

**A JOURNEY  
IN  
PURE MATHEMATICS**



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**PURE MATHEMATICS**  
**MAT3006 & 3040 & 4002 Notebook**

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# Notations and Conventions

$\mathbb{R}^n$	$n$ -dimensional real space
$\mathbb{C}^n$	$n$ -dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
$x_i$	$i$ th entry of column vector $\mathbf{x}$
$a_{ij}$	$(i, j)$ th entry of matrix $\mathbf{A}$
$\mathbf{a}_i$	$i$ th column of matrix $\mathbf{A}$
$\mathbf{a}_i^T$	$i$ th row of matrix $\mathbf{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all $i, j$
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all $i, j$
$\mathbf{A}^T$	transpose of $\mathbf{A}$ , i.e, $\mathbf{B} = \mathbf{A}^T$ means $b_{ji} = a_{ij}$ for all $i, j$
$\mathbf{A}^H$	Hermitian transpose of $\mathbf{A}$ , i.e, $\mathbf{B} = \mathbf{A}^H$ means $b_{ji} = \bar{a}_{ij}$ for all $i, j$
$\text{trace}(\mathbf{A})$	sum of diagonal entries of square matrix $\mathbf{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\mathbf{e}_i$	a unit vector with the nonzero element at the $i$ th entry
$\mathcal{C}(\mathbf{A})$	the column space of $\mathbf{A}$
$\mathcal{R}(\mathbf{A})$	the row space of $\mathbf{A}$
$\mathcal{N}(\mathbf{A})$	the null space of $\mathbf{A}$
$\text{Proj}_{\mathcal{M}}(\mathbf{A})$	the projection of $\mathbf{A}$ onto the set $\mathcal{M}$





# Chapter 7

## Week 7

### 7.1. Monday for MAT3040

**Reviewing.** Define the characteristic polynomial for a linear operator  $T$ :

$$\mathcal{X}_T(x) = \det((T)_{\mathcal{A},\mathcal{A}} - xI)$$

We will use the notation “ $I/I$ ” in two different occasions:

1.  $I$  denotes the identity transformation from  $V$  to  $V$  with  $I(\mathbf{v}) = \mathbf{v}, \forall \mathbf{v} \in V$
2.  $I$  denotes the identity matrix  $(I)_{\mathcal{A},\mathcal{A}}$ , defined based on any basis  $\mathcal{A}$ .

#### 7.1.1. Minimal Polynomial

**Definition 7.1** [Linear Operator Induced From Polynomial] Let  $f(x) := a_mx^m + \cdots + a_0$  be a polynomial in  $\mathbb{F}[x]$ , and  $T: V \rightarrow V$  be a linear operator. Then the mapping

$$f(T) = a_mT^m + \cdots + a_1T + a_0I: V \rightarrow V,$$

is called a linear operator induced from the polynomial  $f(x)$ . ■

**Definition 7.2** [Minimal Polynomial] Let  $T: V \rightarrow V$  be a linear operator. The **minimal polynomial**  $m_T(x)$  is a **nonzero monic polynomial** of least (minimal) degree such that

$$m_T(T) = \mathbf{0}_{V \rightarrow V}.$$



where  $\mathbf{0}_{V \rightarrow V}$  denotes the zero vector in  $\text{Hom}(V, V)$ . ■

■ **Example 7.1** 1. Let  $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , then  $\mathbf{A}$  defines a linear operator:

$$\mathbf{A} : \mathbb{F}^2 \rightarrow \mathbb{F}^2$$

$$\text{with } \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$$

Here  $\mathcal{X}_{\mathbf{A}}(x) = (x - 1)^2$  and  $\mathbf{A} - \mathbf{I} = \mathbf{0}$ , which gives  $m_{\mathbf{A}}(x) = x - 1$ .

2. Let  $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , which implies

$$\mathcal{X}_{\mathbf{B}}(x) = (x - 1)^2,$$

The question is that can we get the minimal polynomial with degree 1?

The answer is no, since  $\mathbf{B} - k\mathbf{I} = \begin{pmatrix} 1-k & 1 \\ 0 & 1-k \end{pmatrix} \neq \mathbf{0}$ .

In fact,  $m_{\mathbf{B}}(x) = (x - 1)^2$ , since

$$(\mathbf{B} - \mathbf{I})^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Two questions naturally arises:

1. Does  $m_T(x)$  exist? If exists, is it unique?
2. What's the relationship between  $m_T(x)$  and  $\mathcal{X}_T(x)$ ?

Regarding to the first question, the minimal polynomial  $m_T(x)$  may not exist, if  $V$  has infinite dimension:

■ **Example 7.2** Consider  $V = \mathbb{R}[x]$  and the mapping

$$T: V \rightarrow V$$

$$p(x) \mapsto \int_0^x p(t) dt$$

In particular,  $T(x^n) = \frac{1}{n+1}x^{n+1}$ . Suppose  $m_T(x)$  is with degree  $n$ , i.e.,

$$m_T(x) = x^n + \cdots + a_1x + a_0,$$

then

$$m_T(T) = T^n + \cdots + a_0I \text{ is a zero linear transformation}$$

It follows that

$$[m_T(T)](x) = \frac{1}{n!}x^n + a_{n-1}\frac{1}{(n-1)!}x^{n-1} + \cdots + a_1x + a_0 = 0_{\mathbb{F}},$$

which is a contradiction since the coefficients of  $x^k$  is nonzero on LHS for  $k = 1, \dots, n$ , but zero on the RHS. ■

**Proposition 7.1** The minimal polynomial  $m_T(x)$  always exists for  $\dim(V) = n < \infty$ .

*Proof.* It's clear that  $\{I, T, \dots, T^n, T^{n+1}, \dots, T^{n^2}\} \subseteq \text{Hom}(V, V)$ . Since  $\dim(\text{Hom}(V, V)) = n^2$ , we imply  $\{I, T, \dots, T^n, T^{n+1}, \dots, T^{n^2}\}$  is linearly dependent, i.e., there exists  $a_i$ 's that are not all zero such that

$$a_0I + a_1T + \cdots + a_{n^2}T^{n^2} = 0$$

i.e., there is a polynomial  $g(x)$  of degree less than  $n^2$  such that  $g(T) = 0$ .

