A FIRST COURSE

IN

ANALYSIS

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MAT2006 Notebook

Lecturer

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Acknowledgments

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Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Wednesday

Recommended Reading.

1. (Springer-Lehrbuch) V. A. Zorich, J. Schüle-Analysis I-Springer (2006).
2. (The Carus mathematical monographs 13) Ralph P. Boas, Harold P. Boas, A primer of real functions-Mathematical Association of America (1996).
3. (International series in pure and applied mathematics) Walter Rudin, Principles of Mathematical Analysis-McGraw-Hill (1976).
4. Terence Tao, Analysis I,II-Hindustan Book Agency (2006)
5. (Cornerstones) Anthony W. Knapp, Basic real analysis-Birkhäuser (2005)

1.1.1. Introduction to Set

For a set $\mathcal{A} = \{1,2,3\}$, we have $2^3 = 8$ subsets of $\mathcal{A}$. We are interested to study the collection of sets.

Definition 1.1 [Collection of Subsets] Given a set $\mathcal{A}$, the the collection of subsets of $\mathcal{A}$ is denoted as $2^{\mathcal{A}}$. ■

We use Candinal to describe the order of number of elements in a set.

Definition 1.2 Given two sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ and $\mathcal{B}$ are said to be **equivalent** (or have the same**candinal**) if there exists a 1-1 onto mapping from $\mathcal{A}$ to $\mathcal{B}$. ■

Definition 1.3 [Countability] The set $\mathcal{A}$ is said to be **countable** if $\mathcal{A} \sim \mathbb{N} = \{1,2,3,\dots\}$; an infinite set $\mathcal{A}$ is **uncountable** if it is not equivalent to $\mathbb{N}$.

R Note that the set of integers, i.e., $\mathbb{Z} = \{\cdots,-2,-1,0,1,2,\cdots\}$ is also countable; the set of rational numbers, i.e., $\mathbb{Q} = \{p/q \mid p,q \in \mathbb{Z}, q \neq 0\}$ is countable.

We skip the process to define real numbers.

Proposition 1.1 The set of real numbers $\mathbb{R}$ is **uncountable**.

For example, $\sqrt{2} \notin \mathbb{Q}$. Some inrational numbers are the roots of some polynomials, such a number is called **algebraic** numbers. However, some inrational numbers are not, such a number is called **transcendental**. For example, π is **not** algebraic. We will show that the collection of algebraic numbers are countable in the future.

There are two steps for the proof for proposition(1.1):

Proof. 1. $2^{\mathbb{N}}$ is **uncountable**:

Assume $2^{\mathbb{N}}$ is countable, i.e.,

$$2^{\mathbb{N}} = \{A_1, A_2, \dots, A_k, \dots\}$$

Define $B := \{k \in \mathbb{N} \mid k \notin A_k\}$, it is a collection of subscripts such that the subscript k does not belong to the corresponding subsets A_k.

It follows that $B \in 2^{\mathbb{N}} \implies B = A_n$ for some n. Then it follows two cases:

- If $n \in A_n$, then $n \notin B = A_n$, which is a contradiction
- Otherwise, $n \in B = A_n$, which is also a contradiction.

The proof for the claim $2^{\mathbb{N}}$ is **uncountable** is complete.

2. $\mathbb{R} \sim 2^{\mathbb{N}}$:

Firstly we have $\mathbb{R} \sim (0,1)$. This can be shown by constructing a one-to-one mapping:

$$f : \mathbb{R} \mapsto (0,1) \quad f(x) = \frac{1}{\pi}\arctan x + \frac{1}{2}, \forall x \in \mathbb{R}$$

Secondly, we show that $2^{\mathbb{N}} \sim (0,1)$**.** We construct a mapping f such that

$$f: 2^{\mathbb{N}} \mapsto (0,1),$$

where for $\forall A \in 2^{\mathbb{N}}$,

$$f(A) = 0.a_1a_2a_3\ldots, \quad a_j = \begin{cases} 2, & \text{if } j \in A \\ 4, & \text{if } j \notin A \end{cases}$$

This function is only 1-1 mapping but not onto mapping.

