A FIRST COURSE

IN

ANALYSIS

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MAT2006 Notebook

Lecturer

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Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i, j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i, j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i, j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 4

Week4

4.1. Wednesday

This lecture will talk about the applications of Baire-Category Theorem and continuity analysis.

4.1.1. Function Analysis

In last lecture we have studied that given a analytic function f, if f can be expressed as a partial sum of its series for each x, then f is a polynomial:

Proposition 4.1 Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ in $|x| < 1$. If for every $x \in (-1,1)$, there exists $n(=n(x))$ such that $\sum_{k=n+1}^{\infty} a_k x^k = 0$, then f is a polynomial, i.e., n is independent of x.

The idea of the proof is to construct a sequence of points such that f coincide with a polynomial over these points, which implies f is indeed a polynomial.

Now we study its stronger version, i.e., f may not be analytic, it only needs to be infinitely differentiable:

Proposition 4.2 Suppose $f \in \mathcal{C}^{\infty}[-1,1]$. If for every $|x| \le 1$, there exists $n(=n(x))$ such that $f^{(n)}(x) = 0$, then f is a polynomial.

R Note that an analytic function (i.e., can be expressed as power series) is always infinitely differentiable, but the reverse direction is necessarily not.

For example, recall we have learnt a function

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases},$$

such that it is infinitely differentiablem but $f^{(n)} = 0$ for $n = 1,2,3,\ldots$. Hence, this function is not analytic at $x = 0$.

Proof. Construct a sequence of set

$$E_n = \{x \in [-1,1] \mid f^{(n)}(x) = 0\} \implies [-1,1] = \bigcup_{n=1}^{\infty} E_n,$$

with E_n closed (Exercise #1). Applying Baire-Category Theorem to $[-1,1]$, at least one E_{N_1} contains a non-empty open interval, say I_1 (Exercise #2).

1. On I_1, $f^{(N_1)} \equiv 0$, which implies f is a polynomial of degree $N_1 - 1$(Exercise #3).
2. If $I_1 = (-1,1)$, the proof is complete.
3. Otherwise, $[-1,1] \setminus I_1 \neq \emptyset$. Applying Baire-Category Theorem on the set $[-1,1] \setminus I_1 := \bigcup_{n=1}^{\infty} E_n \setminus I_1$, we conclude that at least one $E_{N_2} \setminus I_1$ contains a non-empty open interval, say I_2, on which f is a polymial of degree $N_2 - 1$.
4. Each time applying the same trick to construct $I_1, I_2, \ldots$, and make sure these are the **maximal** intervals with the desired properties. Finally, we reach the stage that:

   $f|_{x \in I_j}$ is a polynomial of order $N_j - 1$ for $j = 1, \ldots, \infty$ and $\bigcup_{j=1}^{\infty} I_j$ is dense on $[-1,1]$ (Exercise #4).

5. Construct and claim that

$$H = [-1,1] \setminus \bigcup_{j=1}^{\infty} I_j = \{-1,1\}.\text{(Exercise #5)}$$

6. Combining (4) and (5), we derive f satisfies the condition in Proposition(4.2), and therefore is a polynomial. (Exercise #6)

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Verification. Here we give some hints for the exercises above:

1. Since the inverse image of $\{0\}$ is closed for continuous functions, and $f^{(n)}(\cdot)$ is continuous, we derive E_n's are closed.
2. The Baire-Category Theorem asserts that for a non-empty complete metric space X, or any subsets of X with **non-empty** interior, if it is the countably union of **closed** sets, then one of these **closed** sets has non-empty interior.
3. By integrating $f^{(N_1)}$ for N_1 times, e.g.,
$$f^{(N_1)} = 0 \implies f^{(N_1-1)} = \int f^{(N_1)} \, \mathrm{d}x = a_0 \implies \cdots \implies f = a_{N_1-1}x^{N_1-1} + \cdots + a_0$$
4. Let $I \subseteq [-1,1]$ be any open interval. Its clousure can be expressed as:
$$\overline{I} = \bigcup_{n=1}^{\infty} \overline{I} \bigcap E_n$$
Applying Baire category theorem to $\overline{I}$, at least one $\overline{I} \bigcap E_{n'}$ contains an open interval I'. Thus, $I' \subseteq I$ and $I' \subseteq E_{n'}$, which implies $I' \in \bigcup_{j=1}^{\infty} I_j$ (recall that I_j's are picked maximally). This means that $\bigcup_{j=1}^{\infty} I_j \bigcap I$ is non-empty for arbitrary open interval I, which implies $\bigcup_{j=1}^{\infty} I_j$ is dense.
5. We have seen that $\bigcup_{j=1}^{\infty} I_j$ is an open, dense **proper** subset of $[0,1]$, which means H is **non-empty**, closed, and nowhere dense in $[0,1]$. In order to show $H = \{-1,1\}$, it suffices to show H does not contain open intervals. Otherwise applying Baire-Category Theorem to $H = \bigcup_{n=1}^{\infty} E_n \bigcap H$ again, for some fixed n^*, $E_{n^*} \cap H$ contains an open interval I^*. Thus, $I^* \subseteq E_{n^*}$ and $I^* \subseteq H \implies I^* \subseteq (\bigcup_{j=1}^{\infty} I_j)^c$, which leads to a contradiction as I_j's are picked maximally.
6. Hence, $\bigcup_{j=1}^{\infty} I_j = (-1,1)$, i.e., $f(x) = \sum_{k=0}^{\infty} a_k x^k$ in $|x| < 1$.

A simpler and more clear proof is presented in the website

`https://mathoverflow.net/questions/34059/`

`if-f-is-infinitely-differentiable-then-f-coincides-with-a-polynomial`

We have seen some examples of nowhere differentiable functions. Now we show that almost functions are nowhere differentiable.

Notations. We denote $\mathcal{C}[0,1]$ as the set of all continuous functions on $[0,1]$. One corresponding metric is defined as:

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|, \quad \forall f,g \in \mathcal{C}[0,1].$$

Remember that $(\mathcal{C}[0,1],d)$ is complete.

Theorem 4.1 The set of all nowhere differentiable functions in $(\mathcal{C}[0,1],d)$ is dense, i.e., forms a 2nd Category.

The trick is to show the complement of the set of nowhere differentiable functions, i.e., the set of functions that have a **finite** derivative at some point, forms a 1st Category.

Proof. Construct

$$E_n = \left\{ f \in \mathcal{C}[0,1] \middle| \begin{array}{r} \forall 0 < h < 1-x, \left|\dfrac{f(x+h)-f(x)}{h}\right| \le n, \\ \text{for some } 0 \le x \le 1 - \frac{1}{n} \end{array} \right\}$$

Thus the union of all E_n will contain all functions having a finite **right hand derivative** at some point in $[0,1)$.

Proposition 4.3 E_n is closed, i.e., for a sequence of function $\{f_m\} \subseteq E_n$ such that $f_m \to f$, we have $f \in E_n$.

Proposition 4.4 E_n is nowhere dense, i.e., ($\mathcal{C} \setminus E_n$ is dense):

After showing these two propositions, we conclude that the set of functions, with a right derivatives at some point, is a set of the first category. Similarly, we can repeat these steps for left derivatives. In summary, the set of functions with a well-defined derivatives forms a 1st Category. The proof is complete. ∎

Proof of Proposition(4.3). Since $\{f_m\} \subseteq E_n$, there exists a sequence of $\{x_m\}$ such that for each m,

$$0 \leq x_m \leq 1 - \frac{1}{n}$$
$$|f_m(x_m + h) - f_m(x_m)| \leq hn,$$

for $\forall 0 < h < 1 - x_m$. As $\{x_m\}$ is bounded, there exists a subsequence $\{x_{m,k}\}$ of $\{x_m\}$ with limit $x \in [0, 1 - \frac{1}{n}]$.

