A FIRST COURSE

IN

ANALYSIS

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MAT2006 Notebook

Lecturer

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Acknowledgments

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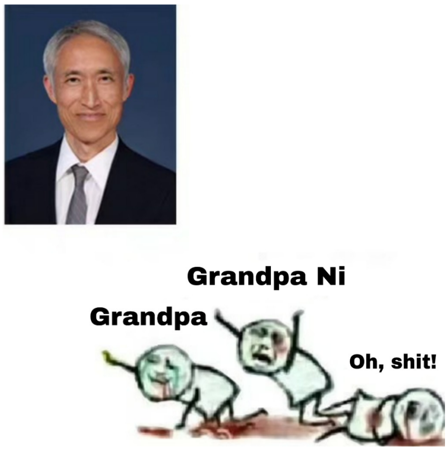
Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

3.2. Friday



"Our first quiz is at 1:30-2:20pm on September 30th. That is next Sunday. There will be around 5 questions." the Grandpa said breezily,

3.2.1. Review

This lecture will review the continuty of function. Let's start with some easy examples:

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ -x, & x \notin \mathbb{Q} \end{cases}$$

This function is continuous nowhere expect for $x = 0$.

Definition 3.8 [Continuous] Given a function $f : D \mapsto \mathbb{R}$,

- we say f is **continuous** at $x_0 \in D$ if for $\forall \varepsilon > 0$, $\exists \delta := \delta(\varepsilon, x_0) > 0$ s.t. $|f(y) - f(x_0)| < \varepsilon$ for $\forall |y - x_0| < \delta$

- f is continuous on D if it is continuous at every point in D.
- If $\delta := \delta(\varepsilon)$, i.e., δ is independent of $x_0 \in D$, then f is said to be **uniformly continuous** on D.

R The following statements are equivalent, you should show by yourself.

1. f is continuous
2. If $\{x_n\} \to x_0$ as $n \to \infty$, then $\{f(x_n)\} \to f(x_0)$ as $n \to \infty$.
3. $f^{-1}(\boldsymbol{A})$ is open/closed if the set $\boldsymbol{A}$ is open/closed.

Definition 3.9 [Compact] A set K is **compact** (cpt) if for evey open cover of K, there exists a **finite** sub-cover.

R The compactness has an important connection with continuity, e.g., the continuous function f maps compact sets to compact sets.

There is a useful way to determine whether a point is continuous at f, which will be discussed in this lecture.

3.2.2. Continuity Analysis

Let's raise some examples first. From these examples we can see that the proof of continuousness is non-trival.

Example 3.1 1. Given a funciton

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

From the graph we can see that f oscillates heavily near zero point.

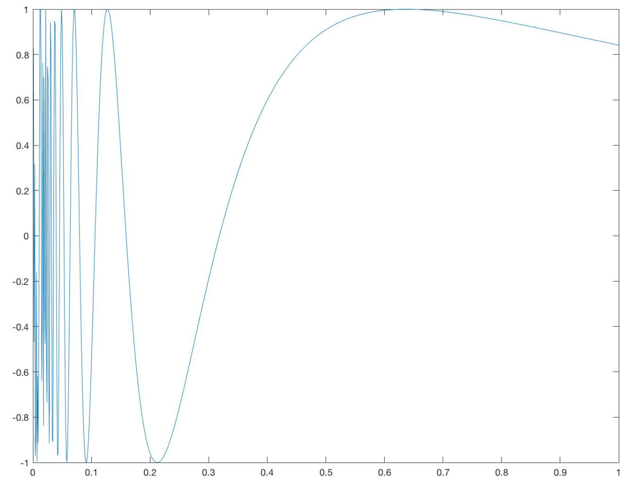


Figure 3.1: Graph for f

It is easy to show that $\sup_{x,y\in N_\delta(0)} |f(x)-f(y)| = 2$ however small δ is.

2. For another function

$$g(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Conversely, it oscillates weakly near the zero point.

