ETH zürich

Lecture «Robot Dynamics»: Intro to Dynamics

151-0851-00 V

lecture: HG F3 Tuesday 10:15 – 12:00, every week

exercise: HG D7.1 Wednesday 8:15 – 10:00, according to schedule

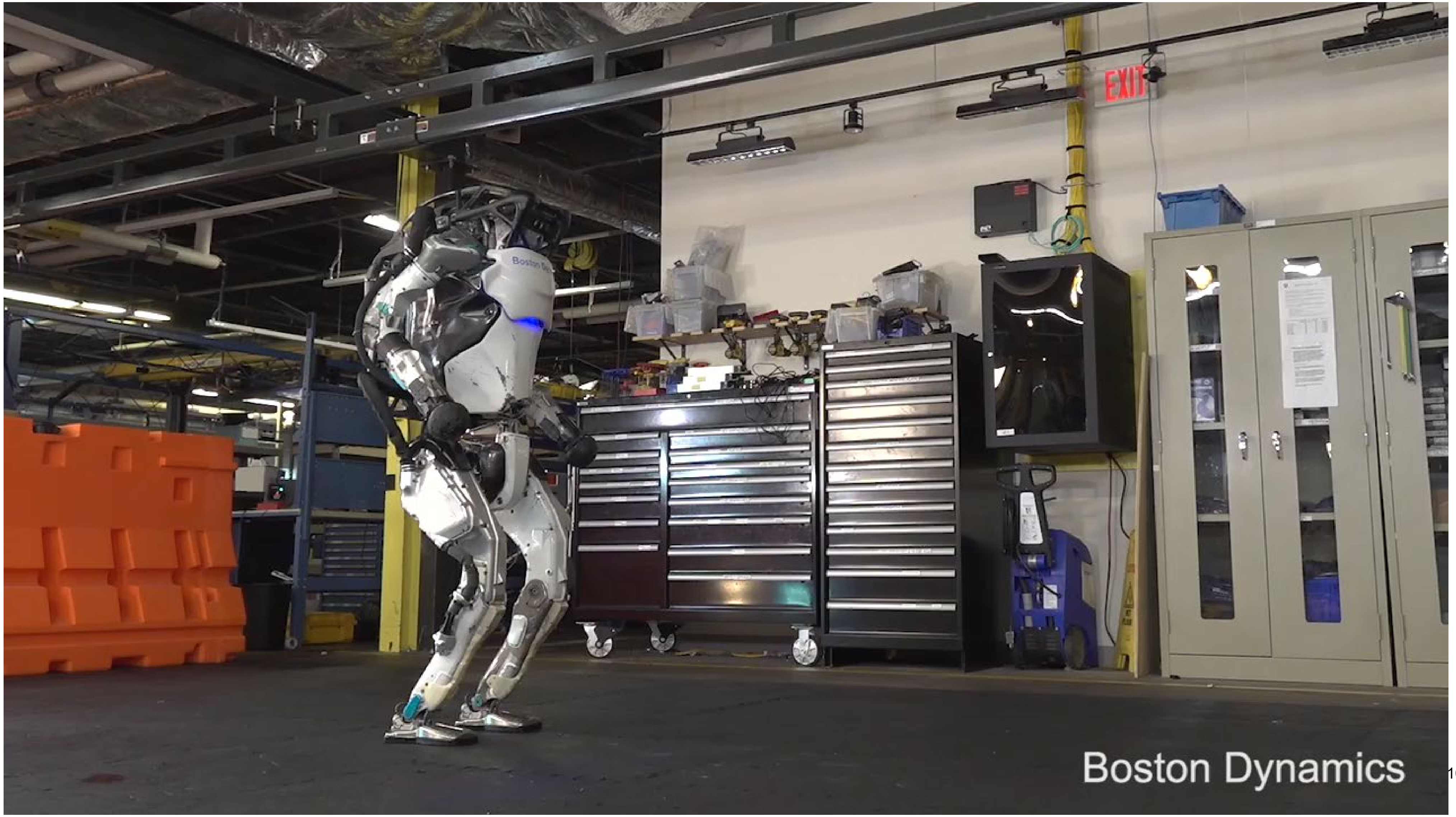
Marco Hutter, Roland Siegwart, and Thomas Stastny

17.09.2019	Intro and Outline	Course Introduction; Recapitulation Position, Linear Velocity			
24.09.2019	Kinematics 1	Rotation and Angular Velocity; Rigid Body Formulation, Transformation	25.09.2019	Exercise 1a	Kinematics Modeling the ABB arm
01.10.2019	Kinematics 2	Kinematics of Systems of Bodies; Jacobians	02.10.2019	Exercise 1a	Differential Kinematics of the ABB arm
08.10.2019	Kinematics 3	Kinematic Control Methods: Inverse Differential Kinematics, Inverse Kinematics; Rotation Error; Multi-task Control	09.10.2019	Exercise 1b	Kinematic Control of the ABB Arm
15.10.2019	Dynamics L1	Multi-body Dynamics	16.10.2019	Midterm 1	Programming kinematics with matlab
22.10.2019	Dynamics L2	Floating Base Dynamics	23.10.2019	Exercise 2a	Dynamic Modeling of the ABB Arm
29.10.2019	Dynamics L3	Dynamic Model Based Control Methods	30.10.2019	Exercise 2b	Dynamic Control Methods Applied to the ABB arm
05.11.2019	Legged Robot	Dynamic Modeling of Legged Robots & Control	06.11.2019	Midterm 2	Programming dynamics with matlab
12.11.2019	Case Studies 1	Legged Robotics Case Study	13.11.2019	Exercise 3	Legged robot
19.11.2019	Rotorcraft	Dynamic Modeling of Rotorcraft & Control	20.11.2019		
26.11.2019	Case Studies 2	Rotor Craft Case Study	27.11.2019	Exercise 4	Modeling and Control of Multicopter
03.12.2019	Fixed-wing	Dynamic Modeling of Fixed-wing & Control	04.12.2019		
10.12.2019	Case Studies 3	Fixed-wing Case Study (Solar-powered UAVs - AtlantikSolar, Vertical Take-off and Landing UAVs – Wingtra)	11.12.2019	Exercise 5	Fixed-wing Control and Simulation
17.12.2019	Summery and Outlook	Summery; Wrap-up; Exam			

Midterm exam tomorrow

- WED 8.15-10
  - Like the exercises, the exam will be done with matlab. Bring your own laptop! We will distribute the exam files at the beginning of the exam through piazza.
  - Additionally, you will get a printout version of the exam to write down your name and to answer a couple of questions.
  - We will ask you to write down numeric results that are generated with matlab on your paper
  - You have to hand in the exam paper and upload the matlab files in a single compressed folder (.zip, .gz, .tar, .rar).
  - We check the matlab files if they produce the correct results.
  - The exam will be open book, which means you can use the script, slides, exercises, etc.
  - You are not allowed to communicate or share results.
  - The use of internet (beside for licenses) is forbidden.
- Due to limited space, the group will be split based on your surname:
  - A-K in HG F3
  - L-Z in HG D7.1

