

**A JOURNEY  
IN  
PURE MATHEMATICS**



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**A JOURNEY**  
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**PURE MATHEMATICS**  
**MAT3006 & 3040 & 4002 Notebook**

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# Notations and Conventions

$\mathbb{R}^n$	$n$ -dimensional real space
$\mathbb{C}^n$	$n$ -dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
$x_i$	$i$ th entry of column vector $\mathbf{x}$
$a_{ij}$	$(i, j)$ th entry of matrix $\mathbf{A}$
$\mathbf{a}_i$	$i$ th column of matrix $\mathbf{A}$
$\mathbf{a}_i^T$	$i$ th row of matrix $\mathbf{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all $i, j$
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all $i, j$
$\mathbf{A}^T$	transpose of $\mathbf{A}$ , i.e, $\mathbf{B} = \mathbf{A}^T$ means $b_{ji} = a_{ij}$ for all $i, j$
$\mathbf{A}^H$	Hermitian transpose of $\mathbf{A}$ , i.e, $\mathbf{B} = \mathbf{A}^H$ means $b_{ji} = \bar{a}_{ij}$ for all $i, j$
$\text{trace}(\mathbf{A})$	sum of diagonal entries of square matrix $\mathbf{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\mathbf{e}_i$	a unit vector with the nonzero element at the $i$ th entry
$\mathcal{C}(\mathbf{A})$	the column space of $\mathbf{A}$
$\mathcal{R}(\mathbf{A})$	the row space of $\mathbf{A}$
$\mathcal{N}(\mathbf{A})$	the null space of $\mathbf{A}$
$\text{Proj}_{\mathcal{M}}(\mathbf{A})$	the projection of $\mathbf{A}$ onto the set $\mathcal{M}$





## 8.5. Wednesday for MAT4002

**Reviewing.** Let  $K' = (V', \Sigma')$  be a **simplicial subcomplex**, then  $K = (V, \Sigma)$ .

$$|K'| = D_{\Sigma'} / \sim_{\Sigma'} \implies |K| = D_{\Sigma} / \sim_{\Sigma}$$

Thus  $D_{\Sigma'} \rightarrow D_{\Sigma} \xrightarrow{p} D_{\Sigma} / \sim_{\Sigma}$ . For all  $x, y \in D_{\Sigma}$ ,

$$x \sim_{\Sigma'} y \iff i(x) \sim_{\Sigma} i(y) \quad (8.6)$$

The whole map  $f$  descends to a continuous map

$$\tilde{f} : D_{\Sigma'} / \sim_{\Sigma'} \rightarrow D_{\Sigma} / \sim_{\Sigma}$$

Indeed, the  $\Leftarrow$  part of (8.2) guarantess that  $\tilde{f}$  is injective.

For  $i : |K'| \rightarrow |K|$  continuous and injective,  $i(|K'|)$  is closed in  $|K|$ .

**Proposition 8.6** For each  $K = (V, \Sigma)$ , and finite  $V$ , there is a continuous injection  $g : |K| \hookrightarrow \mathbb{R}^n$  for some  $n$ .

*Proof.* Consider  $K^p := (V, \Sigma^p)$ , where  $\Sigma^p$  is the power set of  $V$ . Therefore,  $|K^p| = \Delta^{|V|-1} \subseteq \mathbb{R}^{|V|}$ . Also, we have the injection

$$|K'| \xrightarrow{l} |K^p| \xrightarrow{l} \mathbb{R}^{|V|}$$

Since  $K = (V, \Sigma)$  is a simplicial subcomplex of  $K^p = (V, \Sigma^p)$ , the proof is complete. ■

**Proposition 8.7 — Hausdorff.** If  $K = (V, \Sigma)$  with finite  $V$ , then  $|K|$  is Hausdorff.

*Proof.* Let  $g : |K| \xrightarrow{l} \mathbb{R}^n$ . Consider the bijective  $g : |K| \rightarrow g(|K|)$ , which is continuous. Since  $|K|$  is compact,  $g(|K|) \subseteq \mathbb{R}^n$  is Hausdorff. Therefore, we imply that  $|K|$  and  $g(|K|)$  are homeomorphic, i.e.,  $|K|$  is Hausdorff. ■

**Definition 8.13** [Edge Path] An **edge path** of  $K = (V, \Sigma)$  is a sequence of vertices  $(v_1, \dots, v_n), v_i \in V$  such that  $\{v_i, v_{i+1}\} \in \Sigma, \forall i$ . ■

**Proposition 8.8 — Connectedness.** Let  $K = (V, \Sigma)$  be a simplicial complex. TFAE:

1.  $|K|$  is connected
2.  $|K|$  is path-connected
3. Any 2 vertices in  $(V, \Sigma)$  can be joined by an edge path, i.e., for  $\forall u, v \in V$ , there exists  $v_1, \dots, v_k \in V$  such that  $(u, v_1, \dots, v_k, v)$  is an edge path.

