A FIRST COURSE

IN

ANALYSIS

MAT2006 Notebook

Lecturer

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Contents

Acknowledgments

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Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Wednesday

Recommended Reading.

1. (Springer-Lehrbuch) V. A. Zorich, J. Schüle-Analysis I-Springer (2006).
2. (The Carus mathematical monographs 13) Ralph P. Boas, Harold P. Boas, A primer of real functions-Mathematical Association of America (1996).
3. (International series in pure and applied mathematics) Walter Rudin, Principles of Mathematical Analysis-McGraw-Hill (1976).
4. Terence Tao, Analysis I,II-Hindustan Book Agency (2006)
5. (Cornerstones) Anthony W. Knapp, Basic real analysis-Birkhäuser (2005)

1.1.1. Introduction to Set

For a set $\mathcal{A} = \{1,2,3\}$, we have $2^3 = 8$ subsets of $\mathcal{A}$. We are interested to study the collection of sets.

Definition 1.1 [Collection of Subsets] Given a set $\mathcal{A}$, the the collection of subsets of $\mathcal{A}$ is denoted as $2^{\mathcal{A}}$. ■

We use Candinal to describe the order of number of elements in a set.

Definition 1.2 Given two sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ and $\mathcal{B}$ are said to be **equivalent** (or have the same**candinal**) if there exists a 1-1 onto mapping from $\mathcal{A}$ to $\mathcal{B}$. ■

Definition 1.3 [Countability] The set $\mathcal{A}$ is said to be **countable** if $\mathcal{A} \sim \mathbb{N} = \{1,2,3,\dots\}$; an infinite set $\mathcal{A}$ is **uncountable** if it is not equivalent to $\mathbb{N}$.

R Note that the set of integers, i.e., $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ is also countable; the set of rational numbers, i.e., $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$ is countable.

We skip the process to define real numbers.

Proposition 1.1 The set of real numbers $\mathbb{R}$ is **uncountable**.

For example, $\sqrt{2} \notin \mathbb{Q}$. Some inrational numbers are the roots of some polynomials, such a number is called **algebraic** numbers. However, some inrational numbers are not, such a number is called **transcendental**. For example, π is **not** algebraic. We will show that the collection of algebraic numbers are countable in the future.

There are two steps for the proof for proposition(1.1):

Proof.

1. $2^{\mathbb{N}}$ is **uncountable**:

   Assume $2^{\mathbb{N}}$ is countable, i.e.,

   $$2^{\mathbb{N}} = \{A_1, A_2, \dots, A_k, \dots\}$$

   Define $B := \{k \in \mathbb{N} \mid k \notin A_k\}$, it is a collection of subscripts such that the subscript k does not belong to the corresponding subsets A_k.

   It follows that $B \in 2^{\mathbb{N}} \implies B = A_n$ for some n. Then it follows two cases:

   - If $n \in A_n$, then $n \notin B = A_n$, which is a contradiction
   - Otherwise, $n \in B = A_n$, which is also a contradiction.

   The proof for the claim $2^{\mathbb{N}}$ is **uncountable** is complete.

2. $\mathbb{R} \sim 2^{\mathbb{N}}$:

   Firstly we have $\mathbb{R} \sim (0,1)$. This can be shown by constructing a one-to-one mapping:

   $$f : \mathbb{R} \mapsto (0,1) \quad f(x) = \frac{1}{\pi}\arctan x + \frac{1}{2}, \forall x \in \mathbb{R}$$

Secondly, we show that $2^{\mathbb{N}} \sim (0,1)$. We construct a mapping f such that

$$f : 2^{\mathbb{N}} \mapsto (0,1),$$

where for $\forall A \in 2^{\mathbb{N}}$,

$$f(A) = 0.a_1a_2a_3\dots, \quad a_j = \begin{cases} 2, & \text{if } j \in A \\ 4, & \text{if } j \notin A \end{cases}$$

This function is only 1-1 mapping but not onto mapping.

Reversely, we construct a 1-1 mapping from $(0,1)$ to $2^{\mathbb{N}}$. We construct a mapping g such that

$$g : (0,1) \mapsto 2^{\mathbb{N}}$$

where for any real number from $(0,1)$, we can write it into binary expansion:

$$\text{binary form:} \quad 0.a_1a_2\dots \quad \text{where } a_j = 0 \text{ or } 1.$$

Hence, we construct $g(0.a_1a_2\dots) = \{j \in \mathbb{N} \mid a_j = 0\} \subseteq \mathbb{N}$, which implies $g(\cdot) \in 2^{\mathbb{N}}$.

R Our intuition is that two 1-1 mappings in the reverse direction will lead to a 1-1 **onto** mapping. If this is true, then we complete the proof. This intuition is the **Schroder-Berstein Theorem**.

∎

Defining Binary Form. However, during this proof, we must be careful about the binary form of a real number from $(0,1)$. Now we give a clear definition of Binary Form:

For a real number a, to construct its binary form, we define

$$a_1 = \begin{cases} 0, & \text{if } a \in (0, \frac{1}{2}) \\ 1, & \text{if } a \in [\frac{1}{2}, 1). \end{cases}$$

After having chosen $a_1, a_2, \ldots, a_{j-1}$, we define a_j to be the largest integer such that

$$\frac{1}{2}a_1 + \frac{1}{2^2}a_2 + \cdots + \frac{a_j}{2^j} \leq a$$

Then the binary form of a is $a := 0.a_1 a_2 \ldots$.

Theorem 1.1 — **Schroder-Berstein Theorem.** If $f : A \mapsto B$ and $g : B \mapsto A$ are both 1-1 mapping, then there exists a 1-1 onto mapping from A to B, i.e., card $\#A$ equals to card $\#B$.

Exercise: Show that $(0,1]$ and $[0,1)$ have 1-1 onto mapping without applying Schroder-Berstein Theorem.

The next lecture we will take a deeper study into the proof of Schroder-Berstein Theorem and the real number.

