

**A JOURNEY
IN
PURE MATHEMATICS**

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MAT3006 & 3040 & 4002 Notebook

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CUHK(SZ)

Notations and Conventions

\mathbb{R}^n	n -dimensional real space
\mathbb{C}^n	n -dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
x_i	i th entry of column vector \mathbf{x}
a_{ij}	(i, j) th entry of matrix \mathbf{A}
\mathbf{a}_i	i th column of matrix \mathbf{A}
\mathbf{a}_i^T	i th row of matrix \mathbf{A}
\mathbb{S}^n	set of all $n \times n$ real symmetric matrices, i.e., $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all i, j
\mathbb{H}^n	set of all $n \times n$ complex Hermitian matrices, i.e., $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
\mathbf{A}^T	transpose of \mathbf{A} , i.e, $\mathbf{B} = \mathbf{A}^T$ means $b_{ji} = a_{ij}$ for all i, j
\mathbf{A}^H	Hermitian transpose of \mathbf{A} , i.e, $\mathbf{B} = \mathbf{A}^H$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\text{trace}(\mathbf{A})$	sum of diagonal entries of square matrix \mathbf{A}
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
\mathbf{e}_i	a unit vector with the nonzero element at the i th entry
$\mathcal{C}(\mathbf{A})$	the column space of \mathbf{A}
$\mathcal{R}(\mathbf{A})$	the row space of \mathbf{A}
$\mathcal{N}(\mathbf{A})$	the null space of \mathbf{A}
$\text{Proj}_{\mathcal{M}}(\mathbf{A})$	the projection of \mathbf{A} onto the set \mathcal{M}

Chapter 1

Week1

1.1. Monday for MAT3040

1.1.1. Introduction to Advanced Linear Algebra

Advanced Linear Algebra is one of the most important course in MATH major, with pre-request MAT2040. This course will offer the really linear algebra knowledge.

What the content will be covered?.

- In MAT2040 we have studied the space \mathbb{R}^n ; while in MAT3040 we will study the general vector space V .
- In MAT2040 we have studied the *linear transformation* between Euclidean spaces, i.e., $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$; while in MAT3040 we will study the linear transformation from vector spaces to vector spaces: $T : V \rightarrow W$
- In MAT2040 we have studied the eigenvalues of $n \times n$ matrix \mathbf{A} ; while in MAT3040 we will study the eigenvalues of a **linear operator** $T : V \rightarrow V$.
- In MAT2040 we have studied the dot product $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$; while in MAT3040 we will study the **inner product** $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$.

Why do we do the generalization?. We are studying many other spaces, e.g., $\mathcal{C}(\mathbb{R})$ is called the space of all functions on \mathbb{R} , $\mathcal{C}^\infty(\mathbb{R})$ is called the space of all infinitely differentiable functions on \mathbb{R} , $\mathbb{R}[x]$ is the space of polynomials of one-variable.

- **Example 1.1** 1. Consider the Laplace equation $\Delta f = 0$ with linear operator Δ :

$$\Delta : \mathcal{C}^\infty(\mathbb{R}^3) \rightarrow \mathcal{C}^\infty(\mathbb{R}^3) \quad f \mapsto \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f$$

The solution to the PDE $\Delta f = 0$ is the 0-eigenspace of Δ .

2. Consider the Schrödinger equation $\hat{H}f = Ef$ with the linear operator

$$\hat{H} : \mathcal{C}^\infty(\mathbb{R}^3) \rightarrow \mathbb{R}^3, \quad f \mapsto \left[\frac{-\hbar^2}{2\mu} \nabla^2 + V(x, y, z) \right] f$$

Solving the equation $\hat{H}f = Ef$ is equivalent to finding the eigenvectors of \hat{H} . In fact, the eigenvalues of \hat{H} are **discrete**.

