Robot Dynamics HS18

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$$\boxed{\begin{array}{c}(\sin(x))' = -\cos(x)\\ (\cos(x))' = \sin(x)\end{array}}$$

1 Parametrizations

1.1 Position and velocity

For every position parametrization, there is a linear mapping between linear velocities $\dot{\boldsymbol{r}}$ and derivatives of the representation $\dot{\boldsymbol{\chi}}$.
$\dot{\boldsymbol{r}} = \boldsymbol{E}_P(\boldsymbol{\chi}_P)\,\dot{\boldsymbol{\chi}}_P,\ \dot{\boldsymbol{\chi}}_P = \boldsymbol{E}_P(\boldsymbol{\chi}_P)^{-1}\,\dot{\boldsymbol{r}}$

Cartesian Coordinates: $\boldsymbol{E}_{P_c} = \mathbb{I}$
$\boldsymbol{\chi}_{P_c} = [x \quad y \quad z]^T,\ {}_{\mathcal{A}}\boldsymbol{r} = [x \quad y \quad z]^T$
Cylindrical coordinates:
$\boldsymbol{\chi}_{P_z} = [\rho \quad \theta \quad z]^T,$
${}_{\mathcal{A}}\boldsymbol{r} = [\rho\cos\theta \quad \rho\sin\theta \quad z]^T$

$$\boldsymbol{E}_{P_z} = \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & -\rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{E}_{P_z}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Spherical coordinates:
$\boldsymbol{\chi}_{P_s} = [r \quad \theta \quad \phi]^T,$
${}_{\mathcal{A}}\boldsymbol{r} = [r\cos\theta\sin\phi \quad r\sin\theta\cos\phi \quad z]^T$

$$\begin{bmatrix} \cos\theta\sin\phi & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\cos\theta\sin\phi & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta\sin\phi & \sin\phi\sin\theta & \cos\phi \\ -\sin\theta/(r\sin\phi) & \cos\theta/(r\sin\phi) & 0 \\ (\cos\phi\cos\theta)/r & (\cos\phi\sin\theta)/r & -\sin\phi/r \end{bmatrix}$$

1.2 Rotation

$\operatorname{atan2}(y,x) \coloneqq \operatorname{atan}(\frac{y}{x})$, checking for correct quadrant.
${}_{\mathcal{A}}\boldsymbol{u} = \mathbf{C}_{\mathcal{AC}} \cdot {}_{\mathcal{C}}\boldsymbol{u} = \mathbf{C}_{\mathcal{AB}}\mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\boldsymbol{u}$
$\mathbf{C}_{\mathcal{BA}} = \mathbf{C}_{\mathcal{AB}}^{-1} = \mathbf{C}_{\mathcal{AB}}^T$
Elementary rotations:

$$\mathbf{C}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

$$\mathbf{C}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$

$$\mathbf{C}_z = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler ZYZ (proper) angles:

$$\boldsymbol{\chi}_{R,ZYZ} = \begin{pmatrix} \operatorname{atan2}(c_{23}, c_{13}) \\ \operatorname{atan2}(\sqrt{c_{13}^2+c_{23}^2}, c_{33}) \\ \operatorname{atan2}(c_{32}, -c_{31}) \end{pmatrix}$$

Euler ZXZ (proper) angles:

$$\boldsymbol{\chi}_{R,ZXZ} = \begin{pmatrix} \operatorname{atan2}(c_{13}, -c_{23}) \\ \operatorname{atan2}(\sqrt{c_{13}^2+c_{23}^2}, c_{33}) \\ \operatorname{atan2}(c_{31}, c_{32}) \end{pmatrix}$$

Euler ZYX (Tait-Bryan) angles:

$$\boldsymbol{\chi}_{R,ZYX} = \begin{pmatrix} \operatorname{atan2}(c_{21}, c_{11}) \\ \operatorname{atan2}(-c_{31}, \sqrt{c_{32}^2+c_{33}^2}) \\ \operatorname{atan2}(c_{32}, c_{33}) \end{pmatrix}$$

Euler XYZ (Cardan) angles:

$$\boldsymbol{\chi}_{R,XYZ} = \begin{pmatrix} \operatorname{atan2}(-c_{23}, c_{33}) \\ \operatorname{atan2}(c_{13}, \sqrt{c_{11}^2+c_{12}^2}) \\ \operatorname{atan2}(c_{12}, -c_{11}) \end{pmatrix}$$

Angle-axis:

$$\boldsymbol{\chi}_{R,AA} = \begin{pmatrix} \theta \\ \boldsymbol{n} \end{pmatrix},\ \boldsymbol{n} = \frac{1}{2\sin(\theta)} \cdot \begin{pmatrix} c_{32}-c_{23} \\ c_{31}-c_{13} \\ c_{21}-c_{12} \end{pmatrix},$$

$$\theta = \operatorname{acos}(\frac{c_{11}+c_{22}+c_{33}-1}{2}),\ \boldsymbol{\varphi} = \theta \cdot \mathbf{n}$$

Unit Quaternions:
$\boldsymbol{\chi}_{R,quat} = \boldsymbol{\xi} = \begin{pmatrix} \xi \\ \check{\boldsymbol{\xi}} \end{pmatrix},\ \boldsymbol{\xi}^{-1} = \begin{pmatrix} \xi \\ -\check{\boldsymbol{\xi}} \end{pmatrix}$
$\xi_0 = \cos(\theta/2), \quad \check{\boldsymbol{\xi}} = \boldsymbol{n} \cdot \sin(\theta/2)$

$$\boldsymbol{\chi}_{R,quat} = \frac{1}{2}\begin{pmatrix} \sqrt{c_{11}+c_{22}+c_{33}+1} \\ \operatorname{sgn}(c_{32}-c_{23})\sqrt{c_{11}-c_{22}-c_{33}+1} \\ \operatorname{sgn}(c_{13}-c_{31})\sqrt{c_{22}-c_{11}-c_{33}+1} \\ \operatorname{sgn}(c_{21}-c_{12})\sqrt{c_{33}-c_{11}-c_{22}+1} \end{pmatrix}$$

$$\boldsymbol{\xi}_{\mathcal{AB}} \otimes \boldsymbol{\xi}_{\mathcal{BC}} = \begin{bmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix}_{\mathcal{AB}} \begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}_{\mathcal{BC}}$$

$$\begin{pmatrix} 0 \\ {}_{\mathcal{A}}\boldsymbol{r} \end{pmatrix} = \boldsymbol{\xi}_{\mathcal{AB}} \otimes \begin{pmatrix} 0 \\ {}_{\mathcal{B}}\boldsymbol{r} \end{pmatrix} \otimes \boldsymbol{\xi}_{\mathcal{AB}}^{-1}$$

1.3 Angular Velocity

$$[{}_{\mathcal{A}}\boldsymbol{\omega}_{AB}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \dot{\mathbf{C}}_{\mathcal{AB}}\mathbf{C}_{\mathcal{AB}}^T$$

${}_{\mathcal{A}}\boldsymbol{\omega}_{AB} = \boldsymbol{E}_R(\boldsymbol{\chi}_R)\,\dot{\boldsymbol{\chi}}_R$