1.2. Quiz

1. Show that the sequence $\{x_n\}$ is convergent, where

$$x_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \cdots + \frac{\sin n}{2^n}.$$

2. Compute the following limits:

   (a)
$$\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/(1-\cos x)}$$

   (b)
$$\lim_{n \to \infty} \int_0^1 \frac{x^n}{1+\sqrt{x}} \, \mathrm{d}x$$

3. Justify that the natural number e is irrational, where

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

4. Every rational x can be written in the form $x = p/q$, where $q > 0$ and p and q are integers without any common divisors. When $x = 0$, we take $q = 1$. Consider the function f defined on $\mathbb{R}^1$ by

$$f(x) = \begin{cases} 0, & x \text{ is irrational} \\ \dfrac{1}{q}, & x = \frac{p}{q}. \end{cases}$$

   Find:

   (a) all continuities of $f(x)$;
   (b) all discontinuities of $f(x)$

   and prove your results.

1.3. Friday

Before we give a proof of Schroder-Berstein theorem, we'd better review the definitions for one-to-one mapping and onto mapping.

Definition 1.4 [One-to-One/Onto Mapping] If $f : A \mapsto B$, then

- f is said to be **onto** mapping if

$$\forall b \in B, \exists a \in A \text{ s.t. } f(a) = b;$$

- f is said to be **one-to-one** mapping if

$$\forall a, b, \in A, f(a) = f(b) \implies a = b.$$

The Fig.(1.1) shows the examples of one-to-one/onto mappings.

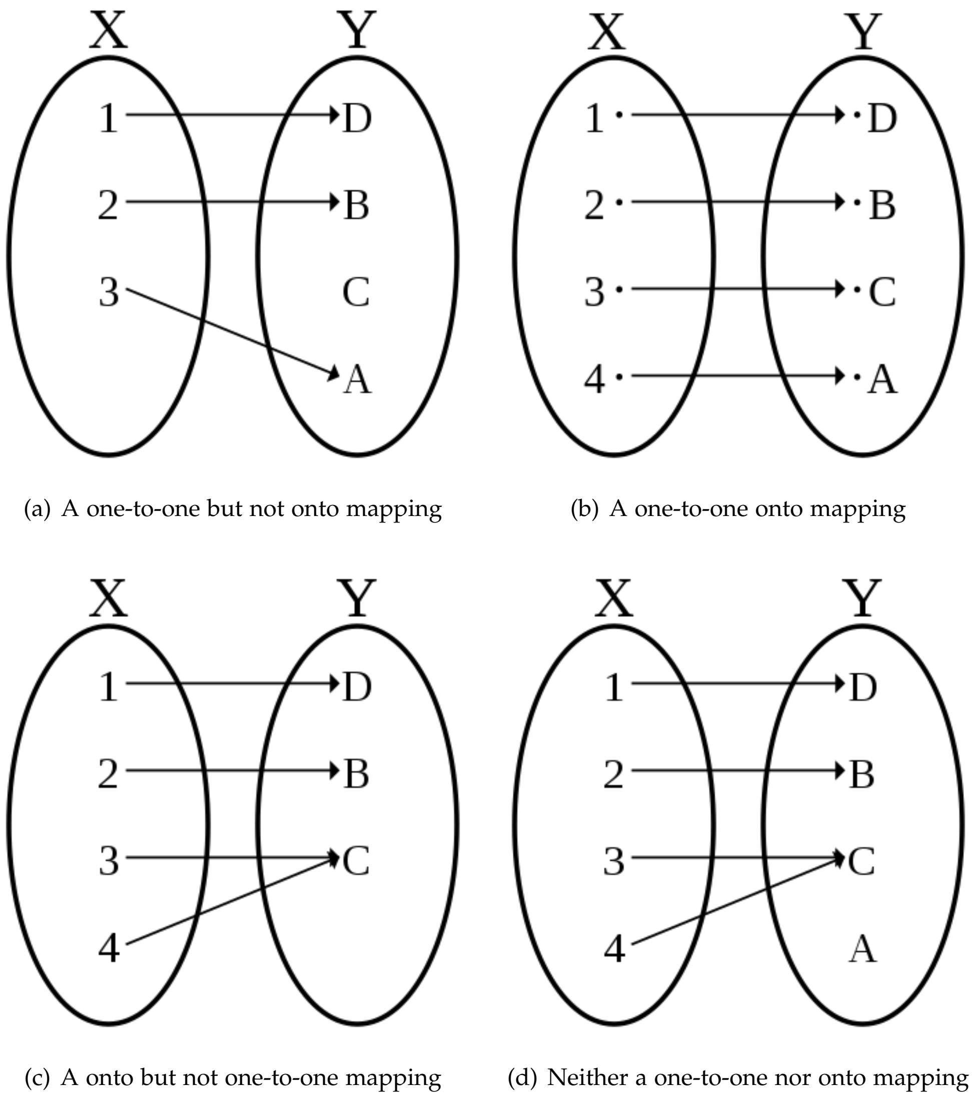
1.3.1. Proof of Schroder-Berstein Theorem

Before the proof, note that in this lecture we abuse the notation fg to denote the composite function $f \circ g$, but in the future fg will refer to other meanings.

Intuition from Fig.(1.2). The proof for this theorem is constructive. Firstly Fig.(1.2) gives us the intuition of the proof for this theorem. Let $f : A \mapsto B$ and $g : B \mapsto A$ be two one-to-one mappings, and D, C are the image from A, B respectively. Note that

> if the set $B \setminus D$ is empty, then $D = B = f(A)$ with f being the one-to-one mapping, which implies f is one-to-one onto mapping. In this case the proof is complete.

Hence it suffices to consider the case $B \setminus D$ is non-empty. Thus $B \setminus D$ is the "**trouble-maker**". To construct a one-to-one onto mapping from A, we should study the subset $g(B \setminus D)$ of A (which can also be viewed as a *trouble-maker*). Moreove, we should study



