ETH zürich

Robot Dynamics

Rotorcrafts: Dynamic Modeling of Rotorcraft & Control

151-0851-00 V

Marco Hutter, **Roland Siegwart** and Thomas Stastny

Contents | Rotorcrafts (Rotary Wing UAV)

Introduction

Rotorcrafts

Rotorcraft: Definition

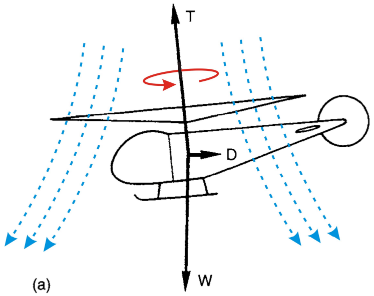
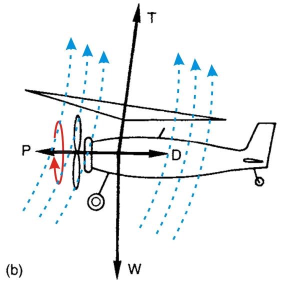
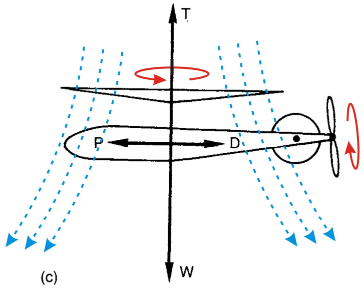
- **Rotorcraft:** Aircraft which produces lift from a rotary wing turning in a plane close to horizontal

"A helicopter is a collection of vibrations held together by differential equations" John Watkinson

Advantage	Disadvantage
Ability to hover	High maintenance costs
Power efficiency during hover	Poor efficiency in forward flight

"If you are in trouble anywhere, an airplane can fly over and drop flowers, but a helicopter can land and save your life" Igor Sikorsky

Rotorcraft | Overview on Types of Rotorcraft

Helicopter	Autogyro	Gyrodyne
		
Power driven main rotor	Un-driven main rotor, tilted away	Power driven main propeller
The air flows from TOP to BOTTOM	The air flows from BOTTOM to TOP	The air flows from TOP to BOTTOM
Tilts its main rotor to fly forward	Forward propeller for propulsion	Main propeller cannot tilt
	No tail rotor required	Additional propeller for propulsion
	Not capable of hovering	

Rotorcraft | Rotor Configuration 1

Single rotor	Multi rotor
	
Most efficient	Reduced efficiency due to multiple rotors and downwash interference
Mass constraint	Able to lift more payload
Need to balance counter-torque	Even numbered rotors can balance counter-torque

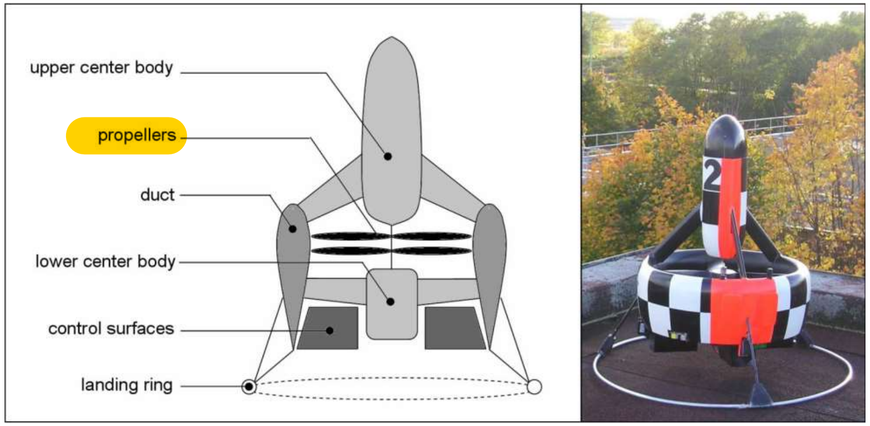
Rotorcraft | at UAS-MAV Size 1

Quadrotor / Multicopter	Std. helicopter
	
Four or more propellers in cross configuration	Very agile
Direct drive (no gearbox)	Most efficient design
Very good torque compensation	Complex to control
High maneuverability	

Rotorcraft | at UAS-MAV Size 2

Ducted fan	Coaxial

Ducted fan



Fix propeller

Torques produced by control surfaces

Heavy

Compact

Coaxial

Complex mechanics

Passively stable

Compact

Suitable for miniaturization

Dynamics Modeling & Control

Rotorcrafts

Modeling | Introduction

- Two main reasons for dynamic modeling
  - **System analysis:** the model allows evaluating the characteristics of the future aircraft in flight or its behavior in various conditions
    - Stability
    - Controllability
    - Power consumption
    - …
  - **Control laws design and simulation:** the model allows comparing various control techniques and tune their parameters
    - Gain of time and money
    - No risk of damage compared to real tests

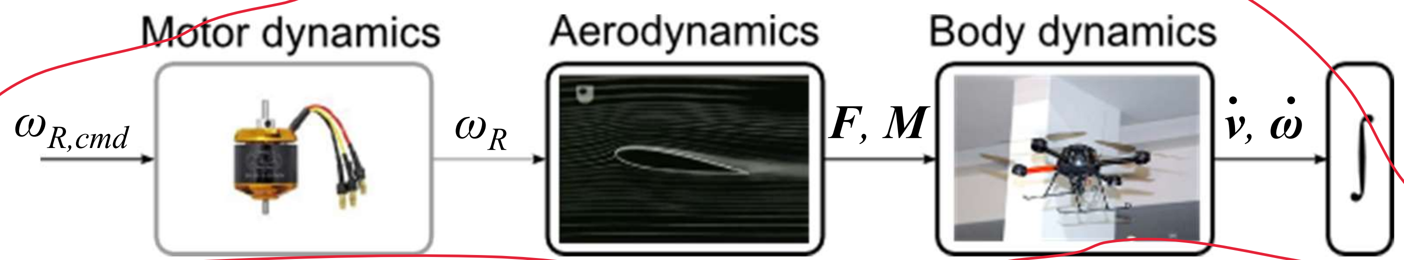
Modeling and simulation are important, but they must be validated in reality

Modeling a Quadrotor | Assumptions

- Main modeling assumptions
  - The CoG and the body frame origin are assumed to coincide
  - Interaction with ground or other surfaces is neglected
  - The structure is supposed rigid and symmetric
  - Propeller is supposed to be rigid
  - Fuselage drag is neglected

Modeling of Rotorcraft | Overview

- The model of the rotorcraft consist of three main parts



  - Input to the system are commanded rotor speeds or the input voltage into the motor
  - Motor dynamics block: Current rotor speeds
    - Due to fast dynamics w.r.t. body dynamics. Not necessary for control design if bandwidth is taken into account
  - Propeller aerodynamics: Aerodynamic forces and moments
  - Body dynamics: Speed and rotational speed

Multi-body Dynamics | General Formulation

$$\mathbf{M}(\vec{\varphi})\ddot{\vec{\varphi}} + \vec{b}(\vec{\varphi},\dot{\vec{\varphi}}) + \vec{g}(\vec{\varphi}) + \mathbf{J}_{ex}^{T}\vec{F}_{ex} = \mathbf{S}^{T}\vec{\tau}_{act}$$

$\ddot{\vec{\varphi}}$	Generalized coordinates
$\mathbf{M}(\vec{\varphi})$	Mass matrix
$\vec{b}(\vec{\varphi},\dot{\vec{\varphi}})$	Centrifugal and Coriolis forces
$\vec{g}(\vec{\varphi})$	Gravity forces
$\vec{\tau}_{act}$	Actuation torque
$\vec{F}_{ex}$	External forces (end-effector, ground contact, propeller thrust, ...)
$\mathbf{J}_{ex}$	Jacobian of external forces
$\mathbf{S}$	Selection matrix of actuated joints

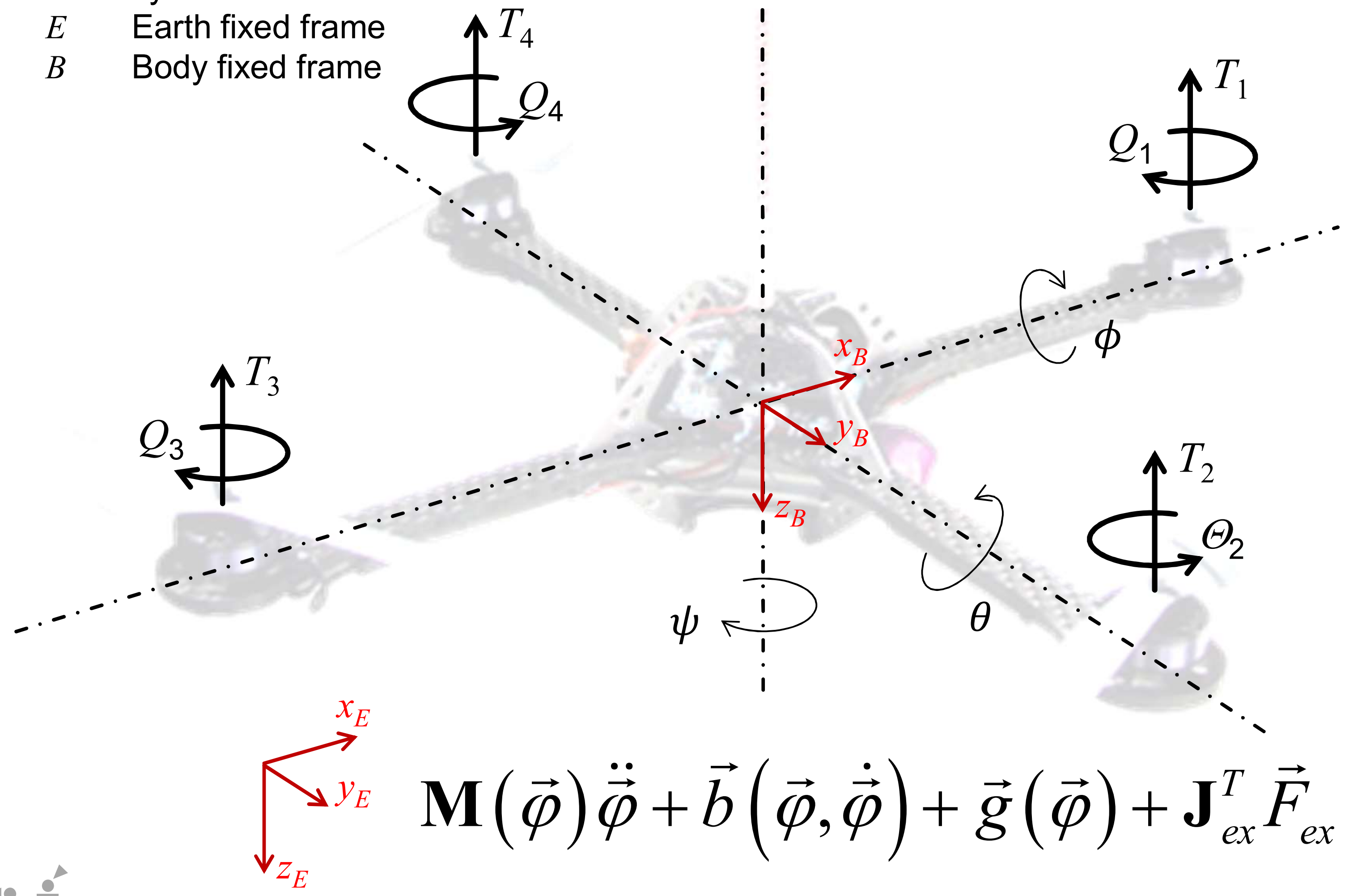
The Quadrotor Example

$$\mathbf{M}(\vec{\varphi})\ddot{\vec{\varphi}} + \vec{b}\left(\vec{\varphi},\dot{\vec{\varphi}}\right) + \vec{g}(\vec{\varphi}) + \mathbf{J}_{ex}^{T}\vec{F}_{ex} = \mathbf{S}^{T}\vec{\tau}_{act}$$

