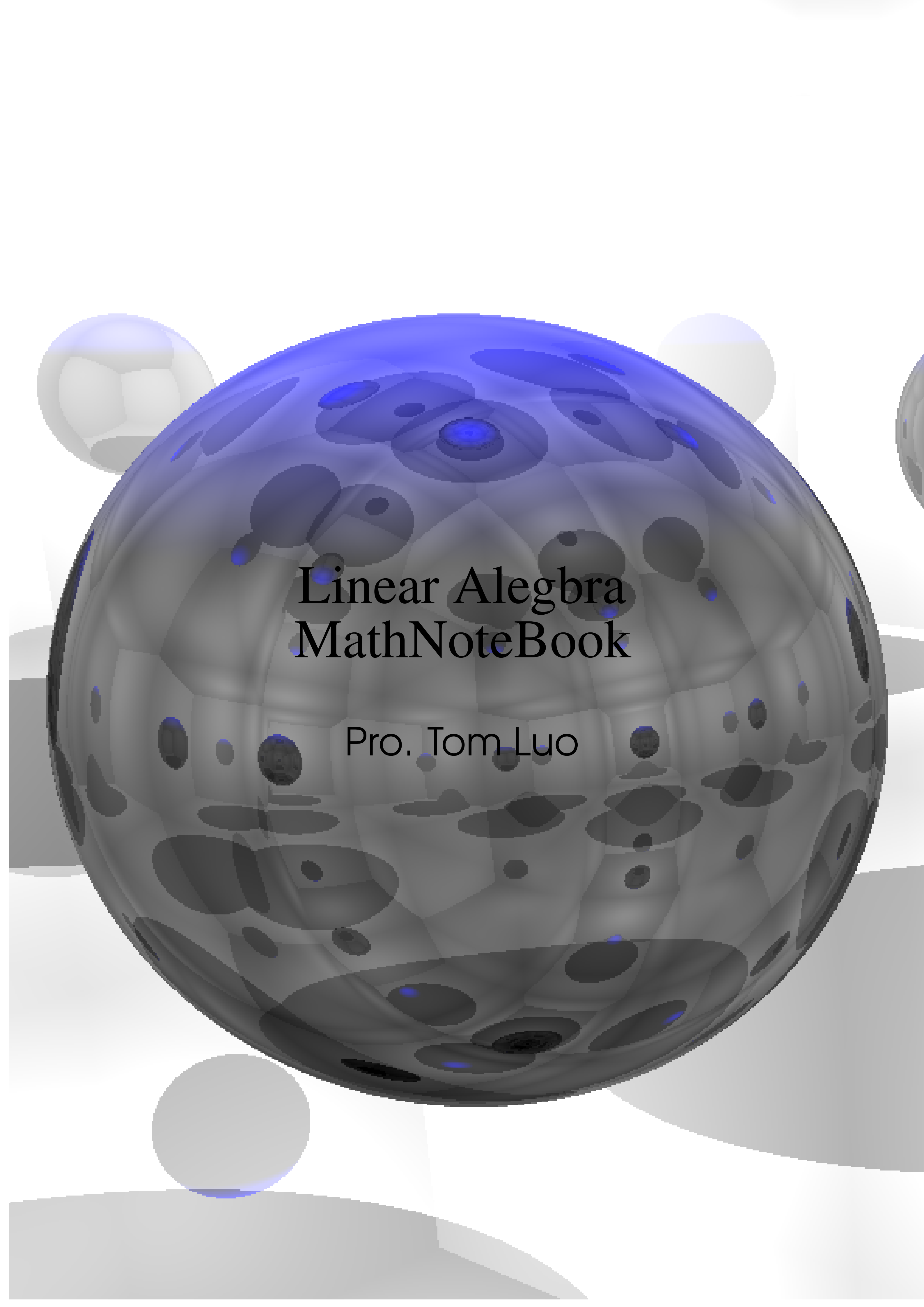

Linear Alegbra MathNoteBook

Pro. Tom Luo

2 — Week 1

2.1 Thursday

2.1.1 *Row-Echelon Form*

Gaussian Elimination don't always derive unique solution

Let's discuss an example to introduce the concept for row-echelon form.

