

**A FIRST COURSE  
IN  
ANALYSIS**



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# A FIRST COURSE IN ANALYSIS

## MAT2006 Notebook

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# Notations and Conventions

$\mathbb{R}^n$	$n$ -dimensional real space
$\mathbb{C}^n$	$n$ -dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
$x_i$	$i$ th entry of column vector $\mathbf{x}$
$a_{ij}$	$(i, j)$ th entry of matrix $\mathbf{A}$
$\mathbf{a}_i$	$i$ th column of matrix $\mathbf{A}$
$\mathbf{a}_i^T$	$i$ th row of matrix $\mathbf{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all $i, j$
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all $i, j$
$\mathbf{A}^T$	transpose of $\mathbf{A}$ , i.e, $\mathbf{B} = \mathbf{A}^T$ means $b_{ji} = a_{ij}$ for all $i, j$
$\mathbf{A}^H$	Hermitian transpose of $\mathbf{A}$ , i.e, $\mathbf{B} = \mathbf{A}^H$ means $b_{ji} = \bar{a}_{ij}$ for all $i, j$
$\text{trace}(\mathbf{A})$	sum of diagonal entries of square matrix $\mathbf{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\mathbf{e}_i$	a unit vector with the nonzero element at the $i$ th entry
$\mathcal{C}(\mathbf{A})$	the column space of $\mathbf{A}$
$\mathcal{R}(\mathbf{A})$	the row space of $\mathbf{A}$
$\mathcal{N}(\mathbf{A})$	the null space of $\mathbf{A}$
$\text{Proj}_{\mathcal{M}}(\mathbf{A})$	the projection of $\mathbf{A}$ onto the set $\mathcal{M}$





# Chapter 12

## Week12

### 12.1. Wednesday

#### 12.1.1. Recap for Rank Theorem

**Inverse Function Theorem.** Given a  $\mathcal{C}^p$  function  $f : E(\subseteq \mathbb{R}^m) \subseteq \mathbb{R}^m$  and  $Df(\mathbf{x}_0)$  is invertible. Then we imply that there is a neighborhood  $U(\mathbf{x}_0) \times V(\mathbf{y}_0)$  of  $(\mathbf{x}_0, f(\mathbf{x}_0))$  such that  $f$  is a  $\mathcal{C}^p$ -diffeomorphism between  $U(\mathbf{x}_0)$  and  $V(\mathbf{y}_0)$ ; moreover,

$$D(f^{-1})(\mathbf{y}_0) = (Df(\mathbf{x}_0))^{-1}$$

**Rank Theorem.** Given a  $\mathcal{C}^p$  function  $f : U(\mathbf{x}_0) \rightarrow \mathbb{R}^n$  of constant rank  $k$  throughout  $U(\mathbf{x}_0)$ . Then there exists a neighborhood  $N(\mathbf{x}_0) \times N(f(\mathbf{x}_0))$  and two  $\mathcal{C}^p$ -diffeomorphisms

$$\mathbf{u} = \phi(\mathbf{x}), \mathbf{x} \in N(\mathbf{x}_0) \quad \mathbf{v} = \psi(\mathbf{y}), \mathbf{y} \in N(\mathbf{y}_0), \mathbf{y}_0 := f(\mathbf{x}_0),$$

such that the composition  $\psi \circ f \circ \phi^{-1}$  takes the form

$$(u_1, \dots, u_k, u_{k+1}, \dots, u_m) \rightarrow (u_1, \dots, u_k, 0, 0, \dots, 0)$$

**Outline of proof.** Step 1:

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_k \\ \vdots \\ f_n \end{pmatrix}, \quad Df = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_m)}$$

w.l.o.g., assume the first  $k \times k$  principal minors to be non-singular.

Step 2: Then construct the map  $\phi(\mathbf{x})$

$$\phi(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_k(\mathbf{x}) \\ x_{k+1} \\ \vdots \\ x_m \end{pmatrix} \implies D\phi = \begin{pmatrix} \frac{\partial(f_1, \dots, f_k)}{\partial(x_1, \dots, x_k)} & \frac{\partial(f_1, \dots, f_k)}{\partial(x_{k+1}, \dots, x_m)} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

which is invertible.

