BUSINESS MATHEMATICS

by

Dr. ANGELIQUE DUKUNDE

Tel: 0788803313

What is a set?

Sets: **A set is a well-defined collection of distinct objects.** We assume that,

- The word set is synonymous with the word collection, aggregate, class and comprises of elements.
- Objects, elements and members of a set are synonymous terms.
- Sets are usually denoted by capital letters A, B, C, ...,.
- Elements of the set are represented by small letters a, b, c, ..., etc.

If 'a' is an element of set A, then we say that 'a' belongs to A. We denote the phrase 'belongs to' by the Greek symbol '$\in$' (epsilon). Thus, we say that $\mathbf{a \in A}$.

If 'b' is an element which does not belong to A, we represent this as $b \notin A$

Cardinal Number of a Set

- The number of distinct elements in a given set A is called the cardinal number of A. It is denoted by n(A). For example: •
  A $\{x : x \in N, x < 5\}$
  $A = \{1, 2, 3, 4\}$; Therefore, $n(A) = 4$, • $B =$ set of letters in the word ALGEBRA
  $B = \{A, L, G, E, B, R\}$
  Therefore, $n(B) = 6$

- **Empty set (Or a Null set)**

The empty set is the set which does not contain any element.

- The unique set with no elements

  is called **empty set** and denoted by $\varnothing$.

- Set Properties that involve $\varnothing$ **.**

  For all sets A,

1. $\varnothing \subseteq A$
2. $A \cup \varnothing = A$
3. $A \cap \varnothing = \varnothing$
4. $A \cap A^c = \varnothing$

- Singleton set (or Unit set) The singleton set is the set that has only one element. That is, if S is a singleton set, then $n(A) = 1$
- Pair set The pair set is the set that has exactly two elements. That is, if P is a pair set, then $n(A) = 2$
- Finite set
- The finite set is the set whose elements are countable. That is if one starts counting elements of the set, one by one, the counting comes to an end.

- .Infinite set
- The infinite set is the set containing an uncountable number of elements. In an infinite set, if the elements are counted, one by one, the counting never comes to an end.
- **Example:** $\boldsymbol{E = \{1, 2, 3, 4, \ldots\}}$

- **Universal set**
- The universal set is the original set that contains all objects under consideration. Or it is the set containing the totality of elements. The symbols used are: U or ε . The universal sets may be finite or infinite.

Examples

- 1. If one considers the set of men and women, then the universal set is probably the set of human beings.
- 2. If one considers sets such as pigs, cows, chickens, or horses, the universal set is probably the set of animals.

- Overlapping sets are sets that have some element in common
- Complement set

The complement of set with respect to a universal is a set that contains all elements in the universal set but not in a given set.

Identical and Equivalent sets

- Two sets A and B are said to be **identical or equal** if they contain the same elements. Every element of A is an element of B and every element of B is an element of A.
  For example: $A = \{p, q, r, s\}$
  $B = \{p, s, r, q\}$
  Therefore, $A = B$
- Two sets A and B are said to be **equivalent** if their cardinal number is same, i.e., $n(A) = n(B)$. The symbol for denoting an equivalent set is '$\leftrightarrow$'.
  For example: $A = \{1, 2, 3\}$ Here $n(A) = 3$
  $B = \{p, q, r\}$ Here $n(B) = 3$; Therefore, $A \leftrightarrow B$

Notation

- $S=\{a, b, c\}$ refers to the set whose elements are a, b and c.
- $a \in S$ means "a is an element of set S".
- $d \notin S$ means "d is *not* an element of set S".
- $\{x \in S \mid P(x)\}$ is the set of all those x from S such that P(x) is true. *E.g.*, $T=\{x \in \mathbf{Z} \mid 0<x<10\}$.
- *Notes:* 1) $\{a,b,c\}$, $\{b,a,c\}$, $\{c,b,a,b,b,c\}$ all represent the same set. 2) Sets can themselves be elements of other sets, *e.g.*, S={ {Mary, John}, {Tim, Ann}, …}

Relations between sets

- **Definition:** Suppose A and B are sets. Then

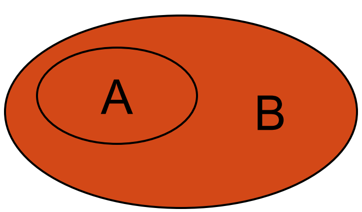
  A is called a **subset** of B: $\mathbf{A \subseteq B}$

  iff every element of A is also an element of B.

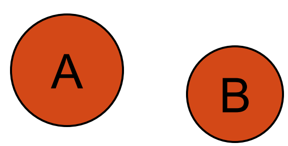
  Symbolically,

  $A \subseteq B \Leftrightarrow \forall x$, if $x \in A$ then $x \in B$.

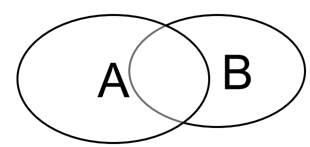
- $A \not\subset B \Leftrightarrow \exists x$ such that $x \in A$ and $x \notin B$.



$A \subseteq B$



$A \not\subset B$



$A \not\subset B$

Cardinal Number of a Set

- The number of distinct elements in a given set A is called the cardinal number of A. It is denoted by n(A). For example: •
  A $\{x : x \in N, x < 5\}$
  A = {1, 2, 3, 4} ; Therefore, n(A) = 4 , • B = set of letters in the word ALGEBRA
  B = {A, L, G, E, B, R}
  Therefore, n(B) = 6

Relations between sets

- **Definition:** Suppose A and B are sets. Then

  A **equals** B: **A = B**

  iff every element of A is in B and

  every element of B is in A.

  Symbolically,

  $A=B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

- **Example:** Let $A = \{m \in \mathbf{Z} \mid m=2k+1$ for some integer $k\}$;

  $B =$ the set of all odd integers.

  Then $A=B$.

Operations on Sets

Definition: Let A and B be subsets of a set U.

1. Union of A and B: $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$
2. Intersection of A and B:

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

3. Difference of B minus A: $B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$
4. Complement of A: $A^c = \{x \in U \mid x \notin A\}$

Ex.: Let $U = \mathbf{R}$, $A = \{x \in \mathbf{R} \mid 3 < x < 5\}$, $B = \{x \in \mathbf{R} \mid 4 < x < 9\}$. Then

1) $A \cup B = \{x \in \mathbf{R} \mid 3 < x < 9\}$.
2) $A \cap B = \{x \in \mathbf{R} \mid 4 < x < 5\}$.
3) $B - A = \{x \in \mathbf{R} \mid 5 \le x < 9\}$, $A - B = \{x \in \mathbf{R} \mid 3 < x \le 4\}$.
4) $A^c = \{x \in \mathbf{R} \mid x \le 3 \text{ or } x \ge 5\}$, $B^c = \{x \in \mathbf{R} \mid x \le 4 \text{ or } x \ge 9\}$

Cardinality of the Union

Let A, B and C be any three sets. It can be shown that:

- ✓ $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- ✓ $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

- In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many people who like both cold drinks and hot drinks?

Set Properties

- Commutative Laws:

$$(a)\ A \cap B = B \cap A$$

$$(b)\ A \cup B = B \cup A$$

- Associative Laws:

$$(a)\ (A \cap B) \cap C = A \cap (B \cap C)$$

$$(b)\ (A \cup B) \cup C = A \cup (B \cup C)$$

Set Properties

- Double Complement Law:

$$(A^c)^c = A$$

- De Morgan's Laws:

$$(a)\ (A \cap B)^c = A^c \cup B^c$$

$$(b)\ (A \cup B)^c = A^c \cap B^c$$

Disjoint Sets

- A and B are called **disjoint** iff $A \cap B = \varnothing$.
- Sets $A_1, A_2, \ldots, A_n$ are called **mutually disjoint**

  iff for all $i,j = 1,2,\ldots, n$

  $$A_i \cap A_j = \varnothing \text{ whenever } i \neq j .$$
- *Examples:*

  1) $A=\{1,2\}$ and $B=\{3,4\}$ are disjoint.

  2) The sets of even and odd integers are disjoint.

  3) $A=\{1,4\}$, $B=\{2,5\}$, $C=\{3\}$ are mutually disjoint.

  4) $A-B$, $B-A$ and $A\cap B$ are mutually disjoint.

Partitions

- **Definition:** A collection of nonempty sets

  $\{A_1, A_2, \ldots, A_n\}$ is a **partition** of a set A *iff*

  1. $A = A_1 \cup A_2 \cup \ldots \cup A_n$
  2. $A_1, A_2, \ldots, A_n$ are mutually disjoint.

- *Examples*:

  1) $\{\mathbf{Z}^+, \mathbf{Z}^-, \{0\}\}$ is a partition of **Z**.

  2) Let $S_0 = \{n \in \mathbf{Z} \mid n = 3k \text{ for some integer } k\}$

     $S_1 = \{n \in \mathbf{Z} \mid n = 3k+1 \text{ for some integer } k\}$

     $S_2 = \{n \in \mathbf{Z} \mid n = 3k+2 \text{ for some integer } k\}$

  Then $\{S_0, S_1, S_2\}$ is a partition of **Z**.

Power Sets

- **Definition:** Given a set A,

  the **power set** of A, denoted $\mathcal{P}(\mathrm{A})$,

  is the set of all subsets of A.
- *Example:* $\mathcal{P}(\{a,b\}) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$.
- **Properties:**

  1) *If* $A \subseteq B$ *then* $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

  2) *If* a set A has n elements

     then $\mathcal{P}(A)$ has 2^n elements.

Exercise: Find the power set of $A=\{1,2, 3, 4\}$

Cartesian product

Definition 32. The Cartesian product $A \times B$ of the sets A and B is the set of all ordered pairs (a, b), where $a \in A$, $b \in B$:

$$A \times B = \{(a, b),\ a \in A \wedge b \in B\}.$$

Example 46. Take $A = \{1, 2\}$, $B = \{x, y, z\}$. Then

$$A \times B = \{(a, b),\ a \in \{1, 2\} \wedge b \in \{x, y, z\}\}$$

thus a can be either 1 or 2, and for each of these 2 values, b can be either x, y or z:

$$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)\}.$$

Note that $A \times B \neq B \times A$, and that a Cartesian product can be formed from n sets $A_1, \ldots, A_n$, which is denoted by $A_1 \times A_2 \times \cdots \times A_n$.

- There are 35 students in art class and 57 students in dance class and 12 students are enrolled in both activities . Find the number of students who are either in art class or in dance class,

a) when two classes meet at different hours

b) when two classes meet at the same hour.

- In a group of 100 persons, 72 people can speak English and 43 can speak French. How many can speak English only? How many can speak French only and how many can speak both English and French?

- In a competition, a school awarded medals in different categories; 36 medals in dance, 12 medals in dramatics and 18 medals in music. If these medals went to a total of 45 persons and only 4 persons got medals in all the three categories, how many received medals in exactly two of these categories?
- Each student in a class of 40 plays at least one of football, volleyball and basketball. 18 play football, 20 play volleyball and 27 play basketball. 7 play football and volleyball, 12 play volleyball and basketball and 4 play football, volleyball and basketball. Find the number of students who play (a) Football and basketball. (b) Football and volleyball but not basketball.

