Optimization Theory and Algorithms

A QCQP Problem (part 1)

(F2018)

Let $A \in \mathbb{R}^{n\times n}$ be symmetric and $b, c \in \mathbb{R}^n$ for $n \geq 2$. Consider the following quadratic-constrained, quadratic program (QCQP):

$$\min_{x,y\in\mathbb{R}^n} \quad f(x,y) = \frac{1}{2}\left(x^TAx + y^TAy\right) - b^Tx - c^Ty \tag{1}$$

$$\text{s.t.} \quad h_i(x,y) = 0,\ i = 1,2,3 \tag{2}$$

where

$$h_1(x,y) = \frac{1}{2}(x^Tx - 1), \quad h_2(x,y) = \frac{1}{2}(y^Ty - 1), \quad h_3(x,y) = x^Ty.$$

Part 1: (100 points)

1. Prove that regularity (constrained qualification) holds at any feasible (x,y).
2. Derive the first-order necessary conditions for this QCQP.
3. Derive expressions for multipliers, $\lambda_1, \lambda_2, \lambda_3$, in terms of (A,b,c) and (x,y).
4. Let $V(x,y) = \left\{w \in \mathbb{R}^{2n} : \nabla h(x,y)^T w = 0\right\}$. We can write $w^T = (u^T, v^T)$. Prove that for any $(u,v) \in V(x,y)$,
$$\begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} u \\ v \end{pmatrix} \perp \begin{pmatrix} y \\ x \end{pmatrix}.$$
5. (CIE only) Let (x,y) be a stationary point and λ be the associated multiplier vector. Prove that if A is positive semidefinite and $(b-c) \perp (x-y)$, then
$$\lambda_3 \geq \frac{\lambda_1 + \lambda_2}{2}.$$

Remarks:

1. Due on November 22, Thursday, 5pm, Dao Yuan 511.
2. Please typeset your answers and derivation.
3. Print clearly your name, student number and course number.