(a) A one-to-one but not onto mapping

(b) A one-to-one onto mapping

(c) A onto but not one-to-one mapping

(d) Neither a one-to-one nor onto mapping

Figure 1.1: Illustrations of one-to-one/onto mappings

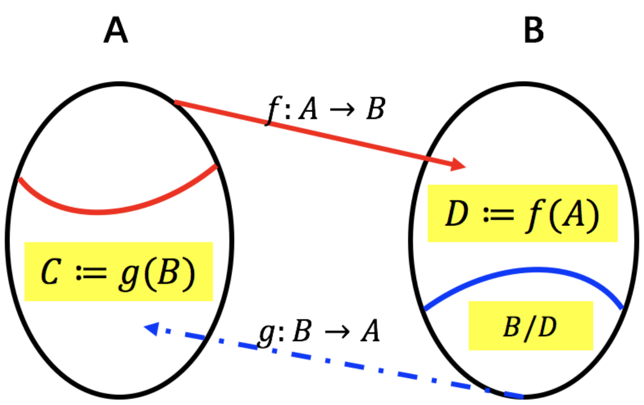


Figure 1.2: Illustration of Schroder-Berstein Theorem

the subset $gf[g(B \setminus D)]$ (which is also a *trouble-maker*)... so on and so forth. Therefore, we should study the *union of these trouble makers*, i.e., we define

$$A_1 := g(B \setminus D), \quad A_2 := gf(A_1), \quad \cdots, \quad A_n := gf(A_{n-1}),$$

Then we study the union of infinite sets

$$S := A_1 \bigcup A_2 \bigcup \cdots \bigcup A_n \bigcup \cdots$$

Define

$$F(a) = \begin{cases} f(a), & a \in A \setminus S \\ g^{-1}(a), & a \in S \end{cases}$$

We claim that $F : A \mapsto B$ is one-to-one onto mapping.

F is onto mapping. Given any element $b \in B$, it follows two cases:

1. $g(b) \in S$. It implies $F(g(b)) = g^{-1}(g(b)) = b$.
2. $g(b) \notin S$. It implies $b \in D$, since otherwise $b \in B \setminus D \implies g(b) \in g(B \setminus D) \subseteq S$, which is a contradiction. $b \in D$ implies that $\exists a \in A$ s.t. $f(a) = b$.

   Then we study the relationship between $gf(S)$ and S. Verify by yourself that

   $$S = g(B \setminus D) \bigcup gf(S)$$

   With this relationship, we claim $a \notin S$, since otherwise $a \in S \implies gf(a) \in S$, but $gf(a) = g(b) \notin S$, which is a contradiction.

   Hence, $F(a) = f(a) = b$.

Hence, for any element $b \in B$, we can find a element from A such that the mapping for which is equal to b, i.e., F is onto mapping.

F is one-to-one mapping. Assume not, verify by yourself that the only possibility is that $\exists a_1 \in A \setminus S$ and $a_2 \in S$ such that $F(a_1) = F(a_2)$, i.e., $f(a_1) = g^{-1}(a_2)$, which follows

$$gf(a_1) = a_2 \in S = A_1 \bigcup A_2 \bigcup \cdots \tag{1.1}$$

We claim that Eq.(1.1) is false. Note that $gf(a_1) \notin A_1 := g(B \setminus D)$, since otherwise $f(a_1) \in B \setminus D$, which is a contradiction; note that $gf(a_1) \notin A_2$, since otherwise $gf(a_1) \in gfg(B \setminus D) \implies a_1 \in g(B \setminus D) = A_1 \subseteq S$, which is a contradiction.

Applying the similar trick, we wil show that $gf(a_1) \notin A_k$ for $k \geq 1$. Hence, Eq.(1.1) is false, the proof is complete.

■ **Example 1.1** Given two sets $A := (0,1]$ and $B := [0,1)$. Now we apply the idea in the proof above to construct a one-to-one onto mapping from A to B:

- Firstly we construct two one-to-one mappings:

$$\begin{aligned} f :A &\mapsto B & g :B &\mapsto A \\ f(x) &= \frac{1}{2}x & g(x) &= x \end{aligned}$$

- It follows that $B \setminus D = (\frac{1}{2},1)$, $gf(B \setminus D) = (\frac{1}{4},1)$, so on and so forth.

$$S = (\frac{1}{2},1) \bigcup (\frac{1}{4},1) \bigcup \cdots$$

- Hence, the one-to-one onto mapping we construct is

$$F(x) = \begin{cases} \frac{1}{2}x, & x \in A \setminus S \\ x, & x \in S \end{cases}$$

- Conversely, to construct the inverse mapping, we define

$$f(x) = x \quad g(x) = \tfrac{1}{2}x$$

- It follows that $D = (0,1)$, $B \setminus D = \{1\}$. Then

$$S = \left\{\frac{1}{2}\right\} \bigcup \cdots = \left\{\frac{1}{2}, \frac{1}{4}, \cdots\right\}$$

- Hence, the function we construct for inverse mapping is

$$F(x) = \begin{cases} x, & x \neq \dfrac{1}{2^m} \\ 2x, & x = \dfrac{1}{2^m} \end{cases} \qquad (m = 1,2,3,\dots)$$

1.3.2. Connectedness of Real Numbers

There are two approaches to construct real numbers. Let's take $\sqrt{2}$ as an example.

1. The first way is to use **Dedekind Cut**, i.e., every non-empty subset has a least upper bound. Therefore, $\sqrt{2}$ is actually the least upper bound of a non-empty subset

$$\{x \in \mathbb{Q} \mid x^2 < 2\}.$$

2. Another way is to use **Cauchy Sequence**, i.e., every Cauchy sequence is convergent. Therefore, $\sqrt{2}$ is actually the limit of the given sequence of decimal approximations below:

$$\{1, 1.4, 1.41, 1.414, 1.4142, \dots\}$$

We will use the second approach to define real numbers. Every real number r essentially represents a collection of cauchy sequences with limit r, i.e.,

$$r \in \mathbb{R} \implies \left\{ \{x_n\}_{n=1}^{\infty} \middle| \lim_{n\to\infty} x_n = r \right\}$$

Let's give a formal definition for cauchy sequence and a formal definition for real number.

Definition 1.5 [Cauchy Sequence]

- Any sequence of rational numbers $\{x_1, x_2, \cdots\}$ is said to be a **cauchy sequence** if for every $\epsilon > 0$, $\exists N$ s.t. $|x_n - x_m| < \epsilon$, $\forall m, n \geq N$

- Two cauchy sequences $\{x_1, x_2, \dots\}$ and $\{y_1, y_2, \dots\}$ are said to be **equivalent** if for every $\epsilon > 0$, there $\exists N$ s.t. $|x_n - y_n| < \epsilon$ for $\forall n \geq N$.
- A real number is a **collection** of **equivalent** cauchy sequences. It can be represented by a cauchy sequence:

$$x \in \mathbb{R} \sim \{x_1, x_2, \dots, x_n, \dots\},$$

where x_j is a rational number.