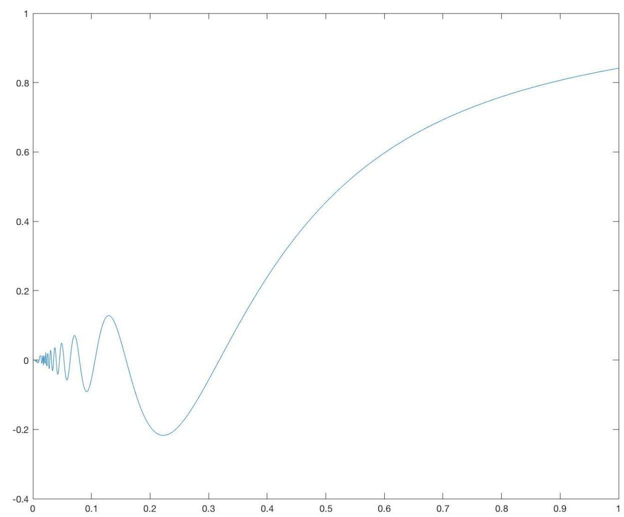


Figure 3.2: Graph for g

It is easy to show that $\sup_{x,y\in N_\delta(0)}|g(x)-g(y)|=0$ as $\delta\to 0$.

Definition 3.10 [oscillation]

- The oscillation of a function f on E is defined as

$$\omega(f;E):=\sup_{x,y\in E}|f(x)-f(y)|$$

- The oscillation of f at a single point x_0 is defined as

$$\lim_{\delta\to 0}\omega(f,N_\delta(x_0)):=w(f;x_0)$$

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- Here we abuse the notation to denote the oscillation at x_0 with $\omega(f;x_0)$, but note that $\omega(f;x_0)\neq\omega(f;\{x_0\})$.
- The well-definedness of $w(f;x_0)$ is because $\omega(f,N_\delta(x_0))$ is non-incresing as δ decreases and has a lower bound 0.
- A function f is continuous at x_0 iff $w(f,x_0)=0$. (verify by yourself)

An classical example that illustrates a function can have continuous points in $\mathbb{R}\setminus\mathbb{Q}$ is shown below. We have faced this example in the diagnostic quiz:

Example 3.2 The Dirichlet function is defined for $\mathbb{R}\setminus\{0\}$:

$$f(x)=\begin{cases}0, & x\notin\mathbb{Q}\\ \frac{1}{q}, & x=\frac{p}{q},q>0,(p,q)=1\end{cases}$$

The function f is continuous at x iff $x\notin\mathbb{Q}$. The set of all discontinuous points of f is $\mathbb{Q}$.

Now the question turns out:

> *Does there exists a function g of which the set of all discontinuous points of g is $\mathbb{R} \setminus \mathbb{Q}$?*

Applying Baire-Category Theorem, we will show the answer to this question is no.

Proposition 3.2 Suppose f is continuous on a dense set in $\mathbb{R}$. Then the set of all discontinous points of f, denoted as T, must form a set of first category, i.e., (a countably union of nowhere dense sets)

R $\mathbb{R} \setminus \mathbb{Q}$ is of second category, otherwise assume

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{i=1}^{\infty} I_i$$

for nowhere dense sets I_i, which implies $\mathbb{R} = \bigcup_{i=1}^{\infty} I_i \bigcup_{q \in \mathbb{Q}} \{q\}$ is a countably union of dense sets. Thus by applying proposition(3.2), the irrational number space cannot be the set of discontinuities.

The idea of the proof is to express T as countably union of sets, and argue that at least one of which must be nowhere dense.

Proof. We construct $D_n = \{x \in \mathbb{R} \mid w(f;x) \geq \frac{1}{n}\}$, which follows that

$$T = \bigcup_{n=1}^{\infty} D_n.$$

It suffices to show that D_n is nowhere dense for every n by contradiction.

Assume for some fixed n, D_n is not nowhere dense, i.e., $\overline{D_n}$ contains an **open** interval $\boldsymbol{I}$. Note that the set of continuous points is dense, we conclude that there exists a point a inside the interval $\boldsymbol{I}$ such that f is continuous at a. (why?) Also, there exists a sequence $\{b_k\} \subseteq D_n$ with limit a. (since you can verify D_n is closed)

Since f is continuous at a, there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \frac{1}{4n} \text{for } |x - a| < \delta. \tag{3.3a}$$

