BUSINESS MATHEMATICS

By

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What is a set?

Sets: **A set is a well-defined collection of distinct objects.** We assume that,

- The word set is synonymous with the word collection, aggregate, class and comprises of elements.
- Objects, elements and members of a set are synonymous terms.
- Sets are usually denoted by capital letters A, B, C, ...,.
- Elements of the set are represented by small letters a, b, c, ..., etc.

If 'a' is an element of set A, then we say that 'a' belongs to A. We denote the phrase 'belongs to' by the Greek symbol '∈' (epsilon). Thus, we say that $\mathbf{a \in A}$.

If 'b' is an element which does not belong to A, we represent this as $b \notin A$

Cardinal Number of a Set

- The number of distinct elements in a given set A is called the cardinal number of A. It is denoted by n(A). For example: •
  A $\{x : x \in N, x < 5\}$
  $A = \{1, 2, 3, 4\}$; Therefore, $n(A) = 4$, • B = set of letters in the word ALGEBRA
  $B = \{A, L, G, E, B, R\}$
  Therefore, $n(B) = 6$

Specific sets

- **Empty set (Or a Null set)**

The empty set is the set which does not contain any element.

- The unique set with no elements

  is called **empty set** and denoted by $\varnothing$.

- Set Properties that involve $\varnothing$ **.**

  For all sets A,

  1. $\varnothing \subseteq A$
  2. $A \cup \varnothing = A$
  3. $A \cap \varnothing = \varnothing$
  4. $A \cap A^c = \varnothing$

- Singleton set (or Unit set) The singleton set is the set that has only one element. That is, if S is a singleton set, then $n(A) = 1$
- Pair set The pair set is the set that has exactly two elements. That is, if P is a pair set, then $n(A) = 2$
- Finite set
- The finite set is the set whose elements are countable. That is if one starts counting elements of the set, one by one, the counting comes to an end.

- .Infinite set
- The infinite set is the set containing an uncountable number of elements. In an infinite set, if the elements are counted, one by one, the counting never comes to an end.
- **Example:** $\boldsymbol{E = \{1, 2, 3, 4, \dots\}}$

- **Universal set**
- The universal set is the original set that contains all objects under consideration. Or it is the set containing the totality of elements. The symbols used are: U or ε . The universal sets may be finite or infinite.

Examples

- 1. If one considers the set of men and women, then the universal set is probably the set of human beings.
- 2. If one considers sets such as pigs, cows, chickens, or horses, the universal set is probably the set of animals.

- Complement set

The complement of set with respect to a universal is a set that contains all elements in the universal set but not in a given set.

Identical and Equivalent sets

- Two sets A and B are said to be **identical or equal** if they contain the same elements. Every element of A is an element of B and every element of B is an element of A.
  For example: $A = \{p, q, r, s\}$
  $B = \{p, s, r, q\}$
  Therefore, $A = B$
- Two sets A and B are said to be **equivalent** if their cardinal number is same, i.e., $n(A) = n(B)$. The symbol for denoting an equivalent set is '$\boxed{\leftrightarrow}$'.
  For example: $A = \{1, 2, 3\}$ Here $n(A) = 3$
  $B = \{p, q, r\}$ Here $n(B) = 3$; Therefore, $A \boxed{\leftrightarrow} B$

Notation

- $S=\{a, b, c\}$ refers to the set whose elements are a, b and c.
- $a \in S$ means "a is an element of set S".
- $d \notin S$ means "d is *not* an element of set S".
- $\{x \in S \mid P(x)\}$ is the set of all those x from S such that P(x) is true. *E.g.*, $T=\{x \in \mathbf{Z} \mid 0<x<10\}$.
- *Notes:* 1) {a,b,c}, {b,a,c}, {c,b,a,b,b,c} all represent the same set. 2) Sets can themselves be elements of other sets, *e.g.*, S={ {Mary, John}, {Tim, Ann}, …}

Relations between sets

- **Definition:** Suppose A and B are sets. Then

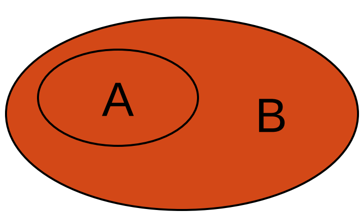
  A is called a **subset** of B: $\mathbf{A \subseteq B}$

  iff every element of A is also an element of B.

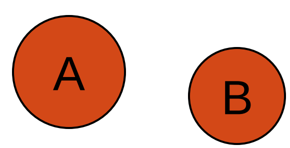
  Symbolically,

  $A \subseteq B \Leftrightarrow \forall x$, if $x \in A$ then $x \in B$.

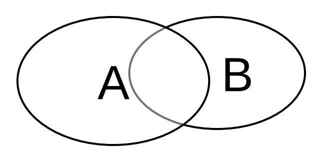
- $A \not\subset B \Leftrightarrow \exists x$ such that $x \in A$ and $x \notin B$.



$A \subseteq B$



$A \not\subset B$



$A \not\subset B$

Cardinal Number of a Set

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  A = {1, 2, 3, 4} ; Therefore, n(A) = 4 , • B = set of letters in the word ALGEBRA
  B = {A, L, G, E, B, R}
  Therefore, n(B) = 6

Relations between sets

- **Definition:** Suppose A and B are sets. Then

  A **equals** B: **A = B**

  iff every element of A is in B and

  every element of B is in A.

  Symbolically,

  $A=B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

- **Example:** Let $A = \{m \in \mathbf{Z} \mid m=2k+1 \text{ for some integer } k\}$;

  B = the set of all odd integers.

  Then $A=B$.

Operations on Sets

Definition: Let A and B be subsets of a set U.

1. Union of A and B: $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$
2. Intersection of A and B:

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

3. Difference of B minus A: $B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$
4. Complement of A: $A^c = \{x \in U \mid x \notin A\}$

Ex.: Let $U = \mathbf{R}$, $A = \{x \in \mathbf{R} \mid 3 < x < 5\}$, $B = \{x \in \mathbf{R} \mid 4 < x < 9\}$. Then

1) $A \cup B = \{x \in \mathbf{R} \mid 3 < x < 9\}$.
2) $A \cap B = \{x \in \mathbf{R} \mid 4 < x < 5\}$.
3) $B - A = \{x \in \mathbf{R} \mid 5 \leq x < 9\}$, $A - B = \{x \in \mathbf{R} \mid 3 < x \leq 4\}$.
4) $A^c = \{x \in \mathbf{R} \mid x \leq 3 \text{ or } x \geq 5\}$, $B^c = \{x \in \mathbf{R} \mid x \leq 4 \text{ or } x \geq 9\}$

Cardinality of the Union

Let A, B and C be any three sets. It can be shown that:

- ✓ $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- ✓ $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

- In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many people who like both cold drinks and hot drinks?

Set Properties

- Commutative Laws:

$$(a)\ A \cap B = B \cap A$$
$$(b)\ A \cup B = B \cup A$$

- Associative Laws:

$$(a)\ (A \cap B) \cap C = A \cap (B \cap C)$$
$$(b)\ (A \cup B) \cup C = A \cup (B \cup C)$$

- Distributive Laws:

$$(a)\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$(b)\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Set Properties

- Double Complement Law:

$$(A^c)^c = A$$

- De Morgan's Laws:

$$(a)\ (A \cap B)^c = A^c \cup B^c$$

$$(b)\ (A \cup B)^c = A^c \cap B^c$$

Disjoint Sets

- A and B are called **disjoint** iff $A \cap B = \emptyset$.
- Sets $A_1, A_2, \ldots, A_n$ are called **mutually disjoint**

  iff for all $i,j = 1,2,\ldots, n$

  $$A_i \cap A_j = \emptyset \quad \text{whenever} \quad i \neq j \ .$$
- *Examples:*

  1) $A=\{1,2\}$ and $B=\{3,4\}$ are disjoint.

  2) The sets of even and odd integers are disjoint.

  3) $A=\{1,4\}$, $B=\{2,5\}$, $C=\{3\}$ are mutually disjoint.

  4) $A-B$, $B-A$ and $A\cap B$ are mutually disjoint.

Application of sets

- There are 35 students in art class and 57 students in dance class. Find the number of students who are either in art class or in dance class. When two classes meet at different hours and 12 students are enrolled in both activities when two classes meet at the same hour.
- In a group of 100 persons, 72 people can speak English and 43 can speak French. How many can speak English only? How many can speak French only and how many can speak both English and French?

- In a competition, a school awarded medals in different categories; 36 medals in dance, 12 medals in dramatics and 18 medals in music. If these medals went to a total of 45 persons and only 4 persons got medals in all the three categories, how many received medals in exactly two of these categories?
- Each student in a class of 40 plays at least one of football, volleyball and basketball. 18 play football, 20 play volleyball and 27 play basketball. 7 play football and volleyball, 12 play volleyball and basketball and 4 play football, volleyball and basketball. Find the number of students who play (a) Football and basketball. (b) Football and volleyball but not basketball.

EXERCISES

- Consider the following data among 110 students in a college dormitory:
- 30 students are on a list A (taking Accounting), 35 students are on a list B (taking Biology), 20 students are on both lists. Find the number of students: (a) on list A or B, (b) on exactly one of the two lists, (c) on neither list.

Ex2

Consider the following data for 120 mathematics students: 65 study French, 20 study French and German,45 study German, 25 study French and Russian, 42 study Russian, 15 study German and Russian, 8 study all three languages

- (a) Fill in the correct number of students in each of the eight regions of the Venn diagram. (b) Find the number k of students studying: (1) exactly one language, (2) exactly two languages.

Ex3

In a survey of 60 people, it was found that: 25 read Newsweek magazine, 9 read both Newsweek and Fortune, 26 read Time, 11 read both Newsweek and Time, 26 read Fortune, 8 read both Time and Fortune, 3 read all three magazines (a) Find the number of people who read at least one of the three magazines (b) Fill in the correct number of people in each of the eight regions of the Venn diagram in where N, T, and F denote the set of people who read Newsweek, Time, and Fortune, respectively.

(c) Find the number of people who read exactly one magazine.

- A group of 40 tourists arrived in Rwanda and visited Akagera National park, Nyungwe forests and Virunga mountains. Results showed that 33 visited Akagera, 21 visited Nyungwe and 23 visited Virunga. 18 visited both Akagera and Nyungwe, 10 visited both Nyungwe and Virunga, and 17 visited both Akagera and Virunga. All tourists visited at least one of the places.
  - (a) Represent the information on a Venn diagram.
  - (b) Find the number of tourists that visited Akagera only.

Ex4

Among the 90 students in a dormitory, 35 own an automobile, 40 own a bicycle, and 10 have both an automobile and a bicycle. Find the number of the students who:

- (a) do not have an automobile.
- (b) have an automobile or a bicycle;
- (c) have neither an automobile nor a bicycle;
- (d) have an automobile or a bicycle, but not both.

Partitions

- **Definition:** A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a **partition** of a set A *iff*
  1. $A = A_1 \cup A_2 \cup \ldots \cup A_n$
  2. $A_1, A_2, \ldots, A_n$ are mutually disjoint.
- *Examples*:
  1) $\{\mathbf{Z}^+, \mathbf{Z}^-, \{0\}\}$ is a partition of $\mathbf{Z}$.
  2) Let $S_0 = \{n \in \mathbf{Z} \mid n = 3k \text{ for some integer } k\}$

     $S_1 = \{n \in \mathbf{Z} \mid n = 3k+1 \text{ for some integer } k\}$

     $S_2 = \{n \in \mathbf{Z} \mid n = 3k+2 \text{ for some integer } k\}$

     Then $\{S_0, S_1, S_2\}$ is a partition of $\mathbf{Z}$.