R Let $\xi_{\mathbb{Q}}$ denote a collection of any cauchy sequences. Then once we have equivalence relation, the whole collection $\xi_{\mathbb{Q}}$ is partitioned into several disjoint subsets, i.e., equivalence classes. Hence, the real number space $\mathbb{R}$ are the equivalence classes of $\xi_{\mathbb{Q}}$.

The real numbers are well-defined, i.e., given two real numbers $x \sim \{x_1, x_2, \dots\}$ $y \sim \{y_1, y_2, \dots\}$, we can define add and multiplication operator.

$$x + y \sim \{x_1 + y_1, x_2 + y_2, \dots\}$$
$$x \cdot y \sim \{x_1 \cdot y_1, x_2 \cdot y_2, \dots\}$$

We will show how to define $x > 0$ in next lecture, this construction essentially leads to the lemma below:

Proposition 1.2 $\mathbb{Q}$ are dense in $\mathbb{R}$.

In the next lecture we will also show the completeness of $\mathbb{R}$:

Theorem 1.2 $\mathbb{R}$ is complete, i.e., every cauchy sequence of real numbers converges.

Recommended Reading:

Prof. Katrin Wehrheim, MIT Open Course, Fall 2010, Analysis I Course Notes, Online avaiable:

https://ocw.mit.edu/courses/mathematics
/18-100b-analysis-i-fall-2010/readings-notes/MIT18_100BF10_Const_of_R.pdf

Chapter 2

Week2

2.1. Wednesday

2.1.1. Review and Announcement

The quiz results will not be posted.

In this lecture we study the number theories.

The office hour is 2 - 4pm, TC606 on Wednesday

2.1.2. Irrational Number Analysis

Definition 2.1 [Algebraic Number] A number $x \in \mathbb{R}$ is said to be an **algebraic number** if it satisfies the following equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{2.1}$$

where $a_n, a_{n-1}, \ldots, a_0$ are integers and not all zero. We say x is **of degree** n if $a_n \neq 0$ and x is not the root of any polynomial with lower degree. ∎

Definition 2.2 A number $x \in \mathbb{R}$ is **transcendental** if it is not an algebraic number. ∎

The first example is that all rational numbers are algebraic, since rational number $\frac{p}{q}$ satisfies $qx - p = 0$. Also, $\sqrt{2}$ is algebraic. We leave an exericse: show that e and π are all transcendental. In history Joseph Liouville (1844) have constructed the first transcendental number. Let's look at the insights of his construction in this lecture:

Proposition 2.1 The set of all algebraic numbers is countable.

Proof.

1. Let $\mathcal{P}_n$ denote the set of all polynomials of degree n (Here we assert polynomials have all integer coefficients by default.), i.e.,

$$\mathcal{P}_n = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mid a_j \in \mathbb{Z}\}$$

The set $\mathcal{P}_n$ have the one-to-one onto mapping to the set $\{(a_n, a_{n-1}, \ldots, a_0) \mid a_j \in \mathbb{Z}\} \subseteq \mathbb{Z}^{n+1}$, which implies $\mathcal{P}_n$ is **countable**.

2. Let $\mathcal{R}_n$ denote the set of all real roots of polynomials in $\mathcal{P}_n$. Since each polynomial of degree n has at most n real roots, the set $\mathcal{R}_n$ is a countable union of finite sets, which is at most countable. It is easy to show $\mathcal{R}_n$ is infinite, and thus countable.
3. Hence, we construct the set of all algebraic numbers $\bigcup_{n=1}^{\infty} \mathcal{R}_n$, which is countable since countably union of countable sets is also countable.

■

How fast to approximate rational numbers using rational numbers? How fast to approximate irrational numbers using rational numbers? How fast to approximate transcendental numbers using rational numbers? We need the definition for the rate of approximation first to answer these questions.

Definition 2.3 A real number ξ is **approximable** by **rational numbers to order** n if $\exists$ a constant $K = K(\xi)$ such that the inequality

$$\left|\frac{p}{q} - \xi\right| \leq \frac{K}{q^n}$$

has **infinitely many** solutions $\frac{p}{q} \in \mathbb{Q}$ with $q > 0$ and p, q are integers without any common divisors. ■

Intuitively, a rational number is approximable by rational numbers. Now we study its rate of approximation by applying this definition.

■ **Example 2.1** Suppose a rational number is apprixmable to oder α (which is a parameter). To calculate the value of α, it suffices to choose (p_k, q_k) such that

$$\left|\frac{p_k}{q_k} - \frac{p}{q}\right| \le \frac{K}{q^\alpha}$$

Note that

$$\left|\frac{p_k}{q_k} - \frac{p}{q}\right| = \left|\frac{p_k q - p q_k}{q_k q}\right| \ge \frac{1}{q q_k} = \frac{1/q}{q_k},$$

- $\frac{p}{q}$ is approximable by rational numbers to order 1:

  If we construct $(p_k, q_k) = (kp - 1, kq)$, it follows that

$$\left|\frac{p_k}{q_k} - \frac{p}{q}\right| = \frac{1}{kq} = \frac{1}{q_k^1}$$

- $\frac{p}{q}$ is approximable by rational numbers to order no higher than 1:

  Otherwise suppose it is approximable to order $n > 1$. The inequality holds for infinitely many (p_k, q_k):

$$\frac{1/q}{q_k} \le \left|\frac{p_k}{q_k} - \frac{p}{q}\right| \le \frac{K}{q_k^n} \implies \frac{1}{q} q_k^{n-1} \le K \tag{2.2}$$

  Since infinite (p_k, q_k) satisfy the inequality (2.2), we can choose a solution such that q_k is arbitrarily large, which falsify (2.2).

In summary, any rational number $\frac{p}{q}$ is approximable by rational numbers to order 1 and no higher than 1. ■

Liouville had shown that the transcendental number has the higher approxiamtion rate than rational and algebraic numbers, which is counter-intuitive. Let's review his process of proof:

Theorem 2.1 — Liouville, 1844. A real algebraic number ξ of degree n is not approximable by rational numbers to any order greater than n.