For $\forall 0 < h < 1 - x$, we have that $0 < h < 1 - x_{m,k}$ for large k. Applying triangle inequality, we obtain:

$$\begin{aligned} |f(x+h) - f(x)| &\leq |f(x+h) - f(x_{m,k}+h)| + |f(x_{m,k}+h) - f_m(x_{m,k}+h)| \\ &\quad + |f_m(x_{m,k}+h) - f_m(x_{m,k})| + |f_m(x_{m,k}) - f(x_{m,k})| + |f(x_{m,k}) - f(x)| \\ &\leq |f(x+h) - f(x_{m,k}+h)| + d(f, f_k) + nh + d(f_k, f) + |f(x_k) - f(x)|. \end{aligned}$$

Taking $k \to \infty$, we find all terms in RHS goes to zero except nh:

$$|f(x+h) - f(x)| \leq nh \implies f \in E_n.$$

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Proof of Proposition(4.4). In order to show E_n is nowhere dense, by using the fact that E_n is closed, it suffices to show that an arbitrary open neighborhood $B(f, \varepsilon)$ will contain elements from the set $\mathcal{C}[0,1] \setminus E_n$, i.e., it suffices to create a function in $B(f, \varepsilon)$ that cannot be in E_n for fixed ε.

- Construct a piecewise linear function $\phi_N(x)$ on $[0,1]$ first:

$$\phi_N(x) = \begin{cases} N(x - \frac{k}{N}), & \frac{k}{N} \leq x \leq \frac{k+1}{N}, k = 0, 2, \ldots, N \\ -N(x + \frac{k}{N}), & \frac{k}{N} \leq x \leq \frac{k+1}{N}, k = 1, 3, \ldots, N-1 \end{cases}$$

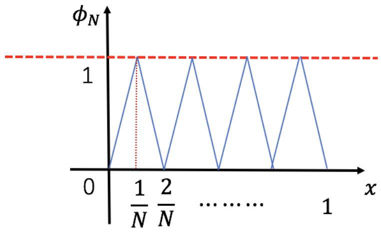

Figure 4.1: Plot of function $\phi_N(x)$

As we can see, N is the maximum slope of the piecewise linear function ϕ_N.

- Let M be the maximum slope of the piecewise linear function f, and pick a positive even integer m such that

$$\frac{1}{2}mN\varepsilon > M + n.$$

Then we construct function

$$g(x) = f(x) + \frac{1}{2}\varepsilon\phi_{mN}(x)$$

As we can see, $d(f,g) = \frac{1}{2}\varepsilon < \varepsilon$, thus $g \in B(f,\varepsilon)$. Also note that

$$\begin{aligned}
\left|\frac{g(x+h)-g(x)}{h}\right| &\geq \left|\frac{1/2\varepsilon(\phi_{mN}(x+h)-\phi_{mN}(x))}{h}\right| - \left|\frac{f(x+h)-f(x)}{h}\right| \\
&\geq \frac{1}{2}\varepsilon\left|\frac{(\phi_{mN}(x+h)-\phi_{mN}(x))}{h}\right| - M \\
&= \frac{1}{2}mN\varepsilon - M > n
\end{aligned}$$

for x in $(0, 1-\frac{1}{mN})$ and some $h \in (0, 1-x)$. Hence, $g \notin E_n$. The proof is complete.

■

4.1.2. Continuity Analysis

Recall the definition for continuity:

- A function f is said to be continuous at $x_0 \in I$ if $\forall \varepsilon > 0$, there exists $\delta > 0$ (δ depends on x_0 and ε) such that

$$|f(x) - f(x_0)| < \varepsilon, \quad \forall |x - x_0| < \delta$$

- A function f is continuous on I if it is continuous at every point in I.

Definition 4.1 [Uniform] We say f is **uniformly continuous** on I if $\forall \varepsilon > 0$, there exists δ (depend only on ε, but independent of $x \in I$) such that

$$|f(y) - f(x)| < \varepsilon, \text{ if } |x - y| < \delta$$

R It is useful to note that the uniform continuity places a upper bound on the growth of the function at every point, i.e., the function cannot grow too fast.