- A marketing manager has identified two groups of customers: Group A (frequent buyers) and Group B (discount seekers). Group A = $\{1, 2, 3, 4, 5\}$, Group B = $\{2, 5, 6, 7, 9\}$. Determine the number of customers in each group and find A∩B and A∪B.
- A survey showed that 60% of customers are satisfied with product A, 50% with product B, and 30% with both. What is the probability that a customer selected at random is satisfied with at least one product?

EXERCISES

- Consider the following data among 110 students in a college dormitory:
- 30 students are on a list A (taking Accounting), 35 students are on a list B (taking Biology), 20 students are on both lists. Find the number of students: (a) on list A or B, (b) on exactly one of the two lists, (c) on neither list.

Ex2

Consider the following data for 120 mathematics students: 65 study French, 20 study French and German,45 study German, 25 study French and Russian, 42 study Russian, 15 study German and Russian, 8 study all three languages

- (a) Fill in the correct number of students in each of the eight regions of the Venn diagram. (b) Find the number k of students studying: (1) exactly one language, (2) exactly two languages.

Ex3

In a survey of 60 people, it was found that: 25 read Newsweek magazine, 9 read both Newsweek and Fortune, 26 read Time, 11 read both Newsweek and Time, 26 read Fortune, 8 read both Time and Fortune, 3 read all three magazines (a) Find the number of people who read at least one of the three magazines (b) Fill in the correct number of people in each of the eight regions of the Venn diagram in where N, T, and F denote the set of people who read Newsweek, Time, and Fortune, respectively.

(c) Find the number of people who read exactly one magazine.

- A group of 40 tourists arrived in Rwanda and visited Akagera National park, Nyungwe forests and Virunga mountains. Results showed that 33 visited Akagera, 21 visited Nyungwe and 23 visited Virunga. 18 visited both Akagera and Nyungwe, 10 visited both Nyungwe and Virunga, and 17 visited both Akagera and Virunga. All tourists visited at least one of the places.
  - (a) Represent the information on a Venn diagram.
  - (b) Find the number of tourists that visited Akagera only.

Ex4

Among the 90 students in a dormitory, 35 own an automobile, 40 own a bicycle, and 10 have both an automobile and a bicycle. Find the number of the students who:

- (a) do not have an automobile.
- (b) have an automobile or a bicycle;
- (c) have neither an automobile nor a bicycle;
- (d) have an automobile or a bicycle, but not both.

SET OF NUMBERs

- **The set of natural numbers ($\mathbb{N}$)**

A natural number is any number used to count the members of finite set. The set of positive integers is called the set of natural numbers denoted "$\mathbb{N}$" and defined by $\mathbb{N}=\{0,1,2,3,4,5,\ldots\}$

Note that:

- The lowest natural number is 0 - There is no highest natural number

- IN is an infinite set. - For any two natural numbers a and b , both the sum $a + b$ and the product $a \times b$ are natural numbers. This is often expressed by saying that the set of natural numbers is closed under the operations of addition and

- The set of integers $\mathbb{Z}$ The set of integers includes positive integer numbers, negative integer numbers and zero. That is: $Z = \{\dots, -3, -2, 1, 0, 1\ 2\ 3, \dots\}$. This leads us to the operation of subtraction (or inverse operation of addition), and for any two integers a and b ; we write $a - b$ or $b - a$
- Any natural number is an integer number. - For any two integer numbers a and b , the sum $a + b$, the difference $a - b$ and the product $a \times b$ are all integer numbers.
- This is often expressed by saying that the set of integer numbers is closed under the operations of addition, subtraction and multiplication.

The set of Rational numbers Q

- A rational number is any number of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. The set of rational numbers leads to the operation of division, or inverse of multiplication, and we write $x = \frac{p}{q}$ where p is the numerator and q the denominator.
- Note that: Any integer number is a rational number as every integer has 1 as the denominator. So, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$

- **The set of Irrational numbers**

The set of irrational numbers is denoted by I and formed by the numbers which are not rational, i.e., they cannot be converted into rational numbers. When it is written in decimal form, an irrational number is a nonterminating and non-recurring (or nonrepeating) decimal.

The set of real numbers $\mathbb{R}$

The set of real numbers is the (disjoint) union of the set of rational numbers with the set of irrational numbers. That is,

$Q \cup I = R$ which leads to $R = (-\infty, +\infty) =]-\infty, +\infty[$

- Graphically, R is represented by the real number line and called the real number system. This means that a rational number is either rational or irrational but not both. The geometric representation of real numbers as point on the real line L is often used. Each real number is represented by one point of L and viceversa $L = R$.
- **Note that**: Every point on the real line represents a real number.

Properties of real numbers

- Commutative property
- Associative Property
- Identity Property
- Symmetric property
- Distributi ve property

EQUATIONS OF 1ST AND 2ND ORDERS AND INEQUALITIES

- Equations and inequations occur very often in economic analysis. The solution of many problems in economics involves the solution of equations intended to determine the values of the variables in the problems. Equations and inequations can either be linear, quadratic or any other non-linear equations or inequations of higher degrees.

Cont'd

- Definition
- An equation is an expression with an equal sign. I n addition to this, in equations, unlike in functions, none of the variables in the expression is designated as the dependent variable or the independent variable, although the variables are explicitly related.
    - Equations of first order or linear equation
- Formulation and examples
- The linear equation has this standard for

Cont'd

$$ax + b = 0$$

Cont'd

- Example: Write the following equations in standard form

$$a)\ 2x + 3 = \frac{x - 3}{4}$$

$$b)\ \frac{2x + 3}{8x + 7} = \frac{x - 3}{4x - 3}$$

- Solution of linear equations

To solve an equation involving a variable is to find the value or roots of the equation, and the set of these values is referred to as the solution set.

Cont'd

$$ax + b = 0 \Leftrightarrow ax = -b$$

$$\Leftrightarrow x = \frac{-b}{a}$$

$$S = \{\frac{-b}{a}\}$$

- Example1:
- Solve the following

$$\frac{3x}{4} = \frac{x}{4} + 9$$

$$\frac{x+3}{16} - \frac{x-1}{4} = \frac{1}{8}$$

- When solving you may have different cases
- 1) $a \neq 0$ and $b \neq 0$

$$S = \left\{\frac{a}{b}\right\}$$

- 2) $a \neq 0$ and $b = 0, S = \{0\}$
- 3) $a = 0$ and $b \neq 0,\ S\ is\ impossible$
- $a = 0$ and $b = 0$ this equation is indeterminate

- **Product equation of the form** $(ax+b)(cx+d)=0$

- Solve for x in the set of real numbers, the equation

$$(2x+4)(x-1)=0$$

Fractional equation of the first degree of the form: $\frac{ax+b}{cx+d}=0$

Existing condition $cx+d \neq 0$

solve the equation $\frac{2x-6}{x+1}=0$

Cont'd

- Example2: The sum of two consecutive odd numbers is 32. What are the two odd numbers?
- Example3: A consumer P spends a monthly average of 23 shillings more on luxury goods than a second consumer Q. Both P and Q spend 139 shillings on luxury goods. What is the average monthly expenditure of each of the two consumers?

Cont'd

- Example4: The national income, Y, of a country is given by: Y=C+I+G Where C, I and G are, respectively, the consumption, investment and government expenditure components of the country's national income. Consumption is the size of investment, but 50,000 shillings less than government expenditure. If the national income of the country is 1,050,000 shillings, find the levels of consumption, investment and government expenditure for the country?

1.3.3. Quadratic equations and applications in business area

A **quadratic equation** is an equation equivalent to one of the form $ax^2 + bx + c = 0$ Where a, b, and c are real numbers and $a \neq 0$ So if we have an equation in x and the highest power is 2, it is quadratic.

Deriving The Quadratic Form

If $ax^2 + bx + c = 0$ (and $a \neq 0$), then: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Divide both sides by ***a***

Cont'd

Divide both sides by a

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Complete the square by adding $(b/2a)^2$ to both sides

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

Factor (left) and find LCD (right)

Combine fractions and take the square root of both sides $\left(x+\frac{b}{2a}\right)=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

Subtract *b*/2*a* and simplify

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Cont'd

If we have a quadratic equation and are considering solutions from the real number system, using the quadratic formula, one of three things can happen.

1. The "stuff" under the square root can be positive and we'd get two unequal real solutions $\text{if } b^2 - 4ac > 0$

2. The "stuff" under the square root can be zero and we'd get one solution (called a repeated or double root because it would factor into two equal factors, each giving us the sam $\text{if } b^2 - 4ac = 0$ solution

Cont’d

if $b^2 - 4ac < 0$

3. The "stuff" under the square root can be negative and we'd get no real solutions.

$$\Delta = b^2 - 4ac$$

The "stuff" under the square root is called the **discriminant**

- Examples:

In , solve the following quadratic equations:

(a) $6x^2 - 7x - 3 = 0$

(b) $18y^2 - 3y + 4 = 2y^2 + 5y + 3$

(c) $m^2 - 2m + 3 = 0$

Cont'd

Vertex

The point on a parabola that represents the absolute minimum or absolute maximum – otherwise known as the turning point.

y coordinate determines the range. (x,y)

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

Axis of symmetry :The vertical line that goes through the vertex of the parabola.

Equation is x = constant , a>0 graph opens upward , a<0 graph opens downward

Cont’d

Application in business

1. A small business’ profits over the last year have been related to the price of the only product. The relationship is $R(p) = -0.4p^2 + 64p - 2400$, where R is the revenue measured in thousands of dollars and p is the price of the product measured in dollars. What price would maximize the revenue? The word maximize screams “FIND THE VERTEX!!”

Cont'd

The answer to this question is the y-value of the vertex

$$p = \frac{-b}{2a}$$

$$p = \frac{-64}{2(-0.4)}$$

$$p = \frac{64}{-0.8}$$

$$p = 80$$

$$p = 80$$

$$R(80) = -0.4(80)^2 + 64(80) - 2400$$

$$R(80) = -0.4(6400) + 5120 - 2400$$

$$R(80) = -2560 + 5120 - 2400$$

$$R(80) = 160$$

The maximum revenue is \$160 000.

Cont'd

2. A small business' profits over the last year have been related to the price of the only product. The relationship is $R(p) = -0.4p^2 + 64p - 2400$, where R is the revenue measured in thousands of dollars and p is the price of the product measured in dollars. How much money would they lose if they gave the product away? This question is talking about a price of 0 or p = 0

$$p = 0$$
$$R(0) = -0.4(0)^2 + 64(0) - 2400$$
$$R(0) = -2400$$

- A company sells a product for a price of p dollars per unit. The number of units sold per week is given by the equation:
- $x = 100 - 5p$
- The cost to produce x units is:
- $C = 200 + 20x$
- and the revenue is $R = p \times x$.
- Write the **profit function** $P(p)$ in terms of p.
- Find the price p that gives the **maximum profit**.
- Find the **maximum profit**.