Power Sets

- **Definition:** Given a set A,

  the **power set** of A, denoted $\mathcal{P}(\mathrm{A})$,

  is the set of all subsets of A.

- *Example:* $\mathcal{P}(\{a,b\}) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$.

- **Properties:**

  1) *If* $A \subseteq B$ *then* $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

  2) *If* a set A has n elements

     then $\mathcal{P}(A)$ has 2^n elements.

Exercise: Find the power set of $A=\{1,2, 3, 4\}$

Cartesian product

Definition 32. The Cartesian product $A \times B$ of the sets A and B is the set of all ordered pairs (a, b), where $a \in A$, $b \in B$:

$$A \times B = \{(a, b),\ a \in A \wedge b \in B\}.$$

Example 46. Take $A = \{1, 2\}$, $B = \{x, y, z\}$. Then

$$A \times B = \{(a, b),\ a \in \{1, 2\} \wedge b \in \{x, y, z\}\}$$

thus a can be either 1 or 2, and for each of these 2 values, b can be either x, y or z:

$$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)\}.$$

Note that $A \times B \neq B \times A$, and that a Cartesian product can be formed from n sets $A_1, \ldots, A_n$, which is denoted by $A_1 \times A_2 \times \cdots \times A_n$.

SET OF NUMBERs

- **The set of natural numbers ($\mathbb{N}$)**

A natural number is any number used to count the members of finite set. The set of positive integers is called the set of natural numbers denoted "$\mathbb{N}$" and defined by $\mathbb{N}=\{0,1,2,3,4,5,\ldots\}$

Note that:

- The lowest natural number is 0 - There is no highest natural number

- IN is an infinite set. - For any two natural numbers a and b , both the sum $a + b$ and the product $a \times b$ are natural numbers. This is often expressed by saying that the set of natural numbers is closed under the operations of addition and

- The set of integers $\mathbb{Z}$ The set of integers includes positive integer numbers, negative integer numbers and zero. That is: $Z = \{\dots, -3, -2, 1, 0, 1\ 2\ 3, \dots\}$. This leads us to the operation of subtraction (or inverse operation of addition), and for any two integers a and b ; we write $a - b$ or $b - a$
- Any natural number is an integer number. - For any two integer numbers a and b , the sum $a + b$, the difference $a - b$ and the product $a \times b$ are all integer numbers.
- This is often expressed by saying that the set of integer numbers is closed under the operations of addition, subtraction and multiplication.

The set of Rational numbers Q

- A rational number is any number of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. The set of rational numbers leads to the operation of division, or inverse of multiplication, and we write $x = \frac{p}{q}$ where p is the numerator and q the denominator.
- Note that: Any integer number is a rational number as every integer has 1 as the denominator. So, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$

- Converting rational to decimal and vice versa All rational numbers can be expressed as either a finite or a recurring decimal. For example, $\frac{1}{2} = 0.5$ is finite decimal and $\frac{1}{3} = 0.\bar{3}$ is a recurring decimal. An irrational number cannot be expressed in this way. Example Express the recurring decimal in the form p/q where p and q are integers and $q \neq 0$.

Example

Express the recurring decimal $0.2\overline{45}$ in the form p/q where p and q are integers and $q \neq 0$.

Solution

Let $x = 0.2\overline{45}$.

Then $100x = 24.5\overline{45}$.

By subtraction we obtain $99x = 24.3$.

Hence, $0.2\overline{45} = \frac{243}{990} = \frac{27}{110}$.

- **The set of Irrational numbers**

The set of irrational numbers is denoted by I and formed by the numbers which are not rational, i.e., they cannot be converted into rational numbers. When it is written in decimal form, an irrational number is a nonterminating and non-recurring (or nonrepeating) decimal.

The set of real numbers $\mathbb{R}$

The set of real numbers is the (disjoint) union of the set of rational numbers with the set of irrational numbers. That is,

$Q \cup I = R$ which leads to $R = (-\infty, +\infty) =]-\infty, +\infty[$

- Graphically, R is represented by the real number line and called the real number system. This means that a rational number is either rational or irrational but not both. The geometric representation of real numbers as point on the real line L is often used. Each real number is represented by one point of L and viceversa $L = R$.
- **Note that**: Every point on the real line represents a real number.

Properties of real numbers

- Commutative property
- Associative Property
- Identity Property
- Symmetric property
- Distributive property

If a, b are two real numbers, then:

(a) Either $a<b$ or $b<a$ or $a=b$

(b) The sum and product of any two positive real numbers are both positive.

(c) If $a<0$ then $-a>0$.

(d) If $a>b$ then $(a-b)>0$. This means that the point on the number line corresponding to a is **to the right of the point** corresponding to b.

(e) The most elementary rules for inequalities are:

i. If $a<b$ and c is any real number; then $a+c<b+c$. That is the inequality does not change when we add (or subtract) any real number to (or from) both sides of an inequality.

ii. - If $a<b$ and $c>0$, then $ac<bc$.

- If $a<b$ and $c<0$, then $ac>bc$.

That is we may multiply (or divide) both sides of an inequality by a positive real number, *but when we multiply (or divide) both sides of an* *inequality by a negative real number we must change* ***the direction of the inequality***.

iii.- If $0 < a < b$ then $0 < a^2 < b^2$

- If $a < b < 0$ then $b^2 < a^2 < 0$

  That is we may square both sides of an inequality if both sides are positive. *However if both sides are negative we may square both sides but we must* ***reverse the direction of the inequality and then both sides become positive***.

- If $0 < a < b$ then $0 < \frac{1}{b} < \frac{1}{a}$

- If $a < b < 0$ then $\frac{1}{a} < \frac{1}{b} < 0$

  That is we may take the inverse of both sides of an inequality only if both sides have the same sign and, *in each possible case*, ***we must reverse the direction of the inequality***.

The most important subsets of real numbers in calculus are intervals. If $a<b$, then:

1. $(a,b)=\{x \in \mathrm{IR} : a<x<b\}$: This set is called "Open interval"
2. $[a,b]=\{x \in \mathrm{IR} : a \le x \le b\}$: This set is called "Closed interval"
3. $[a,b)=\{x \in \mathrm{IR} : a \le x<b\}$: This set is called "Half open or half closed interval"
4. $(a,b]=\{x \in \mathrm{IR} : a<x \le b\}$: This set is called "Half open or half closed interval"
5. $[a,+\infty)=\{x \in \mathrm{IR} : x \ge \mathrm{a}\}$
6. $(a,+\infty)=\{x \in \mathrm{IR} : x > \mathrm{a}\}$
7. $(-\infty,a]=\{x \in \mathrm{IR} : x \le \mathrm{a}\}$
8. $(-\infty,a)=\{x \in \mathrm{IR} : x < \mathrm{a}\}$
9. $(-\infty,+\infty)=\mathrm{IR}$: This set is called "Unbounded interval"

EQUATIONS OF 1ST AND 2ND ORDERS AND INEQUALITIES

- Equations and inequations occur very often in economic analysis. The solution of many problems in economics involves the solution of equations intended to determine the values of the variables in the problems. Equations and inequations can either be linear, quadratic or any other non-linear equations or inequations of higher degrees.

Cont'd

- Definition
- An equation is an expression with an equal sign. I n addition to this, in equations, unlike in functions, none of the variables in the expression is designated as the dependent variable or the independent variable, although the variables are explicitly related.
    - Equations of first order or linear equation
- Formulation and examples
- The linear equation has this standard for

Cont'd

$$ax + b = 0$$

Cont'd

- Example: Write the following equations in standard form

$$a)\ 2x+3=\frac{x-3}{4}$$

$$b)\ \frac{2x+3}{8x+7}=\frac{x-3}{4x-3}$$

- Solution of linear equations

To solve an equation involving a variable is to find the value or roots of the equation, and the set of these values is referred to as the solution set.

Cont'd

$$ax + b = 0 \Leftrightarrow ax = -b$$

$$\Leftrightarrow x = \frac{-b}{a}$$

$$S = \{\frac{-b}{a}\}$$

- Example1:
- Solve the following

$$\frac{3x}{4} = \frac{x}{4} + 9$$

$$\frac{x+3}{16} - \frac{x-1}{4} = \frac{1}{8}$$

- When solving you may have different cases
- 1) $a \neq 0$ and $b \neq 0$

$$S = \left\{\frac{a}{b}\right\}$$

- 2) $a \neq 0$ and $b = 0$, $S = \{0\}$
- 3) $a = 0$ and $b \neq 0,\ S\ is\ impossible$
- $a = 0$ and $b = 0$ this equation is indeterminate

- **Product equation of the form** $(ax+b)(cx+d)=0$
- Solve for x in the set of real numbers, the equation

$$(2x+4)(x-1)=0$$

Fractional equation of the first degree of the form: $\frac{ax+b}{cx+d}=0$

Existing condition $cx+d \neq 0$

solve the equation $\frac{2x-6}{x+1}=0$

Cont’d

- Example2: The sum of two consecutive odd numbers is 32. What are the two odd numbers?
- Example3: A consumer P spends a monthly average of 23 shillings more on luxury goods than a second consumer Q. Both P and Q spend 139 shillings on luxury goods. What is the average monthly expenditure of each of the two consumers?

Cont’d

- Example4: The national income, Y, of a country is given by: Y=C+I+G
- Where C, I and G are, respectively, the consumption, investment and government expenditure components of the country’s national income. Consumption is the size of investment, but 50,000 shillings less than government expenditure. If the national income of the country is 1,050,000 shillings, find the levels of consumption, investment and government expenditure for the country?

1.3.3. Quadratic equations and applications in business area

A **quadratic equation** is an equation equivalent to one of the form $ax^2 + bx + c = 0$ Where a, b, and c are real numbers and $a \neq 0$ So if we have an equation in x and the highest power is 2, it is quadratic.

Deriving The Quadratic Form

If $ax^2 + bx + c = 0$ (and $a \neq 0$), then:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Divide both sides by a

Cont'd

Divide both sides by ***a***

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Complete the square by adding $(b/2a)^2$ to both sides

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

Factor (left) and find LCD (right)

Combine fractions and take the square root of both sides $\left(x+\frac{b}{2a}\right)=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

Subtract $b/2a$ and simplify

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Cont'd

If we have a quadratic equation and are considering solutions from the real number system, using the quadratic formula, one of three things can happen.

1. The "stuff" under the square root can be positive and we'd get two unequal real solutions $\text{if } b^2 - 4ac > 0$

2. The "stuff" under the square root can be zero and we'd get one solution (called a repeated or double root because it would factor into two equal factors, each giving us the sam $\text{if } b^2 - 4ac = 0$ solution

Cont’d

if $b^2 - 4ac < 0$

3. The "stuff" under the square root can be negative and we'd get no real solutions.

$$\Delta = b^2 - 4ac$$

The "stuff" under the square root is called the **discriminant**

- Examples:

In , solve the following quadratic equations:

(a) $6x^2 - 7x - 3 = 0$

(b) $18y^2 - 3y + 4 = 2y^2 + 5y + 3$

(c) $m^2 - 2m + 3 = 0$

Cont'd

Vertex

The point on a parabola that represents the absolute minimum or absolute maximum – otherwise known as the turning point.

y coordinate determines the range. (x,y) $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

Axis of symmetry :The vertical line that goes through the vertex of the parabola.