Dynamics in Robotics



Dynamics in Robotics

Dynamics in Robotics

Dynamics
Outline

- Description of “cause of motion”
  - Input $\boldsymbol{\tau}$ Force/Torque acting on system
  - Output $\ddot{\mathbf{q}}$ Motion of the system
- Principle of virtual work
  - Newton’s law for particles
  - ➢ Conservation of impulse and angular momentum
- 3 methods to get the EoM
  - Newton-Euler: Free cut and conservation of impulse & angular momentum for each body
  - Projected Newton-Euler (generalized coordinates)
  - Lagrange II (energy)
- Introduction to dynamics of floating base systems
  - External forces

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} + \mathbf{J}_c^T \mathbf{F}_c$$

$\ddot{\mathbf{q}}$	Generalized coordinates
$\mathbf{M}(\mathbf{q})$	Mass matrix
$\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})$	Centrifugal and Coriolis forces
$\mathbf{g}(\mathbf{q})$	Gravity forces
$\boldsymbol{\tau}$	Generalized forces
$\mathbf{F}_c$	External forces
$\mathbf{J}_c$	Contact Jacobian

Principle of Virtual Work

- Principle of virtual work (D'Alembert's Principle)
  - Dynamic equilibrium imposes zero virtual work

$$\delta W = \int_{\mathcal{B}} \delta\mathbf{r}^T \cdot (\ddot{\mathbf{r}} \mathrm{d}m - \mathrm{d}\mathbf{F}_{ext}) = 0$$

variational parameter

$\mathrm{d}\mathbf{F}_{ext}$	external forces acting on element i
$\ddot{\mathbf{r}}$	acceleration of element i
dm	mass of element i
$\delta\mathbf{r}$	virtual displacement of element i

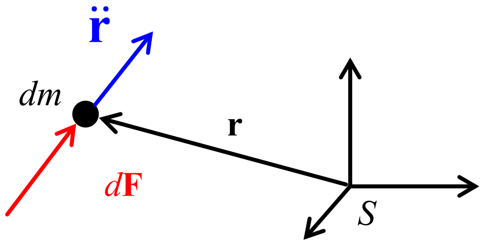
  - Newton's law for every particle in direction it can move

$$\mathbf{F} = m\mathbf{a} \qquad \mathbf{r} \times \mathbf{F} = \mathbf{r} \times m\mathbf{a}$$

$$\mathbf{F} = \frac{d}{dt}\Big(\underbrace{m\mathbf{v}}_{\mathbf{p}}\Big) = \dot{\mathbf{p}}$$

force

Impulse or
linear momentum



$$\mathbf{\Gamma} = \frac{d}{dt}\Big(\underbrace{\mathbf{r} \times m\mathbf{v}}_{\mathbf{N}}\Big) = \dot{\mathbf{N}}$$

moment

angular momentum

Virtual Displacements of Single Rigid Bodies

$$\delta W = \int_{\mathcal{B}} \delta \mathbf{r}^T \cdot (\ddot{\mathbf{r}} dm - d\mathbf{F}_{ext}) = 0$$

- Rigid body Kinematics

$$\mathbf{r} = \mathbf{r}_{OS} + \boldsymbol{\rho}$$

$$\dot{\mathbf{r}} = \mathbf{v}_S + \boldsymbol{\Omega} \times \boldsymbol{\rho} = \begin{bmatrix} \mathbb{I}_{3\times3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{pmatrix} \mathbf{v}_s \\ \boldsymbol{\Omega} \end{pmatrix}$$

$$\ddot{\mathbf{r}} = \mathbf{a}_S + \Psi \times \boldsymbol{\rho} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho}) = \begin{bmatrix} \mathbb{I}_{3\times3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{pmatrix} \mathbf{a}_s \\ \boldsymbol{\Psi} \end{pmatrix} + [\boldsymbol{\Omega}]_\times [\boldsymbol{\Omega}]_\times \boldsymbol{\rho}$$

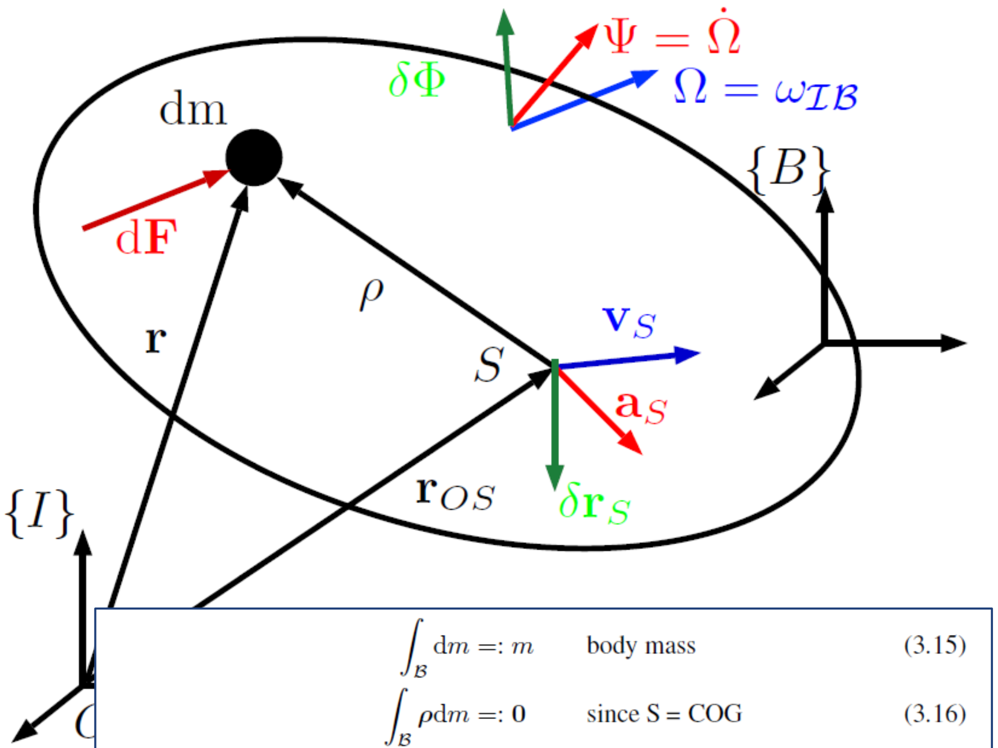
$$\delta \mathbf{r} = \delta \mathbf{r}_S + \boldsymbol{\delta\Phi} \times \boldsymbol{\rho} = \begin{bmatrix} \mathbb{I}_{3\times3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{pmatrix} \delta \mathbf{r}_s \\ \boldsymbol{\delta\Phi} \end{pmatrix}$$