Example 4.1 Given a function $f(x) = x^2$,

1. Is it uniformly continuous on $[0,1]$?

   Yes, intuitively the growth of x^2 is limited within bounded interval.

2. Is it uniformly continuous on $\mathbb{R}$?

   No, intuitively the growth of x^2 tends to infinite as $x \to \infty$.

   Proof: For fixed x, if $|y - x| < \delta$, if we choose $|x| \ge \frac{\varepsilon}{2\delta} + \frac{\delta}{2}$, then

$$\begin{aligned} \underbrace{|f(y) - f(x)|}_{\varepsilon} &= |y^2 - x^2| = |y + x| \underbrace{|y - x|}_{\delta} \\ &\ge (|2x| - |x - y|)|y - x| \ge (\frac{\varepsilon}{\delta} + \delta - \delta)\delta = \varepsilon \end{aligned}$$

   which is a contradiction.

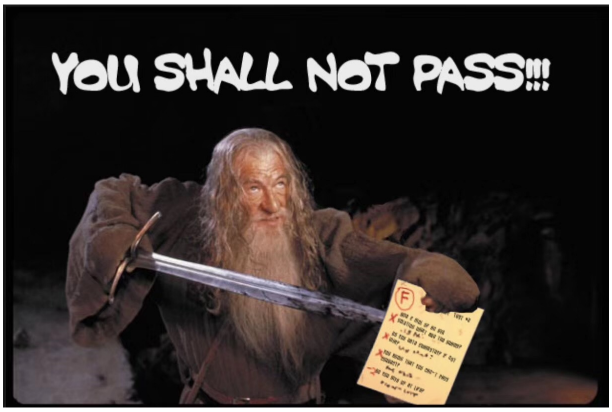

Figure 4.2: The proof and application for the Theorem(4.2) is Mandatory. If you don't know how to do it in the exam, Prof.Ni will fail you without hesitation.

Theorem 4.2 Suppose that f is continuous on a compact set D. Then f is uniformly continuous on D.

Proof. For given $\varepsilon > 0$, since f is continuous at x, there exists $\delta_x > 0$ s.t.

$$|f(y) - f(x)| < \tfrac{\varepsilon}{2}, \quad \text{if } |y - x| < \delta_x.$$

Construct an open cover $\{B_{\delta_x}(x) \mid x \in D\}$ of D with

$$B_{\delta_x}(x) = \{y \in D \mid |y - x| < \frac{1}{2}\delta_x\}.$$

The set D is compact implies there exists a finite subcover:

$$D \subseteq B_{\delta_{x_1}}(x_1) \bigcup B_{\delta_{x_2}}(x_2) \bigcup \cdots \bigcup B_{\delta_{x_k}}(x_k). \tag{4.1}$$

Construct $\delta > 0$ such that $B_\delta(x)$ must be contained entirely in one of the ball, say $B_{\delta_{x_j}}(x_j)$ (Exercise #7)

Therefore given $|y - x| < \delta$ we imply $x, y \in B_{\delta_{x_j}}(x_j)$ for some j, which follows that

$$|f(y) - f(x)| \le |f(y) - f(x_j)| + |f(x_j) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

■

Verification of Exercise. Such a δ is constructed as

$$\delta = \frac{1}{2}\min\{\delta_{x_1}, \delta_{x_2}, \dots, \delta_{x_k}\}.$$

Thus for any x, y with $|y - x| < \delta$, by(4.1), there exists j such that $x \in B_{\delta_{x_j}}(x_j)$, and hence

$$|x - x_j| < \frac{1}{2}\delta_{x_j} \tag{4.2}$$

Also, we have

$$|y - x_j| \leq |y - x| + |x - x_j| \leq \delta + \frac{1}{2}\delta_{x_j} \leq \delta, \tag{4.3}$$

i.e., y is also in $B_{\delta_{x_j}}(x_j)$.

Definition 4.2 [Convex] A real-valued function f defined in (a, b) is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

whenever $a < x < b, a < y < b, 0 < t < 1$.

Check Rudin's book for the proof that a convex function is always continuous.