Equation is x = constant , a>0 graph opens upward , a<0 graph opens downward

Cont'd

Application in business

1. A small business' profits over the last year have been related to the price of the only product. The relationship is R(p) = $-0.4p^2 + 64p - 2400$, where R is the revenue measured in thousands of dollars and p is the price of the product measured in dollars. What price would maximize the revenue? The word maximize screams "FIND THE VERTEX!!"

Cont'd

The answer to this question is the y-value of the vertex

$$p = \frac{-b}{2a}$$

$$p = 80$$

$$p = \frac{-64}{2(-0.4)}$$

$$R(80) = -0.4(80)^2 + 64(80) - 2400$$

$$R(80) = -0.4(6400) + 5120 - 2400$$

$$p = \frac{64}{-0.8}$$

$$R(80) = -2560 + 5120 - 2400$$

$$p = 80$$

$$R(80) = 160$$

The maximum revenue is \$160 000.

Cont'd

2. A small business' profits over the last year have been related to the price of the only product. The relationship is $R(p) = -0.4p^2 + 64p - 2400$, where R is the revenue measured in thousands of dollars and p is the price of the product measured in dollars. How much money would they lose if they gave the product away? This question is talking about a price of 0 or p = 0

$$p = 0$$

$$R(0) = -0.4(0)^2 + 64(0) - 2400$$

$$R(0) = -2400$$

Cont'd

1.3.4. Inequalities of 1st order

An algebraic relation showing that a quantity is greater than or less than another quantity.

Inequality - a mathematical sentence that contains $<, >, \leq, \geq$ or not equal.

$>$ reads as greater than

$<$ reads as less than, $\leq$ reads as less than or equal to, $\geq$ reads as greater than or equal to $\neq$ not equal to

Cont'd

Examples: solve inequalities

1. $x + 8 \geq 19$
2. $-26 > y + 14$
3. $m + 3 > 6$
4. $-5 \leq x - 6$
5. $4 + x < -2$
6. $13 + y \geq 13$

- **Non linear inequalities**
- Product and quotient: Studying signs

i. The sign diagram of a linear factor $ax+b$, $a \neq 0$

The study of the sign of a linear factor $ax+b$ is done as follows:

- Find the critical value of x by solving $ax+b=0 \Leftrightarrow ax=-b \Leftrightarrow x=-\dfrac{b}{a}$
- Make the sign diagram of a linear factor $ax+b$

x	$-\infty$		$-\dfrac{b}{a}$		$+\infty$
$ax+b$		Opposite sign of a	0	Sign of a	

- Examples Study the sign of the following algebraic expressions:
- $(x-1)(x+2)$
- $\frac{3y+1}{1-y}$
- And hence solve: $(x-1)(x+2)<0$

$\frac{3y+1}{1-y}>0$

Quadratic inequalities

The summary of sign diagram for ax^2+bx+c is as follows:

- If $\Delta > 0$, there are two critical values x_1 and x_2 so, in this case, the table of signs of ax^2+bx+c is:

x	$-\infty$		x_1		x_2		$+\infty$
ax^2+bx+c		Sign of a	0	Opposite sign of a	0	Sign of a	

- If $\Delta = 0$, there is a single critical value x_1, so, in this case, the table of signs of ax^2+bx+c is:

x	$-\infty$		x_1		$+\infty$
ax^2+bx+c		Sign of a	0	Sign of a	

- If $\Delta < 0$, there is no critical value, so, in this case, the table of signs of ax^2+bx+c is:

x	$-\infty$		$+\infty$
ax^2+bx+c		Sign of a	

Exercises

- In $\mathbb{R}$, solve the following inequalities:

1) $(1-x)(2x+1)>0$

2) $-2x^2+x+3<0$

3) $x^2+x+1<0$

4) $x(x+4)\geq x-4$

5) $1+2x-x^2<0$

6) $\dfrac{2x-1}{x+2}>2$

7) $\dfrac{3x+2}{2-x}\geq 4$

ABSOLUTE VALUE

- Sometimes we order numbers according to their magnitude. We denote the magnitude or absolute value of the real numbers x by $|x|$, called the modulus of x .

- $|x| = \begin{cases} x, if\ x \geq 0 \\ -x, if\ x < 0 \end{cases}$
- Since the symbol $\sqrt{}$ is used to mean any non-negative square root, we may also define the absolute value of the real number x by: $|x| = \sqrt{x^2}$

- In general, if a is any positive number,
- $|x| < a \Leftrightarrow -a < x < a$ and
- $|x| > a \Leftrightarrow x > a, or\ x < -a$

The following relations are true for all real numbers a and b :

i. $|-a| = |a|$

ii. $-|a| \le a \le |a|$

iii. $|ab| = |a||b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$; $\quad (b \neq 0)$

iv. $|a+b| \le |a| + |b|$

- Solve the following
- $|x-5|=\frac{5}{2}$
- $|3x-2|<1$
- $|-4x+3|\geq 0$

-

(1) $|5-x|=\frac{5}{2}$

(2) $|x|\leq 3$

(3) $|2x+7|\geq 3$ $|3-2x|=$

(4) $|3-2x|=5$

(5) $|x+2|>1$

(6) $|4x-1|<1$

(7) $|9-3x|\geq 0$

Powers and radicals

- Powers
- When a real number is raised to the index m to give a^m, the result is a power of a . When a^m is formed, m is sometimes called the power, but it is more correctly called the index to which is raised. The rules used to manipulate exponential expressions should already be familiar with the reader. These rules may be summarized as follow:

(1) Product Rule: $a^m a^n = a^{m+n}$

(2) Quotient Rule: $\frac{a^m}{a^n} = a^{m-n} \quad ; a \neq 0$

(3) Power of Power Rule: $(a^m)^n = a^{mn}$

(4) Power of Product Rule: $(ab)^m = a^m b^m$

(5) Power of a quotient Rule: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad ; b \neq 0$

For the quotient rule to hold when $m = n$ we must agree that:
$\frac{a^m}{a^m} = a^{m-m} = a^0 \quad ; a \neq 0$

Then it is natural to define $a^0 = 1; a \neq 0$

Now, $\frac{1}{a^n} = \frac{a^0}{a^n} = a^{0-n} = a^{-n} \quad ; a \neq 0$,which gives the meaning to negative exponents.

Rational Exponents

- More generally, if n is a positive integer, $a^{\frac{1}{n}}$ is defined by $a^{\frac{1}{n}} = \sqrt[n]{a}$

Note that:

- \- If n is even, a must be non-negative;
- \- If n is odd, a may be any real number

- (1) Evaluate each of the following operations:
- $\frac{3^4 3^8}{3^{14}}$
- $8^{1-x} = 4^{2x+3}$
- $7^2 \times 7^5$
- $\frac{(a^3)^4 b^6}{a^2 (b^3)^3}$

NUMERICAL FUNCTION

RELATIONS

- Definition: Let A and B be sets. A binary relation or, simply, a relation from A to B is a subset of A x B. Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair a ∈A and b ∈ B, exactly one of the following is true:
- (i) (a, b) ∈ R; we then say "a is R-related to b", written aRb.
- (ii) (a, b) R; we then say" a is not R-related to b", written a b.

Cont’d

- The domain of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R, and so it is a subset of A; and the range of R is the set of all second elements, and so it is a subset of B. Sometimes R is a relation from a set A to itself, that is, R is a subset of $A^2 = A \times A$. In such a case, we say that R is a relation on A

Cont'd

- 3.6 Inverse Relation
- Let R be any relation from a set A to a set B. The inverse of R, denoted by R- 1 , is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,
- $R^{-1} = \{(b, a) : (a, b) \in R\}$ For example: If $R = \{(1, y), (1, z), (3, y)\}$, then $R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$.[Here R is the relation from $A = \{I, 2, 3\}$ to $B = \{x, y, z\}$
- Clearly, if R is any relation, then $(R^{-1})^{-1} = R$.

- 3.7 Functions:
- A function is a relation in which each element of the domain is paired with *exactly one* element of the range. Another way of saying it is that there is one and only one output (y) with each input (x).

- In mathematics, a function is a relation between a set of inputs and a set of permissible outputs. Functions have the property that each input is related to exactly one output.

- For example, in the function $f(x) = x^2$ any input for x will give one output only.
- - Functions are typically named with a single letter, like f .
- - $f(x)$ is read " f of x ", and represents the output of the function f corresponding to an input x . The input variable(s) are sometimes referred to as the argument(s) of the function. Consider the following example: Given the function $f(x) = x^2$ thus, if $x = 3$ then $f(3) = (3)^2 = 9$
- In the example above, the argument is x =3 and the output is 9 . So, we write: $f(3) = 9$.

- In the case of a function with just one input variable, the input and output of the function can be expressed as an ordered pair. The order is such that the first element is ***the argument*** and the second is ***the output***. In the

- *If both the input and output are real numbers then the ordered pair can be viewed as the Cartesian coordinates of a point on the graph of the function.*

Another commonly used notation for a function is: $f : X \to Y$, which reads as saying that f is a function that maps values from the set X onto values of the set Y.

The sets X and Y are called ***the domain*** and ***range*** of f respectively. The domain of f is denoted by $domf$ or Df.

Examples

Let $f(x)=\dfrac{5x}{x+1}$, find the following:

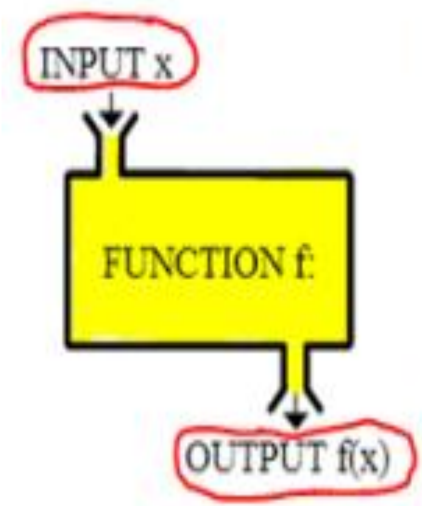
(1) $f(2)$

(2) $f(2x-7)$

(3) $f(x+\mathrm{h})$

(4) $f(x^2)$

- Functions are often described as a machine in a box that is open on two ends. You put something into one end of the box, it gets changed inside of the box, and then the result pops out the other end. The function is the machine inside the box and it is defined by what it does to whatever you put into it.



Function Machine

A function f takes an input x and returns an output $f(x)$. One comparison describes the function as a "***machine***" that ***for each input returns a corresponding output***.

Real valued functions

- Definitions

A real valued function f defined on an interval I of real numbers is ***a rule that assigns to each number*** x ***in exactly one real number*** $f(x)$.

The number $f(x)$, read " f f of x is called "***the value of the function*** f ***at the point*** x**.**"

The real values function f **, is mathematically defined as:**

$$\begin{aligned} f : I &\to IR \\ x &\to y = f(x) \end{aligned}$$ ***Where*** $I \subseteq IR$ (I **is a proper subset of** IR)

- The set I of all those real numbers x for which f(x) has a meaning (or is defined) is called domain or domain of definition of the function f and it is denoted by domf or Df
- If the elements of the domain and range are represented by x and y respectively; then f symbolizes the function and describes the relation between x and y so that x is the independent variable (input variable) and y is the dependent variable (output variable), then y is a function of x .

