A FIRST COURSE

IN

NUMERICAL ANALYSIS

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MAT4001 Notebook

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Contents

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CUHK(SZ)

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Wednesday

1.1.1. Introduction to Imaginary System

Definition 1.1 [Complex Number] A complex number z is a pair of real numbers:

$$z = (x, y),$$

where x is the **real** part and y is the **imaginary part** of z, denoted as

$$\text{Re}z = x \quad \text{Im}z = y$$

R Note that the complex multiplication does not correspond to any standard vector operation. However, $(\mathbb{C}, +)$ and $(\mathbb{C} \setminus \{0\}, \cdot)$ forms a field:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

$$z_1 + z_2 = z_2 + z_1$$

$$z + 0 = 0 + z = z$$

$$z + (-z) = (-z) + z = 0$$

There is no other Euclidean space that can form a field.

Proposition 1.1 $zz' = 0$ if and only if $z = 0$ or $z' = 0$.

Proof. Rewrite the product as a linear system

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and discuss the determinant of the coefficient matrix. ■

Solving quadratic equation with one unknown. We can apply the imaginary number to solve the quadratic equations. For example, to solve $z^2 - 2z + 2 = 0$, the first method is to substitute z with $x + iy$; the second method is to simplify it into standard form to solve it.

Definition 1.2 If $z \neq 0$, then z^{-1} is the complex number satisfying $z \cdot z^{-1} = 1$.

Suppose $z = (x, y)$ and $z^{-1} = (u, v)$. After simplification, we derive

$$\begin{cases} xu - yv = 1 \\ xv + yu = 0 \end{cases} \implies \begin{cases} u = \dfrac{x}{x^2 + y^2} \\ v = \dfrac{-y}{x^2 + y^2} \end{cases}$$

Definition 1.3 [Division] The division between complex numbers is defined as:

$$\tfrac{z_1}{z_2} = z_1 \cdot z_2^{-1}, \quad \text{when } z_2 \neq 0$$

■ **Example 1.1**

$$\frac{3 - 4i}{1 + i} = (3 - 4i)\left(\frac{1}{2} - \frac{1}{2}i\right) = -\frac{1}{2} - \frac{7}{2}i$$

$$\frac{10}{(1 + i)(2 + i)(3 + i)} = \frac{10}{(1 + 3i)(3 + i)} = \frac{10}{10i} = \frac{1}{i} = -i$$

Definition 1.4 [Complex Conjugate] The complex number $x - iy$ is called the **complex conjugate** of $z = x + iy$, which is denoted by $\bar{z}$.

The following properties hold for complex conjugate:

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2. \quad \overline{\frac{z_1}{z_2}} = \frac{\bar{z}_1}{\bar{z}_2}$$

$$\text{Re}z = \frac{z + \bar{z}}{2}, \quad \text{Im}z = \frac{z - \bar{z}}{2i}$$

1.1.2. Algebraic and geometric properties

Definition 1.5 [Algebraic Region]

1. The complex plane: the z-plane, i.e., $\mathbb{C}$
2. Vector in $\mathbb{R}^2$: $(x, y) = x + iy = z \in \mathbb{C}$
3. Modulus of z:

$$|z| = \sqrt{x^2 + y^2} \quad \text{distance to the origin}$$

Note that

$$|z| = 0 \Longleftrightarrow z = 0, \quad |z_1 - z_2| = 0 \Longleftrightarrow z_1 = z_2$$

Definition 1.6 [Circle in plane] A circle with center z_0 and radius R is defined as follows in $\mathbb{C}$:

$$\{z \in \mathbb{C} \mid |z - z_0| = R\}$$

Proposition 1.2 Complex roots of polynomials with real coefficients appear in conjugate pairs.

Proof. Given $P(z_0) = 0$, we derive

$$P(z_0) = \overline{P(z_0)} = 0.$$

■

Note that a polynomial with real coefficients of degree 3 must have at least one real root.

Conjugate Product. Note that the conjugate product leads to the square of modulus:

$$z \cdot \bar{z} = |z|^2 \iff (x + iy)(x - iy) = x^2 + y^2$$

Such a property can be used to simplify quotient of two complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2} = \frac{x_1 x_2 + y_1 y_2 + (y_1 x_2 - x_1 y_2)i}{x_2^2 + y_2^2}$$

■ **Example 1.2**

$$\frac{-1 + 3i}{2 - i} = \frac{(-1 + 3i)(2 + i)}{(2 - i)(2 + i)} = \frac{-5 + 5i}{5} = -1 + i$$

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

■

We can use conjugate to show the **triangle inequality**:

Proposition 1.3 — Triangle Inequality. $|z_1 + z_2| \leq |z_1| + |z_2|$.

Proof.

$$\begin{aligned}|z_1+z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)}\\ &= |z_1|^2+|z_2|^2+z_1\bar{z}_2+\overline{z_1\bar{z}_2}\\ &= |z_1|^2+|z_2|^2+2\text{Re}(z_1\bar{z}_2)\\ &\le |z_1|^2+|z_2|^2+2|z_1\bar{z}_2|\\ &= |z_1|^2+|z_2|^2+2|z_1z_2| = (|z_1|+|z_2|)^2.\end{aligned}$$

■

Corollary 1.1

1. $||z_1|-|z_2|| \le |z_1 \pm z_2|$.
2. If $|z| \le 1$, then $|z^2+z+1| \le 3$

Proof.

