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MAT3006 & 3040 & 4002 Notebook

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Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i, j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i, j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

9.3. Monday for MAT4002

Reviewing.

1. Homotopy: $f \cong g$
2. If $Y \subseteq \mathbb{R}$ is convex, then the set of continuous functions $f : X \to Y$ form a single equivalence class, i.e., $\{\text{continuous functions } f : X \to Y\}/\sim$ has only one element

9.3.1. Remarks on Homotopy

Proposition 9.4 Consider

$$W \to_f X, \quad X \to_g Y, \quad X \to_h Y, \quad Y \to_k Z$$

where f, g, h, k are all continuous. If $g \cong h$, then

$$g \circ f \cong h \circ f, \quad k \circ g \cong k \circ h$$

Proof. Suppose $H : g \cong h$, then $k \circ H : X \times I \to Z$ givs the momotopy between $k \circ g$ and $k \circ h$.

Simiarly, $H \circ (f \times \mathrm{id}_I) : W \times I \to Y$ gives homotopy $g \circ f \simeq h \circ f$.

■

Definition 9.4 [Homotopy Equivalent] Two topological spaces X and Y are **homotopy equivalent** if there are continuous maps $f : X \to Y$, and $g : Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_{X \to X}$$
$$f \circ g \simeq \mathrm{id}_{Y \to Y}$$

denoted as $X \simeq Y$.

Ⓡ

1. If $X \cong Y$ are homeomorphic, then they are homotopic equivalent.
2. $X \simeq Y$ gives a bijection between $\{\phi : \text{continuous } W \to X\}/\sim$ and $\{\phi : \text{continuous } W \to Y\}/\sim$
3. exercise: $X \simeq Y$ forms an equivalence relation between topological spaces

Some properties are lost when we only study spaces up to homotopy equivalence.

Definition 9.5 [Contractible] X is **contractible** if it is homotopy equivalent to a point $\{*\}$.

Equivalently, we need f, g such that

$$\{*\} \to_f X \to_g \{*\},\ g \circ f \simeq \text{id}_{\{*\}}$$

$$X \to_g \{*\} \to_f X,\ f \circ g \simeq \text{id}_X$$

Note that $f \circ g = c_y$ for some $y \in X$, where $c_y : X \to X$ is $c_y(x) = y, \forall x \in X$. Therefore, to check X is contractible, we only need to check $c_y \simeq \text{id}_X, \forall y \in X$

Example 9.1 1. $X = \mathbb{R}^2$ is contractible.

WTS $f(\boldsymbol{x}) = \boldsymbol{x}$ (i.e., $f = \text{id}$) is homotopic to the constant function $g(x) = (0,0), \forall x \in \mathbb{R}^2$, i.e., $g = c_{(0,0)}$

Consider $H(\boldsymbol{x}, t) = tf(\boldsymbol{x})$, which implies

$$H(\boldsymbol{x}, 0) = c_{(0,0)}, \qquad H(\boldsymbol{x}, 1) = \text{id}$$

Therefore, $\text{id} \simeq c_{(0,0)}$, and $c_{(0,0)} \simeq c_y, \forall y \in \mathbb{R}^2$

Therefore, X is contractible.

More ggenerally, any convex $X \subseteq \mathbb{R}^n$ is contractible.

R S^1 is not contractible, and we will see it in 3 weeks' time.

We are not able to construct

$$H : S^1 \to [0,1] \to S^1$$

such that $H(e^{2\pi ix},0) = e^{2\pi ix}$ and $H(e^{2\pi ix},1) = e^{2\pi i(0)} = 1$ (c_1)

But how about $H(e^{2\pi ix},t) = e^{2\pi ixt}$? What's wrong with H? Such a funciton is not well-defined:

$$H(e^{2\pi i(1)},t) = e^{2\pi i(1-t)} = H(e^{2\pi i(0)},t) = 1$$

Therefore, H is not well-defined for $t \neq 0,1$.

Definition 9.6 Let $A \subseteq X$ and $i :\hookrightarrow X$ be an inclusion. We say A is a homotopy retract of X if there exists continuous mapping $r : X \to A$ such that

$$A \hookrightarrow X \to^r A \implies r \circ i = \mathrm{id}_{A \to A}$$

$$X \to^r A \hookrightarrow^i X \implies i \circ r \simeq \mathrm{id}_X$$

In particualr, $A \simeq X$. ■

■ **Example 9.2** S^1 is a homotopy retract of $M =$ Mobius Band

Here $M = [0,1]^2/\sim$ and $S^1 = [0,1]/\sim$. Define

$$i : S^1 \hookrightarrow M, \quad [x] \mapsto [(x, \frac{1}{2})]$$

$$\begin{aligned} r : \quad & M \to S^1 \\ & [(x,y)] \mapsto [x] \end{aligned}$$

Then $r \circ i = \mathrm{id}_{S^1}$; $i \circ r([(x,y)]) = [(x,1/2)]$; $\mathrm{id}_M([(x,y)]) = [(x,y)]$.

Define $H : M \times I \to M$ by

$$H([(x,y)],t) := [(x,(1-t)y + t/2)]$$

We really need to check

$$H([(0,y)],t) = H([(1,1-y)],t), \quad \forall y \in [0,1]$$

Therefore, H gives a homotopy between $i \circ r$ and id_M, i.e., $i \circ r \simeq \mathrm{id}_M$ ■

S^{n-1} is a homotopy retract of $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

We have $i : \mathrm{id} = S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ and

$$\begin{aligned} r : \quad & \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1} \\ & x \mapsto \tfrac{x}{\|x\|} \end{aligned}$$

Therefore, $r \circ i = \mathrm{id}_{S^{n-1}}$ and $i \circ r(x) = x/\|x\|$.

WTS $i \circ r \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ Consider $H(x,t) = t\boldsymbol{x} + (1-t)\boldsymbol{x}/\|\boldsymbol{x}\|$:

$$H(\boldsymbol{x},0) = i \circ r(\boldsymbol{x}), \quad H(\boldsymbol{x},1) = \boldsymbol{x} = \mathrm{id}(\boldsymbol{x})$$

However, we need to check that $H(x,t) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ for all $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $t \in [0,1]$.

Definition 9.7 [Homotopic Relative] Let $A \subseteq X$ be topological spaces. We say $f,g : X \times I \to Y$ are homotopic relative to A if there eixsts $H : X \times I \to Y$ such that

$$\begin{cases} H(x,0) = f(x) \\ H(x,1) = g(x) \end{cases}$$

and

$$H(a,t) = f(a) = g(a), \forall a \in A$$

■