A FIRST COURSE

IN

ANALYSIS

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MAT2006 Notebook

Lecturer

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Acknowledgments

This book is taken notes from the MAT2006 in fall semester, 2018. These lecture notes were taken and compiled in LaTeX by Jie Wang, an undergraduate student in Fall 2018. Prof. Weiming Ni has not edited this document. Students taking this course may use the notes as part of their reading and reference materials. This version of the lecture notes were revised and extended for many times, but may still contain many mistakes and typos, including English grammatical and spelling errors, in the notes. It would be greatly appreciated if those students, who will use the notes as their reading or reference material, tell any mistakes and typos to Jie Wang for improving this notebook.

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all i, j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i, j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Wednesday

Recommended Reading.

1. (Springer-Lehrbuch) V. A. Zorich, J. Schüle-Analysis I-Springer (2006).

2. (The Carus mathematical monographs 13) Ralph P. Boas, Harold P. Boas, A primer of real functions-Mathematical Association of America (1996).

3. (International series in pure and applied mathematics) Walter Rudin, Principles of Mathematical Analysis-McGraw-Hill (1976).

4. Terence Tao, Analysis I,II-Hindustan Book Agency (2006)

5. (Cornerstones) Anthony W. Knapp, Basic real analysis-Birkhäuser (2005)

1.1.1. Introduction to Set

For a set $\mathcal{A} = \{1,2,3\}$, we have $2^3 = 8$ subsets of $\mathcal{A}$. We are interested to study the collection of sets.

> **Definition 1.1** [Collection of Subsets] Given a set $\mathcal{A}$, the the collection of subsets of $\mathcal{A}$ is denoted as $2^{\mathcal{A}}$. ∎

We use Candinal to describe the order of number of elements in a set.

> **Definition 1.2** Given two sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ and $\mathcal{B}$ are said to be **equivalent** (or have the same **candinal**) if there exists a 1-1 onto mapping from $\mathcal{A}$ to $\mathcal{B}$. ∎

> **Definition 1.3** [Countability] The set $\mathcal{A}$ is said to be **countable** if $\mathcal{A} \sim \mathbb{N} = \{1,2,3,\dots\}$; an infinite set $\mathcal{A}$ is **uncountable** if it is not equivalent to $\mathbb{N}$. ∎

R Note that the set of integers, i.e., $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ is also countable; the set of rational numbers, i.e., $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$ is countable.

We skip the process to define real numbers.

Proposition 1.1 The set of real numbers $\mathbb{R}$ is **uncountable**.

For example, $\sqrt{2} \notin \mathbb{Q}$. Some inrational numbers are the roots of some polynomials, such a number is called **algebraic** numbers. However, some inrational numbers are not, such a number is called **transcendental**. For example, π is **not** algebraic. We will show that the collection of algebraic numbers are countable in the future.

There are two steps for the proof for proposition(1.1):

Proof. 1. $2^{\mathbb{N}}$ is **uncountable**:

Assume $2^{\mathbb{N}}$ is countable, i.e.,

$$2^{\mathbb{N}} = \{A_1, A_2, \dots, A_k, \dots\}$$

Define $B := \{k \in \mathbb{N} \mid k \notin A_k\}$, it is a collection of subscripts such that the subscript k does not belong to the corresponding subsets A_k.

It follows that $B \in 2^{\mathbb{N}} \implies B = A_n$ for some n. Then it follows two cases:

- If $n \in A_n$, then $n \notin B = A_n$, which is a contradiction

- Otherwise, $n \in B = A_n$, which is also a contradiction.

The proof for the claim $2^{\mathbb{N}}$ is **uncountable** is complete.

2. $\mathbb{R} \sim 2^{\mathbb{N}}$:

Firstly we have $\mathbb{R} \sim (0,1)$. This can be shown by constructing a one-to-one mapping:

$$f : \mathbb{R} \mapsto (0,1) \quad f(x) = \tfrac{1}{\pi}\arctan x + \tfrac{1}{2}, \forall x \in \mathbb{R}$$

Secondly, we show that $2^{\mathbb{N}} \sim (0,1)$. We construct a mapping f such that

$$f : 2^{\mathbb{N}} \mapsto (0,1),$$

where for $\forall A \in 2^{\mathbb{N}}$,

$$f(A) = 0.a_1 a_2 a_3 \ldots, \quad a_j = \begin{cases} 2, & \text{if } j \in A \\ 4, & \text{if } j \notin A \end{cases}$$

This function is only 1-1 mapping but not onto mapping.

