ETH zürich

Lecture «Robot Dynamics»: Kinematics 2

151-0851-00 V

lecture: HG F3 Tuesday 10:15 – 12:00, every week

exercise: HG D7.1 Wednesday 8:15 – 10:00, according to schedule

Marco Hutter, Roland Siegwart, and Thomas Stastny

ETH zürich

17.09.2019	Intro and Outline	Course Introduction; Recapitulation Position, Linear Velocity			
24.09.2019	Kinematics 1	Rotation and Angular Velocity; Rigid Body Formulation, Transformation	25.09.2019	Exercise 1a	Kinematics Modeling the ABB arm
01.10.2019	Kinematics 2	Kinematics of Systems of Bodies; Jacobians	02.10.2019	Exercise 1a	Differential Kinematics of the ABB arm
08.10.2019	Kinematics 3	Kinematic Control Methods: Inverse Differential Kinematics, Inverse Kinematics; Rotation Error; Multi-task Control	09.10.2019	Exercise 1b	Kinematic Control of the ABB Arm
15.10.2019	Dynamics L1	Multi-body Dynamics	16.10.2019	Midterm 1	Programming kinematics with matlab
22.10.2019	Dynamics L2	Floating Base Dynamics	23.10.2019	Exercise 2a	Dynamic Modeling of the ABB Arm
29.10.2019	Dynamics L3	Dynamic Model Based Control Methods	30.10.2019	Exercise 2b	Dynamic Control Methods Applied to the ABB arm
05.11.2019	Legged Robot	Dynamic Modeling of Legged Robots & Control	06.11.2019	Midterm 2	Programming dynamics with matlab
12.11.2019	Case Studies 1	Legged Robotics Case Study	13.11.2019	Exercise 3	Legged robot
19.11.2019	Rotorcraft	Dynamic Modeling of Rotorcraft & Control	20.11.2019		
26.11.2019	Case Studies 2	Rotor Craft Case Study	27.11.2019	Exercise 4	Modeling and Control of Multicopter
03.12.2019	Fixed-wing	Dynamic Modeling of Fixed-wing & Control	04.12.2019		
10.12.2019	Case Studies 3	Fixed-wing Case Study (Solar-powered UAVs - AtlantikSolar, Vertical Take-off and Landing UAVs – Wingtra)	11.12.2019	Exercise 5	Fixed-wing Control and Simulation
17.12.2019	Summery and Outlook	Summery; Wrap-up; Exam			

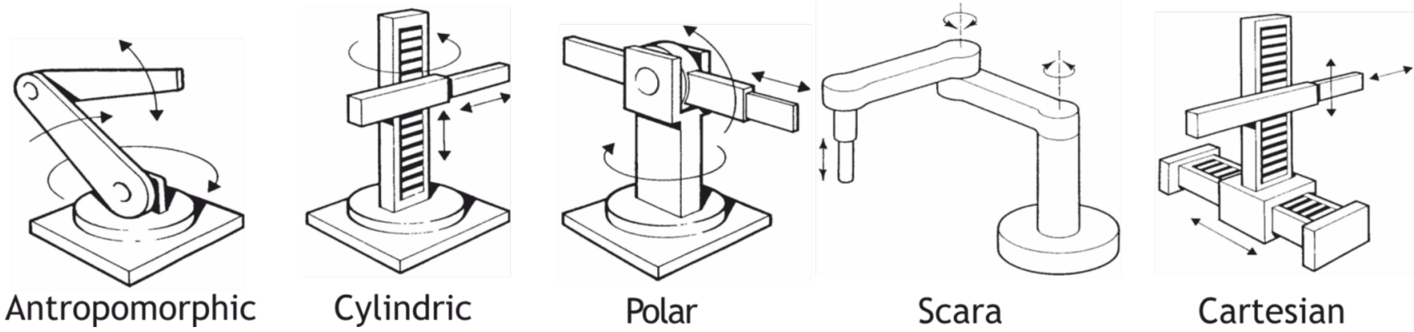
Multi-body Kinematics
Intro

- Machines are built and controlled to
  - achieve extremely accurate positions,
  - independent of the load the robot carries

- ➢ Very stiff structure
- ➢ Play-free gears and transmissions
- ➢ High-accurate joint sensors

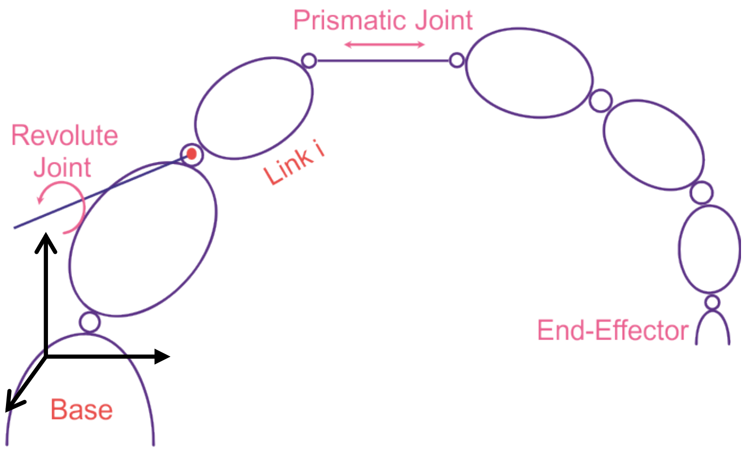
- ➢ End-effector accuracy +/- 0.02mm!

- Large variety of robot arms

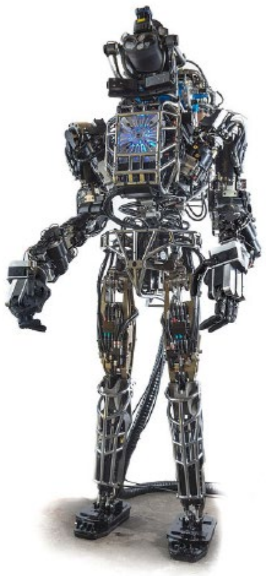


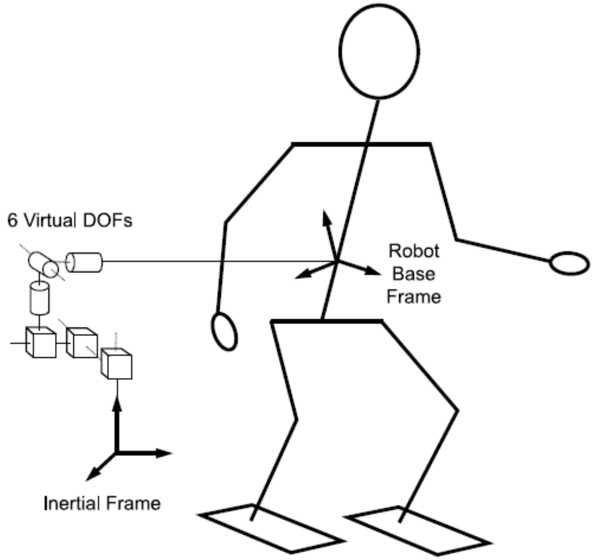
Fixed Base vs. Floating Base Robot

- Base frame is rigidly connected to ground
  - Often indicated as CS 0



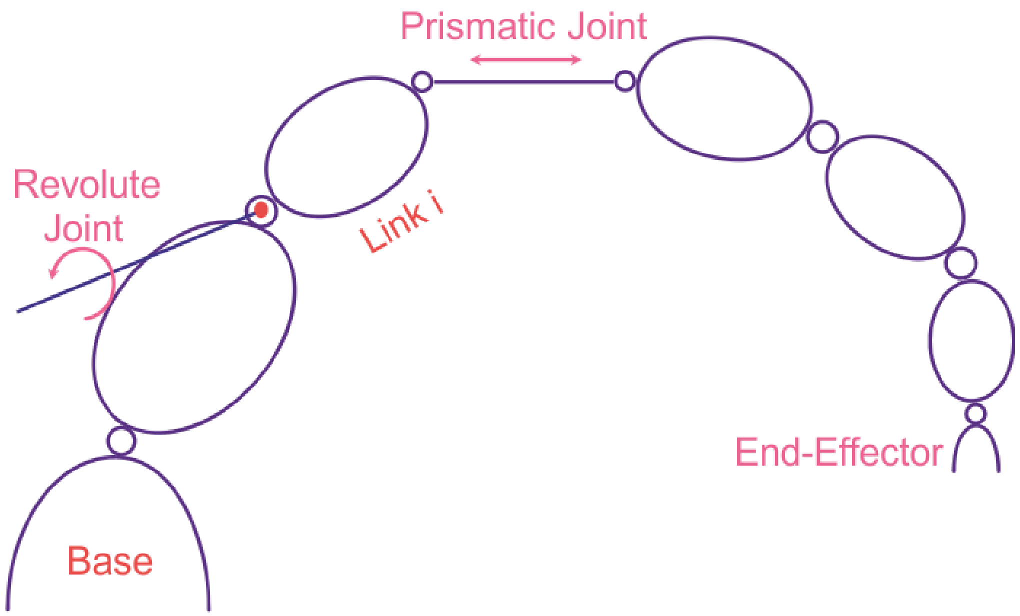
- Base frame is free floating
  - Often indicated as CS $\mathcal{B}$ (base)
  - 6 unactuated DOFs!





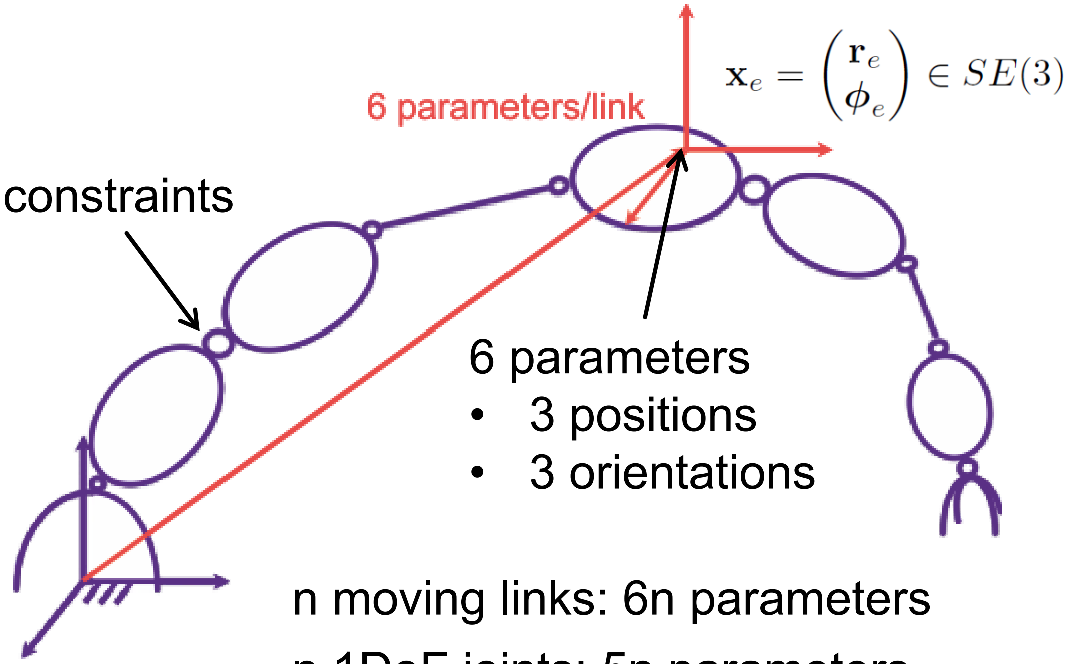
Classical Serial Kinematic Linkages

Generalized robot arm



- n_j joints
  - revolute (1DOF)
  - prismatic (1DOF)
- $n_l = n_j + 1$ links
  - n_j moving links
  - 1 fixed link