- Examples
- (1) Given that: $f(x) = x + 3$ is a function, find $f(2)$.
  Solution: To find " f of 2 "means to find the value of the function when $x = 2$. That is: $f(2) = 2 + 3 = 5$

Then, 5 is the image of 2

- (2) Given that: $f(x) = 3x - 1$ is a function, find $f(-2)$

Classification and combinations of functions

Elementary functions are divided into three categories:

- (1) Algebraic
- (2) Trigonometric, and
- (3) Logarithmic and exponential functions

The most common type of elementary function is a polynomial function

i. Definition of Polynomial function Let n be a nonnegative integer. The

Let n be a ***nonnegative integer***. The function $f(x)=a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ is called a polynomial function of degree n. The numbers a_i are called coefficients, with $a_n \neq 0$ ***the leading coefficient*** and a_0 ***the constant term*** of the polynomial.

Examples:

Degree of the polynomial n	**General form**	**Name of the polynomial**
Zero degree $(n=0)$	$f(x)=a$	Constant function
First degree $(n=1)$	$f(x)=ax+b$	Linear function
Second degree $(n=2)$	$f(x)=ax^2+bx+c$	Quadratic function
Third degree $(n=3)$	$f(x)=ax^3+bx^2+cx+d$	Cubic function

- The domain of a polynomial function is the set of all real numbers and the range is also the set of all real numbers ii. Rational Functions Just as a rational number can be written as the fraction of two integers, so does a rational function. A rational function is written as the fraction of two polynomials. A function $f(x)$ is rational if it has the form:
- $f(x) = \frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$, P(x) and $Q(x)$ are polynomial

- Polynomials functions and rational functions are two examples of a larger class of functions called algebraic functions.
- - An algebraic function is one that can be expressed in terms of finitely many sums, differences, multiples, quotients, and radicals involving the variable. For example the following functions are algebraic:

$$f(x) = \sqrt{x+1} \quad and \quad g(x) = x + \frac{1}{\sqrt[3]{x+1}}$$

- Functions which are not algebraic are called Transcendental.

- Two functions can be combined in various ways to form a new function.

Example

If $f(x)=2x-3$ *and* $g(x)=x^2+1$, we can form the functions:

Operation	**Function result**
Addition	$f(x)+g(x)=(2x-3)+x^2+1=x^2+2x-2$
Subtraction	$f(x)-g(x)=(2x-3)-(x^2+1)=-x^2+2x-4$
Multiplication	$f(x).g(x)=(2x-3)(x^2+1)=2x^3-3x^2+2x-3$
Division	$\dfrac{f(x)}{g(x)}=\dfrac{2x-3}{x^2+1}$

Determination of the domain and range of elementary functions

- Domain of definition Determination of domain of definition a real valued function Domain of a real valued function is the set of all real numbers for which the expression of the function is defined as a real number. In other words, it is all the real numbers for which the expression makes sense.

Domain of definition of polynomial functions

- Since the general form of the polynomial is:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where $a_0, a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ are constants and n is a positive integer called the degree of the polynomial if $a_0 \neq 0$; then, the polynomial function is defined for all real numbers.

Examples

(1) Given that: $f(x) = 6x^5 - 2x^4 + 3x^3 + 5x^2 - x + 7$ then: $domf = IR$

Given that: $g(x) = x$ then: $domf = IR$

Domain of definition of Rational functions

A rational function is a fraction of two polynomial functions:

$$f(x)=\frac{a_n x^n + a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0}{b_n x^n + b_{n-1}x^{n-1}+\cdots b_2x^2+b_1x+b_0}$$

So, the domain is ***all real except the zeros of the denominator.***

Examples:

(1) Given that: $f(x)=\dfrac{x^3-5}{x+4}$; So, the domain of definition of $f(x)$ is:

$domf = IR-\{-4\}$ because $f(x)$ is not defined at $x+4=0$ ***i.e*** at $x=-4$

(2) Given that: $g(x)=\dfrac{2x}{x^2-3x+2}$; So, the domain of definition of $g(x)$ is:

$domf = IR-\{1,2\}$ because $g(x)$ is not defined at $x^2-3x+2=0$ (***i.e*** at $x=1$ and at $x=2$ ***Values obtained after solving the equation*** $x^2-3x+2=0$)

(3) Given that: $h(x)=\dfrac{x^7-2}{x^2+5}$; So, the domain of definition of $h(x)$ is: $domf = IR$ or $domf=(-\infty,+\infty)$ because there is no any real value that can make the denominator of $h(x)$ to be *zero.*

Domain of definition of Irrational functions

- An irrational function is a function the form: $f(x)=\sqrt[n]{g(x)}$

  - **If n is even:** $domf=\{x\in IR: g(x)\geq 0\}$

    Examples

    (a) $f(x)=\sqrt{x}\Rightarrow domf=\{x\in IR: x\geq 0\}$ or $domf=[0,+\infty)$

    (b) $f(x)=\sqrt{x+2}\Rightarrow domf=\{x\in IR: x+2\geq 0\Leftrightarrow x\geq -2\}$ or $domf=[-2,+\infty)$

    (c) $f(x)=\sqrt{-3x+2}\Rightarrow domf=\left\{x\in IR: -3x+2\geq 0\Leftrightarrow \frac{-3x}{-3}\leq\frac{-2}{-3}\Leftrightarrow x\leq\frac{2}{3}\right\}$

    or $domf=\left(-\infty,\frac{2}{3}\right]$

- If n is odd: The domain of definition of $f(x) = \sqrt[n]{g(x)}$ is the domain of the radicand (the real values for which the expression under the root sign($g(x)$) is defined)

Examples

Find the domain of definition of the following functions

- **(a)** $f(x) = \sqrt[5]{6x^2 + 4x - 1}$

(b) $f(x) = \sqrt[13]{\dfrac{4x^3}{x^2 - 1}}$

Solutions:

(a) The domain of definition of $f(x) = \sqrt[5]{6x^2 + 4x - 1}$ is: $dom f = IR$ or $dom f = (-\infty, +\infty)$ as the expression under the root sign with $n = 5$ (***odd***) is a polynomial (It is known that the domain of definition of a polynomial function is IR).

(b) The domain of definition of $f(x) = \sqrt[13]{\dfrac{4x^3}{x^2 - 1}}$ is: $dom f = IR - \{-1 \ \ ,1\}$ as the expression under the root sign with $n = 13$ (***odd***) is a rational function $\dfrac{4x^3}{x^2 - 1}$ whose domain of definition is: $IR - \{x \in IR : x^2 - 1 = 0\}$ or $IR - \{-1, \ 1\}$

EXERCISES

Exercises

Find the domain of the function of each of the following functions below

(1) $f(x) = 3x^3 - 2x^2 + 4x - 6$

(2) $g(x) = \frac{2x+3}{x-3}$

(3) $h(x) = \sqrt{6 - 3x}$

RANGE

Let $f : X \to Y$ be a function. The range of f, denoted by $\operatorname{Im} f$, is the image of X under f, that is, $\operatorname{Im}(f) = f[X]$. **The range consists of all possible values the function f can have.**

Steps to find range of function:

To find the range of the function f described by formula we proceed as it follows:

- Put $y = f(x)$
- Solve for x in terms of y

- The range of f is the set of all real numbers y such that x can have a meaning (is defined).

Examples

For each of the following functions, find the range.

(1) $f(x)=x+5$

(2) $g(x)=\dfrac{8}{2x-4}$

(3) $h(x)=\sqrt{6-3x}$

- Note that: The domain of definition is the set of all values that can be substituted for the independent variable x. When determining the domain of definition, it can be helpful to keep these steps in mind. (1) Ask yourself, "Is there any number that cannot be substituted for x ?" (2) If x is in the denominator of a rational expression, determine what value(s) of x will make the denominator equal 0 by setting the expression equal to zero. Solve for x. This (these) x - values are not in the domain. (3) If the expression contains a radical and x is in the radicand, set up an inequality so that the radicand is greater than 0 . Solve for x . These are the values of x in the domain

Parity of a function

- Even function

The function $f(x)$ **is an even function on interval** I, if

$$\forall x \in I \;\; ; \exists -x \in I : f(-x) = f(x)$$

Examples

(1) $f(x) = x^2$; $dom f = IR$ and $x \in IR; \exists -x \in IR : f(-x) = (-x)^2 = x^2 = f(x)$**;** Thus, f is **even** function.

(2) $f(x) = \cos x$; $dom f = IR$ **and** $x \in IR; \exists -x \in IR : f(-x) = \cos(-x) = \cos x = f(x)$**;** Thus, f is **even** function.

Odd function

- **The function** $f(x)$ **is an odd function on interval** I**,** if $\forall x \in I \;;\exists -x \in I : f(-x) = f(x)$

Examples

(1) $f(x) = x^3$; $domf = IR$ and $x \in IR; \exists -x \in IR : f(-x) = (-x)^3 = -x^3 = -f(x)$**;** Thus, f is **odd** function.

(2) $f(x) = \sin x$; $domf = IR$ and $x \in IR; \exists -x \in IR : f(-x) = \sin(-x) = -\sin x = -f(x)$**;** Thus, f is **odd** function.

Note that:

The function is not necessarily even or odd.

For example

$f(x) = x^3 + 1$

$f(-x) = (-x)^3 + 1 = -x^3 + 1 \neq f(x) \Rightarrow f$ not even.

$\mathrm{f}(\text{-x}) \neq -f(x) = -(x^3 + 1) = -x^3 - 1 \Rightarrow f$ not odd.

Exercises

Verify if $f(x)$ is even or odd or neither:

$a)\ f(x)=x^3+x$ $\quad$ b) $f(x)=x^2+4x$ $\quad$ c) $f(x)=x^4-3x^2+1$

d) $f(x)=\sin 3x$ $\quad$ e) $f(x)=\dfrac{\sin x}{x}$

Composition of functions

The **composition** of g and f, denoted $g \circ f$ is defined by the rule $(g \circ f)(x) = g(f(x))$ provided that $f(x)$ is in the domain of the function g.

Note that:

$f \circ g$ is also read as "f ***compose*** g" and $g \circ f$ is also read as "g ***compose*** f."

The functions $g \circ f$ and $f \circ g$ are called **the composite or compound functions** and in general, $g \circ f \neq f \circ g$.

Examples

Given that the functions defined by $f(x) = 4x + 3$ and $g(x) = 7x$. Find:

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

(c) $(f \circ g)(2)$

(d) $(g \circ f)(2)$

Exercises

-

1. Find $f(g(x))$ if:

a) $f(x) = x^2$ and $g(x) = 2x + 7$

b) $f(x) = 2x + 7$ and $g(x) = x^2$

c) $f(x) = \sqrt{x}$ and $g(x) = 3 - 4x$

d) $f(x) = 3 - 4x$ and $g(x) = \sqrt{x}$

e) $f(x) = \frac{2}{x}$ and $g(x) = x^2 + 3$

f) $f(x) = x^2 + 3$ and $g(x) = \frac{2}{x}$

2. Find $f(x)$ and $g(x)$ given that $f(g(x))$ is:

(a) $(3x + 10)^3$ **(b)** $\frac{1}{2x+4}$ **(c)** $\sqrt{x^2 - 3x}$ **(d)** $\frac{10}{(3x-x^2)^3}$

The inverse of a numerical function

For a one-to-one function defined by $y = f(x)$, the equation of the inverse can be found as follows:

- Replace $\boldsymbol{f(x)}$ by $\boldsymbol{y}$
- Interchange $\boldsymbol{x}$ and $\boldsymbol{y}$.
- Solve for $\boldsymbol{y}$ (or make $\boldsymbol{y}$ as the subject of the formula)
- Replace $\boldsymbol{y}$ by $\boldsymbol{f^{-1}(x)}$

Example

Find the inverse of the one-to-one function defined by: $f(x) = 3 - x^3$

Solution

The inverse function of the function $f(x) = 3 - x^3$ is as follows:

- Let us replace $f(x)$ by y that is: $y = 3 - x^3$
- Let us interchange x and y. That is: $x = 3 - y^3$

- Then, let us make y as the subject of the formula:

$$x = 3 - y^3 \Leftrightarrow -y^3 = 3 - x \Leftrightarrow y^3 = -3 + x \Leftrightarrow y^3 = x - 3 \Leftrightarrow y = \sqrt[3]{x-3}$$

- Finally, replace y found by $f^{-1}(x)$. That is: $f^{-1}(x) = \sqrt[3]{x-3}$

Note that:

$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

Example:

Determine whether $f(x) = \frac{1}{x-3}$ and $g(x) = \frac{3x+1}{x}$ are inverse functions by computing their compositions.