The Quadrotor Example

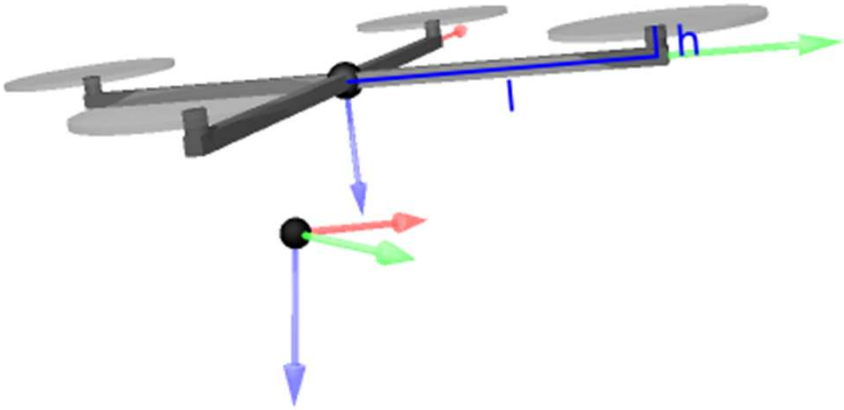
Coordinate systems

E	Earth fixed frame
B	Body fixed frame



$$\mathbf{M}(\vec{\varphi})\ddot{\vec{\varphi}} + \vec{b}\left(\vec{\varphi},\dot{\vec{\varphi}}\right) + \vec{g}(\vec{\varphi}) + \mathbf{J}_{ex}^{T}\vec{F}_{ex} = \mathbf{S}^{T}\vec{\tau}_{act}$$

Modeling a Quadrotor | Properties



- Arm length l
  - Planar distance between CoG and rotor plane
- Rotor height h
  - Height between CoG and rotor plane
- Mass m
  - Total lift of mass
- Inertia $\boldsymbol{I}$
  - Assume symmetry in xz- and yz-plane

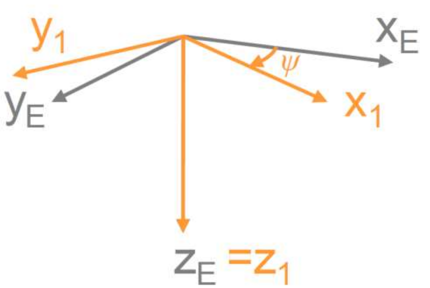
$$\boldsymbol{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

Modeling a Quadrotor | Simplifications

- Quadrotor shall only be considered in near hover condition
- Main forces and moments come from propellers
  - Depend on flight regime (hover or translational flight)
  - In hover: Hub forces and rolling moment small compared to thrust and drag moment
  - Since hover is considered, thus thrust and drag are proportional to propellers‘ squared rotational speed:
    - $T_i = b\omega_{p,i}^2$ b: thrust constant
    - $Q_i = d\omega_{p,i}^2$ d: drag constant

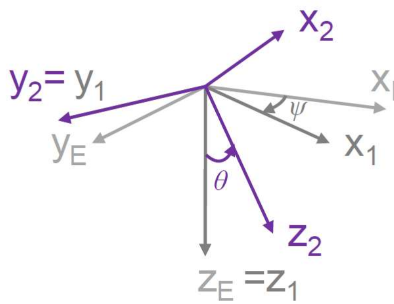
Modeling of Rotorcraft | Representing the Attitude

1 Yaw



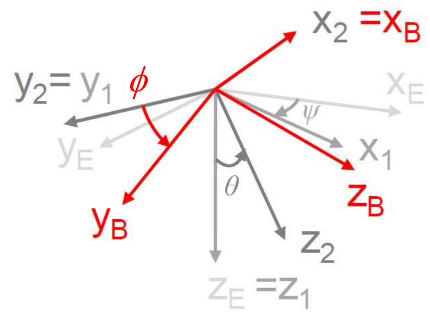
$$C_{E1}(\mathbf{z},\psi)=\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}$$

2 Pitch



$$C_{12}(\mathbf{y},\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$

3 Roll



$$C_{2B}(\mathbf{x},\phi)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi & \cos\phi\end{bmatrix}$$

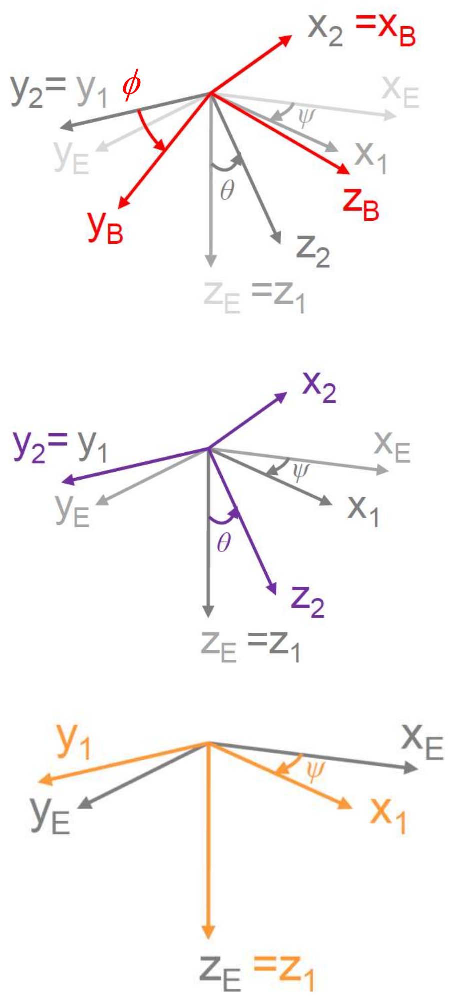
$$C_{EB}=C_{E1}C_{12}C_{2B}=\begin{bmatrix}c(\psi)c(\theta) & c(\psi)s(\theta)s(\phi)-s(\psi)c(\phi) & c(\psi)s(\theta)c(\phi)+s(\psi)s(\phi)\\ s(\psi)c(\theta) & s(\psi)s(\theta)s(\phi)+c(\psi)c(\phi) & s(\psi)s(\theta)c(\phi)-c(\psi)s(\phi)\\ -s(\theta) & c(\theta)s(\phi) & c(\theta)c(\phi)\end{bmatrix}$$

Roll $(-\pi<\phi<\pi)$
Pitch $(-\pi/2<\theta<\pi/2)$
Yaw $(-\pi<\psi<\pi)$

C_{EB}: Rotation matrix from B to E

Modeling of Rotorcraft | Rotational Velocity

- Split angular velocity into the three basic rotations
  - $_B\boldsymbol{\omega} = {_B\boldsymbol{\omega}_{roll}} + {_B\boldsymbol{\omega}_{pitch}} + {_B\boldsymbol{\omega}_{yaw}}$
  - Angular velocity due to change in roll angle
    - $_B\boldsymbol{\omega}_{roll} = (\dot{\phi},0,0)^T$
  - Angular velocity due to change in pitch angle
    - $_B\boldsymbol{\omega}_{pitch} = \mathbf{C}^T{}_{2B}\,(\boldsymbol{x},\phi)\ (0,\ \dot{\theta},0)^T$
  - Angular velocity due to change in yaw angle
    - $_B\boldsymbol{\omega}_{yaw} = [\mathbf{C}_{12}(\boldsymbol{y},\theta)\ \mathbf{C}_{2B}\,(\boldsymbol{x},\ \phi)]^{\,T}\ (0,0,\dot{\psi})^T$



Modeling of Rotorcraft | Rotational Velocity

- Split angular velocity into the three basic rotations
  - ${}_B\boldsymbol{\omega} = {}_B\boldsymbol{\omega}_{roll} + {}_B\boldsymbol{\omega}_{pitch} + {}_B\boldsymbol{\omega}_{yaw}$
  - Angular velocity due to change in roll angle
    - ${}_B\boldsymbol{\omega}_{roll} = (\dot{\phi},0,0)^T$
  - Angular velocity due to change in pitch angle
    - ${}_B\boldsymbol{\omega}_{pitch} = \mathbf{C}^T{}_{2B}\,(\mathbf{x},\phi)\ (0,\ \dot{\theta},0)^T$
  - Angular velocity due to change in yaw angle
    - ${}_B\boldsymbol{\omega}_{yaw} = [\mathbf{C}_{12}(\mathbf{y},\theta)\ \mathbf{C}_{2B}\,(\mathbf{x},\ \phi)]^{\,T}\,(0,0,\dot{\psi})^T$

$${}_B\omega_{roll} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$${}_B\omega_{pitch} = \mathbf{C}_{2B}^T(\mathbf{x},\phi)\cdot\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}\cdot\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\theta}\cdot\cos\phi \\ \dot{\theta}\cdot(-\sin\phi) \end{bmatrix}$$

$${}_B\omega_{yaw} = \left[\mathbf{C}_{12}(\mathbf{y},\theta)\cdot\mathbf{C}_{2B}(\mathbf{x},\phi)\right]^T\cdot\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \mathbf{C}_{2B}^T(\mathbf{x},\phi)\cdot\mathbf{C}_{12}^T(\mathbf{y},\theta)\cdot\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$${}_B\omega_{yaw} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}\cdot\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \dot{\psi}\cdot(-\sin\theta) \\ \dot{\psi}\cdot\sin\phi\cdot\cos\theta \\ \dot{\psi}\cdot\cos\theta\cos\phi \end{bmatrix}$$

Modeling of Rotorcraft | Rotational Velocity

- Relation between Tait-Bryan angles χ and angular velocities

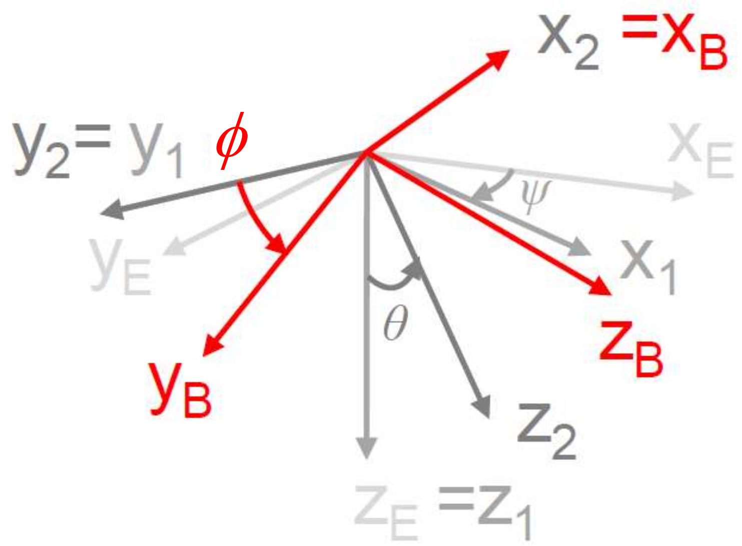
$$E_r \dot{\chi}_r = {}_B\boldsymbol{\omega} \qquad E_r = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}; \qquad \dot{\chi}_r = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\boxed{\Rightarrow \text{ Singularity for } \theta = \pm 90^o}$$

special case:

- Linearized relation at hover

$$E_r\big|_{\phi=0,\theta=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = {}_B\boldsymbol{\omega}$$



Modeling of Rotorcraft | Body Dynamics

$$\mathbf{M}(\vec{\varphi})\ddot{\vec{\varphi}} + \vec{b}\left(\vec{\varphi},\dot{\vec{\varphi}}\right) + \vec{g}(\vec{\varphi}) + \mathbf{J}_{ex}^{T}\vec{F}_{ex} = \mathbf{S}^{T}\vec{\tau}_{act}$$

- Change of momentum and spin in the body frame

$$\begin{bmatrix} mE_{3x3} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} {}_B\dot{\boldsymbol{v}} \\ {}_B\dot{\boldsymbol{\omega}} \end{bmatrix} + \begin{bmatrix} {}_B\boldsymbol{\omega} \times m\ {}_B\boldsymbol{v} \\ {}_B\boldsymbol{\omega} \times \boldsymbol{I}\ {}_B\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} {}_B\boldsymbol{F} \\ {}_B\boldsymbol{M} \end{bmatrix}$$

E_{3x3}: Identity matrix

Torque!!

