A FIRST COURSE

IN

NUMERICAL ANALYSIS

A FIRST COURSE

IN

NUMERICAL ANALYSIS

MAT4001 Notebook

Prof. Yutian Li

The Chinese University of Hongkong, Shenzhen

香港中文大學（深圳）
The Chinese University of Hong Kong, Shenzhen

Contents

Acknowledgments

This book is from the MAT4001 in fall semester, 2018.

CUHK(SZ)

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 3

Week3

3.1. Tuesday

Theorem 3.1 — Optimality Condition.

- primal feasible: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$
- Dual feasible: $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$
- Complementarity: $\boldsymbol{x} \circ \boldsymbol{s} = \boldsymbol{0}$, i.e., $x_i \cdot (c_i - \boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{y}) = \boldsymbol{0}$ for each i.

R (Primal) Simplex method:

1. Always keep primal feasibility:
2. Always keep complementarity:
   Define $\boldsymbol{y} = (\boldsymbol{A}_B^{-1})^{\mathrm{T}}\boldsymbol{c}_B$ as the dual solution. The reduced costs vector is $\boldsymbol{c}^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{A} = \boldsymbol{c} - \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}$
3. Not necessarily keep dual feasible until get the optimal solution, i.e., it will seeks solution that is dual feasible.

Dual Simplex method. Dual Simplex method remains both dual feasiblity and complementarity conditions in each iteration but seeks primal feasibility.

Cases for applying dual simplex method:

- There is a dual BFS available but no primal BFS available.
- $\boldsymbol{b}$ is changed by a large amount or a constraint isadded, i.e., lose the primal feasible solution.

Interior Point Method. Consider the relaxed version of optimality condition:

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \boldsymbol{x} \geq 0 \\ \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{s} &= \boldsymbol{c}, \boldsymbol{s} \geq 0 \\ x_i \cdot s_i &= \mu, \quad \forall i, \text{small } \mu_i > 0 \end{aligned}$$

Keep decreasing μ and finally get the solution to LP.

R

- The optimal solution output from interior point method may not necessarily BFS. If the optimal solution is unique, it is BFS.
- Initial solution for the interior point method can be found by solving the auxiliary problem.
- The complexity for interior point method is $O(n^{3.5})$
- The interior point method gives stable running time compared with simplex method.
- Interior point method always find the optimal solution with maximum possible number of **non-zeros**.
- Interior point method finds high-rank solution (the center of all optimal solutions); but the simplex method finds the low-rank solution.

3.1.1. Reviewing

Linear optimization formulation. Standard Form LP Transformation

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{such that} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Maximin / minimax objective

Absolute values in objective function or constraints.

Theorem 3.2 The BFS for standard LP is equivalent to extreme point.

Theorem 3.3 If there is a feasible solution, then there is a basic feasible solution; If there is a optimal solution, then there is a basic feasible optimal solution.

Care about corollary

Simplex method.

1. Understand how simplex method works, and cases for unbounded, infeasible
2. Apply simplex method to solve small LPs
3. Read and interpret simplex tableau (make use of it to avoid inverse calculation)
4. Apply two-phase method

Duality Theory.

1. Be able to constrauct the dual for any LP.
2. Know the (strong/weak) duality theorems and apply them in different situations.
3. Be able to write down the complentarity conditions and apply them

Sensitivity Analysis. Related to duality theory;

Complexity Theory and interior method. Complexity of LP:

1. No guarntee of simplex method to achieve polynomial time
2. Interior point can achieve polynomial time

Properties of simplex method