A FIRST COURSE

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MAT4500 Notebook

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Contents

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CUHK(SZ)

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 3

Week3

3.1. Tuesday

3.1.1. Uniform Integrability

Definition 3.1 [L_1-convergence] We say $f_n \to f$ in L^1 if

$$\lim_{n\to\infty} \int_S |f_n - f| \, d\mu = 0$$

The **uniform integrability** for a family of integrable random variables is used to handle the convergence of random variables in L^1.

Proposition 3.1 If a random variable X is integrable, i.e., $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, then for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $F \in \mathcal{F}$ with $\mathbb{P}(F) < \delta$, we have

$$\mathbb{E}[|X|;F] := \mathbb{E}[|X| 1_F] = \int_F |X| \, d\mathbb{P} < \varepsilon$$

Proof. Suppose the conclusion is false, then there exists some $\varepsilon_0 > 0$, and a sequence of $\{F_n\}$ with each $F_n \in \mathcal{F}$ such that

$$\mathbb{P}(F_n) < \frac{1}{2^n}, \qquad \mathbb{E}[|X|;F_n] \geq \varepsilon_0.$$

Let $H := \lim_{n\to\infty} \sup F_n$. Note that $\sum_n \mathbb{P}(F_n) < \sum \frac{1}{2^n} < \infty$.

By applying the Borel-Centelli lemma, we have $\mathbb{P}(H) = 0$.

However, with the reverse fatou's lemma, since $1_H(w) = \lim_{n\to\infty} \sup 1_{F_n}(w)$,

$$\int |X| 1_H \, d\mathbb{P} \geq \limsup \int |X| 1_{F_n} \, d\mathbb{P}$$

since $\{|X| 1_{F_n}\}$ is dominated by the integrable random variable $|X|$.

Therefore,

$$\mathbb{E}[|X|; H] \geq \limsup \mathbb{E}[|X|; F_n] \geq \varepsilon_0$$

which contradicts with $\mathbb{P}(H) = 0$. ■

Corollary 3.1 Suppose $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then for any given $\varepsilon > 0$, there exists $K \geq 0$, such that $\mathbb{E}[|X|; |X| > K] := \int_{|X|>K} |X| \, d\mathbb{P} < \varepsilon$.

Proof. Note that

$$\begin{aligned} \mathbb{E}[|X|] &= \mathbb{E}[|X|; |X| > K] + \mathbb{E}[|X|; |X| \leq K] \\ &\geq \mathbb{E}[K; |X| > K] = K\mathbb{E}[1_{|X|>K}] \\ &= K\mathbb{P}(|X| > K) \end{aligned}$$

Therefore, we imply

$$\mathbb{P}(|X| > K) \leq \frac{\mathbb{E}|X|}{K}$$

Applying Proposition (3.1), we choose K large enough such that $\frac{\mathbb{E}|X|}{K} < \delta$.

Therefore, $\mathbb{P}(|X| > K) < \delta$, which implies

$$\int_{|X|>K} |X| \, d\mathbb{P} < \varepsilon.$$

■

Definition 3.2 A class $\mathcal{C}$ of random variables are called **uniform integrable** if and only if for any given $\varepsilon > 0$, there exists $K \geq 0$ such that

$$\mathbb{E}[|X|; |X| > K] < \varepsilon, \qquad \forall X \in \mathcal{C}$$

Note that for such uniform integrable class $\mathcal{C}$, we choose $\varepsilon_1 = 1$, then there exists $K_1 \geq 0$ such that

$$\begin{aligned}\forall X \in \mathcal{C},\ \mathbb{E}[|X|] &= \mathbb{E}[|X|; |X| > K_1] + \mathbb{E}[|X|; X \leq K_1] \\ &\leq \varepsilon_1 + K_1 = 1 + K_1,\end{aligned}$$

i.e., class $\mathcal{C}$ is uniformly bounded in L^1.

The reverse is not true:

Example 3.1 Take $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}[0,1], \text{Leb})$

Let $E_n := (0, \frac{1}{n})$, and set

$$X_n(\omega) = n 1_{E_n}(\omega) = \begin{cases} n, & \text{if } \omega \in E_n \\ 0, & \text{if } \omega \notin E_n \end{cases}$$

Then $\mathbb{E}[X_n] = 1, \forall n$, which implies that $\{X_n\}$ are uniformly bounded in L^1.

However, for any $K \geq 0$, as long as $n > K$,

$$\mathbb{E}[|X_n|; |X_n| > K] = 1$$

Therefore, X_n's are not uniformly integrable.

Ovserve that $X_n \to 0$ a.s., but $1 = \mathbb{E}|X_n|$ not converging to 0.

Question: what about L^p-boundness for $p > 1$?

Theorem 3.1 Suppose a class $\mathcal{C}$ of random variables are uniformly bounded in L^p

$(p > 1)$:

$$\exists A > 0, \text{ s.t. } \mathbb{E}[|X|^p] < A, \forall x \in \mathcal{C}$$

Then the class $\mathcal{C}$ is uniformly integrable (UI).

Proof. Note that

$$\begin{aligned}
\mathbb{E}[|X|;|X| > K] = \int_{|X|>K} |X| \, d\mathbb{P} \leq \int_{|X|>K} \frac{|X|^p}{K^{p-1}} \, d\mathbb{P} = \frac{1}{K^{p-1}} \int_{|X|>K} |X|^p \, d\mathbb{P} \\
\leq \frac{1}{K^{p-1}} \int_\Omega |X|^p \, d\mathbb{P} \\
\leq \frac{1}{K^{p-1}} A, \quad \forall x \in \mathcal{C}
\end{aligned}$$

If $X > K$, then $X^p > K^{p-1}X$.

Therefore, for any given $\varepsilon > 0$, choose K to be such that $\frac{A}{K^{p-1}} \leq \varepsilon$.

■

Theorem 3.2 Suppose that a class $\mathcal{C}$ of random variables are dominated by an integrable random variable Y:

$$|X(\omega)| \leq Y(\omega), \quad \forall \omega \in \Omega, \forall X \in \mathcal{C}, \mathbb{E}|Y| < \infty$$

then the class $\mathcal{C}$ is UI.

Proof. Note that since $|X(\omega)| \leq Y(\omega), \forall \omega$, then

$$\{\omega \mid |X(\omega) > K|\} \subset \{\omega \mid |Y(\omega)| > K\}$$

Therefore,

$$\int_{|X|>K} |X| \, d\mathbb{P} \leq \int_{|Y|>K} |X| \, d\mathbb{P} \leq \int_{|Y|>K} |Y| \, d\mathbb{P}$$

Since Y is integrable, by Corollary 2.5.2, for any given $\varepsilon > 0$, there exists $K \geq 0$ such that

$$\int_{|Y|>K} |Y| \, d\mathbb{P} < \varepsilon.$$

This implies that $\forall X \in \mathcal{C}$,

$$\int_{|X|>K} |X| \, \mathrm{d}\mathbb{P} < \varepsilon.$$

$\blacksquare$

Theorem 3.3 Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{G}_\alpha\}_{\alpha \in \mathcal{A}}$ be a sequence of sub-σ-algebra of f. Denote the class

$$\mathcal{C} := \{\mathbb{E}[X \mid G_\alpha]\}_{\alpha \in \mathcal{A}}$$

Then the class $\mathcal{C}$ is UI.