1.4 Transformations

$$\begin{pmatrix} {}_{\mathcal{A}}\boldsymbol{r}_{AP} \\ 1 \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{C}_{\mathcal{AB}} & {}_{\mathcal{A}}\boldsymbol{r}_{AB} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}}_{\mathbf{T}_{\mathcal{AB}}} \begin{pmatrix} {}_{\mathcal{B}}\boldsymbol{r}_{BP} \\ 1 \end{pmatrix}$$

$$\mathbf{T}_{\mathcal{AB}}^{-1} = \begin{bmatrix} \mathbf{C}_{\mathcal{AB}}^T & \overbrace{-\mathbf{C}_{\mathcal{AB}}^T\,{}_{\mathcal{A}}\boldsymbol{r}_{AB}}^{{}_{\mathcal{B}}\boldsymbol{r}_{BA}} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

2 Kinematics

2.1 Velocity in rigid bodies

- $\boldsymbol{v}_P$: abs. velocity of P
- $\boldsymbol{a}_P$: abs. acceleration of P
- $\boldsymbol{\Omega}_{\mathcal{B}} = {}_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{B}}$: angular vel. of frame $\mathcal{B}$
- $\boldsymbol{\Psi}_{\mathcal{B}} = \dot{\boldsymbol{\Omega}}_{\mathcal{B}}$: angular accel. of frame $\mathcal{B}$

$${}_{\mathcal{A}}\boldsymbol{v}_{AP} = {}_{\mathcal{A}}(\dot{\boldsymbol{r}}_{AP}) = {}_{\mathcal{A}}\boldsymbol{v}_{AB} + {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} \times {}_{\mathcal{A}}\boldsymbol{r}_{BP}$$

In general, unless $\mathcal{C}$ is an inertial frame:
${}_{\mathcal{C}}\boldsymbol{v}_{AP} = {}_{\mathcal{C}}(\dot{\boldsymbol{r}}_{AP}) \neq \frac{\mathrm{d}}{\mathrm{d}t}({}_{\mathcal{C}}\boldsymbol{r}_{AP})$
In rigid body formulation:

$$\boldsymbol{v}_P = \boldsymbol{v}_B + \Omega \times \boldsymbol{r}_{BP}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_B + \Psi \times \boldsymbol{r}_{BP} + \Omega \times (\Omega \times \boldsymbol{r}_{BP})$$

In a kinematic chain:

$${}_{\mathcal{I}}\boldsymbol{v}_{IE} = {}_{\mathcal{I}}\boldsymbol{\omega}_{I1} \times {}_{\mathcal{I}}\boldsymbol{r}_{12} + ... + {}_{\mathcal{I}}\boldsymbol{\omega}_{In} \times {}_{\mathcal{I}}\boldsymbol{r}_{nE}$$

$${}_{\mathcal{I}}\boldsymbol{\omega}_{IE} = {}_{\mathcal{I}}\boldsymbol{\omega}_{I1} + {}_{\mathcal{I}}\boldsymbol{\omega}_{12} + ... + {}_{\mathcal{I}}\boldsymbol{\omega}_{nE}$$

2.2 Forward kinematics

$$\mathbf{T}_{\mathcal{IE}}(\boldsymbol{q}) = \mathbf{T}_{\mathcal{I}0}\left(\prod_{k=1}^{n_j}\mathbf{T}_{k-1,k}(q_k)\right)\mathbf{T}_{n_j\mathcal{E}}$$

2.3 Analytical Jacobian

$$\dot{\chi}(\mathbf{q}) = \frac{\partial\boldsymbol{\chi}}{\partial\mathbf{q}}\dot{\mathbf{q}} = J_A(\mathbf{q}) \cdot \dot{\mathbf{q}} = \begin{bmatrix} \frac{\partial\boldsymbol{\chi}_{pos}}{\partial\mathbf{q}} \\ \frac{\partial\boldsymbol{\chi}_{rot}}{\partial\mathbf{q}} \end{bmatrix}\dot{\mathbf{q}}$$

2.4 Geometric Jacobian

$$\mathbf{w}_E = \begin{bmatrix} \mathbf{v}_E \\ \omega_E \end{bmatrix} = J_0(\mathbf{q})\dot{\mathbf{q}}$$

$$J_0(\mathbf{q}) = \begin{bmatrix} J_{0,P} \\ J_{0,R} \end{bmatrix} = \begin{bmatrix} \mathbf{n}_1 \times \mathbf{r}_{1,E} & \dots & \mathbf{n}_n \times \mathbf{r}_{n,E} \\ \mathbf{n}_1 & \dots & \mathbf{n}_n \end{bmatrix}$$

2.5 Inverse differential kinematics

$$\mathbf{w}_E = J\dot{\mathbf{q}} \Rightarrow \dot{\mathbf{q}} = J^+\mathbf{w}_E$$

where $J^+ = J^T(JJ^T)^{-1}$ (Moore-Penrose). However we risk encountering singular configurations $\mathbf{q}_s$ where $rank(J(\mathbf{q}_s)) < m_0$, m_0 being the number of operational-space coordinates. Here J is badly conditioned. We can mitigate this by using a redundant robot to carefully avoid singularities, and/or by damping the pseudo-inverse:

$$\dot{\mathbf{q}} = J^T(JJ^T + \lambda^2\mathbb{I})^{-1}\mathbf{w}_E$$

Now the pseudo-inverse minimizes $||\mathbf{w}_E^* - J\dot{\mathbf{q}}||^2 + \lambda^2||\dot{\mathbf{q}}||^2$ instead of just $||\mathbf{w}_E^* - J\dot{\mathbf{q}}||^2$, so convergence is slower but more stable for larger λ.

In a redundant configuration q^* where $rank(J(\mathbf{q}^*)) < n$, the pseudoinverse minimizes $||\dot{\mathbf{q}}||^2$ while satisfying $\mathbf{w}_E^* = J\dot{\mathbf{q}}$ by using

$$J(J^+\mathbf{w}_E^* + N\dot{\mathbf{q}}_0) = \mathbf{w}_E^* \quad \forall\dot{\mathbf{q}}_0$$

where $N = \mathbb{I} - J^+J$.

2.6 Multi-task IDK

Given n_t tasks $\{J_i, \mathbf{w}_i^*\}$, we have:

$$\dot{\mathbf{q}} = \begin{bmatrix} J_1 \\ \vdots \\ J_{n_t} \end{bmatrix}^+ \begin{pmatrix} \mathbf{w}_1^* \\ \vdots \\ \mathbf{w}_{n_t}^* \end{pmatrix}$$

In case the row-rank of the stacked Jacobian is greater that the column-rank, we are only minimizing $||\bar{\mathbf{w}} - \bar{J}\dot{\mathbf{q}}||^2$. We can weigh the tasks with

$$\bar{J}^{+W} = (\bar{J}^TW\bar{J})^{-1}\bar{J}W$$

where $W = diag(w_1, ..., w_m)$ and we minimize $||W^{1/2}(\bar{\mathbf{w}} - \bar{J}\dot{\mathbf{q}})||^2$.

