A FIRST COURSE

IN

ANALYSIS

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MAT2006 Notebook

Lecturer

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Contents

Acknowledgments

This book is taken notes from the MAT2006 in fall semester, 2018. These lecture notes were taken and compiled in LaTeX by Jie Wang, an undergraduate student in Fall 2018. Prof. Weiming Ni has not edited this document. Students taking this course may use the notes as part of their reading and reference materials. This version of the lecture notes were revised and extended for many times, but may still contain many mistakes and typos, including English grammatical and spelling errors, in the notes. It would be greatly appreciated if those students, who will use the notes as their reading or reference material, tell any mistakes and typos to Jie Wang for improving this notebook.

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

6.2. Friday

6.2.1. Announcement

This lecture will mainly discuss the integration, but first let's review what we have learnt from last lecture.

The Taylor series has its connection with complex numbers:

■ **Example 6.2** Given a function $f(x) = \frac{1}{1+x^2} \in \mathcal{C}^\infty(\mathbb{R})$. There is a smart way to check the property infinite differentiable:

$$f(x) = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \cdots, \quad \text{holds for } x^2 < 1$$

Why would the function f have taylor series convergent only hold for $x^2 < 1$, but it is infinite differentiable on the whole real line?

The answer is that if extending the domain into complex plane, the function $\frac{1}{1+z^2}$ have poles $\pm i$, and thus have no chance to have taylor expansion beyond $|z| < 1$. Then project the complex plane into real line.

Exercise: find the taylor series of $\frac{1}{1+x^2}$ at $x = 1$ and determine its radius of convergence ($\sqrt{2}$). ■

Taylor series and uniform continuous will be definitely in the mid-term exam.

6.2.2. Riemann Integration

Set Up. Given a bounded function f on the closed (finite) interval $[a,b]$. A partition $\mathcal{P}$ is a set of points $\{x_i\}_{i=0}^n$:

$$a_1 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$$

where the **mesh** of $\mathcal{P}$ is defined to be $\lambda(\mathcal{P}) = \max_{1\leq i\leq n} |\Delta x_i|$.

On each interval $[x_{i-1}, x_i]$, define

$$m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x), \quad M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$$

The lower sum and upper low sum associated with partition $\mathcal{P}$ is defined as:

$$L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} m_i \Delta x_i$$
$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} M_i \Delta x_i$$

Now we define the lower and upper Riemann intergral as:

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup_{\mathcal{P}} L(\mathcal{P}, f)$$
$$\overline{\int_a^b} f(x) = \inf_{\mathcal{P}} U(\mathcal{P}, f)$$

These definitions are well-defined.

Definition 6.2 [integrable] We say that f is (Riemann) integrable if $\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)$.

The set of all **Riemann integrable functions** on $[a, b]$ is denoted as $\mathcal{R}[a, b]$. ■

■ **Example 6.3**

1. $f(x) \equiv 1$ on $[0, 1]$; then $\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x) = 1$
2. Dirichlet function:
$$D(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$$
This function always has lower sum 0 and upper sum 1.
3. Riemann function on $[0, 1]$:
$$R(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ \frac{1}{q},, & x = \frac{p}{q}, q > 0, (p, q) = 1 \end{cases}$$
We will show that it is integrable only by definition.

4. The function defined on $[0,1]$:

$$f(x) = \begin{cases} 0, & x = 0 \\ \sin\frac{1}{x}, & x \neq 0 \end{cases}$$

5.

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]$$

Definition 6.3 [Refinement] Given a partition $\mathcal{P}$, we say $\mathcal{P}^*$ is a refinement of $\mathcal{P}$ if $\mathcal{P}^*$ contains all the sub-division points of $\mathcal{P}$

Proposition 6.4 Let $f:[a,b]\mapsto\mathbb{R}$ with $m\le f(x)\le M$ on $[a,b]$, then

1. $L(\mathcal{P},f)\le L(\mathcal{P}^*,f)$ and $U(\mathcal{P}^*,f)\le U(\mathcal{P},f)$ holds for any refinement $\mathcal{P}^*$ of $\mathcal{P}$
2. $L(\mathcal{P}_1,f)\le U(\mathcal{P}_2,f)$ for any refinements $\mathcal{P}_1,\mathcal{P}_2$.
3. $$m(b-a)\le \underline{\int_a^b} f(x)\,\mathrm{d}x \le \overline{\int_a^b} f(x)\,\mathrm{d}x \le M(b-a)$$
4. f is **Riemann integrable** iff $\forall\varepsilon$, there exists $\mathcal{P}$ s.t. $U(\mathcal{P},f)-L(\mathcal{P},f)\le\varepsilon$.

Proof. For (2), take the $\mathcal{P}^*$ as common refinement for $\mathcal{P}_1,\mathcal{P}_2$, and show that

$$L(\mathcal{P}_1,f)\le L(\mathcal{P}^*,f)\le U(\mathcal{P}^*,f)\le U(\mathcal{P}_2,f)$$

■

Theorem 6.4 If f is continuous on $[a,b]$, then f is Riemann integrable on $[a,b]$.

Proof. f is continuous on $[a,b]$ implies f is uniform continuous, i.e.,$\forall\varepsilon>0,\exists\delta>0$ s.t. for $|x-y|<\delta$,

$$|f(x)-f(y)|<\varepsilon.$$

Pick a partition $\mathcal{P}=\{x_0:=a,x_1:=a+h,x_2:=a+2h,\dots,x_n:=a+nh:=b\}$ with $h=$

$\frac{b-a}{n} < \delta$. It follows that on interval $[x_{i-1}, x_i]$, we have

$$M_i - m_i < \varepsilon \implies U(\mathcal{P}, f) - L(\mathcal{P}, f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i \le \varepsilon \sum_{i=1}^{n} \Delta x_i = \varepsilon(b-a)$$

■

Corollary 6.1 If f is continuous expect for finitely many points on $[a, b]$, then f is Riemann integrable.

does not apply to $f(x) = \sin\frac{1}{x}$

$$f_n(x) = \begin{cases} n, & x \in [0, \frac{1}{n}) \\ 0, & x \notin (0, \frac{1}{n}) \end{cases}$$

then $\int_0^1 f_n(x)\,\mathrm{d}x = 1$ and $\int_0^1 f(x)\,\mathrm{d}x = 0$ since $f = \lim_{n\to\infty} f_n = 0$.

Theorem 6.5 Let $\{f_n\}$ be a sequence of **Riemann integrable** functions on $[a, b]$, and f_n converges uniformly to f. Then f is Riemann integrable and

$$\int_a^b f(x) = \lim_{n\to\infty} \int_a^b f_n$$

Definition 6.4 [Uniform Convergence] Let f be the pointwise limit of f_n, then f_n is said to converge uniformly to f if

$$\sup_{a \le x \le b} |f_n(x) - f(x)| \to 0, \text{ as } n \to \infty.$$

Apply Uniform Convergence Theorem into Dirichlet function.

Converse: $f_n(x) = x^n$ for $x \in [0, 1]$.

Today any function talked is bounded.