An Unified Proof for the Theorems of Alternatives

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Abstract

Theorems of alternatives are very useful in applied mathematics in various ways, which is also a main focus of the mid-term exam for MAT3320. The purpose of the present paper is to prove theorems of alternatives in a unified way by directly making use of the Farkas' Lemma.

1 Introduction

Farkas' Lemma is one of the theorems of alternatives for determining the existence of solutions for linear systems. The purpose of this reflective journal is to prove other theorems of alternatives by directly making use of the Farkas' Lemma.

2 Notations

We denote $\boldsymbol{e}$ as the column vector whose components are all ones. For given two vectors $\boldsymbol{x}, \boldsymbol{y}$, we define

$$\begin{aligned}
\boldsymbol{x} \geq \boldsymbol{y}, &\quad \text{if } x_i \geq y_i, \forall i; \\
\boldsymbol{x} \gneqq \boldsymbol{y}, &\quad \text{if } x_i \geq y_i, \forall i, \boldsymbol{x} \neq \boldsymbol{y}; \\
\boldsymbol{x} > \boldsymbol{y}, &\quad \text{if } x_i > y_i, \forall i.
\end{aligned}$$

Moreover, we use ($\bar{\text{I}}$) or ($\bar{\text{II}}$) to denote the *negation* of the statement (I) or (II), respectively.

3 Theorems of Alternatives

Theorem 3.1 (Farkas' Lemma)**.** *Either*

(I) $\boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ *has a solution* $\boldsymbol{x}$*,*

or

(II) $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} < 0$ *has a solution* $\boldsymbol{y}$*,*

but never both.

We will prove the following theorems, by applying the Farkas' Lemma directly.

Theorem 3.2 (Gordan's Theorem)**.** *Either*

(I) $\boldsymbol{Ax} > \boldsymbol{0}$ *has a solution* $\boldsymbol{x}$*,*

or

(II) $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}, \boldsymbol{y} \gneqq \boldsymbol{0}$ *has a solution* $\boldsymbol{y}$*,*

but never both.

Proof.

- (I) implies ($\bar{\text{II}}$): If (I) holds for $\boldsymbol{x}$, and suppose on the contrary that (II) holds for $\boldsymbol{y}$. Then we imply
$$\boldsymbol{0} = \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y}) = (\boldsymbol{Ax})^{\mathrm{T}}\boldsymbol{y}.$$
Since $\boldsymbol{Ax} \geq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}$, the equality above holds if and only if $\boldsymbol{y} = \boldsymbol{0}$, which is a contradiction.

- ($\bar{\text{I}}$) implies (II): If (I) does not hold, then the linear system below does not have a solution as well:
$$\begin{cases} \boldsymbol{Ax} - \theta\boldsymbol{e} \geq \boldsymbol{0} \\ \theta > 0 \end{cases} \iff \begin{cases} \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{e} \end{bmatrix} \begin{pmatrix} \boldsymbol{x} \\ \theta \end{pmatrix} \geq \boldsymbol{0} \\ \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & -1 \end{bmatrix} \begin{pmatrix} \boldsymbol{x} \\ \theta \end{pmatrix} < 0 \end{cases}$$

By applying the reverse direction of Farkas' Lemma, we imply the linear system below has a solution:

$$\begin{cases} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{0} \\ \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1 \\ \boldsymbol{x} \geq \boldsymbol{0} \end{cases}$$

Therefore, the statement (II) holds.

□

Theorem 3.3 (Stiemke's Theorem)**.** *Either*

(I) $\boldsymbol{A}\boldsymbol{x} \gneqq \boldsymbol{0}$ *has a solution* $\boldsymbol{x}$*,*

or

(II) $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}, \boldsymbol{y} > \boldsymbol{0}$ *has a solution* $\boldsymbol{y}$*,*

but never both.

Proof.