Task Prioritization

$$\dot{\mathbf{q}} = J_1^+\mathbf{w}_1^* + N_1\dot{\mathbf{q}}_0$$

$$\mathbf{w}_2 = J_2\dot{\mathbf{q}} = J_2(J_1^+\mathbf{w}_1^* + N_1\dot{\mathbf{q}}_0)$$

$$\Rightarrow \dot{\mathbf{q}}_0 = (J_2N_1)^+(\mathbf{w}_2^* - J_2J_1^+\mathbf{w}_1^*)$$

$$\Rightarrow \dot{\mathbf{q}} = J_1^+\mathbf{w}_1^* + N_1(J_2N_1)^+(\mathbf{w}_2^* - J_2J_1^+\mathbf{w}_1^*)$$

In general:

$$\dot{\mathbf{q}} = \sum_{i=1}^{n_t}\bar{N}_i\dot{\mathbf{q}}_i$$

$$\dot{\mathbf{q}}_i = (J_i\bar{N}_i)^+\left(\mathbf{w}_i^* - J_i\sum_{k=1}^{i-1}\bar{N}_k\dot{\mathbf{q}}_k\right)$$

2.7 Inverse Kinematics

1. $\mathbf{q} \leftarrow \mathbf{q}^0$
2. While $||\boldsymbol{\chi}_e^* \boxminus \boldsymbol{\chi}_e(\mathbf{q})|| > tol$ do
3. $\quad J_A \leftarrow J_A(\mathbf{q}) = \frac{\partial\boldsymbol{\chi}_e}{\partial\mathbf{q}}(\mathbf{q})$
4. $\quad J_A^+ \leftarrow (J_A)^+$
5. $\quad \Delta\boldsymbol{\chi}_e \leftarrow \boldsymbol{\chi}_e^* \boxminus \boldsymbol{\chi}_e(\mathbf{q})$
6. $\quad \mathbf{q} \leftarrow \mathbf{q} + J_A^+\Delta\boldsymbol{\chi}_e$

One issue is that for very large errors $\Delta\boldsymbol{\chi}_e$, we get too imprecise. We can avoid this by scaling the update with a factor $0 < k < 1$: $\mathbf{q} \leftarrow \mathbf{q} + kJ_A^+\Delta\boldsymbol{\chi}_e$. But we still have issues inverting J_A in singular configurations. An alternative is $\mathbf{q} \leftarrow \mathbf{q} + \alpha J_A^T\Delta\boldsymbol{\chi}_e$, which converges for small α. We must also appropriately compute the difference $\boldsymbol{\chi}_e^* \boxminus \boldsymbol{\chi}_e(\mathbf{q})$ depending on the parametrization. For cartiesian coordinates, this is regular vector subtraction. Also note that with cartesian coordinates $J_{0,P} = J_{A,P}$. For rotational difference we can extract the rotation vector $\Delta\boldsymbol{\varphi}$ from the "rotation difference", and use that for the update:

$$\mathbf{C}_{\mathcal{GS}}(\Delta\boldsymbol{\varphi}) = \mathbf{C}_{\mathcal{GI}}(\boldsymbol{\varphi}^*)\mathbf{C}_{\mathcal{SI}}(\boldsymbol{\varphi}^t)$$

$$\mathbf{q} \leftarrow \mathbf{q} + k_{pR}J_{0,R}^+\Delta\boldsymbol{\varphi}$$

2.8 Trajectory control

Position: with $\Delta\mathbf{r}_e^t = \mathbf{r}_e^*(t) - \mathbf{r}_e(\mathbf{q}^t)$

$$\dot{\mathbf{q}}^* = J_{e0P}^+(\mathbf{q}^t)(\dot{\mathbf{r}}_e^*(t) + k_{pP}\Delta\mathbf{r}_e^t)$$

Orientation: with $\Delta\boldsymbol{\varphi}$ as above,

$$\dot{\mathbf{q}}^* = J_{e0R}^+(\mathbf{q}^t)(\boldsymbol{\omega}_e^*(t) + k_{pR}\Delta\boldsymbol{\varphi})$$

3 Dynamics

$$\boxed{\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} + \mathbf{J}_c(\mathbf{q})^T\mathbf{F}_c}$$

- $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n_q\times n_q}$ Mass matrix ($\perp$).
- $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}} \in \mathbb{R}^{n_q}$ Gen. pos., vel., accel.
- $\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{n_q}$ Coriolis and centrifugal terms
- $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^{n_q}$ Gravity terms
- $\boldsymbol{\tau} \in \mathbb{R}^{n_q}$ External generalized forces
- $\mathbf{F}_c \in \mathbb{R}^{3\times n_c}$ External cartesian forces
- $\mathbf{J}_c(\mathbf{q}) \in \mathbb{R}^{n_c\times n_q}$ Geometric Jacobian of location where external forces apply

$$\begin{pmatrix} \mathbf{v}_s \\ \boldsymbol{\Omega} \end{pmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix}\dot{\mathbf{q}}$$

$$\begin{pmatrix} \mathbf{a}_s \\ \boldsymbol{\Psi} \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{v}}_s \\ \dot{\boldsymbol{\Omega}} \end{pmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix}\ddot{\mathbf{q}} + \begin{bmatrix} \dot{J}_P \\ \dot{J}_R \end{bmatrix}\dot{\mathbf{q}}$$

Newton-Euler method

- m body mass
- $\mathbf{\Theta}_S$ inertia matrix around CoG
- $\mathbf{p}_S = m\mathbf{v}_S$ linear momentum
- $\mathbf{N}_S = \mathbf{\Theta}_S \cdot \mathbf{\Omega}$ angular momentum around CoG
- $\dot{\mathbf{p}} = m\mathbf{a}_S$ change in linear momentum
- $\dot{\mathbf{N}}_S = \mathbf{\Theta}_S \cdot \mathbf{\Psi} + \mathbf{\Omega} \times \mathbf{\Theta}_S \cdot \Omega$ change in angular momentum

Cut each link free as a single rigid body, and introduce constraint forces $\mathbf{F}_i$ acting on the body at the joint. Then apply conservation of linear and angular momentum in all DoFs subject to all external forces (*including* contraints $\mathbf{F}_i$):

$$\dot{\mathbf{p}}_S = \mathbf{F}_{ext,S}$$
$$\dot{\mathbf{N}}_S = \mathbf{T}_{ext}$$

For calculations all quantities must be in the same coordinate system. For the inertia matrix we have ${}_{\mathcal{B}}\mathbf{\Theta} = \mathbf{C}_{\mathcal{BA}} \cdot {}_{\mathcal{A}}\mathbf{\Theta} \cdot \mathbf{C}_{\mathcal{BA}}^T$.

