

**A JOURNEY
IN
PURE MATHEMATICS**

A JOURNEY
IN
PURE MATHEMATICS
MAT3006 & 3040 & 4002 Notebook

Prof. Daniel Wong

The Chinese University of Hong Kong, Shenzhen



香港中文大學(深圳)

The Chinese University of Hong Kong, Shenzhen

Contents

Acknowledgments	xiii
Notations	xv
1 Week1	1
1.1 Monday for MAT3040	1
1.1.1 Introduction to Advanced Linear Algebra	1
1.1.2 Vector Spaces	2
1.2 Monday for MAT3006	5
1.2.1 Overview on uniform convergence	5
1.2.2 Introduction to MAT3006	6
1.2.3 Metric Spaces	7
1.3 Monday for MAT4002	10
1.3.1 Introduction to Topology	10
1.3.2 Metric Spaces	11
1.4 Wednesday for MAT3040	14
1.4.1 Review	14
1.4.2 Spanning Set	14
1.4.3 Linear Independence and Basis	16
1.5 Wednesday for MAT3006	20
1.5.1 Convergence of Sequences	20
1.5.2 Continuity	24
1.5.3 Open and Closed Sets	25
1.6 Wednesday for MAT4002	27
1.6.1 Forget about metric	27
1.6.2 Topological Spaces	30

1.6.3	Closed Subsets	31
-------	----------------	----

2 Week2 33

2.1 Monday for MAT3040 33

2.1.1	Basis and Dimension	33
-------	---------------------	----

2.1.2	Operations on a vector space	36
-------	------------------------------	----

2.2 Monday for MAT3006 39

2.2.1	Remark on Open and Closed Set	39
-------	-------------------------------	----

2.2.2	Boundary, Closure, and Interior	43
-------	---------------------------------	----

2.3 Monday for MAT4002 46

2.3.1	Convergence in topological space	46
-------	----------------------------------	----

2.3.2	Interior, Closure, Boundary	48
-------	-----------------------------	----

2.4 Wednesday for MAT3040 52

2.4.1	Remark on Direct Sum	52
-------	----------------------	----

2.4.2	Linear Transformation	53
-------	-----------------------	----

2.5 Wednesday for MAT3006 60

2.5.1	Compactness	60
-------	-------------	----

2.5.2	Completeness	65
-------	--------------	----

2.6 Wednesday for MAT4002 67

2.6.1	Remark on Closure	67
-------	-------------------	----

2.6.2	Functions on Topological Space	69
-------	--------------------------------	----

2.6.3	Subspace Topology	71
-------	-------------------	----

2.6.4	Basis (Base) of a topology	73
-------	----------------------------	----

3 Week3 75

3.1 Monday for MAT3040 75

3.1.1	Remarks on Isomorphism	75
-------	------------------------	----

3.1.2	Change of Basis and Matrix Representation	76
-------	---	----

3.2	Monday for MAT3006	83
3.2.1	Remarks on Completeness	83
3.2.2	Contraction Mapping Theorem	84
3.2.3	Picard Lindelof Theorem	87
3.3	Monday for MAT4002	89
3.3.1	Remarks on Basis and Homeomorphism	89
3.3.2	Product Space	92
3.4	Wednesday for MAT3040	94
3.4.1	Remarks for the Change of Basis	94
3.5	Wednesday for MAT3006	100
3.5.1	Remarks on Contraction	100
3.5.2	Picard-Lindelof Theorem	101
3.6	Wednesday for MAT4002	105
3.6.1	Remarks on product space	105
3.6.2	Properties of Topological Spaces	108
4	Week4	111
4.1	Monday for MAT3040	111
4.1.1	Quotient Spaces	111
4.1.2	First Isomorphism Theorem	114
4.2	Monday for MAT3006	117
4.2.1	Generalization into System of ODEs	117
4.2.2	Stone-Weierstrass Theorem	119
4.3	Monday for MAT4002	123
4.3.1	Hausdorffness	123
4.3.2	Connectedness	124
4.4	Wednesday for MAT3040	128
4.4.1	Dual Space	133