Configuration Parameters
Generalized coordinates



n moving links: 6n parameters

n 1DoF joints: 5n parameters

6n – 5n = n DoFs

Generalized coordinates

A set of scalar parameters **q** that describe the robot's configuration

- Must be **complete**
- (Must be **independent**)
  => minimal coordinates
- Is **not unique**

$$\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_{n_j} \end{pmatrix} \in \mathbb{R}^{n_j}$$

Degrees of Freedom

- Nr of minimal coordinates

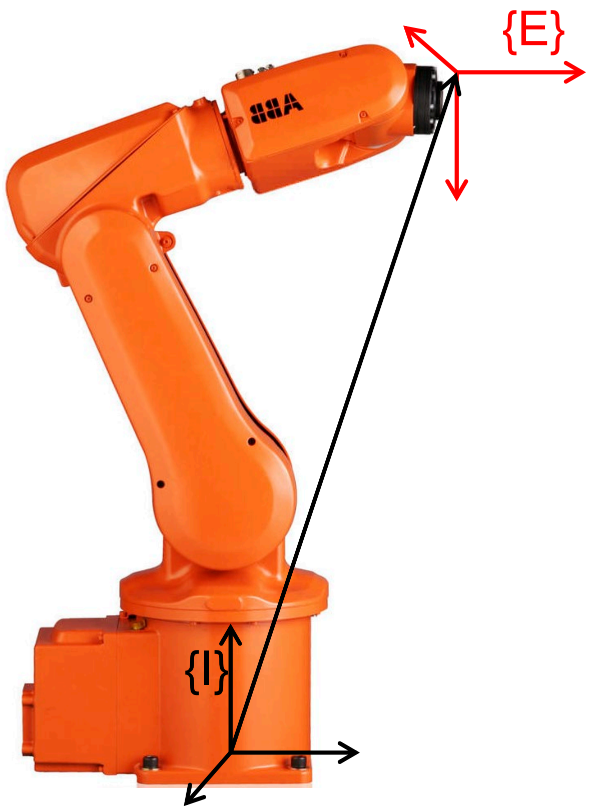
End-effector Configuration Parameters

- End-effector configuration parameters
  - A set of m parameters that completely specify the end-effector position and orientation with respect to $\mathcal{I}$

$$\chi_e = \begin{pmatrix} \chi_{eP} \\ \chi_{eR} \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_m \end{pmatrix} \in \mathbb{R}^m$$

- Operational space coordinates
  - the m_0 configuration parameters are independent => m_0 number of degrees of freedom of end-effector

$$\chi_o = \begin{pmatrix} \chi_{oP} \\ \chi_{oR} \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{m_0} \end{pmatrix}$$



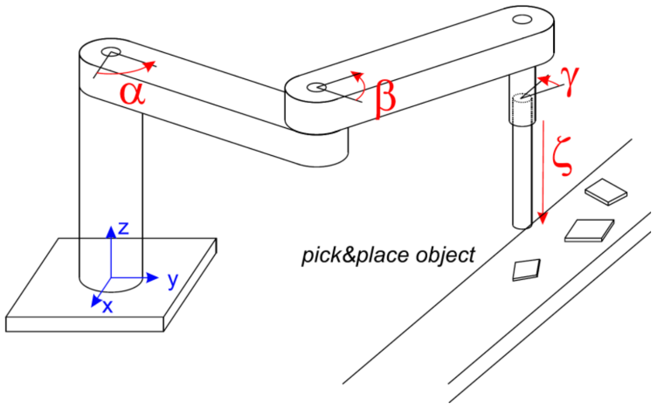
5

5

Quiz
3min
1min

End-effector Configuration Parameters
Example

- Most general robot arm:
  - **q** =
  - m_e= $\quad\quad\quad\quad$ m_o=
  - χ_e= $\quad\quad\quad\quad$ χ_o=
- SCARA robot arm
  - **q** =
  - m_e= $\quad\quad\quad\quad$ m_o=
  - χ_e= $\quad\quad\quad\quad$ χ_o=

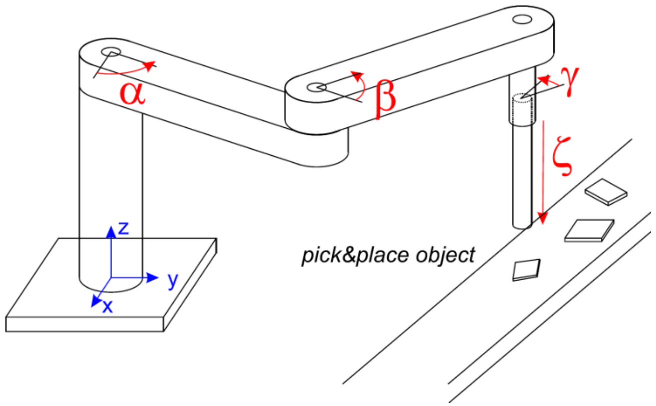


- ANYpulator: robot arm with 4 rotational joints
  - **q** =
  - m_e= $\quad\quad\quad\quad$ m_o=
  - χ_e= $\quad\quad\quad\quad$ χ_o=

Quiz
3min
1min

End-effector Configuration Parameters
Example

- Most general robot arm:
  - $\mathbf{q} = (q_1 \ldots q_6)$
  - $m_e = 6$ $m_o = 6$
  - $\chi_e = (x, y, z, \alpha_x, \beta_y, \gamma_z)$ $\chi_o = (x, y, z, \alpha_x, \beta_y, \gamma_z)$
- SCARA robot arm
  - $\mathbf{q} = (\alpha, \beta, \gamma, \zeta)$
  - $m_e = 6$ $m_o = 4$
  - $\chi_e = (x, y, z, \alpha_x, \beta_y, \gamma_z)$ $\chi_o = (x, y, z, \gamma_z)$
- ANYpulator: robot arm with 4 rotational joints
  - $\mathbf{q} = (q_1, q_2, q_3, q_4)$
  - $m_e = 6$ $m_o = 4$
  - $\chi_e = (x, y, z, \alpha_x, \beta_y, \gamma_z)$ $\chi_o =$

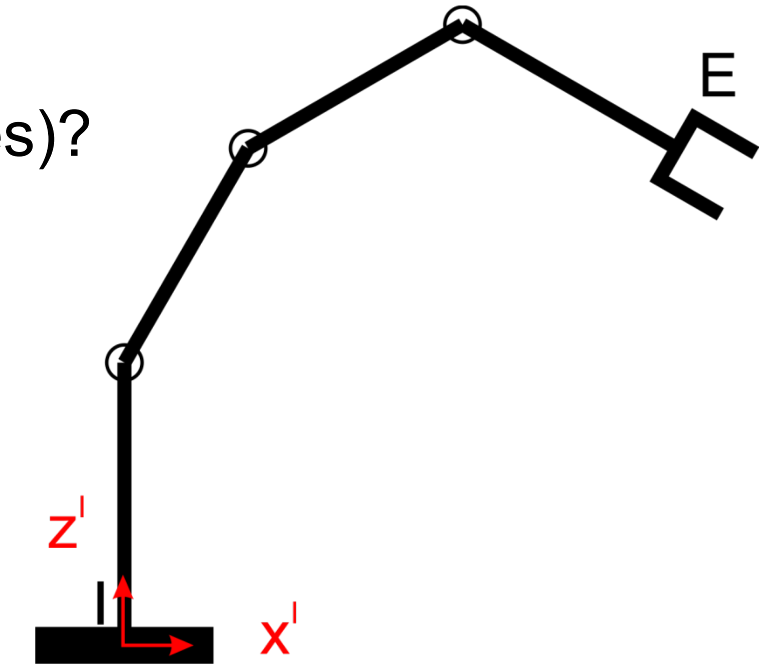


End-effector Configuration Parameters
Simple example

- Planar robot arm
  - 3 revolute joints
  - 1 end-effector (gripper) <= *don't consider this for the moment*

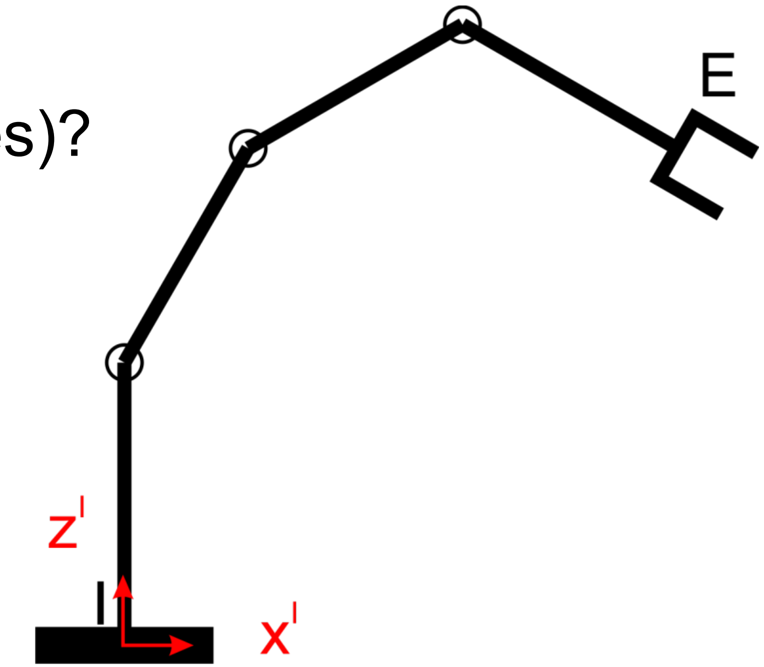
- What are the joint coordinates (generalized coordinates)?

- What are the end-effector parameters?



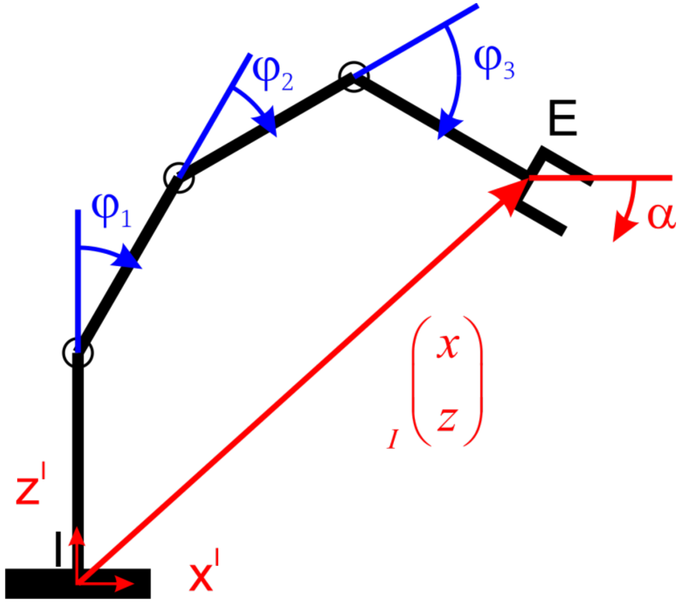
End-effector Configuration Parameters
Simple example

- Planar robot arm
  - 3 revolute joints
  - 1 end-effector (gripper) <= *don't consider this for the moment*
- What are the joint coordinates (generalized coordinates)?
  - **q** = $\left(q_1 \ldots q_{n_j}\right)$
- What are the end-effector parameters?
  - m_e= 3
  - χ_e= (x, z, α)