- Applied to principle of virtual work

$$0 = \delta W = \int_{\mathcal{B}} \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \boldsymbol{\Phi} \end{pmatrix}^T \begin{bmatrix} \mathbb{I}_{3\times3} \\ [\boldsymbol{\rho}]_\times \end{bmatrix} \left(\begin{bmatrix} \mathbb{I}_{3\times3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{pmatrix} \mathbf{a}_s \\ \boldsymbol{\Psi} \end{pmatrix} dm + [\boldsymbol{\Omega}]_\times^2 \boldsymbol{\rho} dm - d\mathbf{F}_{ext} \right)$$

$$= \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \boldsymbol{\Phi} \end{pmatrix}^T \int_{\mathcal{B}} \left(\begin{bmatrix} \mathbb{I}_{3\times3} dm & [\boldsymbol{\rho}]_\times^T dm \\ [\boldsymbol{\rho}]_\times dm & -[\boldsymbol{\rho}]_\times^2 dm \end{bmatrix} \begin{pmatrix} \mathbf{a}_s \\ \boldsymbol{\Psi} \end{pmatrix} + \begin{pmatrix} [\boldsymbol{\Omega}]_\times^2 \boldsymbol{\rho} dm \\ [\boldsymbol{\rho}]_\times [\boldsymbol{\Omega}]_\times^2 \boldsymbol{\rho} dm \end{pmatrix} - \begin{pmatrix} d\mathbf{F}_{ext} \\ [\boldsymbol{\rho}]_\times d\mathbf{F}_{ext} \end{pmatrix} \right)$$

$$\boxed{0 = \delta W = \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \boldsymbol{\Phi} \end{pmatrix}^T \left(\begin{bmatrix} \mathbb{I}_{3\times3} m & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Theta}_S \end{bmatrix} \begin{pmatrix} \mathbf{a}_s \\ \boldsymbol{\Psi} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ [\boldsymbol{\Omega}]_\times \boldsymbol{\Theta}_S \boldsymbol{\Omega} \end{pmatrix} - \begin{pmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext} \end{pmatrix} \right) \quad \forall \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \boldsymbol{\Phi} \end{pmatrix}}$$



$\int_{\mathcal{B}} dm =: m$	body mass	(3.15)
$\int_{\mathcal{B}} \rho dm =: 0$	since S = COG	(3.16)
$\int_{\mathcal{B}} -[\rho]_\times^2 dm = \int_{\mathcal{B}} [\rho]_\times [\rho]_\times^T dm =: \boldsymbol{\Theta}_S$	Inertia matrix around COG	(3.17)

Impulse and angular momentum

- Use the following definitions

$$\mathbf{p}_S = m\mathbf{v}_S \qquad \text{linear momentum}$$

$$\mathbf{N}_S = \mathbf{\Theta}_S \mathbf{\Omega}_S \qquad \text{angular momentum}$$

$$\dot{\mathbf{p}}_S = m\mathbf{a}_S \qquad \text{change in linear momentum}$$

$$\dot{\mathbf{N}}_S = \mathbf{\Theta}_S \mathbf{\Psi} + \mathbf{\Omega} \times \mathbf{\Theta}_S \mathbf{\Omega} \qquad \text{change in angular momentum}$$

- Conservation of impulse and angular momentum

Newton

$$0 = \delta W = \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix}^T \left(\begin{pmatrix} \dot{\mathbf{p}}_S \\ \dot{\mathbf{N}}_S \end{pmatrix} - \begin{pmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext} \end{pmatrix} \right) \quad \forall \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix}$$

Euler

A free body can move
In all directions

$$\boxed{\begin{aligned} \dot{\mathbf{p}}_S &= \mathbf{F}_{ext} \\ \dot{\mathbf{N}}_S &= \mathbf{T}_{ext} \end{aligned}}$$

External forces and moments

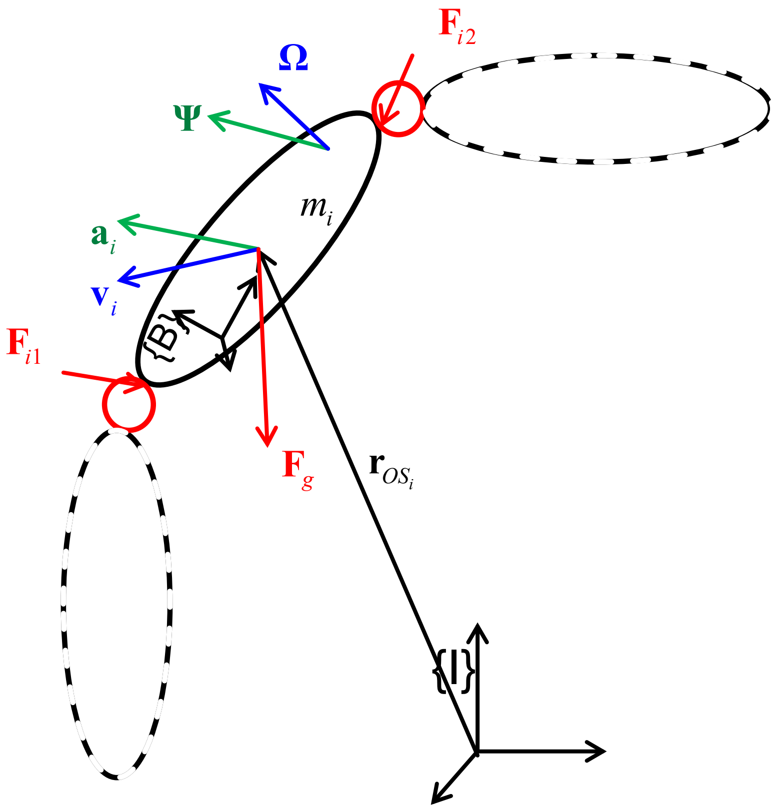
Change in impulse and angular momentum

1^{st} Method for EoM
Newton-Euler for single bodies

- Cut all bodies free
- Introduction of constraining force
- Apply conservation $\mathbf{p}$ and $\mathbf{N}$ to individual bodies

➢ System of equations
  - 6n equation
  - Eliminate all constrained forces (5n)

- Pros and Cons
  + Intuitively clear
  + Direct access to constraining forces
  − Becomes a huge combinatorial problem for large MBS



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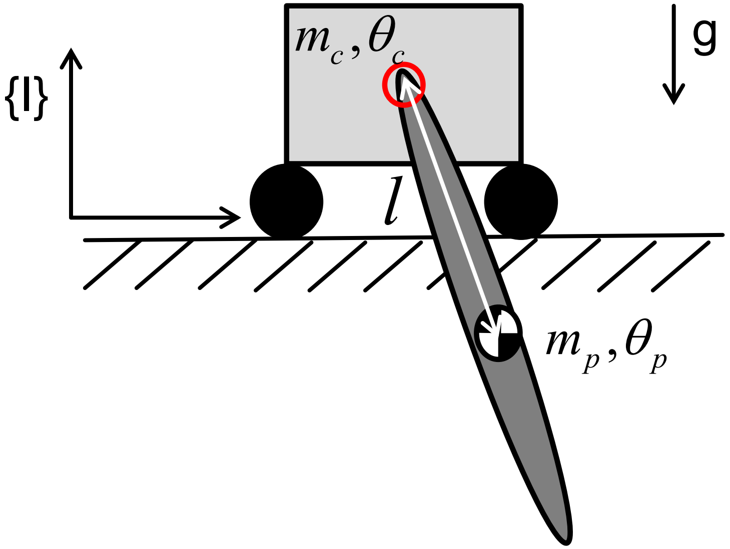
Quiz
3min
1min

