# A FIRST COURSE

# IN

# ANALYSIS

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# MAT2006 Notebook


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# Contents

# Acknowledgments

This book is taken notes from the MAT2006 in fall semester, 2018. These lecture notes were taken and compiled in LaTeX by Jie Wang, an undergraduate student in Fall 2018. Prof. Weiming Ni has not edited this document. Students taking this course may use the notes as part of their reading and reference materials. This version of the lecture notes were revised and extended for many times, but may still contain many mistakes and typos, including English grammatical and spelling errors, in the notes. It would be greatly appreciated if those students, who will use the notes as their reading or reference material, tell any mistakes and typos to Jie Wang for improving this notebook.

# Notations and Conventions

| | |
|---|---|
| $\mathbb{R}^n$ | $n$-dimensional real space |
| $\mathbb{C}^n$ | $n$-dimensional complex space |
| $\mathbb{R}^{m\times n}$ | set of all $m \times n$ real-valued matrices |
| $\mathbb{C}^{m\times n}$ | set of all $m \times n$ complex-valued matrices |
| $x_i$ | $i$th entry of column vector $\boldsymbol{x}$ |
| $a_{ij}$ | $(i,j)$th entry of matrix $\boldsymbol{A}$ |
| $\boldsymbol{a}_i$ | $i$th column of matrix $\boldsymbol{A}$ |
| $\boldsymbol{a}_i^{\mathrm{T}}$ | $i$th row of matrix $\boldsymbol{A}$ |
| $\mathbb{S}^n$ | set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all $i,j$ |
| $\mathbb{H}^n$ | set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all $i,j$ |
| $\boldsymbol{A}^{\mathrm{T}}$ | transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all $i,j$ |
| $\boldsymbol{A}^{\mathrm{H}}$ | Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all $i,j$ |
| $\mathrm{trace}(\boldsymbol{A})$ | sum of diagonal entries of square matrix $\boldsymbol{A}$ |
| $\mathbf{1}$ | A vector with all 1 entries |
| $\mathbf{0}$ | either a vector of all zeros, or a matrix of all zeros |
| $\boldsymbol{e}_i$ | a unit vector with the nonzero element at the $i$th entry |
| $\mathcal{C}(\boldsymbol{A})$ | the column space of $\boldsymbol{A}$ |
| $\mathcal{R}(\boldsymbol{A})$ | the row space of $\boldsymbol{A}$ |
| $\mathcal{N}(\boldsymbol{A})$ | the null space of $\boldsymbol{A}$ |
| $\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$ | the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$ |

# Chapter 7

# Week7

## 7.1. Wednesday

**Announcement.** Our mid-term is on next Wednesday in Liwen Building, from 8:00am to 10:00am. We will cover everything until this Friday.

### 7.1.1. Integrable Analysis

**Recap.** Given a sequence of functions $\{f_n\}$ with pointwise limit $f$, we are curious about whether the equation holds:

$$\lim_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x$$

Let's give a counter-example to show this equaiton may not necessarily true.

**Example 7.1** Let $\{f_n\}$ defined on $[0,1]$ with

$$f_n(x) = \begin{cases} n, & \text{if } x \in (0, \frac{1}{n}) \\ 0, & \text{otherwise} \end{cases}$$

We find that $\int_0^1 f_n\,\mathrm{d}x = 1$, and $f_n \to 0$ as $n \to \infty$. Thus

$$\int_0^1 \left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x = 0 \neq \lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x$$

There is a **sufficient** condition that guarantees the equation holds:

**Theorem 7.1** Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a,b]$. If $f_n$ converges to $f$ uniformly as $n \to \infty$, then $f$ is also **Riemann integrable**, and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

**Definition 7.1** We say that $f_n$ converges to $f$ uniformly as $n \to \infty$ on $[a,b]$ if for every $\varepsilon > 0$, there exists $N$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [a,b]$ and for all $n \geq N$. ■

*Proof.*

- **Step 1:** First we need to show that both $\int_a^b f_n(x)\,\mathrm{d}x$ and $\int_a^b f(x)\,\mathrm{d}x$ is well-defined, i.e., $f$ and $f_n$ is **uniformly bounded**, i.e., there exists $M, M' > 0$ such that $|f(x)| \leq M$ and $|f_n(x)| \leq M', \forall n$. First show that $\{f_n\}$ is uniformly bounded:

$$\begin{aligned} |f_n(x) - f_k(x)| &= |f_n(x) - f(x) + f(x) - f_k(x)| && (7.1a) \\ &\leq |f_n(x) - f(x)| + |f(x) - f_k(x)| && (7.1b) \end{aligned}$$

Due to the uniform convergence of $\{f_n\}$, we choose $\varepsilon := 1$, then there exists $N > 0$ s.t.

$$|f_m(x) - f(x)| < 1, \qquad \forall m \geq N. \tag{7.1c}$$

Therefore, we give a bound on (7.1a):

$$|f_n(x) - f_k(x)| < 2, \quad \forall n, k \geq N \tag{7.1d}$$

In particular, take $k = N$, thus

$$|f_n(x) - f_N(x)| < 2 \implies |f_n(x)| < |f_N(x)| + 2, \qquad \forall n \geq N, \tag{7.1e}$$

i.e., every $f_n$ for $n \geq N$ is bounded from $|f_N(x)|$ as 2. Therefore, we have $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded by $M$. (just set $M = \max\{|f_1(x)|, \ldots, |f_{N-1}(x)|, |f_N| + 2\}$.)

