3.3. Assignment 1

1. We mark all the points on a circle obtained from a fixed point by rotations of the circle through angles of n radians, where $n \in \mathbb{Z}$ ranges over all integers. Describe all the limit points of the set so constructed.

   Solution. The inituition is the limit point is the whole circle. We describe the angles for the set of points as:

   $$\boldsymbol{A} = \left\{ n - \left\lfloor \frac{n}{2\pi} \right\rfloor * 2\pi : n \in \mathbb{Z} \right\}$$

   It suffices to show $\overline{\boldsymbol{A}} = [0, 2\pi]$. We set $a_n = \frac{n}{2\pi} - \lfloor \frac{n}{2\pi} \rfloor$. It suffices to show that for any $x \in [0, 1]$, there exists a sub-sequence a_{n_k} converging to x. The idea is to show that there exists a sequence of $\{q_n\}$ such that a_{q_n} converges to zero first, and then index n_k is of appropriately multiple of q_n.

   - Note that there exists a sequence of $\{p_n, q_n\}$ such that

     $$\frac{1}{2\pi} - \left\lfloor \frac{1}{2\pi} \right\rfloor - \frac{p_n}{q_n} \le \frac{1}{q_n^2} \implies \frac{q_n}{2\pi} - q_n \left\lfloor \frac{1}{2\pi} \right\rfloor - p_n \le \frac{1}{q_n}$$

     Or equivalently,

     $$a_{q_n} = \frac{q_n}{2\pi} - \left\lfloor \frac{q_n}{2\pi} \right\rfloor = \min_{l \in \mathbb{Z}, 2\pi l \le q_n} \left(\frac{q_n}{2\pi} - l \right) \le \frac{q_n}{2\pi} - q_n \left\lfloor \frac{1}{2\pi} \right\rfloor - p_n \le \frac{1}{q_n}$$

     We can change the inequality into equality.

   - Therefore, for any $x \in [0, 1]$, choose j such that $|ja_{q_n} - x| \le \frac{1}{q_n}$. Note that $\{ja_{q_n}\}$ is a subsequence of $\{a_n\}$, say, $a_{j,q,n}$.

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2. Show that if we take only the set $\mathbb{Q}$ of rational numbers instead of the complete set of real numbers, taking closed interval, open interval, and neighborhood of a point $r \in \mathbb{Q}$ to mean respectively the corresponding subsets of $\mathbb{Q}$, then none of the **nested interval lemma**, **finite covering lemma**, **limit point lemma** remains true.

Proof. (a) Construct our intervals around irrational, e.g.,

$$\boldsymbol{I}_n = \left(\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}\right) \bigcap \mathbb{Q} \implies \bigcap_{i=1}^{\infty} \boldsymbol{I}_n = \varnothing$$

(b) Construct our collection of open covers with total out-measure (interval width) as a series, e.g., suppose

$$[0,1] \bigcap \mathbb{Q} = \bigcup_{i=1}^{\infty} \{q_i\}, \qquad \boldsymbol{I}_n = \left(q_n - \frac{1}{2^{n+1}}, q_n + \frac{1}{2^{n+1}}\right)$$

Thus $m(\bigcup_{n=1}^{\infty} \boldsymbol{I}_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, while any sub-collection has total out-measure strictly less than one, i.e., the subcover does not exist.

(c) The idea is to construct a sequence of bounded rational numbers with limit point $x \notin \mathbb{Q}$, as the neighborhood of irrational numbers is not defined, e.g.,

$$q_n \in \left(\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}\right)$$

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3. Show that a number $x \in \mathbb{R}$ is rational if and only if its q-ary expression in any base q is periodic, i.e., from some rank on it consists of periodically repeating digits.

Proof. The reverse direction is easy to show. For the necessity, for a given $\frac{m}{n} \in \mathbb{Q}$, suppose it can be written as

$$\frac{m}{n} = s.d_1 d_2 \cdots d_n + \frac{r_n}{q^n m}$$

You can show that recursively we have

$$q r_n = m d_{n+1} + r_{n+1} \tag{3.5}$$

All the remainders $\{r_n\}$ are between 0 and $m-1$, there must exist some $k < n$ such that $r_k = r_n$. Verify from (3.5) that $d_{k+1} = d_{n+1}$ and $r_{k+1} = r_{n+1}$, and thus by applying the same trick, we can show that when the remainder repeats for the first time, the expansion form repeats. ■