Reversely, we construct a 1-1 mapping from $(0,1)$ to $2^{\mathbb{N}}$. We construct a mapping g such that

$$g: (0,1) \mapsto 2^{\mathbb{N}}$$

where for any real number from $(0,1)$, we can write it into binary expansion:

$$\text{binary form:} \quad 0.a_1a_2\ldots \quad \text{where } a_j = 0 \text{ or } 1.$$

Hence, we construct $g(0.a_1a_2\ldots) = \{j \in \mathbb{N} \mid a_j = 0\} \subseteq \mathbb{N}$, which implies $g(\cdot) \in 2^{\mathbb{N}}$.

R Our intuition is that two 1-1 mappings in the reverse direction will lead to a 1-1 **onto** mapping. If this is true, then we complete the proof. This intuition is the **Schroder-Berstein Theorem**.

∎

Defining Binary Form. However, during this proof, we must be careful about the binary form of a real number from $(0,1)$. Now we give a clear definition of Binary Form:

For a real number a, to construct its binary form, we define

$$a_1 = \begin{cases} 0, & \text{if } a \in (0, \frac{1}{2}) \\ 1, & \text{if } a \in [\frac{1}{2}, 1). \end{cases}$$

After having chosen $a_1, a_2, \ldots, a_{j-1}$, we define a_j to be the largest integer such that

$$\frac{1}{2}a_1 + \frac{1}{2^2}a_2 + \cdots + \frac{a_j}{2^j} \le a$$

Then the binary form of a is $a := 0.a_1a_2\ldots$.

Theorem 1.1 — Schroder-Berstein Theorem. If $f : A \mapsto B$ and $g : B \mapsto A$ are both 1-1 mapping, then there exists a 1-1 onto mapping from A to B, i.e., card $\#A$ equals to card $\#B$.

Exercise: Show that $(0,1]$ and $[0,1)$ have 1-1 onto mapping without applying Schroder-Berstein Theorem.

The next lecture we will take a deeper study into the proof of Schroder-Berstein Theorem and the real number.

1.2. Quiz

1. Show that the sequence $\{x_n\}$ is convergent, where

$$x_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \cdots + \frac{\sin n}{2^n}.$$

2. Compute the following limits:

   (a)
   $$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/(1-\cos x)}$$

   (b)
   $$\lim_{n\to\infty}\int_0^1 \frac{x^n}{1+\sqrt{x}}\,\mathrm{d}x$$

3. Justify that the natural number e is irrational, where

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

4. Every rational x can be written in the form $x = p/q$, where $q > 0$ and p and q are integers without any common divisors. When $x = 0$, we take $q = 1$. Consider the function f defined on $\mathbb{R}^1$ by

$$f(x) = \begin{cases} 0, & x \text{ is irrational} \\ \dfrac{1}{q}, & x = \frac{p}{q}. \end{cases}$$

   Find:

   (a) all continuities of $f(x)$;
   (b) all discontinuities of $f(x)$

   and prove your results.

1.3. Friday

Before we give a proof of Schroder-Berstein theorem, we'd better review the definitions for one-to-one mapping and onto mapping.

Definition 1.4 [One-to-One/Onto Mapping] If $f : A \mapsto B$, then

- f is said to be **onto** mapping if

$$\forall b \in B, \exists a \in A \text{ s.t. } f(a) = b;$$

- f is said to be **one-to-one** mapping if

$$\forall a, b, \in A, f(a) = f(b) \implies a = b.$$

The Fig.(1.1) shows the examples of one-to-one/onto mappings.