Lagrange method

Define the *Lagrangian function*:

$$\mathcal{L} \coloneqq \mathcal{T} - \mathcal{U}$$

Where $\mathcal{T}$ is the kinetic energy and $\mathcal{U}$ the potential energy. Then the *Euler-Lagrange equation of the second kind* holds for the total external generalized forces $\boldsymbol{\tau}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}}\right) = \boldsymbol{\tau}$$

The kinetic energy for a system of n_b bodies is defined as:

$$\begin{aligned}\mathcal{T} &\coloneqq \sum_{i=1}^{n_b} \left(\tfrac{1}{2} m_i \, {}_{\mathcal{A}}\dot{\boldsymbol{r}}_{S_i}^T \, {}_{\mathcal{A}}\dot{\boldsymbol{r}}_{S_i} + \tfrac{1}{2} \, {}_{\mathcal{B}}\boldsymbol{\Omega}_{S_i}^T \cdot {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\boldsymbol{\Omega}_{S_i} \right) \\ &= \frac{1}{2}\dot{\mathbf{q}}^T \underbrace{\left(\sum_{i=1}^{n_b} (J_{S_i}^T m J_{S_i} + J_{R_i}^T \mathbf{\Theta}_{S_i} J_{R_i}) \right)}_{\mathbf{M}(\mathbf{q})} \dot{\mathbf{q}} \\ &= \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}\end{aligned}$$

The potential energy is typically in the form of gravitational and elastic terms:

$$\mathcal{U} = \underbrace{-\sum_{i=1}^{n_b} \boldsymbol{r}_{S_i}^T (m_i g \cdot \mathbf{e}_g)}_{\text{gravitational}} + \underbrace{\sum_{j=1}^{n_E} \frac{1}{2} k_j (d(\mathbf{q}) - d_{0,j})^2}_{\text{elastic}}$$

Here we have n_E elastic components with coefficients k_j and rest configuration $d_{0,j}$.

Projected Newton-Euler Method

$$\mathbf{M} = \sum_{i=1}^{n_b} ({}_{\mathcal{A}}\boldsymbol{J}_{S_i}^T m \, {}_{\mathcal{A}}\boldsymbol{J}_{S_i} + {}_{\mathcal{B}}\boldsymbol{J}_{R_i}^T \, {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \, {}_{\mathcal{B}}\boldsymbol{J}_{R_i})$$

$$\begin{aligned}\mathbf{b} = \sum_{i=1}^{n_b} ({}_{\mathcal{A}}\boldsymbol{J}_{S_i}^T m \, {}_{\mathcal{A}}\dot{\boldsymbol{J}}_{S_i}\dot{\mathbf{q}} + {}_{\mathcal{B}}\boldsymbol{J}_{R_i}^T ({}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \, {}_{\mathcal{B}}\dot{\boldsymbol{J}}_{R_i}\dot{\mathbf{q}} \\ + {}_{\mathcal{B}}\boldsymbol{\Omega}_{S_i} \times {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \, {}_{\mathcal{B}}\boldsymbol{\Omega}_{S_i}))\end{aligned}$$

$$\mathbf{g} = \sum_{i=1}^{n_b} (-\,{}_{\mathcal{A}}\boldsymbol{J}_{S_i}^T \, {}_{\mathcal{A}}\boldsymbol{F}_{g,i})$$

4 Floating-base dynamics

Generalized coordinates are now $\mathbf{q} = [\mathbf{q}_b^T \; \mathbf{q}_j^T]^T$, where $\mathbf{q}_b$ are the generalized coordinates of the base (position and orientation). The generalized velocities are therefore no longer $\dot{\mathbf{q}}$, but are denoted $\mathbf{u} = [{}_{\mathcal{I}}\boldsymbol{v}_B^T \;\; {}_{\mathcal{B}}\boldsymbol{\omega}_{IB}^T \;\; \dot{\mathbf{q}}_j^T]^T$.

$$\boxed{\mathbf{M}(\mathbf{q})\dot{\mathbf{u}} + \mathbf{b}(\mathbf{q},\mathbf{u}) + \mathbf{g}(\mathbf{q}) = \mathbf{S}^T\boldsymbol{\tau} + \mathbf{J}_{ext}^T\mathbf{F}_{ext}}$$

- $\mathbf{u}, \dot{\mathbf{u}} \in \mathbb{R}^{n_u}$ Gen. vel., accel.
- $\mathbf{S}$ selection matrix of actuated joints
- $\mathbf{F}_{ext} \in \mathbb{R}^{3 \times n_c}$ External cartesian forces acting on robot
- $\mathbf{J}_{ext}(\mathbf{q}) \in \mathbb{R}^{n_c \times n_u}$ Geometric Jacobian of location where external forces apply

Position and velocity of a point Q on the robot:

$${}_{\mathcal{I}}\boldsymbol{r}_{IQ}(\mathbf{q}) = {}_{\mathcal{I}}\boldsymbol{r}_{IB}(\mathbf{q}) + \mathbf{C}_{\mathcal{IB}}(\mathbf{q}) \cdot {}_{\mathcal{B}}\boldsymbol{r}_{BQ}(\mathbf{q})$$
$${}_{\mathcal{I}}\boldsymbol{v}_Q = \underbrace{\left[\mathbb{I}_{3\times3} \;\; -\mathbf{C}_{\mathcal{IB}} \cdot [{}_{\mathcal{B}}\boldsymbol{r}_{BQ}]_\times \;\; \mathbf{C}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\boldsymbol{J}_{P_{q_j}}(\mathbf{q}_j) \right]}_{= {}_{\mathcal{I}}\boldsymbol{J}_Q(\mathbf{q})} \cdot \mathbf{u}$$

Contact kinematics

The point of contant C is not allowed to move: $\mathbf{r}_C = \mathit{const.}$ and $\dot{\mathbf{r}}_C = \ddot{\mathbf{r}}_C = \mathbf{0}$. Written in generalized coordinates these are:

$${}_{\mathcal{I}}\boldsymbol{J}_{C_i}\mathbf{u} = \mathbf{0}, \quad {}_{\mathcal{I}}\boldsymbol{J}_{C_i}\dot{\mathbf{u}} + {}_{\mathcal{I}}\dot{\boldsymbol{J}}_{C_i}\mathbf{u} = \mathbf{0}$$

We can therefore stack the constraint Jacobians:

$$\mathbf{J}_c = \begin{bmatrix} {}_{\mathcal{I}}\boldsymbol{J}_{C_1} \\ \vdots \\ {}_{\mathcal{I}}\boldsymbol{J}_{C_{n_c}} \end{bmatrix} \in \mathbb{R}^{3n_c \times (n_b + n_j)}$$

By using the nullspace projection $\mathbf{N}_c$ of $\mathbf{J}_c$ we can still move the system:

$$\mathbf{0} = \dot{\mathbf{r}} = \mathbf{J}_c\dot{\mathbf{q}} \quad \Rightarrow \quad \dot{\mathbf{q}} = \mathbf{J}_c^+\mathbf{0} + \mathbf{N}_c\dot{\mathbf{q}}_0$$
$$\mathbf{0} = \ddot{\mathbf{r}} = \mathbf{J}_c\ddot{\mathbf{q}} + \dot{\mathbf{J}}_c\dot{\mathbf{q}} \quad \Rightarrow \quad \ddot{\mathbf{q}} = \mathbf{J}_c^+(-\dot{\mathbf{J}}_c\dot{\mathbf{q}}) + \mathbf{N}_c\ddot{\mathbf{q}}_0$$

The contact Jacobian tells us how the system can move. If we partition it into the part relating to the base and the part relating to the joints:

- $\mathbf{J}_c = [\mathbf{J}_{c,b} \; \mathbf{J}_{c,j}]$
- $rank(\mathbf{J}_{c,b})$ is the number of constraints on the base $\rightarrow$ the number of controllable base DoFs.
- $rank(\mathbf{J}_c) - rank(\mathbf{J}_{c,b})$ is the number of contraints on the actuators.