Configuration Space ⇔ Joint Space

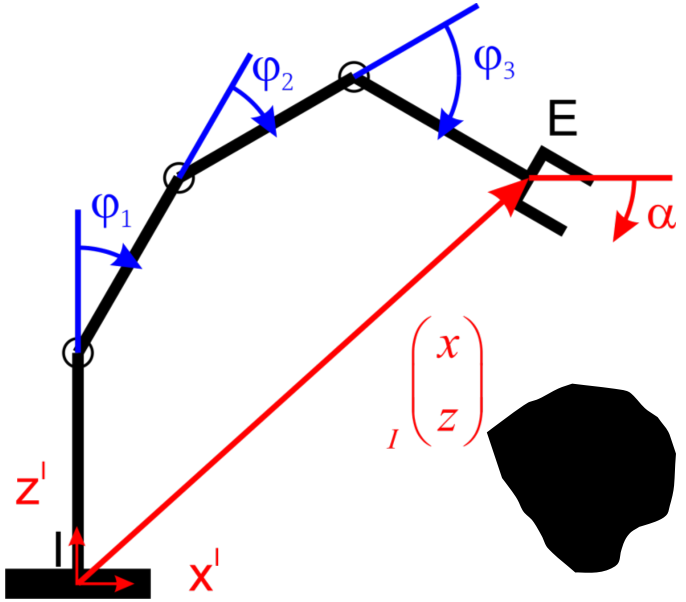
- Joint Coordinates

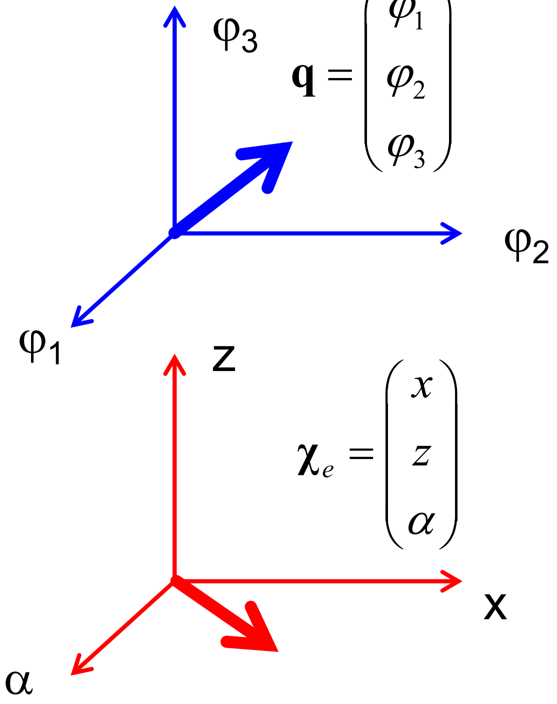


- Task Coordinates

Configuration Space ⇔ Joint Space

- Joint Coordinates => Joint Space





$$\mathbf{q} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

$$\chi_e = \begin{pmatrix} x \\ z \\ \alpha \end{pmatrix}$$

- Task Coordinates => Task Space

Forward Kinematics

- End-effector configuration as a function of generalized coordinates

$$\chi_e = \chi_e(\mathbf{q}) \in \mathbb{R}^{n_e}$$

- For multi-body system, use transformation matrices

$$\mathbf{T}_{\mathcal{I}\mathcal{E}}(\mathbf{q}) = \mathbf{T}_{\mathcal{I}0} \cdot \left(\prod_{k=1}^{n_j} \mathbf{T}_{k-1,k}(q_k) \right) \cdot \mathbf{T}_{n_j\mathcal{E}} = \begin{bmatrix} \mathbf{C}_{\mathcal{I}\mathcal{E}}(\mathbf{q}) & {}_{\mathcal{I}}\mathbf{r}_{IE}(\mathbf{q}) \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

Forward Kinematics
Simple example

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

- What is the end-effector configuration as a function of generalized coordinates?

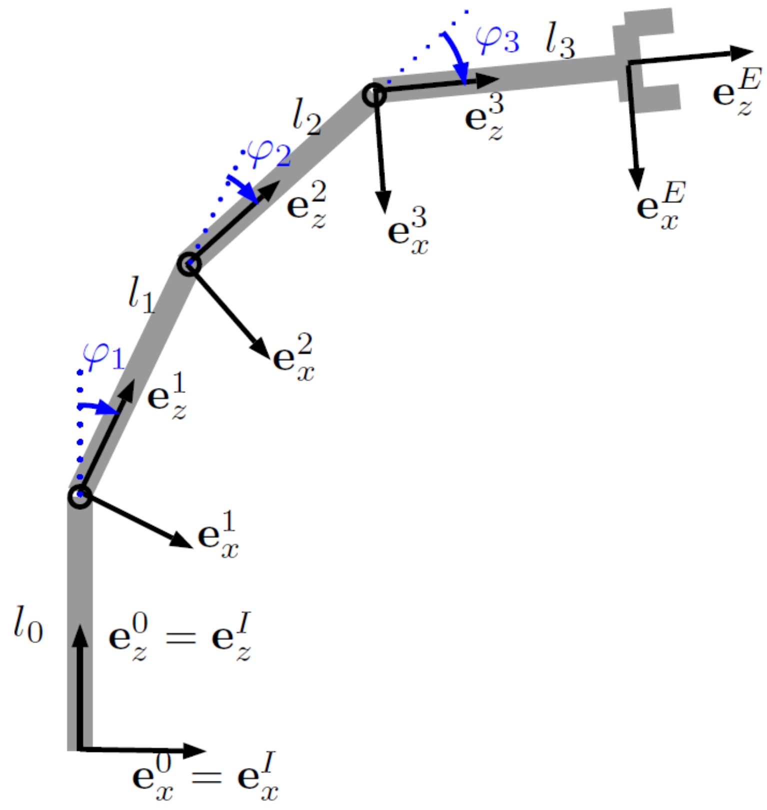
$$\mathbf{T}_{IE} = \mathbf{T}_{I0} \cdot \mathbf{T}_{01} \cdot \mathbf{T}_{12} \cdot \mathbf{T}_{23} \cdot \mathbf{T}_{3E}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & 0 \\ -s_1 & 0 & c_1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s_2 & 0 & c_2 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & 0 & s_3 & 0 \\ 0 & 1 & 0 & 0 \\ -s_3 & 0 & c_3 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \ldots = \begin{bmatrix} c_{123} & 0 & s_{123} & l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 & 1 & 0 & 0 \\ -s_{123} & 0 & c_{123} & l_0 + l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\chi_{eP}(\mathbf{q}) = \begin{pmatrix} x \\ z \end{pmatrix} = \boxed{\begin{pmatrix} l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) + l_3 \sin(q_1 + q_2 + q_3) \\ l_0 + l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) + l_3 \cos(q_1 + q_2 + q_3) \end{pmatrix}}$$

$$\chi_{eR}(\mathbf{q}) = \chi_{eR}(\mathbf{q}) = \boxed{q_1 + q_2 + q_3}$$



Forward Differential Kinematics
Analytical Jacobian

- Forward Kinematics $\chi_e = \begin{pmatrix} \chi_{e_P} \\ \chi_{e_R} \end{pmatrix} = \chi_e(\mathbf{q})$ $\quad \mathbf{T}_{\mathcal{IE}}(\mathbf{q}) = \mathbf{T}_{\mathcal{I}0} \cdot \left(\prod_{k=1}^{n_j} \mathbf{T}_{k-1,k}(q_k) \right) \cdot \mathbf{T}_{n_j \mathcal{E}} = \begin{bmatrix} \mathbf{C}_{\mathcal{IE}}(\mathbf{q}) & {}_{\mathcal{I}}\mathbf{r}_{IE}(\mathbf{q}) \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$

- Forward **Differential** Kinematics
  - Analytic: $\chi_e + \delta\chi_e = \chi_e(\mathbf{q} + \delta\mathbf{q}) = \chi_e(\mathbf{q}) + \frac{\partial \chi_e(\mathbf{q})}{\partial \mathbf{q}} \delta\mathbf{q} + O(\delta\mathbf{q}^2)$

$$\delta\chi_e \approx \frac{\partial \chi_e(\mathbf{q})}{\partial \mathbf{q}} \delta\mathbf{q} = \mathbf{J}_{eA}(\mathbf{q})\,\delta\mathbf{q} \quad \text{with} \quad \mathbf{J}_{eA} = \frac{\partial \chi_e}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial \chi_1}{\partial q_1} & \cdots & \frac{\partial \chi_1}{\partial q_{n_j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \chi_m}{\partial q_1} & \cdots & \frac{\partial \chi_m}{\partial q_{n_j}} \end{bmatrix}$$

$$\boxed{\dot{\chi}_e = \mathbf{J}_{eA}(\mathbf{q})\,\dot{\mathbf{q}} \quad \text{with } \mathbf{J}_{eA}(\mathbf{q}) \in \mathbb{R}^{m_e \times n_j}}$$

5

5

Quiz
3min
1min

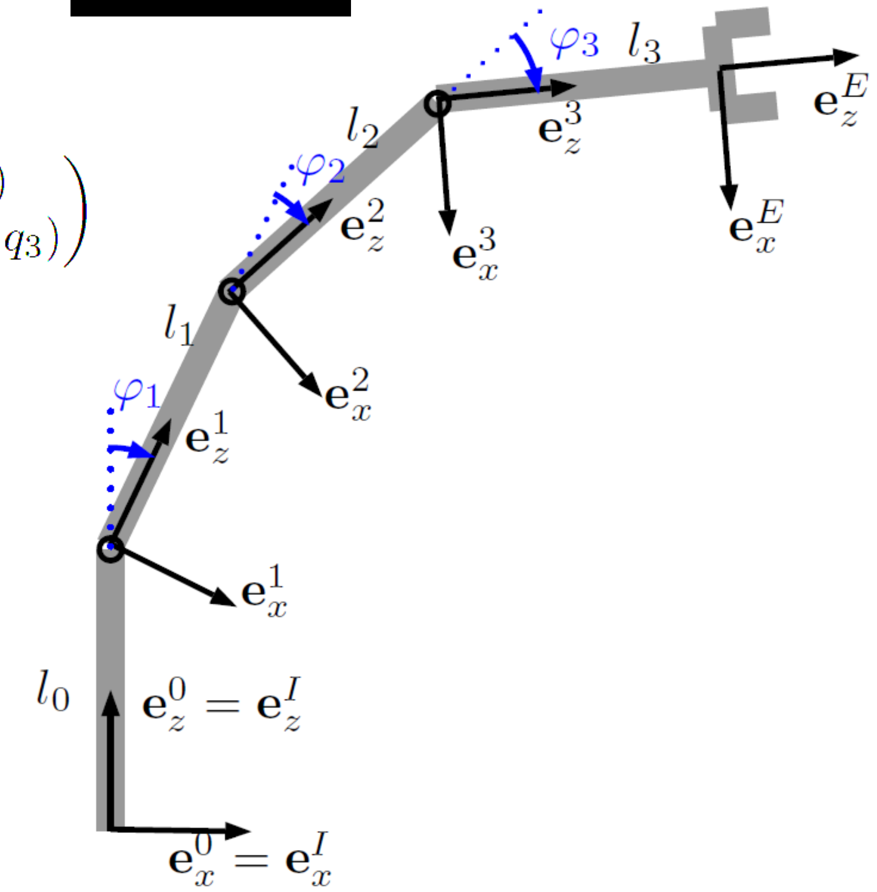
Analytical Jacobian
Planar robot arm

- Given (from last example)

$$\chi_{eP}(\mathbf{q}) = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} l_1 \sin(q_1) + l_2 \sin(q_1+q_2) + l_3 \sin(q_1+q_2+q_3) \\ l_0 + l_1 \cos(q_1) + l_2 \cos(q_1+q_2) + l_3 \cos(q_1+q_2+q_3) \end{pmatrix}$$

$$\chi_{eR}(\mathbf{q}) = \chi_{eR}(\mathbf{q}) = q_1 + q_2 + q_3$$

- Determine the analytical Jacobian



Quiz
3min
1min

Analytical Jacobian
Planar robot arm

- Given (from last example)