We can show some numbers is not algebraic, i.e., transcendental by applying this theorem:

■ Example 2.2 [1st Constructed Transcendental Number] Given a number

$$\xi := \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots,$$

we aim to show it is transcendental. Assume that it is an algebraic number of order n, then we construct the first n tails of ξ:

$$\xi_n = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{n!}}$$

It follows that

$$\begin{aligned}
|\xi_n - \xi| &= \frac{1}{10^{(n+1)!}} + \frac{1}{10^{(n+2)!}} + \cdots \\
&= \frac{1}{10^{(n+1)!}}\left[1 + \frac{1}{10^{n+2}} + \frac{1}{10^{(n+2)(n+3)}} + \cdots\right] \\
&\le \frac{1}{10^{(n+1)!}} \cdot 2 = \frac{2}{(10^{n!})^{n+1}} = \frac{2}{q^{n+1}}
\end{aligned}$$

which implies $|\xi - \frac{p}{q}| \le \frac{K}{q^{n+1}}$ has one solution ξ_n.

We can construct infinitely many solutions from this solution:

$$\xi_{n,1} = \xi_n + \frac{1}{10^{n+2}}, \quad \xi_{n,2} = \xi_n + \frac{1}{10^{(n+2)(n+3)}}, \quad \cdots,$$

Hence, this number is approximable by rational numbers to order $n+1$, which contradicts the fact that it is an algebraic number of degree n. ■

Proof. Given an algebraic number ξ of degree n, there exists a polynomial whose roots contain ξ:

$$f(x) \equiv a_n x^n + \cdots + a_1 x + a_0 = 0.$$

We fix an interval around ξ, i.e, $I_\lambda = [\xi - \lambda, \xi + \lambda]$ (with $\lambda = \lambda(\xi) \in (0,1)$) such that I_λ contains no other root of f except ξ.

Hence, the value of f at any rational number $\frac{p}{q}$ inside I_λ is given by:

$$\left|f(\frac{p}{q})\right| = \left|a_n\frac{p^n}{q^n} + \cdots + a_1\frac{p}{q} + a_0\right| = \left|\frac{a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_0 q^n}{q^n}\right| \neq 0$$

Hence, $\left|f(\frac{p}{q})\right| \geq \frac{1}{q^n}$, which implies that

$$\frac{1}{q^n} \leq \left|f(\frac{p}{q})\right| \tag{2.3a}$$

$$= \left|f(\frac{p}{q}) - f(\xi)\right| \tag{2.3b}$$

$$\leq |f'(\eta)||\xi - \frac{p}{q}| \tag{2.3c}$$

$$\leq M|\xi - \frac{p}{q}| \tag{2.3d}$$

with $M := \max_{\eta \in I_\lambda} f(\eta)$. Note that from (2.3b) to (2.3c) is due to mean value theorem.

Or equivalently, $|\xi - \frac{p}{q}| \geq \frac{1/M}{q^n}$ applies for any rational number $\frac{p}{q}$ inside the interval I_λ.

- Verify by yourself that ξ is not approximable by rational numbers inside the interval I_λ to any order greater than n.

- For any rational number $\frac{p}{q} \notin I_\lambda$, we have

$$\left|\frac{p}{q} - \xi\right| \geq \lambda(\xi) \geq \frac{\lambda(\xi)}{q^n}$$

  for $q \geq 1, n \geq 1$. It is obvious that ξ is not approximable by rational numbers outside the interval I_λ to any order greater than n.

The two cases above complete the proof. ■

It's hard to determine which order the transcendental number is approximable by rational numbers. However, we can assrt that there is a "fast" approximation to transcendental numbers by applying countinued faction expansion.

Continued Fraction Expansion. Let x be irrational, then intuitively x could be represented as an infinite continued fraction as below:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{2.4}$$

We denote the continued fraction (2.4) as $[a_0; a_1, a_2, \ldots]$. Let's define the rigorous process of continued fraction expansion, i.e., how to find a_i:

- We set $a_0 = \lfloor x \rfloor$, which implies that

$$x := a_0 + \xi_0 = a_0 + \frac{1}{\frac{1}{\xi_0}} \qquad \text{for } 0 < \xi_0 < 1.$$

- We set $a_1 = \lceil \frac{1}{\xi_0} \rceil$, which implies that

$$x := a_0 + \frac{1}{a_1 + \xi_1} = a_0 + \frac{1}{a_1 + \frac{1}{\frac{1}{\xi_1}}}$$

- After $n+1$ steps we obtain the continued fraction of x:

$$[a_0; a_1, a_2, \ldots, a_n + \xi_n]$$

  We continue this process iteratively with

$$\frac{1}{\xi_n} = a_{n+1} + \xi_{n+1}$$

  with $\xi_{n+1} \in (0,1)$.

Such a process will continue without end as x is irrational. After $n+1$ steps alternatively, we write

$$x = [a_0, a_1, \ldots, a_n, a'_{n+1}] \quad \text{with } a'_{n+1} := a_{n+1} + \xi_{n+1}.$$

Observations from continued fraction expansion.

1. For $x = [a_0; a_1, \dots] \notin \mathbb{Q}$, consider its nth convergent term

$$\tfrac{p_n}{q_n} := [a_0, a_1, \dots, a_n], \quad n \geq 0$$

note that p_n and q_n can be computed iteratively:

$$\begin{aligned} p_0 &= a & \qquad q_0 &= 1 \\ p_1 &= a_1 a_0 + 1 & q_1 &= a_1 \\ &\vdots & &\vdots \\ p_n &= a_n p_{n-1} + p_{n-2} & q_n &= a_n q_{n-1} + q_{n-2} \end{aligned}$$

Note that (p_n, q_n) have no common divisors. (exercise)

> **Corollary 2.1** $q_n \geq n$ for $\forall n$.