Free Cut
Cart pendulum example

- Find the equation of motion



Quiz
3min
1min

Free Cut
Cart pendulum

- Impulse / angular momentum cart

$$m_c \ddot{x}_c = F_x \quad (1)$$
$$m_c \ddot{y}_c = F_y + F_l + F_r - m_c g \quad (2)$$
$$\theta_c \ddot{\varphi}_c = F_r b - F_l b \quad (3)$$

- Impulse / angular momentum pendulum

$$m_p \ddot{x}_p = -F_x \quad (4)$$
$$m_p \ddot{y}_p = -F_y - m_p g \quad (5)$$
$$\theta_p \ddot{\varphi}_p = F_x l \cos(\varphi_p) + F_y l \sin(\varphi_p) \quad (6)$$

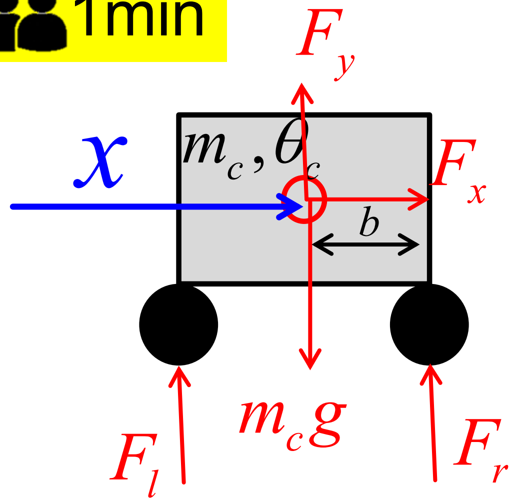
- Kinematics

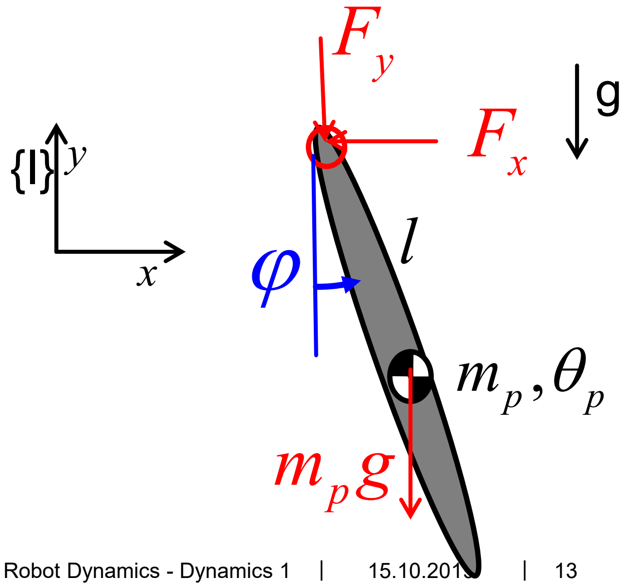
$$x_c = x \quad (7)$$
$$y_c = 0 \text{ (constraint)} \quad (8)$$
$$\varphi_c = 0 \text{ (constraint)} \quad (9)$$

$$x_p = x + l\sin(\varphi) \quad (10a) \qquad \ddot{x}_p = \ddot{x} + \ddot{\varphi} l \cos(\varphi) - \dot{\varphi}^2 l \sin(\varphi) \quad (10)$$
$$y_p = -l\cos(\varphi) \quad (11a) \qquad \ddot{y}_p = \ddot{\varphi} l \sin(\varphi) + \dot{\varphi}^2 l \cos(\varphi) \quad (11)$$
$$\varphi_p = \varphi \quad (12a) \qquad \ddot{\varphi}_p = \ddot{\varphi} \quad (12)$$

6 equations, 6 unknowns resp.
12 equations, 12 unknowns

How many dimensions does the EoM have?





Free Cut
Cart pendulum

- (7),(10-12) in (1) and (4-6)

$$m_c\ddot{x} = F_x \tag{13}$$

$$m_p\left(\ddot{x} + \ddot{\varphi} l\cos(\varphi) - \dot{\varphi}^2 l\sin(\varphi)\right) = -F_x \tag{14}$$

$$m_p\left(\ddot{\varphi} l\sin(\varphi) + \dot{\varphi}^2 l\cos(\varphi)\right) = -F_y - m_p g \tag{15}$$

$$\theta_p\ddot{\varphi} = F_x l\cos(\varphi) + F_y l\sin(\varphi) \tag{16}$$

- From (13) and (14) remove F_x

$$\left(m_p + m_c\right)\ddot{x} + m_p l\cos(\varphi)\ddot{\varphi} - \dot{\varphi}^2 l m_p\sin(\varphi) = 0$$

- Insert (13) and (15) in (16) to remove F_x and F_y

$$\left(\theta_p + m_p l^2\right)\ddot{\varphi} + m_p l\cos(\varphi)\ddot{x} + g l m_p\sin(\varphi) = 0$$

$$m_c\ddot{x}_c = F_x \tag{1}$$

$$m_c\ddot{y}_c = F_y + F_l + F_r - m_c g \tag{2}$$

$$\theta_c\ddot{\varphi}_c = F_r b - F_l b \tag{3}$$

$$m_p\ddot{x}_p = -F_x \tag{4}$$

$$m_p\ddot{y}_p = -F_y - m_p g \tag{5}$$

$$\theta_p\ddot{\varphi}_p = F_x l\cos(\varphi_p) + F_y l\sin(\varphi_p) \tag{6}$$

$$x_c = x \tag{7}$$

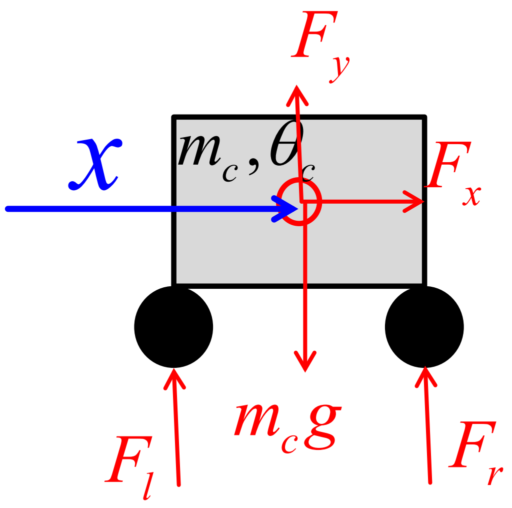
$$y_c = 0 \text{ (constraint)} \tag{8}$$

