A FIRST COURSE

IN

ANALYSIS

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MAT2006 Notebook

Lecturer

Prof. Weiming Ni

The Chinese University of Hongkong, Shenzhen

Tex Written By

Mr. Jie Wang

The Chinese University of Hongkong, Shenzhen

香港中文大學（深圳）
The Chinese University of Hong Kong, Shenzhen

Contents

Acknowledgments

This book is taken notes from the MAT2006 in fall semester, 2018. These lecture notes were taken and compiled in LaTeX by Jie Wang, an undergraduate student in Fall 2018. Prof. Weiming Ni has not edited this document. Students taking this course may use the notes as part of their reading and reference materials. This version of the lecture notes were revised and extended for many times, but may still contain many mistakes and typos, including English grammatical and spelling errors, in the notes. It would be greatly appreciated if those students, who will use the notes as their reading or reference material, tell any mistakes and typos to Jie Wang for improving this notebook.

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 12

Week12

12.1. Wednesday

12.1.1. Recap for Rank Theorem

Inverse Function Theorem. Given a $\mathcal{C}^p$ function $f : E(\subseteq \mathbb{R}^m) \subseteq \mathbb{R}^m$ and $Df(\boldsymbol{x}_0)$ is invertible. Then we imply that there is a neighborhood $U(\boldsymbol{x}_0) \times V(\boldsymbol{y}_0)$ of $(\boldsymbol{x}_0, f(\boldsymbol{x}_0))$ such that f is a $\mathcal{C}^p$-diffeomorphism between $U(\boldsymbol{x}_0)$ and $V(\boldsymbol{y}_0)$; moreover,

$$D(f^{-1})(\boldsymbol{y}_0) = (Df(\boldsymbol{x}_0))^{-1}$$

Rank Theorem. Given a $\mathcal{C}^p$ function $f : U(\boldsymbol{x}_0) \to \mathbb{R}^n$ of constant rank k throughout $U(\boldsymbol{x}_0)$. Then there exists a neighborhood $N(\boldsymbol{x}_0) \times N(f(\boldsymbol{x}_0))$ and two $\mathcal{C}^p$-diffeomorphisms

$$\boldsymbol{u} = \phi(\boldsymbol{x}), \boldsymbol{x} \in N(\boldsymbol{x}_0) \quad \boldsymbol{v} = \psi(\boldsymbol{y}), \boldsymbol{y} \in N(\boldsymbol{y}_0), \boldsymbol{y}_0 := f(\boldsymbol{x}_0),$$

such that the composition $\psi \circ f \circ \phi^{-1}$ takes the form

$$(u_1, \dots, u_k, u_{k+1}, \dots, u_m) \to (u_1, \dots, u_k, 0, 0, \dots, 0)$$

Outline of proof. Step 1:

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_k \\ \vdots \\ f_n \end{pmatrix}, \qquad Df = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_m)}$$

w.l.o.g., assume the first $k \times k$ principal minors to be non-singular.

Step 2: Then construct the map $\phi(\boldsymbol{x})$

$$\phi(\boldsymbol{x}) = \begin{pmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_k(\boldsymbol{x}) \\ x_{k+1} \\ \vdots \\ x_m \end{pmatrix} \implies D\phi = \begin{pmatrix} \frac{\partial(f_1,\ldots,f_k)}{\partial(x_1,\ldots,x_k)} & \frac{\partial(f_1,\ldots,f_k)}{\partial(x_{k+1},\ldots,x_m)} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix}$$

which is invertible.

Step 3: define $g := f \circ \phi^{-1} : \phi(N(\boldsymbol{x}_0)) \to \mathbb{R}^n$, then rewrite g as

$$\begin{pmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_k \\ g_{k+1}(\boldsymbol{u}) \\ \vdots \\ g_n(\boldsymbol{u}) \end{pmatrix} \implies Dg = \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ & \frac{\partial(g_{k+1},\ldots,g_n)}{\partial(u_{k+1},\ldots,u_n)} \end{pmatrix},$$

which implies the lower right corner should be zero matrix, i.e., $(g_{k+1}, \ldots, g_n)(\boldsymbol{u})$

depends only on the first k variables. Thus rewrite g as:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_k \\ g_{k+1}(u_1, \dots, u_k) \\ \vdots \\ g_n(u_1, \dots, u_k) \end{pmatrix}$$