1.1.2. Vector Spaces

Definition 1.1 [Vector Space] A **vector space** over a field \mathbb{F} (in particular, $\mathbb{F} = \mathbb{R}$ or \mathbb{C}) is a set of objects V equipped with vector addition and scalar multiplication such that

1. the vector addition $+$ is closed with the rules:

- (a) **Commutativity**: $\forall \mathbf{v}_1, \mathbf{v}_2 \in V, \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$.
- (b) **Associativity**: $\mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3$.
- (c) **Additive Identity**: $\exists \mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}, \forall \mathbf{v} \in V$.

2. the **scalar multiplication** is closed with the rules:

- (a) **Distributive**: $\alpha(\mathbf{v}_1 + \mathbf{v}_2) = \alpha\mathbf{v}_1 + \alpha\mathbf{v}_2, \forall \alpha \in \mathbb{F} \text{ and } \mathbf{v}_1, \mathbf{v}_2 \in V$
- (b) **Distributive**: $(\alpha_1 + \alpha_2)\mathbf{v} = \alpha_1\mathbf{v} + \alpha_2\mathbf{v}$
- (c) $0\mathbf{v} = \mathbf{0}, 1\mathbf{v} = \mathbf{v}$.

Here we study several examples of vector spaces:

■ **Example 1.2** For $V = \mathbb{F}^n$, we can define

1. Addictive Identity:

$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

2. Scalar Multiplication:

$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

3. Vector Addiction:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

■ **Example 1.3** 1. It is clear that the set $V = M_{n \times n}(\mathbb{F})$ (the set of all $m \times n$ matrices) is a vector space as well.

2. The set $V = \mathcal{C}(\mathbb{R})$ is a vector space:

(a) Vector Addiction:

$$(f + g)(x) = f(x) + g(x), \forall f, g \in V$$

(b) Scalar Multiplication:

$$(\alpha f)(x) = \alpha f(x), \forall \alpha \in \mathbb{R}, f \in V$$

(c) Addictive Identity is a zero function, i.e., $\mathbf{0}(x) = 0$ for all $x \in \mathbb{R}$.

Definition 1.2 A sub-collection $W \subseteq V$ of a vector space V is called a **vector subspace** of V if W itself forms a vector space, denoted by $W \leq V$. ■

- **Example 1.4**
1. For $V = \mathbb{R}^3$, we claim that $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \leq V$
 2. $W = \{(x, y, 1) \mid x, y \in \mathbb{R}\}$ is not the vector subspace of V . ■

Proposition 1.1 $W \subseteq V$ is a **vector subspace** of V iff for $\forall \mathbf{w}_1 + \mathbf{w}_2 \in W$, we have $\alpha \mathbf{w}_1 + \beta \mathbf{w}_2 \in W$, for $\forall \alpha, \beta \in \mathbb{F}$.

- **Example 1.5**
1. For $V = M_{n \times n}(\mathbb{F})$, the subspace $W = \{A \in V \mid \mathbf{A}^T = \mathbf{A}\} \leq V$
 2. For $V = \mathcal{C}^\infty(\mathbb{R})$, define $W = \{f \in V \mid \frac{d^2}{dx^2}f + f = 0\} \leq V$. For $f, g \in W$, we have

$$(\alpha f + \beta g)'' = \alpha f'' + \beta g'' = \alpha(-f) + \beta(-g) = -(\alpha f + \beta g),$$

which implies $(\alpha f + \beta g)'' + (\alpha f + \beta g) = 0$. ■

1.2. Monday for MAT3006

1.2.1. Overview on uniform convergence

Definition 1.3 [Convergence] Let $f_n(x)$ be a sequence of functions on an interval $I = [a, b]$. Then $f_n(x)$ converges **pointwise** to $f(x)$ (i.e., $f_n(x_0) \rightarrow f(x_0)$) for $\forall x_0 \in I$, if

$$\forall \varepsilon > 0, \exists N_{x_0, \varepsilon} \text{ such that } |f_n(x_0) - f(x_0)| < \varepsilon, \forall n \geq N_{x_0, \varepsilon}$$