Step 3: define  $g := f \circ \phi^{-1} : \phi(N(\mathbf{x}_0)) \rightarrow \mathbb{R}^n$ , then rewrite  $g$  as

$$\begin{pmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_k \\ g_{k+1}(\mathbf{u}) \\ \vdots \\ g_n(\mathbf{u}) \end{pmatrix} \implies Dg = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \frac{\partial(g_{k+1}, \dots, g_n)}{\partial(u_{k+1}, \dots, u_n)} \end{pmatrix},$$

which implies the lower right corner should be zero matrix, i.e.,  $(g_{k+1}, \dots, g_n)(\mathbf{u})$

depends only on the first  $k$  variables. Thus rewrite  $g$  as:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_k \\ g_{k+1}(u_1, \dots, u_k) \\ \vdots \\ g_n(u_1, \dots, u_k) \end{pmatrix}$$

Step 4: Define the map  $\mathbf{v} = \psi(\mathbf{y})$ :

$$\begin{pmatrix} v_1 \\ \vdots \\ v_k \\ v_{k+1} \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} - g_{k+1}(y_1, \dots, y_k) \\ \vdots \\ y_{k+1} - g_n(y_1, \dots, y_k) \end{pmatrix}$$

flatten out

■ **Example 12.1** 1. Define  $f(t) = (\cos t, \sin t), t \in \mathbb{R}$ . Define  $t_0 = \frac{\pi}{4}$ . Can we flatten out the curve near  $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ ? Note that

$$Df\left(\frac{\pi}{4}\right) = (-\sqrt{2}/2, \sqrt{2}/2) \neq 0,$$

with rank 1. The answer is yes.

Choose  $\phi(t) = \cos t$  and  $\phi^{-1}(u) = t = \cos^{-1} u$ , which follows that

$$g(u) = f(\phi^{-1}(u)) = \begin{pmatrix} \cos(\phi^{-1}u) \\ \sin(\phi^{-1}(u)) \end{pmatrix} = \begin{pmatrix} u \\ \sin(\cos^{-1} u) \end{pmatrix}$$



Choose  $\psi(y) = \begin{pmatrix} y_1 \\ y_2 - \sin(\cos^{-1} y_1) \end{pmatrix}$ , which follows that

$$\begin{aligned} \psi \circ f \circ \phi^{-1}(u) &= \psi \circ f(\cos^{-1} u) \\ &= \psi \begin{pmatrix} \cos \cos^{-1} u \\ \sin \cos^{-1} u \end{pmatrix} = \psi \begin{pmatrix} u \\ \sin \cos^{-1} u \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} \end{aligned}$$

2.  $f(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_1 x_2)$ . Can we flatten out the curve of  $f$  near  $(0,0)$ ?

$$Df(x_1, x_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ x_2 & x_1 \end{pmatrix},$$

which is of rank 2 throughout  $\mathbb{R}^2$ .

Note that

$$\phi(x_1, x_2) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

and

$$g = f \circ \phi^{-1}(u_1, u_2) = f \begin{pmatrix} \frac{u_1 + u_2}{2} \\ \frac{u_1 - u_2}{2} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \frac{u_1^2 - u_2^2}{4} \end{pmatrix}$$

and define

$$\psi(y) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 - \frac{y_1^2 - y_2^2}{4} \end{pmatrix}.$$

Thus in summary, we have

$$\psi \circ f \circ \phi^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \psi \begin{pmatrix} u_1 \\ u_2 \\ \frac{u_1^2 - u_2^2}{4} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$$

## 12.1.2. Functional Dependence

In linear algebra we have talked about the linear independence. Given  $n$  vectors  $v_1, \dots, v_n$ , they are linear dependent if  $\exists a_1, \dots, a_n$  not all zero such that

$$a_1 v_1 + \dots + a_n v_n = 0$$

Then we talk about the dependence between functions.

**Definition 12.1** [Dependence] A set of **continuous** functions  $f_1, \dots, f_n : U \rightarrow \mathbb{R}$ , where  $U \subseteq \mathbb{R}^m$  is a neighborhood of  $\mathbf{x}_0 \in \mathbb{R}^m$ , is said to be **functionally independent** if for any continuous function

$$F(y) = F(y_1, \dots, y_n)$$

in a neighborhood  $V$  of  $\mathbf{y}_0 = f(\mathbf{x}_0) = (f_1(\mathbf{x}_0), \dots, f_n(\mathbf{x}_0))$ , the relation  $F(f_1(\mathbf{x}), \dots, f_n(\mathbf{x})) \equiv 0$  for  $\forall \mathbf{x} \in U$  is the only possible when  $F \equiv 0$  in  $V$ . ■

**Proposition 12.1** Let  $\{f_1, \dots, f_n\}$  be  $\mathcal{C}^1$  and the rank of

$$\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_m)}$$

is  $k$  at every  $\mathbf{x} \in U$ , then

1.  $k = n$  implies  $\{f_1, \dots, f_n\}$  is functionally independent
2.  $k < n$  implies there exists a neighborhood of  $\mathbf{x}_0$  and  $k$  functions  $f_1, \dots, f_k$  such that the rest of  $(n - k)$  functions can be written as

$$f_j(\mathbf{x}) = g_i(f_1(\mathbf{x}), \dots, f_k(\mathbf{x}))$$

for  $\forall i = k + 1, \dots, n$ , where  $g_i$  are  $\mathcal{C}^1$  functions of  $k$  variables.