- Position in the inertial frame and the attitude

$$_E\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = C_{EB}\ {}_B v = C_{EB}\ {}_B\begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad E_r\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = {}_B\omega = {}_B\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

- Forces and moments
  - Aerodynamics and gravity

$$_B\boldsymbol{F} = {}_B\boldsymbol{F}_G + {}_B\boldsymbol{F}_{Aero}$$

$$_B\boldsymbol{M} = {}_B\boldsymbol{M}_{Aero}$$

$$_B\boldsymbol{F}_G = \boldsymbol{C}_{EB}^T\ {}_E\begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}$$

Modeling a Quadrotor | Aerodynamic Forces

- Hover forces
  - Thrust forces in the shaft direction

$$_B F_{Aero} = \sum_{i=1}^{4} {}_B\begin{bmatrix} 0 \\ 0 \\ -T_i \end{bmatrix}$$

$$T_i = b_i \omega_{p,i}^2$$

- Additional Forces:
  - Can be neglected
  - Hub forces along the horizontal speed
  - $T_i = \rho/2 A C_H (\Omega_i R_{prop})^2$
  - $v_h = [u\ v\ 0]^T$

$$_B\mathbf{H} = \sum_{i=1}^{4} H_i \frac{_B\mathbf{v}_h}{\left\| _B\mathbf{v}_h \right\|}$$

Modeling a Quadrotor | Aerodynamic Moments

- Hover moments
  - Thrust induced moment
  - Drag torques

$$ {}_{B}M_{Aero} = {}_{B}M_{T} + {}_{B}Q = {}_{B}\begin{bmatrix} l(T_4 - T_2) \\ l(T_1 - T_3) \\ 0 \end{bmatrix} + {}_{B}\begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{4} Q_i (-1)^{(i-1)} \end{bmatrix} $$

$$ T_i = b_i \omega_{p,i}^{\ 2} \qquad Q_i = d_i \omega_{p,i}^{\ 2} $$

- Additional moments
  - Inertial counter torques
  - Propeller gyro effect
  - Rolling moments
  - Hub moments

$$ {}_{B}\boldsymbol{M}_{CT} = I_{\mathrm{Prop}}\, {}_{B}\begin{bmatrix} 0 \\ 0 \\ \dot{\omega}_P \end{bmatrix} \qquad {}_{B}\boldsymbol{M}_{G} = {}_{B}\begin{bmatrix} 0 \\ 0 \\ I_{\mathrm{Prop}}\omega_P \end{bmatrix} \times {}_{B}\boldsymbol{\omega} $$

$$ {}_{B}\boldsymbol{R} = \sum_{i=1}^{4} R_i (-1)^{i-1} \frac{{}_{B}\boldsymbol{v}_h}{\left\| {}_{B}\boldsymbol{v}_h \right\|} \qquad {}_{B}\boldsymbol{M}_{H} = \sum_{i=1}^{4} H_i \; {}_{B}\boldsymbol{p}_{P_i} \times \frac{{}_{B}\boldsymbol{v}_h}{\left\| {}_{B}\boldsymbol{v}_h \right\|} $$

Control of a Quadrotor | Analysis of the Dynamics 1

$$\begin{bmatrix} mE_{3x3} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} {}_B\dot{\boldsymbol{v}} \\ {}_B\dot{\boldsymbol{\omega}} \end{bmatrix}+\begin{bmatrix} {}_B\boldsymbol{\omega}\times m\ {}_B\boldsymbol{v} \\ {}_B\boldsymbol{\omega}\times \boldsymbol{I}\ {}_B\boldsymbol{\omega} \end{bmatrix}=\begin{bmatrix} {}_B\boldsymbol{F} \\ {}_B\boldsymbol{M} \end{bmatrix}$$

- Rotational dynamics near hover

$$I_{xx}\dot{p} = q\cdot r(I_{yy}-I_{zz})+l\cdot b(\omega_{p,4}^2-\omega_{p,2}^2)$$

$$I_{yy}\dot{q} = r\cdot p(I_{zz}-I_{xx})+l\cdot b(\omega_{p,1}^2-\omega_{p,3}^2)$$

$$I_{zz}\dot{r} = d(-\omega_{p,1}^2+\omega_{p,2}^2-\omega_{p,3}^2+\omega_{p,4}^2)$$

$$_B\omega\times I\ {}_B\omega=\begin{bmatrix} p \\ q \\ r \end{bmatrix}\times\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix}=\begin{bmatrix} q\cdot r\left(I_{zz}-I_{yy}\right) \\ r\cdot p\left(I_{xx}-I_{zz}\right) \\ \boxed{p\cdot q\left(I_{yy}-I_{xx}\right)} \end{bmatrix}$$

0

$$_B\omega = {}_B\begin{bmatrix} p \\ q \\ r \end{bmatrix} = E_r\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

b : thrust constant

d : drag constant

l : distance of propeller from the CoG

$\omega_{p,i}$: rotational speed of propellor i

- Full control over all rotational speeds
- Not dependent on position states

Control of a Quadrotor | Analysis of the Dynamics 2

$$\begin{bmatrix} mE_{3x3} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} {}_B\dot{\boldsymbol{v}} \\ {}_B\dot{\boldsymbol{\omega}} \end{bmatrix} + \begin{bmatrix} {}_B\boldsymbol{\omega} \times m \ {}_B\boldsymbol{v} \\ {}_B\boldsymbol{\omega} \times \boldsymbol{I} \ {}_B\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} {}_B\boldsymbol{F} \\ {}_B\boldsymbol{M} \end{bmatrix}$$

$$_B\omega \times m \ _B v = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times m \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} m(q \cdot w - r \cdot v) \\ m(r \cdot u - p \cdot w) \\ m(p \cdot v - q \cdot u) \end{bmatrix}$$

- Translational dynamics in hover

$$_B F_G = C_{EB}^T \ {}_E\begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \sin(\phi)\cos(\theta) \\ \cos(\phi)\cos(\theta) \end{bmatrix} mg$$

$$m\dot{u} = m(r \cdot v - q \cdot w) - \sin\theta \ mg$$

$$m\dot{v} = m(p \cdot w - r \cdot u) + \sin\phi \cos\theta \ mg$$

$$m\dot{w} = m(q \cdot u - p \cdot v) + \cos\phi \cos\theta \ mg - b(\omega_{p,1}^2 + \omega_{p,2}^2 + \omega_{p,3}^2 + \omega_{p,4}^2)$$

$$C_{EB} \ {}_B v = C_{EB} \ {}_B\begin{bmatrix} u \\ v \\ w \end{bmatrix} = {}_E\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

b: thrust constant

$\omega_{p,i}$: rotational speed of propellor i

- Only direct control of the body z velocity
  - Use attitude to control full position

Control of a Quadrotor | General Overview

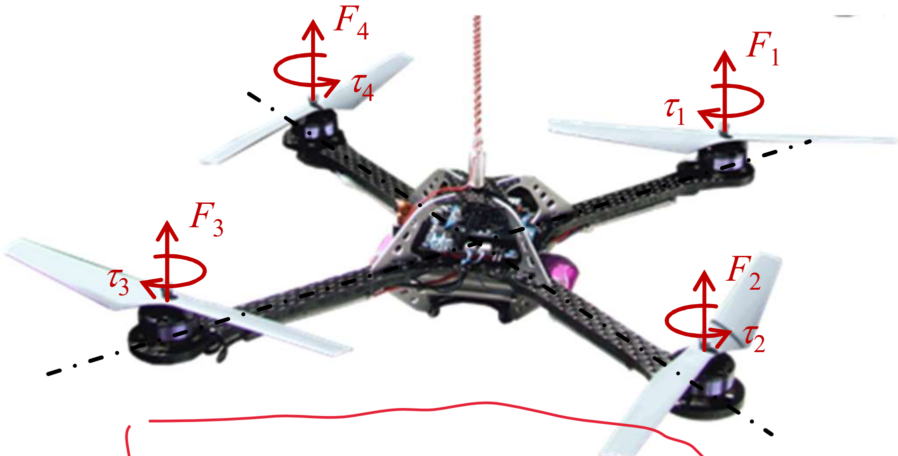
- Goal: Move the Quadrotor in space
  - Six degree of freedom
  - Four motor speeds as system input
  - Under-actuated system
    - Not all states are independently controllable
    - Fly maneuvers
- Control using full state feedback
  - Assume all states are known
    - Needs an additional state estimator
- Assumption
  - Neglect motor dynamics
  - Low speeds

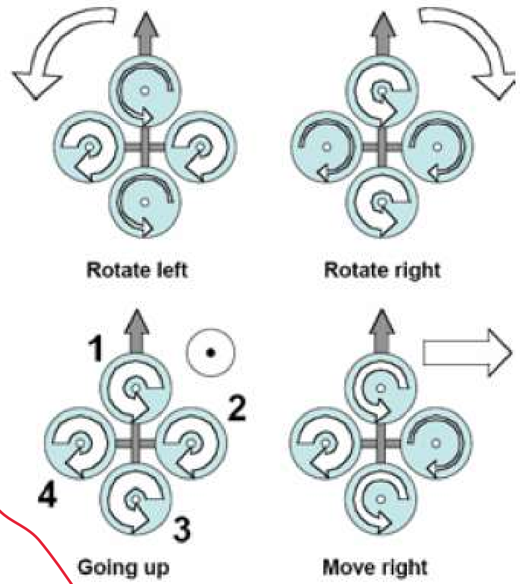
  → only dominate aerodynamic forces considered

  → linearization around small attitude angles

Modeling a Quadrotor | Control Overview

- Four propellers in cross configuration
  - Two pairs (1,3) and (2,4), turning in opposite directions
  - Vertical control by simultaneous change in rotor speed
  - Directional control (Yaw) by rotor speed imbalance between the two rotor pairs
  - Longitudinal control by converse change of rotor speeds 1 and 3
  - Lateral control by converse change of rotor speeds 2 and 4





Control of a Quadrotor | Virtual Control Input

- Define virtual control inputs to simplify the equations
  - Get decoupled and linear inputs
    - Moments along each axis

$$I_{xx}\dot{p} = q \cdot r(I_{yy} - I_{zz}) + U_2 \qquad U_2 = l \cdot b(\omega_{p,4}^2 - \omega_{p,2}^2)$$

$$I_{yy}\dot{q} = r \cdot p(I_{zz} - I_{xx}) + U_3 \qquad U_3 = l \cdot b(\omega_{p,1}^2 - \omega_{p,3}^2)$$

$$I_{zz}\dot{r} = U_4 \qquad U_4 = d(-\omega_{p,1}^2 + \omega_{p,2}^2 - \omega_{p,3}^2 + \omega_{p,4}^2)$$