We say $f_n(x)$ converges **uniformly** to $f(x)$, (i.e., $f_n(x) \Rightarrow f(x)$) for $\forall x_0 \in I$, if

$$\forall \varepsilon > 0, \exists N_\varepsilon \text{ such that } |f_n(x_0) - f(x_0)| < \varepsilon, \forall n \geq N_\varepsilon$$

■ **Example 1.6** It is clear that the function $f_n(x) = \frac{n}{1+nx}$ converges pointwise into $f(x) = \frac{1}{x}$ on $[0, \infty)$, and it is uniformly convergent on $[1, \infty)$. ■

Proposition 1.2 If $f_n(x)$ is continuous on I , $\forall n$, and $f_n(x) \Rightarrow f(x)$. Then

1. $f(x)$ is continuous on I .
2. $\int_a^b f_n(x) dx \rightarrow \int_a^b f(x) dx$.
3. Suppose furthermore that $f_n(x)$ is **continuously differentiable**, and $f'_n(x) \Rightarrow g(x)$, then $f(x)$ is differentiable, and $f'_n(x) \rightarrow f'(x)$.

Proposition 1.3 Putting the discussions above into the content of series, i.e., $f_n(x) = \sum_{k=1}^n S_k(x)$. If $S_k(x)$ is continuous for $\forall k$, and $f_n(x) \Rightarrow f(x) := \sum_{k=1}^\infty S_k(x)$, then

1. $f(x) = \sum_{k=1}^\infty S_k(x)$ is continuous,
2. $\sum_{k=1}^\infty \int_a^b S_k(x) dx = \int_a^b \sum_{k=1}^\infty S_k(x) dx$
3. blabla, then

$$\left(\sum_{k=1}^\infty S_k(x) \right)' = \sum_{k=1}^\infty S'_k(x)$$

Proposition 1.4 For power series, i.e., $S_k(x) = a_k x^k$. Suppose $f(x) = \sum_{k=1}^\infty a_k x^k$ has

radius of convergence R , then

$$\sum_{k=1}^n a_k x^k \Rightarrow f(x)$$

for any $[-L, L]$ with $L < R$. Then $f(x)$ is continuous, and

$$\int_0^x f(t) dt = \sum_{k=1}^{\infty} \frac{a_k}{k+1} x^{k+1}$$

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

1.2.2. Introduction to MAT3006

What are we going to do.

1. (a) Generalize our study of (sequence, series, functions) on \mathbb{R}^n into a metric space.
(b) We will study spaces outside \mathbb{R}^n .

Remark:

- For (a), different metric may yield different kind of convergence of sequences. For (b), one important example we will study is $X = \mathcal{C}[a, b]$ (all continuous functions defined on $[a, b]$.) We will generalize X into $\mathcal{C}_b(E)$, which means the set of bounded continuous functions defined on $E \subseteq \mathbb{R}^n$.
- The insights of analysis is to find a **unified** theory to study sequences/series on a metric space X , e.g., $X = \mathbb{R}^n, \mathcal{C}[a, b]$. In particular, for $\mathcal{C}[a, b]$, we will see that
 - most functions in $\mathcal{C}[a, b]$ are nowhere differentiable. (repeat part of content in MAT2006)
 - We will prove the existence and uniqueness of ODEs.
 - the set $\text{poly}[a, b]$ (the set of polynomials on $[a, b]$) is dense in $\mathcal{C}[a, b]$. (analogy: $\mathbb{Q} \subseteq \mathbb{R}$ is dense)

2. Introduction to the Lebesgue Integration.

For convergence of integration $\int_a^b f_n(x) dx \rightarrow \int_a^b f(x)$, we need the pre-conditions

(a) $f_n(x)$ is continuous, and (b) $f_n(x) \Rightarrow f(x)$. The natural question is that Can we relax the condition to

- (a) $f_n(x)$ is integrable?
- (b) $f_n(x) \rightarrow f(x)$ pointwisely?

The answer is yes, by using the tool of Lebesgue integration. If $f_n(x) \rightarrow f(x)$ and $f_n(x)$ is Lebesgue integrable, then $\int_a^b f_n(x) dx \rightarrow \int_a^b f(x) dx$, which is so called the **dominated convergence**.