4.5	Wednesday for MAT3006	136
4.5.1	Stone-Weierstrass Theorem	137
4.6	Wednesday for MAT4002	142
4.6.1	Remark on Connectedness	142
4.6.2	Completeness	144
5	Week5	147
5.1	Monday for MAT3040	147
5.1.1	Remarks on Dual Space	148
5.1.2	Annihilators	150
5.2	Monday for MAT3006	154
5.2.1	Stone-Weierstrass Theorem in \mathbb{C}	155
5.2.2	Baire Category Theorem	157
5.3	Monday for MAT4002	159
5.3.1	Continuous Functions on Compact Space	159
5.4	Wednesday for MAT3040	162
5.4.1	Adjoint Map	163
5.4.2	Relationship between Annihilator and dual of quotient spaces	166
5.5	Wednesday for MAT3006	167
5.5.1	Remarks on Baire Category Theorem	167
5.5.2	Compact subsets of $\mathcal{C}[a, b]$	169
5.6	Wednesday for MAT4002	171
5.6.1	Remarks on Compactness	171
5.6.2	Quotient Spaces	172
6	Week6	177
6.1	Monday for MAT3040	177
6.1.1	Polynomials	177

6.2	Monday for MAT3006	180
6.2.1	Compactness in Functional Space	180
6.2.2	An Application of Ascoli-Arzela Theorem	181
6.3	Monday for MAT4002	182
6.3.1	Quotient Topology	182
6.3.2	Properties in quotient spaces	184
6.2	Wednesday for MAT3040	180
6.2.1	Eigenvalues & Eigenvectors	183
6.3	Wednesday for MAT4002	186
6.3.1	Remarks on Compactness	186
7	Week7	187
7.1	Monday for MAT3040	187
7.1.1	Minimal Polynomial	187
7.1.2	Minimal Polynomial of a vector	191
7.2	Monday for MAT3006	193
7.2.1	Remarks on the outer measure	193
7.3	Monday for MAT4002	197
7.3.1	Quotient Map	197
7.3.2	Simplicial Complex	198
7.4	Wednesday for MAT3040	201
7.4.1	Cayley-Hamilton Theorem	201
7.5	Wednesday for MAT4002	206
7.5.1	Simplicial Sub-complex	206
7.5.2	Some properties of simplicial complex	208

8	Week8	211
8.1	Monday for MAT3040	211
8.1.1	Cayley-Hamilton Theorem	213
8.1.2	Primary Decomposition Theorem	216
8.2	Monday for MAT3006	218
8.2.1	Remarks for Outer Measure	218
8.2.2	Lebesgue Measurable	219
7.3	Monday for MAT4002	197
7.3.1	Quotient Map	197
7.3.2	Simplicial Complex	198
8.3	Wednesday for MAT3040	216
8.3.1	Cayley-Hamilton Theorem	216
8.4	Wednesday for MAT3006	223
8.4.1	Remarks on Lebesgue Measurability	223
8.4.2	Measures In Probability Theory	224
8.5	Wednesday for MAT4002	228
8.5.1	Homotopy	230
9	Week9	221
9.1	Monday for MAT3040	221
9.2	Monday for MAT3006	226
9.2.1	Measurable Functions	226
9.3	Monday for MAT4002	230
9.3.1	Remarks on Homotopy	230
8.3	Wednesday for MAT3040	216
8.3.1	Cayley-Hamilton Theorem	216
8.4	Wednesday for MAT3006	223
8.4.1	Remarks on Lebesgue Measurability	223

8.4.2	Measures In Probability Theory	224
8.3	Wednesday for MAT4002	216
8.3.1	Homotopy	218

Acknowledgments

This book is from the MAT3006, MAT3040, MAT4002 in spring semester, 2018-2019.

