

**A JOURNEY  
IN  
PURE MATHEMATICS**



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**PURE MATHEMATICS**  
**MAT3006 & 3040 & 4002 Notebook**

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# Notations and Conventions

$\mathbb{R}^n$	$n$ -dimensional real space
$\mathbb{C}^n$	$n$ -dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
$x_i$	$i$ th entry of column vector $\mathbf{x}$
$a_{ij}$	$(i, j)$ th entry of matrix $\mathbf{A}$
$\mathbf{a}_i$	$i$ th column of matrix $\mathbf{A}$
$\mathbf{a}_i^T$	$i$ th row of matrix $\mathbf{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all $i, j$
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all $i, j$
$\mathbf{A}^T$	transpose of $\mathbf{A}$ , i.e, $\mathbf{B} = \mathbf{A}^T$ means $b_{ji} = a_{ij}$ for all $i, j$
$\mathbf{A}^H$	Hermitian transpose of $\mathbf{A}$ , i.e, $\mathbf{B} = \mathbf{A}^H$ means $b_{ji} = \bar{a}_{ij}$ for all $i, j$
$\text{trace}(\mathbf{A})$	sum of diagonal entries of square matrix $\mathbf{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\mathbf{e}_i$	a unit vector with the nonzero element at the $i$ th entry
$\mathcal{C}(\mathbf{A})$	the column space of $\mathbf{A}$
$\mathcal{R}(\mathbf{A})$	the row space of $\mathbf{A}$
$\mathcal{N}(\mathbf{A})$	the null space of $\mathbf{A}$
$\text{Proj}_{\mathcal{M}}(\mathbf{A})$	the projection of $\mathbf{A}$ onto the set $\mathcal{M}$



## 5.2. Monday for MAT3006

Our first quiz will be held on this Wednesday.

**Reviewing.** We have shown that the algebra  $\mathcal{A} \subseteq \mathcal{C}(X)$  with separation, non-vanishing property implies  $\overline{\mathcal{A}} = \mathcal{C}(X)$ .

Now we show that if  $\overline{\mathcal{A}} = \mathcal{C}(X)$ , then the algebra  $\mathcal{A}$  has separation, non-vanishing property:

1. Suppose on the contrary that  $\mathcal{A}$  is not separating, i.e., there exists  $x_1, x_2 \in X$  such that  $\phi(x_1) = \phi(x_2), \forall \phi \in \mathcal{A}$ .

By the definition of closure, it's clear that for given  $S \subseteq (X, d), \forall x \in \overline{S}$ , there exists a sequence  $\{S_n\}$  in  $S$  such that  $S_n \rightarrow x$ .

Construct  $f \in \mathcal{C}(X)$  defined by  $f(x) = d(x, x_1)$ . It follows that

$$f(x_1) = 0, \quad f(x_2) = d(x_2, x_1) := k > 0$$

Now we claim that  $f \notin \overline{\mathcal{A}}$ , since otherwise there exists  $\{\phi_n\}$  in  $\mathcal{A}$  such that  $\phi_n \rightarrow f$ , i.e.,

$$\phi_n(x_1) \rightarrow f(x_1), \quad \phi_n(x_2) \rightarrow f(x_2), \quad \phi_n(x_1) = \phi_n(x_2), \forall n,$$

i.e.,  $0 = f(x_1) = f(x_2) > 0$ .

2. Suppose on the contrary that  $\mathcal{A}$  is not non-vanishing, i.e., there exists some  $x_0 \in X$  such that  $\phi(x_0) = 0, \forall \phi \in \mathcal{A}$ . Construct  $g \in \mathcal{C}(X)$  defined by  $g(x) = d(x, x_0) + 1$ . Following the similar idea, we can show that there does not exist  $\phi_n \in \mathcal{A}$  such that  $\phi_n \rightarrow g$ , i.e.,  $g \notin \overline{\mathcal{A}}$ , which is a contradiction.

■ **Example 5.4** 1. Let  $X \subseteq \mathbb{R}^n$  be a compact space. Then the polynomial ring

$$\mathbb{R}[x_1, \dots, x_n] = \{\text{Polynomials in } n \text{ variables with coefficients in } \mathbb{R}\}$$

forms a dense set in  $\mathcal{C}(X)$ .

It's clear that the set  $\mathbb{R}[x_1, \dots, x_n]$  satisfies the separating and non-vanishing property.

For the special case  $n = 1$  and  $X = [a, b]$ , we get the Weierstrass Approximation Theorem.

2. In particular, when  $X = S^1 \subseteq \mathbb{R}^2$ , we imply  $\mathbb{R}[x, y]$  is dense in  $\mathcal{C}(S^1)$ .

### 5.2.1. Stone-Weierstrass Theorem in $\mathbb{C}$

Consider the circle  $S^1 \subseteq \mathbb{C}$  and the mappings

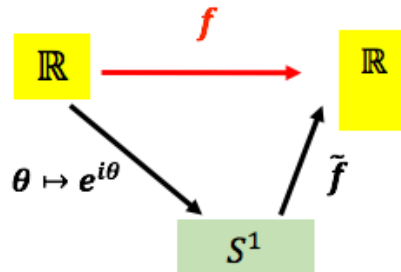
$$\begin{aligned} c : S^1 &\rightarrow \mathbb{R} & s : S^1 &\rightarrow \mathbb{R} \\ \text{with } e^{i\theta} &\rightarrow \cos \theta & \text{with } e^{i\theta} &\rightarrow \sin \theta \end{aligned}$$

are both continuous.

The algebra formed by  $s$  and  $c$  is given by

$$\mathcal{J} := \langle c, s \rangle = \text{span}\{\cos^m \theta \sin^n \theta \mid m, n \in \mathbb{N}\}$$

1. The  $\mathcal{J}$  satisfies both separating and non-vanishing property, which implies  $\overline{\mathcal{J}} = \mathcal{C}(S^1)$ .
2. Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous,  $2\pi$ -periodic mapping. It's easy to construct a continuous mapping  $\tilde{f} : S^1 \rightarrow \mathbb{R}$  such that the diagram below commutes:



Or equivalently,  $f(\theta) = \tilde{f}(e^{i\theta})$  for some  $\tilde{f} \in \mathcal{C}(S^1)$ . Since  $\overline{\mathcal{J}} = \mathcal{C}(S^1)$ , we can approximate  $\tilde{f} \in \mathcal{C}(S^1)$  by  $\langle \cos \theta, \sin \theta \rangle$ , which implies that the  $f(\theta)$  can be approximated

by


$$\sum_{m,n \in \mathbb{N}} a_{m,n} \cos^m \theta \sin^n \theta.$$

Since  $\text{span}\{\cos^m \theta \sin^n \theta\}_{m,n \in \mathbb{N}} = \text{span}\{\cos(m\theta), \sin(n\theta), 1\}_{m,n \in \mathbb{N}}$ , we imply  $f(\theta)$  can be approximated by

$$\sum_{m,n \in \mathbb{N}} a_m \cos(m\theta) + b_n \sin(n\theta).$$

Or equivalently, for any  $\varepsilon > 0$ , there exists  $N > 0$  and  $a_m, a_n \in \mathbb{R}$  such that

$$\left| f(\theta) - \left( a_0 + \sum_{m=1}^N a_m \cos(m\theta) + \sum_{n=1}^N b_n \sin(n\theta) \right) \right| < \varepsilon, \quad \forall \theta \in [0, 2\pi]. \quad (5.1)$$

 The natural question is that do we have the following equation hold:

$$f(\theta) = a_0 + \sum_{m=1}^{\infty} a_m \cos(m\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta) \quad (5.2)$$

It seems that Eq.(5.2) above is equivalent to the expression in (5.1). However, unlike the Taylor expansion, the values of  $a_m, a_n, M, N$  may change once we switch the number  $\varepsilon > 0$ .

Therefore, Eq.(5.2) does not hold for most functions, but only for some functions with nice structure.

**Fourier Analysis.** Given the condition that the Eq.(5.2) holds. How can we get the values of  $a_m$  and  $b_n$ ? The way is to take “inner product” between  $f(\theta)$  and trigonometric functions. For example, by taking the inner product with  $\cos(k\theta)$  for Eq.(5.2) both sides, we have

$$\begin{aligned} \int_{-\pi}^{\pi} f(\theta) \cos(k\theta) d\theta &= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos(k\theta) d\theta \\ &+ \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos(m\theta) \cos(k\theta) d\theta + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(n\theta) \cos(k\theta) d\theta \\ &= \pi \cdot a_k \end{aligned}$$

Following the same trick, we obtain:

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(k\theta) d\theta \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(k\theta) d\theta \end{aligned} \quad (5.3)$$

Naturally, we define the fourier expansion for general  $f(\theta)$ , even though we don't verify whether (5.2) holds or not:

$$g_N(\theta) = \frac{a_0}{2} + \sum_{n=1}^N a_n \cos(n\theta) + \sum_{n=1}^N b_n \sin(n\theta),$$

where the term  $a_m$  and  $b_n$  follow the definition in (5.3). The natural question is that whether  $g_N(\theta) \rightarrow f(\theta)$  as  $N \rightarrow \infty$ ?

## 5.2.2. Baire Category Theorem

**Motivation.** The set  $\mathcal{P}[a, b] \subseteq \mathcal{C}[a, b]$  is dense by Weierstrass Approximation. However, it is not “abundant” in  $\mathcal{C}[a, b]$ , just like  $\mathbb{Q} \subseteq \mathbb{R}$  is dense in  $\mathbb{R}$ . (Every  $r \in \mathbb{R}$  is a limit of a sequence in  $\mathbb{Q}$ )

The set  $\mathbb{Q}$  is countable yet  $\mathbb{R} \setminus \mathbb{Q}$  is uncountable, i.e., there are many more holes in  $\mathbb{R} \setminus \mathbb{Q}$ .

**Definition 5.2** [Nowhere Dense] A subset  $S \subseteq (X, d)$  is **nowhere dense** if  $\bar{S}$  does not contain any open ball, i.e.,

$$X \setminus \bar{S} \text{ is dense in } X$$

For example, a single point is nowhere dense.

**Theorem 5.1** Let  $\{E_i\}_{i=1}^{\infty}$  be a collection of nowhere dense sets in a complete metric space  $(X, d)$ . Then the set

$$\bigcup_{i=1}^{\infty} \bar{E}_i$$



also does not contain any open ball.

*Proof.* I have no time to review and modify the proof during the lecture. Therefore, we encourage the reader to go through the proof in the note

W, Ni & J. Wang (January, 2019). Lecture Notes for MAT2006. Retrieved from <https://walterbabyrudin.github.io/information/information.html>

Of course, I will also add the proof in this note during this week. ■