- A manufacturer finds that the cost C (in dollars) of producing x items is given by:
- $C = 5x^2 - 200x + 5000$
- Find the number of items x that minimizes the cost.
- Find the **minimum cost**.

- A bakery sells cakes for p dollars each.
  It's found that at a price of \$10, they sell 40 cakes per day, and for each \$1 increase in price, they sell 5 fewer cakes.

a) Express the bakery's **daily revenue** $R(p)$ as a quadratic function.

b) Find the price that **maximizes revenue**.

c) What is the **maximum revenue**?

- Let the number of cakes sold be x. When price increases by \$1, sales decrease by 5 cakes. So, the relationship between x and p is:
- $x = 40 - 5(p - 10)$
- We want an equation that starts at $x = 40$ when $p = 10$, and decreases by 5 for every +1 in p. We can express that idea as:
- $x = 40 - 5(\textit{change in price from 10})$. But the "change in price from 10" is just $(p - 10)$. So: $x = 40 - 5(p - 10)$

Simplify: $x = 40 - 5p + 50$ it gives x=90−5p

- Revenue = (price per cake) × (number of cakes sold)
- $R(p) = p \times x = p(90 - 5p)$

Simplify: $R(p) = -5p^2 + 90p$

So the **revenue function** is: $\boxed{R(p) = -5p^2 + 90p}$

- Law of remainder: If you divide a polynomial $P(x)$ of order n by a binomial of type $x + a$ then the remainder is equal to $P(-a)$
- Solve over IR the equation

$$2x^4 + 5x^3 - 5x - 2 = 0$$

Cont'd

1.3.4. Inequalities of 1st order

An algebraic relation showing that a quantity is greater than or less than another quantity.

Inequality - a mathematical sentence that contains $<, >, \leq, \geq$ or not equal.

$>$ reads as greater than

$<$ reads as less than, $\leq$ reads as less than or equal to, $\geq$ reads as greater than or equal to $\neq$ not equal to

Cont'd

Examples: solve inequalities

1. $x + 8 \geq 19$
2. $-26 > y + 14$
3. $m + 3 > 6$
4. $-5 \leq x - 6$
5. $4 + x < -2$
6. $13 + y \geq 13$

- **Non linear inequalities**
- Produc
- t and quotient: Studying signs

i. The sign diagram of a linear factor $ax+b$ **,** $a \neq 0$

The study of the sign of a linear factor $ax+b$ is done as follows:

- Find the critical value of x by solving $ax+b=0 \Leftrightarrow ax=-b \Leftrightarrow x=-\dfrac{b}{a}$
- Make the sign diagram of a linear factor $ax+b$

x	$-\infty$		$-\dfrac{b}{a}$		$+\infty$
$ax+b$		Opposite sign of a	0	Sign of a	

- Examples Study the sign of the following algebraic expressions:
- $(x-1)(x+2)$
- $\frac{3y+1}{1-y}$
- And hence solve: $(x-1)(x+2)<0$

$\frac{3y+1}{1-y}>0$

Quadratic inequalities

The summary of sign diagram for ax^2+bx+c is as follows:

- If $\Delta > 0$, there are two critical values x_1 and x_2 so, in this case, the table of signs of ax^2+bx+c is:

x	$-\infty$		x_1		x_2		$+\infty$
ax^2+bx+c		Sign of a	0	opposite sign of a	0	Sign of a	

- If $\Delta = 0$, there is a single critical value x_1, so, in this case, the table of signs of ax^2+bx+c is:

x	$-\infty$		x_1		$+\infty$
ax^2+bx+c		Sign of a	0	Sign of a	

- If $\Delta < 0$, there is no critical value, so, in this case, the table of signs of ax^2+bx+c is:

x	$-\infty$		$+\infty$
ax^2+bx+c		Sign of a	

Exercises

In $\mathbb{R}$, solve the following inequalities:

1) $(1-x)(2x+1)>0$

2) $-2x^2+x+3<0$

3) $x^2+x+1<0$

4) $x(x+4)\geq x-4$

5) $1+2x-x^2<0$

6) $\dfrac{2x-1}{x+2}>2$

7) $\dfrac{3x+2}{2-x}\geq 4$

CHAP 3: MATRICES

CHAPTER 2. MATRICES ALGEBRA

2.1 Matrices (types, operations, determinants methods, inverse)

2.1.1. Matrix definition and type of matrix

Definition: Let a field $(\mathbb{R}, +, .)$ and p , q ϵ $\mathbb{N}$;

The matrix A of the type p x q, defined on a field $\mathbb{R}$ is a table with p lines (rows) and q columns whose entries are elements of the field $\mathbb{R}$ denoted by:

Cont'd

$$A=\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_q^1 \\ a_1^2 & a_2^2 & \cdots & a_q^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^p & a_2^p & \cdots & a_q^p \end{pmatrix}$$

The element a_j^i of the matrix is written at the intersection of the i^{th} row and the j^{th} column.(The upper index shows the numbering of the row while the lower index indicates the numbering of the column)

Cont’d

Examples: The matrix M=$\begin{pmatrix} 6 & -5 \\ 4 & 7 \end{pmatrix}$ is a 2x2 matrix and D= $\begin{pmatrix} 4 & 1 & -3 \\ 0 & 2 & 8 \end{pmatrix}$ is a 2x3 matrix.

We denote by M (pxq , $\mathbb{R}$) the set of the matrices of the type pxq.

Cont'd

-The matrix B is said "a square matrix of order m" if B has m rows and m columns (the number of lines and the number of columns are equals)

$$B=\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_m^1 \\ a_1^2 & a_2^2 & \cdots & a_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \cdots & a_m^m \end{pmatrix}$$ is a square matrix of order m.

We denote M(m, $\mathbb{R}$) the set of the square matrix of order m.

Cont'd

-A square matrix D is known as a diagonal matrix if its elements are zero except those of the

$$\text{diagonal } D=\begin{pmatrix} a_1^1 & 0 & 0 & \dots & 0 \\ 0 & a_2^2 & 0 & \dots & 0 \\ 0 & 0 & a_3^3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_m^m \end{pmatrix}$$

Cont’d

-The square matrices whose form are $L=\begin{pmatrix} a_1^1 & 0 & 0 & \cdots & 0 \\ a_1^2 & a_2^2 & 0 & \cdots & 0 \\ a_1^3 & a_2^3 & a_3^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & a_3^m & \cdots & a_m^m \end{pmatrix}$ and

$U=\begin{pmatrix} a_1^1 & a_2^1 & a_3^1 & \cdots & a_m^1 \\ 0 & a_2^2 & a_3^2 & \cdots & a_m^2 \\ 0 & 0 & a_3^3 & \cdots & a_m^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_m^m \end{pmatrix}$ are called the triangular matrices.

Cont'd

-The matrix B of the type 1xn is called a row – matrix and has 1 row and n columns.

Example:

P=$(a_1^1 \quad a_2^1 \quad \dots \quad \dots \quad a_m^1)$

-The matrix M of the type nx1 is called a column – matrix and it has m rows and 1 column:

Example:

K=$\begin{pmatrix} a_1^1 \\ a_1^2 \\ a_1^3 \\ \vdots \\ a_1^m \end{pmatrix}$

Cont'd

2.1.2. Operation on matrices

a) Equality and transpose

- Two matrices (a_j^i) and (b_j^i) are equal if they are at the same type pxq and if $a_j^i = b_j^i$

$\forall i \epsilon \{1,2, \ldots, p\}; \forall j \epsilon \{1,2, \ldots, q\}.$

Cont'd

Example: The matrices A = $\begin{pmatrix} x-4 & 23 & 9 \\ 5 & y+8 & -12 \end{pmatrix}$ and B=$\begin{pmatrix} 3y+x+14 & 23 & 9 \\ 5 & 4y+2x & -12 \end{pmatrix}$ are equal if x – 4 =3y + x+14 and if y+8=4y+2x.It means when y = - 6 , and x=13.

-The transpose of the matrix A is the matrix denoted by t_A obtained from the matrix A by interchanging the role of the k^{th} rows and the k^{th}columns where 1≤k≤p (or 1≤k≤q).

Remark: $t_{(A_j^i)} = A_i^j$ and $t(t_M)$=M

Cont'd

b) Addition of matrices

Let A = (a_j^i) and B=(b_j^i) two matrices of the same type p x q . The sum of A and B is a matrix C

of the type p x q obtained by adding entry by entry ($ie : c_j^i = a_j^i + b_j^i$).

Example: If we use A and B which are above we find

C=A+B=$\begin{pmatrix} 2x+3y+10 & 46 & 18 \\ 10 & 2x+5y+8 & -24 \end{pmatrix}$

Cont'd

Consequences:

The matrix O in which every entry is zero is called a zero matrix and it is an identity element of the addition of matrices of same types.

-The addition of matrices of same types is closure, associative, commutative and every matrix (a_j^i) has its inverse over addition which is $(-a_j^i)$; then the set of matrix of same type is a commutative group.

Cont'd

c) Multiplication of matrices

To multiply a matrix by a scalar k, we multiply each entry of the matrix by the scalar k.

Example: - If we use A and B which are above we find

$$W=2C=\begin{pmatrix} 4x+6y+20 & 92 & 36 \\ 20 & 4x+10y+16 & -48 \end{pmatrix}$$

Exercise: Calculate M=2A – 5B if $A = \begin{pmatrix} 4 & 5 & 1 \\ 0 & 7 & 2 \\ -3 & 6 & 10 \end{pmatrix}$ and $B=\begin{pmatrix} 0 & 5 & 1 \\ 0 & 9 & 2 \\ -1 & 6 & 8 \end{pmatrix}$

Cont'd

-The multiplication A.B of two matrices A and B is possible if the number of columns in the first matrix A is the same as the number of rows in the second matrix B .

Example: Let A=$\begin{pmatrix} 4 & 2 & 3 \\ 0 & 6 & -1 \end{pmatrix}$ and B=$\begin{pmatrix} 1 & 0 \\ 3 & -1 \\ 2 & 1 \end{pmatrix}$. Find the matrix A.B and B.A

Cont'd

2.1.3. Determinant of a square matrix

At a square matrix A , we associate a real number called « determinant of the matrix A » denoted det A or $|A|$.If detA=0 the matrix is called " a singular matrix", but if det A≠0 then "A is a regular matrix."

2.1.3.1. Determinant of 2x2 matrices

Let A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a square matrix of order 2, then det A= $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$=a .d − b .c

Cont'd

2.1.3.2. Determinant of 3x3 matrix by the Sarrus' rule

The Sarrus'rule used to calculate the determinant of a 3x3 matrix only is given by the following way :

If $A=\begin{pmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{pmatrix}$, then determinant of the matrix A is given by the following formula:

$$|A| = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{vmatrix} = (a_1^1.a_2^2.a_3^3 + a_2^1.a_3^2.a_1^3 + a_3^1.a_1^2.a_2^3) - (a_1^3.a_2^2.a_3^1 + a_2^3.a_3^2.a_1^1 + a_3^3.a_1^2.a_1^2)$$

Cont’d

Example: Calculate the determinant of the matrix A=$\begin{pmatrix} 2 & 3 & -2 \\ 4 & 1 & 5 \\ 2 & -1 & -3 \end{pmatrix}$ and det B if B=$\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$

We show that detA=82 and we deduce it that A is a regular matrix.