A typical quadruped has 18 DoF (6 for base, 12 actuators). Each foot in contact with the ground adds 3 total constraints. One foot on the ground allows us to control 3 base DoFs, two feet 5, and three or more allow us to control all base DoFs.

Support-consistent dynamics

If we use **soft contacts** to model the contact, we simply introduce an external force acting on the robot:

$$\mathbf{F}_c = k_p(\mathbf{r}_c - \mathbf{r}_{c0}) + k_d\dot{\mathbf{r}}_c$$

However such problems are hard to accurately solve numerically (slow system dynamics, fast contact dynamics).

Instead it works better to use **hard contacts**. We impose the kinematic constraint ${}_{\mathcal{I}}\boldsymbol{J}_{C_i}\dot{\mathbf{u}} + {}_{\mathcal{I}}\dot{\boldsymbol{J}}_{C_i}\mathbf{u} = \mathbf{0}$ from above and calculate the resulting force and null-space matrix:

$$\mathbf{F}_c = (\mathbf{J}_c\mathbf{M}^{-1}\mathbf{J}_c^T)^{-1}(\mathbf{J}_c\mathbf{M}^{-1}(\mathbf{S}^T\tau - \mathbf{b} - \mathbf{g}) + \dot{\mathbf{J}}_c\mathbf{u})$$
$$\mathbf{N}_c = \mathbb{I} - \mathbf{M}^{-1}\mathbf{J}_c^T(\mathbf{J}_c\mathbf{M}^{-1}\mathbf{J}_c^T)^{-1}\mathbf{J}_c$$
$$\Rightarrow \boxed{\mathbf{N}_c^T(\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g}) = \mathbf{N}_c^T\mathbf{S}^T\boldsymbol{\tau}, \quad \mathbf{J}_c\mathbf{N}_c = \mathbf{0}}$$

By defining the *end-effector inertia* $\mathbf{\Lambda}_c = (\mathbf{J}_c\mathbf{M}_c^{-1}\mathbf{J}_c^T)^{-1}$ we can write the kinetic energy loss on impact:

$$\mathbf{u}^+ = \mathbf{N}_c\mathbf{u}^-$$
$$E_{loss} = \Delta E_{kin} = -\frac{1}{2}\Delta\mathbf{u}^T\mathbf{M}\Delta\mathbf{u} = -\frac{1}{2}\dot{\mathbf{r}}^{-T}\mathbf{M}\dot{\mathbf{r}}^-$$

5 Dynamic control

Joint impedance control

$$\boxed{\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}}$$

Torque as a function of position and velocity error:

$$\boldsymbol{\tau}^* = k_p(\mathbf{q}^* - \mathbf{q}) + k_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}})$$

Compensate for gravity by adding an estimated gravity term:

$$\boldsymbol{\tau}^* = k_p(\mathbf{q}^* - \mathbf{q}) + k_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}}) + \hat{\mathbf{g}}(\mathbf{q})$$

Inverse dynamics control

Compensate for system dynamics:

$$\boldsymbol{\tau} = \hat{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{q}}^* + \hat{\mathbf{b}}(\mathbf{q},\dot{\mathbf{q}}) + \hat{\mathbf{g}}(\mathbf{q})$$

If the model is exact, we have $\mathbb{I}\ddot{\mathbf{q}} = \ddot{\mathbf{q}}^*$, meaning we can perfectly control system dynamics. We could apply a PD-control law, making each joint behave like a mass-spring-damper with unitary mass:

$$\ddot{\mathbf{q}}^* = k_p(\mathbf{q}^* - \mathbf{q}) + k_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}})$$
$$\omega = \sqrt{k_p}, \quad D = \frac{k_d}{2\sqrt{k_p}}$$

Operational space control

Generalized framework to control motion and force. End-effector dynamics:

$$\begin{aligned}\mathbf{\Lambda}\dot{\mathbf{w}}_e + \boldsymbol{\mu} + \mathbf{p} &= \mathbf{F}_e \\ \mathbf{\Lambda} &= (\mathbf{J}_e\mathbf{M}^{-1}\mathbf{J}_e^T)^{-1} \\ \boldsymbol{\mu} &= \mathbf{\Lambda}\mathbf{J}_e\mathbf{M}^{-1}\mathbf{b} - \mathbf{\Lambda}\dot{\mathbf{J}}_e\dot{\boldsymbol{q}} \\ \mathbf{p} &= \mathbf{\Lambda}\mathbf{J}_e\mathbf{M}^{-1}\mathbf{g} \\ \boldsymbol{\tau} &= \mathbf{J}_e^T\mathbf{F}_e \\ \Rightarrow \boldsymbol{\tau}^* &= \hat{\mathbf{J}}^T(\hat{\mathbf{\Lambda}}\dot{\mathbf{w}}_e^* + \hat{\boldsymbol{\mu}} + \hat{\mathbf{p}})\end{aligned}$$

Hence we can steer the robot along any trajectory by determining the desired task-space end-effector acceleration:

$$\dot{\mathbf{w}}_e^* = k_p\mathbf{E}(\boldsymbol{\chi}_e^* \boxminus \boldsymbol{\chi}_e) + k_d(\mathbf{w}_e^* - \mathbf{w}_e) \quad \underbrace{+\dot{\mathbf{w}}_e^*(t)}_{\text{trajectory control}}$$

In order to also control the contact force, we use selection matrices:

$$\underline{\mathbf{F}_c} + \mathbf{\Lambda}\dot{\mathbf{w}}_e + \boldsymbol{\mu} + \mathbf{p} = \mathbf{F}_e$$
$$\boldsymbol{\tau}^* = \hat{\mathbf{J}}^T(\hat{\mathbf{\Lambda}}\underline{\mathbf{S}_M}\dot{\mathbf{w}}_e^* + \underline{\mathbf{S}_F\mathbf{F}_c} + \hat{\boldsymbol{\mu}} + \hat{\mathbf{p}})$$

Let $\mathbf{C}$ represent the rotation from the inertial frame to the contact force frame. The selection matrices can be calculated as:

$$\mathbf{\Sigma}_p = \begin{bmatrix} \sigma_{px} & 0 & 0 \\ 0 & \sigma_{py} & 0 \\ 0 & 0 & \sigma_{pz} \end{bmatrix}, \; \mathbf{\Sigma}_r = \begin{bmatrix} \sigma_{rx} & 0 & 0 \\ 0 & \sigma_{ry} & 0 \\ 0 & 0 & \sigma_{rz} \end{bmatrix}$$
$$\mathbf{S}_M = \begin{bmatrix} \mathbf{C}^T\mathbf{\Sigma}_p\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^T\mathbf{\Sigma}_r\mathbf{C} \end{bmatrix}$$
$$\mathbf{S}_F = \begin{bmatrix} \mathbf{C}^T(\mathbb{I} - \mathbf{\Sigma}_p)\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^T(\mathbb{I} - \mathbf{\Sigma}_r)\mathbf{C} \end{bmatrix}$$

Floating-base inverse dynamics

From the support-consistent dynamics:

$$\boldsymbol{\tau}^* = (\mathbf{N}_c^T\mathbf{S}^T)^+\mathbf{N}_c^T(\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g}) \underbrace{+\mathcal{N}(\mathbf{N}_c^T\mathbf{S}^T)\boldsymbol{\tau}_0^*}_{\text{multiple solutions}}$$

OSC with multiple objectives

Example: quadruped with three stationary legs and one in swing.