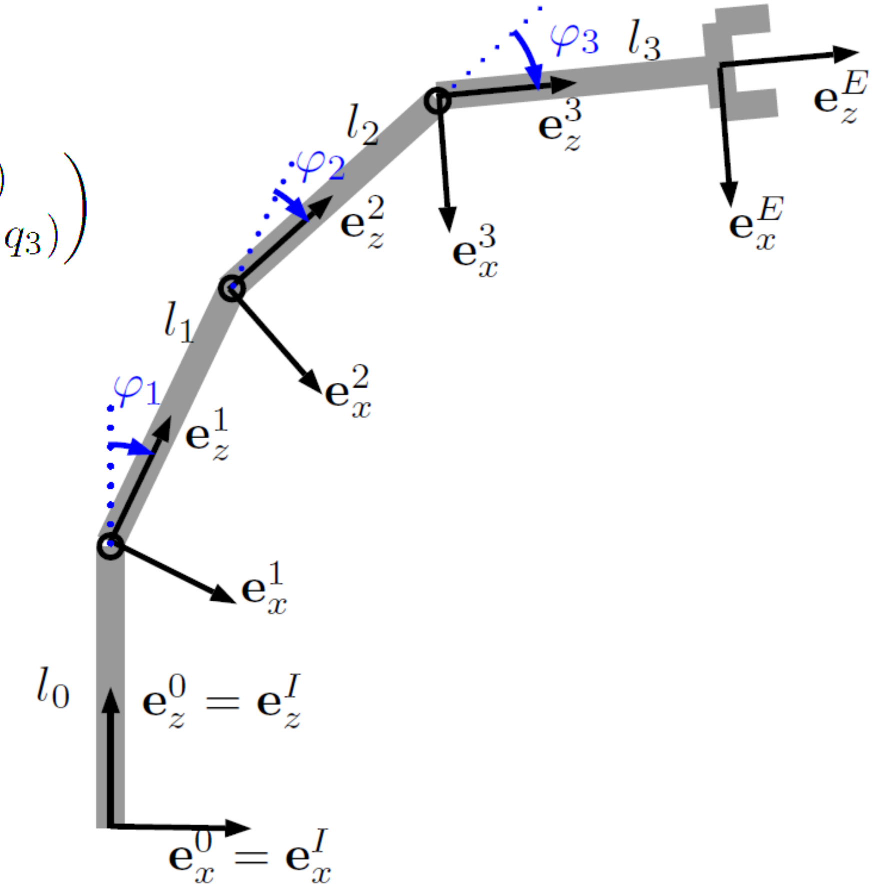
$$\chi_{eP}(\mathbf{q}) = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} l_1 \sin(q_1) + l_2 \sin(q_1+q_2) + l_3 \sin(q_1+q_2+q_3) \\ l_0 + l_1 \cos(q_1) + l_2 \cos(q_1+q_2) + l_3 \cos(q_1+q_2+q_3) \end{pmatrix}$$

$$\chi_{eR}(\mathbf{q}) = \chi_{eR}(\mathbf{q}) = q_1 + q_2 + q_3$$

- Determine the analytical Jacobian

$$\mathbf{J}_{eAP}(\mathbf{q}) = \frac{\partial \chi_{eP}}{\partial \mathbf{q}} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{213} & l_3 c_{213} \\ -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{213} & -l_3 s_{213} \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

$$\mathbf{J}_{eAR}(\mathbf{q}) = \frac{\partial \chi_{eR}}{\partial \mathbf{q}} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{1\times 3}$$



Forward Differential Kinematics

- Analytic: $\delta\chi_e \approx \frac{\partial \chi_e(\mathbf{q})}{\partial \mathbf{q}} \delta\mathbf{q} = \mathbf{J}_{eA}(\mathbf{q})\,\delta\mathbf{q}$ with $\mathbf{J}_{eA} = \frac{\partial \chi_e}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial \chi_1}{\partial q_1} & \cdots & \frac{\partial \chi_1}{\partial q_{n_j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \chi_m}{\partial q_1} & \cdots & \frac{\partial \chi_m}{\partial q_{n_j}} \end{bmatrix}$

$$\boxed{\dot{\chi}_e = \mathbf{J}_{eA}(\mathbf{q})\,\dot{\mathbf{q}} \quad \text{with } \mathbf{J}_{eA}(\mathbf{q}) \in \mathbb{R}^{m_e \times n_j}}$$

Depending on parameterization!!

- Geometric: $$\boxed{\mathbf{w}_e = \begin{pmatrix} \mathbf{v}_e \\ \boldsymbol{\omega}_e \end{pmatrix} = \mathbf{J}_{e0}(\mathbf{q})\,\dot{\mathbf{q}} \quad \text{with } \mathbf{J}_{e0}(\mathbf{q}) \in \mathbb{R}^{6 \times n_j}}$$

Independent of parameterization

$$\mathbf{w}_e = \mathbf{E}_e(\chi_e)\,\dot{\chi}_e$$
$$\downarrow$$
$$\mathbf{J}_{e0}(\mathbf{q}) = \mathbf{E}_e(\chi)\,\mathbf{J}_{eA}(\mathbf{q})$$

- Algebra: $$\mathbf{w}_C = \begin{pmatrix} \mathbf{v}_C \\ \boldsymbol{\omega}_C \end{pmatrix} = \mathbf{w}_B + \mathbf{w}_{BC}$$

$$\mathbf{J}_C\dot{\mathbf{q}} = \mathbf{J}_B\dot{\mathbf{q}} + \mathbf{J}_{BC}\dot{\mathbf{q}}$$

$${}_{\mathcal{A}}\mathbf{J}_C = {}_{\mathcal{A}}\mathbf{J}_B + {}_{\mathcal{A}}\mathbf{J}_{BC}$$

Velocity in Moving Bodies

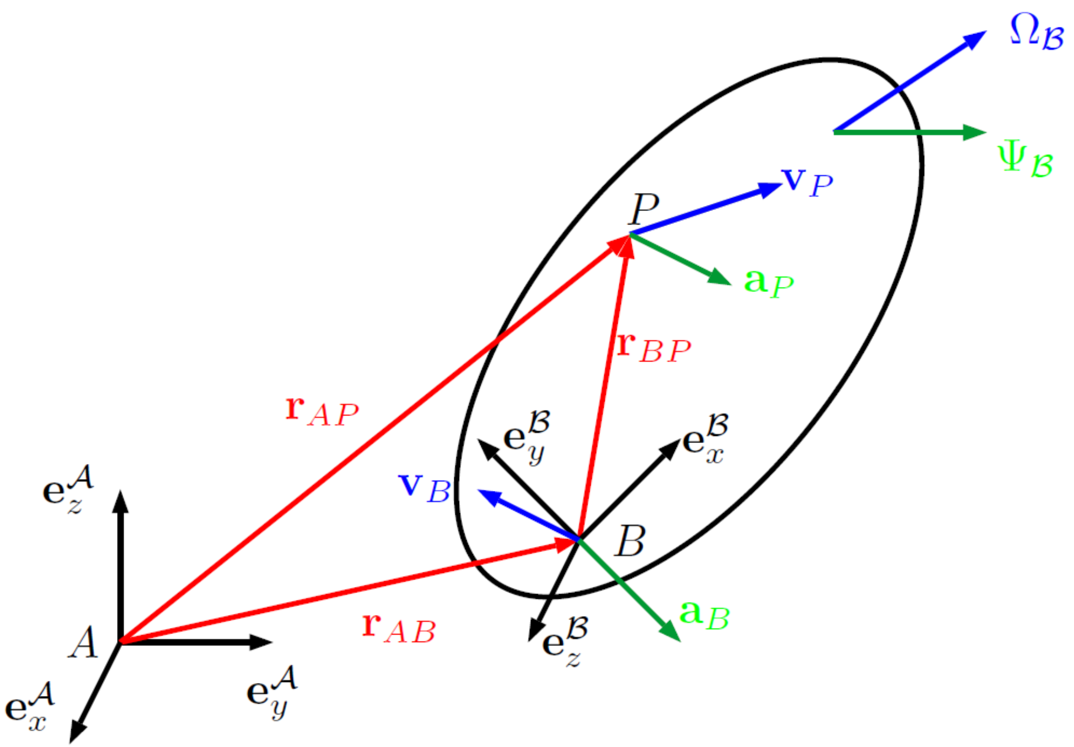
- Definitions

  $\mathbf{v}_P$: the absolute velocity of P

  $\mathbf{a}_P = \dot{\mathbf{v}}_P$: the absolute acceleration of P

  $\boldsymbol{\Omega}_\mathcal{B} = \boldsymbol{\omega}_{\mathcal{AB}}$: (absolute) angular velocity of body $\mathcal{B}$

  $\boldsymbol{\Psi}_\mathcal{B} = \dot{\boldsymbol{\Omega}}_\mathcal{B}$: (absolute) angular acceleration of body $\mathcal{B}$



- Remember the difference:
  - Velocity
  - Time derivative of coordinates:

Vector Differentiation in Moving Frame
Euler differentiation rule

- For non-moving reference frames: $_\mathcal{A}\mathbf{v}_P = {}_\mathcal{A}\dot{\mathbf{r}}_{AP}$
- For moving reference frames: $\mathbf{v}_P \neq \dot{\mathbf{r}}_{AP}$
- Vector differentiation in moving frames ($\mathcal{A}$ = inertial/reference frame):

$$
\begin{aligned}
{}_\mathcal{B}\mathbf{v}_P &= \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot \frac{d}{dt}\left(\mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_\mathcal{B}\mathbf{r}_{AP}\right) \\
&= \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot \left(\mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_\mathcal{B}\dot{\mathbf{r}}_{AP} + \dot{\mathbf{C}}_{\mathcal{A}\mathcal{B}} \cdot {}_\mathcal{B}\mathbf{r}_{AP}\right) \\
&= \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot \left(\mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_\mathcal{B}\dot{\mathbf{r}}_{AP} + [{}_\mathcal{A}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_\times \cdot \mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_\mathcal{B}\mathbf{r}_{AP}\right) \\
&= {}_\mathcal{B}\dot{\mathbf{r}}_{AP} + \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot [{}_\mathcal{A}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_\times \cdot \mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_\mathcal{B}\mathbf{r}_{AP} \\
&= {}_\mathcal{B}\dot{\mathbf{r}}_{AP} + {}_\mathcal{B}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}} \times {}_\mathcal{B}\mathbf{r}_{AP}
\end{aligned}
$$

$$[{}_\mathcal{A}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_\times = \dot{\mathbf{C}}_{\mathcal{A}\mathcal{B}}\mathbf{C}_{\mathcal{A}\mathcal{B}}^T$$

$$[{}_\mathcal{B}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_\times = \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot [{}_\mathcal{A}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_\times \cdot \mathbf{C}_{\mathcal{A}\mathcal{B}}$$

Velocity in Moving Bodies
Rigid body formulation

- Apply transformation rule as learned before

$$ {}_{\mathcal{A}}\mathbf{r}_{AP} = {}_{\mathcal{A}}\mathbf{r}_{AB} + {}_{\mathcal{A}}\mathbf{r}_{BP} = {}_{\mathcal{A}}\mathbf{r}_{AB} + \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{BP} $$