Det B=4 +6=10

Cont'd

2.1.3.2. General method (Minor and Cofactor method)

Let A a n x n matrix below: A=$\begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^n & \cdots & a_n^n \end{pmatrix}$.

− *The determinant of the matrix* M_j^i Which is obtained from A by omitting the i^{th} row and the j^{th} column is called the minor of the element a_j^i.

The cofactor of the entry a_j^i is the number $A_j^i = (-1)^{i+j} \det M_j^i$

Cont’d

Examples: Calculate the determinant of the following matrix: $A = \begin{pmatrix} 2 & 0 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 7 & 4 & 3 & 5 \\ -6 & 2 & 2 & 4 \end{pmatrix}$

Cont'd

Properties of the determinant

(1)detA=det(t_A)

(2)det(A.B) = detA . detB

(3)If we interchange two columns (or two rows) of a matrix then its determinant is multiplied by -1.

(4)If two columns (or two rows) of a matrix are identical then its determinant is nil.

(5)If we multiply a column (or a row) of a matrix by a number k,then its determinant is multiplied by k.

(6)If a column(or a row)is a linear combination of other columns(rows),then its determinant is nil.

Cont'd

(7)The determinant of a matrix does not change if a column (or a row) is replaced by a linear combination of others columns (or others rows).

2.1.4. Calculation of the inverse matrix

Let A a nxn matrix which is regular.

-The inverse matrix of A is a matrix denoted A^{-1}such that A . A^{-1}=I where I is a nxn unit matrix.

Cont'd

-We call the cofactor matrix of A ; the matrix $\tilde{A}$ obtained by replacing each entry a_j^i by its cofactor A_j^i

-We call the adjoint matrix of A denoted adj(A),the transpose matrix of of the cofactor matrix.

I e adj (A) = ${}^t\tilde{A}$

The inverse matrix of a regular matrix A is given by $A^{-1} = \frac{adj(A)}{\det A}$

Cont'd

Example: Calculate the inverse matrix of $A=\begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ and verify the result.

Cont'd

2.2. SYSTEM OF LINEAR EQUATIONS

2.2.1. Definitions and examples: A system of p linear equations with n unknowns is defined

by: $$\begin{cases} a_1^1 x_1 + a_2^1 x_2 + \cdots + a_n^1 x_n = b_1 \\ a_1^2 x_1 + a_2^2 x_2 + \cdots + a_n^2 x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_1^p x_1 + a_2^p x_2 + \cdots + a_n^p x_n = b_p \end{cases} \quad (\&)$$

The real numbers a_j^i *and* b_i are called the coefficients of the system and the variables x_j are the

unknowns of the system.

Cont'd

Remark: Every system of equations can be written matricially. For this we consider the matrices

$$A=(a_j^i)=\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^p & a_2^p & \cdots & a_n^p \end{pmatrix} ; B=(b_j)=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix} ; X=(x_i)=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} ;$$

the system (&) is written A.X=B(&&).

In (&&) the matrix A is called the matrix of the system.

If B is null, then the system is called a homogeneous system.

Resolution of the system

- Resolution of the system by Gauss’s method
- Resolution of the system by Cramer’s method
- . Resolution of the system by using inverse matrix

Cont'd

- Example: Use all method to solve the following system of equation:

$$\begin{cases} 2X + 3Y + Z = -1 \\ 4X + 5Y + 2Z = 0 \\ 3X + 2Y - Z = -4 \end{cases}$$

- A woman invested different amounts at 8%, $8\frac{3}{4}\%$ and 9%, all at simple interest. Altogether she invested Rs. 40,000 and earns Rs. 3,455 per year. How much did she invest at each rate if she has Rs. 4,000 more invested at 9% than at 8%?

- Mrs. Rina Patel, a retired teacher, decided to invest her total savings of FRW 60,000 into three different fixed deposit schemes offered by a national bank. The bank provided her with annual simple interest rates of 7%, $8\frac{1}{4}$%, and 9%. To maintain a balance between safety and returns, Mrs. Patel planned her investments carefully. She decided that the amount invested at 9% would be FRW 6,000 more than the amount invested at 7%. At the end of the year, the total simple interest she earned from all three investments amounted to FRW 4,980.
- Based on this information, determine how much Mrs. Patel invested at each rate.
- Additionally, calculate what percentage of her total investment was placed in the $8\frac{1}{4}$% scheme.
- Finally, if the bank offers her an extra 1% bonus interest on her 9% deposit in the following year, find out how much additional income she would earn from that investment.

- Find the **domain of definition** of the following functions:

i) $f(x) = 2x^2 - 8x + 6$

ii) $g(x) = \frac{5x^2-4}{(4x-3)(2x+5)}$

- iii) $\sqrt{(3x-1)(7x-6)}$
- Find the **power set** of $A = \{k, l, m\}$.
- Find the **inverse** of the matrix: $A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & 2 \\ 1 & -2 & 1 \end{pmatrix}$

APLICATION IN BUSINESS

2.3. APPLICATIONS IN BUSINESS AREA

2. 3.1. Market Model

1. A single commodity linear model

The single commodity market involves the consideration of one commodity alone. This model includes only 3 variables namely.

The quantity demanded: $d = d(p)$;The quantity supplied: $s = s(p)$

The price of the commodity: p ;$d(p)$ is a decreasing function

$s(p)$ is an increasing function

Cont'd

For equilibrium condition to hold, $d(p)$ and $s(p)$ must be equal.

Translating these assumptions into mathematical form we have:

$$d = ap + a_0 \qquad (3.1)$$

$$s = bp + b_0 \qquad (3.2)$$

$$d = s \qquad (3.3)$$

Where $a_{0,} b \succ 0$ and $a_{0,} b \prec 0$

Solving the model involves the determination of the equilibrium values p_0 and q_0.

Cont'd

Solving the model involves the determination of the equilibrium values p_0 and q_0.

In order to solve the model, use is made of the equilibrium condition (3.3) to equate demand and supply components of the model, thereby obtaining one equation in one unknown:

$$ap + a_0 = bp + b_0 \qquad (3.4)$$

Solution of this equation yields the equilibrium value of price: $p_0 = \dfrac{b_0 - a_0}{a - b}$ (3.5)

Cont'd

Equilibrium quantity, q_0 is obtained by substituting p_0 into either (3.1) or (3.2)

$$q_0 = \frac{ab_0 - ba_0}{a - b}$$

Cont'd

2. The two commodity market linear model

The equilibrium for a 2 commodity market model will involve two equations, representing the equality between demand and supply functions for each one of the 2 commodities. Consider the following 2 commodity market model in which both demand and supply functions are assumed to be linear:

Cont'd

$d_1 = a_{11}p_1 + a_{12}p_2 + c_0 (3.1);$

$s_1 = b_1 p_1 + d_1 (3.2)$

$d_1 = s_1 (2.3)$

$d_2 = a_{21}p_1 + a_{22}p_2 + c_2 (3.4);$

$s_2 = b_2 p_2 + d_2 (3.5)$

$d_2 = s_2 (3.6)$

The quantity supplied of commodity 1, depends on the price, p_1, of that commodity alone. s_2 ,

the quantity of commodity 2 supplied also depend only p_2 .

Cont'd

The quantity supplied of commodity 1, depends on the price, p_1, of that commodity alone. s_2, the quantity of commodity 2 supplied also depend only p_2.

However, d_1, the quantity of commodity 1 demanded depends of the price of that commodity as well as of the price of commodity 2. This is also true for s_2, and it suggests that the 2 commodities are related in consumption.

The equilibrium model is determined by making use of the 2 equilibrium conditions (3.3) and (3.6).

Cont'd

By these conditions we have the following system of 2 simultaneous equations in the 2 unknowns p_{10} and p_{20}.

$$\begin{cases} a_{11}p_{10} + a_{12}p_{20} + c_1 = b_1 p_{10} + d_1 \\ a_{21}p_{10} + a_{22}p_{20} + c_2 = b_2 p_{20} + d_2 \end{cases}$$

The equilibrium prices are obtained by solving the above system.

Cont'd

The corresponding equilibrium quantities are obtained by substituting p_{10} and p_{20} into (3.1) and (3.4) or by substituting p_{10} into (3.3) and p_{20} into (3.5).

Example3.1. A two commodity market model is defined by the following

$$d_1 = -p_1 + \frac{1}{2}p_2 + 4; d_2 = p_1 - p_2 + 10$$

$$s_1 = 4p_1 - 3; s_2 = 4p_2 - 18$$

Cont'd

Determine the equilibrium prices and quantities for the 2 commodities.

Solution

Equilibrium for the 2 commodity market model requires that $d_1 = s_1; d_2 = s_2$.

We have

$$\begin{cases} -p_{10} + \frac{1}{2} p_{20} + 4 = 4p_{10} - 3 \\ p_{10} - p_{20} + 10 = 4p_{20} - 18 \end{cases} \Leftrightarrow \begin{cases} 5p_{10} - \frac{1}{2} p_{20} = 7 \\ p_{10} - 5p_{20} = -28 \end{cases}$$

Solving for p_{10} and p_{20} gives $p_{10} = 2; p_{20} = 6$

Cont'd

Example3.2 Consider a two commodity market represented by:

$$d_1 = -p_1 + \frac{1}{2}p_2 + 4; s_1 = \frac{3}{2}p_1 - \frac{5}{2}$$

$$d_2 = 2p_1 - 2p_2 + 8; s_2 = 6p_2 - 3$$

Calculate the equilibrium prices and quantities for the market

CHAPTER 4: Relation and Function

- **3.4 RELATIONS**
- Definition: Let A and B be sets. A binary relation or, simply, a relation from A to B is a subset of A x B. Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair a ∈A and b ∈ B, exactly one of the following is true:
- (i) (a, b) ∈ R; we then say "a is R-related to b", written aRb.
- (ii) (a, b) R; we then say" a is not R-related to b", written a b.

Cont'd

- The domain of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R, and so it is a subset of A; and the range of R is the set of all second elements, and so it is a subset of B. Sometimes R is a relation from a set A to itself, that is, R is a subset of $A^2 = A \times A$. In such a case, we say that R is a relation on A

Cont’d

- 3.6 Inverse Relation
- Let R be any relation from a set A to a set B. The inverse of R, denoted by R- 1 , is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,
- $R^{-1} = \{(b, a) : (a, b) \in R\}$ For example: If $R = \{(1, y), (1, z), (3, y)\}$, then $R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$.[Here R is the relation from $A = \{I, 2, 3\}$ to $B = \{x, y, z\}$
- Clearly, if R is any relation, then $(R^{-1})^{-1} = R$.