Another application of (7.1c) gives the uniform boundness of $f$:

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + |f_N(x)|.$$

- **Step 2:** Argue the Riemann integrability of $f$. Define $\varepsilon_n = \sup_{a \leq x \leq b} |f_n(x) - f(x)|$, and $\varepsilon_n \to 0$ as $n \to \infty$. Therefore, we give bounds on $f$:

$$-\varepsilon_n + f_n(x) \leq f(x) \leq \varepsilon_n + f_n(x) \tag{7.2a}$$

So that the lower and upper integrals of $f$ satisfy:

$$\underline{\int_a^b}[-\varepsilon_n + f_n(x)]\,\mathrm{d}x \leq \underline{\int_a^b} f(x)\,\mathrm{d}x \leq \overline{\int_a^b} f(x)\,\mathrm{d}x \leq \overline{\int_a^b}[\varepsilon_n + f_n(x)]\,\mathrm{d}x \tag{7.2b}$$

Note that $f_n$ is integrable, so we can remove the upper and lower integral symbols of $f_n \pm \varepsilon_n$:

$$\int_a^b f_n(x) - \varepsilon_n\,\mathrm{d}x \leq \underline{\int_a^b} f(x)\,\mathrm{d}x \leq \overline{\int_a^b} f(x)\,\mathrm{d}x \leq \int_a^b f_n(x) - \varepsilon_n\,\mathrm{d}x \tag{7.2c}$$

Hence we give a bound on the difference of upper and lower integrals of $f$:

$$0 \leq \overline{\int_a^b} f(x)\,\mathrm{d}x - \underline{\int_a^b} f(x)\,\mathrm{d}x \leq 2(b-a)\varepsilon_n, \tag{7.2d}$$

Since $\varepsilon_n \to 0$ as $n \to \infty$, the upper and lower integrals of $f$ are equal. Thus $f \in \mathcal{R}[a,b]$.

- Another application of (7.2c) now yields

$$\left|\int_a^b f(x) - f_n(x)\,\mathrm{d}x\right| \leq \int_a^b \varepsilon_n\,\mathrm{d}x = \varepsilon_n(b-a), \tag{7.3}$$

which implies $\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$.

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R The sequence of functions also remains the question that:

Would the equation (7.4) holds?

$$\lim_{n\to\infty} f_n'(x) = \left[\lim_{n\to\infty} f_n(x)\right]' \tag{7.4}$$

Equation (7.4) holds also depends on the uniform convergence of $\{f_n\}$.

### 7.1.2. Elementary Calculus Analysis

**Theorem 7.2 — Fundamental Theorem of Calculus.** If $f : [a,b] \mapsto \mathbb{R}$ is continuous, then the function $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is **differentiable** with $F' = f$.

*Proof.* The proof is simply by definition, keep in mind that difference quotient is useful in proofs related to differentiation.

$$\begin{aligned}
\frac{F(x+h)-F(x)}{h} - f(x) &= \frac{1}{h}\left[\int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t\right] - f(x) && \text{(7.5a)}\\
&= \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t - f(x) && \text{(7.5b)}\\
&= \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t - \frac{1}{h}\left[\int_x^{x+h} 1\,\mathrm{d}t\right] f(x) && \text{(7.5c)}\\
&= \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t - \frac{1}{h}\int_x^{x+h} f(x)\,\mathrm{d}t && \text{(7.5d)}\\
&= \frac{1}{h}\int_x^{x+h} [f(t) - f(x)]\,\mathrm{d}t, && \text{(7.5e)}
\end{aligned}$$

which implies that

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \le \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,\mathrm{d}t,$$

Then apply continuity condition to give a bound on $|f(t) - f(x)|$:

Since $f$ is continuous at $x$, for $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ if $|y - x| < \delta$. Therefore,

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \le \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,\mathrm{d}t \le \frac{1}{h}\int_x^{x+h} \varepsilon\,\mathrm{d}t = \varepsilon,$$

If $h < \delta$, we imply

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

■

The integraiton by parts is an important part from Calculus, the core idea is from the product rule for differentiation.

**Theorem 7.3 — Integration by Parts.** Given two functions $f, g \in \mathcal{C}^1[a,b]$, (similar to $(fg)' = f'g + fg'$), we have

$$\int_a^b (fg)' \,\mathrm{d}x = \int_a^b f'g \,\mathrm{d}x + \int_a^b fg' \,\mathrm{d}x,$$

or equivalently,

$$(fg)(a) - (fg)(b) = \int_a^b f'g \,\mathrm{d}x + \int_a^b fg' \,\mathrm{d}x,$$

i.e.,

$$\int_a^b fg' \,\mathrm{d}x = (fg)|_a^b - \int_a^b f'g \,\mathrm{d}x$$

There are two versions of change of variables in Calculus. We will discuss the difference of these.