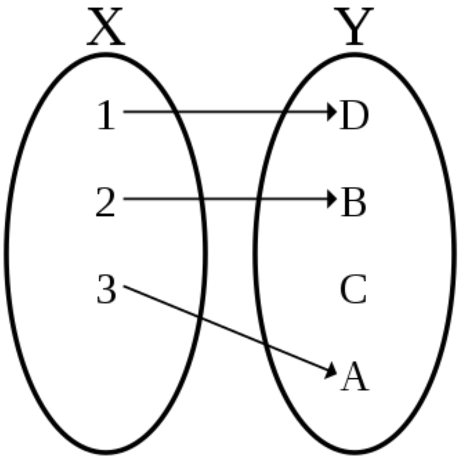
1.3.1. Proof of Schroder-Berstein Theorem

Before the proof, note that in this lecture we abuse the notation fg to denote the composite function $f \circ g$, but in the future fg will refer to other meanings.

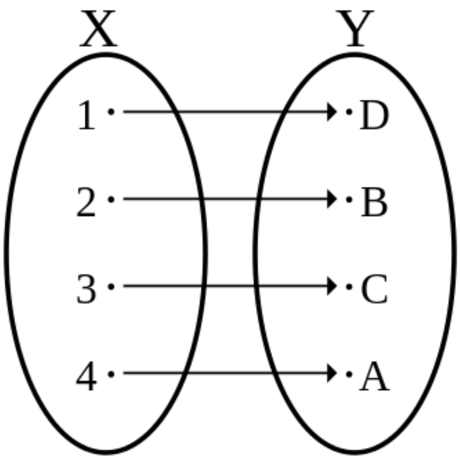
Intuition from Fig.(1.2). The proof for this theorem is constructive. Firstly Fig.(1.2) gives us the intuition of the proof for this theorem. Let $f : A \mapsto B$ and $g : B \mapsto A$ be two one-to-one mappings, and D, C are the image from A, B respectively. Note that

> if the set $B \setminus D$ is empty, then $D = B = f(A)$ with f being the one-to-one mapping, which implies f is one-to-one onto mapping. In this case the proof is complete.

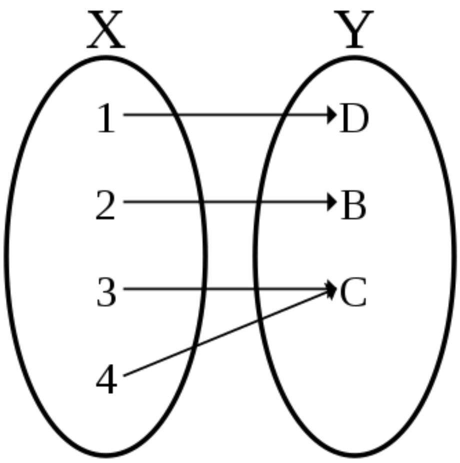
Hence it suffices to consider the case $B \setminus D$ is non-empty. Thus $B \setminus D$ is the "**trouble-maker**". To construct a one-to-one onto mapping from A, we should study the subset $g(B \setminus D)$ of A (which can also be viewed as a *trouble-maker*). Moreove, we should study