1.2.3. Metric Spaces

Normed Space. We will study the distance and length of elements in an arbitrary set X .

Definition 1.4 [Normed Space] Let X be a vector space. A **norm** on X is a function $\|\cdot\| : X \rightarrow \mathbb{R}$ such that

1. $\|\mathbf{x}\| \geq 0$ for $\forall \mathbf{x} \in X$, with equality iff $\mathbf{x} = \mathbf{0}$
2. $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$, for $\forall \alpha \in \mathbb{R}$ and $\mathbf{x} \in X$.
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangular inequality)

Any vector space equipped with $\|\cdot\|$ is called a **normed space**. ■

■ **Example 1.7** 1. For $X = \mathbb{R}^n$, define

$$\|\mathbf{x}\|_2 = (\sum_{i=1}^n x_i^2)^{1/2} \quad (\text{Euclidean Norm})$$

$$\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p} \quad (p\text{-norm})$$

2. For $X = \mathcal{C}[a, b]$, define

$$\|f\|_\infty = \max_{x \in [a, b]} |f(x)|$$

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

Exercise: check the norm defined above are well-defined. ■

Here raise a question: can we define the distance in an arbitrary set?

Definition 1.5 A set X is a **metric space** with metric (X, d) if there exists a (distance) function $d : X \times X \rightarrow \mathbb{R}$ such that

1. $d(\mathbf{x}, \mathbf{y}) \geq 0$ for $\forall \mathbf{x}, \mathbf{y} \in X$, with equality iff $\mathbf{x} = \mathbf{y}$.
 2. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
 3. $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.
-

- **Example 1.8**
1. If X is a normed space, then define $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$, which is so called the metric induced from the norm $\|\cdot\|$.
 2. Let X be any (non-empty) set with $\mathbf{x}, \mathbf{y} \in X$, the discrete metric is given by:

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \text{if } \mathbf{x} = \mathbf{y} \\ 1, & \text{if } \mathbf{x} \neq \mathbf{y} \end{cases}$$

Exercise: check the metric space defined above are well-defined. ■

So we have defined a metric on $\mathcal{C}[a, b]$ by

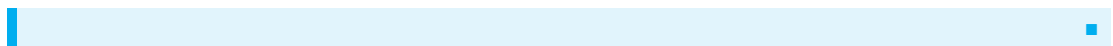
$$d_\infty(f, g) = \|f - g\|_\infty := \max_{x \in [a, b]} |f(x) - g(x)|$$

This is the correct metric to study the uniform convergence for $\{f_n\} \subseteq \mathcal{C}[a, b]$.

Definition 1.6 Let (X, d) be a metric space. An **open ball** centered at $\mathbf{x} \in X$ of radius r is

$$B_r(\mathbf{x}) = \{\mathbf{y} \in X \mid d(\mathbf{x}, \mathbf{y}) < r\}.$$
■

- **Example 1.9**
1. $X = \mathbb{R}^2$, and $d_2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2$. Graph:
 $d_1(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_1$. Graph:



1.3. Monday for MAT4002

1.3.1. Introduction to Topology

We will study global properties of a geometric object, i.e., the distance between 2 points in an object is totally ignored.

For example, consider the topology illustrated below.

Therefore, the difference relies on the connectedness. Why the last graph is intrinsically different?

In this course, we are going to introduce the mathematics to describe the phenomenon above.

First you learn about distances, and then you forget about distances.

1.3.2. Metric Spaces

Definition 1.7 [Metric Space] Metric space is a set X where one can measure distance between any two objects in X .