Proof. Note that $q_{n-1} \leq q_n$ for $\forall n \geq 1$; and that $q_{n-1} < q_n$ for $\forall n > 1$. ■

2. From the first observation, $x := [a_0; a_1, \dots, a'_{n+1}]$ can be written as

$$x = \frac{p_{n+1}}{q_{n+1}} = \frac{a'_{n+1} p_n + p_{n-1}}{a'_{n+1} q_n + q_{n-1}}$$

> **Corollary 2.2** If $\frac{p_n}{q_n} (n \geq 0)$ is the nth convergent term of x, then
>
> $$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$$

Proof. First note that for $k \geq 2$,

$$\begin{aligned} p_{k-1} q_k - p_k q_{k-1} &= p_{k-1}(a_k q_{k-1} + q_{k-2}) - (a_k p_{k-1} + p_{k-2}) q_{k-1} \\ &= -(p_{k-2} q_{k-1} - p_{k-1} q_{k-2}) \end{aligned}$$

After computation,

$$
\begin{aligned}
\left| x - \frac{p_n}{q_n} \right| &= \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \\
&= \left| \frac{a'_{n+1} p_n + p_{n-1}}{a'_{n+1} q_n + q_{n-1}} - \frac{p_n}{q_n} \right| = \left| \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n (a'_{n+1} q_n + q_{n-1})} \right| \\
&= \left| \frac{(-1)^n p_1 q_0 - p_0 q_1}{q_n (a'_{n+1} q_n + q_{n-1})} \right| = \frac{1}{q_n (a'_{n+1} q_n + q_{n-1})} \\
&< \frac{1}{q_n q_{n+1}}
\end{aligned}
$$

■

Corollary 2.3 Furthermore, for the convergent term $\frac{p_n}{q_n} (n \geq 0)$ of x, we have

$$
\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}
$$

3. The sequence $\{[a_0, a_1, \ldots, a_n]\}$ is a Cauchy sequence. (Exercise)

In next lecture, we will apply Liouville Theorem to construct the first transcdental number and discuss the completeness of real numbers.

btw, π is apprixmable by rational number of order 42.

2.2. Friday

2.2.1. Set Analysis

This lecture will discuss different kinds of sets. Now recall our common sense:

Definition 2.4 [Interval]

- Open interval:
$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$$
- Closed interval:
$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$
- Half open intervals:
$$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$
$$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

Definition 2.5 [Open sets] A set $\boldsymbol{A}$ is open if $\forall x \in \boldsymbol{A}$, there exists $(a,b) \subseteq \boldsymbol{A}$ such that $x \in (a,b)$.

Theorem 2.2
1. An open set in $\mathbb{R}$ is a **disjoint** union of finitely many or countably many open intervals.
2. The union of any collection of open sets is open.
3. The intersection of **finitely** many open sets is open.

The proof is omitted, check Rudin's book for reference.

R Note that the intersection of **countably** many open sets may be open.

$$\bigcup_{n=1}^{\infty}\left(-\frac{1}{n},1+\frac{1}{n}\right) = [0,1]$$

Definition 2.6 [Neighborhood] A **neighborhood** N of a point $a \in \mathbb{R}$ is an open set containing a.

Definition 2.7 [Limit Point] x is a **limit point** of the set $\boldsymbol{A}$ if for any neighborhood N of x, N contains a point $a \in A$ such that $a \neq x$.

Definition 2.8 [Closed Set] A set $\boldsymbol{A}$ is **closed** if $\boldsymbol{A}$ contains all of its limit points.

Proposition 2.2 $\boldsymbol{A}$ is **closed** of and only if $\mathbb{R} \setminus \boldsymbol{A}$ is open.

2.2.2. Set Analysis Meets Sequence

Definition 2.9 [Limit Point of sequence] Given a sequence $\{a_n\}$, i.e.,

$$a_1, a_2, a_3, \ldots,$$

a point x is said to be the **limit point** of $\{a_n\}$ if there exists a subsequence $\{x_{n_1}, x_{n_2}, \ldots\}$ converging to x.

Does there exist a sequence of rational numbers such that every irrational number is a limit point? Yes, and we use an example as illustration.

Example 2.3 $\{q_1, q_2, \ldots\}$ is a sequence of all rational numbers. For example, to construct a subsequence with limit $\sqrt{2}$, we pick:

$$\begin{aligned}
&q_{m_1} \in (\sqrt{2}-1, \sqrt{2}+1) \setminus (\sqrt{2}-\frac{1}{2}, \sqrt{2}+\frac{1}{2}) \\
&q_{m_2} \in (\sqrt{2}-\frac{1}{2}, \sqrt{2}+\frac{1}{2}) \setminus (\sqrt{2}-\frac{1}{3}, \sqrt{2}+\frac{1}{3}) \\
&\cdots \\
&q_{m_k} \in (\sqrt{2}-\frac{1}{k}, \sqrt{2}+\frac{1}{k}) \setminus (\sqrt{2}-\frac{1}{k+1}, \sqrt{2}+\frac{1}{k+1})
\end{aligned}$$

The same argument works for all irrational numbers, also for all rational numbers. ■

2.2.3. Completeness of Real Numbers

Now we use Cauchy sequence to construct the completeness of real numbers. First let's give a proof of three important theorems. Note that the proof and applications of these theorems are mandatory.

Theorem 2.3 — Bolzano-Weierstrass. Every bounded sequence has a convergent subsequence.

Theorem 2.4 — Cantor's Nested Interval Lemma. A sequence of nested closed bounded intervals $I_1 \supseteq I_2 \supseteq \cdots$ has a non-empty intersection, i.e., $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$.

Theorem 2.5 — Heine-Borel. Any open cover $\{\mathcal{U}\}$ of a bounded closed set $\boldsymbol{E}$ consists of a finite sub-cover, i.e, $\boldsymbol{E} \subseteq$ the union of $\{\mathcal{U}\}$.

Proof for Bolzano-Weierstrass Theorem.

- Suppose $\{a_1, a_2, \dots\}$ is a bounded sequence, w.l.o.g., $\{a_1, a_2, \dots\} \subseteq [-M, M]$. We pick $a_{n_1} = a_1$.
- w.l.o.g., assume that $[0, M] \bigcap \{a_1, a_2, \dots\}$ is infinite (otherwise $[-M, 0] \bigcap \{a_1, a_2, \dots\}$ is infinite), then we pick $a_{n_2} \neq a_{n_1}$ such that $a_{n_2} \in [0, M]$.
- w.l.o.g., assume that $[0, \frac{M}{2}] \bigcap \{a_1, a_2, \dots\}$ is infinite, then we pick $a_{n_3} \neq a_{n_1}, a_{n_2}$ such that $a_{n_3} \in [0, \frac{M}{2}]$.