- (II) implies ($\bar{\text{I}}$): If (II) holds for $\boldsymbol{y}$, and suppose on the contrary that (I) holds for $\boldsymbol{x}$. Then we imply

$$\boldsymbol{0} = \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y}) = (\boldsymbol{A}\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y}.$$

Since $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{y} > \boldsymbol{0}$, the equality above holds if and only if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, which is a contradiction.

- ($\bar{\text{II}}$) implies (I): If (II) does not hold, then the linear system below does not have a solution as well:

$$\begin{cases} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{0} \\ -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{0} \\ \boldsymbol{y} - \theta\boldsymbol{e} \geq \boldsymbol{0} \\ \theta > 0 \end{cases} \iff \begin{cases} \begin{bmatrix} \boldsymbol{A}^{\mathrm{T}} & \boldsymbol{0} \\ -\boldsymbol{A}^{\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{I} & -\boldsymbol{e} \end{bmatrix} \begin{pmatrix} \boldsymbol{y} \\ \theta \end{pmatrix} \geq \boldsymbol{0} \\ \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & -1 \end{bmatrix} \begin{pmatrix} \boldsymbol{y} \\ \theta \end{pmatrix} < 0 \end{cases}$$

By applying the reverse direction of Farkas' Lemma, we imply the linear system below has a solution:

$$\begin{cases} \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{A} & \boldsymbol{I} \\ \boldsymbol{0}^{\mathrm{T}} & \boldsymbol{0}^{\mathrm{T}} & -\boldsymbol{e}^{\mathrm{T}} \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \geq \boldsymbol{0} \end{cases} \implies \boldsymbol{A}(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \boldsymbol{x}_3 \gneqq \boldsymbol{0},$$

i.e., the statement (I) holds for $\boldsymbol{x}_2 - \boldsymbol{x}_1$.

$\square$

Theorem 3.4 (Gale's Theorem)**.** *Assuming $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ is feasible. Either*

(I) $\boldsymbol{A}\boldsymbol{x} \lneqq \boldsymbol{b}$ *has a solution* $\boldsymbol{x}$*,*

or

(II) $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}, \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}, \boldsymbol{y} > \boldsymbol{0}$ *has a solution* $\boldsymbol{y}$*,*

but never both.

Proof.

- (II) implies ($\bar{\text{I}}$): If (II) holds for $\boldsymbol{y}$, and suppose on the contrary that (I) holds for $\boldsymbol{x}$. Then we imply

$$\boldsymbol{0} = \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y}) = (\boldsymbol{A}\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} < \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0},$$

where the inequality is strict since $\boldsymbol{y} > \boldsymbol{0}$ and $\boldsymbol{A}\boldsymbol{x} \lneqq \boldsymbol{b}$. Therefore, we derive a contradiction.

- ($\bar{\text{II}}$) implies (I): If (II) does not hold, then the linear system below does not have a solution as well:

$$\begin{cases} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{0} \\ -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{0} \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{0} \\ -\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{0} \\ \boldsymbol{y} - \theta\boldsymbol{e} \geq 0 \\ \theta > 0 \end{cases} \iff \begin{cases} \begin{bmatrix} \boldsymbol{A}^{\mathrm{T}} & 0 \\ -\boldsymbol{A}^{\mathrm{T}} & 0 \\ \boldsymbol{b}^{\mathrm{T}} & 0 \\ -\boldsymbol{b}^{\mathrm{T}} & 0 \\ \boldsymbol{I} & -\boldsymbol{e} \end{bmatrix} \begin{pmatrix} \boldsymbol{y} \\ \theta \end{pmatrix} \geq \boldsymbol{0} \\ \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & -1 \end{bmatrix} \begin{pmatrix} \boldsymbol{y} \\ \theta \end{pmatrix} < 0 \end{cases}$$