- Differentiate with respect to time

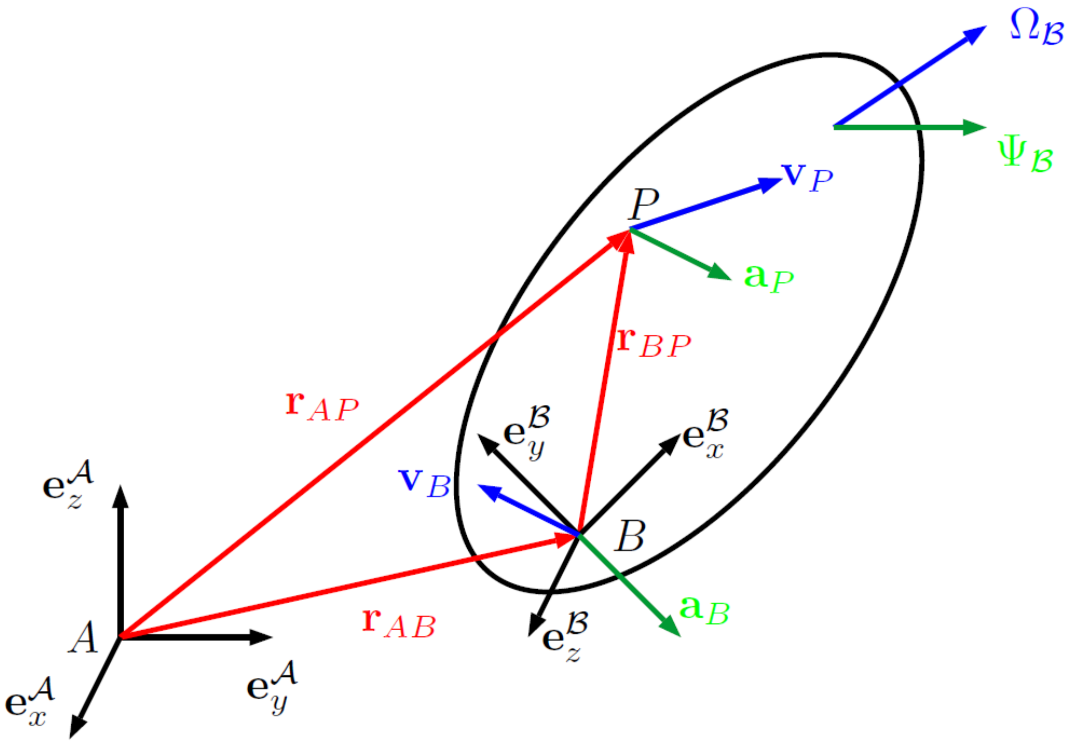
$$ {}_{\mathcal{A}}\dot{\mathbf{r}}_{AP} = {}_{\mathcal{A}}\dot{\mathbf{r}}_{AB} + \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{BP} + \dot{\mathbf{C}}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{BP} $$

- Substitute $\dot{\mathbf{C}}_{\mathcal{AB}} = [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_{\times} \cdot \mathbf{C}_{\mathcal{AB}}$

- Rigid body formulation

$$ \boxed{\begin{aligned} {}_{\mathcal{A}}\dot{\mathbf{r}}_{AP} &= {}_{\mathcal{A}}\dot{\mathbf{r}}_{AB} + [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_{\times} \cdot \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{BP} \\ &= {}_{\mathcal{A}}\dot{\mathbf{r}}_{AB} + {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} \times {}_{\mathcal{A}}\mathbf{r}_{BP} \end{aligned}} $$

$$ \mathbf{v}_P = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{BP} $$



Geometric Jacobian Derivation

- Rigid body formulation at a single element

$$\dot{\mathbf{r}}_{Ik} = \dot{\mathbf{r}}_{I(k-1)} + \boldsymbol{\omega}_{\mathcal{I}(k-1)} \times \mathbf{r}_{(k-1)k}$$

- Apply this to all bodies

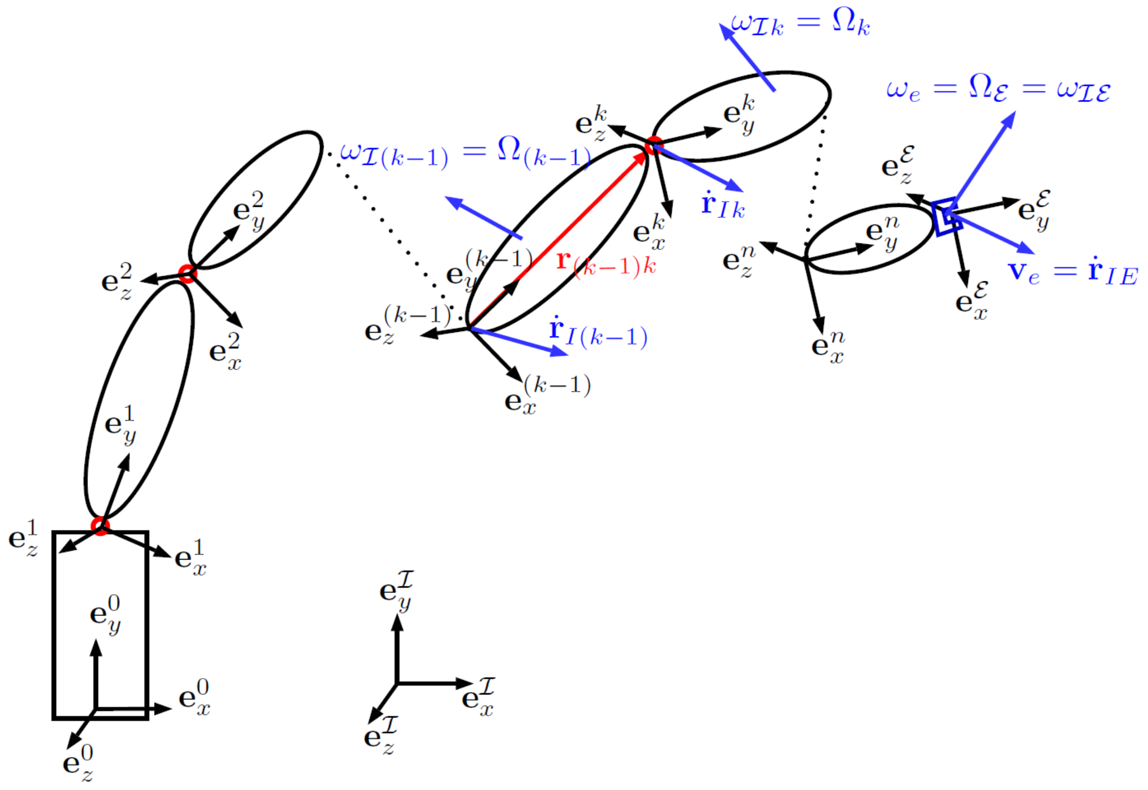
$$\dot{\mathbf{r}}_{IE} = \sum_{k=1}^{n} \boldsymbol{\omega}_{\mathcal{I},k} \times \mathbf{r}_{k(k+1)}$$

- Angular velocity propagation

$$\left.\begin{aligned} \boldsymbol{\omega}_{\mathcal{I}(k)} &= \boldsymbol{\omega}_{\mathcal{I}(k-1)} + \boldsymbol{\omega}_{(k-1)k} \\ \text{with} \quad \boldsymbol{\omega}_{(k-1)k} &= \mathbf{n}_k \dot{q}_k \end{aligned}\right\} \quad \boldsymbol{\omega}_{\mathcal{I}k} = \sum_{i=1}^{k} \mathbf{n}_i \dot{q}_i$$

- …get the end-effector velocity

$$\dot{\mathbf{r}}_{IE} = \sum_{k=1}^{n} \left\{ \sum_{i=1}^{k} (\mathbf{n}_i \dot{q}_i) \times \mathbf{r}_{k(k+1)} \right\} = \sum_{k=1}^{n} \mathbf{n}_k \dot{q}_k \times \mathbf{r}_{k(n+1)}$$



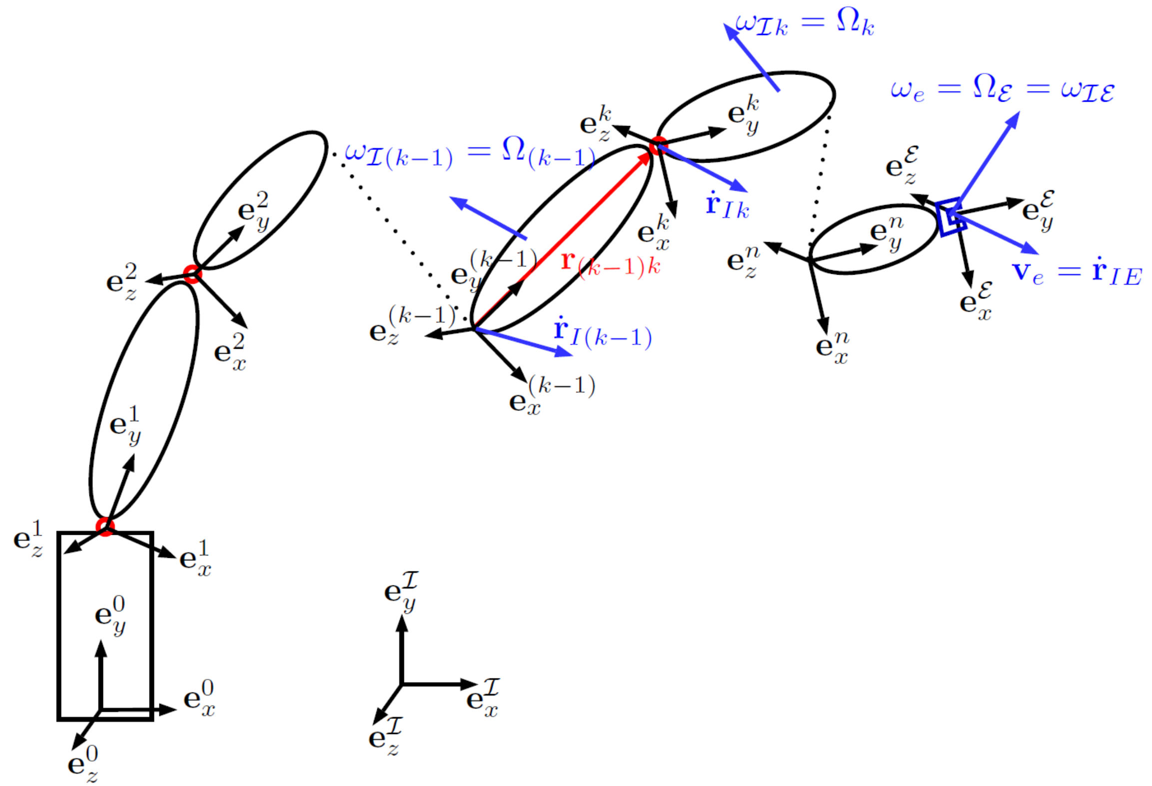
Geometric Jacobian Derivation

- Position Jacobian $\dot{\mathbf{r}}_{IE} = \sum_{k=1}^{n} \mathbf{n}_k \dot{q}_k \times \mathbf{r}_{k(n+1)}$

$$\dot{\mathbf{r}}_{IE} = \underbrace{\begin{bmatrix} \mathbf{n}_1 \times \mathbf{r}_{1(n+1)} & \mathbf{n}_2 \times \mathbf{r}_{2(n+1)} & \dots & \mathbf{n}_n \times \mathbf{r}_{n(n+1)} \end{bmatrix}}_{\mathbf{J}_{e0_P}} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

- Rotation Jacobian from $\boldsymbol{\omega}_{\mathcal{I}k} = \sum_{i=1}^{k} \mathbf{n}_i \dot{q}_i$

$$\boldsymbol{\omega}_{\mathcal{I}\mathcal{E}} = \sum_{i=1}^{n} \mathbf{n}_i \dot{q}_i = \underbrace{\begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \dots & \mathbf{n}_n \end{bmatrix}}_{\mathbf{J}_{e0_R}} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$



$$\Rightarrow \quad {}_{\mathcal{I}}\mathbf{J}_{e0} = \begin{bmatrix} {}_{\mathcal{I}}\mathbf{J}_{e0_P} \\ {}_{\mathcal{I}}\mathbf{J}_{e0_R} \end{bmatrix} = \begin{bmatrix} {}_{\mathcal{I}}\mathbf{n}_1 \times {}_{\mathcal{I}}\mathbf{r}_{1(n+1)} & {}_{\mathcal{I}}\mathbf{n}_2 \times {}_{\mathcal{I}}\mathbf{r}_{2(n+1)} & \dots & {}_{\mathcal{I}}\mathbf{n}_n \times {}_{\mathcal{I}}\mathbf{r}_{n(n+1)} \\ {}_{\mathcal{I}}\mathbf{n}_1 & {}_{\mathcal{I}}\mathbf{n}_2 & \dots & {}_{\mathcal{I}}\mathbf{n}_n \end{bmatrix}$$

$$\boxed{{}_{\mathcal{I}}\mathbf{n}_k = \mathbf{C}_{\mathcal{I}(k-1)} \cdot {}_{(k-1)}\mathbf{n}_k}$$