Solution

Since $(f \circ g)(x) = f[g(x)] = f\left(\frac{3x+1}{x}\right) = \frac{1}{\frac{3x+1}{x} - 3} = \frac{1}{\frac{3x+1-3x}{x}} = \frac{1}{\frac{1}{x}} = x$ and

$$(g \circ f)(x) = g[f(x)] = g\left(\frac{1}{x-3}\right) = \frac{3\left(\frac{1}{x-3}\right) + 1}{\frac{1}{x-3}} = \frac{\frac{3 + x - 3}{x-3}}{\frac{1}{x-3}} = \frac{\frac{x}{x-3}}{\frac{1}{x-3}} = x$$

Thus the functions f and g are a pair of inverse functions.

LOGARITHM AND EXPONENTIAL FUNCTIONS

1.2.5. Exponents and logarithms

1.2.5.1. Exponents

The product of n factors, each of which is x is given by:

$$x^n = \underbrace{x \cdot x \cdot x \cdot \cdots x}_{n\ factors}$$

In the expression, x^n is the n^{th}power of x, when x is the base and n is the exponent.

Laws of exponents

Multiplication law: $X^{\alpha}.X^{\beta} = X^{\alpha+\beta}$ Example: $X^2.X^4 = X^{2+4} = X^6$

Cont'd

Quotient law: $\frac{X^{\alpha}}{X^{\beta}} = X^{\alpha-\beta}$

Example: $\frac{X^4}{X^2} = X^{4-2} = X^2$

Zero Power law: $X^0 = 1$

Example: $\frac{X^4}{X^4} = X^{4-4} = X^0 = 1$

The Power law: $(X^{\alpha})^{\beta} = X^{\alpha\beta}$

Example: $(X^2)^3 = X^2 \cdot X^2 \cdot X^2 = X^{2+2+2} = X^6$

Reciprocal law: i) $\frac{1}{X^{\beta}} = X^{-\beta}$ ii) $\frac{1}{X^{-\beta}} = X^{\beta}$

Example: $\frac{1}{X^4} = X^{-4}$

Solving Exponential equations

For solving many exponential equations we use this property:

If $a^x = a^y$, then $x = y$. $(a > 0, a \neq 1)$

Example: solve each equation

$a)\ 2^x = 8$ b) $49^{c+3} = 7^{3c}$ c) $9^{6n} = 27^{n-4}$

Exercises

Solve each equation: $a)\ (12)^x = 144$ b) $6^{t-5} = 36^{t+4}$ c) $32^{2w} = 8^{4w-1}$

Example: solve each equation

$a)\ 2^x = 8$ b) $49^{c+3} = 7^{3c}$ c) $9^{6n} = 27^{n-4}$

Cont'd

1.2.5.2. Logarithms

Consider the following exponential expression:

$y = a^x$

a is the base, and x is the exponent of the exponential expression.

The logarithm of y to the base a is the exponent to which a must be raised to obtain y.

Cont'd

Example: Evaluate

i) $\log 10{,}000$

ii) $\log_3 81$

Answer:

i) $\log 10{,}000 = \log 10^4 = 4$

ii) $\log_3 81 = \log_3 3^4 = 4$

Cont'd

Laws of logarithmic operations

Multiplication law: $\log_a(x \cdot y) = \log_a x + \log_a y$

Example: $\log_3(9 \cdot 27) = \log_3 9 + \log_3 27 = 2 + 3 = 5$

Quotient law: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

Example: $\log_3\left(\frac{27}{9}\right) = \log_3 27 - \log_3 9 = 3 - 2 = 1$

Zero Power law: $\log_a(x^\alpha) = \alpha \log_a x$

Example: $\log_5(5^{-3}) = -3\log_5 5 = -3$

$\log_a a = 1$

EXERCISES 1.2

1. Evaluate the following

i) $\log_m m$ ii) $\log_2 81 + \log_2 16$ iii) $4\log_4 3 + 2\log_2\left(\frac{1}{9}\right)$

2. Simplify

i) $7^5 \times 7^{16}$ ii) $\frac{(a^3)^4 \cdot b^6}{a^2 \cdot (b^3)^3}$

Cont'd

Exercises

$$\log_2(32) =$$

$$\log_2(64) =$$

$$\log_2(16) =$$

$$\log_2(\sqrt[3]{4}) =$$

Change of base formula

- While there are several useful identities, the most important for calculator use lets one find logarithms with bases other than those built into the calculator (usually $\log_e$ and $\log_{10}$).
- To find a logarithm with base b, using any other base b

$$\log_r(x) = \frac{\log_b(x)}{\log_b(r)} \text{ for any base } b \text{ or simply } \log_r(x) = \frac{\log(x)}{\log(r)}$$

LIMITS AND CONTINUITY

The function f(x) is said to tend to a **limit L as x tends to** x_0 **and we write** $\lim_{x \to \infty} f(x) = l$ **,** if the difference $|f(x) - l| \to 0$, as $x \to x_0$.

Using mathematical denotation, we have:

Let $f(x)$ be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then, we say that, $\lim_{x \to a} f(x) = L$ if for every number $\varepsilon > 0$ there is some number $\delta(\varepsilon) > 0$ such that:

$|f(x) - L| < \varepsilon$ Whenever $0 < |x - a| < \delta$.

In words, this means that **,** the number L is the limit of $f(x)$ as x approaches a and write $\lim_{x \to a} f(x) = L$ if for any positive number ε (however small) we can find some positive number δ (usually depending on ε) such that $|f(x) - L| < \varepsilon$ Whenever $0 < |x - a| < \delta$.

Cont'd

Example: Prove that $\lim_{x\to 3}(2x+1)=7$

Solution: Given $\varepsilon > 0$, we must find $\delta > 0$ such that

$$|(2x+1)-7| < \varepsilon \text{ if } 0 < |x-3| < \delta \text{ ,Now}$$

$|(2x+1)-7| = |2x-6| = 2|x-3|$ so, $0 < |x-3| < \frac{\varepsilon}{2}$ implies that $|(2x+1)-7| < 2\cdot\frac{\varepsilon}{2} = \varepsilon.$

Hence, given $\varepsilon > 0$, it suffices to choose $\delta = \frac{\varepsilon}{2}$. This illustrates the observation that the required number δ is generally a function of the given number ε.

Examples

Use the definition of the limit to prove the following limits:

(1) $\lim_{x \to 2} 5x - 4 = 6$

(2) $\lim_{x \to 4} x^2 + x - 11 = 9$

Right- and left – hand limits (One – side limits).

In the definition of limit no restriction was made as to how x should approach x_0. Considering x and x_0 is fixed and x is moving, then x can approach x_0 from the right or from the left. We indicate these respective approaches by writing $x \to x_0^-$ and $x \to x_0^+$.

If $\lim\limits_{x\to x_0^-} f(x) = l_1$ and $\lim\limits_{x\to x_0^+} f(x) = l_2$, we call l_1 and l_2, respectively, the left-and right – hand limits of f at x_0.

Example

Evaluate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ if $f(x)=\dfrac{x}{|x|}$

Solution

Note that:

$$|x| = \begin{cases} x; & x \geq 0 \\ -x & x < 0 \end{cases} \qquad \text{So, } \frac{x}{|x|} = \begin{cases} 1; & x \geq 0 \\ -1; & x < 0 \end{cases}$$

Therefore,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{|x|} = -1$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{|x|} = 1$$

Evaluation of algebraic limits by direct substitution

- If f is a polynomial or a rational function and x_0 is in the domain of f , then: $\lim_{x \to x_0} f(x) = f(x_0)$

We can direct put the limiting value of in the function, provided the function does not assume an indeterminate form.

Examples

Evaluate the following limits:

a. $\lim_{x \to 3} 5x$

b. $\lim_{x \to 4} \frac{3x}{2}$

c. $\lim_{x \to -2} x^2$

Infinity limits

- Let f be a function defined on both sides of a , except possibly at a itself. Then $\lim_{x\to a} f(x) = 0$

Which means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a , but not equal to a .

Example

Evaluate

$$\lim_{x\to 0^-} f(x) \text{ and } \lim_{x\to 0^+} f(x) \quad \text{if } f(x) = \frac{1}{x}$$

Solution

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{x} = +\infty$$

Limits at infinity

Let f be a function defined on some interval $(a,+\infty)$. Then, $\lim_{x\to+\infty} f(x)=L$

Which means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Let f be a function defined on some interval $(-\infty,a)$. Then, $\lim_{x\to-\infty} f(x)=L$

Which means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently small.

Note that: Don't confuse ∞ and $+\infty$. Generally, saying ∞ is like saying $-\infty$ or $+\infty$ thus, we write $\infty=\pm\infty$

To compute the limit of a function as $x \to \pm\infty$, we use the following properties

$\forall a \in IR; \quad a \neq 0$

Operation	Property
(1) Addition and subtraction	**(a)** $+\infty \pm a = +\infty$ **(b)** $-\infty \pm a = -\infty$ **(c)** $+\infty + \infty = +\infty$ **(d)** $-\infty - \infty = -\infty$
(2) Multiplication	**(1) For** $a > 0$ **(a)** $+\infty \times a = +\infty$ **(b)** $-\infty \times a = -\infty$

(2) For $a < 0$

(a) $+\infty \times a = -\infty$

(b) $-\infty \times a = +\infty$

(3) Division	**(1)** $\frac{a}{\infty}=0$ or $\frac{a}{-\infty}=\frac{a}{+\infty}=0$
	(2) For $a>0$ **(a)** $\frac{+\infty}{a}=+\infty$ **(b)** $\frac{-\infty}{a}=-\infty$
	(3) For $a<0$ **(a)** $\frac{+\infty}{a}=-\infty$ **(b)** $\frac{-\infty}{a}=+\infty$
(4) Power and nth root	**(1) If** $n\in IN$ **and** n **is even** **(a)** $(+\infty)^n=+\infty$ (b) $(-\infty)^n=+\infty$
	(2) If $n\in IN$ **and** n **is odd** **(a)** $(+\infty)^n=+\infty$ (b) $(-\infty)^n=-\infty$

	(3) $\forall a \in \mathbb{R}^+$ and $a \neq 1$; **(a)** If $a > 1$ i. $a^{+\infty} = +\infty$ ii. $a^{-\infty} = 0$ **(b)** If $0 < a < 1$ i. $a^{+\infty} = 0$ ii. $a^{-\infty} = +\infty$
	(4) If $n \in IN \mid \{0,1\}$ **and if** $n \in IN$ **and** n **is even** **(a)** $\sqrt[n]{+\infty} = +\infty$ **(b)** $\sqrt[n]{-\infty}$ does not exist
	(5) If $n \in IN \mid \{0,1\}$ **and if** $n \in IN$ **and** n **is odd** **(a)** $\sqrt[n]{+\infty} = +\infty$ **(b)** $\sqrt[n]{-\infty} = -\infty$

