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MAT3006 & 3040 & 4002 Notebook

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Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i, j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i, j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i, j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 3

Week3

3.1. Monday for MAT3040

Reviewing.

1. Complementation. Suppose $\dim(V) = n < \infty$, then $W \le V$ implies $\exists W'$ such that
$$W \oplus W' = V.$$
2. Given the linear transformation $T : V \to W$, define the set $\ker(T)$ and $\mathrm{Im}(T)$
3. Isomorphism of vector spaces:
$$T : V \cong W$$
4. Rank-Nullity Theorem

R On isomorphism T on vector spaces,

1. The set $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}$ is linearly independent in V if and only if $\{T\boldsymbol{v}_1,\dots,T\boldsymbol{v}_k\}$ is linearly independent.
   Question 8 in Homework 1: if $T : V \to W$ is injective, then $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}$ is linearly independent in V implies $\{T\boldsymbol{v}_1,\dots,T\boldsymbol{v}_k\}$ is linearly independent.
2. The same goes if we replace the linearly independence by spans.
3. If $\dim(V) = n$, then $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_n\}$ forms a basis of V if and only if $\{T\boldsymbol{v}_1,\dots,T\boldsymbol{v}_n\}$ forms a basis of W. In particular, $\dim(V) = \dim(W)$.
   Actually, if $\dim(V) = \dim(W) = n$, then $V \cong W$.

3.1.1. Change of Basis and Matrix Representation

Definition 3.1 Let V be a finite dimensional vector space and $B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ an **ordered** basis of V. The **coordinate vector** of a vector $\boldsymbol{v} \in V$ is given by:

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n \implies \mapsto [\boldsymbol{v}]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Note that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} \neq \{\boldsymbol{v}_2, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ w.r.t. ordered basis.

Example 3.1 Given $V = M_{2\times 2}(\mathbb{F})$ and

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \right\}$$

any matrix has the coordinate vector w.r.t. B:

$$\left[\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \right]_B = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 3 \end{pmatrix}$$

Howver, given another ordered basis

$$B = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \right\}$$

any matrix has the coordinate vector w.r.t. B_1:

$$\left[\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\right]_{B_1} = \begin{pmatrix} 4 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

Theorem 3.1 The mapping

$$[]_B : V \mapsto \mathbb{F}^n$$

is an isomorphism of vector spaces.

Proof.

1. The operator $[]_B$ is well-defined: suppose

$$[\boldsymbol{v}]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \qquad [\boldsymbol{v}]_B = \begin{pmatrix} \alpha_1' \\ \vdots \\ \alpha_n' \end{pmatrix},$$

then

$$\begin{aligned} \boldsymbol{v} &= \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n \\ &= \alpha_1' \boldsymbol{v}_1 + \cdots + \alpha_n' \boldsymbol{v}_n \end{aligned}$$

By the uniqueness of coordinates, we imply $\alpha_i = \alpha_i'$ for $i = 1, \dots, n$.

2. The operator $[]_B$ is a linear transformation, i.e.,

$$[p\boldsymbol{v} + q\boldsymbol{w}]_B = p[\boldsymbol{v}]_B + q[\boldsymbol{w}]_B, \qquad p, q \in \mathbb{F}$$

3. The operator $[]_B$ is surjective: suppose

$$[\boldsymbol{v}]_B = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{F}^n,$$

then $\boldsymbol{v} = 0\boldsymbol{v}_1 + \cdots + 0\boldsymbol{v}_n = \mathbf{0}$.

4. The injective is clear.

Therefore, $[]_B$ is an isomorphism. ■

Exercise: if $\dim(V) = \dim(W)$, and $T : V \to W$ is injective, then $V \cong W$.

■ **Example 3.2** For $V = P_3[x]$ and its basis $B = \{1, x, x^2, x^3\}$.

To check if the set $\{1 + x^2, 3 - x^3, x - x^3\}$ is linearly independent, it suffices to check the corresponding coordinate vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

is linearly independent (Do Gaussian Elimination and check the number of pivots). ■

■ **Example 3.3** Questions: if B_1, B_2 form two basis of V, then how are $[\boldsymbol{v}]_{B_1}, [\boldsymbol{v}]_{B_2}$ related to each other.

Suppose $V = \mathbb{R}^n$ and $B_1 = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$. For any $\boldsymbol{v} \in V$,

$$\boldsymbol{v} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_n \boldsymbol{e}_1 + \cdots + \alpha_n \boldsymbol{e}_n \implies [\boldsymbol{v}]_{B_1} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Also, it is clear that the B_2 forms a basis as well:

$$B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

which implies

$$[\boldsymbol{v}]_{B_2} = \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_2 - \alpha_3 \\ \vdots \\ \alpha_{n-1} - \alpha_n \\ \alpha_n \end{pmatrix}$$