By applying the reverse direction of Farkas' Lemma, we imply the linear system below has a solution:

$$\begin{cases} \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{A} & \boldsymbol{b} & -\boldsymbol{b} & \boldsymbol{I} \\ \boldsymbol{0}^{\mathrm{T}} & \boldsymbol{0}^{\mathrm{T}} & \boldsymbol{0}^{\mathrm{T}} & \boldsymbol{0}^{\mathrm{T}} & -\boldsymbol{e}^{\mathrm{T}} \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \\ \boldsymbol{x}_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5 \geq \boldsymbol{0} \end{cases} \implies \boldsymbol{A}(\boldsymbol{x}_1 - \boldsymbol{x}_2) + \boldsymbol{b}(x_3 - x_4) = -\boldsymbol{x}_5 \lneqq \boldsymbol{0},$$

Suppose we have a feasible solution $\boldsymbol{x}^0$ such that $\boldsymbol{A}\boldsymbol{x}^0 \leq \boldsymbol{b}$, which implies for any $N > 0$,

$$\boldsymbol{A}(\boldsymbol{x}_1 - \boldsymbol{x}_2 + N\boldsymbol{x}^0) + \boldsymbol{b}(x_3 - x_4 - N) \lneqq \boldsymbol{0}$$

Therefore, we take sufficient large N to make $x_3 - x_4 - N < 0$, which follows that

$$\boldsymbol{A}\left(\frac{1}{N - x_3 - x_4}\boldsymbol{x}_1 - \boldsymbol{x}_2 + N\boldsymbol{x}^0\right) - \boldsymbol{b} \lneqq \boldsymbol{0},$$

i.e., the statement (I) holds for $(\boldsymbol{x}_1 - \boldsymbol{x}_2 + N\boldsymbol{x}^0)/(N - x_3 - x_4)$.

$\square$

Theorem 3.5 (Tucker's Theorem)**.** *Assuming that $\boldsymbol{A} \neq \boldsymbol{0}$. Either*

(I) $\boldsymbol{A}\boldsymbol{x} \gneqq \boldsymbol{0}, \boldsymbol{B}\boldsymbol{x} \geq 0, \boldsymbol{C}\boldsymbol{x} = \boldsymbol{0}$ *has a solution $\boldsymbol{x}$,*

or

(II) $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{v} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{0}, \boldsymbol{u} > \boldsymbol{0}, \boldsymbol{v} \geq \boldsymbol{0}$ *has a solution $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$,*

but never both.

Proof.

- (II) implies ($\bar{\text{I}}$): If (II) holds for $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$, and suppose on the contrary that (I) holds for $\boldsymbol{x}$. Then we imply

$$\boldsymbol{0} = \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{v} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{w}) = (\boldsymbol{A}\boldsymbol{x})^{\mathrm{T}}\boldsymbol{u} + (\boldsymbol{B}\boldsymbol{x})^{\mathrm{T}}\boldsymbol{v} + (\boldsymbol{C}\boldsymbol{x})^{\mathrm{T}}\boldsymbol{w} > \boldsymbol{0}$$

where the inequality is strict since $\boldsymbol{A}\boldsymbol{x} \gneqq \boldsymbol{0}$ and $\boldsymbol{u} > 0$. Therefore, we derive a contradiction.

- ($\bar{\text{II}}$) implies (I): If (II) does not hold, then the linear system below does not have a solution as well:

$$\begin{cases} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{u} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{v} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{w} \geq 0 \\ -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{v} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{w} \geq 0 \\ \boldsymbol{v} \geq 0 \\ \boldsymbol{u} - \theta\boldsymbol{e} \geq 0 \\ \theta > 0 \end{cases} \Longleftrightarrow \begin{cases} \begin{bmatrix} \boldsymbol{A}^{\mathrm{T}} & \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{C}^{\mathrm{T}} & 0 \\ -\boldsymbol{A}^{\mathrm{T}} & -\boldsymbol{B}^{\mathrm{T}} & -\boldsymbol{C}^{\mathrm{T}} & 0 \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & 0 \\ \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{e} \end{bmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \\ \theta \end{pmatrix} \geq \boldsymbol{0} \\ \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \\ \theta \end{pmatrix} < 0 \end{cases}$$