1. Note that

$$|z_1| = |z_1 \pm z_2 \mp z_2| \le |z_1 \pm z_2| + |z_2| \implies |z_1| - |z_2| \le |z_1 \pm z_2|$$

Similarly, $|z_2| - |z_1| \le |z_1 \pm z_2|$.

2. $$|z^2+z+1| \le |z^2| + |z+1| \le |z|^2+|z|+1 \le 1+1+1 = 3.$$

■

Proposition 1.4 — **Cauchy-Schwarz inequality.** If $z_1,\ldots,z_n$ and $w_1,\ldots,w_n$ are complex numbers, then

$$\left[\sum_{k=1}^{n} z_k w_k\right]^2 \le \left[\sum_{k=1}^{n}|z_k|^2\right]\left[\sum_{k=1}^{n}|w_k|^2\right]$$

1.1.3. Polar and exponential forms

Definition 1.7 [Polar Form] The polar form of a nonzero complex number z is:

$$z = r(\cos\theta + i\sin\theta)$$

where (r,θ) is the polar coordinates of (x,y).

$$(r,\theta) \implies (x,y) : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$(x,y) \implies (r,\theta) : \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

Note that θ is said to be the **argument** of z, i.e., $\theta = \arg z$. The augument is not unique, i.e.,

$$z = r(\cos\theta + i\sin\theta) r(\cos(\theta+2\pi) + i\sin(\theta+2\pi))$$

■

If given an argument of z, then we form the set of arguments of z:

$$\{\theta + 2n\pi \mid n \in \mathbb{Z}\}$$

Definition 1.8 [Principal Value] The principal value of $\arg z$, denoted by $\mathrm{Arg} z$, is the unique value of $\arg z$ such that $-\pi < \arg z \leq \pi$ ■

■ **Example 1.3**

1. $\mathrm{Arg} z = \pi$ implies $z = r(\cos\pi + i\sin\pi) = -r < 0$, which is a negative real number.
2. $\mathrm{Arg} z = 0$ implies $z = r(\cos 0 + i\sin 0) = r > 0$m which is a positive real number.
3. $\mathrm{Arg} z = -\frac{\pi}{2}$ implies $z = r(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})) = -ri$
4. $\mathrm{Arg} z = \frac{\pi}{2}$ implies $z = ri$
5. Particularly, $\pm i = \cos(\pm\frac{\pi}{2}) + i\sin(\pm\frac{\pi}{2})$

■

Product in polar form. Given $z_i = r_i(\cos\theta_i + i\sin\theta_i)$ for $i = 1,2$, we can compute its product:

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\ &= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

Thus, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$.

R Note that $\mathrm{Arg}(z_1 z_2) \neq \mathrm{Arg} z_1 + \mathrm{Arg} z_2$. ($\mathrm{Arg}(z_1 z_2)$ should be restricted to be within the interval $(-\pi, \pi]$)

Inverse in Polar form. Given $z = r(\cos\theta + i\sin\theta)$, we aim to find the inverse such that $zz^{-1} = 1$. Hence, $z^{-1} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$.

If we obtain the inverse, we can compute the division $\frac{z_1}{z_2}$:

$$\frac{z_1}{z_2} = r_1(\cos\theta_1 + i\sin\theta_1)\frac{1}{r_2}(\cos(-\theta_2) + i\sin(-\theta_2)) = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

Thus, $\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$.

Euler Identity. The Euler Identity is given by:

$$e^{ix} = \cos x + i\sin x$$

The proof requires Taylor's expansion.

Exponential Form. The exponential form of z in polar form is given by:

$$z = re^{i\theta}$$

Then it is convenient to define produt, inverse, and division:

$$
\begin{aligned}
(r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2}) &= r_1 r_2 e^{i(\theta_1+\theta_2)} \\
\frac{1}{re^{i\theta}} &= \frac{1}{r} e^{i(-\theta)} \\
\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} &= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}
\end{aligned}
$$

Nonuniqueness. $z = re^{i\theta} = re^{i(\theta+2n\pi)}$

Equality. Two complex numbers are equal means that:

$$
r_1 e^{i\theta_1} = r_2 e^{i\theta_2} \Longleftrightarrow \begin{cases} r_1 = r_2 \\ \theta_1 = \theta_2 + 2k\pi, k \in \mathbb{Z} \end{cases}
$$

Circle. The circle centered at the origin with radius R can be described as:

$$
|z| = R \Longleftrightarrow z = Re^{i\theta}, \quad 0 \leq \theta < 2\pi
$$

The circle centered at z_0 with radius R can be described as:

$$
|z - z_0| = R \Longleftrightarrow z = z_0 + Re^{i\theta}, \quad 0 \leq \theta < 2\pi
$$

Neighborhoold. The ϵ-neighborhood of the point z_0 is given by:

$$
|z - z_0| < \epsilon
$$

If delete the center, it is given by:

$$
0 < |z - z_0| < \epsilon
$$

1.2. Powers and Roots

Powers. The powers of $z = re^{i\theta}$ is given by:

$$z^n = r^n e^{in\theta}$$
$$z^{-n} = r^{-n} e^{i(-n)\theta}$$

Thus we derive the **De Moiver's Formula**:

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = \cos n\theta + i\sin n\theta.$$

It is useful for computing powers tha contains complex number. For example,

$$(1+i)^n = (\sqrt{2}e^{i\frac{\pi}{4}})^n = 2^{n/2}e^{\frac{in\pi}{4}}$$

Proposition 1.5

$$\sin(n\theta) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k+1} (-1)^k \cos^{n-2k-1}\theta \sin^{2k+1}\theta,$$

where $\lfloor x \rfloor$ denotes the largest integer that not exceeds x.

Solving high order equations. The powers of complex can also be used to solve high order equations.

■ **Example 1.4** To sovle the equation $z^n = 1$, we express $z = re^{i\theta}$. It follows that

$$(re^{i\theta})^n = 1e^{i0} \implies \begin{cases} r^n = 1 \\ n\theta = 2k\pi \end{cases} \implies \begin{cases} r = 1 \\ \theta = \frac{2k\pi}{n} \end{cases}$$

Thus, the distinct n-th roots(of unity) are given by:

$$\exp(i\frac{2k\pi}{n}) = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \qquad k = 0,1,2,\dots,n-1.$$

If we denote $w_n = \exp(i\frac{2\pi}{n})$, we derive the roots:

$$1, w_n, w_n^2, \dots, w_n^{n-1}.$$

Roots of high order equations. Suppose $z_0 = r_0 e^{i\theta_0}$, we aim to sovle $z^n = z_0$:

$$r^n e^{in\theta} = r_0 e^{i\theta_0} \implies \begin{cases} r = r_0^{1/n} \\ \theta = \dfrac{\theta_0 + 2k\pi}{n} \end{cases}$$

Thus the distinct nth roots are given by:

$$r_0^{1/n} \exp(i\frac{\theta_0 + 2k\pi}{n}), \qquad k = 0, 1, 2, \dots, n-1.$$

If c is any particular n-th roots of z_0, then

$$(cw)^n = z_0 \implies c^n w^n = z_0 \implies w_n = 1.$$

Hence, the distinct n-th roots of z_0 are

$$c, cw_n, cw_n^2, \dots, cw_n^{n-1}$$

R

- There are n of the n-th roots of a complex number, all the roots are equally spaced about a circle that is centered at origin with radius $|z_0|^{1/n}$.
- Let $z_0^{1/n}$ denote the set of all n-th roots of z_0. If $\theta_0 = \text{Arg} z_0$, then

  $$c_0 = r_0^{1/n} \exp(i\frac{\theta_0}{n})$$

  is called the principal n-th root of z_0.

- The distinct n-th roots of z_0 are:

$$c_0, c_0 w_n, c_o w_n^2, \ldots, c_0 w_n^{n-1},$$

or equivalently,

$$z_0^{1/n} = r_0^{1/n} \exp(i \frac{\theta_0 + 2k\pi}{n})$$

■ **Example 1.5** For $z_0 = -8i$, we write $z_0 = 8e^{i(-\pi/2)}$. It follows that

$$z_0^{1/3} == 2\exp(i\frac{-\pi/2 + 2k\pi}{3}) = 2\exp(-\frac{\pi}{6}i), 2\exp(\frac{\pi}{2}i), 2\exp(\frac{7\pi}{6}i)$$

■

Chapter 2

Week2

2.1. Error

Definition 2.1 [Decimal floating-point number]

$$y = 0.d_1 d_2 \cdots d_k \times 10^n, \quad 1 \le d_1 \le 9, 0 \le d_2, \ldots, d_k \le 9$$

Definition 2.2 [Rounding] Adds $5 \times 10^{n-(k+1)}$ to y and then chps the result to obtain a number of form

$$fl(y) = 0.\delta_1 \cdots \delta_k \times 10^n$$

Consider a decimal number x with rounding to $\tilde{x}$ with n digits.

- If $(n+1)$th digit of x is $5, \ldots, 9$, $\tilde{x} = \hat{x} + 10^{-n}$, where $\hat{x}$ is a number with the same n digits as x and all the other digits beyond the nth are 0.

- If $(n+1)$th digit of x is $0, \ldots, 4$, then $x = \tilde{x} + \varepsilon$ with $\varepsilon < \frac{1}{2} \times 10^{-n}$.

Thus $|x - \tilde{x}| \le \frac{1}{2} * 10^{-n}$.

Definition 2.3 [erros] p^* is the approximation to p, actual error is $p - p^*$; absolute error is $|p - p^*|$; relative error is $\frac{|p-p^*|}{|p|}$; p^* is said to approximate p to k significant digits if k is the largest non-negative integer for which the relative error is no more than 5×10^{-k}.

2.1.1. Bisection

Theorem 2.1 If f is continuous in the interval $[a,b]$, and $f(a)f(b)<0$, then there exists at least one solution $x^* \in (a,b)$ such that $f(x^*)=0$.

■ **Example 2.1** [Bisection Algorithm] Input: $a,b,\ \varepsilon,\delta$. Assume $f(a)<0<f(b)$

- Set $a_0=a,\ b_0=b;\ p_0=\frac{a_0+b_0}{2}$
- If $f(p_0)>0$, set $a_{k+1}=a_k; b_{k+1}=p_k$;
- If $f(\frac{a_k+b_k}{2})<0$, set $a_{k+1}=p_k; b_{k+1}=b_k$;
- If $\frac{|p_k-p_{k-1}|}{|p_k|}<\varepsilon$, terminate and output p_k.

■

Theorem 2.2 — **Convergence Rate of bisection.**

$$|x^k-x^*|\le\frac{1}{2}(b_k-a_k)=2^{-(k+1)}(b-a)$$

Proof. The zero point x^* satisfies $x^*=\lim_{k\to\infty}a_k=\lim_{k\to\infty}b_k$ (use $f(a_k)f(b_k)<0$).

Let $x_k=\frac{a_k+b_k}{2}$, which implies $|x_k-x^*|\le\frac{1}{2}(b_k-a_k)$. ■

Theorem 2.3 — **Existence of fixed point.** Let $g\in\mathcal{C}[a,b]$ and $g\in[a,b]$ for all $x\in[a,b]$, then g has a fixed point.

Proof. Define $h=g(x)-x$ and consider $h(a)h(b)$ ■

Theorem 2.4 — **Uniqueness.** If g' exists on $[a,b]$ and $|g'(x)|\le k<1$, then g has a unique fixed point. The sequence $p_n=g(p_{m-1})$ will converge to unique fixed point p. The convergence rate is

$$|p_n-p|\le\frac{k^n}{1-k}|p_1-p_0|$$

Proof. • For two fixed points p, q, we have $|p-q| = |g'(\xi)||p-q| < |p-q|$.

- Note that $|p_n - p| \le k|p_{n-1} - p| \le k^n |p_0 - p|$

- Since $|p_{n+1} - p_n| \le k^n |p-1-p_0|$ and thus

$$|p_m - p_n| \le k^n(1 + k + k^2 + \cdots + k^{m-n-1})|p_1 - p_0|,$$

taking $m \to \infty$.

■

Ⓡ If $\phi'(x) = \phi''(x) = \cdots = \phi^{(p-1)}(x) = 0$, then

$$x_{k+1} = \phi(x_k) = \phi(x^*) + \cdots + \frac{\phi^{(p)}(\xi_k)}{p!}(x_l - x^*)^p \implies x_{k+1} - x^* = \frac{\phi^{(p)}(\xi_k)}{p!}(x_k - x^*)^p$$

Definition 2.4 [Newton's method] Consider $f(p) = 0$ with $f(p) = f(p_0) + (p-p_0)f'(p_0) + \frac{f''(\xi_p)}{2!}(p-p_0)^2$, we derive

$$p = p_0 - f(p_0)/f'(p_0)$$

stop criteria: $\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon$

Theorem 2.5 — **convergence rate of Newton's method.** There exists a $\delta > 0$ s.t. the sequence $\{p_n\}$ converges to p for any initial guess $p_0 \in [p-\delta, p+\delta]$.

Proof. Define $x_{k+1} = \phi(x_k)$ and therefore $e_{k+1} = \phi'(\xi_k)e_k$. It suffices to show $|\phi'(\xi_k)| \le \frac{1}{2}$, it is true since

$$\phi'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}, \quad \phi'(x^*) = 0.$$

■

Proof. We apply Taylor's expansion

$$f(x^*) = f(x_k) + f'(x_k)(x^* - x_k) + \frac{1}{2}f''(\xi_k)(x^* - x_k)^2$$

and therefore

$$x_{k+1} - x^* = \frac{(x_k - x^*)f'(x_k) - f(x_k)}{f'(x_k)} = \frac{f''(\xi_k)(x^* - x_k)^2}{2f'(x_k)}$$

and therefore

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^2} = \frac{f''(x^*)}{2f'(x^*)}$$

■

Definition 2.5 [Secant Method]

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})},$$

■ **Example 2.2** [Secant with False Position Method] $q_0 = f(p_0); q_1 = f(p_1)$.

Secant Method

- Set $p = p_1 - \frac{q_1(p_1 - p_0)}{q_1 - q_0}$
- Set $p_0 = p_1$; $q_0 = q_1$; $p_1 = p$; $q_1 = f(p)$

False Position Method

- Set $p = p_1 - \frac{q_1(p_1 - p_0)}{q_1 - q_0}$
- $q = f(p)$. If $q \cdot q_1 < 0$, then set $p_0 = p_1$; $q_0 = q_1$
- Set $p_1 = p$; $q_1 = q$.

Theorem 2.6 — **Weierstrass Approximation Theorem.** f continuous on $[a,b]$. For $\forall \epsilon > 0$, $\exists$ polynomial $P(x)$s.t. $|f(x) - P(x)| < \varepsilon$, $\forall x \in [a,b]$

Theorem 2.7 — **Mean Value Theorem.** Suppose $f \in C^n[a,b]$ and $a = x_0 < x_1 < \cdots < x_n = b$. There exists some $\xi \in (a,b)$ such that

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

Proof. Let $P(x)$ be the newton form interpolation and $g = f(x) - P(x)$, which has $n+1$ distinct zeros. By roll's theorem, there exists ξ such that $p^{(n)}(\xi) = n! f[x_0, \ldots, x_n] = f^{(n)}(\xi)$, ■

Definition 2.6 [Lagrange interpolation] Choose basis $\{L_{n,k}(x) \mid k = 0, \ldots, n\}$ such that

$$L_{n,k}(x_k) = 1, \quad L_{n,k}(x_l) = 0, \forall l \neq k$$

Lagrange interpolation: $P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$ with

$$L_{n,k}(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_n)}$$

Theorem 2.8 — **Error Bound for Lagrange polynomial.** For $f \in C^{n+1}[a,b]$, we have

- $$f(x) = P(x) + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$

  with $P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$ is the interpolation of f at $\{x_0 = a, x_1, \ldots, x_n = b\}$.

- Hermite's interpolation error:

  $$f(x) - H(x) = \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!}(x - x_0)^2 (x - x_1)^2 \cdots (x - x_n)^2$$

Proof. For fixed $x \notin \{x_0, \ldots, x_n\}$, introduce

$$\phi(t) = f(t) - P(t) - \lambda(t - x_0)(t - x_1) \cdots (t - x_n)$$

Choose λ s.t. $\phi(x) = 0$, then $\phi(x) = 0, \phi(x_0) = \cdots = \phi(x_n) = 0$.

By Rolle's theorem, $\phi^{(n+1)}(t)$ has at least one zero, say $\xi_x \in (a, b)$. Therefore,

$$0 = \phi^{(n+1)}(\xi_x) = f^{(n+1)}(\xi_x) - \underbrace{p^{(n+1)}(\xi_x)}_{0} - \lambda(n+1)! \implies \lambda = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}$$

Note that $\phi(x) = 0$ for such λ, the proof is complete. ■

Definition 2.7 [Cubic spline interpolations] The cubic spline interpolation on $\{x_0, x_1, \ldots, x_n\}$ has the form

$$S(x) = S_k(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3, \qquad x \in [x_k, x_{k+1}], \quad k = 0, 1, \ldots, n-1$$

1. $S(x_i) = f_i$ for $i = 0, 1, \ldots, n$; which gives 2 condition on each $[x_k, x_{k+1}]$, total $2n$
2. $S'(x)$ and $S''(x)$ must be continuous on $[x_0, x_n]$, i.e., $S'_{i+1}(x_{i+1}) = S'_i(x_{i+1})$ for $i = 0, 1, \ldots, n-1$.
3. Boundary conditions:
   - $S''(x_0) = S''(x_n) = 0$ (natural cubic spline, free)
   - $S'(x_0) = f(x_0)$ and $S'(x_n) = f(x_n)$ (clamped boundary)

Definition 2.8 [Constructing natural spline interpolant] First sovle system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ with $c_n = 0$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & \ddots & \ddots & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix},$$

and

$$\boldsymbol{b} = \begin{pmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Then solve for b_j, d_j, i.e., for $j = 0, 1, \ldots, n-1$

$$\begin{aligned} b_j &= \frac{a_{j+1} - a_j}{h_j} - h_j \frac{(c_{j+1} + 2c_j)}{3} \\ d_j &= \frac{c_{j+1} - c_j}{3h_j} \end{aligned}$$

Equally spaced integral.

$$\int_0^{nh} S(x)\,\mathrm{d}x = \sum_{j=0}^{n-1} a_j h_j + \frac{1}{2} b_j h_j^2 + \frac{1}{3} c_j h_j^3 + \frac{1}{4} d_j h_j^4$$

Definition 2.9 [Clamped Spline Interpolant: Linear system $\boldsymbol{Ax} = \boldsymbol{b}$]

$$\boldsymbol{A} = \begin{bmatrix} 2h_0 & h_0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & \ddots & \ddots & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & h_{n-1} & 2h_{n-1} \end{bmatrix},$$

$$\boldsymbol{b} = \begin{pmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \end{pmatrix}$$

Definition 2.10 [Divided difference & Newtpn form]

$$f[x_0, \ldots, x_k] = \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}$$

Therefore

$$P(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \ldots, x_n](x - x_0) \cdots (x - x_{n-1})$$

Hermite polynomial:

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \ldots, z_k](x - z_0)(x - z_1) \cdots (x - z_{k-1})$$

Theorem 2.9 — Spline Error. For $f \in \mathcal{C}^4[a,b]$ with $\max_{a \le x \le b} |f^{(4)}(x)| = M$, we have

$$|f(x) - S_{\text{clamped}}(x)| \le \frac{5M}{384} \max_{0 \le j \le n-1} (x_{j+1} - x_j)^4$$

Chapter 3

Week3

3.1. Tuesday

Theorem 3.1 — Optimality Condition.

- primal feasible: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$
- Dual feasible: $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$
- Complementarity: $\boldsymbol{x} \circ \boldsymbol{s} = \boldsymbol{0}$, i.e., $x_i \cdot (c_i - \boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{y}) = \boldsymbol{0}$ for each i.

R (Primal) Simplex method:

1. Always keep primal feasibility:
2. Always keep complementarity:
   Define $\boldsymbol{y} = (\boldsymbol{A}_B^{-1})^{\mathrm{T}}\boldsymbol{c}_B$ as the dual solution. The reduced costs vector is $\boldsymbol{c}^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{A} = \boldsymbol{c} - \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}$
3. Not necessarily keep dual feasible until get the optimal solution, i.e., it will seeks solution that is dual feasible.

Dual Simplex method. Dual Simplex method remains both dual feasiblity and complementarity conditions in each iteration but seeks primal feasibility.

Cases for applying dual simplex method:

- There is a dual BFS available but no primal BFS available.
- $\boldsymbol{b}$ is changed by a large amount or a constraint isadded, i.e., lose the primal feasible solution.

Interior Point Method. Consider the relaxed version of optimality condition:

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \boldsymbol{x} \geq 0 \\ \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{s} &= \boldsymbol{c}, \boldsymbol{s} \geq 0 \\ x_i \cdot s_i &= \mu, \quad \forall i, \text{small } \mu_i > 0 \end{aligned}$$

Keep decreasing μ and finally get the solution to LP.

R

- The optimal solution output from interior point method may not necessarily BFS. If the optimal solution is unique, it is BFS.
- Initial solution for the interior point method can be found by solving the auxiliary problem.
- The complexity for interior point method is $O(n^{3.5})$
- The interior point method gives stable running time compared with simplex method.
- Interior point method always find the optimal solution with maximum possible number of **non-zeros**.
- Interior point method finds high-rank solution (the center of all optimal solutions); but the simplex method finds the low-rank solution.

3.1.1. Reviewing

Linear optimization formulation. Standard Form LP Transformation

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{such that} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Maximin / minimax objective

Absolute values in objective function or constraints.

Theorem 3.2 The BFS for standard LP is equivalent to extreme point.

Theorem 3.3 If there is a feasible solution, then there is a basic feasible solution; If there is a optimal solution, then there is a basic feasible optimal solution.

Care about corollary

Simplex method.

1. Understand how simplex method works, and cases for unbounded, infeasible
2. Apply simplex method to solve small LPs
3. Read and interpret simplex tableau (make use of it to avoid inverse calculation)
4. Apply two-phase method

Duality Theory.

1. Be able to constrauct the dual for any LP.
2. Know the (strong/weak) duality theorems and apply them in different situations.
3. Be able to write down the complentarity conditions and apply them

Sensitivity Analysis. Related to duality theory;

Complexity Theory and interior method. Complexity of LP:

1. No guarntee of simplex method to achieve polynomial time
2. Interior point can achieve polynomial time

Properties of simplex method

Chapter 4

Week4

4.1. Convergence

Definition 4.1 [Convergent] An infinite sequence $\{z_n\}$ of complex numbers has a limit z_0, if for $\forall\ \varepsilon > 0$, there exists a positive integer n_0 such that

$$|z_n - z| < \varepsilon, \quad \text{whenever } n > n_0$$

We say the sequence z_n converges to z and write as

$$\lim_{n\to\infty} z_n = z$$

When the sequence does not have a limit, then it diverges.

The uniqueness of limit of a seuqnece is guaranteed.

Proposition 4.1 For $z_n = x_n + iy_n$, we have

$$\lim_{n\to\infty} z_n = x + iy$$

if and only if

$$\lim_{n\to\infty} x_n = x, \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

Definition 4.2 [Convergent Series] An infinite **series** $\sum_{n=1}^{\infty} z_n$ of complex numbers converges to the sum S if the partial sum sequences

$$S_N = \sum_{n=1}^{N} z_n$$

converges to S, then we write

$$\sum_{n=1}^{\infty} z_n = S.$$

Proposition 4.2 For $z_n = x_n + iy_n$, we have

$$\sum_{n=1}^{\infty} z_n = X + iY$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y$$

Proposition 4.3 The series $\sum_{n=1}^{\infty} z_n$ converges implies that $\lim_{n\to\infty} z_n = 0$.

Definition 4.3 [Absolute Convergence] The series $\sum_{n=1}^{\infty} z_n$ is said to be **absolutely convergent** if

$$\sum_{n=1}^{\infty} |z_n|$$

converges, i.e., $\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ converge.

Proposition 4.4 Absolute convergence implies convergence

Definition 4.4 [Remainder] The **remainder** ρ_N of a series after N terms is defined by:

$$\rho_N = \sum_{n=N+1}^{\infty} S_n$$

Proposition 4.5 A series converges to a number S iff the sequence of remainders tends to zero.

It's easy to verifty that

$$\sum_{n=0}^{\infty} z_n = \frac{1}{1-z}, \qquad \text{whenever } |z| < 1$$

with the aid of partial sums and remainders.

4.1.1. Taylor Series

Definition 4.5 [Power Series] The power series has the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Theorem 4.1 — Convergent of Taylor Series. Suppose f is **analytic** on $|z - z_0| < R$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \tag{4.1}$$

for $|z - z_0| < R$, i.e., $f(z)$ admits its Taylor expansion at $z = z_0$ in this rigion.

R Typically, when $z_0 = 0$, we say this series is the **Maclaurin series**.

Proof. **Step 1: Applying Cauchy Integral Formula.** For fixed z, let $r := |z - z_0| < R$ and take r_0 such that $r < r_0 < R$. Construct a contour $C_0 : \{z \in \mathbb{C} \mid |z - z_0| = r_0\}$ in the positive sense, which follows that

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s - z} \, \mathrm{d}s \tag{4.2a}$$

Step 2: Expand $1/(s-z)$**.** With some calculation, we obtain

$$\frac{1}{s-z} = \frac{1}{s-z_0} \cdot \frac{1}{1-\frac{z-z_0}{s-z_0}} \tag{4.2b}$$

$$= \frac{1}{s-z_0} \left\{ 1 + \frac{z-z_0}{s-z_0} + \cdots + (\frac{z-z_0}{s-z_0})^{N-1} + \frac{(\frac{z-z_0}{s-z_0})^N}{1-\frac{z-z_0}{s-z_0}} \right\} \tag{4.2c}$$

$$= \frac{1}{s-z_0} + \frac{z-z_0}{(s-z_0)^2} + \cdots + \frac{(z-z_0)^{N-1}}{(s-z_0)^N} + \frac{(z-z_0)^N}{(s-z)(s-z_0)^N} \tag{4.2d}$$

where $(4.2c)$ is because that

$$\frac{1}{1-c} = 1 + c + c^2 + \cdots + c^{N-1} + \frac{c^N}{1-c}.$$

Substituting (4.2e) into (4.2a), we obtain

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \left\{ \frac{f(s)}{(s-z_0)} + \frac{f(s)(z-z_0)}{(s-z_0)^2} + \cdots + \frac{f(s)(z-z_0)^{N-1}}{(s-z_0)^N} + \frac{f(s)(z-z_0)^N}{(s-z)(s-z_0)^N} \right\}$$

$$= f(z_0) + f'(z_0)(z-z_0) + \cdots + \frac{f^{(N-1)}(z_0)}{(N-1)!}(z-z_0)^{N-1} + \rho_N(z)$$

with

$$\rho_N(z) = \frac{(z-z_0)^B}{2\pi i} \int_{C_0} \frac{f(s)}{(s-z)(s-z_0)^N} \, \mathrm{d}s \tag{4.2e}$$

Step 3: Show that $\rho_N(z)$ **is convergent.**

$$|\rho_N(z)| \le \frac{|z-z_0|^N}{2\pi} \int_{C_0} \frac{|f(s)|}{|s-z||s-z_0|^N} |\,\mathrm{d}s| \tag{4.2f}$$

$$\le \frac{r^N}{2\pi} \int_{C_0} \frac{M}{(r_0-r)r_0^N} |\,\mathrm{d}s| \tag{4.2g}$$

$$= \frac{Mr_0}{r_0-r} \left(\frac{r}{r_0} \right)^N \tag{4.2h}$$

where we suppose $|f(s)| \le M$ on C_0; and $|s-z| \ge |s-z_0| - |z-z_0| = r_0 - r$. Therefore,

$$\rho_N(z) \to 0$$

since $r < r_0$ and $(r/r_0)^N \to 0$.

■

■ **Example 4.1**

1. For $f(z) = e^z$, which is analytic for $|z - 0| < \infty$, thus we have

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad |z| < \infty$$

2. For $f(z) = \sin z = \frac{e^{iz} - e^{-i}}{2i}$, which is analytic for $|z - 0| < \infty$, thus we have $f^{(2n)}(0) = 0; f^{(2n+1)}(0) = (-1)^n$, and therefore

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \qquad |z| < \infty$$

3. For $f(z) = \frac{1}{1-z}$, which is analytic for $|z - 0| < 1$, we have $f^{(0)} = n!$, and therefore

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1$$

4. For $f(z) = \frac{1}{z} \cdot \frac{1}{1+z}$, we have

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n, \qquad |z| < 1,$$

and therefore

$$\frac{1}{z+z^2} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n z^{n-1}, \qquad 0 < |z| < 1$$

■

4.1.2. Laurent Series

We cannot apply Taylor expansion at a non-analytic point. Fortunately, we can find another series representation for $f(z)$ that involving positive and negative powers of $(z - z_0)$

Definition 4.6 [Laurent Series] The **Laurent series** has the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Theorem 4.2 Suppose f is analytic throughout an **annular domain** $R_1 < |z-z_0| < R_2$. Let C be any **positively oriented simple closed contour** around z_0 and lying in that domain. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \qquad R_1 < |z-z_0| < R_2 \tag{4.3a}$$

with

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z, \qquad n = 0,1,2,\ldots \tag{4.3b}$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} \, \mathrm{d}z, \qquad n = 1,2,\ldots \tag{4.3c}$$

R The Laurent series is often written as the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n, \qquad R_1 < |z-z_0| < R_2,$$

where

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z, \qquad n = 0, \pm 1, \pm 2, \ldots$$

When f is analytic on $|z-z_0| < R_2$, we have $b_n = 0, a_n = \frac{f^{(n)}(z_0)}{n!}$, i.e., the Laurent series reduces to the Taylor series.

Proof.

- For fixed z in the domain, let $r = |z-z_0|$, and construct two positively oriented contours $C_i : \{z \in \mathbb{C} \mid |z-z_0| = r_i\}, i = 1,2$ such that $R_1 < r_1 < r < r_2 < R_2$. (The reaon why we don't use the boundary is that the function is not analytic on the boundary but only interior to)
- Construct a circle $\gamma : \{s \in \mathbb{C} \mid s = z + \delta e^{i\theta}, 0 \le \theta \le 2\pi\}$, where the δ is picked such that γ is contained in the interior between C_1, C_2. By Cauchy Integral Formula,

$$\int_{C_2} \frac{f(s)\,\mathrm{d}s}{s-z} = \int_{C_1} \frac{f(s)\,\mathrm{d}s}{s-z} + \int_{\gamma} \frac{f(s)\,\mathrm{d}s}{s-z} \tag{4.4}$$

