A FIRST COURSE

IN

NUMERICAL ANALYSIS

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MAT4001 Notebook

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Contents

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CUHK(SZ)

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Tuesday

The Markov Chains course mainly focus on the performance analysis for which the Markov decision is made.

1.1.1. News Vendor Problems

A store sells perishable iterms (newpapers) with:

- Selling price $c_p = 1$
- Variable cost $c_v = 0.25$
- Salvage value $c_s = 0$

The aim is to decide how many copies should be ordered. Before making the decision, we need to estimate the demand.

Suppose the demand D has the following distribution:

d	10	15	20	25	30
$\mathbb{P}$	1/4	1/8	1/8	1/4	1/4

Hence, the profit for the day i is given by:

$$\text{profit per day} = \min(q, D_i)c_p - qc_v,$$

with q being the number of copies ordered.

More generally, for $c_s \neq 0$, the profit for the day i is

$$\text{profit per day} = \min(q, D_i)c_p - qc_v + \max(q - D_i, 0)c_s,$$

Hence, our objective is to maximize the expected profit for a day

$$\begin{aligned} h(q) &= \mathbb{E}_D \text{Profit}(q, D) \\ &= c_p \mathbb{E}_D \min(q, D) - c_v q + c_s \mathbb{E}_D \max(q - D, 0) \end{aligned} \tag{1.1}$$

Chapter 2

Week2

2.1. Tuesday

2.1.1. Optimal order quantity

The optimal y^* is the smallest y such that

$$F(y) \geq \frac{c_p - c_v}{c_p - c_s}$$

Discrete Case Optimization. Given the expected profit function $h(y)$, we want to find the optimal y. Note that y cannot be optimal if $h(y+1) - h(y) > 0$.

$$\begin{aligned} h(y+1) - h(y) &= \mathbb{E}\text{profit}(y+1, D) - \text{profit}(y, D) \\ &= \mathbb{E}\text{revenue}(y+1, D) - \text{revenue}(y, D) - c_v \end{aligned}$$

If $D \geq y+1$, there will be no leftover iterms in both systems:

$$\text{revenue}(y+1, D) - \text{revenue}(y, D) = c_p$$

If $D \leq y$,

$$\text{revenue}(y+1, D) - \text{revenue}(y, D) = c_s$$

It follows that

$$\begin{aligned} h(y+1) - h(y) &= c_p \mathbb{P}[D \geq y+1] + c_s \mathbb{P}[D \leq y] - c_v \\ &= c_p - c_v - (c_p - c_s)\mathbb{P}[D \leq y] \end{aligned}$$

Hence, $h(y+1) > h(y)$ iff

$$\mathbb{P}[D \le y] < \frac{c_p - c_v}{c_p - c_s}$$

In this case y cannot be optimal. Hence, y^* is the smallest y s.t.

$$\mathbb{P}[D \le y] \ge \frac{c_p - c_v}{c_p - c_s}$$

Holding Cost for $h = .1$. If each leftover item cost h, we have

$$\mathbb{E}\text{Profit}(q, D) = \mathbb{E}\min(q, D)c_p - c_v q - h\mathbb{E}(q-D)^+$$

In addiction, if given a fixed cost of order c_f, we derive

$$\mathbb{E}\text{Profit}(q, D) = \mathbb{E}\min(q, D)c_p - c_v q - h\mathbb{E}(q-D)^+ - c_f$$

Confidence Interval. The expected profit in next three months is a summation of 90 i.i.d. RVs:

$$R(25) = P_1(25) + \cdots + P_{90}(25)$$

Using central limit theorem, this RV is approximatly normal. It follows that

$$R(25) \sim \mathcal{N}(11812, 90\sigma^2)$$

where $\sigma^2 = Var(R(25))$. Hence,

$$\mathbb{P}\left(\left| \frac{R(25) - 11812}{\sqrt{90}\sigma} \right| < 1.96 \right) = 0.95$$

With 95% level of confidence, $R(25)$ is between $11812 + 1.96 \cdot \sqrt{90}\sigma$ and $11812 - 1.96 \cdot \sqrt{90}\sigma$

2.1.2. Non-perishable Products

1. For $X_n \ge s$, do not order anything

2. Otherwise, order enough to bring the inventory level to S at the beginning of the next period.

For example, $(s,S) = (20,30)$, determine the probability $\mathbb{P}\{X_10 = 10 \mid X_9 = 10\}$ and $\mathbb{P}\{X_10 = 10 \mid X_9 = 20\}$:

$$\mathbb{P}\{X_{10} = 10 \mid X_9 = 10\} = \mathbb{P}(D_{10} = 20) = \frac{1}{8}$$

$$\mathbb{P}\{X_{10} = 10 \mid X_9 = 20\} = \mathbb{P}(D_{10} = 10) = \frac{1}{4}.$$

Actually, we can create a matrix to describe these conditional probabilities:

$$S_{ij} = \mathbb{P}\{X_{10} = j \mid X_9 = i\}$$

Knowing current state, past states are irrelevant to predict future states.

2.2. Thursday

Earning & learning

2.2.1. Discrte Time Markov Chains

Given

- time index $T = \{0, 1, 2, \dots\}$
- discrte set of states (alphabet): $\mathcal{S}$
- X_n denotes the state at time n
- Transition probabilities (Time homogeneous):

$$P_{ij} = \mathbb{P}\{X_{n+1} = j \mid X_n = i\} \quad i, j \in \mathcal{S}$$

$X = \{X_n : n = 0, 1, \dots\}$ satisfies the Markov property, i.e., for each $n \geq 1$ for $i_0, i_1, \dots, i, j \in \mathcal{S}$,

$$\mathbb{P}\{X_{n+1} = j \mid X_n = i, X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}\} = P_{ij}$$

Given the current information, the past information is irralevant for future information.

Example 1: simple random walk. Suppose toss a coin at each time, and you go right if get a head; go left if get a tail. i denotes the location. We have the conditional probability

$$\mathbb{P}\{X_{n+1} = i+1 \mid X_n = i, X_{n-1}, \dots, X_0\} = \mathbb{P}\{X_{n+1} = i+1 \mid X_n = i\} = p$$

In this case the transition probability matrix has infinite dimension.

Given $\mathcal{S} = \{0, 1\}$ and the transition probability matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$$

Theorem 2.1 Given a function

$$f : (i, u) \in \mathcal{S} \times \mathbb{R}^+, \quad f(i, u) \in \mathcal{S}$$

$\{U_n : n = 1, 2, \dots\}$ is an i.i.d. sequence. $X_{n+1} = f(X_n, U_{n+1})$. Then $\{X_n : n = 1, 2, \dots\}$ is a DTMC.

Application 1. Suppose U_n is a coin toss at time n, i.e.,

$$\mathbb{P}(U_n = 1) = p, \quad \mathbb{P}(U_n = -1) = q$$

and define $f : (i, u) \in \mathbb{Z} \times \{-1, 1\} = i + u \in \mathbb{Z}$