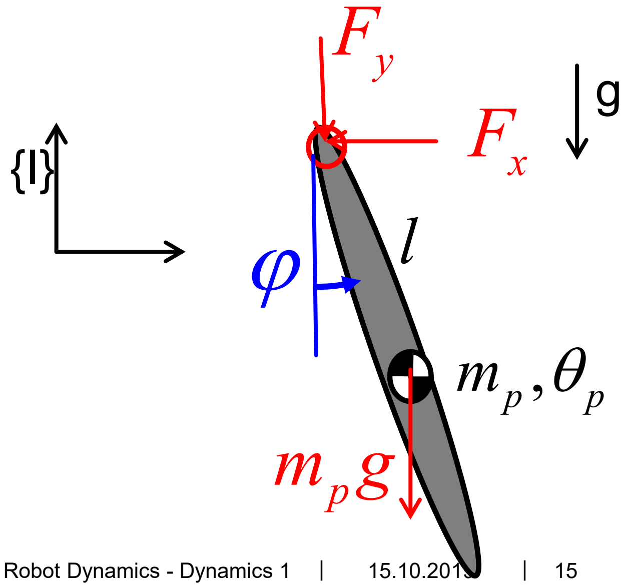
$$\varphi_c = 0 \text{ (constraint)} \tag{9}$$

$$\ddot{x}_p = \ddot{x} + \ddot{\varphi} l\cos(\varphi) - \dot{\varphi}^2 l\sin(\varphi) \tag{10}$$

$$\ddot{y}_p = \ddot{\varphi} l\sin(\varphi) + \dot{\varphi}^2 l\cos(\varphi) \tag{11}$$

$$\ddot{\varphi}_p = \ddot{\varphi} \tag{12}$$





Newton-Euler in Generalized Motion Directions

- For multi-body systems $$0 = \delta W = \sum_{i=1}^{n_b} \begin{pmatrix} \delta \mathbf{r}_{S_i} \\ \delta \mathbf{\Phi}_{S_i} \end{pmatrix}^T \left(\begin{pmatrix} \dot{\mathbf{p}}_{S_i} \\ \dot{\mathbf{N}}_{S_i} \end{pmatrix} - \begin{pmatrix} \mathbf{F}_{ext,i} \\ \mathbf{T}_{ext,i} \end{pmatrix} \right) \qquad \boxed{\forall \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi}_{S_i} \end{pmatrix}_{\text{consistent}}}$$
- Express the impulse/angular momentum in generalized coordinates

$$\begin{pmatrix} \mathbf{v}_s \\ \mathbf{\Omega} \end{pmatrix} = \begin{bmatrix} \mathbf{J}_P \\ \mathbf{J}_R \end{bmatrix} \dot{\mathbf{q}} \qquad \begin{pmatrix} \dot{\mathbf{p}}_{S_i} \\ \dot{\mathbf{N}}_{S_i} \end{pmatrix} = \begin{pmatrix} m\mathbf{a}_{S_i} \\ \mathbf{\Theta}_{S_i} \mathbf{\Psi}_{S_i} + \mathbf{\Omega}_{S_i} \times \mathbf{\Theta}_{S_i} \mathbf{\Omega}_{S_i} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{a}_s \\ \mathbf{\Psi} \end{pmatrix} = \begin{bmatrix} \mathbf{J}_P \\ \mathbf{J}_R \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} \dot{\mathbf{J}}_P \\ \dot{\mathbf{J}}_R \end{bmatrix} \dot{\mathbf{q}} \qquad = \begin{pmatrix} m\mathbf{J}_{S_i} \\ \mathbf{\Theta}_{S_i} \mathbf{J}_{R_i} \end{pmatrix} \ddot{\mathbf{q}} + \begin{pmatrix} m\dot{\mathbf{J}}_{S_i} \dot{\mathbf{q}} \\ \mathbf{\Theta}_{S_i} \dot{\mathbf{J}}_{R_i} \dot{\mathbf{q}} + \mathbf{J}_{R_i} \dot{\mathbf{q}} \times \mathbf{\Theta}_{S_i} \mathbf{J}_{R_i} \dot{\mathbf{q}} \end{pmatrix}$$

- Virtual displacement in generalized coordinates

$$\begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix} = \begin{bmatrix} \mathbf{J}_P \\ \mathbf{J}_R \end{bmatrix} \delta \mathbf{q}$$

- With this, the principle of virtual work transforms to

$$0 = \delta \mathrm{W} = \delta \mathbf{q}^T \sum_{i=1}^{n_b} \underbrace{\begin{pmatrix} \mathbf{J}_{S_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \begin{pmatrix} m\mathbf{J}_{S_i} \\ \mathbf{\Theta}_{S_i} \mathbf{J}_{R_i} \end{pmatrix}}_{\mathbf{M}(\mathbf{q})} \ddot{\mathbf{q}} + \underbrace{\begin{pmatrix} \mathbf{J}_{S_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \begin{pmatrix} m\dot{\mathbf{J}}_{S_i} \dot{\mathbf{q}} \\ \mathbf{\Theta}_{S_i} \dot{\mathbf{J}}_R \dot{\mathbf{q}} + \mathbf{J}_{R_i} \dot{\mathbf{q}} \times \mathbf{\Theta}_{S_i} \mathbf{J}_{R_i} \dot{\mathbf{q}} \end{pmatrix}}_{\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}})} - \underbrace{\begin{pmatrix} \mathbf{J}_{P_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \begin{pmatrix} \mathbf{F}_{ext,i} \\ \mathbf{T}_{ext,i} \end{pmatrix}}_{\mathbf{g}(\mathbf{q})} \qquad \boxed{\forall \delta \mathbf{q}}$$

Projected Newton-Euler

- Equation of motion $\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}+\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})+\mathbf{g}(\mathbf{q})=\mathbf{0}$
- Directly get the dynamic properties of a multi-body system with *n* bodies

$$\mathbf{M} = \sum_{i=1}^{n_b} \left({}_{\mathcal{A}}\mathbf{J}_{S_i}^T \cdot m \cdot {}_{\mathcal{A}}\mathbf{J}_{S_i} + {}_{\mathcal{B}}\mathbf{J}_{R_i}^T \cdot {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\mathbf{J}_{R_i} \right)$$

$$\mathbf{b} = \sum_{i=1}^{n_b} \left({}_{\mathcal{A}}\mathbf{J}_{S_i}^T \cdot m \cdot {}_{\mathcal{A}}\dot{\mathbf{J}}_{S_i} \cdot \dot{\mathbf{q}} + {}_{\mathcal{B}}\mathbf{J}_{R_i}^T \cdot \left({}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\dot{\mathbf{J}}_{R_i} \cdot \dot{\mathbf{q}} + {}_{\mathcal{B}}\mathbf{\Omega}_{S_i} \times {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\mathbf{\Omega}_{S_i} \right) \right)$$

$$\mathbf{g} = \sum_{i=1}^{n_b} \left(-{}_{\mathcal{A}}\mathbf{J}_{S_i}^T \, {}_{\mathcal{A}}\mathbf{F}_{g,i} \right)$$

- For actuated systems, include actuation force as external force for each body
  - If actuators act in the direction of generalized coordinates, $\boldsymbol{\tau}$ corresponds to stacked actuator commands

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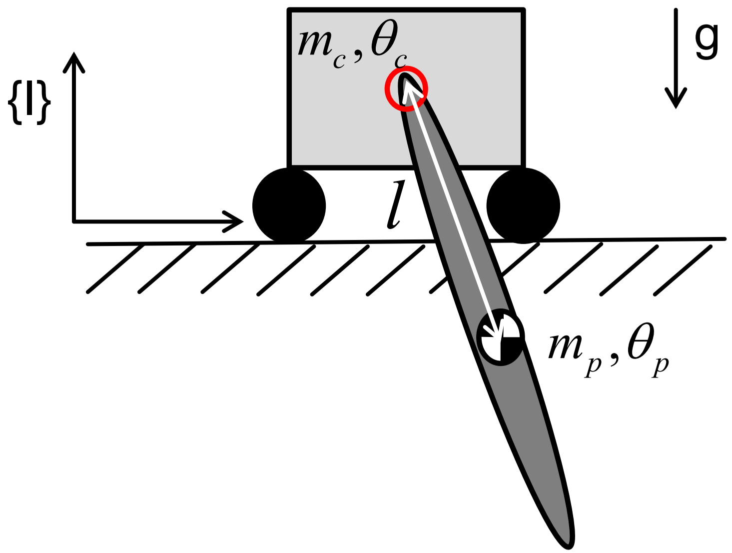
Quiz
3min
1min

Projected Newton-Euler
Cart pendulum example

- Find the equation of motion



Projected Newton-Euler
Cart pendulum example

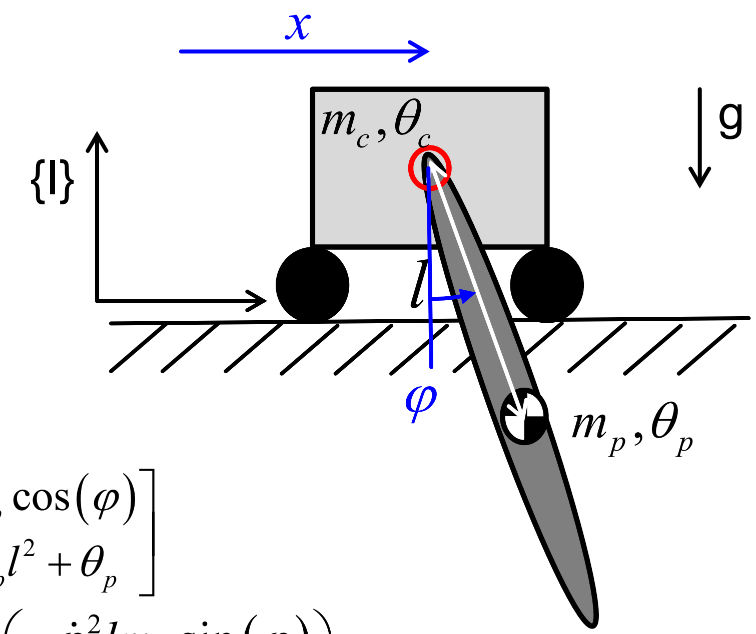
- Kinematics cart and pendulum

$$\mathbf{q} = \begin{pmatrix} x \\ \varphi \end{pmatrix}$$

$$\mathbf{r}_{OS_c} = \begin{pmatrix} x \\ 0 \end{pmatrix} \qquad \mathbf{r}_{OS_p} = \begin{pmatrix} x + l\sin(\varphi) \\ -l\cos(\varphi) \end{pmatrix}$$

$$\mathbf{J}_{Pc} = \frac{\partial \mathbf{r}_{OS_c}}{\partial \mathbf{q}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{J}_{Pp} = \frac{\partial \mathbf{r}_{OS_p}}{\partial \mathbf{q}} = \begin{bmatrix} 1 & l\cos(\varphi) \\ 0 & l\sin(\varphi) \end{bmatrix} \qquad \omega_p = \dot{\varphi} \qquad \mathbf{J}_{Rp} = \frac{\partial \omega_p}{\partial \dot{\mathbf{q}}} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\dot{\mathbf{J}}_{Pc} = \frac{d\mathbf{J}_{Pc}}{dt} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \dot{\mathbf{J}}_{Pp} = \frac{d\mathbf{J}_{Pp}}{dt} = \begin{bmatrix} 0 & -l\sin(\varphi)\dot{\varphi} \\ 0 & l\cos(\varphi)\dot{\varphi} \end{bmatrix}$$



- Equation of motion

$$\mathbf{M} = \sum \mathbf{J}_{P_i}^T m_i \mathbf{J}_{P_i} + \mathbf{J}_{R_i}^T \Theta_i \mathbf{J}_{R_i} = \mathbf{J}_{Pc}^T m_c \mathbf{J}_{Pc} + \mathbf{J}_{Pp}^T m_p \mathbf{J}_{Pp} + \mathbf{J}_{Rp}^T \theta_p \mathbf{J}_{Rp} = \begin{bmatrix} m_c + m_p & l m_p \cos(\varphi) \\ l m_p \cos(\varphi) & m_p l^2 + \theta_p \end{bmatrix}$$

$$\mathbf{b} = \sum \mathbf{J}_{P_i}^T m_i \dot{\mathbf{J}}_{P_i} \dot{\mathbf{q}} + \mathbf{J}_{R_i}^T \Theta_i \dot{\mathbf{J}}_{R_i} \dot{\mathbf{q}} + \overbrace{\cancel{\left(\mathbf{J}_{R_i}\dot{\mathbf{q}}\right) \times \Theta_i \mathbf{J}_{R_i} \dot{\mathbf{q}}}}^{=0 \text{ (planar system)}} = \mathbf{J}_{Pp}^T m_p \dot{\mathbf{J}}_{Pp} \dot{\mathbf{q}} = \begin{pmatrix} -\dot{\varphi}^2 l m_p \sin(\varphi) \\ 0 \end{pmatrix}$$

$$\mathbf{g} = \sum_{i=1}^{n} -\mathbf{J}_{s_i}^T \mathbf{F}_i^g = -\mathbf{J}_{P_c}^T \begin{pmatrix} 0 \\ -m_c g \end{pmatrix} - \mathbf{J}_{P_p}^T \begin{pmatrix} 0 \\ -m_p g \end{pmatrix} = \begin{pmatrix} 0 \\ m_p g l \sin(\varphi) \end{pmatrix}$$

3rd Method for EoM
Lagrange II

- Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{U}$

kinetic energy

potential energy

- Lagrangian equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}}\right) = \boldsymbol{\tau}$

Lagrangian equation

- Since $\mathcal{U} = \mathcal{U}(\mathbf{q})$, $\mathcal{T} = \mathcal{T}(\mathbf{q}, \dot{\mathbf{q}})$

inertial forces

gravity vector

$$\underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{T}}{\partial \mathbf{q}}}_{\text{inertial forces}} + \underbrace{\frac{\partial \mathcal{U}}{\partial \mathbf{q}}}_{\text{gravity vector}} = \boldsymbol{\tau}$$

$$\mathcal{T} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}$$

with $\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}} = \mathbf{M}\dot{\mathbf{q}}$

$$\mathbf{M}\ddot{\mathbf{q}} + \dot{\mathbf{M}}\dot{\mathbf{q}} - \frac{1}{2}\begin{pmatrix} \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}}{\partial q_1}\dot{\mathbf{q}} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}}{\partial q_n}\dot{\mathbf{q}} \end{pmatrix} + \mathbf{g} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$$

Lagrange II
Kinetic energy

- Kinetic energy in joint space $\mathcal{T} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}\dot{\mathbf{q}}$

form 1:

- Kinetic energy for all bodies

$$\mathcal{T} = \sum_{i=1}^{n_b} \left(\frac{1}{2} m_i \,{}_{\mathcal{A}}\dot{\mathbf{r}}_{S_i}^T \,{}_{\mathcal{A}}\dot{\mathbf{r}}_{S_i} + \frac{1}{2} \,{}_{\mathcal{B}}\mathbf{\Omega}_{S_i}^T \cdot {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\mathbf{\Omega}_{S_i} \right)$$

- From kinematics we know that

$$\dot{\mathbf{r}}_{S_i} = \mathbf{J}_{S_i}\dot{\mathbf{q}}$$
$$\mathbf{\Omega}_{S_i} = \mathbf{J}_{R_i}\dot{\mathbf{q}}$$

form 2

- Hence we get

$$\mathcal{T}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T \underbrace{\left(\sum_{i=1}^{n_b} \left(\mathbf{J}_{S_i}^T m \mathbf{J}_{S_i} + \mathbf{J}_{R_i}^T \mathbf{\Theta}_{S_i} \mathbf{J}_{R_i} \right) \right)}_{\mathbf{M}(\mathbf{q})} \dot{\mathbf{q}}$$

Lagrange II
Potential energy

- Two sources for potential forces

  - Gravitational forces $\mathbf{F}_{g_i} = m_i \, g \, {}_I\mathbf{e}_g \longrightarrow \mathcal{U}_g = -\sum_{i=1}^{n_b} \mathbf{r}_{S_i}^T \, \mathbf{F}_{g_i}$

  - Spring forces $\mathbf{F}_E = k_j \left(\|\mathbf{r} - \mathbf{r}_0\| - d_0 \right) \frac{\mathbf{r} - \mathbf{r}_0}{\|\mathbf{r} - \mathbf{r}_0\|} \longrightarrow \mathcal{U}_{E_j} = \frac{1}{2} k_j \left(d(\mathbf{q}) - d_0 \right)^2$

ETH zürich

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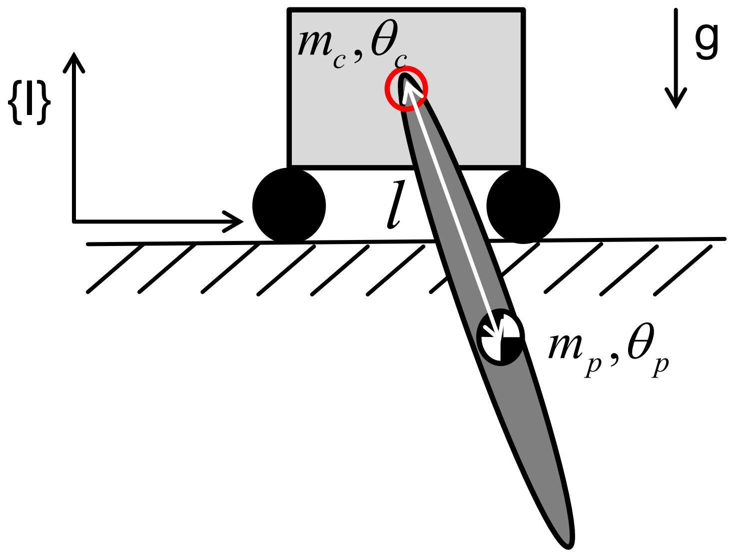
Quiz
3min
1min

Lagrange II
Cart pendulum example

- Find the equation of motion

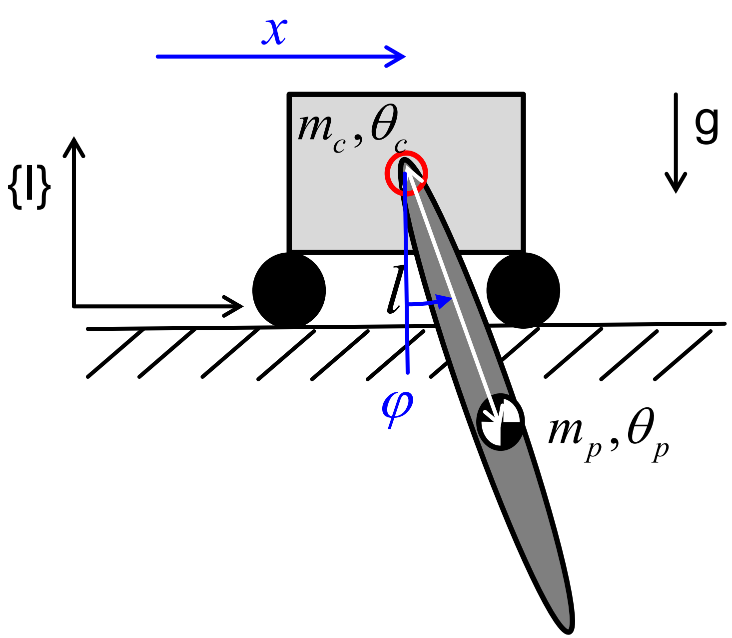


Lagrange II
Cart pendulum example

- Kinematics cart and pendulum

$$\mathbf{r}_{OS_c} = \begin{pmatrix} x \\ 0 \end{pmatrix} \qquad \mathbf{r}_{OS_p} = \begin{pmatrix} x + l\sin(\varphi) \\ -l\cos(\varphi) \end{pmatrix} \qquad \varphi_p = \varphi$$

$$\dot{\mathbf{r}}_{S_c} = \begin{pmatrix} \dot{x} \\ 0 \end{pmatrix} \qquad \dot{\mathbf{r}}_{S_p} = \begin{pmatrix} \dot{x} + l\cos(\varphi)\dot{\varphi} \\ l\sin(\varphi)\dot{\varphi} \end{pmatrix} \qquad \omega_p = \dot{\varphi}$$



- Kinetic and potential energy

$$\mathcal{T} = \sum \frac{1}{2}\dot{\mathbf{r}}_{S_i}^T m_i \dot{\mathbf{r}}_{S_i} + \frac{1}{2}\boldsymbol{\omega}_i^T \Theta_i \boldsymbol{\omega}_i = \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_p\dot{x}^2 + \frac{1}{2}m_p l^2\dot{\varphi}^2 + m_p\dot{x}l\cos(\varphi)\dot{\varphi} + \frac{1}{2}\theta\dot{\varphi}^2$$

$$\mathcal{U} = -m_p g l\cos(\varphi)$$

0-level can be chosen

- Equation of motion

$$\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}} = \begin{pmatrix} m_c\dot{x} + m_p\dot{x} + m_p l\cos(\varphi)\dot{\varphi} \\ m_p l^2\dot{\varphi} + m_p\dot{x}l\cos(\varphi) + \theta\dot{\varphi} \end{pmatrix}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{T}}{\partial \mathbf{q}} + \frac{\partial \mathcal{U}}{\partial \mathbf{q}} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}}\right) = \begin{pmatrix} m_c\ddot{x} + m_p\ddot{x} + m_p l\cos(\varphi)\ddot{\varphi} - m_p l\sin(\varphi)\dot{\varphi}^2 \\ m_p l^2\ddot{\varphi} + m_p\ddot{x}l\cos(\varphi) - m_p\dot{x}l\sin(\varphi)\dot{\varphi} + \theta\ddot{\varphi} \end{pmatrix}$$

