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MAT3006 & 3040 & 4002 Notebook

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Contents

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Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

8.5. Wednesday for MAT4002

Reviewing. Let $K' = (V', \Sigma')$ be a **simplicial subcomplex**, then $K = (V, \Sigma)$.

$$|K'| = D_{\Sigma'} / \sim_{\Sigma'} \implies |K| = D_{\Sigma} / \sim_{\Sigma}$$

Thus $D_{\Sigma'} \to D_{\Sigma} \to^{P} D_{\Sigma} / \sim_{\Sigma}$. For all $x, y \in D_{\Sigma}$,

$$x \sim_{\Sigma'} y \Longleftrightarrow i(x) \sim_{\Sigma} i(y) \tag{8.6}$$

The whole map f descends to a continuous map

$$\tilde{f} : D_{\Sigma'} / \sim_{\Sigma'} \to D_{\Sigma} / \sim_{\Sigma}$$

Indeed, the $\Longleftarrow$ part of (8.2) guarantess that $\tilde{f}$ is injective.

For $i : |K'| \to |K|$ continuous and injective, $i(|K'|)$ is closed in $|K|$.

Proposition 8.6 For each $K = (V, \Sigma)$, and finite V, there is a continuous injection $g : |K| \hookrightarrow \mathbb{R}^n$ for some n.

Proof. Consider $K^p := (V, \Sigma^p)$, where Σ^p is the power set of V. Therefore, $|K^p| = \Delta^{|V|-1} \subseteq \mathbb{R}^{|V|}$. Also, we have the injection

$$|K'| \xrightarrow{l} |K^p| \xrightarrow{l} \mathbb{R}^{|V|}$$

Since $K = (V, \Sigma)$ is a simplicial subcomplex of $K^p = (V, \Sigma^p)$, the proof is complete. ■

Proposition 8.7 — **Hausdorff.** If $K = (V, \Sigma)$ with fintie V, then $|K|$ is Hausdorff.

Proof. Let $g : |K| \xrightarrow{l} \mathbb{R}^n$. Consider the bijective $g : |K| \to g(|K|)$, which is continuous. Sicne $|K|$ is compact, $g(|K|) \subseteq \mathbb{R}^n$ is Hausdorff. Therefore, we imply that $|K|$ and $g(|K|)$ are homeomorphic, i.e., $|K|$ is Hausdorff. ■

Definition 8.13 [Edge Path] An **edge path** of $K=(V,\Sigma)$ is a sequence of vertices $(v_1,\dots,v_n), v_i \in V$ such that $\{v_i,v_{i+1}\}\in\Sigma, \forall i$.

Proposition 8.8 — **Connectedness.** Let $K=(V,\Sigma)$ be a simplicial complex. TFAE:

1. $|K|$ is connected
2. $|K|$ is path-connected
3. Any 2 vertices in (V,Σ) can be joined by an edge path, i.e., for $\forall u,v\in V$, there exists $v_1,\dots,v_k\in V$ such that $(u,v_1,\dots,v_k,v)$ is an edge path.

Sketch of Proof.

1. (3) implies (2): For every $x,y\in|K|$,
$$\begin{cases} x\in\Delta_{\sigma_1} \text{ for some } \sigma_1\in\Sigma. \\ y\in\Delta_{\sigma_2} \text{ for some } \sigma_2\in\Sigma. \end{cases}$$
Take a path joining x to a vertex $v_1\in\sigma_1$ and a path joining y to a vertex $v_2\in\sigma_2$. By (3), we have a path joninig v_1 and v_2.
2. (1) implies (3): Suppose on the contrary that there is a vertex v not satisfying (3). Take V' as the set of vertexs that can be joined with v; and V'' as the set of vertexs that cannot be joinied with v.
Then $V',V''\neq\emptyset$. Consider K',K'' be simplicial subcomplexes of K, spanned by V' and V''. Then $|K'|,|K''|$ are disjoint, closed in $|K|$.
$|K|=|K'|\cup|K''|$. If there exists $x\in|K|\setminus(|K'|\cup|K''|)$, then for any $\sigma\in\Sigma$ such that $x\in\Delta_\sigma$, we imply $\Delta_\sigma\not\subseteq|K'|$ or $|K''|$.
Therefore, σ consists of vertices in both V' and V''. Then there is $v',v''\in\sigma$ joining V' and V''.
Therefore, there is no such x and hence $|K|=|K'|\cup|K''|$ is a disjoint union of two closed sets, i.e., not connected.

■

8.5.1. Homotopy

Yoneda's "philosophy". To understand an object X (in our focus, X denotes topological space), we should understand functions

$$f : A \to X, \quad \text{or} \quad g : X \to B$$

One special example is to let $B = \mathbb{R}$.

There are many continuous functions $g : X \to Y$. We will group all these functions into equivalence classes.

Definition 8.14 [Homotopy] A **Homotopy** between two continuous maps $f, g : X \to Y$ is a continuous map

$$H : X \times [0,1] \to Y$$

such that

$$H(x,0) = f(x), \quad H(x,1) = g(x)$$

If such H exists, we say f and g are **homotopic**, denoted as $f \cong g$.

Example 8.7 Let $Y \subseteq \mathbb{R}^2$ be a convex subset. Consider two continuous maps $f : X \to Y$ and $g : X \to Y$. They are always homotopic since we can define the homotopy

$$H(x,t) = tg(x) + (1-t)f(x)$$

Proposition 8.9 Homotopy is an equivalent relation

1. $f \cong f$ is obvious: let $H(x,t) = f(x)$, for $\forall 0 \le t \le 1$
2. If $f \cong g$, then $g \cong f$: For homotopic from f to g, say $H(x,t)$, construct

$$H'(x,t) := H(x, 1-t)$$

   Therefore, $H'(x,0) = g(x)$ and $H'(x,1) = f(x)$.

3. If $f \cong g$ and $g \cong h$, then $f \cong h$: suppose $H : f \cong g$, and $K : g \cong h$. Consider

$$J(x,t) = \begin{cases} H(x,2t), & 0 \leq t \leq 1/2 \\ K(x,2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

Note that $J(x,1/2)$ are well-defined. Then J is continuous, since for all closed $V \subseteq Y$,

$$J^{-1}(V) = (J^{-1}(V) \cap (X \times [0,1/2])) \cup (J^{-1}(V) \cap (X \times [1/2,1])) = H^{-1}(V) \cup K^{-1}(V)$$

Since $H^{-1}(V)$ and $K^{-1}(V)$ are both closed, we imply $J^{-1}(V)$ is closed.

Therefore, there is only one equivalence class in example (1). This reflects the fact that $Y \subseteq \mathbb{R}^2$ is a "simple" object.

Proof. Tkae $y_0 \in Y$. Consider $C_y : X \to Y$ by $C_y(x) = y_0, \forall x$. For all continuous maps $f : X \to Y$, $f \cong C_y$.

Therefore, there is only one equivalence class since every continuous map is homotopic to C_y ■