- 3.7 Functions:
- A function is a relation in which each element of the domain is paired with *exactly one* element of the range. Another way of saying it is that there is one and only one output (y) with each input (x).
- 3.8 Types of functions
- 3.8.1 One-to-One Functions
- A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff every element of its range has only one pre-image.
- Only one element of the domain is mapped to any given one element of the range.

Cont'd

- 3.8.2 Onto (Surjective) Functions
- A function $f: A \rightarrow B$ is onto or surjective or a surjection iff its range is equal to its codomain ($\forall b \in B, \exists a \in A: f(a)=b$). An onto function maps the set A onto (over, covering) the entirety of the set B, not just over a piece of it.
- 3.9 Bijections
- A function f is *a one-to-one correspondence*, or *a bijection*, or *reversible*, or *invertible*, iff it is both one-to-one and onto

Cont'd

- 3.10 Function Composition
- For functions $g:A\rightarrow B$ and $f:B\rightarrow C$, there is a special operator called compose ("$\circ$").
- It composes (i.e., creates) a new function out of f,g by applying f to the result of g.
- $(f\circ g):A\rightarrow C$, where $(f\circ g)(a) = f(g(a))$. Note $g(a)\in B$, so $f(g(a))$ is defined and $\in C$.
- **The range of g must be a subset of f's domain**
- ***INVERSE OF FUNCTION***

1) A company produces and sells notebooks.

The cost (in dollars) to make x notebooks is: $C = 50 + 2x$

- The revenue is: $R = 5x$
- Find the inequality that represents the condition for **profit**, and solve it to find how many notebooks must be sold to make a profit.

2) A factory charges \$20 per item plus a one-time setup fee of \$100. A customer wants the **average cost per item** to be **less than \$25**.

- Find the **minimum number of items** the customer must order.
- Hint: Average cost per item $= \dfrac{\text{Total } Cost}{Number\ of\ items} = \dfrac{100+20x}{x}$

3) A company spends x dollars on advertising.
The estimated number of new customers gained is given by:

$$N = -0.1x^2 + 200x$$

The company wants at least **8000 new customers**.

- Write and solve an inequality to find the **range of advertising budgets** that meet this goal.

DOMAIN OF THE FUNCTIONS

a) Rational function

Let h and g be two polynomial functions.

A rational function$x \to y$: $\frac{h(x)}{g(x)}$; is defined for all x ∈ ℝ such that g(x)#0

$Df = \{x \in \mathbb{R}, g(x) \neq 0\}$ or $Df = \{g(x) = 0\}$.

Examples:

1) $f(x) = \frac{x+1}{2x-4}$, Here we have $2x \ -4 \neq 0 \ \Leftrightarrow x = 2$

$Df = \ \mathbb{R} \setminus \{2\} or \ f =]-\infty, 2[\cup] + \infty.$

Cont'd

2) $f(x) = \frac{2x+3}{x^3+2x^2-x-2}$ we have $x^3 + 2x^2 - x - 2 = 0$

$\Leftrightarrow (x^2 - 1)(x + 2) = 0$

$\Leftrightarrow x = \pm 1 \;\; x = -2$

$Df = \mathbb{R} \setminus \{-2, -1, 1\}$ or $Df = \{-\infty, -2[\cup] - 2, -1[\cup[-1, 1[\cup]1, +\infty[.$

3) $f(x) = \frac{x}{x^2+1}, D = -4 < 0 \Rightarrow -4 < 0 \Rightarrow x2 + 1 \neq 0$

Cont'd

$\forall x \in \mathbb{R}, so\ Df = \mathbb{R}.$

b) Square root function

Let f be a function defined by x $\rightarrow f(x) = \sqrt{t(x)}$ where t (x) is a polynomial, a rational function,

Df = {x ∈ ℝ : t(x)≥ 0}.

Notice:

In general, when f: x$\rightarrow f(x) = \sqrt[n]{t(x)}$. We have the following cases which need a discussion

$n \in \mathbb{N} \setminus \{0,1\}$.

Exercises

a) $f(x)=\dfrac{\sqrt{2x+4}}{(5-x)(x-2)}$

b) $f(x)=\dfrac{\sqrt[4]{x^2-2x+15}}{\sqrt[5]{x^2-2x-8}}$

c) $f(x)=\dfrac{\sqrt[5]{x^2-3x-1.75}}{x^3-8}$

d) compare the domain of the following function defined by $f(x)$: $\sqrt{\dfrac{x^2-2x}{x^2-2x-3}}$ and

$$g(x)=\frac{\sqrt{x^2-2x}}{\sqrt{x^2-2x-3}}$$

Cont'd

Exercises:

Find the domain of the following functions defined by.

a) $f(x) = \sqrt[5]{x^2 - 3x + 2}$

b) $g(x) = \sqrt[5]{\dfrac{x+1}{x^2-3x+2}}$

Cont'd

c) $h(x) = \sqrt{x^2 - 3x - 10}$

d) $p(x) = \sqrt[4]{x^2 - 2x + 3}$

e) $d(x) = \sqrt[6]{\dfrac{x^2 - x - 2}{9 - x^2}}$

The linear function

- Definition: The linear function is a polynomial of the form: whereby represents – intercept i.e value of when is zero, represents the slope or gradient i.e the value that changes when changes by one unit
- **Characteristic of a straight line**
- It has only one solution
- It has no turning point. The turning point is also known as critical or stationary point
- It is completely defined when critical: i) either two points on the line are given, or ii) One point and the slope are specified.

Examples

1. Determine the linear function that goes through the following points

X	4	-2
Y	1	3

2. What is the straight line which has slope $b = -0.5$ and goes through the point $(x, y) =$ (10,18)

Equations

- Let R = total sales revenue in monetary terms

$$P = \text{unit price}$$

$$V = \text{unit variable costs}$$

$$f = \text{fixed cost}$$

$$x = \text{sales in physical units}$$

$$Vx = \text{total variable cost}$$

$$C = \text{total cost} = f + Vx$$

$$\Pi = \text{profit.}$$

1. Profit function

$\Pi = R - C, Where\ R = Px\ and\ C = Vx + f$

$\Pi = Px - (Vx + f) \leftrightarrow \Pi = Px - Vx - f$

$\boldsymbol{\Pi = (P - V)x - f}$

Contribution margin $(C.M) = \boldsymbol{P - V}$, then we can determine the profit in term of C.M as:

$\boldsymbol{\Pi} = CM\boldsymbol{x - f}$

2. Unit sales $\boldsymbol{x}$ determination for target profit Rwf $\boldsymbol{T}$

$$T = \Pi = CMx - f \rightarrow \boldsymbol{x} = \frac{\boldsymbol{T} + \boldsymbol{f}}{\boldsymbol{CM}}$$

3. Break even sales units, $\boldsymbol{X_{ble}}$

At Break Even Point (BEP), $R = C$ so that $\Pi = 0$

$$0 = CMx - f \rightarrow \boldsymbol{X_{ble}} = \frac{\boldsymbol{f}}{\boldsymbol{CM}}$$

Sales in monetary terms e.g Rwf $\boldsymbol{R}$

Cont'd

Examples

1. Consider a product with the following data.

$P = Rwf\ 200;\ \ V = Rwf\ 140;\ \ f = Rwf 800{,}0$

Required

Cont'd

a) Determine the break even sales units

b) The profit if sales are 10,000 units

c) Sales units required to make profit of Rwf 2,000,000

Cont'd

Solution

a) $X = \frac{f}{CM} = \frac{f}{P-V} = \frac{800,000}{200-140} = \frac{800,000}{60} = 13,333\, units$

b) $\pi = CMX - f = 60 \times 10,000 - 800,000 = Rwf\,200,000$

c) If Target profit = Rwf 2,000,000, then sales unit $X = \frac{T+f}{CM} = \frac{2,000,000+800,000}{60} = 46,667 units$

a) Let the system (S)

- $\begin{cases} x + y - 2z = 2 \\ 2x - y + 3z = -6 \\ y - 2x + 4z = -1 \end{cases}$

- Let A be the matrix of the system (S),

a) write the matrix A.

b) Compute the det A.

c) Compute A^{-1} if exist.

d) Solve the system (S).

- A company manufactures a certain product and the profit (P) from selling xx units of the product is given by the quadratic expression: $p(x) = -2x^2 + 12x - 10$

a) Calculate the value of x that will yield this maximum profit.

b) What is the maximum profit the company can achieve?

c) Calculate the profit when the company produces 0, 2, 4, 6, and 8 units.

d) How does the profit change with respect to the number of units produced?

- Determine the number of units that need to be sold to break even (i.e., when profit P(x)=0)

1) Find the domain of definition of the following

i) $f(x) = x^2 + 2x + 1$

ii) $\frac{(2x^2-1)}{(3x-1)(x-1)}$

- $\sqrt{(\boldsymbol{x} = \mathbf{3})(\boldsymbol{x} - \mathbf{2})}$

2)

a) : Find the power set of A= {1,2, 3, 4}.

b) A company's revenue R is modeled by the equation $R = 50x - 200$, where x is the number of units sold. Find the break-even point (where $R = 0$).

c) Find the inverse of the matrix:$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \\ 2 & 4 & 1 \end{pmatrix}$.

a) Given that $f(x) = 3x^2 + 2x - 1$ and $g = x - 2$ find:

i) $fog(x)$

ii) $gof(x)$

- $g^{-1}(x)$

EXPONENT AND LOGARITHM

1.2.5. Exponents and logarithms

1.2.5.1. Exponents

The product of n factors, each of which is x is given by:

$$x^n = \underbrace{x \cdot x \cdot x \cdot \cdots x}_{n\ factors}$$

In the expression, x^n is the n^{th}power of x, when x is the base and n is the exponent.

Laws of exponents

Multiplication law: $X^{\alpha}.X^{\beta} = X^{\alpha+\beta}$ Example: $X^2.X^4 = X^{2+4} = X^6$

Cont'd

Quotient law: $\frac{x^{\alpha}}{x^{\beta}} = X^{\alpha-\beta}$

Example: $\frac{x^{4}}{x^{2}} = X^{4-2} = X^{2}$

Zero Power law: $X^{0} = 1$

Example: $\frac{x^{4}}{x^{4}} = X^{4-4} = X^{0} = 1$

The Power law: $(X^{\alpha})^{\beta} = X^{\alpha\beta}$

Example: $(X^{2})^{3} = X^{2} \cdot X^{2} \cdot X^{2} = X^{2+2+2} = X^{6}$

Reciprocal law: i) $\frac{1}{x^{\beta}} = X^{-\beta}$ ii) $\frac{1}{x^{-\beta}} = X^{\beta}$

Example: $\frac{1}{x^{4}} = X^{-4}$

Cont’d

1.2.5.2. Logarithms

Consider the following exponential expression:

$y = a^x$

a is the base, and x is the exponent of the exponential expression.

The logarithm of y to the base a is the exponent to which a must be raised to obtain y.