Theorem on limits

$$If \ \lim_{x\to x_0} f(x) = A \text{ and} \lim_{x\to x_0} g(x) = B, then$$

$$1. \lim_{x\to x_0}\left(f(x)+g(x)\right) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = A + B$$

$$\text{Example: } \lim_{x\to 2}\left(2x^2+3x+1\right) = \lim_{x\to 2} 2x^2 + \lim_{x\to 2} 3x + \lim_{x\to 2} 1 = 15$$

$$2. \lim_{x\to x_0}\left(f(x)-g(x)\right) = \lim_{x\to x_0} f(x) - \lim_{x\to x_0} g(x) = A - B$$

$$\text{Example: } \lim_{x\to 3}\left(3x^2-2x-1\right) = \lim_{x\to 3} 3x^2 - \lim_{x\to 3} 2x - \lim_{x\to 3} 1 = 20$$

$$3. \lim_{x\to x_0}\left(f(x)g(x)\right) = \left(\lim_{x\to x_0} f(x)\right)\left(\lim_{x\to x_0} g(x)\right) = AB$$

$$\text{Example: } \lim_{x\to 3}\left(2\text{x}^2.x\right) = \lim_{x\to 3} 2\text{x}^2 . \lim_{x\to 3} x = 54$$

$$4. \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{A}{B} \text{ if B} \neq 0$$

$$\text{Example: } \lim_{x\to 0} \frac{3x+1}{x+3} = \frac{\lim_{x\to 0}(3x+1)}{\lim_{x\to 0}(x+3)} = \frac{1}{3}$$

5. $\lim_{x \to x_0} cf(x) = c \lim_{x \to x_0} f(x)$

Example: $\lim_{x \to 2} 2x^3 = 2 \lim_{x \to 2} x^3 = 16$

6. $\lim_{x \to x_0} c = c$

Example: $\lim_{x \to 0} 5 = 5$

7. $\lim_{x \to x_0} ((x))^n = \left(\lim_{x \to x_0} f(x) \right)^n$

Example: $\lim_{x \to -1} (3x)^5 = \left(\lim_{x \to -1} 3x \right)^5 = -3^5$

8. $\lim_{x \to x_0} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to x_0} f(x)}$

Example: $\lim_{x \to 3} \sqrt{x^2 + x + 2} = \sqrt{\lim_{x \to 3} \left(x^2 + x + 2 \right)} = \sqrt{14}$

9. $\lim_{x \to +\infty} \left(a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots a_0 \right) = \lim_{x \to +\infty} a_n x^n = \pm\infty$

example $\lim_{x \to \infty} \left(3x^3 + 5x^2 + 2x + 5 \right) = +\infty$

10. $\lim_{x \to +\infty} \frac{b}{x} = 0$ car $\frac{0}{\infty} = 0$

example: $\lim_{x \to \infty} \frac{3}{x} = 0$

Limit of a polynomial function at infinity

Limit of a polynomial function $f(x)$ at infinity, is equal to the limit at infinity of its leading term. That is:

$$\lim_{x\to\infty}\left(a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0\right) = \lim_{x\to\infty} a_n x^n = \pm\infty$$

Examples

Evaluate the following limits

(a) $\lim_{x\to-\infty}\left(3x^2 - x - 2\right)$

(b) $\lim_{x\to+\infty}\left(2 + 5x - 3x^5 + 10x^2\right)$

Exercises

Calculate the following limits:

1. $\lim_{x \to 2} x\left(\frac{1}{x} + \frac{x-2}{3}\right)$

2. $\lim_{x \to -1} (x-1)(x+6)$

3. $\lim_{x \to 2}\left(\frac{1}{x+1} + \frac{3}{x^2-4}\right)$

4. $\lim_{x \to \pm\infty} \left(-3x^5 + 2x^2 - 6x\right)$

5. $\lim_{x \to 3} \sqrt[3]{6x+3}$

6. $\lim_{x \to 3^-} \frac{x^2+x+1}{x-3}$

7. $\lim_{x \to -1} \frac{x^5+1}{x^2+1}$

8. $\lim_{x \to 3^+} \left(x^2 + x + 1\right)$

9. $\lim_{x \to +\infty} \left(3x^2 + x^2 - 2\right)$

Indeterminate forms

An indeterminate form is a certain type of expression with a limit that is not evident by inspection.

There are several types of indeterminate forms: $\frac{0}{0}, \frac{\infty}{\infty}, \infty, \infty - \infty, 0^0, \infty^0, 1^\infty$.

Suppose that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$.

The product $f(x) \cdot g(x)$ has the indeterminate form $0 \cdot \infty$ at $x = a$.

To find the limit of $f(x) \cdot g(x)$ at $x = a$, we can change the problem to one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in this way: $f(x) \cdot g(x) = \dfrac{f(x)}{1/g(x)} = \dfrac{g(x)}{1/f(x)}$.

Cont'd

i. Indeterminate form of type $\frac{0}{0}$.

a) $\lim_{x\to 1}\frac{x^2-1}{x-1}=\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}\lim_{x\to 1}(x+1)=2$

ii. Indeterminate form of type $\frac{\infty}{\infty}$

$$\lim_{x\to+\infty}\frac{\sum_{k=0}^{n}a_k x^k}{\sum_{i=0}^{m}b_i x^i}=\begin{cases}\frac{a_n}{b_n},\ n=m\\ 0,\ n<m\\ \infty,\ n>m\end{cases}$$

Examples

Find the following limits

(1) $\lim_{x \to 3} \dfrac{x^3 - 3x^2 - 9x + 27}{2x^3 - 11x^2 + 12x + 9}$

(2) $\lim_{x \to -1} \dfrac{x^2 - 1}{x^2 - 3x - 4}$

- More exercises on page 82

Evaluate the following limits:

(1) $\lim_{x \to 0} \dfrac{\sqrt{1+x} - 1}{x}$

(2) $\lim_{x \to -2} \dfrac{\sqrt[3]{x-6} + 2}{3 - \sqrt{x+11}}$

- When $x \to x_0$ with x_0 the real number, it is very simple to transform $0 \times \infty$ to $\frac{0}{0}$
- When $x \to \infty$, it is very simple to transform $0 \times \infty$ to $\frac{\infty}{\infty}$

Examples

Evaluate the following limits

(1) $\lim_{x \to -2}\left((x+2)\cdot\frac{1}{x^2-4}\right)$ **(2)** $\lim_{x \to -\infty}\left((x+2)\cdot\frac{1}{x^2-4}\right)$

Solutions

(1) $\lim_{x \to -2}\left((x+2)\cdot\frac{1}{x^2-4}\right) = ((-2)+2)\cdot\frac{1}{(-2)^2-4} = 0 \times \frac{1}{0} = 0 \times \infty \quad (IF)$

To remove this indeterminate form, firstly; we transform the type $0 \times \infty$ into the type $\frac{0}{0}$ as it follows:

$$\lim_{x \to -2}\left((x+2)\cdot\frac{1}{x^2-4}\right) = \lim_{x \to -2}\frac{x+2}{x^2-4} = \frac{-2+2}{(-2)^2-4} = \frac{0}{4-4} = \frac{0}{0} \quad (IF)$$

Continuity
Continuity at a point

A function $f(x)$ is continuous at point $x = x_0$ if $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Note that:

This implies three conditions which must be met in order that $f(x)$ be continued at $x = x_0$.

(1) $f(x)$ must exist. It means that $f(x)$ is defined at x_0 and $x_0 \in IR$

(2) $\lim\limits_{x \to x_0} f(x) = L$ must exist and $L \in IR$

(3) $\lim\limits_{x \to x_0} f(x) = f(x_0)$

If at least one condition from the three listed conditions is not satisfied then $f(x)$ is said **discontinuous** at x_0.

Continuity at on an interval

A function $f(x)$ is said to be continuous in an interval $a \le x \le b$ or $[a,b]$, if it is continuous at all points of the interval, we write $f(x) \in C^0[a,b]$.

Examples

1. Analyse the continuity of f(x) at point x = 2

$$f(x) = \begin{cases} \dfrac{(x^2-4)(x-3)}{x^3-8}, x \ne 2 \\ \dfrac{-1}{3}, x = 2 \end{cases}$$

Solution:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x^2-4)(x-3)}{x^3-8} = \lim_{x\to 2} \frac{(x-2)(x+2)(x-3)}{(x-2)(x^2+x+4)} = -\frac{1}{3} \text{ and}$$

$f(2)=-\frac{1}{3}$. As $\lim_{x\to 2} f(x)=f(2)=-\frac{1}{3}$ then f is continuous at x=2.

2. Analyze the continuity of $f(x)=x^3+x^2-1$ at point x=1.

f(x) is continuous at x=1 since f(1)=1 and

$$\lim_{x\to 1} f(x)=\lim_{x\to 0} f(x)=1=f(1)$$

3. For which value of m f(x) is it continuous at point x=3

$$f(x)=\begin{cases}\dfrac{6x^2-54}{\text{x-3}}, & x\neq 3\\ \text{m}, & \text{x=3}\end{cases}$$

Cont'd

Continuity

A function $f(x)$ is continuous at point a if $\lim_{x \to a} f(x) = f(a)$.

The definition implies that,

i. $f(x)$ is defined at a;

ii. The left-hand and the right-hand limit exist and,

iii. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.

If any of above three conditions is not satisfied, then $f(x)$ is said to be discontinuous at a.

DERIVATIVES

DIFFERENTIATIONS

- Increment

If a variable x is given an increment Δx from $x = x$, an arbitrary but fixed value of x in its range, a function $y = f(x)$ will be given in turn an increment,

$$\Delta y = \Delta f(x) = f(x + \Delta x) - f(x).$$

Cont'd

The average rate of change of a function $f(x)$ with respect to x is given by $\dfrac{\Delta f(x)}{\Delta x}$.

- The derivative

The derivative of a function $f(x)$ with respect to x is denoted by $f'(x)$ and defined as

$$f'(x) \overset{def}{=} \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$, provided the limit exists.

- The equation of the slope of the tangent line T of $y = f(x)$ at (x_0, y_0) is given by

$$T \equiv y - y_0 = f'(x_0)(x - x_0).$$

If a function $f(x)$ is differentiable at point a, then it is continuous at that point.

4.1.1. Rules for differentiation

1. $c' = 0$, where c is a constant

2. $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

3. $\left(cf(x)\right)' = cf'(x)$

4. $\left(f(x)g(x)\right)' = f'(x)g(x) + f(x).g'(x)$

5. $$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

6. The chain rule (differentiation of a function of function) $\left[f\left(u(x)\right)\right]' = f'\left(u(x)\right).u'(x)$

7. The differentiation of the inverse function $y'_x = \dfrac{1}{x'_y}$

Properties

1. $dc = 0$

2. $d\left(f(x) \pm g(x)\right) = df(x) \pm dg(x)$

3. $d\left(f(x)g(x)\right) = g(x)df(x) + f(x) \cdot dg(x)$

4. $d\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)df(x) - f(x)dg(x)}{g^2(x)}$

Higher differentials

Higher differentials are defined is similarly ways as higher derivatives.