    - Total thrust

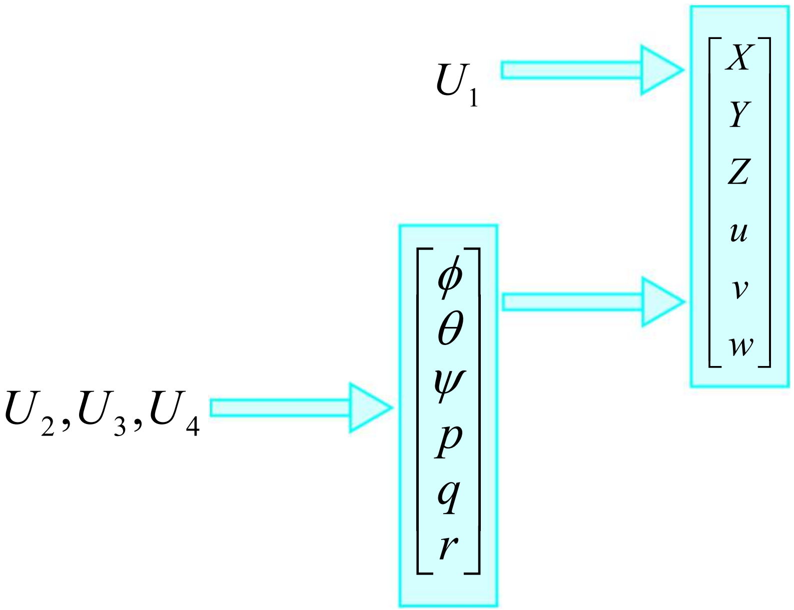
$$m\dot{u} = m(r \cdot v - q \cdot w) - \sin\theta\; mg$$

$$m\dot{v} = m(p \cdot w - r \cdot u) + \sin\phi\cos\theta\; mg$$

$$m\dot{w} = m(q \cdot u - p \cdot v) + \cos\phi\cos\theta\; mg - U_1 \qquad U_1 = b(\omega_{p,1}^2 + \omega_{p,2}^2 + \omega_{p,3}^2 + \omega_{p,4}^2)$$

Control of a Quadrotor | Hierarchical Control

- Rotational dynamics independent of translational dynamics
  - Use a cascaded control structure
    - Inner loop: Control of the attitude
      - Input: U_2, U_3, U_4
    - Outer loop: Control of the position
      - Input: U_1, ϕ, θ



Control of a Quadrotor | Linearization of Attitude Dynamics

- Linearize attitude dynamic at hover
  - Equilibrium point: $\phi = \theta = p = q = r = 0 \;\; ; \;\; U_2 = U_3 = U_4 = 0 \;\; ; \;\; U_1 = mg$

$$I_{xx}\dot{p} = qr(I_{yy} - I_{zz}) + U_2$$

$$I_{yy}\dot{q} = pr(I_{zz} - I_{xx}) + U_3$$

$$I_{zz}\dot{r} = U_4$$

$$E_r \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = {}_B\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\ddot{\phi} = \frac{1}{I_{xx}} U_2$$

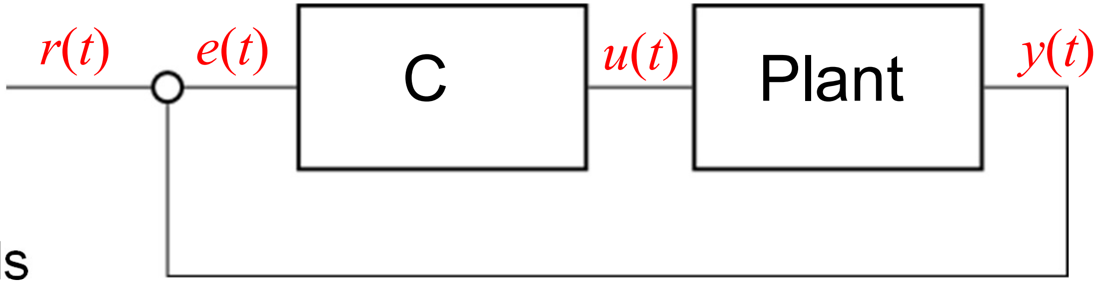
$$\ddot{\theta} = \frac{1}{I_{yy}} U_3$$

$$\ddot{\psi} = \frac{1}{I_{zz}} U_4$$

This model has no coupling
→ We use 3 individual controller

Control of a Quadrotor | PID Control

- PID control strategy
  - Generic non model based control
  - Consist of three terms
    - P - Proportional
    - I - Integral
    - D - Derivative
  - Different tuning methods
    - Ziegler-Nichols method
    - Pole placement
  - Widely used



$$u(t) = k_p e(t) + k_i \int_0^t e(t)dt + k_d \frac{d\ e(t)}{dt}$$

Control of a Quadrotor | PID Attitude Control

- Consider roll subsystem

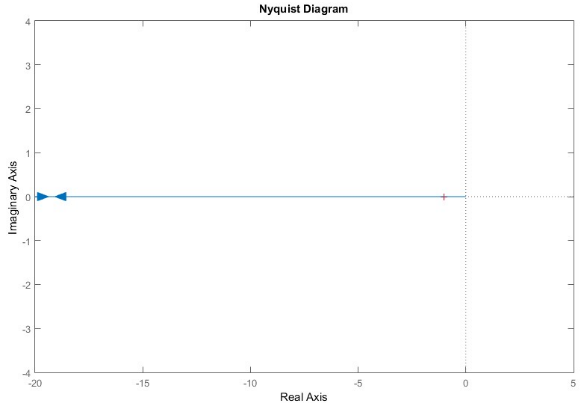
$$\ddot{\phi} = \frac{1}{I_{xx}} U_2$$

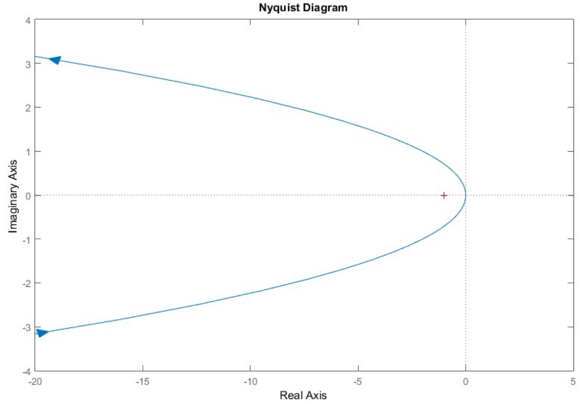
  - P controller can‘t stabilize the system
    - Harmonic oscillation

$$\ddot{\phi} = \frac{1}{I_{xx}} k_p (\phi_{des} - \phi)$$

  - Use a PD controller

$$\ddot{\phi} = \frac{1}{I_{xx}} k_p (\phi_{des} - \phi) + k_d (\dot{\phi}_{des} - \dot{\phi})$$





Control of a Quadrotor | Attitude Control Design

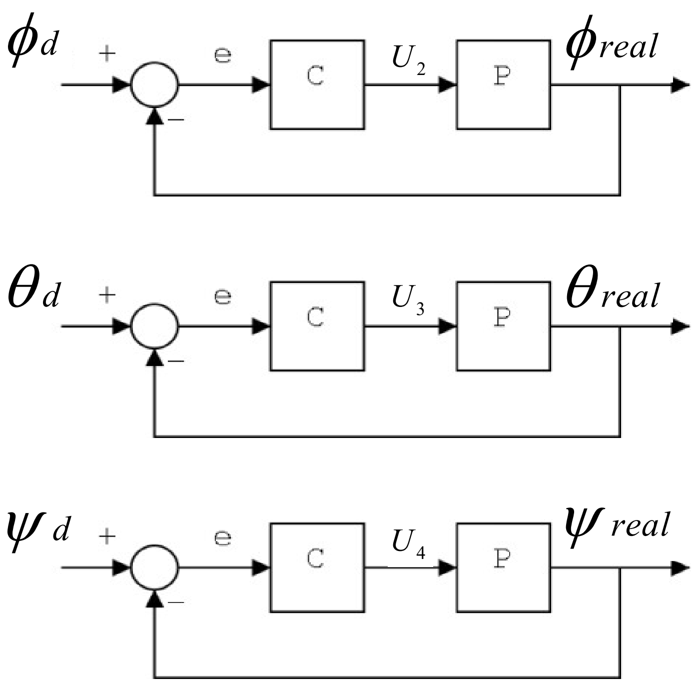
- 3 separate control loops
  - 1 additional thrust input

$$U_1 = T_{des}$$

$$U_2 = (\phi_{des} - \phi)k_{pRoll} - \dot{\phi}k_{dRoll}$$

$$U_3 = (\theta_{des} - \theta)k_{pPitch} - \dot{\theta}k_{dPitch}$$

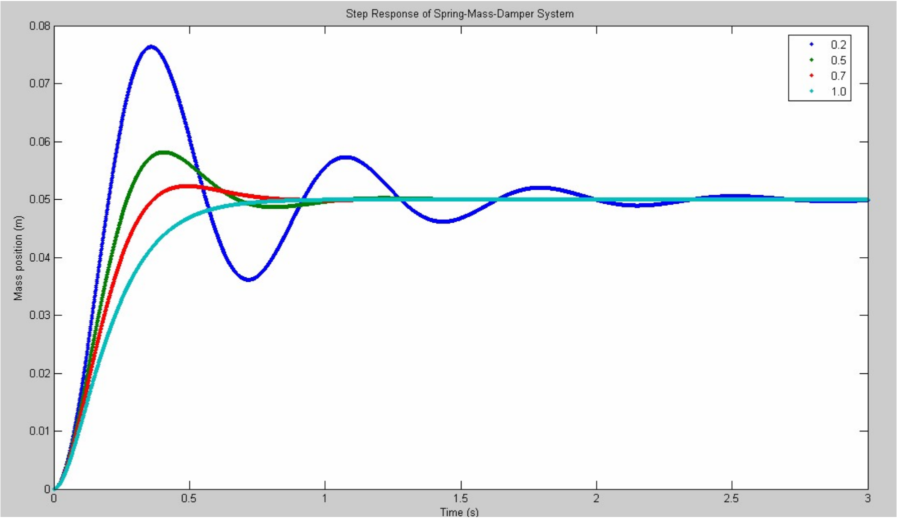
$$U_4 = (\psi_{des} - \psi)k_{pYaw} - \dot{\psi}k_{dYaw}$$



- Aspect on the implementation
  - Get angle derivatives from transformation of the body angular rates
  - Avoid integral element in controller

Control of a Quadrotor | Parameter Tuning

- k_p is chosen to meet desired convergence time
- k_d is chosen to get desired damping
- In theory any convergence time can be reached
  - Actuator dynamics allow for a limited control signal bandwidth
    - Closed loop system dynamics must respect the bandwidth of actuator dynamics
    - Actuator signals shall not saturate the motors



Control of a Quadrotor | Control Allocation

- Transformation of virtual control inputs to motor speeds
  - Rewrite virtual control inputs in matrix form
  - Invert matrix to get motor speeds