- Leg swing: $\ddot{\mathbf{r}}_{OF} = \mathbf{J}_F\ddot{\mathbf{q}}_F + \dot{\mathbf{J}}_F\dot{\mathbf{q}}_F = \ddot{\mathbf{r}}_{OF,des}(t) = k_p(\mathbf{q}\mathbf{r}^* - \mathbf{r}) + k_d(\dot{\mathbf{r}}^* - \dot{\mathbf{r}}) + \ddot{\mathbf{r}}^*$
- Body movement (translation and orientation): $\dot{\mathbf{w}}_B = \mathbf{J}_B\ddot{\mathbf{q}}_B + \dot{\mathbf{J}}_B\dot{\mathbf{q}}_B = \dot{\mathbf{w}}_{OB,des}(t) = k_p\begin{pmatrix} \mathbf{r}^* - \mathbf{r} \\ \boldsymbol{\varphi}^* \boxminus \boldsymbol{\varphi} \end{pmatrix} + k_d(\mathbf{w}^* - \mathbf{w}) + \dot{\mathbf{w}}^*$
- Enforce contact constraints: $\ddot{\mathbf{r}}_c = \mathbf{J}_c\ddot{\mathbf{q}}_c + \dot{\mathbf{J}}_c\dot{\mathbf{q}}_c = \mathbf{0}$

Solve for generalized acceleration and torque giving each task **equal priority**:

$$\ddot{\mathbf{q}}^* = \begin{bmatrix} \mathbf{J}_F \\ \mathbf{J}_B \\ \mathbf{J}_c \end{bmatrix}^+ \left(\begin{pmatrix} \ddot{\mathbf{r}}_{OF,des}(t) \\ \dot{\mathbf{w}}_{B,des}(t) \\ \mathbf{0} \end{pmatrix} - \begin{bmatrix} \dot{\mathbf{J}}_F \\ \dot{\mathbf{J}}_B \\ \dot{\mathbf{J}}_c \end{bmatrix} \dot{\mathbf{q}} \right)$$

Solve **with prioritization**:

$$\ddot{\mathbf{q}}^* = \sum_{i=1}^{n_t} \mathbf{N}_i\ddot{\mathbf{q}}_i,$$
$$\ddot{\mathbf{q}}_i \coloneqq (\mathbf{J}_j\mathbf{N}_i)^+ \left(\mathbf{w}_i^* - \dot{\mathbf{J}}_i\dot{\mathbf{q}} - \mathbf{J}\sum_{k=1}^{i-1} \mathbf{N}_k\dot{\mathbf{q}}_k \right)$$

Where $\mathbf{N}_i$ is the nullspace projection of $\mathbf{J}_i \coloneqq [\mathbf{J}_1^T \dots \mathbf{J}_i^T]^T$.

Quadratic minimization

Least squares problems can be expressed in form of quadratic minimization problems. We can also perform multiple tasks with or without prioritization:

$$\rightarrow \mathbf{Ax} - \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{A}^{+}\mathbf{b}$$

$$\Leftrightarrow \min_{\mathbf{x}} ||\mathbf{Ax} - \mathbf{b}||_2; \ \min_{\mathbf{x}} ||\mathbf{x}||_2$$

$$\rightarrow \mathbf{A_1 x_2} - \mathbf{b} = \mathbf{A_2 x_2} \Rightarrow \begin{pmatrix}\mathbf{x_1}\\ \mathbf{x_2}\end{pmatrix} = \begin{bmatrix}\mathbf{A_1} & \mathbf{A_2}\end{bmatrix}^{+}\mathbf{b}$$

$$\Leftrightarrow \min_{\mathbf{x_1},\mathbf{x_2}} \left\| \begin{bmatrix}\mathbf{A_1} & \mathbf{A_2}\end{bmatrix} \begin{pmatrix}\mathbf{x_1}\\ \mathbf{x_2}\end{pmatrix} - b \right\|_2; \ \min_{\mathbf{x_1},\mathbf{x_2}} \left\| \begin{pmatrix}\mathbf{x_1}\\ \mathbf{x_2}\end{pmatrix} \right\|_2$$

$$\rightarrow \begin{cases}\mathbf{A_1 x} = \mathbf{b_1}\\ \mathbf{A_2 x} = \mathbf{b_2}\end{cases} \text{Equal priority} \Rightarrow \mathbf{x} = \begin{bmatrix}\mathbf{A_1}\\ \mathbf{A_2}\end{bmatrix} \begin{pmatrix}\mathbf{b_1}\\ \mathbf{b_2}\end{pmatrix}$$

$$\Leftrightarrow \min_{\mathbf{x}} \left\| \begin{bmatrix}\mathbf{A_1}\\ \mathbf{A_2}\end{bmatrix}\mathbf{x} - \begin{pmatrix}\mathbf{b_1}\\ \mathbf{b_2}\end{pmatrix} \right\|_2; \ \min_{\mathbf{x}} ||\mathbf{x}||_2$$

$$\rightarrow \begin{cases}\mathbf{A_1 x} = \mathbf{b_1}\\ \mathbf{A_2 x} = \mathbf{b_2}\end{cases} \text{Hierarchy} \Rightarrow \text{(nullspace projections)}$$

$$\Leftrightarrow \min_{\mathbf{x}} ||\mathbf{A_1 x} - \mathbf{b_1}||_2; \ \begin{cases}\min_{\mathbf{x}} ||\mathbf{A_2 x} - \mathbf{b_2}||_2\\ \text{s.t.} \ \ ||\mathbf{A_1 x} - \mathbf{b_1}||_2 = c_1\end{cases}$$

OSC as quadratic program

Rewrite the equations of motion and subsequent tasks as a prioritized sequence of quadratic minimization problems:

$$\min_{\mathbf{x}} ||\mathbf{A}_i \mathbf{x} - \mathbf{b}_i||_2 \quad \mathbf{x} = \begin{pmatrix}\dot{\mathbf{u}}\\ \mathbf{F}_c\\ \boldsymbol{\tau}\end{pmatrix}$$

6 Rotorcraft

Propeller thrust and drag proportional to squared rotational speed (b: thrust constant; d: drag constant):

$$T_i = b\omega_{p,i}^2, \quad Q_i = d\omega_{p,i}^2$$

Representation of rotation

Use Tait-Bryan angles, consisting of yaw ψ (Z-axis), pitch θ (Y-axis) and roll ϕ (X-axis).

$$\mathbf{C}_{EB} = \mathbf{C}_{E1}(\mathbf{z}, \psi) \cdot \mathbf{C}_{12}(\mathbf{y}, \theta) \cdot \mathbf{C}_{2B}(\mathbf{x}, \phi)$$