Thus $d^2 f(x) = d(df); d^3 f(x) = d\left(d^2 f(x)\right)$, etc

We can also write: $f''(x) = \dfrac{d^2 f}{dx^2}; f'''(x) = \dfrac{d^3 f}{dx^3}; \ldots; f^{(n)}(x) = \dfrac{d^n f(x)}{dx^n}$

APPLICATION

a) Elimination of Indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

L'Hospital's Rule

If for $x = a$ (a being finite or infinite) $f(x)$ and $g(x)$ both vanish or become infinite, and

therefore $\frac{f(x)}{g(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided the

latter limit exists as x approaches a from one or both sides.

Cont'd

ii) Increasing and decreasing functions. Extrema values

The rate of change of a function $f(x)$ with respect to x, is given by $f'(x)$.

If $f'(x) > 0, \forall x \in \left]a, b\right[$, then f is an increasing function $[a,b](f)$

If $f'(x) < 0, \forall x \in \left]a, b\right[$ then f is a decreasing function $[a,b](f)$.

Critical values for function $f(x)$ are values of x for which the function is defined and for which $f'(x) = 0$ or becomes infinity.

Cont'd

2. Second Derivative Method

i. Find $f'(x)$ and critical values ;ii. Find $f''(x)$

iii. For a critical value $x = x_0$,

. $f(x)$ has a maximum value $[= f(x_0)]$ if $f''(x_0) < 0$

. $f(x)$ has a minimum value $[= f(x_0)]$ if $f''(x_0) > 0$

. The test fails if $f''(x) = 0$ or becomes infinite

Optimization

Example

An apartment complex has 250 apartments to rent. If they rent x apartments then their monthly profit, in dollars, is given by,

$$P(x) = -8x^2 + 3200x - 80{,}000$$

How many apartments should they rent in order to maximize their profit?

INTEGRATIONS(RULES: INDEFINITE AND DEFINITE FOR SIMPLE FUNCTIONS)

- ***1. Indefinite integral***
- The study of integral calculus is an inverse problem of the differentiation. This study relies heavily on notions of derivative and differential, so in the following we assume that these concepts are already known.
- **1.1. Definitions**
- Let $f(x)$ be a differentiable function of one real variable. We already know how to calculate the derivative . Assume now the opposite problem: Given the function $f(x)$, the derivative of a function $F(x)$, i.e. $= f(x)$, reconstruct the function $F(x)$.

Cont’d

Definition

The function $F(x)$ is called an **antiderivative** function of the function $f(x)$ on the interval $[a,b]$ if at any point on this interval the function $F(x)$ is differentiable and $F'(x) = f(x)$.

In other words, we know that given a function $F(x)$, $f(x) = F'(x)$ is its derivative. Given now a function $f(x)$, the inverse problem of the derivation is the determination of a function $F(x)$ such that its derivative is equal to $f(x)$, i.e. $F'(x) = f(x)$.

Cont'd

Example 1

Find an antiderivative function of $f(x) = x^2$.

We check immediately after the definition, that the antiderivative sought is $F(x) = \dfrac{x^3}{3}$.

$$\text{Indeed} \left(\frac{x^3}{3}\right)' = \frac{3x^2}{3} = x^2.$$

Cont'd

Definition

The set of all antiderivatives of a function $f(x)$ is called an indefinite integral of $f(x)$ and we denote $\int f(x)dx$.

The symbol $\int (sum)$ represents the integral and the function $f(x)$ is called the integrand or function under the sign of integration.

Note that any function does not have an antiderivative but it is assumed, without demonstration, that any continuous function on an interval $[a, b]$ has a antiderivative.

Cont'd

Definition

The process of finding the antiderivative of a function $f(x)$ is called **antidifferentiation** or **integration** of the function $f(x)$.

1.2. Properties

We mention the following evident properties of the indefinite integral:

1. The indefinite integral of the sum of two or more functions is equal to the sum of their integrals i.e. if $f_1(x)$ and $f_2(x)$ are two continuous functions on $[a,b]$, then

$$\int [f_1(x) + f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx.$$

Cont'd

2. We can output a constant factor under the sign of integration, i.e. if a is a constant, then

$$\int af(x)dx = a\int f(x)dx.$$

During the calculation of indefinite integrals, it is sometimes useful to remember that:

- If $\int f(x)dx = F(x) + C$, then $\int f(ax)dx = \frac{1}{a}F(ax) + C$
- If $\int f(x)dx = F(x) + C$, then $\int f(x+b)dx = F(x+b) + C$
- If $\int f(x)dx = F(x) + C$, then $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$

These properties are proved using the rules of derivation.

Cont'd

- ***1.3. Methods of integration***
- During integration, there are different methods used to find an antiderivative function. All techniques or transformations carried out during this process is to ensure we get the expressions for which the antiderivatives are immediate. Before we go through all those different integration methods we give a list of antiderivatives of functions known as **immediate antiderivative**s

Cont'd

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$ (where n is a relative integer different of -1)

2. $\int \dfrac{dx}{x} = \ln|x| + C$

$\int e^x dx = e^x + C$

$\int a^x dx = \dfrac{a^x}{\ln a} + C$

Cont’d

$$\int \frac{dx}{1-x^2} = \ln\left|\frac{1+x}{1-x}\right| + C$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

Cont'd

1.3.1. Integration by change of variable

We want to calculate the indefinite integral

$$I = \int f(x)dx.$$

Assume that this integral exists even if it is not immediate. To get there, make a change of variable as follows.

Let $x = \varphi(t)$. If a function is monotone and admits a continuous derivative, it is particularly invertible. Then:

$$x = \varphi(t) \Rightarrow dx = \varphi'(t)dt.$$

Putting the expression of x and dx into the expression of I, we arrive at

$$I = \int f[\varphi(\mathrm{t})]\varphi'(t)dt.$$

Cont'd

Example 1

Calculate $\int \frac{xdx}{1+x^2}$

$$\text{Let } 1+x^2=t \Rightarrow 2xdx=dt \Rightarrow xdx=\frac{dt}{2} \text{ and } \int\frac{xdx}{1+x^2}=\int\frac{dt}{2t}=\frac{1}{2}\int\frac{dt}{t}=\frac{1}{2}\ln|t|+c=\frac{1}{2}\ln\left|1+x^2\right|+C$$

Cont'd

1.3.2. Integration by parts

Let u and v be two differentiable functions of x. According to the rule of differentiation of a product of two functions, we have

$$d(uv) = udv + vdu.$$

By integrating both sides of this equality, we find $uv = \int udv + \int vdu$.

This gives

$$\int udv = uv - \int vdu.$$

Cont'd

Calculate $\int x^2 \ln x dx$

Assume $u = \ln x \Rightarrow du = \frac{dx}{x}$, $dv = x^2 dx \Rightarrow v = \frac{x^3}{3}$ and

$$\int x^2 \ln x dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3} dx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$$

Calculate $\int xe^x dx$.

Let $u = x \Rightarrow du = dx$, $dv = x^2 dx \Rightarrow v = e^x$ and $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$

Cont'd

Calculate the integral $\int \frac{dx}{x^2-5x+6} = \int \frac{dx}{(x-2)(x-3)}$.

The expression under the sign of integration can be decomposed as follows

$$\frac{1}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$\Leftrightarrow \frac{1}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} \Leftrightarrow \frac{1}{(x-2)(x-3)} = \frac{(A+B)x-(3A+2B)}{(x-2)(x-3)}$$

By identifying one has:

$$\begin{cases} A+B=0 \\ 3A+2B=-1 \end{cases} \Rightarrow \begin{cases} B=1 \\ A=-1 \end{cases} \text{ and } \frac{1}{(x-3)(x-2)} = \frac{-1}{x-2} + \frac{1}{x-3} .$$

MATRICES

CHAPTER 2. MATRICES ALGEBRA

2.1 Matrices (types, operations, determinants methods, inverse)

2.1.1. Matrix definition and type of matrix

Definition: Let a field $(\mathbb{R}, +, .)$ and $p\ ,\ q \in \mathbb{N}$;

The matrix A of the type p x q, defined on a field $\mathbb{R}$ is a table with p lines (rows) and q columns

whose entries are elements of the field $\mathbb{R}$ denoted by:

Cont'd

$$A=\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_q^1 \\ a_1^2 & a_2^2 & \cdots & a_q^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^p & a_2^p & \cdots & a_q^p \end{pmatrix}$$

The element a_j^i of the matrix is written at the intersection of the i^{th} row and the j^{th} column.(The upper index shows the numbering of the row while the lower index indicates the numbering of the column)

Cont'd

Examples: The matrix M=$\begin{pmatrix} 6 & -5 \\ 4 & 7 \end{pmatrix}$ is a 2x2 matrix and D= $\begin{pmatrix} 4 & 1 & -3 \\ 0 & 2 & 8 \end{pmatrix}$ is a 2x3 matrix.

We denote by M (pxq , $\mathbb{R}$) the set of the matrices of the type pxq.

Cont'd

-The matrix B is said "a square matrix of order m" if B has m rows and m columns (the number of lines and the number of columns are equals)

$$B=\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_m^1 \\ a_1^2 & a_2^2 & \cdots & a_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \cdots & a_m^m \end{pmatrix}$$ is a square matrix of order m.

We denote $M(m, \mathbb{R})$ the set of the square matrix of order m.

Cont'd

-A square matrix D is known as a diagonal matrix if its elements are zero except those of the

diagonal D= $\begin{pmatrix} a_1^1 & 0 & 0 & \dots & 0 \\ 0 & a_2^2 & 0 & \dots & 0 \\ 0 & 0 & a_3^3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_m^m \end{pmatrix}$

Cont'd

-The square matrices whose form are $L=\begin{pmatrix} a_1^1 & 0 & 0 & \dots & 0 \\ a_1^2 & a_2^2 & 0 & \dots & 0 \\ a_1^3 & a_2^3 & a_3^3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & a_3^m & \dots & a_m^m \end{pmatrix}$ and

$U=\begin{pmatrix} a_1^1 & a_2^1 & a_3^1 & \dots & a_m^1 \\ 0 & a_2^2 & a_3^2 & \dots & a_m^2 \\ 0 & 0 & a_3^3 & \dots & a_m^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_m^m \end{pmatrix}$ are called the triangular matrices.

Cont'd

-The matrix B of the type 1xn is called a row – matrix and has 1 row and n columns.

Example:

P=$(a_1^1 \quad a_2^1 \quad \ldots \quad \ldots \quad a_m^1)$

-The matrix M of the type nx1 is called a column – matrix and it has m rows and 1 column:

Example:

K=$\begin{pmatrix} a_1^1 \\ a_1^2 \\ a_1^3 \\ \vdots \\ a_1^m \end{pmatrix}$

Cont'd

2.1.2. Operation on matrices

a) Equality and transpose

- Two matrices (a_j^i) and (b_j^i) are equal if they are at the same type pxq and if $a_j^i = b_j^i$

$\forall i \epsilon \{1,2, \dots, p\}; \forall j \epsilon \{1,2, \dots, q\}$.

Cont’d

Example: The matrices A = $\begin{pmatrix} x-4 & 23 & 9 \\ 5 & y+8 & -12 \end{pmatrix}$ and B=$\begin{pmatrix} 3y+x+14 & 23 & 9 \\ 5 & 4y+2x & -12 \end{pmatrix}$ are

equal if $x - 4 = 3y + x + 14$ and if $y+8=4y+2x$.It means when y = - 6 , and x=13.