$$-\frac{\partial \mathcal{T}}{\partial \mathbf{q}} = \begin{pmatrix} 0 \\ m_p\dot{x}l\sin(\varphi)\dot{\varphi} \end{pmatrix}$$

$$\frac{\partial \mathcal{U}}{\partial \mathbf{q}} = \begin{pmatrix} 0 \\ m_p g l\sin(\varphi) \end{pmatrix}$$

$$\begin{pmatrix} m_c\ddot{x} + m_p\ddot{x} + m_p l\cos(\varphi)\ddot{\varphi} - m_p l\sin(\varphi)\dot{\varphi}^2 \\ m_p l^2\ddot{\varphi} + m_p\ddot{x}l\cos(\varphi) + \theta\ddot{\varphi} + m_p g l\sin(\varphi) \end{pmatrix} = 0$$

External Forces

- Given: $n_{f,ext}$ external forces $\mathbf{F}_j$
- Generalized forces are calculated as: $$\boldsymbol{\tau}_{F,ext} = \sum_{j=1}^{n_{f,ext}} \mathbf{J}_{P,j}{}^T \mathbf{F}_j$$
- Given: $n_{m,ext}$ external torques $\mathbf{T}_k$
- Generalized forces are calculated $$\boldsymbol{\tau}_{T,ext} = \sum_{k=1}^{n_{m,ext}} \mathbf{J}_{R,k}{}^T \mathbf{T}_{ext,k}$$
- For actuator torques: $$\boldsymbol{\tau}_{a,k} = \left(\mathbf{J}_{S_k} - \mathbf{J}_{S_{k-1}}\right)^T \mathbf{F}_{a,k} + \left(\mathbf{J}_{R_k} - \mathbf{J}_{R_{k-1}}\right)^T \mathbf{T}_{a,k}$$

External Forces
Cart pendulum example

- Equation of motion without actuation

$$\underbrace{\begin{bmatrix} m_c+m_p & lm_p\cos(\varphi) \\ lm_p\cos(\varphi) & m_pl^2+\theta_p \end{bmatrix}}_{\mathbf{M}}\ddot{\mathbf{q}}+\underbrace{\begin{pmatrix} -\dot{\varphi}^2 lm_p\sin(\varphi) \\ 0 \end{pmatrix}}_{\mathbf{b}}+\underbrace{\begin{pmatrix} 0 \\ m_pgl\sin(\varphi) \end{pmatrix}}_{\mathbf{g}}=\boxed{\mathbf{0}}$$

$\tau = J^T F$

- Add actuator for the pendulum τ_a
  - Action on pendulum $T_p=\tau_a$ $\quad \mathbf{J}_{Rp}=[0 \quad 1]$
  - Reaction on cart $T_c=-\tau_a$ $\quad \mathbf{J}_{Rc}=[0 \quad 0]$

$$\boldsymbol{\tau}=\sum \mathbf{J}_{R,i}^T\mathbf{T}_i=\mathbf{J}_{Rc}^T\mathbf{T}_c+\mathbf{J}_{Rp}^T\mathbf{T}_p=\begin{pmatrix} 0 \\ \tau_a \end{pmatrix}$$

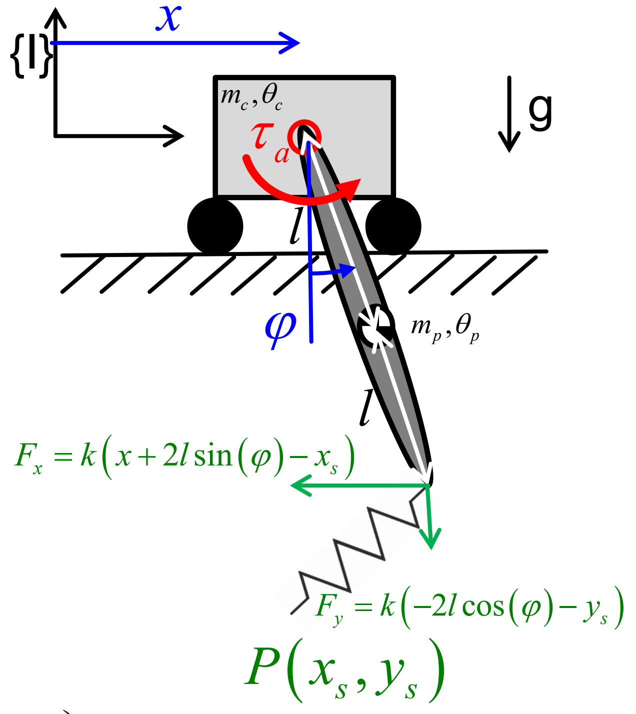
- Add spring to the pendulum
  - (world attachment point P, zero length 0, stiffness k)
  - Action on pendulum

$$\mathbf{F}_s=\begin{pmatrix} -F_x \\ -F_y \end{pmatrix}$$

$$\mathbf{r}=\begin{pmatrix} x+2l\sin(\varphi) \\ -2l\cos(\varphi) \end{pmatrix}$$

$$\mathbf{J}_s=\frac{\partial \mathbf{r}_s}{\partial \mathbf{q}}=\begin{bmatrix} 1 & 2l\cos(\varphi) \\ 0 & 2l\sin(\varphi) \end{bmatrix}$$

$$\boldsymbol{\tau}=\mathbf{J}^T\mathbf{F}_s=\begin{pmatrix} -F_x \\ -2l\left(F_x\cos(\varphi)+F_y\sin(\varphi)\right) \end{pmatrix}$$



External Forces
Cart pendulum example

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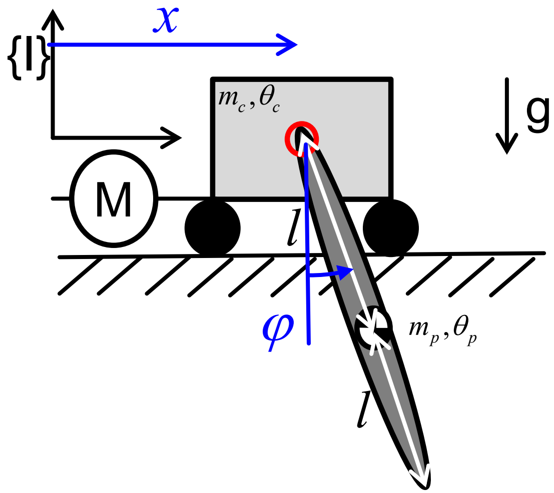
Quiz
2min
1min

- What is the external force coming from the motor and how does it influence the EoM



External Forces
Cart pendulum example

- What is the external force coming from the motor and how does it influence the EoM
  - Action on cart F_{act}

$$F_c = F_{act} \qquad \mathbf{J}_{Pc} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \boldsymbol{\tau} = \mathbf{J}^T F_c = \begin{pmatrix} F_{act} \\ 0 \end{pmatrix}$$