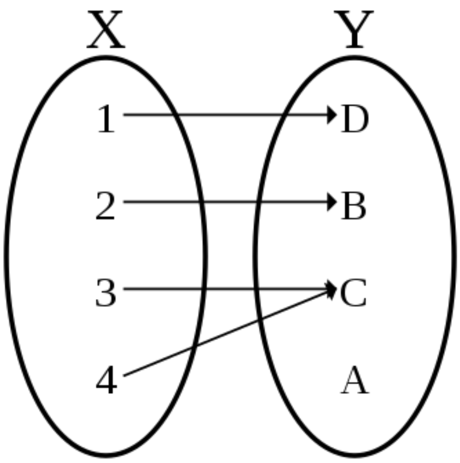
(a) A one-to-one but not onto mapping



(b) A one-to-one onto mapping



(c) A onto but not one-to-one mapping



(d) Neither a one-to-one nor onto mapping

Figure 1.1: Illustrations of one-to-one/onto mappings

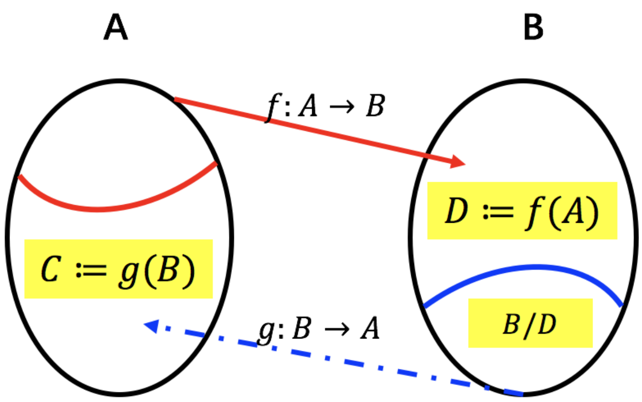


Figure 1.2: Illustration of Schroder-Berstein Theorem

the subset $gf[g(B \setminus D)]$ (which is also a *trouble-maker*)... so on and so forth. Therefore, we should study the *union of these trouble makers*, i.e., we define

$$A_1 := g(B \setminus D), \quad A_2 := gf(A_1), \quad \cdots, \quad A_n := gf(A_{n-1}),$$

Then we study the union of infinite sets

$$S := A_1 \bigcup A_2 \bigcup \cdots \bigcup A_n \bigcup \cdots$$

Define

$$F(a) = \begin{cases} f(a), & a \in A \setminus S \\ g^{-1}(a), & a \in S \end{cases}$$

We claim that $F : A \mapsto B$ is one-to-one onto mapping.

F is onto mapping. Given any element $b \in B$, it follows two cases:

1. $g(b) \in S$. It implies $F(g(b)) = g^{-1}(g(b)) = b$.
2. $g(b) \notin S$. It implies $b \in D$, since otherwise $b \in B \setminus D \implies g(b) \in g(B \setminus D) \subseteq S$, which is a contradiction. $b \in D$ implies that $\exists a \in A$ s.t. $f(a) = b$.

   Then we study the relationship between $gf(S)$ and S. Verify by yourself that

   $$S = g(B \setminus D) \bigcup gf(S)$$

   With this relationship, we claim $a \notin S$, since otherwise $a \in S \implies gf(a) \in S$, but $gf(a) = g(b) \notin S$, which is a contradiction.

   Hence, $F(a) = f(a) = b$.

Hence, for any element $b \in B$, we can find a element from A such that the mapping for which is equal to b, i.e., F is onto mapping.

F is one-to-one mapping. Assume not, verify by yourself that the only possibility is that $\exists a_1 \in A \setminus S$ and $a_2 \in S$ such that $F(a_1) = F(a_2)$, i.e., $f(a_1) = g^{-1}(a_2)$, which follows

$$gf(a_1) = a_2 \in S = A_1 \bigcup A_2 \bigcup \cdots \tag{1.1}$$

We claim that Eq.(1.1) is false. Note that $gf(a_1) \notin A_1 := g(B \setminus D)$, since otherwise $f(a_1) \in B \setminus D$, which is a contradiction; note that $gf(a_1) \notin A_2$, since otherwise $gf(a_1) \in gfg(B \setminus D) \implies a_1 \in g(B \setminus D) = A_1 \subseteq S$, which is a contradiction.

Applying the similar trick, we wil show that $gf(a_1) \notin A_k$ for $k \geq 1$. Hence, Eq.(1.1) is false, the proof is complete.

■ **Example 1.1** Given two sets $A := (0,1]$ and $B := [0,1)$. Now we apply the idea in the proof above to construct a one-to-one onto mapping from A to B:

- Firstly we construct two one-to-one mappings:

$$\begin{aligned} f : A &\mapsto B & g : B &\mapsto A \\ f(x) &= \frac{1}{2}x & g(x) &= x \end{aligned}$$

- It follows that $B \setminus D = (\frac{1}{2}, 1)$, $gf(B \setminus D) = (\frac{1}{4}, 1)$, so on and so forth.

$$S = (\frac{1}{2}, 1) \bigcup (\frac{1}{4}, 1) \bigcup \cdots$$

- Hence, the one-to-one onto mapping we construct is

$$F(x) = \begin{cases} \frac{1}{2}x, & x \in A \setminus S \\ x, & x \in S \end{cases}$$