At the same time $\{b_k\} \subseteq (a-\delta, a+\delta)$ for all large k, i.e., $\omega(f; b_k) \geq \frac{1}{n}$. Hence, there exists a sequence $\{c_{kl}\}$with limit b_k, such that the difference $|f(c_{kl}) - f(b_k)|$ is at least greater than $\frac{1}{2n}$ (why not $\frac{1}{n}$?), i.e.,

$$|f(c_{kl} - f(b_k)| \geq \frac{1}{2n} \tag{3.3b}$$

Meanwhile, for large l, note that c_{kl} is close to a, i.e., from (3.3a) we have

$$|f(c_{kl}) - f(a)| < \frac{1}{4n} \tag{3.3c}$$

Also, note that b_k is close to a for large k, i.e.,from (3.3a) we have

$$|f(b_k) - f(a)| < \frac{1}{4n} \tag{3.3d}$$

Three inequalities (3.3b) to (3.3d) show a contradiction:

$$|f(c_{kl}) - f(b_k)| \leq |f(c_{kl}) - f(a)| + |f(b_k) - f(a)| < \frac{1}{4n} + \frac{1}{4n} = \frac{1}{2n}$$

■

Theorem 3.3 Let f be the **pointwise** limit of a sequence of continuous functions $\{f_n\}$, i.e., $f(x) = \lim_{n\to\infty} f_n(x)$. Then the set if all discontinuous points of f must be a set of first category.

R Review the uniform limit version of this theorem;

Proof. We claim $D_\varepsilon = \{x \in \mathbb{R} \mid \omega(f, x) \geq \varepsilon\}$ is nowhere dense for any $\varepsilon > 0$. (fixed ε).

Assume $\overline{D_\varepsilon}$ contains an open set, or equivalently, D_ε contains an open set $\mathcal{U}$(since D_ε is closed, i.e., $\overline{D_\varepsilon} = D_\varepsilon$). Define

$$A_{mn} = \left\{x \in U \mid |f_m(x) - f_n(x)| \leq \frac{\varepsilon}{4}\right\}.$$

Proposition 3.3 The set A_{mn} is closed.

The proof of proposition is moved in the end.

Set $A_m = \bigcap_{n\geq m} A_{mn}$, which is also closed, and

$$A_m \subseteq \left\{x \in U \mid |f_m(x) - f(x)| \leq \frac{\varepsilon}{4}\right\}$$

For every $x \in U$, as $\lim_{n\to\infty} f_n(x) = f(x)$, we have $x \in \bigcup_{m=1}^{\infty} A_m$, which implies $U \subseteq \bigcup_{m=11}^{\infty} A_m$. Applying the Baire Category Theorem, there exists one A_m containing an open set W.

For $x_0 \in W \subseteq \mathcal{U} \subseteq D_\varepsilon$, pick $\{x_n\} \subseteq W$ with limit x_0 such that

$$|f(x_n) - f(x_0)| \geq \frac{3}{4}\varepsilon \tag{3.4a}$$

At the same time, since $x_n, x_0 \in W$, it follows that

$$|f_m(x_n) - f(x_n)| \leq \frac{\varepsilon}{4} \tag{3.4b}$$

$$|f_m(x_0) - f(x_0)| \leq \frac{\varepsilon}{4} \tag{3.4c}$$

From (3.4a) to (3.4c), we conclude that

$$\begin{aligned}\frac{3}{4}\varepsilon &\leq |f(x_n) - f(x_0)| \\ &\leq |f_m(x_n) - f(x_n)| + |f_m(x_n) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\ &\leq \frac{1}{2}\varepsilon + |f_m(x_n) - f_m(x_0)|\end{aligned}$$

Or equivalently,

$$|f_m(x_n) - f_m(x_0)| \geq \frac{\varepsilon}{4}, \forall n \tag{3.4d}$$

which implies $\omega(f_m; x_0) \geq \frac{\varepsilon}{4}$, which implies f_m is discontinuous at x_0, which is a contradiction. ∎

R

- The proof for A_{mn} is closed is easy: pick any sequence $\{x_k\} \subseteq A_{mn}$ with limit x, it suffices to show $x \in A_{mn}$, i.e.,

$$\lim_{k \to \infty} |f_m(x_k) - f_m(x_k)| \leq \frac{\varepsilon}{4}$$

- The applications of Baire Category Theorem give us an estimation of how large and how small a set is. In next lecture we will see how large is the set of continuous but nowhere differential functions.