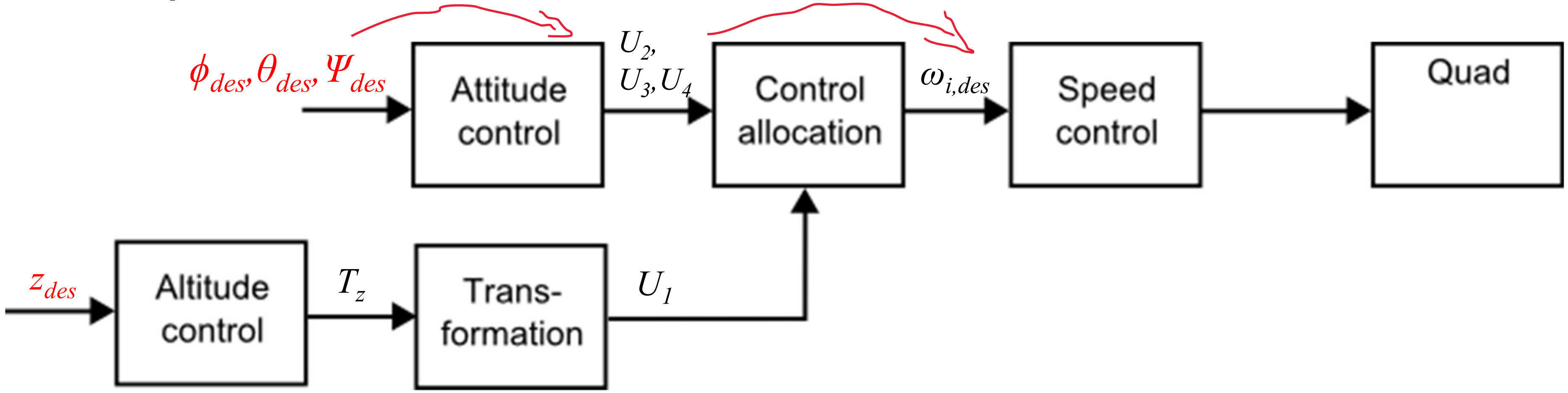
$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} b & b & b & b \\ 0 & -lb & 0 & lb \\ lb & 0 & -lb & 0 \\ -d & d & -d & d \end{bmatrix} \begin{bmatrix} \omega_{p,1}^2 \\ \omega_{p,2}^2 \\ \omega_{p,3}^2 \\ \omega_{p,4}^2 \end{bmatrix}$$

$$\begin{bmatrix} \omega_{p,1}^2 \\ \omega_{p,2}^2 \\ \omega_{p,3}^2 \\ \omega_{p,4}^2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \frac{1}{4}\frac{1}{b} & 0 & \frac{1}{2}\frac{1}{lb} & -\frac{1}{4}\frac{1}{d} \\ \frac{1}{4}\frac{1}{b} & -\frac{1}{2}\frac{1}{lb} & 0 & \frac{1}{4}\frac{1}{d} \\ \frac{1}{4}\frac{1}{b} & 0 & -\frac{1}{2}\frac{1}{lb} & -\frac{1}{4}\frac{1}{d} \\ \frac{1}{4}\frac{1}{b} & \frac{1}{2}\frac{1}{lb} & 0 & \frac{1}{4}\frac{1}{d} \end{bmatrix} \\ \begin{array}{cccc} \text{Thrust} & \text{Roll} & \text{Pitch} & \text{Yaw} \end{array} \end{array} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

- Limit motor speeds to feasible (positive) values

Control of a Quadrotor | Altitude Control

- A possible control flow for altitude control



  - Control attitude and the altitude
    - Attitude controller to control the angles
    - Parallel control loop. Altitude controller to control the horizontal thrust and nonlinear transformation to calculate the corresponding thrust input

Control of a Quadrotor | Altitude Control 2

- Rewrite translational dynamics in the inertial frame

$$\dot{\boldsymbol{X}} = \boldsymbol{V}$$

$$\dot{\boldsymbol{V}} = {}_E\begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} + \boldsymbol{C}_{EB}\frac{1}{m}\,{}_B\begin{bmatrix} 0 \\ 0 \\ U_1 \end{bmatrix}$$

  - Only look at the altitude dynamics

$$\ddot{z} = g + \cos\varphi\cos\theta\ \frac{1}{m}U_1$$

  - Virtual Control

$$\ddot{z} = g + \frac{1}{m}T_Z \qquad T_Z = \cos\varphi\cos\theta\ U_1$$

  - PD controller for the height

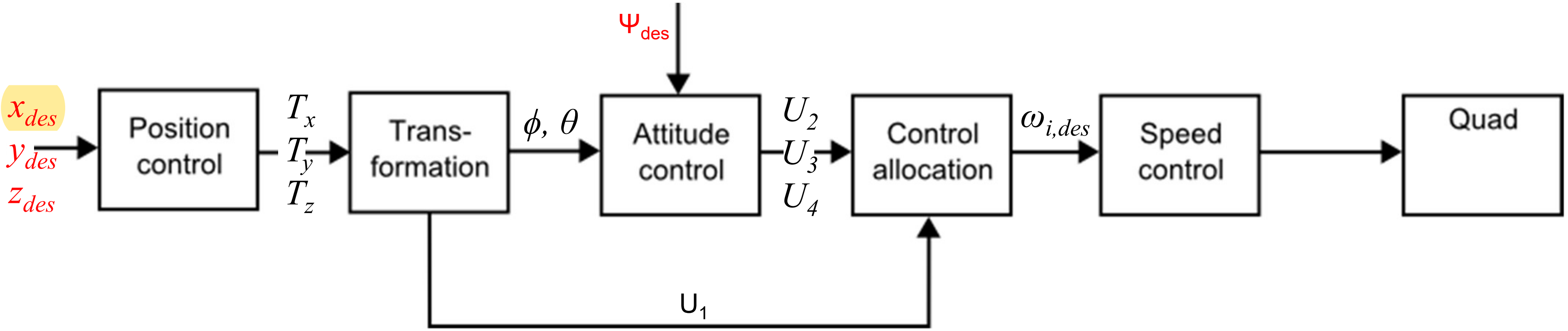
$$T_z = k_p(z_{des} - z) - k_d\dot{z} - mg$$

  - Transformation to the body frame

$$U_1 = \frac{-T_z}{\cos\phi\cos\theta}$$

Control of a Quadrotor | Position Control

- Possible control structure for full position control



  - Control position and yaw angle
    - Hierarchical controller
      - Outer loop: Position control
        - Thrust vector control. Desired thrust vector in inertial frame
        - Transform to thrust and roll and pitch angles
      - Inner loop: Attitude control

Control of a Quadrotor | Position Control 2

- Thrust vector control
  - Assume that the control input is the thrust in the inertial frame
  - Inertial position system dynamics

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \frac{1}{m} \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

  - Use 3 separate PD controller for each direction
  - Transform to desired thrust, roll and pitch

$$T = \sqrt{T_x^2 + T_y^2 + T_z^2} \qquad \frac{1}{T} C_{E1}^T (z, \psi) \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\varphi \\ -\sin\varphi \\ \cos\theta \cos\varphi \end{bmatrix}$$

Thrust Force of Propeller / Rotor

Rotorcrafts

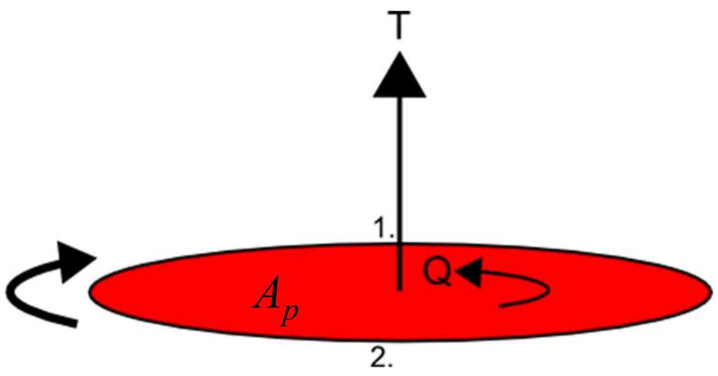
Modeling of Rotorcraft | Rotor/Propeller Aerodynamics

- Highly depends on the structure
  - Propeller: Generation of forces and torques
    - Thrust force $\boldsymbol{T}$
    - Hub force $\boldsymbol{H}$ (orthogonal to $\boldsymbol{T}$)
    - Drag torque $\boldsymbol{Q}$
    - Rolling torque $\boldsymbol{R}$
  - Rotor: Generation of forces and torques. Additional tilt dynamics of rotor disc
    - Modeling of forces and moments similar to propeller
    - Modeling the orientation of the TPP (Tip Path Plan)

Modeling a Quadrotor | Propeller Aerodynamics 1

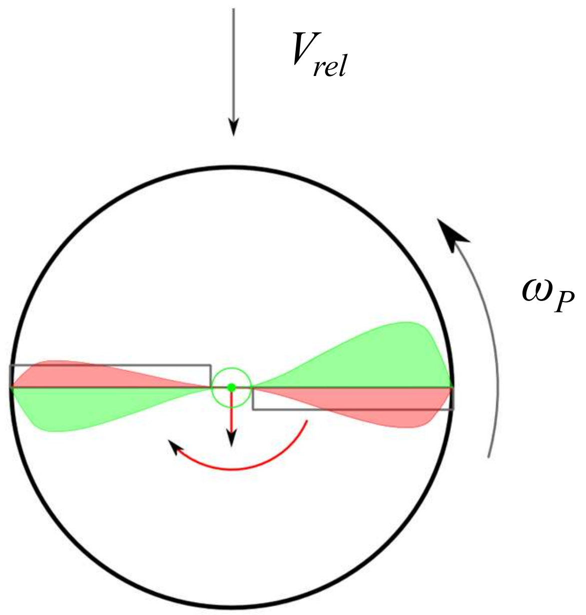
- Propeller in hover
  - Thrust force $\boldsymbol{T}$
    - Aerodynamic force perpendicular to propeller plane
    - $|\boldsymbol{T}| = \rho/_2 A_P C_T (\omega_P R_P)^2$
  - Drag torque $\boldsymbol{Q}$
    - Torque around rotor plane
    - $|\boldsymbol{Q}| = \rho/_2 A_P C_Q (\omega_P R_P)^2 R_P$
  - C_T and C_Q are depending on
    - Blade pitch angle (propeller geometry)
    - Reynolds number (propeller speed, velocity, rotational speed)
    - …

Similar to blade element



Modeling a Quadrotor | Propeller Aerodynamics 2

- Propeller in forward flight
  - Additional forces due to force unbalance at position 1. and 2.
  - Hub force $\boldsymbol{H}$
    - Opposite to horizontal flight direction V_H
    - $|\boldsymbol{H}| = {}^{\rho}/_{2} A_P C_H (\omega_P R_P)^2$
  - Rolling moment $\boldsymbol{R}$
    - Around flight direction
    - $|\boldsymbol{R}| = {}^{\rho}/_{2} A_P C_R (\omega_P R_P)^2 R_P$
  - C_H and C_R are depending on the advance ratio μ
    - $\mu = {}^{V}/_{\omega_P R_P}$
    - …



The Propeller Thrust Force | Momentum Theory

- Analysis of an ideal propeller/rotor
  - Power put into fluid to change its momentum downwards
    - Actio-reactio: Thrust force at the propeller/rotor
- Assumptions
  - Infinitely thin propeller/rotor disc area A_R
  - Thrust and velocity distribution is uniform over disc area
    - One dimensional flow analysis
  - Quasi-static airflow
    - Flow properties do not change over time
  - No viscous effects
    - No profile drag
  - Air is incompressible

Momentum Theory | Recall of Fluid Dynamics

- Conservation of fluid mass

$$\iint \rho \vec{V} \cdot \vec{n} dA = 0 \qquad (1)$$

ρ : density of fluid

$\vec{V}$: flow speed at surface element

$\vec{n}$: suface normal unity vector

dA : patch of control surface area

  - Mass flow inside and outside control volume must be equal (quasi-static flow)

- Conservation of fluid momentum

$$\iint p\ \vec{n} dA + \iint \rho \vec{V}\vec{V}\ \vec{n} dA = \vec{F} \qquad (2)$$

p : surface pressure

  - The net force on the fluid is the change of momentum of the fluid

- Conservation of energy

$$\iint \frac{1}{2} \rho V^2\ \vec{V}\ \vec{n} dA = \frac{dE}{dt} = P \qquad (3)$$