- Conversely, to construct the inverse mapping, we define

$$f(x) = x \quad g(x) = \tfrac{1}{2}x$$

- It follows that $D = (0,1)$, $B \setminus D = \{1\}$. Then

$$S = \left\{\frac{1}{2}\right\} \bigcup \cdots = \left\{\frac{1}{2}, \frac{1}{4}, \cdots\right\}$$

- Hence, the function we construct for inverse mapping is

$$F(x) = \begin{cases} x, & x \neq \dfrac{1}{2^m} \\ 2x, & x = \dfrac{1}{2^m} \end{cases} \qquad (m = 1,2,3,\dots)$$

1.3.2. Connectedness of Real Numbers

There are two approaches to construct real numbers. Let's take $\sqrt{2}$ as an example.

1. The first way is to use **Dedekind Cut**, i.e., every non-empty subset has a least upper bound. Therefore, $\sqrt{2}$ is actually the least upper bound of a non-empty subset
$$\{x \in \mathbb{Q} \mid x^2 < 2\}.$$
2. Another way is to use **Cauchy Sequence**, i.e., every Cauchy sequence is convergent. Therefore, $\sqrt{2}$ is actually the limit of the given sequence of decimal approximations below:
$$\{1, 1.4, 1.41, 1.414, 1.4142, \dots\}$$

We will use the second approach to define real numbers. Every real number r essentially represents a collection of cauchy sequences with limit r, i.e.,

$$r \in \mathbb{R} \implies \left\{ \{x_n\}_{n=1}^{\infty} \middle| \lim_{n\to\infty} x_n = r \right\}$$

Let's give a formal definition for cauchy sequence and a formal definition for real number.

Definition 1.5 [Cauchy Sequence]

- Any sequence of rational numbers $\{x_1, x_2, \cdots\}$ is said to be a **cauchy sequence** if for every $\epsilon > 0$, $\exists N$ s.t. $|x_n - x_m| < \epsilon$, $\forall m, n \geq N$

- Two cauchy sequences $\{x_1, x_2, \dots\}$ and $\{y_1, y_2, \dots\}$ are said to be **equivalent** if for every $\epsilon > 0$, there $\exists N$ s.t. $|x_n - y_n| < \epsilon$ for $\forall n \geq N$.
- A real number is a **collection** of **equivalent** cauchy sequences. It can be represented by a cauchy sequence:

$$x \in \mathbb{R} \sim \{x_1, x_2, \dots, x_n, \dots\},$$

where x_j is a rational number.

R Let $\xi_{\mathbb{Q}}$ denote a collection of any cauchy sequences. Then once we have equivalence relation, the whole collection $\xi_{\mathbb{Q}}$ is partitioned into several disjoint subsets, i.e., equivalence classes. Hence, the real number space $\mathbb{R}$ are the equivalence classes of $\xi_{\mathbb{Q}}$.

The real numbers are well-defined, i.e., given two real numbers $x \sim \{x_1, x_2, \dots\}$ $y \sim \{y_1, y_2, \dots\}$, we can define add and multiplication operator.

$$x + y \sim \{x_1 + y_1, x_2 + y_2, \dots\}$$
$$x \cdot y \sim \{x_1 \cdot y_1, x_2 \cdot y_2, \dots\}$$

We will show how to define $x > 0$ in next lecture, this construction essentially leads to the lemma below:

Proposition 1.2 $\mathbb{Q}$ are dense in $\mathbb{R}$.

In the next lecture we will also show the completeness of $\mathbb{R}$:

Theorem 1.2 $\mathbb{R}$ is complete, i.e., every cauchy sequence of real numbers converges.

Recommended Reading:

Prof. Katrin Wehrheim, MIT Open Course, Fall 2010, Analysis I Course Notes, Online avaiable:

https://ocw.mit.edu/courses/mathematics

/18-100b-analysis-i-fall-2010/readings-notes/MIT18_100BF10_Const_of_R.pdf