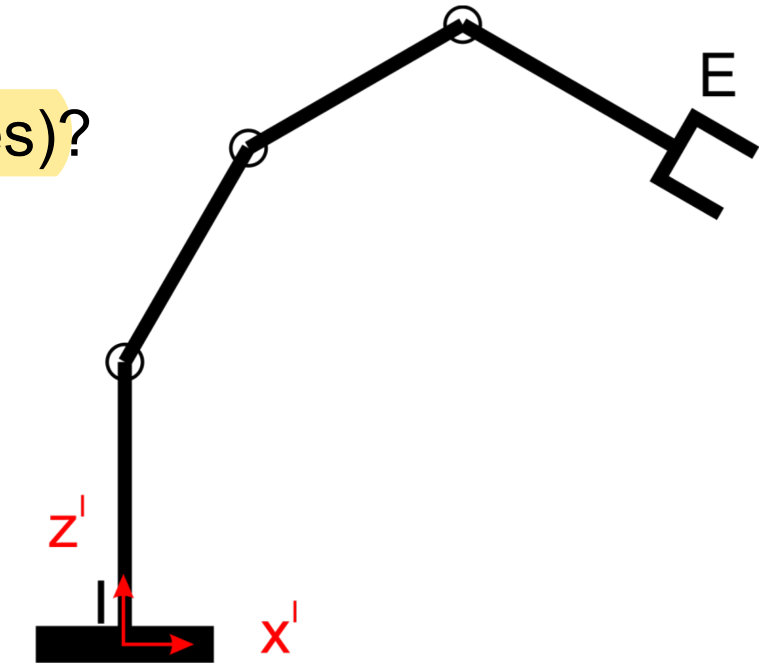
- Nr of minimal coordinates

End-effector Configuration Parameters
Simple example

- Planar robot arm
  - 3 revolute joints
  - 1 end-effector (gripper) <= *don't consider this for the moment*

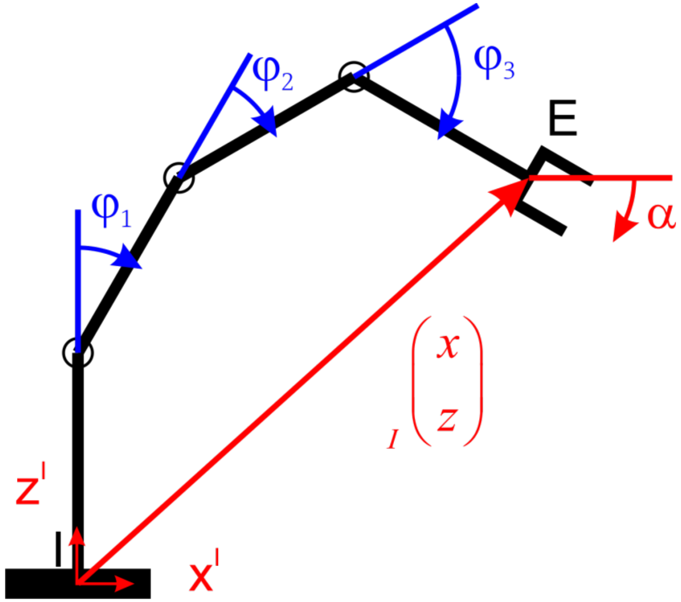
- What are the joint coordinates (generalized coordinates)?

- What are the end-effector parameters?



Configuration Space ⇔ Joint Space

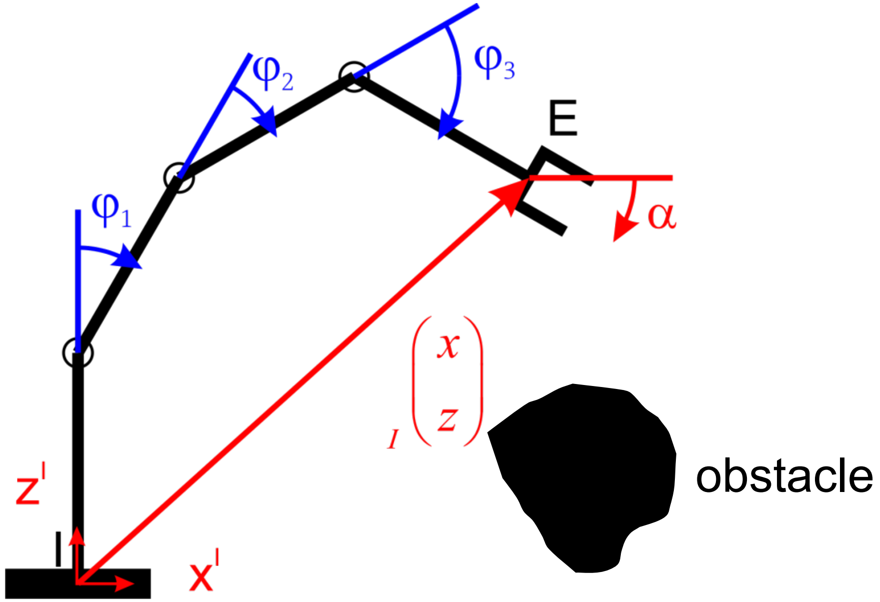
- Joint Coordinates

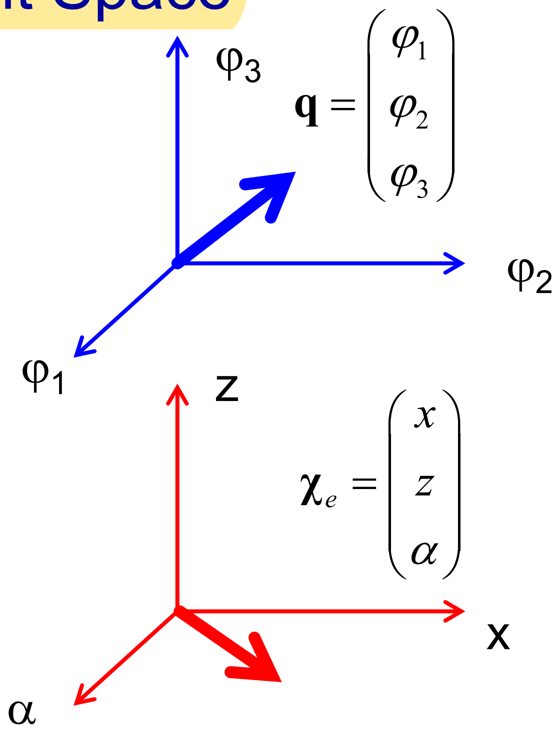


- Operational Coordinates

Joint Space ⇔ Task Space

- Joint Coordinates => Joint Space





- Task Coordinates => Task Space

Forward Kinematics

- End-effector configuration as a function of generalized coordinates

$$\chi_e = \chi_e(\mathbf{q}) \in \mathbb{R}^{n_e}$$

- For multi-body system, use transformation matrices

$$\mathbf{T}_{\mathcal{I}\mathcal{E}}(\mathbf{q}) = \mathbf{T}_{\mathcal{I}0} \cdot \left(\prod_{k=1}^{n_j} \mathbf{T}_{k-1,k}(q_k)\right) \cdot \mathbf{T}_{n_j\mathcal{E}} = \begin{bmatrix} \mathbf{C}_{\mathcal{I}\mathcal{E}}(\mathbf{q}) & {}_{\mathcal{I}}\mathbf{r}_{IE}(\mathbf{q}) \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

- Note: depending on the selected end-effector parameterization, it is not possible to analytically write down end-effector parameters!

3:00

60

Quizz
3min
1min

Forward Kinematics
Simple example

- What is the end-effector configuration as a function of generalized coordinates?

$\mathbf{T}_{IE} =$

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$