■ **Example 2.1**

We use Gaussian Elimination to try to transfrom a Augmented matrix:

Here in step one we choose the first row as pivot row (the first nonzero entry is the pivot):

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right) \xrightarrow[\text{Add } 1\times \text{ row 1 to row 2; Add } 2\times \text{ row 1 to row 3}]{\text{Add } (-1)\times \text{ row 1 to row 5}}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right]$$

And we choose second row as pivot row to continue elimination:

$$\xrightarrow[\text{Add } (-2)\times \text{ row 2 to row 3; Add } (-1)\times \text{ row 2 to row 4}]{\text{Add } (-1)\times \text{ row 2 to row 5}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

And we choose the third row as pivot row to continue elimination:

$$\xrightarrow[\text{Add } (-1)\times \text{ row 3 to row 5}]{\text{Add } (-1)\times \text{ row 3 to row 1; Add } (-1)\times \text{ row 3 to row 4}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right] \tag{2.1}$$

And matrix (2.1) is of **Row Echlon form**. And we set second row as pivot row then set third row as pivot row to do elimination:

$$\xrightarrow[\text{Add } 2\times \text{ row 3 to row 1; Add } (-2)\times \text{ row 3 to row 2}]{\text{Add } (-1)\times \text{ row 2 to row 1}} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right] \tag{2.2}$$

The matrix (2.2) is of **Reduced Row Echelon form**. And it is *singular matrix*. (Don't worry, we will introduce the definition for singular matrix in the future.)
You may find there exist many solutions to this system of equation, which means Gaussian Elimination **don't** always derive **unique** solution. ■

So let's give the definition for Row-Echelon Form.

Definition 2.1 — Row Echelon Form.
A matrix is said to be in **row echelon form** if

- **(i)** The first nonzero entry in each nonzero row is 1.
- **(ii)** If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.
- **(iii)** If there are rows whose entries are all zero, they are below the rows having nonzero entries.

■

R You should notice that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is also of *Row Echelon Form*! Moreover, it is of *Reduced Row Echelon Form*.

Definition 2.2 — Reduced Row Echelon Form.
A matrix is said to be in **Reduced row echelon form** if

- **(i)** The matrix is in *row echelon form*.
- **(ii)** The first nonzero entry in each row is the only nonzero entry in its column.

■

2.1.2 *Matrix Multiplication*

Matrix Multiplied by Vector

Here we give the definition for inner product of vector:

Definition 2.3 — inner product. Given two vectors $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$, the inner product between x and y is given by

$$< x, y >= x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

And the notation can also be written as $x^{\mathrm{T}}y$ or $x \centerdot y$. ■

R Pro. Tom Luo highly recommends you to write *inner procuct* as $< x, y >$. For myself, I also try to avoid using notation $x \centerdot y$ to avoid misunderstanding.

Let's see an example of matrix multiply a vector:

■ **Example 2.2**

For the system of equation $\begin{cases} 2x_1 + x_2 + x_3 = 5 \\ 4x_1 - 6x_2 = -2 \\ -2x_2 + 7x_2 + 2x_3 = 9 \end{cases}$, we define

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}.$$

Here $\boldsymbol{x}$ and a_1, a_2, a_3 are all column vector (3×1 matrix). More specifically,

$$a_1 = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}, a_2 = \begin{pmatrix} 1 \\ -6 \\ 7 \end{pmatrix}, a_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Then we multiple matrix $\boldsymbol{A}$ with vector $\boldsymbol{x}$:

$$\boldsymbol{Ax} = \begin{pmatrix} 2x_1 + x_2 + x_3 \\ 4x_1 - 6x_2 \\ -2x_1 + 7x_2 + 2x_3 \end{pmatrix} = \begin{pmatrix} < a_1, \boldsymbol{x} > \\ < a_2, \boldsymbol{x} > \\ < a_3, \boldsymbol{x} > \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Hence we finally write the system equation as:

$$\boldsymbol{Ax} = \boldsymbol{b} \qquad (\boldsymbol{Compactmatrixform})$$

And also, if we regard $\boldsymbol{x}$ as a scalar, we can also write:

$$\boldsymbol{b} = \boldsymbol{Ax} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \boldsymbol{x} = a_1\boldsymbol{x} + a_2\boldsymbol{x} + a_3\boldsymbol{x}$$