Angular velocity:

$$_{\mathcal{B}}\boldsymbol{\omega} = {_{\mathcal{B}}\boldsymbol{\omega}_{\text{roll}}} + {_{\mathcal{B}}\boldsymbol{\omega}_{\text{pitch}}} + {_{\mathcal{B}}\boldsymbol{\omega}_{\text{yaw}}}$$

$$_{\mathcal{B}}\boldsymbol{\omega}_{\text{roll}} = (\dot{\psi}, 0, 0)^T$$

$$_{\mathcal{B}}\boldsymbol{\omega}_{\text{pitch}} = \mathbf{C}_{2B}^T (0, \dot{\theta}, 0)^T$$

$$_{\mathcal{B}}\boldsymbol{\omega}_{\text{yaw}} = [\mathbf{C}_{12} \cdot \mathbf{C}_{2E}]^T (0, 0, \dot{\phi})^T$$

$$_{\mathcal{B}}\boldsymbol{\omega} = J_r \dot{\chi}_r = J_r \begin{pmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{pmatrix}$$

$$J_r = \begin{bmatrix}1 & 0 & -\sin\theta\\ 0 & \cos\phi & \sin\phi\cos\theta\\ 0 & -\sin\phi & \sin\phi\cos\theta\end{bmatrix}$$

NB: singularity for $\theta = \pm 90°$ (gimbal lock).

Body Dynamics

Change of momentum and spin in the body frame ($\mathbf{M}$ = total moment/torque):

$$\begin{bmatrix}mE & 0\\ 0 & \mathbf{I}\end{bmatrix} \begin{bmatrix}_{\mathcal{B}}\dot{\boldsymbol{v}}\\ _{\mathcal{B}}\dot{\boldsymbol{\omega}}\end{bmatrix} + \begin{bmatrix}_{\mathcal{B}}\boldsymbol{\omega} \times m \, _{\mathcal{B}}\boldsymbol{v}\\ _{\mathcal{B}}\boldsymbol{\omega} \times \mathbf{I} \, _{\mathcal{B}}\boldsymbol{\omega}\end{bmatrix} = \begin{bmatrix}_{\mathcal{B}}\boldsymbol{F}\\ _{\mathcal{B}}\boldsymbol{M}\end{bmatrix}$$

Forces and moments come from gravity and aerodynamics:

$$_{\mathcal{B}}\boldsymbol{F} = {_{\mathcal{B}}\boldsymbol{F}_G} + {_{\mathcal{B}}\boldsymbol{F}_{Aero}}$$

$$_{\mathcal{B}}\boldsymbol{M} = {_{\mathcal{B}}\boldsymbol{M}_{Aero}}$$

$$_{\mathcal{B}}\boldsymbol{F}_G = \mathbf{C}_{EB}^T \begin{bmatrix}0\\ 0\\ mg\end{bmatrix}$$

$$_{\mathcal{B}}\boldsymbol{F}_{Aero} = \sum_{i=1}^{4} \begin{bmatrix}0\\ 0\\ -T_i = -b\omega_{p,i}^2\end{bmatrix}$$

$$_{\mathcal{B}}\boldsymbol{M}_{Aero} = {_{\mathcal{B}}\boldsymbol{M}_T} + {_{\mathcal{B}}\boldsymbol{Q}} = \begin{bmatrix}l(T_4 - T_2)\\ l(T_1 - T_3)\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \sum_{i=1}^4 Q_i(-\end{bmatrix}$$

The quadrotor automatically has full control over all rotational speeds, independently of the current position state. On the other hand, it can only directly control vertical cartesian velocity - attitude control must be used for full position control.

Propeller aerodynamics

Propeller in hover:

- Thrust force T normal to prop. plane, $|T| = \frac{\rho}{2} A_P C_T (\omega_p R_p)^2$
- Drag torque Q, around rotor plane $|Q| = \frac{\rho}{2} A_P C_Q (\omega_p R_p)^2 R_p$
- C_T and C_Q depend on blade pitch angle (prop geometry), Reynolds number (prop speed, velocity, rotational speed).

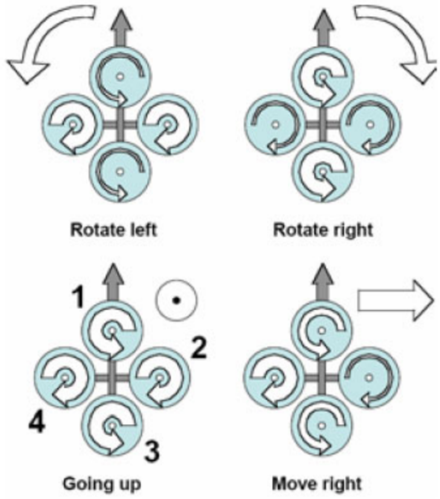
Propeller in forward flight: additional forces due to force unbalance between forward- and backward-moving props.

- Hub force H (orthogonal to T, opposite to horizontal flight direction V_H), $|H| = \frac{\rho}{2} A_P C_H (\omega_p R_p)^2 R_p$
- Rolling torque R around flight direction $|R| = \frac{\rho}{2} A_P C_R (\omega_p R_p)^2 R_p$
- C_R and C_H depend on advance ratio $\mu = \frac{V}{\omega_P R_P}$

Ideal power consumption at hover: $P = \frac{F_{Thrust}^{3/2}}{\sqrt{2\rho A_R}} = \frac{(mg)^{3/2}}{\sqrt{2\rho A_R}}$. The prop efficiency is measured with the Figure of Merit FM:

$$FM = \frac{\text{Ideal power to hover}}{\text{Actual power to hover}} < 1$$

Blade Elemental and Momentum Theory (BEMT): blade shape determines drag and lift coefficients c_D, c_L.



- Define virtual control inputs to simplify the equations
  - Get decoupled and linear inputs
    - Moments along each axis

$$I_{xx}\dot{p} = qr(I_{yy} - I_{zz}) + U_2 \qquad U_2 = lb(\omega_{p,4}^2 - \omega_{p,2}^2)$$

$$I_{yy}\dot{q} = rp(I_{zz} - I_{xx}) + U_3 \qquad U_3 = lb(\omega_{p,1}^2 - \omega_{p,3}^2)$$

$$I_{zz}\dot{r} = U_4 \qquad U_4 = d(-\omega_{p,1}^2 + \omega_{p,2}^2 - \omega_{p,3}^2 + \omega_{p,4}^2)$$

- Total thrust

$$m\dot{u} = m(rv - qw) - \sin\theta \ mg$$

$$m\dot{v} = m(pw - ru) + \sin\phi\cos\theta \ mg$$

$$m\dot{w} = m(qu - pv) + \cos\phi\cos\theta \ mg - U_1 \qquad U_1 = b(\omega_{p,1}^2 + \omega_{p,2}^2 + \omega_{p,3}^2 + \omega_{p,4}^2)$$