Cont'd

Example: Evaluate

i) $\log 10{,}000$

ii) $\log_3 81$

Answer:

i) $\log 10{,}000 = \log 10^4 = 4$

ii) $\log_3 81 = \log_3 3^4 = 4$

Cont'd

Laws of logarithmic operations

Multiplication law: $\log_a(x \cdot y) = \log_a x + \log_a y$

Example: $\log_3(9 \cdot 27) = \log_3 9 + \log_3 27 = 2 + 3 = 5$

Quotient law: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

▪

Example: $\log_3\left(\frac{27}{9}\right) = \log_3 27 - \log_3 9 = 3 - 2 = 1$

Zero Power law: $\log_a(x^\alpha) = \alpha \log_a x$

Example: $\log_5(5^{-3}) = -3 \log_5 5 = -3$

$\log_a a = 1$

EXERCISES 1.2

1. Evaluate the following

i) $\log_m m$ ii) $\log_2 81 + \log_2 16$ iii) $4\log_4 3 + 2\log_2\left(\frac{1}{9}\right)$

2. Simplify

i) $7^5 \times 7^{16}$ ii) $\frac{(a^8)^4 \cdot b^6}{a^2 \cdot (b^8)^8}$

Cont'd

Exercises

$$\log_2(32) =$$

$$\log_2(64) =$$

$$\log_2(16) =$$

$$\log_2(\sqrt[3]{4}) =$$

Change of base formula

- While there are several useful identities, the most important for calculator use lets one find logarithms with bases other than those built into the calculator (usually $\log_e$ and $\log_{10}$).
- To find a logarithm with base b, using any other base b

$$\log_r(x) = \frac{\log_b(x)}{\log_b(r)} \text{ for any base } b \text{ or simply } \log_r(x) = \frac{\log(x)}{\log(r)}$$

- how long it will take for an investment of \$5,000 to grow to \$10,000 at an annual interest rate of 8%, compounded yearly.
- Monthly sales revenue, S (in \$), and monthly advertising expenditure, A (in \$), are modelled by the linear relation, S = 9000 + 12A.

a) If the firm does not spend any money on advertising, what is the expected sales revenue that month?

b) If the firm spends \$800 on advertising one month, what is the expected sales revenue?

c) How much does the firm need to spend on advertising to achieve monthly sales revenue of \$15 000?

d) If the firm increases monthly expenditure on advertising by \$1, what is the corresponding increase in sales revenue?

- An airline charges \$300 for a flight of 2000 km and \$700 for a flight of 4000 km.

a) Plot these points on graph paper with distance on the horizontal axis and cost on the vertical axis.

Assuming a linear model, estimate:

i) the cost of a flight of 3200 km.

ii) the distance travelled on a flight costing \$400.

LIMITS AND CONTINUITY

Limit definition

The function $f(x)$ tends to limit A as x tends to a and we write $\lim_{x \to a} f(x) = A$, if the difference between $f(x)$ and A remains small as we please so long as x remains sufficiently near to while remaining district from a.

Using mathematical denotation, we have:

$\lim_{x \to a} f(x) = A iff \forall \varepsilon > 0 \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow |f(x) - A| < \varepsilon$, or in terms of intervals, we can write: $\lim_{x \to a} f(x) = A iff \forall \varepsilon > 0 \exists \delta > 0 : x \in \left]a - \delta, a + \delta\right[\Rightarrow f(x) \in \left]A - \varepsilon, A + \varepsilon\right[$

Cont'd

Example: Prove that $\lim_{x \to 3}(2x+1) = 7$

Solution: Given $\varepsilon > 0$, we must find $\delta > 0$ such that

$$|(2x+1)-7| < \varepsilon \text{ if } 0 < |x-3| < \delta \text{ ,Now}$$

$|(2x+1)-7| = |2x-6| = 2|x-3|$ so, $0 < |x-3| < \dfrac{\varepsilon}{2}$ implies that $|(2x+1)-7| < 2 \cdot \dfrac{\varepsilon}{2} = \varepsilon.$

Hence, given $\varepsilon > 0$, it suffices to choose $\delta = \dfrac{\varepsilon}{2}$. This illustrates the observation that the required number δ is generally a function of the given number ε.

Cont'd

The limit laws

1. Constant Law

If $f(x) \equiv C$, where C is a constant, then $\lim_{x \to a} f(x) = \lim_{x \to a} C = C$.

2. Sun and difference Law

If both limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$.

3. Product Law

If both limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$.

4. Quotient Law

If both limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ and if $\lim_{x\to a} g(x)\neq 0$ exist, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

Cont'd

5. Substitution Law

Suppose $\lim_{x\to a} g(x) = A$ and that $\lim_{x\to A} f(x) = f(A)$, then $\lim_{x\to a} f\left(g(x) = f\left(\lim_{x\to a} g(x)\right)\right) = f(A)$.

6. Squeeze Law (Sandwich Law)

Suppose that $f(x) \le g(x) \le h(x)$ in some deleted neighborhood of a and that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$,

then $\lim_{x\to a} g(x)$ exists and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x)$.

Cont'd

Two remarkable limits

1. The basic trigonometric limit: $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

2. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

Cont'd

An indeterminate form is a certain type of expression with a limit that is not evident by inspection.

There are several types of indeterminate forms: $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 0^0, \infty^0, 1^\infty$.

Suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$.

The product $f(x)\cdot g(x)$ has the indeterminate form $0\cdot\infty$ at $x = a$.

To find the limit of $f(x)\cdot g(x)$ at $x = a$, we can change the problem to one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in this way: $f(x)\cdot g(x) = \dfrac{f(x)}{1/g(x)} = \dfrac{g(x)}{1/f(x)}$.

Cont'd

i. Indeterminate form of type $\frac{0}{0}$.

a) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \left(\frac{0}{0}\right) = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} \lim_{x \to 1}(x+1) = 2$

ii. Indeterminate form of type $\frac{\infty}{\infty}$

$$\lim_{x \to +\infty} \frac{\sum_{k=0}^{n} a_k x^k}{\sum_{i=0}^{m} b_i x^i} = \begin{cases} \frac{a_n}{b_n}, n = m \\ 0, n < m \\ \infty, n > m \end{cases}$$

Cont'd

iii. Indeterminate form of type 1^{∞}

a) $\lim\limits_{x \to +\infty} \left(1 + \dfrac{2}{x}\right)^{x} = \left(1^{\infty}\right)$

Cont'd

Continuity

A function $f(x)$ is continuous at point a if $\lim_{x \to a} f(x) = f(a)$.

The definition implies that,

i. $f(x)$ is defined at a;

ii. The left-hand and the right-hand limit exist and,

iii. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.

If any of above three conditions is not satisfied, then $f(x)$ is said to be discontinuous at a.

DERIVATIVES

DIFFERENTIATIONS

- Increment

If a variable x is given an increment Δx from $x = x$, an arbitrary but fixed value of x in its range, a function $y = f(x)$ will be given in turn an increment,

$$\Delta y = \Delta f(x) = f(x + \Delta x) - f(x).$$

Cont'd

The average rate of change of a function $f(x)$ with respect to x is given by $\dfrac{\Delta f(x)}{\Delta x}$.

- The derivative

The derivative of a function $f(x)$ with respect to x is denoted by $f'(x)$ and defined as

$$f'(x) \overset{\text{def}}{=} \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$, provided the limit exists.

▪

- The equation of the slope of the tangent line T of $y = f(x)$ at (x_0, y_0) is given by

$$T \equiv y - y_0 = f'(x_0)(x - x_0).$$

If a function $f(x)$ is differentiable at point a, then it is continuous at that point.

4.1.1. Rules for differentiation

1. $c' = 0$, where c is a constant

2. $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

3. $\left(cf(x)\right)' = cf'(x)$

4. $\left(f(x)g(x)\right)' = f'(x)g(x) + f(x).g'(x)$

5. $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

6. The chain rule (differentiation of a function of function) $\left[f\left(u(x)\right)\right]' = f'\left(u(x)\right).u'(x)$

7. The differentiation of the inverse function $y'_x = \dfrac{1}{x'_y}$

▪

Properties

1. $dc = 0$

2. $d\left(f(x) \pm g(x)\right) = df(x) \pm dg(x)$

3. $d\left(f(x)g(x)\right) = g(x)df(x) + f(x) \cdot dg(x)$

4. $d\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)df(x) - f(x)dg(x)}{g^2(x)}$

Higher differentials

Higher differentials are defined is similarly ways as higher derivatives.

Thus $d^2 f(x) = d(df); d^3 f(x) = d\left(d^2 f(x)\right)$, etc

We can also write: $f''(x) = \dfrac{d^2 f}{dx^2}; f'''(x) = \dfrac{d^3 f}{dx^3}; \ldots; f^{(n)}(x) = \dfrac{d^n f(x)}{dx^n}$

-
- Given $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \\ 15 \end{pmatrix}$ find x_1, x_2, x_3
- A book club has 50 members. The following information is available: 60 members read fiction., 40 members read non-fiction, 20 members read both fiction and non-fiction.

  ii) Draw a Venn diagram to represent the membership data. Label all relevant regions.

  iii) How many members read at least one type of book?

  iv) How many members read only fiction?

- You are a financial analyst at a company that operates in three different regions: North, South, and East. The company has two primary sources of revenue: Product A and Product B. You have gathered the revenue data for each region and product for the current quarter, which is organized into the following matrix

  - $R = \begin{pmatrix} 1500 & 2000 & 2500 \\ 1200 & 1800 & 2200 \end{pmatrix}$

- Where, The rows represent products: Product A (first row) and Product B (second row). The columns represent regions: North (first column), South (second column), and East (third column).

i) Calculate the total revenue for each product across all regions. Represent this as a column matrix.

ii) Determine the total revenue for each region across all products. Represent this as a row matrix.

iii) Suppose there is a 5% increase in revenue for each product in all regions. Compute the new revenue matrix after applying this increase.

- ii) Calculate number of members who attended none of the above trainings

1. Let the system (S)

- $$\begin{cases} x + y - 2z = 2 \\ 2x - y + 3z = -6 \\ y - 2x + 4z = -1 \end{cases}$$
- Let A be the matrix of the system (S),
- a) write the matrix A.
- b) Compute the det A.
- c) Compute A^{-1} if exist.

1. d) Solve the system (S).

a) **Solve for x**

i) $x^2 + 5x + 6 \leq 0$

ii) $x^2 - 5x + 6 = 0$

APPLICATION

a) Elimination of Indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

L'Hospital's Rule

If for $x = a$ (a being finite or infinite) $f(x)$ and $g(x)$ both vanish or become infinite, and

therefore $\frac{f(x)}{g(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided the

latter limit exists as x approaches a from one or both sides.

Cont'd

ii) Increasing and decreasing functions. Extrema values

The rate of change of a function $f(x)$ with respect to x, is given by $f'(x)$.

If $f'(x) > 0, \forall x \in \left]a,b\right[$, then f is an increasing function $\left[a,b\right](f)$

If $f'(x) < 0, \forall x \in \left]a,b\right[$ then f is a decreasing function $\left[a,b\right](f)$.