  By applying the reverse direction of Farkas' Lemma, we imply the linear system below has a solution:

$$\begin{cases} \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{B} & -\boldsymbol{B} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{C} & -\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} \\ 0 & 0 & 0 & -\boldsymbol{e}^{\mathrm{T}} \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \\ \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4 \geq \boldsymbol{0} \end{cases} \Longrightarrow \begin{cases} \boldsymbol{A}(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \boldsymbol{x}_4 \gneqq \boldsymbol{0} \\ \boldsymbol{B}(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \boldsymbol{x}_2 \geq \boldsymbol{0} \\ \boldsymbol{C}(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \boldsymbol{0} \end{cases},$$

  i.e., the statement (I) holds for $\boldsymbol{x}_2 - \boldsymbol{x}_1$.

□

Theorem 3.6 (Motzkin's Theorem)**.** *Assuming that $\boldsymbol{A} \neq \boldsymbol{0}$. Either*

(I) $\boldsymbol{A}\boldsymbol{x} > 0, \boldsymbol{B}\boldsymbol{x} \geq 0, \boldsymbol{C}\boldsymbol{x} = \boldsymbol{0}$ *has a solution* $\boldsymbol{x}$*,*

or

(II) $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{v} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{0}, \boldsymbol{u} \gneqq \boldsymbol{0}, \boldsymbol{v} \geq \boldsymbol{0}$ *has a solution* $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$*,*

but never both.

Proof.

- (I) implies ($\bar{\text{II}}$): If (I) holds for $\boldsymbol{x}$, and suppose on the contrary that (II) holds for $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$. Then we imply

$$\boldsymbol{0} = \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{v} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{w}) = (\boldsymbol{A}\boldsymbol{x})^{\mathrm{T}}\boldsymbol{u} + (\boldsymbol{B}\boldsymbol{x})^{\mathrm{T}}\boldsymbol{v} + (\boldsymbol{C}\boldsymbol{x})^{\mathrm{T}}\boldsymbol{w} > \boldsymbol{0}$$

  where the inequality is strict since $\boldsymbol{A}\boldsymbol{x} > \boldsymbol{0}$ and $\boldsymbol{u} \gneqq 0$. Therefore, we derive a contradiction.

- $(\bar{\mathrm{I}})$ implies (II): If (I) does not hold, then the linear system below does not have a solution as well:

$$\begin{cases} \boldsymbol{A}\boldsymbol{x} - \theta \boldsymbol{e} \geq 0 \\ \boldsymbol{B}\boldsymbol{x} \geq \boldsymbol{0} \\ \boldsymbol{C}\boldsymbol{x} \geq \boldsymbol{0} \\ -\boldsymbol{C}\boldsymbol{x} \geq \boldsymbol{0} \\ \theta > 0 \end{cases} \Longleftrightarrow \begin{cases} \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{e} \\ \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} \\ -\boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{x} \\ \theta \end{pmatrix} \geq \boldsymbol{0} \\ \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{pmatrix} \boldsymbol{x} \\ \theta \end{pmatrix} < 0 \end{cases}$$

By applying the reverse direction of Farkas' Lemma, we imply the linear system below has a solution:

$$\begin{cases} \begin{bmatrix} \boldsymbol{A}^{\mathrm{T}} & \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{C}^{\mathrm{T}} & -\boldsymbol{C}^{\mathrm{T}} \\ -\boldsymbol{e}^{\mathrm{T}} & \boldsymbol{0}^{\mathrm{T}} & \boldsymbol{0}^{\mathrm{T}} & \boldsymbol{0}^{\mathrm{T}} \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4 \geq \boldsymbol{0} \end{cases} \implies \begin{cases} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{x}_1 + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{x}_2 + \boldsymbol{C}^{\mathrm{T}}(\boldsymbol{x}_3 - \boldsymbol{x}_4) = 0 \\ \boldsymbol{x}_1 \gneqq \boldsymbol{0}, \boldsymbol{x}_2 \geq \boldsymbol{0} \end{cases},$$

i.e., the statement (II) holds for $(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 - \boldsymbol{x}_4)$.

$\square$