*Sketch of Proof.* 1. (3) implies (2): For every  $x, y \in |K|$ ,

$$\begin{cases} x \in \Delta_{\sigma_1} \text{ for some } \sigma_1 \in \Sigma. \\ y \in \Delta_{\sigma_2} \text{ for some } \sigma_2 \in \Sigma. \end{cases}$$

Take a path joining  $x$  to a vertex  $v_1 \in \sigma_1$  and a path joining  $y$  to a vertex  $v_2 \in \sigma_2$ . By (3), we have a path joining  $v_1$  and  $v_2$ .

2. (1) implies (3): Suppose on the contrary that there is a vertex  $v$  not satisfying (3). Take  $V'$  as the set of vertices that can be joined with  $v$ ; and  $V''$  as the set of vertices that cannot be joined with  $v$ .

Then  $V', V'' \neq \emptyset$ . Consider  $K', K''$  be simplicial subcomplexes of  $K$ , spanned by  $V'$  and  $V''$ . Then  $|K'|, |K''|$  are disjoint, closed in  $|K|$ .

$|K| = |K'| \cup |K''|$ . If there exists  $x \in |K| \setminus (|K'| \cup |K''|)$ , then for any  $\sigma \in \Sigma$  such that  $x \in \Delta_\sigma$ , we imply  $\Delta_\sigma \not\subseteq |K'|$  or  $|K''|$ .

Therefore,  $\sigma$  consists of vertices in both  $V'$  and  $V''$ . Then there is  $v', v'' \in \sigma$  joining  $V'$  and  $V''$ .

Therefore, there is no such  $x$  and hence  $|K| = |K'| \cup |K''|$  is a disjoint union of two closed sets, i.e., not connected. ■

## 8.5.1. Homotopy

**Yoneda's "philosophy".** To understand an object  $X$  (in our focus,  $X$  denotes topological space), we should understand functions

$$f : A \rightarrow X, \quad \text{or} \quad g : X \rightarrow B$$

One special example is to let  $B = \mathbb{R}$ .

There are many continuous functions  $g : X \rightarrow Y$ . We will group all these functions into equivalence classes.

**Definition 8.14** [Homotopy] A **Homotopy** between two continuous maps  $f, g : X \rightarrow Y$  is a continuous map

$$H : X \times [0, 1] \rightarrow Y$$

such that

$$H(x, 0) = f(x), \quad H(x, 1) = g(x)$$

If such  $H$  exists, we say  $f$  and  $g$  are **homotopic**, denoted as  $f \cong g$ . ■

■ **Example 8.7** Let  $Y \subseteq \mathbb{R}^2$  be a convex subset. Consider two continuous maps  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$ . They are always homotopic since we can define the homotopy

$$H(x, t) = tg(x) + (1 - t)f(x)$$

**Proposition 8.9** Homotopy is an equivalent relation

1.  $f \cong f$  is obvious: let  $H(x, t) = f(x)$ , for  $\forall 0 \leq t \leq 1$
2. If  $f \cong g$ , then  $g \cong f$ : For homotopic from  $f$  to  $g$ , say  $H(x, t)$ , construct

$$H'(x, t) := H(x, 1 - t)$$

Therefore,  $H'(x, 0) = g(x)$  and  $H'(x, 1) = f(x)$ .

3. If  $f \cong g$  and  $g \cong h$ , then  $f \cong h$ : suppose  $H : f \cong g$ , and  $K : g \cong h$ . Consider

$$J(x, t) = \begin{cases} H(x, 2t), & 0 \leq t \leq 1/2 \\ K(x, 2t - 1), & 1/2 \leq t \leq 1 \end{cases}$$

Note that  $J(x, 1/2)$  are well-defined. Then  $J$  is continuous, since for all closed  $V \subseteq Y$ ,

$$J^{-1}(V) = (J^{-1}(V) \cap (X \times [0, 1/2])) \cup (J^{-1}(V) \cap (X \times [1/2, 1])) = H^{-1}(V) \cup K^{-1}(V)$$

Since  $H^{-1}(V)$  and  $K^{-1}(V)$  are both closed, we imply  $J^{-1}(V)$  is closed.

Therefore, there is only one equivalence class in example (1). This reflects the fact that  $Y \subseteq \mathbb{R}^2$  is a “simple” object.

*Proof.* Take  $y_0 \in Y$ . Consider  $C_y : X \rightarrow Y$  by  $C_y(x) = y_0, \forall x$ . For all continuous maps  $f : X \rightarrow Y$ ,  $f \cong C_y$ .

Therefore, there is only one equivalence class since every continuous map is homotopic to  $C_y$  ■