Critical values for function $f(x)$ are values of x for which the function is defined and for which $f'(x) = 0$ or becomes infinity.

Cont’d

2. Second Derivative Method

i. Find $f'(x)$ and critical values ;ii. Find $f''(x)$

iii. For a critical value $x = x_0$,

. $f(x)$ has a maximum value $[=f(x_0)]$ if $f''(x_0)<0$

. $f(x)$ has a minimum value $[=f(x_0)]$ if $f''(x_0)>0$

. The test fails if $f''(x)=0$ or becomes infinite

Optimization

Example

An apartment complex has 250 apartments to rent. If they rent x apartments then their monthly profit, in dollars, is given by,

$$P(x)=-8x^2+3200x-80{,}000$$

How many apartments should they rent in order to maximize their profit?

a) A company's **cost function** and **revenue function** for producing and selling a certain product are given as:

- **Cost Function:** $C(x) = 5000 + 30x + 0.2x^2$
- **Revenue Function:** $R(x) = 80x$

- **<u>Required</u>**

i) Find the marginal cost and marginal revenue functions.

ii) Determine the production level at which marginal cost equals marginal revenue (profit-maximizing quantity).

a) Find the power set of A= {a,b, c, d}.

b) A company's revenue R is modeled by the equation $R = 25x - 100$, where x is the number of units sold. Find the break-even point (where $R = 0$).

- Find the inverse of the matrix:$A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 1 \end{pmatrix}$.

a) Given that $f(x) = 2x^2 + 3x - 4$ and $g = x - 3$ find:

i) $fog(x)$

ii) $gof(x)$

iii) $g^{-1}(x)$

- If $f(x) = x^2 - 2x - 1$, and $g(x) = \frac{x+1}{x-1}$ find their limit at $x = 2$.

a) A company produces and sells a product, and its **total revenue (R)** and **total cost (C)** functions are given by:

- **Revenue Function:** $R(x) = 150x - 0.5x^2$
- **Cost Function:** $C(x) = 50x + 2000$
  - Where:
- **Required**

1. Find the profit function P(x) and determine the number of units that maximize profit.

- Compute the maximum profit using the second derivative test.

- A firm has analyse its operating condition, prices and costs and has developed the following functions: $R = 400Q - 4Q^2 \quad C = Q^2 + 10Q + 30$ where Q is the number of units sold, R is revenue and C is the cost. The firm wishes to maximize profit and wish to know:

a) What quantity should be sold.

b) At what price.

c) What should be the amount of profit.

You are a business analyst at a company that sells a product. The revenue R(x)generated from selling xx units of the product is given by the function: $R(x) = -2x^2 + 40x + 500$ where R(x) is the revenue in dollars and xx is the number of units sold.

a) Determine the number of units x that maximizes the revenue.

b) Calculate the maximum revenue.

c) Find the rate at which the revenue changes when $x = 10$ units are sold.

1. A firm faces the demand schedule $p = 184 - 4q$ and the Total Cost (TC) function

- $TC = q^3 - 21q^2 + 160q + 40$ where q is the quantity produced . What output will maximize the profit?

2. A management company is going to build a new apartment complex. They know that if the complex contains x apartments the maintenance costs for the building, landscaping etc. will be $C(x) = 400 + 14x - 0.02x^2$. The land they have purchased can hold a complex of at most 500 apartments. How many apartments should the complex have in order to minimize the

-

3. Find the value of a and b so that $\begin{pmatrix} a+4 & 3b \\ 8 & -6 \end{pmatrix} = \begin{pmatrix} 2a+2 & b^2+2 \\ 8 & b^2-5b \end{pmatrix}$

Given $A = \begin{bmatrix} 1 & a & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & a \end{bmatrix}$. Find all possible values of a such that $ABC^T = 0$

5. The total cost of producing x set of shoes it $T(x) = \frac{1}{4}x^2 + 35x + 25$ and the price set at which they may be sold is $R(x) = x(50 - \frac{1}{2}x)$

i) What should be the daily output to obtain a maximum total profit?

ii) Show that the cost of producing a set is a relative minimum

a) A company invests **$10,000** in a savings account that offers an **annual interest rate of 5%,** compounded **monthly.**

i) Write an **exponential function** to model the **investment growth** over **t years.**

ii) Calculate the **value of the investment** after **3 years**.

iii) How many **years** will it take for the investment to **double**?

b) A construction company buys a **new machine** for **$50,000**. The machine **depreciates** at a rate of **10% per year**, following an **exponential decay model**: $V(t) = 50000 \cdot (0.9)^t$ where V(t) is the **value of the machine** after **t years**.

i) Calculate the **value of the machine** after **5 years**.

ii) Determine how many **years** it will take for the machine's **value** to drop below **$10,000**.

- Akazi and Habirama are Certified Public Accountants who have recently started to give business pieces of advice to their clients. Acting as consultants at Kigali city, they have estimated the demand curve of a client's firm to be AR=200-8Q. Where AR is average revenue in millions of FRW in millions and Q is the output in units. Investigation of the client firm's cost profile in millions shows that marginal cost (MC) is given by: $MC = Q^2 - 28Q + 211$
- Further investigations have shown that the firm's cost when not producing output is FRW10,000,000

a) **Formulate the equation of total cost.**

b) **Compute the equation of total revenue**

INTEGRATIONS(RULES: INDEFINITE AND DEFINITE FOR SIMPLE FUNCTIONS)

- ***1. Indefinite integral***
- The study of integral calculus is an inverse problem of the differentiation. This study relies heavily on notions of derivative and differential, so in the following we assume that these concepts are already known.
- **1.1. Definitions**
- Let $f(x)$ be a differentiable function of one real variable. We already know how to calculate the derivative . Assume now the opposite problem: Given the function $f(x)$, the derivative of a function $F(x)$, i.e. $= f(x)$, reconstruct the function $F(x)$.

Cont'd

Definition

The function $F(x)$ is called an **antiderivative** function of the function $f(x)$ on the interval $[a,b]$ if at any point on this interval the function $F(x)$ is differentiable and $F'(x) = f(x)$.

In other words, we know that given a function $F(x)$, $f(x) = F'(x)$ is its derivative. Given now a function $f(x)$, the inverse problem of the derivation is the determination of a function $F(x)$ such that its derivative is equal to $f(x)$, i.e. $F'(x) = f(x)$.

Cont'd

Example 1

Find an antiderivative function of $f(x) = x^2$.

We check immediately after the definition, that the antiderivative sought is $F(x) = \dfrac{x^3}{3}$.

$$\text{Indeed } \left(\frac{x^3}{3}\right)' = \frac{3x^2}{3} = x^2 .$$

Cont'd

Definition

The set of all antiderivatives of a function $f(x)$ is called an indefinite integral of $f(x)$ and we denote $\int f(x)dx$.

The symbol $\int (sum)$ represents the integral and the function $f(x)$ is called the integrand or function under the sign of integration.

Note that any function does not have an antiderivative but it is assumed, without demonstration, that any continuous function on an interval $[a, b]$ has a antiderivative.

Cont'd

Definition

The process of finding the antiderivative of a function $f(x)$ is called **antidifferentiation** or **integration** of the function $f(x)$.

1.2. Properties

We mention the following evident properties of the indefinite integral:

1. The indefinite integral of the sum of two or more functions is equal to the sum of their integrals i.e. if $f_1(x)$ and $f_2(x)$ are two continuous functions on $[a,b]$, then $\int [f_1(x)+f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$.

Cont'd

2. We can output a constant factor under the sign of integration, i.e. if a is a constant, then

$$\int af(x)dx = a\int f(x)dx.$$

During the calculation of indefinite integrals, it is sometimes useful to remember that:

- If $\int f(x)dx = F(x) + C$, then $\int f(ax)dx = \frac{1}{a}F(ax) + C$
- If $\int f(x)dx = F(x) + C$, then $\int f(x+b)dx = F(x+b) + C$
- If $\int f(x)dx = F(x) + C$, then $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$

These properties are proved using the rules of derivation.

Cont'd

- ***1.3. Methods of integration***
- During integration, there are different methods used to find an antiderivative function. All techniques or transformations carried out during this process is to ensure we get the expressions for which the antiderivatives are immediate. Before we go through all those different integration methods we give a list of antiderivatives of functions known as **immediate antiderivative**s

Cont'd

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$ (where n is a relative integer different of -1)

2. $\int \dfrac{dx}{x} = \ln|x| + C$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

- A marketing agency conducted a survey to analyze customer preferences for two product categories, **A** and **B**. Among 500 customers, 300 expressed interest in Product A, 250 in Product B, and 100 customers were interested in both products. The agency wants to understand how to target customers effectively using this information.
- Using the concept of sets and the principle of inclusion and exclusion, the agency can calculate the total number of customers in different groups and refine their marketing strategy to cater to each segment effectively.

i) How many customers are interested in only Product A or only Product B?

ii) How many customers are interested in at least one of the two products?

iii) How many customers are not interested in either product?

a) The matrices A , B and C are given below in terms of the scalar constants a , b , c and d , by

- $A = \begin{pmatrix} -2 & 3 \\ 1 & a \end{pmatrix}$, $B = \begin{pmatrix} b & -1 \\ 2 & -4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & c \\ d & 4 \end{pmatrix}$, Given that A +B =C, find the value of a, b, c and d.
- Given $A = \begin{pmatrix} 1 & a & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 & a \end{pmatrix}$. Find all possible values of a such that $ABC^t = 0$.

- a) A company's **marginal cost function** (the rate of change of total cost with respect to the number of units produced) is given by: $C'(x) = 20 + x$
- where x is the number of units produced. The fixed cost is **4000**.

i) Find the total cost function C(x).

ii) Determine the total cost of producing **25 units**.

b) A company's **marginal revenue function** is given by: $R'(x) = 200 - 4x$

- where x is the number of units sold. The company has no revenue when **no units are sold** (i.e., $R(0) = 0$.
- Required

i) Find the total revenue function R(x

- Determine the total revenue generated when **40 units** are sold.

- A manufacturing company produces and sells **smartwatches**. The company's **profit function**, $P(x)$, in **thousands of dollars**, is given by: $P(x) = -4x^2 + 80x - 200$ where xxx is the **number of smartwatches sold** (in **hundreds**).

a) Calculate the **marginal profit** when **150 smartwatches** are sold.

b) Determine the **sales level** that will **maximize profit**.

c) What is the **maximum profit** the company can achieve?

d) Explain how the **marginal profit** can guide the company's **production decisions**, and discuss the **business implications** of the **maximum profit point**.

Cont’d

$$\int \frac{dx}{1-x^2} = \ln\left|\frac{1+x}{1-x}\right| + C$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

Cont'd

1.3.1. Integration by change of variable

We want to calculate the indefinite integral

$$I = \int f(x)dx.$$

Assume that this integral exists even if it is not immediate. To get there, make a change of variable as follows.