In this case, $\{a_{n_1}, a_{n_2}, \dots\}$ is Cauchy (by showing $|a_{n_k} - a_{n_l}| < \epsilon$ for large k, l), hence converges. ■

Proof for Cantor's Nested Interval Lemma.

1. Pick $a_k \in I_k$ for $k = 1, 2, \dots$, thus the sequence $\{a_1, \dots, a_k, \dots\}$ is bounded. By Theorem (2.3), there exists a convergent sub-sequence $\{a_{k_l}\}$ (with limit a). It suffices to show $a \in \bigcup_{m=1}^{\infty} I_k$.

2. For fiexed m, there exists index j such that $a_{k_l} \in I_m$ for all $l \geq m$. Since I_m is closed, it must contain a_{k_l}'s limit point, i.e., $a \in I_m$.
3. Our choice is arbitrary m and hence a belongs to the intersection of all nested closed intervals. The proof is complete.

■

Before the proof of third theorem, let's have a review for open cover definitions:

> **Definition 2.10** [Open Cover] Let $\boldsymbol{E}$ be a subset of a metric space X. An open cover $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ of $\boldsymbol{E}$ is a collection of open sets in X whose union contains $\boldsymbol{E}$, i.e., $\boldsymbol{E} \subseteq \bigcup_{\alpha \in A} \mathcal{U}_\alpha$. A finite **subcover** of $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ is a **finite** sub-collection of $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ whose union still contains $\boldsymbol{E}$.

For example, consider $\boldsymbol{E} := [\frac{1}{2}, 1)$ in metric space $\mathbb{R}$. Then the collection

$$\{I_n\}_{n=3}^{\infty}, \qquad \text{where } I_n := (\frac{1}{n}, 1 - \frac{1}{n})$$

is a open cover of $\boldsymbol{E}$. Note that the finite subcover may not necessarily exist. In this example, the finite subcover of $\{I_n\}_{n=3}^{\infty}$ does not exist.

Proof for Heine-Borel Theorem.

Suppose $\boldsymbol{E} := [0, M]$ is a bounded closed interval with an open cover $\{\mathcal{U}\}$. The trick of this proof is to construct a sequence of nested closed bounded intervals.

- **Base case** We choose $I_1 = \boldsymbol{E} = [0, M]$
- **Inductive step** For example, Assume that $\boldsymbol{E}$ cannot be covered by finitely many open sets from $\{\mathcal{U}\}$, then at least one sub-interval $[0, \frac{M}{2}]$ or $[\frac{M}{2}]$ cannot be covered. Let I_2 be one of these sub-intervals that cannot be covered by finitely many elements of $\{\mathcal{U}\}$.

Repeating this process, we attain a nested bouned closed intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq$, which implies $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$ (suppose $a \in \bigcap_{k=1}^{\infty}$), and $|I_k| = \frac{M}{2^k} \to 0$.

Note that $a \in \boldsymbol{E}$ implies that there exists an open set ξ in $\{\mathcal{U}\}$ such that $a \in \xi$. Thus $(a-\epsilon, a+\epsilon) \in \xi$ for small ϵ. Note that there exists sufficiently large k such that $\frac{M}{2^k} < 2\epsilon$, and $a \in I_k$, which implies $I_k \subseteq \xi$, which is a contradiction. ■

These theorems have simple applications:

Proposition 2.3 Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with the series convergent for $|x| < 1$. If for $\forall x \in [0,1)$, there exists $n := n(x)$ such that $\sum_{k=n}^{\infty} a_k x^k = 0$, then f is a polynomial (that is independent from x, i.e., n does not depend on x.)

In next lecture we will continue to study the completeness of real numbers and will speed up.

Chapter 3

Week3

3.1. Tuesday

3.1.1. Application of Heine-Borel Theorem

Theorem 3.1 Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ which converges in $|x| < 1$. If for every $x \in [0,1)$, there exists $n(= n(x))$ such that $\sum_{n+1}^{\infty} a_k x^k = 0$, then f is a polynomial, i.e., n does not depend on x.

The idea is to construct a sequence of points $\{x_n\}$ satisfying $f(x_k) = a_0 + \cdots + a_m x_k^m$, i.e., infinite points coincide $f(x)$ with a polynomial, which implies f is a polynomial.

Proof. Construct $E_N := \{x \in [0, \frac{1}{2}] \mid \sum_{k=N+1}^{\infty} a_k x^k = 0\}$. It follows that

$$[0, \frac{1}{2}] = \bigcup_{N=1}^{\infty} E_N,$$

which implies that at least one E_N is uncountable, say, E_m is uncountable. In particular, E_m is infinite

By Bolzano-Weierstrass Theorem, there exists a sequence $\{x_k\} \subset E_m$ with limit x_0 in E_m as E_m is closed. Hence, $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ holds for the sequence $\{x_m\}$. Intuitively we conclude the power series and the analytics function coincide each other for every point $x \in (-1,1)$.

$$f(x) \equiv a_0 + a_1 x + \cdots + a_m x^m$$

∎

However, the proof above does not show why a sequence coincide $f(x)$ with a polynomial could imply f is a polynomial for every point. We summarize this induction as the proposition(3.1) and give a proof below. Before that we formulate what we want to prove precisely:

> Let f be analytic, i.e., $f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$ on $(-1,1)$; and $f(x_k) = \sum_{i=1}^m a_i x_k^i$ for all $k \geq 1$, where $\{x_k\}$ is a sequence with limit x_0. Then $f(x) = \sum_{i=1}^m a_i x^i$ on $(-1,1)$.

To show this statement, we construct

$$g(x) = f(x) - \sum_{i=1}^m a_i x^i \implies g(x_k) = 0, \forall k \geq 1$$

It suffices to show $g \equiv 0$ on $(-1,1)$. Moreover, if we construct $y_k := x_k - x_0$, and set $f(x) = a_0 + a_1(x - x_0) + \cdots$, then it suffices to prove the proposition given below:

Proposition 3.1 Let g be analytic, i.e., $g(x) = b_0 + b_1 x + \cdots + b_n x^n + \cdots$ on $(-1,1)$; and $g(x_k) = 0$ for all $k \geq 1$, where $\{x_k\} \to 0$. Then $g \equiv 0$ on $(-1,1)$ (i.e., $b_0 = b_1 = \cdots = 0$)

Proof.