Proposition 3.1 — Change of Basis. Let $A = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ and $A' = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ be two basis of a vector space V. Suppose $\boldsymbol{v}_j = \sum_{i=1}^n \alpha_{ij} \boldsymbol{w}_i$ for $j = 1, \dots, n$. Then the **change of basis** matrix

$$\mathcal{C}_{A',A} = \begin{pmatrix} \alpha_{ij} \end{pmatrix}_{i,j=1,\dots,n}$$

satisfies the following:

$$\mathcal{C}_{A',A}[\boldsymbol{v}]_A = [\boldsymbol{v}]_{A'} \tag{3.1}$$

Also, the matrix $\mathcal{C}_{A',A}$ is invertible with the inverse

$$(\mathcal{C}_{A',A})^{-1} = \mathcal{C}_{A,A'}$$

where $\mathcal{C}_{A,A'} = (\beta_{ij})$, with β_{ij} satisfying

$$\boldsymbol{w}_j = \sum_{i=1}^n \beta_{ij} \boldsymbol{v}_i$$

Proof. Consider $\boldsymbol{v} = \boldsymbol{v}_j$, then LHS of (3.1) is

$$(\alpha_{ij})\boldsymbol{e}_j = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix}$$

the RHS of (3.1) is

$$[\boldsymbol{v}_j]_{A'} = [\sum_{i=1}^{n} \alpha_i \boldsymbol{w}_i]_{A'} = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix} = \text{LHS}$$

Therefore, $\mathcal{C}_{A',A}[\boldsymbol{v}_j]_A = [\boldsymbol{v}_j]_{A'}$ for $\forall j = 1, \ldots, n$.

Then for all $\boldsymbol{v} = r_1\boldsymbol{v}_1 + \cdots + r_n\boldsymbol{v}_n$,

$$\begin{aligned}
\mathcal{C}_{A',A}[\boldsymbol{v}]_A &= \mathcal{C}_{A',A}[r_1\boldsymbol{v}_1 + \cdots + r_n\boldsymbol{v}_n]_A \\
&= \mathcal{C}_{A',A}\left[r_1[\boldsymbol{v}_1]_A + \cdots + r_n[\boldsymbol{v}_n]_A\right] \\
&= \sum_{j=1}^{n} r_j \mathcal{C}_{A',A}[\boldsymbol{v}_j]_A \\
&= \sum_{j=1}^{n} r_j [\boldsymbol{v}_j]_{A'} \\
&= \left[\sum_{j=1}^{n} r_j \boldsymbol{v}_j\right]_{A'} \\
&= (\boldsymbol{v})_{A'}
\end{aligned}$$

Noew, suppose

$$\begin{aligned}
\boldsymbol{v}_j &= \sum_{i=1}^{n} \alpha_{ij} \boldsymbol{w}_i \\
&= \sum_{i=1}^{n} \alpha_{ij} \sum_{k=1}^{n} \beta_{ki} \boldsymbol{v}_k \\
&= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \beta_{ki} \alpha_{ij} \right) \boldsymbol{v}_i
\end{aligned}$$

By the uniqueness of coordinates, we imply

$$\left(\sum_{i=1}^{n}\beta_{ki}\alpha_{ij}\right)=\begin{cases}1, & j=k\\ 0, & j\neq k\end{cases}$$

where

$$\left(\sum_{i=1}^{n}\beta_{ki}\alpha_{ij}\right)=(\mathcal{C}_{AA'}\mathcal{C}_{A'A})$$

Therefore, $(\mathcal{C}_{AA'}\mathcal{C}_{A'A})=\boldsymbol{I}_n$. ■

■ **Example 3.4** Back to Example (3.3), suppose

$$B_1=\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\},\qquad B_2=\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_n\}$$

and suppose $\boldsymbol{w}_i=\boldsymbol{e}_1+\cdots+\boldsymbol{e}_i$. Therefore,

$$\mathcal{C}_{B_1,B_2}=\begin{pmatrix}1&1&\cdots&1\\0&1&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}$$

and

$$\mathcal{C}_{B_1,B_2}[\boldsymbol{v}]_{B_2}=\begin{pmatrix}1&1&\cdots&1\\0&1&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\begin{pmatrix}\alpha_1-\alpha_2\\\vdots\\\alpha_{n-1}-\alpha_n\\\alpha_n\end{pmatrix}=\begin{pmatrix}\alpha_1\\\vdots\\\alpha_n\end{pmatrix}=[\boldsymbol{v}]_{B_2}$$

■

Definition 3.2 Let $T:V\to W$ be a linear transformation, and

$$\mathcal{A}=\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\},\quad \mathcal{B}=\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_m\}$$

be bases of V and W, respectively. The **matrix representation** of T with respect to (w.r.t.) $\mathcal{A}$ and $\mathcal{B}$ is given by:

$$T(\boldsymbol{v}_j) = \sum_{i=1}^{m} \alpha_{ij} \boldsymbol{w}_j \implies (T)_{\mathcal{B}\mathcal{A}} = (\alpha_{ij})_{i,j=1,\ldots,m},$$

where $(T)_{\mathcal{B}\mathcal{A}} \in M_{m\times m}(\mathbb{F})$. ▪