**Proposition 7.1 — Change of variables,version 1.** Let $\phi : [\alpha, \beta] \mapsto [a,b]$ be a continuously differentiable function such that

$$\phi(\alpha) = a, \quad \phi(\beta) = b.$$

Then for every continuous function $f : [a,b] \mapsto \mathbb{R}$, we have

$$\int_a^b f(x) \,\mathrm{d}x = \int_\alpha^\beta f(\phi(t))\phi'(t) \,\mathrm{d}t$$

*Proof.* Define $F(x) = \int_a^x f(t) \,\mathrm{d}t$, which implies

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = f(x), \qquad \int_a^b f(x) \,\mathrm{d}x = F(b).$$

Observe that

$$\frac{\mathrm{d}F(\phi(t))}{\mathrm{d}t} = \frac{\mathrm{d}F(\phi(t))}{\phi(t)} \frac{\phi(t)}{\mathrm{d}t} = f(\phi(t))\phi'(t)$$

Or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}(F \circ \phi)(t) = f(\phi(t))\phi'(t)$$

Therefore,

$$\begin{aligned}
\int_\alpha^\beta (F \circ \phi)'(t)\,\mathrm{d}t &= \int_\alpha^\beta f(\phi(t))\phi'(t)\,\mathrm{d}t && (7.6)\\
&= (F \circ \phi)(\beta) - (F \circ \phi)(\alpha) = F(\phi(\beta)) - F(\phi(\alpha)) && (7.7)\\
&= F(b) - F(a) = F(b) && (7.8)\\
&= \int_a^b f(x)\,\mathrm{d}x && (7.9)
\end{aligned}$$

■

**Proposition 7.2 — Change of variables,version 2.** Let $\phi : [\alpha, \beta] \mapsto [a, b]$ be continuously differentiable and **strictly monotone**. Then for any $f \in \mathcal{R}[a, b]$, we have

1. $f(\phi(t))\phi'(t) \in \mathcal{R}[\alpha, \beta]$
2. $$\int_\alpha^\beta f(\phi(t))\phi'(t) = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,\mathrm{d}x$$

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- Comparing proposition(7.2) to (7.1), note that we **relax** $f$ from being continuously differentiable to being Riemann integrable; but **restrict** $\phi$ to be **strictly monotone**.
- The proof for proposition(7.2) is messy. For most time functions we have faced is *not continuous*, but we can break into finite sub-intervals and apply proposition(7.1). Thus the benifit for proposition(7.2) is not such huge. In practice, proposition(7.1) is enough.

Last, let's discuss a initutive fact of Riemann sum, i.e., as the mesh goes to zero, Riemann sums always converges to their corresponding integration

**Theorem 7.4** Let $f \in \mathcal{R}[a,b]$. Then a Riemann sum $S(\mathcal{P}, f)$ converges to $\int_a^b f(x)\,\mathrm{d}x$ as the mesh $\lambda(\mathcal{P}) \to 0$, i.e.,

$$\sum_{i=1}^{n} f(t_i)\Delta x_i \to \int_a^b f(x)\,\mathrm{d}x, \qquad \text{as } \max_{1\le i\le n} \Delta x_i \to 0,$$

where $t_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$.

We apply this theorem to evaluate some limits:

■ **Example 7.2** 1. Evaluate the limit

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right].$$

$$\begin{aligned}
x_n &= \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} \\
&= \frac{1}{n}\left[\frac{n}{n+1}+\frac{n}{n+2}+\cdots+\frac{n}{2n}\right] \\
&= \frac{1}{n}\left[\frac{1}{1+1/n}+\frac{1}{1+2/n}+\cdots+\frac{1}{1+n/n}\right] \\
&= \Delta x_i\left[f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n}{n})\right]
\end{aligned}$$

which is essentially the Riemann sum of function $f(x) = \frac{1}{1+x}$ over interval $[0,1]$. Therefore, as $n \to \infty$,

$$x_n \to \int_0^1 \frac{1}{1+x}\,\mathrm{d}x$$

2. Evaluate the limit

$$\lim_{n\to\infty}\frac{1^\alpha+\cdots+n^\alpha}{n^\alpha}$$

$$\begin{aligned}
x_n &= \frac{1}{n}\frac{1^\alpha+\cdots+n^\alpha}{n^\alpha} \\
&= \frac{1}{n}\left[(\frac{1}{n})^\alpha+(\frac{2}{n})^\alpha+\cdots+(\frac{n}{n})^\alpha\right] \\
&= \Delta x_i\left[f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n}{n})\right]
\end{aligned}$$

As $n \to \infty$,

$$x_n \to \int_0^1 x^\alpha \, \mathrm{d}x = \frac{1}{\alpha+1} x^{\alpha+1} \Big|_0^1 = \frac{1}{\alpha+1}$$

■