- Note that $g(0) = 0$ due to continuity property. Also, $g(0) = b_0 = 0$, which follows that

$$g(x) = x(b_1 + b_2 x + \cdots + b_n x^{n-1} + \cdots \tag{3.1}$$

- Substituting x with x_k in Eq.(3.1), we derive

$$0 = g(x_k) = x_k(b_1 + b_2 x_k + \cdots + b_n x_k^{n-1} + \cdots \tag{3.2}$$

  Taking limit both sides for (3.2), we derive $b_1 = 0$.

- By applying the same trick, we conclude $b_0 = b_1 = \cdots = 0$ (the rigorous proof requires induction).

∎

Now we talk about some advanced topics in Analysis.

3.1.2. Set Structure Analysis

Definition 3.1 [Nowhere Dense] A set $\boldsymbol{B}$ is said to be **nowhere dense** if its closure $\overline{B}$ contains no non-empty open set. ■

For example,

$$B = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, ..\} \implies \overline{B} = B \bigcup \{0\},$$

which contains no non-empty open set.

Definition 3.2 [1st category] A set of $\boldsymbol{B}$ is said to be of 1st category if it can be written as the **union** of **finitely** many or **countably** many **nowhere** dense sets. ■

Definition 3.3 [2rd category] A set is said to be of 2rd category if it is **not** of 1st category ■

Theorem 3.2 — **Baire-Category Theorem.**
- $\mathbb{R}$ is of 2rd category, i.e.,
- $\mathbb{R}$ cannot be written as the union of countably many nowhere dense sets, i.e.,
- if $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$, then at least one A_n whose closure contains a non-empty open set.

Proof.
- Assume $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$ such that all A_n's are nowhere dense. It follows that

$$\mathbb{R} \setminus \overline{A_1} \text{ is open,}$$

since $\overline{A_1}$ is closed and its complement is open.

- We construct an open set N_1 such that $\overline{N_1} \subseteq \mathbb{R} \setminus \overline{A_1}$. (e.g., there exists ε and $x \in \mathbb{R} \setminus \overline{A_1}$ such that $N_1 := B(x, \varepsilon) \subseteq \overline{N_1} \subseteq \mathbb{R} \setminus \overline{A_1}$.)

- Since A_2 is nowhere dense, we imply $\overline{A_2}$ does not contain N_1, i.e., $N_1 \setminus \overline{A_2}$ is open.

- By applying similar trick, we obtain a sequence of nested sets

$$\overline{N_1} \supseteq N_1 \supset \overline{N_2} \supset N_2 \cdots$$

The cantor's theorem implies that $\bigcap_{k=1}^{\infty} \overline{N_k} \neq \emptyset$.

- On the other hand, $\bigcap_{k=1}^{\infty} \overline{N_k} \subseteq \mathbb{R} \setminus \bigcup_{n=1}^{m} A_n$ for any finite m.
- Therefore, $\emptyset \neq \bigcup_{k=1}^{\infty} \overline{N_k} \subseteq \mathbb{R} \setminus \bigcup_{n=1}^{\infty} A_n = \emptyset$, which is a contradiction.

■

R $\mathbb{R}$ is of 2nd category, i.e., if $\mathbb{R} = \bigcup_{n=1}^{\infty} A_m$, then at least A_n whose closure contains a **non-empty** open sets; The theorem also holds if we replace $\mathbb{R}$ by a **complete** metric space (essentially the same proof).

Most proof for $\mathbb{R}$ can be generalized into metric space, the proof for which is essentially the same. Now let's introduce the metric space informally.

Metric Space. A metric space is an ordered pair (M, d), where M is a set and d is a metric on M, i.e., d is a distance function defined for two points on M. Here we list several examples:

The Real Line. For $\mathbb{R}$, $d(x,y) = |x-y|$. Note that $(\mathbb{Q}, d)$ and $(\mathbb{R} \setminus \mathbb{Q}, d)$ are also metric spaces, but not complete.

n-Cell Real Space. $\mathbb{R}^n$, with $d(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}|$ is a metric space.

Bounded Sequences. The set of all bounded sequences on $\mathbb{R}$ is a metric space, with d defined as:

$$d(\{x_n\}, \{y_n\}) = \sup\{|x_i - y_i| \mid i = 1, 2, \dots\}$$

Bounded Functions. Similarly, the set of all bounded continuous functions on $\mathbb{R}$ (different domains), with

$$d_1(f,g) = \sup\{|f(x) - g(x)| \mid x \in \mathbb{R}\},$$

or

$$d_2(f,g) = (\int_0^1 |f(x) - g(x)|^2 \, dx)^{1.2}$$

is a metric space. Note that $(\xi[0,1], d_1)$ is complete, and $(\xi[0,1], d_2)$ is not complete. (exercise)

R

Different distance definition corresponds to different metric spaces.

Recall that a metric space is complete if all Cauchy sequence of which converge.

3.1.3. Reviewing

Definition 3.4 [Sequence] A sequence is defined as a kind of function $f : \mathbb{N} \to \mathbb{R}$, denoted as $\{f(0), f(1), \dots\}$. Conventionally we denote it as $x_1, x_2, \dots$ ■

Definition 3.5 [Limit] A number α is the limit of $\{x_1, x_2, \dots\}$ if $\forall \epsilon > 0$, there $\exists N = N(\epsilon)$ such that $|x_k - \alpha| < \epsilon$ for $\forall k \geq N$, denoted by $\alpha_n \to \alpha$ ■

Definition 3.6 [liminf & limsup]

$$\liminf_{k\to\infty} x_k := \lim_{n\to\infty} \inf_{k\geq n} x_k$$

which is the smallest limit point of the sequence

$$\limsup_{k\to\infty} x_k := \lim_{n\to\infty} \sup_{k\geq n} x_k$$

which is the largest limit point of the sequence. ■

A sequence always has liminf and limsup.

Definition 3.7 [Partial Sum] Given the sequence $\{a_n\}$, its n-th partial sum are defined as:

$$s_n = a_1 + \cdots + a_n,$$

the series $\sum_i a_i$ is defined as the limit of the partial sum, ▪

Next lecture we will show that most continuous function is nowhere differentiable, by applying the Baire Category Theorem on $(\xi[0,1], d_1)$