CUHK(SZ)

Notations and Conventions

\mathbb{R}^n	n -dimensional real space
\mathbb{C}^n	n -dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
x_i	i th entry of column vector \mathbf{x}
a_{ij}	(i, j) th entry of matrix \mathbf{A}
\mathbf{a}_i	i th column of matrix \mathbf{A}
\mathbf{a}_i^T	i th row of matrix \mathbf{A}
\mathbb{S}^n	set of all $n \times n$ real symmetric matrices, i.e., $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all i, j
\mathbb{H}^n	set of all $n \times n$ complex Hermitian matrices, i.e., $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
\mathbf{A}^T	transpose of \mathbf{A} , i.e., $\mathbf{B} = \mathbf{A}^T$ means $b_{ji} = a_{ij}$ for all i, j
\mathbf{A}^H	Hermitian transpose of \mathbf{A} , i.e., $\mathbf{B} = \mathbf{A}^H$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\text{trace}(\mathbf{A})$	sum of diagonal entries of square matrix \mathbf{A}
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
\mathbf{e}_i	a unit vector with the nonzero element at the i th entry
$\mathcal{C}(\mathbf{A})$	the column space of \mathbf{A}
$\mathcal{R}(\mathbf{A})$	the row space of \mathbf{A}
$\mathcal{N}(\mathbf{A})$	the null space of \mathbf{A}
$\text{Proj}_{\mathcal{M}}(\mathbf{A})$	the projection of \mathbf{A} onto the set \mathcal{M}

9.3. Monday for MAT4002

Reviewing.

1. Homotopy: $f \cong g$
2. If $Y \subseteq \mathbb{R}$ is convex, then the set of continuous functions $f : X \rightarrow Y$ form a single equivalence class, i.e., $\{\text{continuous functions } f : X \rightarrow Y\} / \sim$ has only one element

9.3.1. Remarks on Homotopy

Proposition 9.4 Consider

$$W \xrightarrow{f} X, \quad X \xrightarrow{g} Y, \quad X \xrightarrow{h} Y, \quad Y \xrightarrow{k} Z$$

where f, g, h, k are all continuous. If $g \cong h$, then

$$g \circ f \cong h \circ f, \quad k \circ g \cong k \circ h$$

Proof. Suppose $H : g \cong h$, then $k \circ H : X \times I \rightarrow Z$ gives the momotopy between $k \circ g$ and $k \circ h$.

Simiarly, $H \circ (f \times \text{id}_I) : W \times I \rightarrow Y$ gives homotopy $g \circ f \simeq h \circ f$.

■

Definition 9.4 [Homotopy Equivalent] Two topological spaces X and Y are **homotopy equivalent** if there are continuous maps $f : X \rightarrow Y$, and $g : Y \rightarrow X$ such that

$$g \circ f \simeq \text{id}_{X \rightarrow X}$$

$$f \circ g \simeq \text{id}_{Y \rightarrow Y}$$

denoted as $X \simeq Y$.

■

1. If $X \cong Y$ are homeomorphic, then they are homotopic equivalent.
2. $X \simeq Y$ gives a bijection between $\{\phi : \text{continuous } W \rightarrow X\} / \sim$ and $\{\phi : \text{continuous } W \rightarrow Y\} / \sim$
3. exercise: $X \simeq Y$ forms an equivalence relation between topological spaces

Some properties are lost when we only study spaces up to homotopy equivalence.

Definition 9.5 [Contractible] X is **contractible** if it is homotopy equivalent to a point $\{*\}$.

Equivalently, we need f, g such that

$$\begin{aligned} \{*\} &\rightarrow_f X \rightarrow_g \{*\}, \quad g \circ f \simeq \text{id}_{\{*\}} \\ X &\rightarrow_g \{*\} \rightarrow_f X, \quad f \circ g \simeq \text{id}_X \end{aligned}$$

Note that $f \circ g = c_y$ for some $y \in X$, where $c_y : X \rightarrow X$ is $c_y(x) = y, \forall x \in X$. Therefore, to check X is contractible, we only need to check $c_y \simeq \text{id}_X, \forall y \in X$

■ **Example 9.1** 1. $X = \mathbb{R}^2$ is contractible.