E : energy

P : power

  - Work done on the fluid results in a gain of kinetic energy

Momentum Theory | Derivation 1

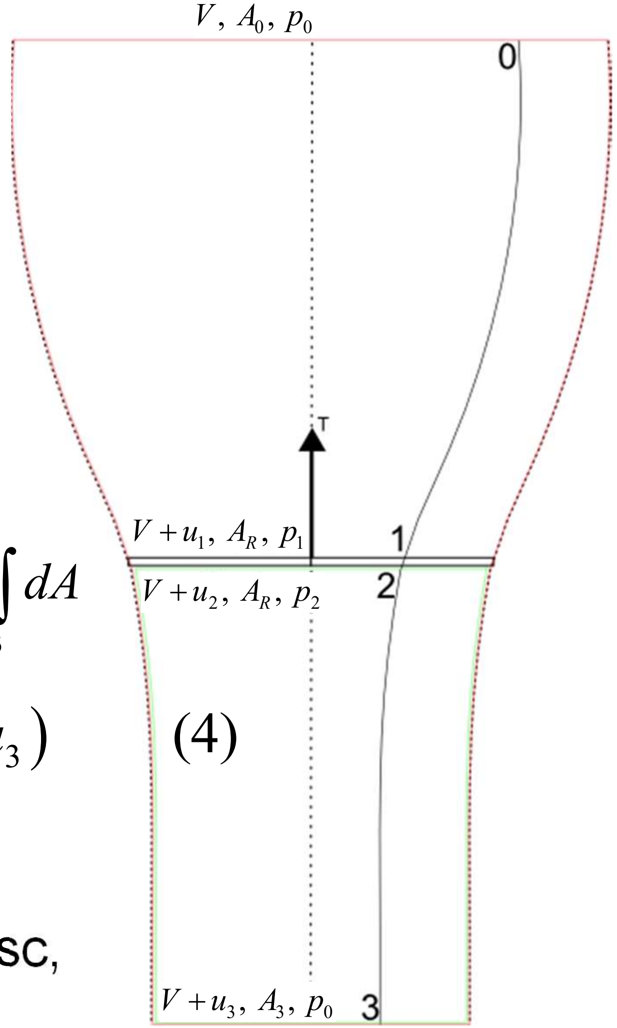
- One-dimensional analysis
  - Area of interest 0,1,2 and 3
  - Area 0 and 3 are on the far field with atmospheric pressure
- Conservation of mass

$$0 = -\rho V \iint_0 dA + \rho(V+u_1)\iint_1 dA$$

$$= -\rho V \iint_0 dA + \rho(V+u_2)\iint_2 dA = -\rho V \iint_0 dA + \rho(V+u_3)\iint_3 dA$$

$$\Rightarrow \quad \rho A_0 V = \rho A_R (V+u_1) = \rho A_R (V+u_2) = \rho A_3 (V+u_3) \qquad (4)$$

$$\Rightarrow \quad u_1 = u_2 \qquad (5)$$



  - No change of speed across rotor/propeller disc, but change in pressure

Momentum Theory | Derivation 2

- Conservation of momentum
  - Unconstrained flow:
    → net pressure force $\iint p\,\vec{n}dA$ is zero

$$F_{Thrust} = -\rho V^2 \iint_0 dA + \rho (V+u_3)^2 \iint_3 dA$$

$$= -\rho A_0 V^2 + \rho A_3 (V+u_3)^2$$

With eq. (4) and (5)

$$= \rho A_R (V+u_1) u_3 \qquad (6)$$

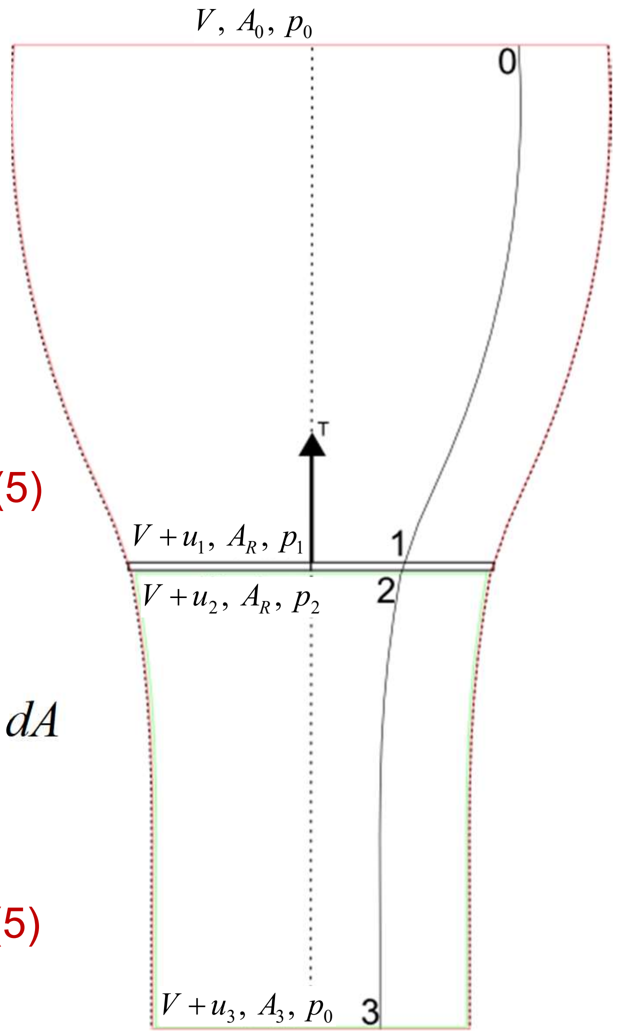
- Conservation of energy

$$P_{Thrust} = F_{Thrust}(V+u_1) = -\frac{1}{2}\rho V^3 \iint_0 dA + \frac{1}{2}\rho (V+u_3)^3 \iint_3 dA$$

$$= -\frac{1}{2}\rho A_0 V^3 + \frac{1}{2}\rho A_3 (V+u_3)^3$$

With eq. (4) and (5)

$$= \frac{1}{2}\rho A_R (V+u_1)(2V+u_3) u_3 \qquad (7)$$



Momentum Theory | Derivation 3

- Comparison between momentum and energy
  - Far wake slipstream velocity is twice the induced velocity

$$u_3 = 2u_1 \qquad (8)$$

With eq. (6) and (7)

  - Thrust force formula

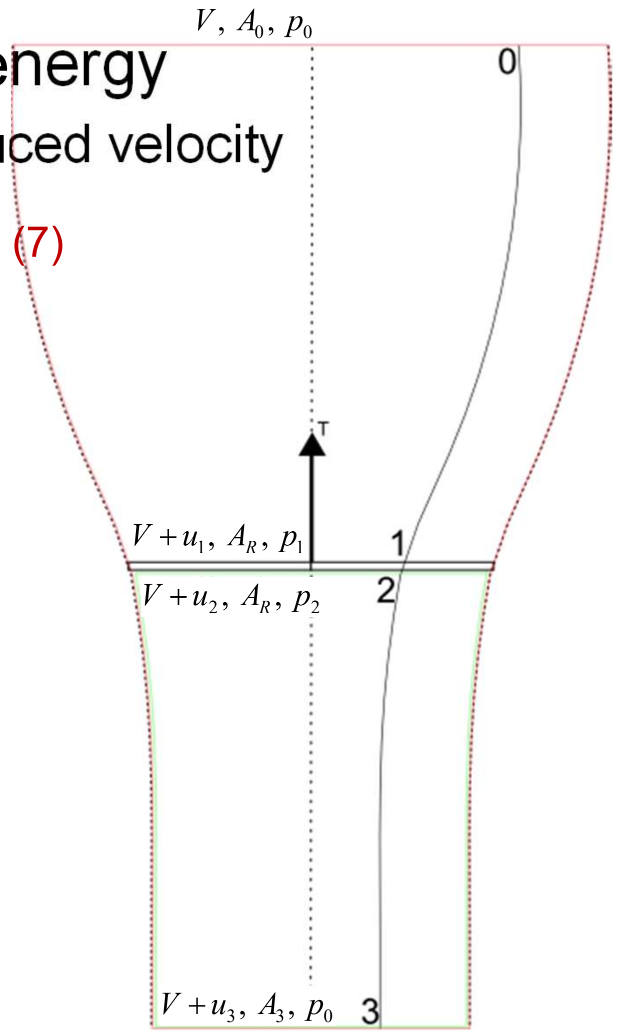
$$F_{Thrust} = 2\rho A_R \left(V + u_1\right) u_1 \qquad (9)$$

- Hover case ($V = V_0 = 0$)
  - Thrust force

$$F_{Thrust} = 2\rho A_R u_1^2 \qquad (10)$$

  - Slipstream tube

$$A_0 = \infty \quad ; \quad A_3 = \frac{A_R}{2} \qquad (11)$$



Momentum Theory | Power consumption in hover

- Ideal power used to produce thrust

$$P = F_{Thrust}\left(V + u_1\right) \qquad (11)$$

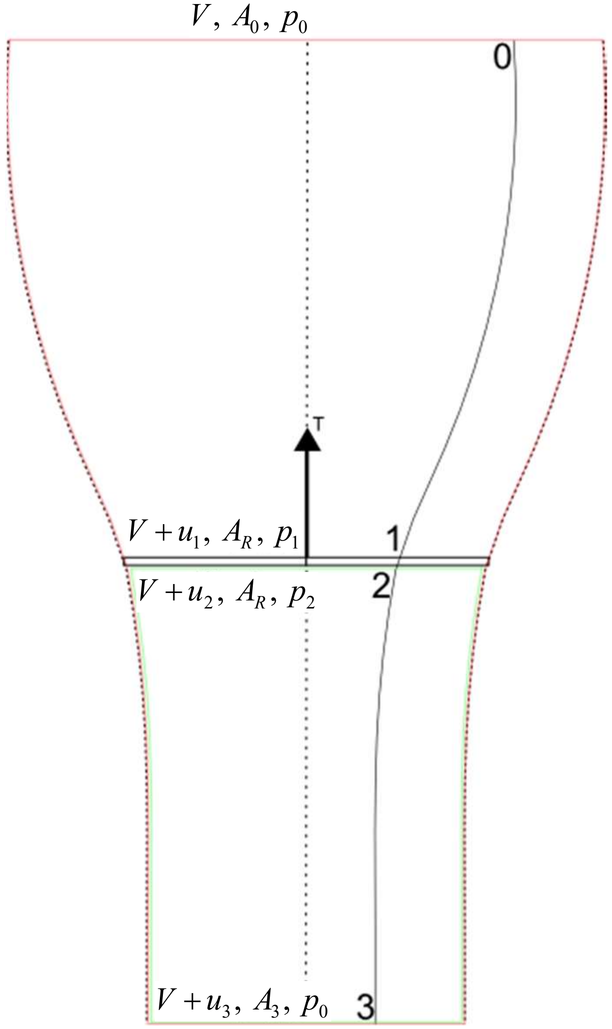
With eq. (10)

  - Hover case

$$P = \frac{F_{Thrust}^{3/2}}{\sqrt{2\rho A_R}} = \frac{(mg)^{3/2}}{\sqrt{2\rho A_R}} \qquad (12)$$

$$F_{Thrust} = mg \qquad (13)$$

    - Power depends on disc loading F_{Thrust} / A_R
  - Decreasing disc loading reduces power
    - Mechanical constraint: Tip *Mach Number*
    - More profile/structural drag
    - Longer tail
    - …



Propeller/Rotor Efficiency

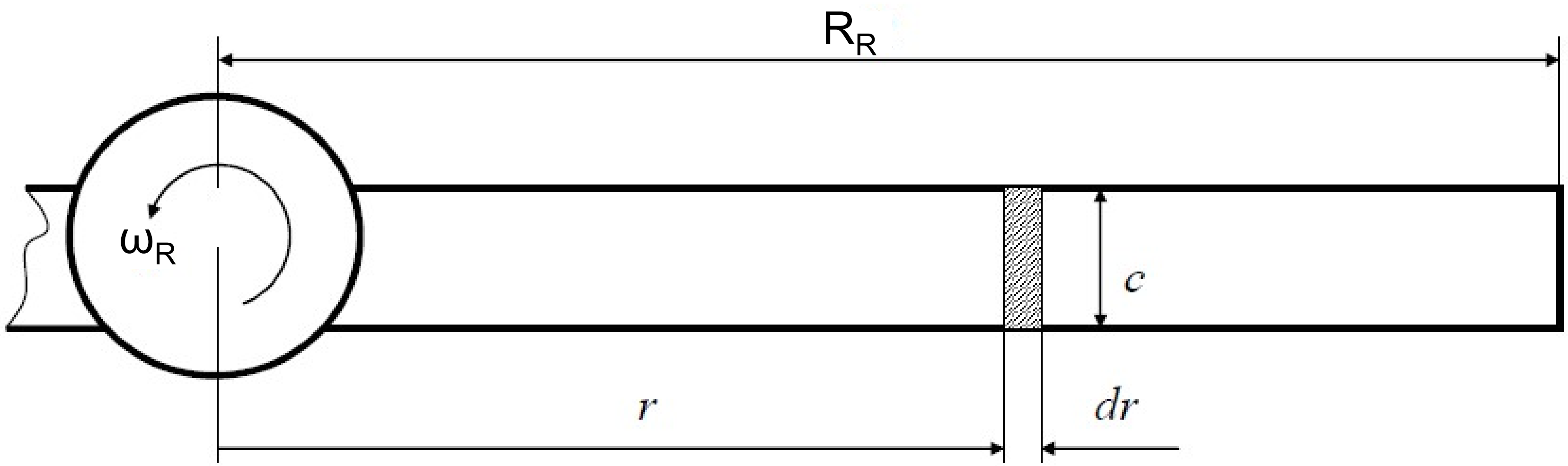
- Defining the efficiency of a rotor
  - Figure of Merit FM

$$FM = \frac{\text{Ideal power required to hover}}{\text{Actual power required to hover}} < 1$$