Step 4: Define the map $\boldsymbol{v} = \psi(\boldsymbol{y})$:

$$\begin{pmatrix} v_1 \\ \vdots \\ v_k \\ v_{k+1} \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} - g_{k+1}(y_1, \dots, y_k) \\ \vdots \\ y_{k+1} - g_n(y_1, \dots, y_k) \end{pmatrix}$$

flatten out

■ **Example 12.1** 1. Define $f(t) = (\cos t, \sin t), t \in \mathbb{R}$. Define $t_0 = \frac{\pi}{4}$. Can we flatten out the curve near $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$? Note that

$$Df(\frac{\pi}{4}) = (-\sqrt{2}/2, \sqrt{2}/2) \neq 0,$$

with rank 1. The answer is yes.

Choose $\phi(t) = \cos t$ and $\phi^{-1}(u) = t = \cos^{-1} u$, which follows that

$$g(u) = f(\phi^{-1}(u)) = \begin{pmatrix} \cos(\phi^{-1}u) \\ \sin(\phi^{-1}(u)) \end{pmatrix} = \begin{pmatrix} u \\ \sin(\cos^{-1} u) \end{pmatrix}$$

Choose $\psi(y) = \begin{pmatrix} y_1 \\ y_2 - \sin(\cos^{-1} y_1) \end{pmatrix}$, which follows that

$$\begin{aligned}\psi \circ f \circ \phi^{-1}(u) &= \psi \circ f(\cos^{-1} u) \\ &= \psi \begin{pmatrix} \cos\cos^{-1} u \\ \sin\cos^{-1} u \end{pmatrix} = \psi \begin{pmatrix} u \\ \sin\cos^{-1} u \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}\end{aligned}$$

2. $f(x_1,x_2) = (x_1 + x_2, x_1 - x_2, x_1x_2)$. Can we flatten out the curve of f near $(0,0)$?

$$Df(x_1,x_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ x_2 & x_1 \end{pmatrix},$$

which is of rank 2 throughout $\mathbb{R}^2$.

Note that

$$\phi(x_1,x_2) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

and

$$g = f \circ \phi^{-1}(u_1,u_2) = f \begin{pmatrix} \frac{u_1+u_2}{2} \\ \frac{u_1-u_2}{2} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \frac{u_1^2-u_2^2}{4} \end{pmatrix}$$

and define

$$\psi(y) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 - \frac{y_1^2-y_2^2}{4} \end{pmatrix}.$$

Thus in summary, we have

$$\psi \circ f \circ \phi^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \psi \begin{pmatrix} u_1 \\ u_2 \\ \frac{u_1^2-u_2^2}{4} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$$

∎

12.1.2. Functional Dependence

In linear algebra we have talked about the linear independence. Given n vectors $v_1,\dots,v_n$, they are linear dependent if $\exists\ a_1,\dots,a_n$ not all zero such that

$$a_1v_1 + \cdots + a_nv_n = 0$$

Then we talk about the dependence between functions.

Definition 12.1 [Dependence] A set of **continuous** functions $f_1,\dots,f_n : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^m$ is a neighborhood of $\boldsymbol{x}_0 \in \mathbb{R}^m$, is said to be **functionally independent** if for any continuous function

$$F(y) = F(y_1,\dots,y_n)$$

in a neighborhood V of $\boldsymbol{y}_0 = f(\boldsymbol{x}_0) = (f_1(\boldsymbol{x}_0),\dots,f_n(\boldsymbol{x}_0))$, the relation $F(f_1(\boldsymbol{x}),\dots,f_n(\boldsymbol{x})) \equiv 0$ for $\forall \boldsymbol{x} \in U$ is the only possible when $F \equiv 0$ in V. ■

Proposition 12.1 Let $\{f_1,\dots,f_n\}$ be $\mathcal{C}^1$ and the rank of

$$\frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_m)}$$

is k at every $\boldsymbol{x} \in U$, then

1. $k = n$ implies $\{f_1,\dots,f_n\}$ is functionally independent
2. $k < n$ implies there exists a neighborhood of $\boldsymbol{x}_0$ and k functions $f_1,\dots,f_k$ such that the rest of $(n-k)$ functions can be written as

$$f_j(\boldsymbol{x}) = g_i(f_1(\boldsymbol{x}),\dots,f_k(\boldsymbol{x}))$$

for $\forall i = k+1,\dots,n$, where g_i are $\mathcal{C}^1$ funcitions of k variables.