WTS $f(\mathbf{x}) = \mathbf{x}$ (i.e., $f = \text{id}$) is homotopic to the constant function $g(x) = (0,0), \forall x \in \mathbb{R}^2$, i.e., $g = c_{(0,0)}$

Consider $H(\mathbf{x}, t) = tf(\mathbf{x})$, which implies

$$H(\mathbf{x}, 0) = c_{(0,0)}, \quad H(\mathbf{x}, 1) = \text{id}$$

Therefore, $\text{id} \simeq c_{(0,0)}$, and $c_{(0,0)} \simeq c_y, \forall y \in \mathbb{R}^2$

Therefore, X is contractible.

More generally, any convex $X \subseteq \mathbb{R}^n$ is contractible.

Ⓡ S^1 is not contractible, and we will see it in 3 weeks' time.

We are not able to construct

$$H : S^1 \rightarrow [0,1] \rightarrow S^1$$

such that $H(e^{2\pi ix}, 0) = e^{2\pi ix}$ and $H(e^{2\pi ix}, 1) = e^{2\pi i(0)} = 1$ (c_1)

But how about $H(e^{2\pi ix}, t) = e^{2\pi ixt}$? What's wrong with H ? Such a function is not well-defined:

$$H(e^{2\pi i(1)}, t) = e^{2\pi i(1-t)} = H(e^{2\pi i(0)}, t) = 1$$

Therefore, H is not well-defined for $t \neq 0, 1$.

Definition 9.6 Let $A \subseteq X$ and $i : A \hookrightarrow X$ be an inclusion. We say A is a homotopy retract of X if there exists continuous mapping $r : X \rightarrow A$ such that

$$A \hookrightarrow X \xrightarrow{r} A \implies r \circ i = \text{id}_{A \rightarrow A}$$

$$X \xrightarrow{r} A \hookrightarrow^i X \implies i \circ r \simeq \text{id}_X$$

In particular, $A \simeq X$. ■

■ **Example 9.2** S^1 is a homotopy retract of $M = \text{Mobius Band}$

Here $M = [0,1]^2 / \sim$ and $S^1 = [0,1] / \sim$. Define

$$i : S^1 \hookrightarrow M, \quad [x] \mapsto [(x, \frac{1}{2})]$$

$$r : M \rightarrow S^1 \\ [(x, y)] \mapsto [x]$$

Then $r \circ i = \text{id}_{S^1}$; $i \circ r([(x, y)]) = [(x, 1/2)]$; $\text{id}_M([(x, y)]) = [(x, y)]$.

Define $H : M \times I \rightarrow M$ by

$$H([(x, y)], t) := [(x, (1-t)y + t/2)]$$

We really need to check

$$H([(0,y)],t) = H([(1,1-y)],t), \quad \forall y \in [0,1]$$

Therefore, H gives a homotopy between $i \circ r$ and id_M , i.e., $i \circ r \simeq \text{id}_M$ ■

S^{n-1} is a homotopy retract of $\mathbb{R}^n \setminus \{0\}$.

We have $i : \text{id} = S^{n-1} \rightarrow \mathbb{R}^n \setminus \{0\}$ and

$$\begin{aligned} r : \mathbb{R}^n \setminus \{0\} &\rightarrow S^{n-1} \\ x &\mapsto \frac{x}{\|x\|} \end{aligned}$$

Therefore, $r \circ i = \text{id}_{S^{n-1}}$ and $i \circ r(x) = x/\|x\|$.

WTS $i \circ r \simeq \text{id}_{\mathbb{R}^n \setminus \{0\}}$ Consider $H(x,t) = tx + (1-t)x/\|x\|$:

$$H(\mathbf{x},0) = i \circ r(\mathbf{x}), \quad H(\mathbf{x},1) = \mathbf{x} = \text{id}(\mathbf{x})$$

However, we need to check that $H(x,t) \in \mathbb{R}^n \setminus \{0\}$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ and $t \in [0,1]$.

Definition 9.7 [Homotopic Relative] Let $A \subseteq X$ be topological spaces. We say $f, g : X \times I \rightarrow Y$ are homotopic relative to A if there exists $H : X \times I \rightarrow Y$ such that

$$\begin{cases} H(x,0) = f(x) \\ H(x,1) = g(x) \end{cases}$$

and

$$H(a,t) = f(a) = g(a), \forall a \in A$$