3 separate control loops

- 1 additional thrust input

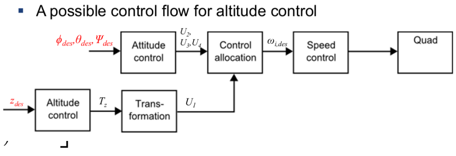
$$U_1 = T_{des}$$

$$U_2 = (\phi_{des} - \phi)k_{pRoll} - \dot{\phi}k_{dRoll}$$

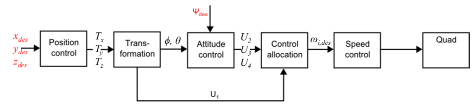
$$U_3 = (\theta_{des} - \theta)k_{pPitch} - \dot{\theta}k_{dPitch}$$

$$U_4 = (\psi_{des} - \psi)k_{pYaw} - \dot{\psi}k_{dYaw}$$

- A possible control flow for altitude control

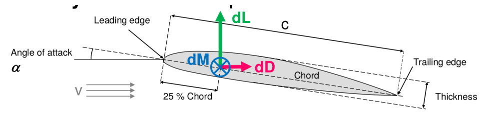


- Possible control structure for full position control



7 Fixed wing aerodynamics

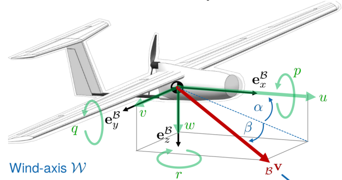
For aircraft rotation, use Tait-Bryan angles (like for copters).



Pressure distribution can be reduced to two forces and one moment ***per unit length***:

Lift force	$dL = C_l \frac{\rho}{2} c \cdot dy \cdot V^2$	ρ	: Density of fluid (air) [kg/m³]
		c	: Chord length [m]
		V	: Flight speed (w.r.t. air) [m/s]
Drag force	$dD = C_d \frac{\rho}{2} c \cdot dy \cdot V^2$	C_l	: Airfoil lift coefficient [-]
		C_d	: Airfoil drag coefficient [-]
Moment	$dM = C_m \frac{\rho}{2} c^2 \cdot dy \cdot V^2$	C_m	: Airfoil moment coefficient [-]

Aircraft Kinematics | Reference axes



Body-axis $\mathcal{B}$

Body velocity:

$$_{\mathcal{B}}\mathbf{v}_a = (u, v, w)^T$$

- Air-mass relative speed (airspeed):

$$V = \sqrt{u^2 + v^2 + w^2}$$

- V is always positive

Body rates:

$$_{\mathcal{B}}\omega = (p, q, r)^T$$

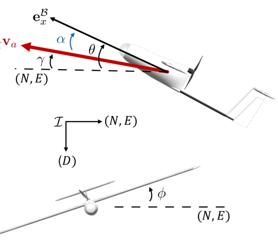
Wind-axis $\mathcal{W}$

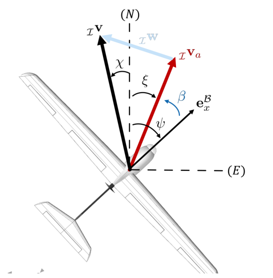
Airflow angles:

- Angle of attack $\alpha = \tan^{-1}(w/u)$
- Sideslip angle $\beta = \sin^{-1}(v/V)$
- Wind frame is opposite to "free-stream" velocity, or "wind" (i.e. the air-mass)

Longitudinal Inertial Frame (*side view)

- γ : Flight path angle, defined from horizon to $_{\mathcal{I}}\mathbf{v}_a$
- θ : Pitch angle, from horizon to x-body axis
- ϕ : Roll angle, rotation about x-body axis (note: pointing out of slide)
- $_{\mathcal{I}}\mathbf{v}_a = \mathbf{C}_{\mathcal{IB}} \, _{\mathcal{B}}\mathbf{v}_a$





Lateral-directional Inertial Frame (*top view)

- ξ : Heading angle, defined from North to $_{\mathcal{I}}\mathbf{v}_a$
- ψ: Yaw angle, from North to x-body axis

Adding a constant, horizontal wind:

- χ : Course angle, defined from North to $_{\mathcal{I}}\mathbf{v}$
- $_{\mathcal{I}}\mathbf{v}$: Ground-based inertial velocity (or "ground speed")
- $_{\mathcal{I}}\mathbf{w}$: Wind velocity
- Note: the constant wind assumption effects only position dynamics.

Translation

$$\dot{u} = rv - qw + \frac{1}{m}\left(F_T\cos\varepsilon - D\cos\alpha + L\sin\alpha\right) - g\sin\theta$$

$$\dot{v} = pw - ru + \frac{1}{m}Y + g\sin\phi\cos\theta$$

$$\dot{w} = qu - pv + \frac{1}{m}\left(F_T\sin\varepsilon - D\sin\alpha - L\cos\alpha\right) + g\cos\phi\cos\theta$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \mathbf{C}_{IB}\begin{bmatrix}u\\ v\\ w\end{bmatrix} + {_I}\mathbf{w}$$

Rotation (simplified with $I_{xz} \approx 0$):

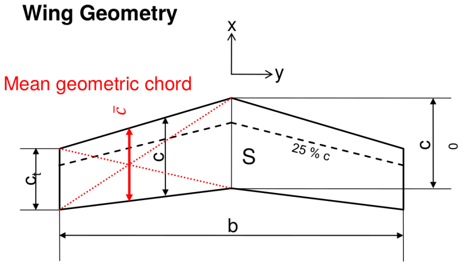
$$\dot{p} = \frac{1}{I_{xx}}\left[L_m + L_{m_T} - qr\left(I_{zz} - I_{yy}\right)\right]$$

$$\dot{q} = \frac{1}{I_{yy}}\left[M_m + M_{m_T} - pr\left(I_{xx} - I_{zz}\right)\right]$$

$$\dot{r} = \frac{1}{I_{zz}}\left[N_m + N_{m_T} - pq\left(I_{yy} - I_{xx}\right)\right]$$

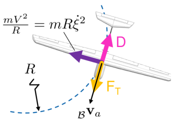
$$\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = \mathbf{J}_r^{-1}\begin{bmatrix}p\\ q\\ r\end{bmatrix} = \begin{bmatrix}p + q\tan\theta\sin\phi + r\tan\theta\cos\phi\\ q\cos\phi - r\sin\phi\\ q\frac{\sin\phi}{\cos\theta} + r\frac{\cos\phi}{\cos\theta}\end{bmatrix}$$

Wing Geometry



Fixed-wing Control | Steady level turning flight

Assuming NO sideslip, i.e. $\xi = \psi$



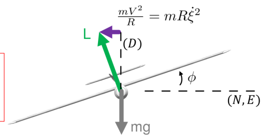
- Turning (not straight)

  ${}_{\mathcal{B}}\dot{\mathbf{v}}_a = \mathbf{0}, {}_{\mathcal{B}}\dot{\omega} = \mathbf{0}$ <- steady

  $\theta = \alpha \rightarrow \gamma = 0$ <- level

  $\phi = \text{const.}$ <- turning

  - Demand for **coordinated** turn: Y=0
  - L increases with $\frac{1}{\cos\phi}$
  - V_{min} increases with $\sqrt{\frac{1}{\cos\phi}}$



Recall Y is composed of only aerodynamic forces, which must be zero, thus the lateral force here only comes from centripetal acceleration.