Primal-Dual Interior-Point Algorithms for LP

Problem

Given $A \in \mathbb{R}^{m\times n}$, $m < n$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, solve

$$F(x,y,z) = \begin{pmatrix} A^T y + z - c \\ Ax - b \\ x \circ z \end{pmatrix} = 0, \quad (x,z) \geq 0.$$

A Basic Algorithm

Input: $A, b, c, tol, maxit, ...$
Output: $x, y, z, iter, ...$
Find ordering: `p = symamd(abs(A)*abs(A)');` (see handout for `chol`)
Initialize: `x = z = n*ones(n,1); y = zeros(m,1);`
for $k = 1$, *maxit* **do**
  Set $\mu = \sigma \frac{x^T z}{n}$ where $\sigma = \min(0.2, 10x^T z/n)$;
  Compute: $r_d = c - A^T y - z,\ r_p = b - Ax,\ r_c = \mu - x \circ z$;
  Form $M = A\,\mathrm{diag}(x./z)A^T$ and the reduced rhs for dy;
  Solve $M(p,p)\,dy(p) = rhs(p)$ by **chol()**;
  Solve for dz and dx by back substitutions;
  Compute the primal and dual step sizes α_p, α_d to the boundary;
  Set $\tau = \max(.9995, 1 - 10x^T z/n)$;
  Update $x = x + \min(\tau\alpha_p, 1)dx$;
  Update $(y,z) = (y,z) + \min(\tau\alpha_d, 1)(dy, dz)$;
  if *stopping criteria are met* **then**
    break the loop and exit;
  end
end

Remarks

- The stopping criterion should be

$$\frac{\|Ax-b\|_2}{1+\|b\|_2} + \frac{\|A^T y + z - c\|_2}{1+\|c\|_2} + \frac{|c^T x - b^T y|}{1+|b^T y|} < tol.$$

- Efficient Matlab implementations are usually not direct translations of mathematical formulas. Do not use any dense matrices (especially there is no need for storing any diagonal matrices because they can be stored as vectors). Do not repeat nontrivial calculations (save and reuse calculated quantities). All involved constant values can be further adjusted by you.

- To see how the algorithm progresses, you may set the switch `prt = 1` in the test scripts.

The handout solver from the instructor, `yz_pdipm.p`, is an implementation of the following, slightly more sophisticated version. You may or may not be required to do it (see the assignment).

A Predictor-Corrector Algorithm

Input: $A, b, c, tol, maxit$, ...
Output: $x, y, z, iter$, ...
Find ordering: `p = symamd(abs(A)*abs(A)');`
Initialize: `x = z = n*ones(n,1), y = zeros(m,1);`
for $k = 1$, *maxit* **do**

(**Predictor step**)
Compute: $r_d = c - A^T y - z,\ r_p = b - Ax,\ r_c = -x \circ z$;
Form $M = A\,\mathrm{diag}(x./z)A^T$ and the reduced rhs for dy part;
Solve $M(p,p)\,dy_1(p) = rhs(p)$ by **chol()**;
Solve for dz_1 and dx_1 by back substitutions;

(**Corrector step**)
Compute the primal and dual step sizes α_p, α_d to the boundary;
Set $\mu = \sigma \frac{x^T z}{n}$ where $\sigma = \min\left(0.2, ((x + \alpha_p dx_1)^T (z + \alpha_p dz_1)/x^T z)^3\right)$;
Set $r_d = 0,\ r_p = 0,\ r_c = \mu e - dx_1 \circ dz_1$ and the rhs for dy part;
Solve $M(p,p)\,dy_2(p) = rhs(p)$ by **chol()**;
Solve for dz_2 and dx_2 by back substitutions;

(**Combined step**)
Set $(dx, dy, dz) = (dx_1, dy_1, dz_1) + (dx_2, dy_2, dz_2)$;
Compute the primal and dual step sizes α_p, α_d to the boundary;
Set $\tau = \max(.9995, 1 - 10x^T z/n)$;
Update $x = x + \min(\tau\alpha_p, 1)dx$;
Update $(y, z) = (y, z) + \min(\tau\alpha_d, 1)(dy, dz)$;
if *stopping criteria are met* **then**
break the loop and exit;
end

end

Remarks

- The step (dx, dy, dz) is the sum of two terms: the predictor step and the corrector step.
- The predictor step and the corrector step solve linear systems of equations that share the same coefficient matrix with different right-hand sides. Only one Cholesky factorization is necessary. It is a good idea to write a function to perform the linear solving that can be used by both steps.
- Other things remain the same.