■

Matrix Multiply Matrix

R Note that an $m \times n$ matrix $\boldsymbol{A}$ can be written as (a_{ij}), where a_{ij} is the entry of ith row, jth column of $\boldsymbol{A}$.

Notice that matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ can do multiplication operatr if and only if the # for column of $\boldsymbol{A}$ equal to the # for row of $\boldsymbol{B}$. And moreover, for $m \times n$ matrix $\boldsymbol{A}$ and $n \times k$ matrix $\boldsymbol{B}$, we can do multiplication as follows:

$$\boldsymbol{AB} = \boldsymbol{A} \begin{pmatrix} b_1 & b_2 & \dots & b_k \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}b_1 & \boldsymbol{A}b_2 & \dots & \boldsymbol{A}b_k \end{pmatrix}$$

And the result is the $m \times k$ matrix. Then we only need to calculate matrix multiplied by vector.

■ **Example 2.3** We want to calculate the result for $m \times n$ matrix $\boldsymbol{A}$ multiply $n \times k$ matrix $\boldsymbol{B}$, which is written as

$$\boldsymbol{AB} = \boldsymbol{C} = \begin{pmatrix} \boldsymbol{A}b_1 & \boldsymbol{A}b_2 & \dots & \boldsymbol{A}b_k \end{pmatrix}$$

Hence for the the entry of ith row, jth column of $\boldsymbol{C}$ is given by

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj} = < a_i^{\boldsymbol{T}} b_j >$$

You should understand this result, this means the ith row, jth column entry of $\boldsymbol{C}$ is given by the ith row of $\boldsymbol{A}$ multiply the jth row of $\boldsymbol{B}$. ■

R **Time Complexity Analysis**

- To Calculate the single entry of $\boldsymbol{C}$ you need to do n times multiplication.
- There exists n^2 entries in $\boldsymbol{C}$
- Hence it takes $n \times n^2 \sim O(n^3)$ operations to compute $\boldsymbol{C}$. (Moreover, using Strassen Algorithm, the time complexity is reduced to $O(n^{log_2^7})$)

2.1.3 Special Matrices

Here we introduce several special matrices:

Definition 2.4 — Identity Matrix. The $n \times n$ identity matrix is the matrix $\boldsymbol{I} = (m_{ij})$, where

$$m_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

■

Proposition 2.1 — Identity. It has the properties:

$$\boldsymbol{IB} = \boldsymbol{B}$$

$$\boldsymbol{AI} = \boldsymbol{A}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are all matrix.

Definition 2.5 — Elementary Matrix of type *III*. An elementary matrix $\boldsymbol{E}_{ij}$ of type *III* is a matrix that its diagonal entries are all 1 and the ith row j th column is a scalar, and the remaining entries are all zero. ■

For example, the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$ is elementary matrix of type *III*

Proposition 2.2 If $\boldsymbol{A}$ is a matrix, postmultiplying $\boldsymbol{E}_{ij}$ has the same effect of performing row operation on a matrix. For example, $\boldsymbol{E}_{21}$ is elementary matrix of type *III* and A is a matrix given by:

Then the effect of $\boldsymbol{EA}$ has the same effect of adding $(-2)\times$ row 1 to row 2:

$$\boldsymbol{E}_{21}A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{pmatrix}$$

And if define $\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, then postmultiplying $\boldsymbol{E}_{31}$ is just like do Gaussian Elimination: