A FIRST COURSE

IN

ANALYSIS

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MAT2006 Notebook

Lecturer

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Acknowledgments

This book is taken notes from the MAT2006 in fall semester, 2018. These lecture notes were taken and compiled in LaTeX by Jie Wang, an undergraduate student in Fall 2018. Prof. Weiming Ni has not edited this document. Students taking this course may use the notes as part of their reading and reference materials. This version of the lecture notes were revised and extended for many times, but may still contain many mistakes and typos, including English grammatical and spelling errors, in the notes. It would be greatly appreciated if those students, who will use the notes as their reading or reference material, tell any mistakes and typos to Jie Wang for improving this notebook.

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 7

Week7

7.1. Wednesday

Announcement. Our mid-term is on next Wednesday in Liwen Building, from 8:00am to 10:00am. We will cover everything until this Friday, i.e., the improper integral.

7.1.1. Integrable Analysis

Recap. Given a sequence of functions $\{f_n\}$ with pointwise limit f, we are curious about whether the equation holds:

$$\lim_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x$$

■ **Example 7.1** Let $\{f_n\}$ defined on $[0,1]$ with

$$f_n(x) = \begin{cases} n, & \text{if } x\in(0,\frac{1}{n}) \\ 0, & \text{otherwise} \end{cases}$$

We find that $\int_0^1 f_n\,\mathrm{d}x = 1$, and $f_n\to 0$ as $n\to\infty$. Thus

$$\int_0^1 \left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x = 0 \neq \lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x$$

There is a sufficient condition that guarantees the equation holds:

Theorem 7.1 Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a,b]$. If f_n converges to f uniformly as $n \to \infty$, then f is also **Riemann integrable**, and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Definition 7.1 We say that f_n converges to f uniformly as $n \to \infty$ on $[a,b]$ if for every $\varepsilon > 0$, there exists N such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [a,b]$ and for all $n \geq N$.

Proof. **Step 1:** Show that the limit f is **uniformly bounded**:

$$|f_n(x) - f_k(x)| = |f_n(x) - f(x) + f(x) - f_k(x)| \tag{7.1a}$$

$$\leq |f_n(x) - f(x)| + |f(x) - f_k(x)| \tag{7.1b}$$

Choose ε, then there exists $N > 0$ s.t. $|f_n(x) - f_k(x)| < 2$ if $n,k \geq N$. In particular,

$$|f_n(x) - f_N(x)| < 2 \implies |f_n(x)| < |f_N(x)| + 2, \qquad \forall n \geq N, \tag{7.1c}$$

i.e., every $|f_n|$ for $n \geq N$ is bounded from $|f_N(x)|$ as 2. Therefore, we have $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded by M. Therefore we have

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + |f_N(x)|,$$

i.e., f is also uniformly bounded.

Step 2: Let $\varepsilon_n = \sup_{a\leq x\leq b} |f_n(x) - f(x)|$, thus $\varepsilon_n \to 0$ as $n \to \infty$. Therefore

$$-\varepsilon_n \leq f(x) - f_n(x) \leq \varepsilon_n \implies -\varepsilon_n + f_n(x) \leq f(x) \leq \varepsilon_n + f_n(x)$$

Consider the lower and upper Riemann integral:

$$\underline{\int_a^b} f_n(x) - \varepsilon_n \, dx \leq \underline{\int_a^b} f(x) \, dx \leq \overline{\int_a^b} f(x) \, dx \leq \overline{\int_a^b} f_n(x) - \varepsilon_n \, dx$$

Since f_n is integrable, we have

$$\int_a^b f_n(x) - \varepsilon_n \, dx \leq \underline{\int_a^b} f(x) \, dx \leq \overline{\int_a^b} f(x) \, dx \leq \int_a^b f_n(x) - \varepsilon_n \, dx$$

The difference between the lower and upper integral of f is

$$0 \leq \overline{\int_a^b} f(x) \, dx - \underline{\int_a^b} f(x) \, dx \leq 2(b-a)\varepsilon_n,$$

which holds for every n. Taking $n \to \infty$, we imply $\overline{\int_a^b} f(x) \, dx = \underline{\int_a^b} f(x) \, dx$.

Now we have

$$\int_a^b f_n(x) - \varepsilon_n \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b f_n(x) + \varepsilon_n \, dx$$

and therefore

$$\int_a^b -\varepsilon_n \, dx \leq \leq \int_a^b f(x) - f_n(x) \, dx \leq \int_a^b \varepsilon_n \, dx,$$

which implies

$$\left| \int_a^b f(x) - f_n(x) \, dx \right| \leq \int_a^b \varepsilon_n \, dx = \varepsilon_n(b-a) \to 0,$$

i.e., $\lim_{n\to\infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx$. ■

R Would derivatives approach the limit function? When can we exchange these two limiting process?

7.1.2. Elementary Calculus Analysis

Theorem 7.2 — **Fundamental Theorem of Calculus.** If $f:[a,b]\mapsto\mathbb{R}$ is continuous, then the function $F(x)=\int_a^x f(t)\,\mathrm{d}t$ is **differentiable** with $F'=f$.

Proof. The proof is simply by definition: (difference quotient is useful in proofs related to differentiation)

$$
\begin{aligned}
\frac{F(x+h)-F(x)}{h}-f(x) &= \frac{1}{h}\left[\int_a^{x+h} f(t)\,\mathrm{d}t-\int_a^x f(t)\,\mathrm{d}t\right]-f(x) && \text{(7.2a)}\\
&= \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t-f(x) && \text{(7.2b)}\\
&= \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t-\frac{1}{h}\left[\int_x^{x+h} 1\,\mathrm{d}t\right]f(x) && \text{(7.2c)}\\
&= \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t-\frac{1}{h}\int_x^{x+h} f(x)\,\mathrm{d}t && \text{(7.2d)}\\
&= \frac{1}{h}\int_x^{x+h} [f(t)-f(x)]\,\mathrm{d}t, && \text{(7.2e)}
\end{aligned}
$$

which implies that

$$\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|\le\frac{1}{h}\int_x^{x+h}|f(t)-f(x)|\,\mathrm{d}t,$$

Since f is continuous at x, for $\varepsilon>0$, there exists $\delta>0$ such that $|f(y)-f(x)|<\varepsilon$ if $|y-x|<\delta$. Therefore,

$$\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|\le\frac{1}{h}\int_x^{x+h}|f(t)-f(x)|\,\mathrm{d}t\le\frac{1}{h}\int_x^{x+h}\varepsilon\,\mathrm{d}t=\varepsilon,$$

if $h<\delta$, i.e.,

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=f(x)$$

■

Theorem 7.3 — **Integration by Parts.** Given two functions $f,g\in\mathcal{C}^1[a,b]$, (similar to $(fg)'=f'g+fg'$), we have

$$\int_a^b (fg)'\,\mathrm{d}x=\int_a^b f'g\,\mathrm{d}x+\int_a^b fg'\,\mathrm{d}x,$$

or equivalently,

$$(fg)(a) - (fg)(b) = \int_a^b f'g \, \mathrm{d}x + \int_a^b fg' \, \mathrm{d}x,$$

i.e.,

$$\int_a^b fg' \, \mathrm{d}x = (fg)|_a^b - \int_a^b f'g \, \mathrm{d}x$$

Proposition 7.1 — **Change of variables.** Let $\phi : [\alpha, \beta] \mapsto [a, b]$ be a continuously differentiable function such that

$$\phi(\alpha) = a, \quad \phi(\beta) = b.$$

Then for every continuous function $f : [a, b] \mapsto \mathbb{R}$, we have

$$\int_a^b f(x) \, \mathrm{d}x = \int_\alpha^\beta f(\phi(t))\phi'(t) \, \mathrm{d}t$$

Proof. Define $F(x) = \int_a^x f(t) \, \mathrm{d}t$, which implies

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = f(x), \qquad \int_a^b f(x) \, \mathrm{d}x = F(b).$$

Observe that

$$\frac{\mathrm{d}F(\phi(t))}{\mathrm{d}t} = \frac{\mathrm{d}F(\phi(t))}{\phi(t)} \frac{\phi(t)}{\mathrm{d}t} = f(\phi(t))\phi'(t)$$

Or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}(F \circ \phi)(t) = f(\phi(t))\phi'(t)$$

Therefore,

$$\begin{aligned}
\int_\alpha^\beta (F \circ \phi)'(t) \, \mathrm{d}t &= \int_\alpha^\beta f(\phi(t))\phi'(t) \, \mathrm{d}t && (7.3)\\
&= (F \circ \phi)(\beta) - (F \circ \phi)(\alpha) = F(\phi(\beta)) - F(\phi(\alpha)) && (7.4)\\
&= F(b) - F(a) = F(b) && (7.5)\\
&= \int_a^b f(x) \, \mathrm{d}x && (7.6)
\end{aligned}$$

■

Proposition 7.2 Let $\phi : [\alpha, \beta] \mapsto [a, b]$ be continuously differentiable and **strictly mono-**

tone. Then for any $f \in \mathcal{R}[a,b]$, we have

1. $f(\phi(t))\phi'(t) \in \mathcal{R}[\alpha,\beta]$
2. $$\int_{\alpha}^{\beta} f(\phi(t))\phi'(t) = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,\mathrm{d}x$$

R We relax f from being continuously differentiable to being Riemann integrable; but restrict ϕ to be **strictly monotone**.

The proof for this proposition is messy. For most time functions we have faced is not continuous, but we can break into finite sub-intervals and apply the former proposition. Thus the benifit for this proposition is not such huge. For practical, the former proposition is useful enough.

Theorem 7.4 Let $f \in \mathcal{R}[a,b]$. Then a Riemann sum $S(\mathcal{P},f)$ converges to $\int_a^b f(x)\,\mathrm{d}x$ as the mesh $\lambda(\mathcal{P}) \to 0$, i.e.,

$$\sum_{i=1}^{n} f(t_i)\Delta x_i \to \int_a^b f(x)\,\mathrm{d}x, \qquad \text{as } \max_{1\le i\le n} \Delta x_i \to 0,$$

where $t_i \in [x_{i-1}, x_i]$, $i = 1,\dots,n$.

■ **Example 7.2** 1. Evaluate the limit

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right].$$

Let

$$\begin{aligned}
x_n &= \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} \\
&= \frac{1}{n}\left[\frac{n}{n+1}+\frac{n}{n+2}+\cdots+\frac{n}{2n}\right] \\
&= \frac{1}{n}\left[\frac{1}{1+1/n}+\frac{1}{1+2/n}+\cdots+\frac{1}{1+n/n}\right] \\
&= \Delta x_i\left[f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n}{n})\right]
\end{aligned}$$

which is essentially the Riemann sum of function $f(x) = \frac{1}{1+x}$ over interval $[0,1]$.

Therefore, as $n \to \infty$,

$$x_n \to \int_0^1 \frac{1}{1+x}\,\mathrm{d}x$$

2. Evaluate the limit

$$\lim_{n\to\infty} \frac{1^\alpha + \cdots + n^\alpha}{n^\alpha}$$

Let

$$\begin{aligned}
x_n &= \frac{1}{n}\frac{1^\alpha + \cdots + n^\alpha}{n^\alpha} \\
&= \frac{1}{n}\left[(\frac{1}{n})^\alpha + (\frac{2}{n})^\alpha + \cdots + (\frac{n}{n})^\alpha\right] \\
&= \Delta x_i \left[f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n}{n})\right]
\end{aligned}$$

As $n \to \infty$,

$$x_n \to \int_0^1 x^\alpha\,\mathrm{d}x = \frac{1}{\alpha+1}x^{\alpha+1}\Big|_0^1 = \frac{1}{\alpha+1}$$

■