The proof is complete. ■

**Proposition 7.2** The minimal polynomial  $m_T(x)$ , if exists, then it exists uniquely.

*Proof.* Suppose  $f_1, f_2$  are two distinct minimal polynomials with  $\deg(f_1) = \deg(f_2)$ . It follows that

- $\deg(f_1 - f_2) < \deg(f_1)$ .
- $f_1 - f_2 \neq 0$
- $(f_1 - f_2)(T) = f_1(T) - f_2(T) = 0_{V \rightarrow V}$

By scaling  $f_1 - f_2$ , there is a monic polynomial  $g$  with lower degree satisfying  $g(T) = 0$ , which contradicts the definition for minimal polynomial. ■

**Proposition 7.3** Suppose  $f(x) \in \mathbb{F}[x]$  satisfying  $f(T) = 0$ , then

$$m_T(x) \mid f(x).$$

*Proof.* It's clear that  $\deg(f) \geq \deg(m_T)$ . The division algorithm gives

$$f(x) = q(x)m_T(x) + r(x).$$

Therefore, for any  $\mathbf{v} \in V$

$$[r(T)](\mathbf{v}) = [f(T)](\mathbf{v}) - [q(T)m_T(T)](\mathbf{v}) = \mathbf{0}_V - q(T)\mathbf{0}_V = \mathbf{0}_V - \mathbf{0}_V = \mathbf{0}_V$$

Therefore,  $r(T) = \mathbf{0}_{V \rightarrow V}$ . By definition of minimal polynomial, we imply  $r(x) \equiv 0$ . ■

**Proposition 7.4** If  $\mathbf{A}, \mathbf{B} \in \mathbb{F}^{n \times n}$  are similar to each other, then  $m_A(x) = m_B(x)$ .

*Proof.* Suppose that  $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ , then

$$m_A(x) = x^k + \cdots + a_1x + a_0, \quad m_B(x) = x^\ell + \cdots + b_0$$

It follows that

$$\begin{aligned} m_A(\mathbf{B}) &= \mathbf{B}^k + \cdots + a_0\mathbf{I} \\ &= \mathbf{P}^{-1}\mathbf{A}^k\mathbf{P} + \cdots + a_0\mathbf{P}^{-1}\mathbf{P} \\ &= \mathbf{P}^{-1}(\mathbf{A}^k + \cdots + a_0\mathbf{I})\mathbf{P} \\ &= \mathbf{P}^{-1}(m_A(\mathbf{A}))\mathbf{P} \end{aligned}$$

By proposition,  $m_B(x) \mid m_A(x)$ . Similarly,  $m_A(x) \mid m_B(x)$ . Since  $m_A(x)$  and  $m_B(x)$  are monic, we imply  $m_A(x) = m_B(x)$ . ■

Recall that we also have  $\mathcal{X}_A(x) = \mathcal{X}_B(x)$ .

We now focus on vanishing of a single vector  $\mathbf{v} \in V$ .

**Proposition 7.5** Let  $T : V \rightarrow V$  be a linear operator and  $\mathbf{v} \in V$ . Consider  $m_{T,\mathbf{v}}(x)$ , the unique monic polynomial of least possible degree such that  $m_{T,\mathbf{v}}(T)(\mathbf{v}) = 0$ . Then

$$\deg(m_{T,\mathbf{v}}(x)) \leq \dim(V).$$

*Proof.* Let  $\dim(V) = n$ , and since

$$\{\mathbf{v}, T\mathbf{v}, \dots, T^n\mathbf{v}\} \subseteq V,$$

we imply there exists  $a_i$ 's that are not all zero such that

$$a_n T^n \mathbf{v} + \dots + a_0 I = 0$$

i.e.,  $\deg(m_{T,\mathbf{v}}(x)) \leq n$ . ■

**Proposition 7.6** Suppose that  $m_{T,\mathbf{v}}(x) = f_1(x)f_2(x)$ , where  $f_1, f_2$  are both monic. Let  $\mathbf{w} = f_1(T)\mathbf{v}$ , then

$$m_{T,\mathbf{w}}(x) = f_2(x)$$

*Proof.* Note that

$$f_2(T)\mathbf{w} = f_2(T)f_1(T)\mathbf{v} = m_{T,\mathbf{v}}(T)\mathbf{v} = \mathbf{0}$$

Therefore,  $m_{T,\mathbf{w}} \mid f_2$ .

On the other hand,

$$\mathbf{0} = m_{T,\mathbf{w}}(T)(\mathbf{w}) = m_{T,\mathbf{w}}(T)(f_1(T)\mathbf{v})$$

Therefore,  $\mathbf{0} = f_1(T)m_{T,w}(T)\mathbf{v}$ , which implies that

$$m_{T,\mathbf{v}}(x) \mid f_1(x)m_{T,w}(x),$$

i.e.,

$$f_1 \cdot f_2 \mid f_1 \cdot m_{T,w} \implies f_2 \mid m_{T,w}.$$

The proof is complete. ■