Geometric Jacobian
Planar robot arm

- Preparation: determine the rotation matrices

$$\mathbf{C}_{\mathcal{I}1} = \begin{bmatrix} c_1 & 0 & s_1 \\ 0 & 1 & 0 \\ -s_1 & 0 & c_1 \end{bmatrix} \qquad \mathbf{C}_{\mathcal{I}2} = \mathbf{C}_{\mathcal{I}1} \cdot \begin{bmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{bmatrix} = \begin{bmatrix} c_{12} & 0 & s_{12} \\ 0 & 1 & 0 \\ -s_{12} & 0 & c_{12} \end{bmatrix} \qquad \mathbf{C}_{\mathcal{I}3} = \ldots$$

- Determine the rotation axes
  - Locally $\quad {}_0\mathbf{n}_1 = {}_1\mathbf{n}_2 = {}_2\mathbf{n}_3 = \mathbf{e}_y$

Inertial frame

$$\begin{aligned} {}_{\mathcal{I}}\mathbf{n}_1 &= {}_0\mathbf{n}_1 = \mathbf{e}_y \\ {}_{\mathcal{I}}\mathbf{n}_2 &= \mathbf{C}_{\mathcal{I}1} \cdot {}_1\mathbf{n}_2 = \mathbf{e}_y \\ {}_{\mathcal{I}}\mathbf{n}_3 &= \mathbf{C}_{\mathcal{I}2} \cdot {}_2\mathbf{n}_3 = \mathbf{e}_y \end{aligned}$$

- Determine the position vectors

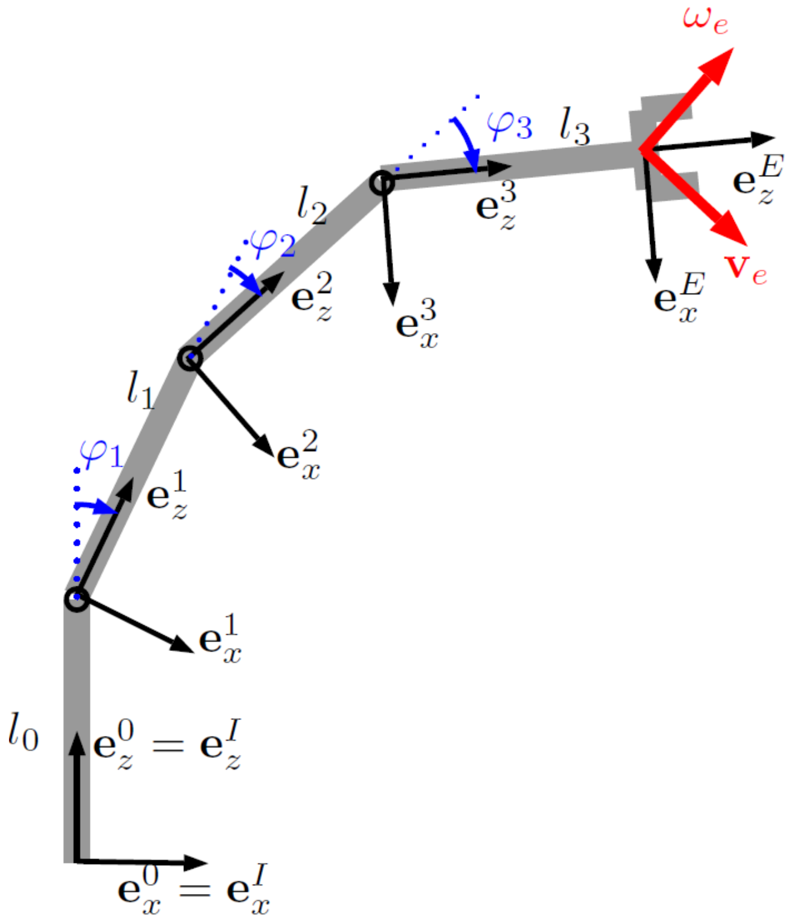
$${}_{\mathcal{I}}\mathbf{r}_{1E} = {}_{\mathcal{I}}\mathbf{r}_{12} + {}_{\mathcal{I}}\mathbf{r}_{23} + {}_{\mathcal{I}}\mathbf{r}_{3E} = \mathbf{C}_{\mathcal{I}1} \cdot {}_1\mathbf{r}_{12} + \mathbf{C}_{\mathcal{I}2} \cdot {}_2\mathbf{r}_{23} + \mathbf{C}_{\mathcal{I}3} \cdot {}_3\mathbf{r}_{3E} = l_1 \begin{pmatrix} s_{q_1} \\ 0 \\ c_{q_1} \end{pmatrix} + l_2 \begin{pmatrix} s_{12} \\ 0 \\ c_{12} \end{pmatrix} + l_3 \begin{pmatrix} s_{123} \\ 0 \\ c_{123} \end{pmatrix}$$

$${}_{\mathcal{I}}\mathbf{r}_{2E} = {}_{\mathcal{I}}\mathbf{r}_{23} + {}_{\mathcal{I}}\mathbf{r}_{3E} = \ldots \qquad {}_{\mathcal{I}}\mathbf{r}_{3E} = \ldots$$

- Get the Jacobian

$$\begin{aligned} {}_{\mathcal{I}}\mathbf{J}_{e0_P} &= \begin{bmatrix} {}_{\mathcal{I}}\mathbf{n}_1 \times {}_{\mathcal{I}}\mathbf{r}_{1E} & {}_{\mathcal{I}}\mathbf{n}_2 \times {}_{\mathcal{I}}\mathbf{r}_{2E} & {}_{\mathcal{I}}\mathbf{n}_3 \times {}_{\mathcal{I}}\mathbf{r}_{3E} \end{bmatrix} \\ &= \begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ -l_1 s_1 - l_2 s_{12} - l_3 c_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} {}_{\mathcal{I}}\mathbf{J}_{e0_R} &= \begin{bmatrix} {}_{\mathcal{I}}\mathbf{n}_1 & {}_{\mathcal{I}}\mathbf{n}_2 & {}_{\mathcal{I}}\mathbf{n}_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$



5

5

Quiz
3min
1min

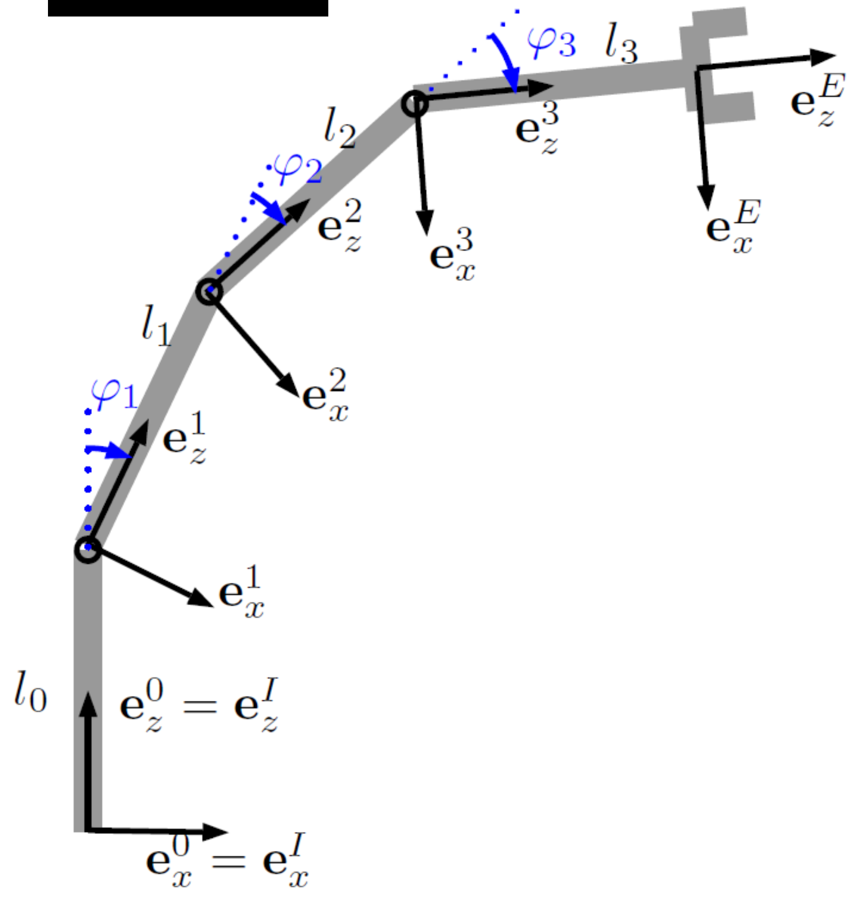
Geometric Jacobian
Planar robot arm

- Given the end-effector velocity

$$\dot{\mathbf{r}}_P = \begin{pmatrix} l_1 c_1 \dot{q}_1 + l_1 c_{12}\left(\dot{q}_1 + \dot{q}_2\right) + l_1 c_{123}\left(\dot{q}_1 + \dot{q}_2 + \dot{q}_3\right) \\ -l_1 s_1 \dot{q}_1 - l_1 s_{12}\left(\dot{q}_1 + \dot{q}_2\right) - l_1 s_{123}\left(\dot{q}_1 + \dot{q}_2 + \dot{q}_3\right) \end{pmatrix}$$

$$\omega = \dot{q}_1 + \dot{q}_2 + \dot{q}_3$$

- Determine the geometric Jacobian



Geometric Jacobian
Planar robot arm

Quiz
3min
1min

- Given the end-effector velocity

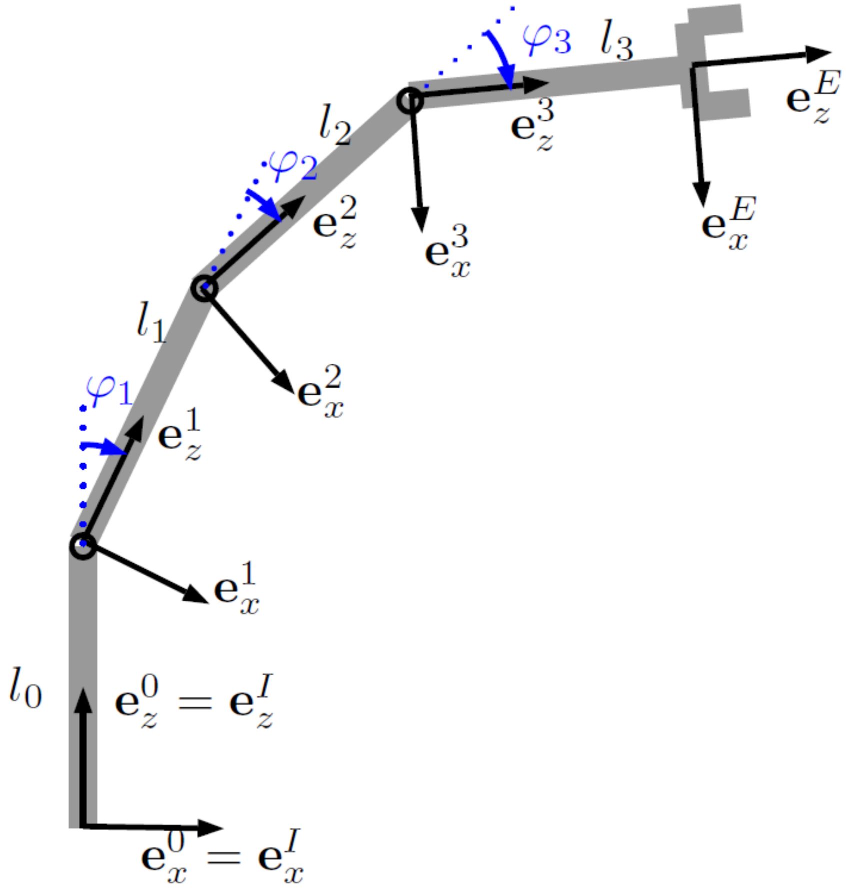
$$\dot{r}_P = \begin{pmatrix} l_1 c_1 \dot{q}_1 + l_1 c_{12}(\dot{q}_1 + \dot{q}_2) + l_1 c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ -l_1 s_1 \dot{q}_1 - l_1 s_{12}(\dot{q}_1 + \dot{q}_2) - l_1 s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \end{pmatrix}$$

$$\omega = \dot{q}_1 + \dot{q}_2 + \dot{q}_3$$

- Determine the geometric Jacobian

$$\mathbf{J}_P = \begin{bmatrix} l_1 c_1 + l_1 c_{12} + l_1 c_{123} & l_1 c_{12} + l_1 c_{123} & l_1 c_{123} \\ -l_1 s_1 - l_1 s_{12} - l_1 s_{123} & -l_1 s_{12} - l_1 s_{123} & -l_1 s_{123} \end{bmatrix}$$

$$\mathbf{J}_R = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$



Recapitulation
Analytical and Kinematic Jacobian

- Analytical Jacobian

$$\dot{\chi}_e = \mathbf{J}_{eA}(\mathbf{q})\,\dot{\mathbf{q}}$$

- Geometric (or basic) Jacobian

$$\mathbf{w}_e = \begin{pmatrix} \mathbf{v}_e \\ \boldsymbol{\omega}_e \end{pmatrix} = \mathbf{J}_{e0}(\mathbf{q})\,\dot{\mathbf{q}}$$

$$\mathbf{J}_{e0}(\mathbf{q}) = \mathbf{E}_e(\chi)\,\mathbf{J}_{eA}(\mathbf{q})$$

Analytical Jacobian:

- Relates time-derivatives of config. parameters to generalized velocities
- Depending on selected parameterization (mainly rotation) in 3D $\Delta\chi \Leftrightarrow \Delta\mathbf{q}$
  Note: there exist no "rotation angle"
- Mainly used for numeric algorithms

Geometric (or basic) Jacobian:

- Relates end-effector velocity to generalized velocities
- Unique for every robot
- Used in most cases

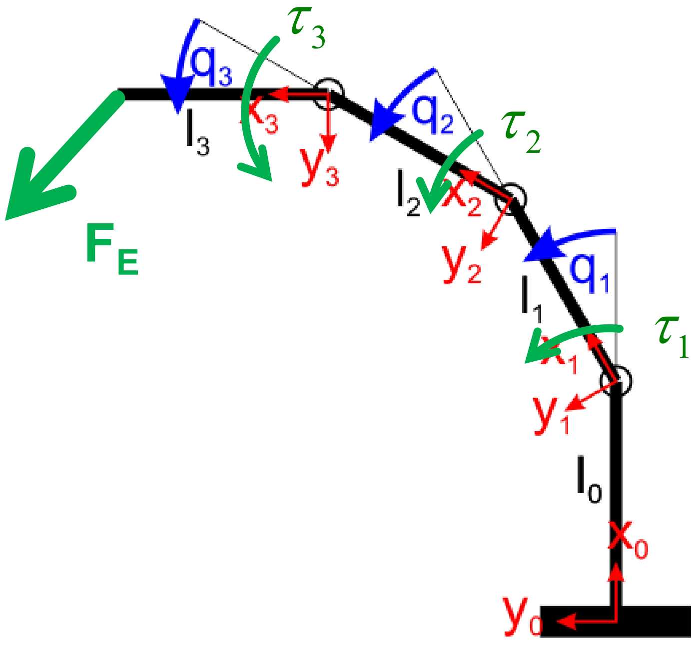
Importance of Jacobian

- Kinematics (mapping of changes from joint to task space)
  - Inverse kinematics control
  - Resolve redundancy problems
  - Express contact constraints
- Statics (and later also dynamics)
  - Principle of virtual work
    - Variations in work must cancel for all virtual displacement
    - Internal forces of ideal joint don’t contribute

$$\delta W = \sum_i \mathbf{f}_i \mathbf{x}_i = \boldsymbol{\tau}^T \delta \mathbf{q} + \left(-\mathbf{F}_E\right)^T \delta \mathbf{x}_E$$
$$= \boldsymbol{\tau}^T \delta \mathbf{q} + \left(-\mathbf{F}_E\right)^T \mathbf{J} \delta \mathbf{q} = 0 \quad \forall \delta \mathbf{q}$$

➢ Dual problem from principle of virtual work

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{J}\dot{\mathbf{q}} \\ \boldsymbol{\tau} &= \mathbf{J}^T \mathbf{F} \end{aligned}}$$



ETH zürich

Floating Base Kinematics

151-0851-00 V

lecture:	HG F3	Tuesday 10:15 – 12:00, every week
exercise:	HG D7.1	Wednesday 8:15 – 10:00, according to schedule

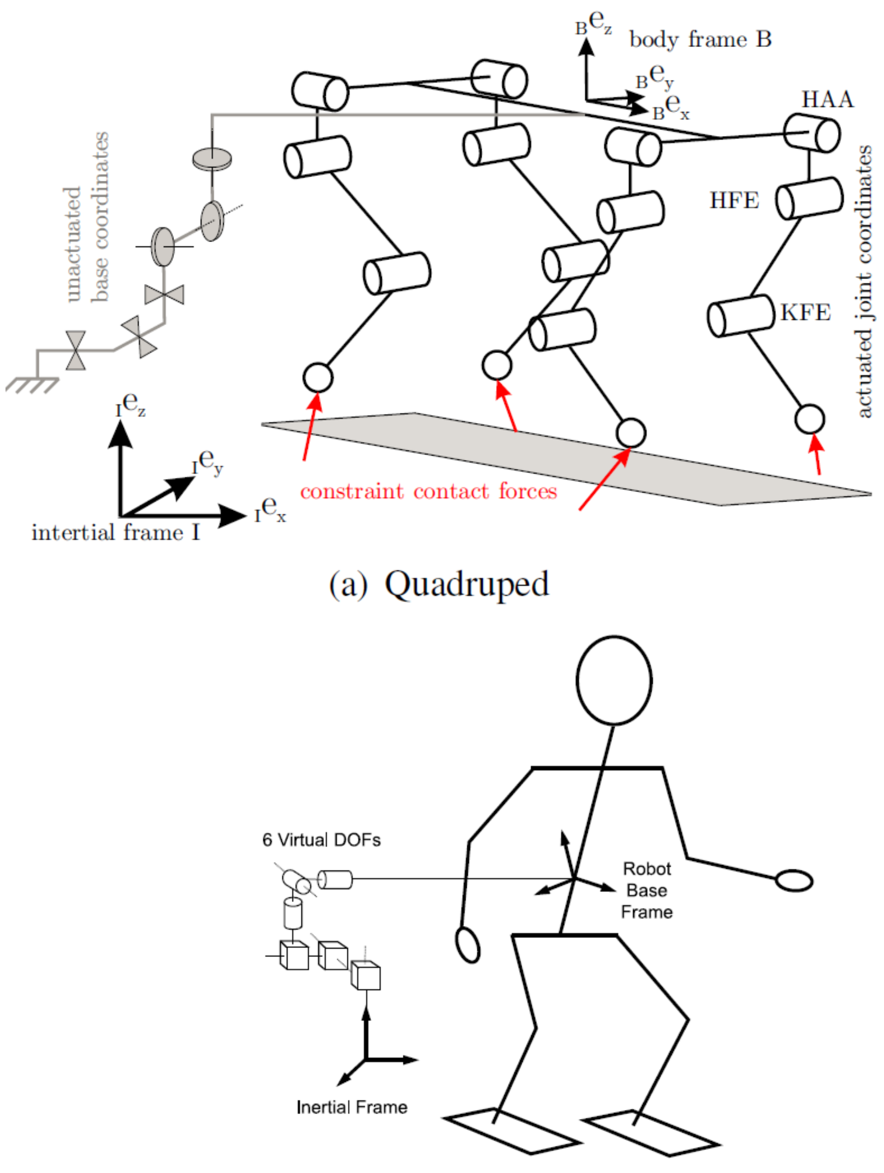
Marco Hutter, Roland Siegwart, and Thomas Stastny

Floating Base Systems
Kinematics

- Generalized coordinates

$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_b \\ \mathbf{q}_j \end{pmatrix} \quad \text{with} \quad \mathbf{q}_b = \begin{pmatrix} \mathbf{q}_{b_P} \\ \mathbf{q}_{b_R} \end{pmatrix} \in \mathbb{R}^3 \times SO(3)$$

- Generalized velocities and accelerations?
  - Time derivatives $\dot{\mathbf{q}}, \ddot{\mathbf{q}}$ depend on parameterization
  - Often $\mathbf{u} = \begin{pmatrix} {}_I\mathbf{v}_B \\ \boxed{{}_B}\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \qquad \dot{\mathbf{u}} = \begin{pmatrix} {}_I\mathbf{a}_B \\ \boxed{{}_B}\boldsymbol{\psi}_{IB} \\ \ddot{\varphi}_1 \\ \vdots \\ \ddot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$
  - Linear mapping $\mathbf{u} = \mathbf{E}_{fb} \cdot \dot{\mathbf{q}}$, with $\mathbf{E}_{fb} = \begin{bmatrix} \mathbb{I}_{3\times 3} & 0 & 0 \\ 0 & \mathbf{E}_{\chi_R} & 0 \\ 0 & 0 & \mathbb{I}_{n_j \times n_j} \end{bmatrix}$



(a) Quadruped

(b) Humanoid

Floating Base Systems
Differential kinematics

- Position of an arbitrary point on the robot

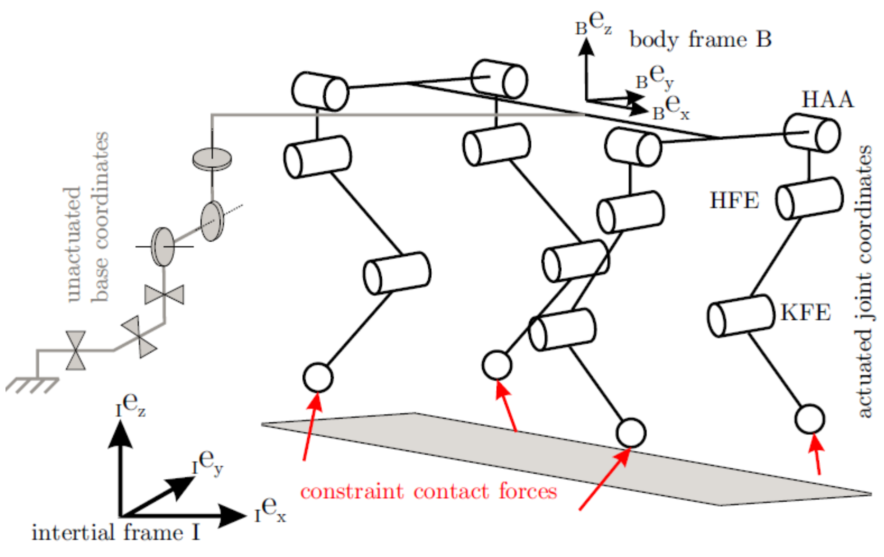
$$
{}_{\mathcal{I}}\mathbf{r}_{IQ}(\mathbf{q}) = \boxed{{}_{\mathcal{I}}\mathbf{r}_{IB}(\mathbf{q})} + \boxed{\mathbf{C}_{\mathcal{IB}}(\mathbf{q})} \cdot \boxed{{}_{\mathcal{B}}\mathbf{r}_{BQ}(\mathbf{q})}
$$

$$
{}_{\mathcal{I}}\mathbf{r}_{IB}(\mathbf{q}_b) \qquad \mathbf{C}_{\mathcal{IB}}(\mathbf{q}_b) \qquad {}_{\mathcal{B}}\mathbf{r}_{BQ}(\mathbf{q}_j)
$$

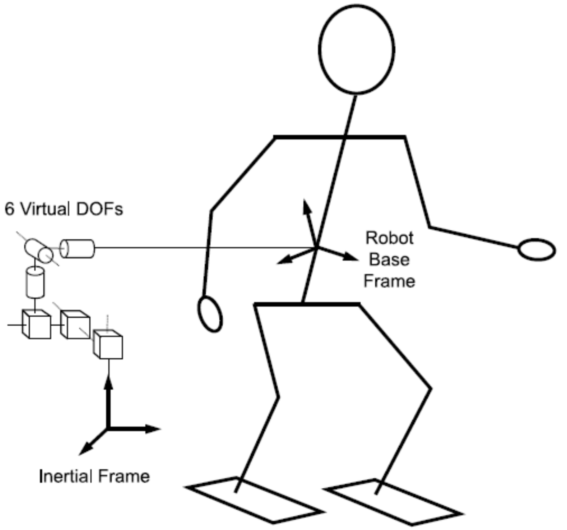
- Velocity of this point

$$
\begin{aligned}
{}_{\mathcal{I}}\mathbf{v}_Q &= {}_{\mathcal{I}}\mathbf{v}_B + \dot{\mathbf{C}}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{BQ} + \mathbf{C}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{BQ} \\
&= {}_{\mathcal{I}}\mathbf{v}_B + \mathbf{C}_{\mathcal{IB}} \cdot [{}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}}]_\times \cdot {}_{\mathcal{B}}\mathbf{r}_{BQ} + \mathbf{C}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{BQ} \\
&= {}_{\mathcal{I}}\mathbf{v}_B - \mathbf{C}_{\mathcal{IB}} \cdot [{}_{\mathcal{B}}\mathbf{r}_{BQ}]_\times \cdot {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}} + \mathbf{C}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{BQ} \\
&= {}_{\mathcal{I}}\mathbf{v}_B - \mathbf{C}_{\mathcal{IB}} \cdot [{}_{\mathcal{B}}\mathbf{r}_{BQ}]_\times \cdot {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}} + \mathbf{C}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \cdot \dot{\mathbf{q}}_j \\
&= \underbrace{\boxed{\begin{bmatrix} \mathbb{I}_{3\times3} & -\mathbf{C}_{\mathcal{IB}} \cdot [{}_{\mathcal{B}}\mathbf{r}_{BQ}]_\times & \mathbf{C}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \end{bmatrix}}}_{{}_{\mathcal{I}}\mathbf{J}_Q(\mathbf{q})} \cdot \mathbf{u} \quad \text{with} \quad \mathbf{u} = \begin{pmatrix} {}_{I}\mathbf{v}_B \\ {}_{B}\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix}
\end{aligned}
$$

$$
[{}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}}]_\times = \mathbf{C}_{\mathcal{BI}} [{}_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{IB}}]_\times \mathbf{C}_{\mathcal{BI}}^T = \mathbf{C}_{\mathcal{IB}}^T \dot{\mathbf{C}}_{\mathcal{IB}} \mathbf{C}_{\mathcal{IB}}^T \mathbf{C}_{\mathcal{IB}} = \mathbf{C}_{\mathcal{IB}}^T \dot{\mathbf{C}}_{\mathcal{IB}}
$$



(a) Quadruped



(b) Humanoid

Contact Constraints

- A contact point C_i is not allowed to move:

$$_{\mathcal{I}}\mathbf{r}_{IC_i} = const, \quad _{\mathcal{I}}\dot{\mathbf{r}}_{IC_i} = {_{\mathcal{I}}\ddot{\mathbf{r}}_{IC_i}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Constraint as a function of generalized coordinates:

$$_{\mathcal{I}}\mathbf{J}_{C_i}\mathbf{u} = \mathbf{0}, \qquad _{\mathcal{I}}\mathbf{J}_{C_i}\dot{\mathbf{u}} + {_{\mathcal{I}}\dot{\mathbf{J}}_{C_i}}\mathbf{u} = \mathbf{0}$$

- Stack of constraints

$$\mathbf{J}_c = \begin{bmatrix} \mathbf{J}_{C_1} \\ \vdots \\ \mathbf{J}_{C_{n_c}} \end{bmatrix} \in \mathbb{R}^{3n_c \times n_n}$$

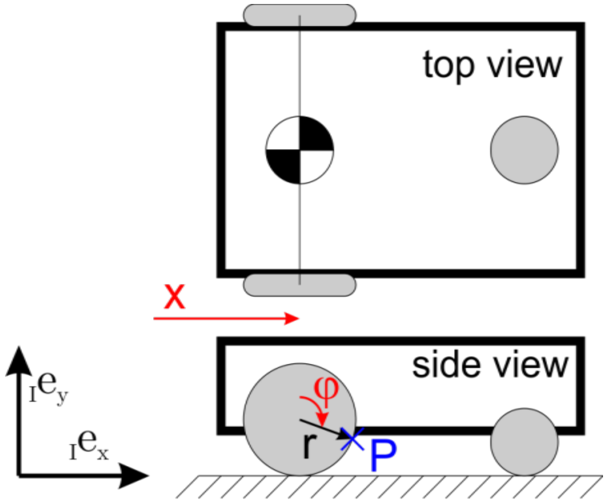
5

5

Quiz
3min
1min

Contact Constraint
Wheeled vehicle simple example

- Contact constraints
  - Point on wheel
  - Jacobian
  - Contact constraints



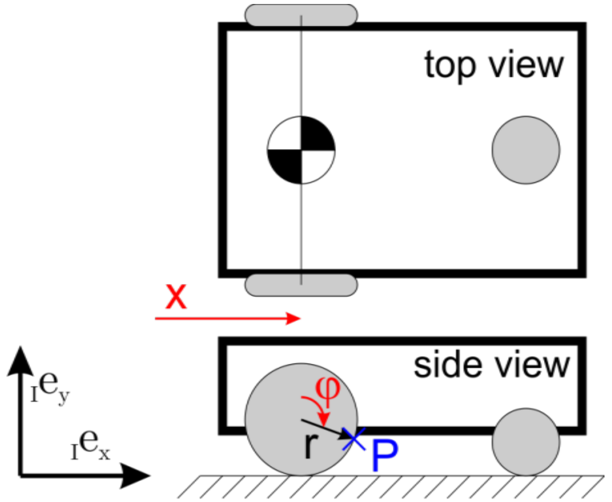
Quiz
3min
1min

Contact Constraint
Wheeled vehicle simple example

- Contact constraints
  - Point on wheel $_{\mathcal{I}}\mathbf{r}_{IP} = \begin{pmatrix} x + r\sin(\varphi) \\ r + r\cos(\varphi) \\ 0 \end{pmatrix}$
  - Jacobian $_{\mathcal{I}}\mathbf{J}_{P} = \begin{bmatrix} 1 & r\cos(\varphi) \\ 0 & -r\sin(\varphi) \\ 0 & 0 \end{bmatrix}$
  - Contact constraints

$$_{\mathcal{I}}\dot{\mathbf{r}}_{IP}\big|_{\varphi=\pi} = {}_{\mathcal{I}}\mathbf{J}_{P}\big|_{\varphi=\pi}\dot{\mathbf{q}} = \begin{bmatrix} 1 & -r \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix} = \mathbf{0}$$

=> Rolling condition $\dot{x} - r\dot{\varphi} = 0$



$$\mathbf{q} = \begin{pmatrix} x \\ \varphi \end{pmatrix}$$

Un-actuated base
Actuated joints

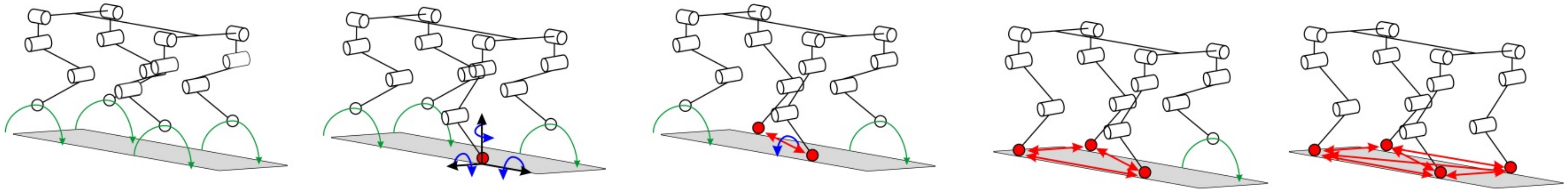
Properties of Contact Jacobian

- Contact Jacobian tells us, how a system can move.
  - Separate stacked Jacobian $\mathbf{J}_c = \begin{bmatrix} \boxed{\mathbf{J}_{c,b}} & \mathbf{J}_{c,j} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{r}_c}{\partial \mathbf{q}_b} & \frac{\partial \mathbf{r}_c}{\partial \mathbf{q}_j} \end{bmatrix} \in \mathbb{R}^{n_c \times (n_b + n_j)}$

    relation between base motion and constraints
  - Base is fully controllable if $rank(\mathbf{J}_{c,b}) = 6$
  - Nr of kinematic constraints for joint actuators: $rank(\mathbf{J}_c)$ - $rank(\mathbf{J}_{c,b})$
- Generalized coordinates DON'T correspond to the degrees of freedom
  - Contact constraints!
- Minimal coordinates (= correspond to degrees of freedom)
  - Require to switch the set of coordinates depending on contact state (=> never used)

Quadrupedal Robot with Point Feet

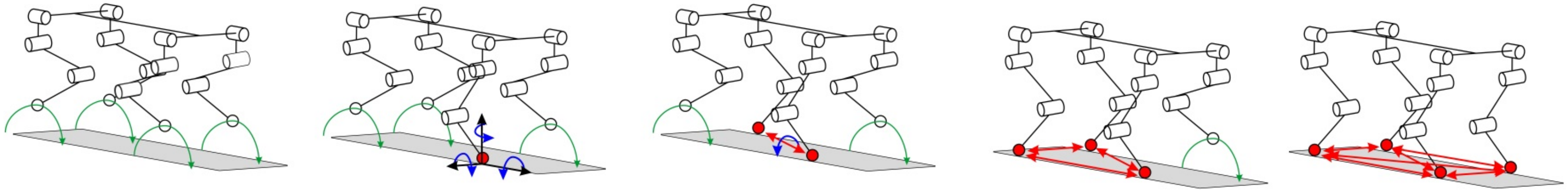
- Floating base system with 12 actuated joint and 6 base coordinates (18DoF)



Total constraints
Internal constraints
Uncontrollable DoFs

Quadrupedal Robot with Point Feet

- Floating base system with 12 actuated joint and 6 base coordinates (18DoF)



Total constraints	0	3	6	9	12
Internal constraints	0	0	1	3	6
Uncontrollable DoFs	6	3	1	0	0

Outlook

- Exercise TOMORROW
  - Differential Kinematics
  - Use it as extended office hour!

- Next Lecture
  - Script Section 2.9 (Kinematic Control Methods)
  - Inverse Kinematics
  - Inverse Differential Kinematics