  - Ideal power: Can be calculated using the momentum theory
  - Actual power: Includes profile drag, blade-tip vortex, …
- FM can be used to compare different propellers with the same disc loading

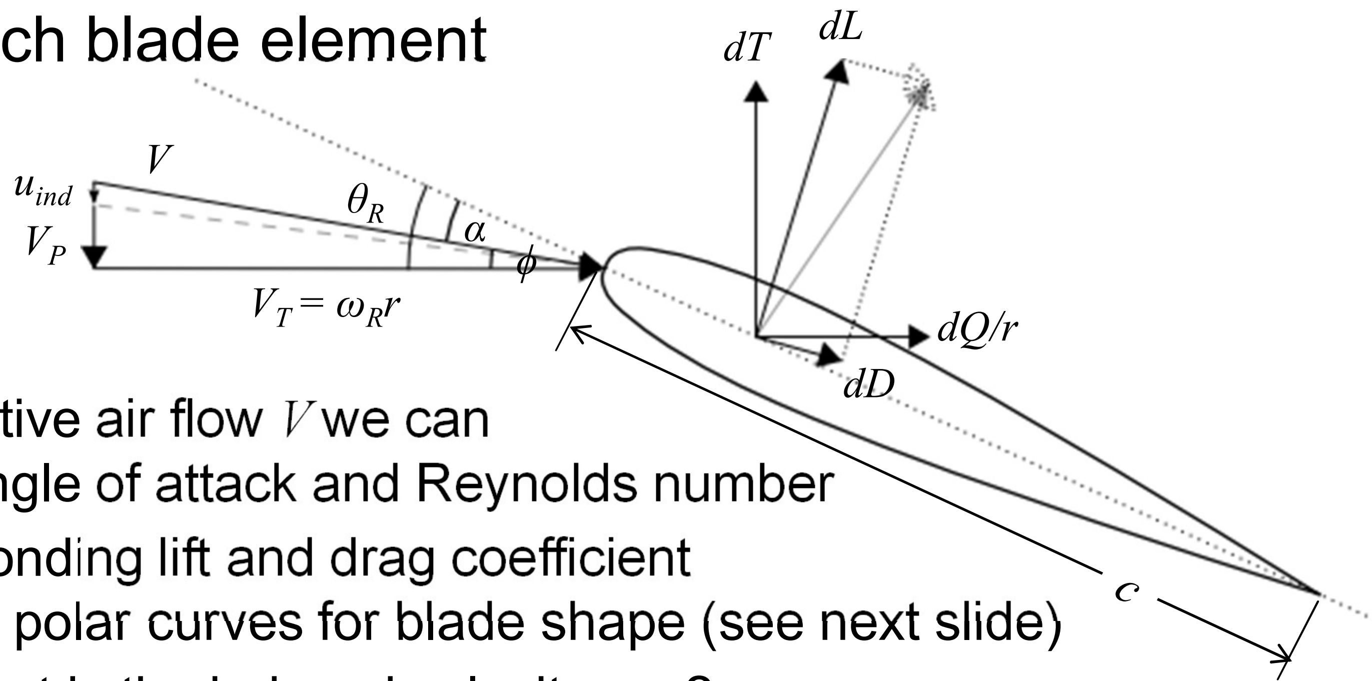
BEMT (Blade Elemental and Momentum Theory)

- Combined Blade Elemental and Momentum Theory
  - Used for axial flight analysis
  - Divide rotor into different blade elements (dr)
    - Use 2D blade element analysis
  - Calculate forces for each element and sum them up



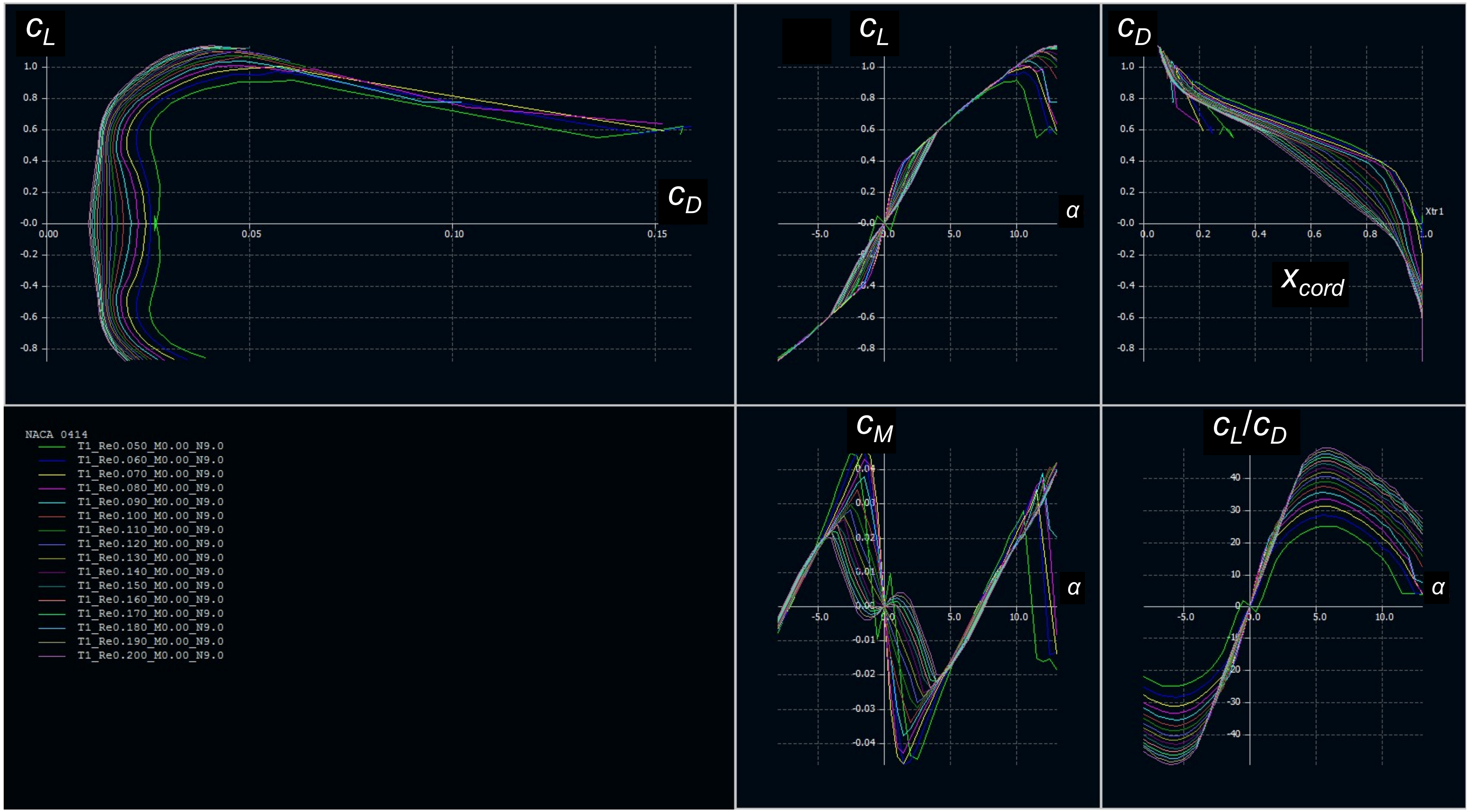
BEMT | introduction

- Forces at each blade element

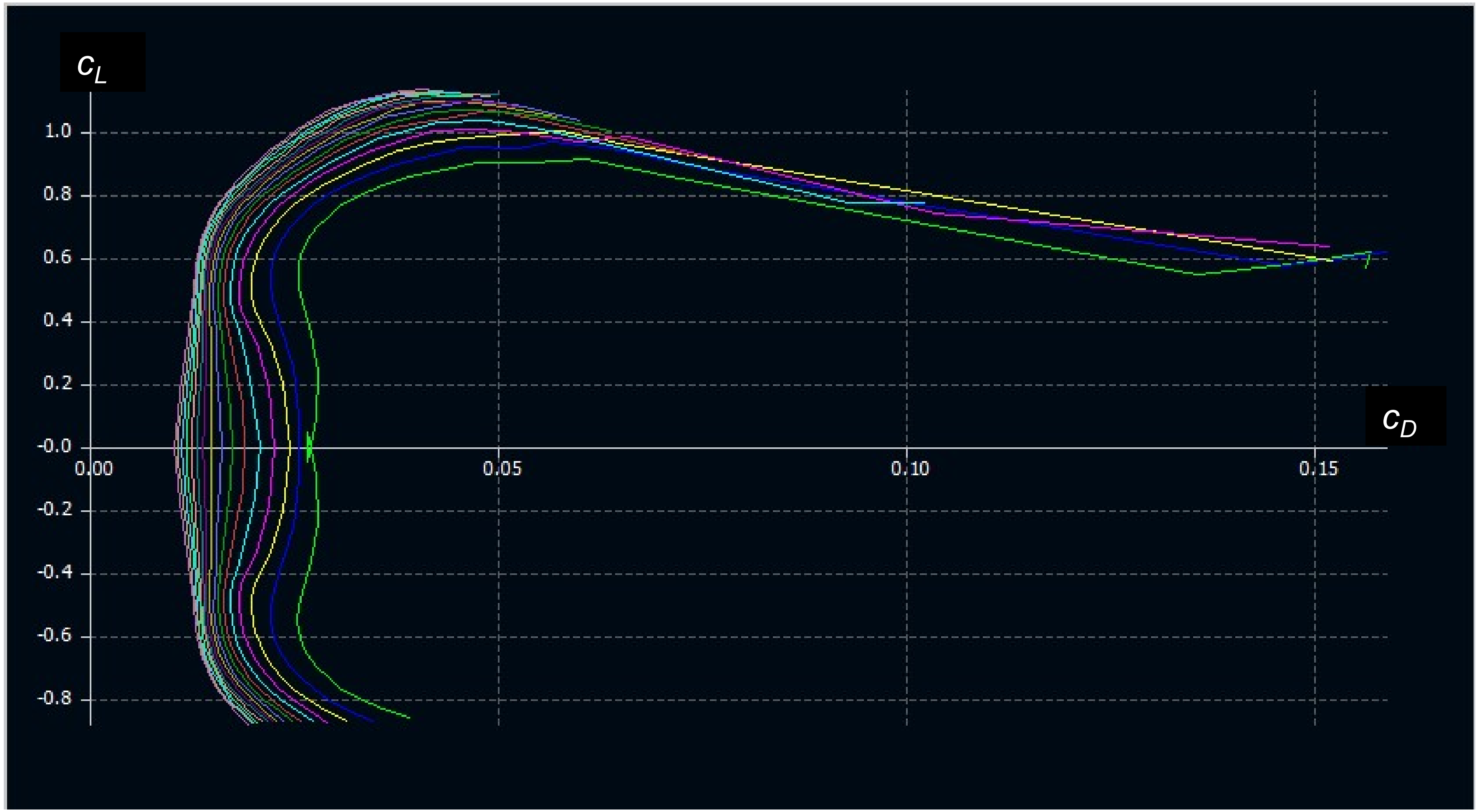


- With the relative air flow V we can determine angle of attack and Reynolds number
- The corresponding lift and drag coefficient are found on polar curves for blade shape (see next slide)
- Problem: What is the induced velocity u_{ind}?
  - Calculation of angle of attack (AoA) depends on axial velocity V_P and induced velocity u_{ind} (influence of profile)
  - Axial velocity $V_P' = V_P + u_{ind}$
  - Can be calculated with the help of the Momentum Theory