Or equivalently,

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)\,\mathrm{d}s}{s-z} + \frac{1}{2\pi i}\int_{C_1} \frac{f(s)\,\mathrm{d}s}{z-s}$$

- By applying the same trick as (4.2b), we have

$$f(z) = \sum_{n=0}^{N-1} a_n (z-z_0)^n + \rho_N(z) + \sum_{n=1}^{N} \frac{b_n}{(z-z_0)^n} + \sigma_N(z) \tag{4.5a}$$

with

$$\begin{aligned}
a_n &= \frac{1}{2\pi i}\int_{C_2} \frac{f(s)\,\mathrm{d}s}{(s-z_0)^{n+1}} && \text{(4.5b)}\\
b_n &= \frac{1}{2\pi i}\int_{C_1} \frac{f(s)\,\mathrm{d}s}{(s-z_0)^{-n+1}} && \text{(4.5c)}\\
\rho_N(z) &= \frac{(z-z_0)^N}{2\pi i}\int_{C_2} \frac{f(s)\,\mathrm{d}s}{(s-z)(s-z_0)^N} && \text{(4.5d)}\\
\sigma_N(z) &= \frac{(z-z_0)^{-N}}{2\pi i}\int_{C_1} \frac{f(s)\,\mathrm{d}s}{(z-s)(s-z_0)^{-N}} && \text{(4.5e)}
\end{aligned}$$

- Then we bound the term $\rho_N(z)$ and $\sigma_N(z)$. Suppose $|f(s)| \leq M$ on C_1, C_2, and note that $|s-z| \geq r_2 - r$ for $s \in C_2$; $|z-s| \geq r - r_1$ for $s \in C_1$:

$$\begin{aligned}
\rho_N(z) &\leq \frac{Mr_2}{r_2 - r}\left(\frac{r}{r_2}\right)^N \\
\sigma_N(z) &\leq \frac{Mr_1}{r - r_1}\left(\frac{r_1}{r}\right)^N
\end{aligned}$$

- Finally, note that

$$\begin{aligned}
a_n &= \frac{1}{2\pi i}\int_{C} \frac{f(s)\,\mathrm{d}s}{(s-z_0)^{n+1}} \\
b_n &= \frac{1}{2\pi i}\int_{C} \frac{f(s)\,\mathrm{d}s}{(s-z_0)^{-n+1}}
\end{aligned}$$

■

Example 4.2 1.

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

This function has two singular points 1,2. We take $z_0 = 0$.

- When $z \in D_1 = \{z : |z| < 1\}$, we obtain the Taylor expansion:

$$f(z) = -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n$$

- When $z \in D_2 = \{z : 1 < |z| < 2\}$, we obtain the Laurent series:

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n}$$

- When $z \in D_3 = \{z : |z| > 2\}$, we obtain

$$f(z) = \sum_{n=1}^{\infty} \frac{1 - 2^{n-1}}{z^n}$$

2. Expand $f(z) = \frac{-z}{(z-1)(z-3)}$ near $z_0 = 1$, and find the domain of expansion.

The expansion should be the laurent series with domain of expansion $0 < |z-1| < 2$.

$$f(z) = \frac{1/2}{z-1} - \frac{3/2}{z-3} = \frac{1/2}{z-1} + \sum_{n=0}^{\infty} \frac{3}{2^{n+2}} (z-1)^n$$

4.1.3. Power Series

For power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{4.6}$$

, first we study the range of its convergence.

Theorem 4.3 If the power series (4.6) converges at $z = z_1 (\neq z_0)$, then it is **absolutely convergent** at each point in the disk $|z - z_0| < |z_1 - z_0|$.

Proof. For any point z around the disk, we have, we have

$$\left|\frac{z - z_0}{z_1 - z_0}\right| := q < 1,$$

which follows that

$$|a_n(z - z_0)^n| = |a_n(z_1 - z_0)^n| \left|\frac{z - z_0}{z_1 - z_0}\right|^n \leq Mq^n,$$

where $|a_n(z_1 - z_0)^n| \leq M$ since z_1 makes the series convergent. By Comparison test, we conclude (4.6) is absolutely convergent. ∎

Definition 4.7 [Uniform convergence] The series (4.6) is said to be **uniformly convergent** for $|z - z_0| < R$ if as $N \to \infty$,

$$\sup_{|z-z_0|<R} |\rho_N(z)| \to 0$$

Theorem 4.4 If the power series (4.6) converges at $z = z_1 (\neq z_0)$, then it must be uniformly convergent for any closed circle $|z - z_0| \leq \rho$ $(\rho < |z_1 - z_0|)$.

Proof. Notice that for any $\rho < |z_1 - z_0|$, for any z in that closed circle, we have

$$|a_n(z - z_0)^n| \leq |a_n \rho^n|$$

Due to the conclusion in Theorem(4.3), we conclude $\sum_{n=1}^{\infty} |a_n| \rho^n$ is convergent, and therefore (4.6) is uniformly convergent.

∎

Now we are curious about whether the power series is analytic. First we show under which condition does the power series is continuous.

Theorem 4.5 The series (4.6) represents a continuous function at each point inside the circle of convergence.

Theorem 4.6 The sum $S(z)$ of power series is analytic at each point z interior to the circle convergence of that series.