Specifically speaking, a metric space X is a non-empty set endowed with a function (distance function) $d : X \times X \rightarrow \mathbb{R}$ such that

1. $d(\mathbf{x}, \mathbf{y}) \geq 0$ for $\forall \mathbf{x}, \mathbf{y} \in X$ with equality iff $\mathbf{x} = \mathbf{y}$
2. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$
3. $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ (triangular inequality)

■ **Example 1.10** 1. Let $X = \mathbb{R}^n$, with

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{i=1, \dots, n} |x_i - y_i|$$

2. Let X be any set, and define the discrete metric

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Homework: Show that (1) and (2) defines a metric. ■

Definition 1.8 [Open Ball] An **open ball** of radius r centered at $\mathbf{x} \in X$ is defined as

$$B_r(\mathbf{x}) = \{\mathbf{y} \in X \mid d(\mathbf{x}, \mathbf{y}) < r\}$$

■

- **Example 1.11**
1. Under the metric $(X = \mathbb{R}^2, d_2)$, the set $B_1(0,0)$ defines an open ball
 2. Under the metric $(X = \mathbb{R}^2, d_\infty)$, the set $B_1(0,0)$ defines an open ball
 3. Under the metric $(X = \mathbb{R}^2, \text{discrete metric})$, the set $B_1(0,0)$ is one single point, also defines an open ball.

■

Definition 1.9 [Open Set] Let X be a metric space, $U \subseteq X$ is an open set in X if $\forall u \in U$, there exists $\epsilon_u > 0$ such that $B_{\epsilon_u}(u) \subseteq U$. ■

We denote \mathcal{T} to be the collection of all open sets in (X, d) , which is so called the **topology** induced from (X, d) .

- **Example 1.12**
1. All open balls $B_r(\mathbf{x})$ are open in (X, d) , e.g., for $X = \mathbb{R}$ and d_2 , we have $B_r(x) = (x - r, x + r)$. Take $\mathbf{y} \in B_r(\mathbf{x})$ such that $d(\mathbf{x}, \mathbf{y}) = q < r$ and consider $B_{(r-q)/2}(\mathbf{y})$, then for all $z \in B_{(r-q)/2}(\mathbf{y})$, we have

$$d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) < q + \frac{r-q}{2} < r,$$

which implies $z \in B_r(x)$. ■

Proposition 1.5 Let (X, d) be metric space, and \mathcal{T} is the topology induced from (X, d) , then

1. let the set $\{G_\alpha \mid \alpha \in \mathcal{A}\}$ be a collection of (uncountable) open sets, i.e., $G_\alpha \in \mathcal{T}$, then $\bigcup_{\alpha \in \mathcal{A}} G_\alpha \in \mathcal{T}$.
2. Let $G_1, \dots, G_n \in \mathcal{T}$, then $\bigcap_{i=1}^n G_i \in \mathcal{T}$. The finite intersection of open sets is open.

Proof. 1. Take $x \in \bigcup_{\alpha \in \mathcal{A}} G_\alpha$, then $x \in G_\beta$ for some $\beta \in \mathcal{A}$. Since G_β is open, there exists $\epsilon_x > 0$ s.t.

$$B_{\epsilon_x}(x) \subseteq G_\beta \subseteq \bigcup_{\alpha \in \mathcal{A}} G_\alpha$$

2. Take $x \in \bigcap_{i=1}^n G_i$, i.e., $x \in G_i$ for $i = 1, \dots, n$, i.e., there exists $\epsilon_i > 0$ such that $B_{\epsilon_i}(x) \subseteq G_i$ for $i = 1, \dots, n$. Take $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$, then

$$B_\epsilon(x) \subseteq B_{\epsilon_i}(x) \subseteq G_i, \forall i$$

which implies $B_\epsilon(x) \subseteq \bigcap_{i=1}^n G_i$ ■

Exercise: let $\mathcal{T}_2, \mathcal{T}_\infty$ be topologies induced from the metrics d_2, d_∞ in \mathbb{R}^2 . Then $J_2 = J_\infty$, i.e., every open set in (\mathbb{R}^2, d_2) is open in (\mathbb{R}^2, d_∞) , and every open set in (\mathbb{R}^2, d_∞) is open in (\mathbb{R}^2, d_2) .

Let \mathcal{T} be topology induced from the discrete metric (X, d_{discrete}) . What is \mathcal{T} ?