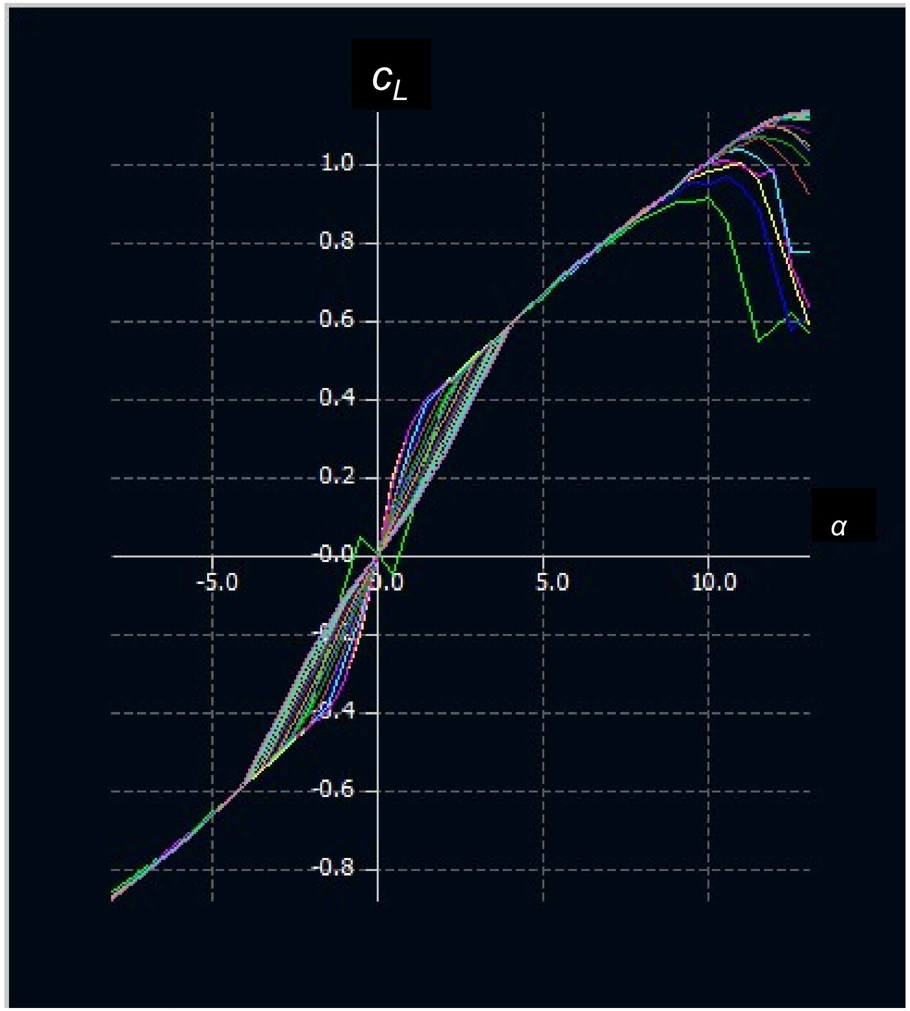
BEMT | Example of Lift and Drag Coefficient

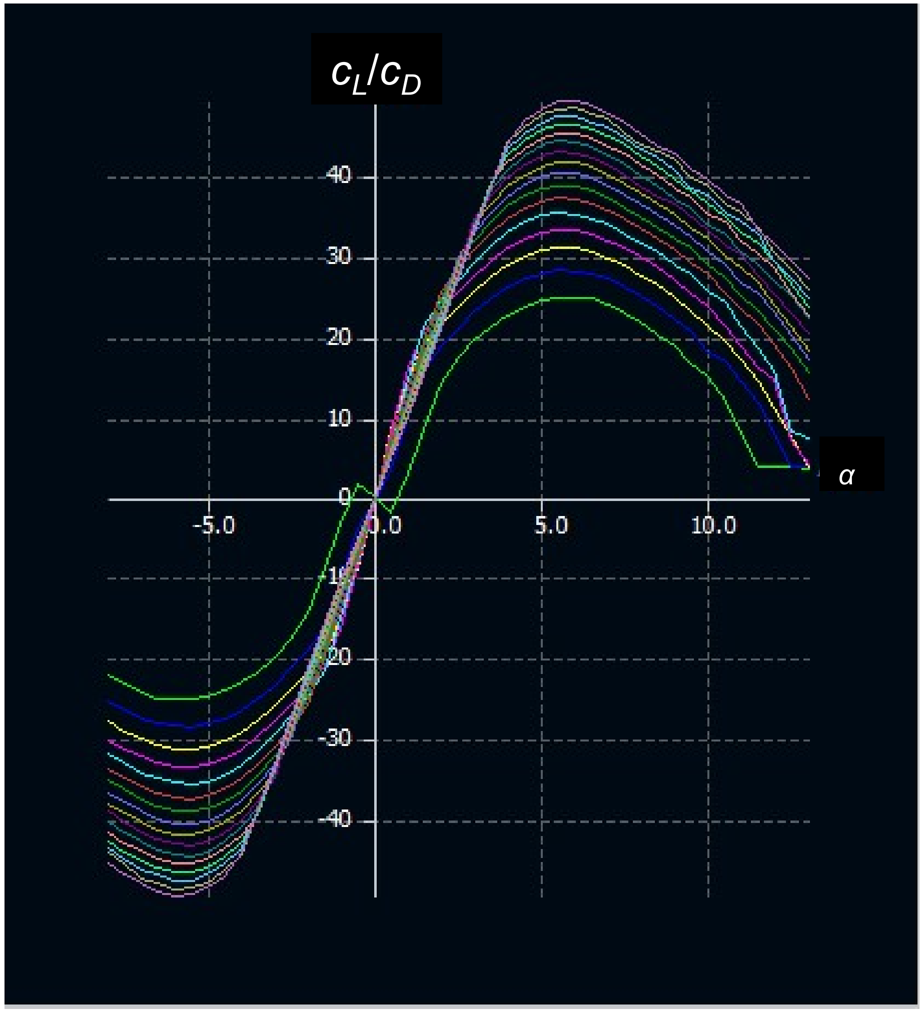


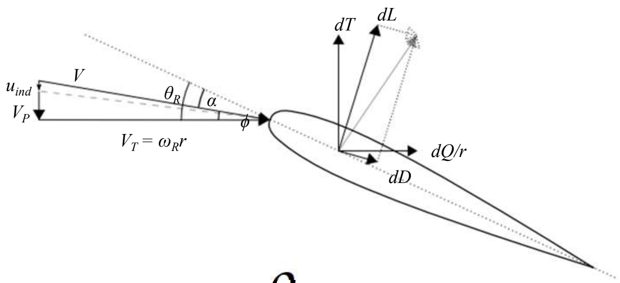
BEMT | Example of Lift and Drag Coefficient



BEMT | Example of Lift and Drag Coefficient







BEMT | blade element

- Lift and drag at blade element $$dL = \frac{\rho}{2} C_L c dr V^2 \qquad dD = \frac{\rho}{2} C_L c dr V^2$$
- Thrust and drag torque element $$dT = N_b (dL \cos\phi - dD \sin\phi)$$ $$dQ = N_b (dL \sin\phi - dD \cos\phi) r$$
  - N_b: Number of blades
  - V_T: $\omega_R r$
  - c : *cord length*
- Approximation (small angles) $$V \approx V_T \qquad \phi \approx \frac{V_P'}{V_T}$$ $$dT \approx N_b dL$$
- Describing the lift coefficient
  - Assume a linear relationship with AoA (angle of attack) $$C_L = C_{L\alpha}(\alpha - \alpha_0)$$ α_0: zero lift angle of attack. Used for asymmetric profiles
    - Thin airfoil theory (plate) $$C_{L\alpha} = 2\pi$$
    - Experimental results $$C_{L\alpha} = 5.7$$
    - Linearization of polar — Linearize polar for Reynolds number at 2/3 R
- Thrust at blade element $$dT_{be} = N_b \frac{\rho}{2} C_{L\alpha}\left(\theta_R - \frac{V_P'}{V_T} - \alpha_0\right) c dr V_T^2 \qquad (14)$$

BEMT | Momentum Theory

- $V_p{}'$ is still unknown
  - Use momentum theory at each blade annulus to estimate the induced velocity
  - Equate thrust from blade element theory and momentum theory
    - Solve equation for $V_p{}'$
  - With solved $V_p{}'$ calculate the thrust and drag torque with the blade element theory
  - Integrate dT and dQ over the whole blade

$$dT_{mt} = 2\rho(V_P + u_{ind})u_{ind}dA = 4\pi\rho V_P^{'}(V_P^{'} - V_P)rdr \quad (15)$$

$$dT_{mt} = dT_{be}$$

$$x = \frac{r}{R_R}, \quad \lambda = \frac{V_P^{'}}{\omega_R R_R}, \quad \lambda_V = \frac{V_P}{\omega_R R_R}, \quad \sigma = \frac{N_b c}{\pi R_R}$$

With eq. (14) and (15)

$$\lambda^2 + \lambda\left(\frac{\sigma C_{L\alpha}}{8} - \lambda_V\right) - \frac{\sigma C_{L\alpha}}{8}(\theta_R - \alpha_0)x = 0$$

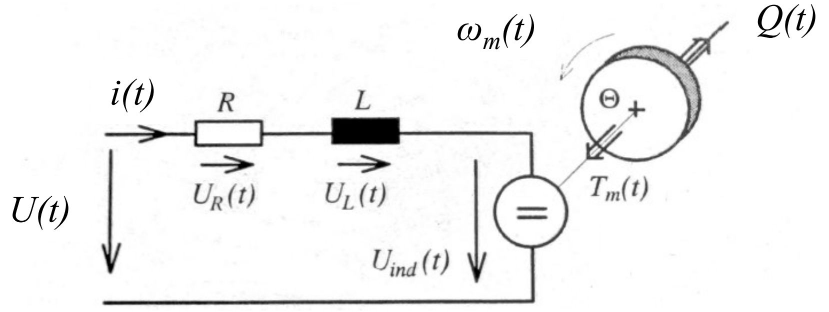
$$\Rightarrow \alpha \text{ and/or } u_{ind}$$

$$T = \sum dT = \sum N_b(dL\cos\phi - dD\sin\phi)$$

$$Q = \sum dQ = \sum N_b(dL\sin\phi - dD\cos\phi)r$$

Modeling of Rotorcraft | Motor Dynamics

- Use DC motor model
  - Consists of a mechanical and an electrical system
  - Mechanical system
    - Change of rotational speed depends on load Q and generated motor torque T_m
      - $I_m d\omega_m/dt = T_m - Q$
    - Generated torque due to electromagnetic field in coil
      - $T_m = k_t i$
      - k_t: Torque constant



Modeling of Rotorcraft | Motor Dynamics

- Electrical System
  - Voltage balance
    - $L di/dt = U - Ri - U_{ind}$
  - Induced voltage due to back electromagnetic field from the rotation of the static magnet
    - $U_{ind} = k_t \omega_m$
  - Motor is a second order system
    - Torque constant k_t given by the manufacturer
    - Electrical dynamics are usually much faster than the mechanical
      - Approximate as first order system