Let $x = \varphi(t)$. If a function is monotone and admits a continuous derivative, it is particularly invertible. Then:

$$x = \varphi(t) \Rightarrow dx = \varphi'(t)dt.$$

Putting the expression of x and dx into the expression of I, we arrive at

$$I = \int f[\varphi(\mathrm{t})]\varphi'(t)dt.$$

Cont’d

Example 1

Calculate $\int \frac{xdx}{1+x^2}$

$$\text{Let } 1+x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{dt}{2} \text{ and } \int \frac{xdx}{1+x^2} = \int \frac{dt}{2t} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + c = \frac{1}{2}\ln\left|1+x^2\right| + C$$

Cont'd

1.3.2. Integration by parts

Let u and v be two differentiable functions of x. According to the rule of differentiation of a product of two functions, we have

$$d(uv) = udv + vdu.$$

By integrating both sides of this equality, we find $uv = \int udv + \int vdu$.

This gives

$$\int udv = uv - \int vdu.$$

You are managing a production facility where the profit from producing x units is governed by the marginal profit function $100 - 2x$. This function represents the additional profit earned from producing one more unit beyond the fixed costs and initial setup.

Required:

Calculate and analyze the increase in total profit when the production level increases from 10 units to 15 units. Explain how understanding the marginal profit function helps in decision-making regarding production levels.

Cont'd

Calculate $\int x^2 \ln x dx$

Assume $u = \ln x \Rightarrow du = \dfrac{dx}{x}$, $dv = x^2 dx \Rightarrow v = \dfrac{x^3}{3}$ and

$$\int x^2 \ln x dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3} dx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$$

Calculate $\int xe^x dx$.

Let $u{=}x \Rightarrow du = dx$, $dv = x^2 dx \Rightarrow v = e^x$ and $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$

- Maniraguha company manufactures and sells leather bags. After a careful study of the market, the company concluded that demand is sensitive to price and production costs change based on the scale of production. The selling price per bag (in FRW) is $p(x) = 100 – 2x$ and the cost of producing x bags (000) is $C(x) = 20x + 0.5x^2 +500$. Due to material discounts and labour patterns, the total cost of producing a bag behaves quadratically.
- **Establish the profit function, marginal profit function and output that maximizes profit.**
- **Workout the maximum profit for Maniraguha company.**
- **Derive marginal cost and marginal revenue functions for Maniraguha company.**
- **Plot the graph of the total cost function for all value ranging from 1 to 4 bags.**

- A technology company, **Rwanda SmartTech Ltd.**, produces and sells advanced wearable health devices. Market research shows that the price per unit decreases as more units are sold, but not in a simple linear way. The firm's pricing is influenced by both consumer adoption patterns (which follow an exponential trend) and brand reputation effects (which follow a logarithmic trend). Based on an econometric analysis, the company's **price–demand function** is modeled as: $p(x) = 800e^{-0.01x} + 50\ln(x+1)$, where: $p(x)$ is the **price per unit** (in thousands of Rwandan francs, FRW), x is the **number of units sold** (in hundreds).
- Management wants to estimate the **total revenue** generated when sales increase from **0 to 200 hundred units** (that is,

- A company, manufactures and sells products. Market research shows that as the company increases its production, the price it can charge per table decreases due to competition and demand behavior. The **price–demand function** is given by: $p(x) = 500 - 2x$ where $p(x)$ =price per table (in FRW), x= number of tables sold. The management wants to calculate the **total revenue** obtained from selling tables when sales increase from 0 to 200 tables.
- **Formulate the total revenue function** as the integral of the price–demand function with respect to quantity. **Interpret the result in business terms (explain what the total revenue represents for the company).**

Cont'd

Calculate the integral $\int \frac{dx}{x^2-5x+6} = \int \frac{dx}{(x-2)(x-3)}$.

The expression under the sign of integration can be decomposed as follows

$$\frac{1}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$\Leftrightarrow \frac{1}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} \Leftrightarrow \frac{1}{(x-2)(x-3)} = \frac{(A+B)x-(3A+2B)}{(x-2)(x-3)}$$

By identifying one has:

$$\begin{cases} A+B=0 \\ 3A+2B=-1 \end{cases} \Rightarrow \begin{cases} B=1 \\ A=-1 \end{cases} \text{ and } \frac{1}{(x-3)(x-2)} = \frac{-1}{x-2} + \frac{1}{x-3} .$$

- In business, integration is widely used to optimize costs, forecast revenues, and analyze growth trends. One key area where integration is applied is in demand and revenue forecasting, where companies determine total revenue generated over a given period by integrating marginal revenue functions. XYZ Retail Company is a leading e-commerce platform that tracks its revenue growth based on the demand for its products. The company's economists have estimated that the marginal revenue function for a particular product line is given by: 50-2x where x represents the number of units sold. Determine the total revenue of the company earned by selling a certain number of units if the fixed revenue is 3000.
- ABC Manufacturing Company produces electronic components. The company's economists estimate that the marginal cost function for producing a unit is given by: $MC(x) = x^2 + e^x + \frac{3}{x}$ where x represents the number of units produced. Determine the total cost of the company.

- a) A company's **marginal cost function** (the rate of change of total cost with respect to the number of units produced) is given by: $C'(x) = 10 + 0.5x$
- where xxx is the number of units produced. The fixed cost is **2000**.
  - **<u>Required</u>**

i) Find the total cost function C(x).

ii) Determine the total cost of producing **50 units**.

b) A company's **marginal revenue function** is given by: $R'(x) = 100 - 2x$

- where x is the number of units sold. The company has no revenue when **no units are sold** (i.e., $R(0) = 0$.

- a) The matrices A , B and C are given below in terms of the scalar constants a , b , c and d , by

- $A = \begin{pmatrix} -2 & 3 \\ 1 & a \end{pmatrix}$, $B = \begin{pmatrix} b & -1 \\ 2 & -4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & c \\ d & 4 \end{pmatrix}$,

Given that A +B =C, find the value of a, b, c and d.

(5 marks)

- Given $A = \begin{pmatrix} 1 & a & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{pmatrix}$ and

$C = \begin{pmatrix} 1 & 2 & a \end{pmatrix}$. Find all possible values of a such

1) A roll of 32 bills contains only \$5 bills and \$10 bills. If the value of the roll is \$220, then how many of each bill are in the roll? Express this problem mathematically and solve it graphically.

2) The demand curve of a given product is quadratic and passes through points A(0,6), B(2,8) and C(4,26). Determine the demand curve in the standard form p(q). Find the total revenue function of this product.

- You are a financial analyst at a company that operates in three different regions: North, South, and East. The company has two primary sources of revenue: Product A and Product B. You have gathered the revenue data for each region and product for the current quarter, which is organized into the following matrix

- $R = \begin{pmatrix} 1500 & 2000 & 2500 \\ 1200 & 1800 & 2200 \end{pmatrix}$

- Where, The rows represent products: Product A (first row) and Product B (second row). The columns represent regions: North (first column), South (second column), and East (third column).

- **Required**

i) Calculate the total revenue for each product across all regions. Represent this as a column matrix.

ii) Determine the total revenue for each region across all products. Represent this as a row matrix.

iii) Suppose there is a 5% increase in revenue for each product in all regions. Compute the new revenue matrix after applying this increase.

a) A small business models its revenue R from selling x units of a product with the equation $R(x) = -4x^2 + 200x$. Determine the values of x for which gives a maximum the revenue and find out the maximum revenue.

b) **Solve for x**

- $x^2 - 5x + 6 = 0$

- Let the system (S)

$$(S)\begin{cases} 2x_1 \quad + 4x_2 - 2x_3 = \\ 4x_1 - 6x_2 + 2x_3 = -8 \\ 10x_1 - 2x_2 - 4x_3 = - \end{cases}$$

- Let A be the matrix of the system (S),

i) write the matrix A.
ii) Compute the det A.
iii) Compute A^{-1} if exist .
iv) Solve the system (S).

4) A straight line is given by $L_1 \equiv 3y - x = 3$.

Find the equation of line passing through the point (-1,3) and is

i) Parallel to L_1.

ii) Perpendicular to L_1.

iii) Find the equation of a line passes through the point A(4, 3) and B(1, 2)

5) You are managing a production facility where the profit from producing xx units is governed by the marginal profit function 100 - 2x. This function represents the additional profit earned from producing one more unit beyond the fixed costs and initial setup.

Required:

Calculate and analyze the increase in total profit when the production level increases from 10 units to 15 units. Explain how understanding the marginal profit function helps in decision-making regarding production levels.

6) You are analyzing the enrollment data for three extracurricular clubs at a school: the Science Club, the Art Club, and the Drama Club. The following sets represent the students enrolled in each club:

- Science Club (S): {Alice, Bob, Charlie, David, Eva}
- Art Club (A): {Bob, Charlie, Fiona, George}
- Drama Club (D): {Alice, David, Fiona, Henry}

Required

i) Determine the number of students who are members of at least one club.

ii) Find the number of students who are members of all three clubs.

iii) Identify the students who are members of exactly two clubs.

a) The demand function for a product is given by D(x)=40−x. Find the total revenue generated when selling from 0 to 20 units.

b) A local coffee shop is evaluating its sales over a week to understand its revenue trends better. The shop owner models the daily revenue as a function $R(t) = 100t^2 + 400$, where t represents the number of days since the shop opened. To find the total revenue generated over the first 10 days, the owner sets up a definite integral $\int_0^{10} R(t)dt$. By calculating this integral, the owner can determine the total revenue earned during that period, helping to inform decisions about inventory, staffing, and potential promotions for the upcoming week.

- A firm's profit P(x) from selling x units is given by $P(x) = 100x - 5x^2 - 200$ Determine the marginal profit when 10 units are sold.
- The cost function for manufacturing x items is $C(x) = 2x^3 - 3x^2 + 5x + 100$

Find the derivative to determine the rate of change of cost at $x = 3$.

THANKS

- Peter, started a consultant firm to provide advisory services his clients on business operations. The firm estimated demand curve and cost functions of one of the steel processing companies to be R = $21x - x^2$ and C = $\frac{x^3}{3} - 3x^2 + 9x + 16$ (in million Rwandan Francs) respectively. Where R is the sales revenue, C is the total cost function and x represents quantity produced in tonnes. The consultant has approached you and asked you to calculate:

i) the output when revenue is maximum and find the total revenue at this point.

ii) the marginal cost at a minimum level.

iii) the output that maximizes profit and the maximum profit.

A small company is analyzing its production processes and has established the following system of linear equations to represent the relationship between the quantities of products they produce and the associated costs. The equations are represented in matrix form as follows: $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \\ 15 \end{pmatrix}$

- Where x_1 = number of units of Product A, x_2= number of units of Product B, x_3 = number of units of Product C

- Let the system (S)

- $$\begin{cases} x + y - 2z = 2 \\ 2x - y + 3z = -6 \\ y - 2x + 4z = -1 \end{cases}$$

- Let A be the matrix of the system (S),

a) write the matrix A.

b) Compute the det A.

c) Compute A^{-1} if exist.

d) Solve the system (S).