-The transpose of the matrix A is the matrix denoted by t_A obtained from the matrix A by

interchanging the role of the k^{th} rows and the k^{th}columns where $1 \leq k \leq p$ (or $1 \leq k \leq q$).

Remark: $t_{(A_j^i)} = A_i^j$ and $t(t_M)$=M

Cont'd

b) Addition of matrices

Let A = (a_j^i) and B=(b_j^i) two matrices of the same type p x q . The sum of A and B is a matrix C

of the type p x q obtained by adding entry by entry ($ie : c_j^i = a_j^i + b_j^i$).

Example: If we use A and B which are above we find

C=A+B=$\begin{pmatrix} 2x+3y+10 & 46 & 18 \\ 10 & 2x+5y+8 & -24 \end{pmatrix}$

Cont'd

Consequences:

The matrix O in which every entry is zero is called a zero matrix and it is an identity element of the addition of matrices of same types.

-The addition of matrices of same types is closure, associative, commutative and every matrix (a_j^i) has its inverse over addition which is $(-a_j^i)$; then the set of matrix of same type is a commutative group.

Cont'd

c) Multiplication of matrices

To multiply a matrix by a scalar k, we multiply each entry of the matrix by the scalar k.

Example: - If we use A and B which are above we find

W=2C= $\begin{pmatrix} 4x+6y+20 & 92 & 36 \\ 20 & 4x+10y+16 & -48 \end{pmatrix}$

Exercise: Calculate M=2A – 5B if A = $\begin{pmatrix} 4 & 5 & 1 \\ 0 & 7 & 2 \\ -3 & 6 & 10 \end{pmatrix}$ and B= $\begin{pmatrix} 0 & 5 & 1 \\ 0 & 9 & 2 \\ -1 & 6 & 8 \end{pmatrix}$

Cont’d

-The multiplication A.B of two matrices A and B is possible if the number of columns in the first matrix A is the same as the number of rows in the second matrix B.

Example: Let $A=\begin{pmatrix} 4 & 2 & 3 \\ 0 & 6 & -1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 \\ 3 & -1 \\ 2 & 1 \end{pmatrix}$. Find the matrix A.B and B.A

Cont’d

2.1.3. Determinant of a square matrix

At a square matrix A , we associate a real number called « determinant of the matrix A » denoted det A or $|A|$.If detA=0 the matrix is called “ a singular matrix”, but if det A≠0 then “A is a regular matrix.”

2.1.3.1. Determinant of 2x2 matrices

Let A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a square matrix of order 2, then det A= $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$=a .d − b .c

Cont'd

2.1.3.2. Determinant of 3x3 matrix by the Sarrus' rule

The Sarrus'rule used to calculate the determinant of a 3x3 matrix only is given by the following way :

If A=$\begin{pmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{pmatrix}$, then determinant of the matrix A is given by the following formula:

$$|A| = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{vmatrix} = (a_1^1.a_2^2.a_3^3 + a_2^1.a_3^2.a_1^3 + a_3^1.a_1^2.a_2^3) - (a_1^3.a_2^2.a_3^1 + a_2^3.a_3^2.a_1^1 + a_3^3.a_1^2.a_1^2)$$

Cont’d

Example: Calculate the determinant of the matrix A= $\begin{pmatrix} 2 & 3 & -2 \\ 4 & 1 & 5 \\ 2 & -1 & -3 \end{pmatrix}$ and det B if B= $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$

We show that detA=82 and we deduce it that A is a regular matrix.

Det B=4 +6=10

Cont'd

2.1.3.2. General method (Minor and Cofactor method)

Let A a n x n matrix below: A=$\begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^n & \cdots & a_n^n \end{pmatrix}$.

– *The determinant of the matrix* M_j^i Which is obtained from A by omitting the i^{th} row and the j^{th} column is called the minor of the element a_j^i.

The cofactor of the entry a_j^i is the number $A_j^i = (-1)^{i+j} \det M_j^i$

Cont'd

Examples: Calculate the determinant of the following matrix: $A = \begin{pmatrix} 2 & 0 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 7 & 4 & 3 & 5 \\ -6 & 2 & 2 & 4 \end{pmatrix}$

Cont'd

Properties of the determinant

(1)detA=det(t_A)

(2)det(A.B) = detA . detB

(3)If we interchange two columns (or two rows) of a matrix then its determinant is multiplied by -1.

(4)If two columns (or two rows) of a matrix are identical then its determinant is nil.

(5)If we multiply a column (or a row) of a matrix by a number k,then its determinant is multiplied by k.

(6)If a column(or a row)is a linear combination of other columns(rows),then its determinant is nil.

Cont'd

(7)The determinant of a matrix does not change if a column (or a row) is replaced by a linear combination of others columns (or others rows).

2.1.4. Calculation of the inverse matrix

Let A a nxn matrix which is regular.

-The inverse matrix of A is a matrix denoted A^{-1}such that A . A^{-1}=I where I is a nxn unit matrix.

Cont'd

-We call the cofactor matrix of A ; the matrix $\tilde{A}$ obtained by replacing each entry a_j^i by its cofactor A_j^i

-We call the adjoint matrix of A denoted adj(A),the transpose matrix of of the cofactor matrix.

I e adj (A) = $t_{\tilde{A}}$

The inverse matrix of a regular matrix A is given by $A^{-1} = \frac{adj(A)}{\det A}$

Cont'd

Example: Calculate the inverse matrix of $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ and verify the result.

Cont'd

2.2. SYSTEM OF LINEAR EQUATIONS

2.2.1. Definitions and examples: A system of p linear equations with n unknowns is defined

by: $$\begin{cases} a_1^1 x_1 + a_2^1 x_2 + \cdots + a_n^1 x_n = b_1 \\ a_1^2 x_1 + a_2^2 x_2 + \cdots + a_n^2 x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_1^p x_1 + a_2^p x_2 + \cdots + a_n^p x_n = b_p \end{cases} \quad (\&)$$

The real numbers a_j^i *and* b_i are called the coefficients of the system and the variables x_j are the unknowns of the system.

Cont'd

Remark: Every system of equations can be written matricially. For this we consider the matrices

$$A=(a_j^i)=\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^p & a_2^p & \cdots & a_n^p \end{pmatrix} ; B=(b_j)=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix} ;X=(x_i) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} ;$$

the system (&) is written A.X=B(&&).

In (&&) the matrix A is called the matrix of the system.

If B is null, then the system is called a homogeneous system.

Resolution of the system

- Resolution of the system by Gauss's method
- Resolution of the system by Cramer's method
- . Resolution of the system by using inverse matrix

Cont'd

- Example: Use all method to solve the following system of equation:

$$\begin{cases} 2X + 3Y + Z = -1 \\ 4X + 5Y + 2Z = 0 \\ 3X + 2Y - Z = -4 \end{cases}$$

- A woman invested different amounts at 8%, $8\frac{3}{4}\%$ and 9%, all at simple interest. Altogether she invested Rs. 40,000 and earns Rs. 3,455 per year. How much did she invest at each rate if she has Rs. 4,000 more invested at 9% than at 8%?

APLICATION IN BUSINESS

2.3. APPLICATIONS IN BUSINESS AREA

2. 3.1. Market Model

1. A single commodity linear model

The single commodity market involves the consideration of one commodity alone. This model includes only 3 variables namely.

The quantity demanded: $d = d(p)$;The quantity supplied: $s = s(p)$

The price of the commodity: p ;$d(p)$ is a decreasing function

$s(p)$ is an increasing function

Cont'd

For equilibrium condition to hold, $d(p)$ and $s(p)$ must be equal.

Translating these assumptions into mathematical form we have:

$d = ap + a_0$ (3.1)

$s = bp + b_0$ (3.2)

$d = s$ (3.3)

Where $a_0, b \succ 0$ and $a_0, b \prec 0$

Solving the model involves the determination of the equilibrium values p_0 and q_0.

Cont'd

Solving the model involves the determination of the equilibrium values p_0 and q_0.

In order to solve the model, use is made of the equilibrium condition (3.3) to equate demand and supply components of the model, thereby obtaining one equation in one unknown:

$$ap + a_0 = bp + b_0 \qquad (3.4)$$

Solution of this equation yields the equilibrium value of price: $p_0 = \dfrac{b_0 - a_0}{a - b}$ (3.5)

Cont’d

Equilibrium quantity, q_0 is obtained by substituting p_0 into either (3.1) or (3.2)

$$q_0 = \frac{ab_0 - ba_0}{a - b}$$

Cont’d

2. The two commodity market linear model

The equilibrium for a 2 commodity market model will involve two equations, representing the equality between demand and supply functions for each one of the 2 commodities. Consider the following 2 commodity market model in which both demand and supply functions are assumed to be linear:

Cont’d

$d_1 = a_{11}p_1 + a_{12}p_2 + c_0(3.1);$

$s_1 = b_1p_1 + d_1(3.2)$

$d_1 = s_1(2.3)$

$d_2 = a_{21}p_1 + a_{22}p_2 + c_2(3.4);$

$s_2 = b_2p_2 + d_2(3.5)$

$d_2 = s_2(3.6)$

The quantity supplied of commodity 1, depends on the price, p_1, of that commodity alone. s_2 ,

the quantity of commodity 2 supplied also depend only p_2 .

Cont'd

The quantity supplied of commodity 1, depends on the price, p_1, of that commodity alone. s_2 , the quantity of commodity 2 supplied also depend only p_2 .

However, d_1, the quantity of commodity 1 demanded depends of the price of that commodity as well as of the price of commodity 2. This is also true for s_2, and it suggests that the 2 commodities are related in consumption.

The equilibrium model is determined by making use of the 2 equilibrium conditions (3.3) and (3.6).

Cont'd

By these conditions we have the following system of 2 simultaneous equations in the 2 unknowns p_{10} and p_{20}.

$$\begin{cases} a_{11}p_{10} + a_{12}p_{20} + c_1 = b_1 p_{10} + d_1 \\ a_{21}p_{10} + a_{22}p_{20} + c_2 = b_2 p_{20} + d_2 \end{cases}$$

The equilibrium prices are obtained by solving the above system.

Cont'd

The corresponding equilibrium quantities are obtained by substituting p_{10} and p_{20} into (3.1) and (3.4) or by substituting p_{10} into (3.3) and p_{20} into (3.5).

Example3.1. A two commodity market model is defined by the following

$$d_1 = -p_1 + \frac{1}{2} p_2 + 4; d_2 = p_1 - p_2 + 10$$

$$s_1 = 4p_1 - 3; s_2 = 4p_2 - 18$$

Cont'd

Determine the equilibrium prices and quantities for the 2 commodities.

Solution

Equilibrium for the 2 commodity market model requires that $d_1 = s_1; d_2 = s_2$.

We have

$$\begin{cases} -p_{10} + \frac{1}{2} p_{20} + 4 = 4p_{10} - 3 \\ p_{10} - p_{20} + 10 = 4p_{20} - 18 \end{cases} \Leftrightarrow \begin{cases} 5p_{10} - \frac{1}{2} p_{20} = 7 \\ p_{10} - 5p_{20} = -28 \end{cases}$$

Solving for p_{10} and p_{20} gives $p_{10} = 2; p_{20} = 6$

Cont’d

Example3.2 Consider a two commodity market represented by:

$$d_1 = -p_1 + \frac{1}{2}p_2 + 4; s_1 = \frac{3}{2}p_1 - \frac{5}{2}$$

$$d_2 = 2p_1 - 2p_2 + 8; s_2 = 6p_2 - 3$$

Calculate the equilibrium prices and quantities for the market

THANKS