Reversely, we construct a 1-1 mapping from $(0,1)$ to $2^{\mathbb{N}}$. We construct a mapping g such that

$$g : (0,1) \mapsto 2^{\mathbb{N}}$$

where for any real number from $(0,1)$, we can write it into binary expansion:

$$\text{binary form:} \quad 0.a_1 a_2 \ldots \quad \text{where } a_j = 0 \text{ or } 1.$$

Hence, we construct $g(0.a_1 a_2 \ldots) = \{j \in \mathbb{N} \mid a_j = 0\} \subseteq \mathbb{N}$, which implies $g(\cdot) \in 2^{\mathbb{N}}$.

(R) Our intuition is that two 1-1 mappings in the reverse direction will lead to a 1-1 **onto** mapping. If this is true, then we complete the proof. This intuition is the **Schroder-Bernstein Theorem**.

∎

Defining Binary Form. However, during this proof, we must be careful about the binary form of a real number from $(0,1)$. Now we give a clear definition of Binary Form:

For a real number a, to construct its binary form, we define

$$a_1 = \begin{cases} 0, & \text{if } a \in (0, \frac{1}{2}) \\ 1, & \text{if } a \in [\frac{1}{2}, 1). \end{cases}$$

After having chosen $a_1, a_2, \ldots, a_{j-1}$, we define a_j to be the largest integer such that

$$\frac{1}{2}a_1 + \frac{1}{2^2}a_2 + \cdots + \frac{a_j}{2^j} \leq a$$

Then the binary form of a is $a := 0.a_1 a_2 \ldots$.

Theorem 1.1 — **Schroder-Bernstein Theorem.** If $f : A \mapsto B$ and $g : B \mapsto A$ are both 1-1 mapping, then there exists a 1-1 onto mapping from A to B, i.e., card #A equals to card #B.

Exercise: Show that $(0,1)$ and $[0,1]$ have 1-1 onto mapping without applying Schroder-Bernstein Theorem.

The next lecture we will take a deeper study into the proof of Schroder-Bernstein Theorem and the real number.

1.2. Quiz

1. Show that the sequence $\{x_n\}$ is convergent, where

$$x_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \cdots + \frac{\sin n}{2^n}.$$

2. Compute the following limits:

   (a)
   $$\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/(1 - \cos x)}$$

   (b)
   $$\lim_{n \to \infty} \int_0^1 \frac{x^n}{1 + \sqrt{x}} \, dx$$

3. Justify that the natural number e is irrational, where

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

4. Every rational x can be written in the form $x = p/q$, where $q > 0$ and p and q are integers without any common divisors. When $x = 0$, we take $q = 1$. Consider the function f defined on $\mathbb{R}^1$ by

$$f(x) = \begin{cases} 0, & x \text{ is irrational} \\ \dfrac{1}{q}, & x = \frac{p}{q}. \end{cases}$$

   Find:

   (a) all continuities of $f(x)$;

   (b) all discontinuities of $f(x)$

   and prove your results.

1.3. Friday

Before we give a proof of Schroder-Bernstein theorem, we'd better review the definitions for one-to-one mapping and onto mapping.

Definition 1.4   [One-to-One/Onto Mapping] If $f : A \mapsto B$, then

- f is said to be **onto** mapping if

$$\forall b \in B, \exists a \in A \text{ s.t. } f(a) = b;$$

- f is said to be **one-to-one** mapping if

$$\forall a, b, \in A, f(a) = f(b) \implies a = b.$$

■

The Fig.(1.1) shows the examples of one-to-one/onto mappings.

1.3.1. Proof of Schroder-Bernstein Theorem

Before the proof, note that in this lecture we abuse the notation fg to denote the composite function $f \circ g$, but in the future fg will refer to other meanings.

Intuition from Fig.(1.2). The proof for this theorem is constructive. Firstly Fig.(1.2) gives us the intuition of the proof for this theorem. Let $f : A \mapsto B$ and $g : B \mapsto A$ be two one-to-one mappings, and D, C are the image from A, B respectively. Note that

> if the set $B \setminus D$ is empty, then $D = B = f(A)$ with f being the one-to-one mapping, which implies f is one-to-one onto mapping. In this case the proof is complete.

Hence it suffices to consider the case $B \setminus D$ is non-empty. Thus $B \setminus D$ is the "**trouble-maker**". To construct a one-to-one onto mapping from A, we should study the subset $g(B \setminus D)$ of A (which can also be viewed as a *trouble-maker*). Moreove, we should study
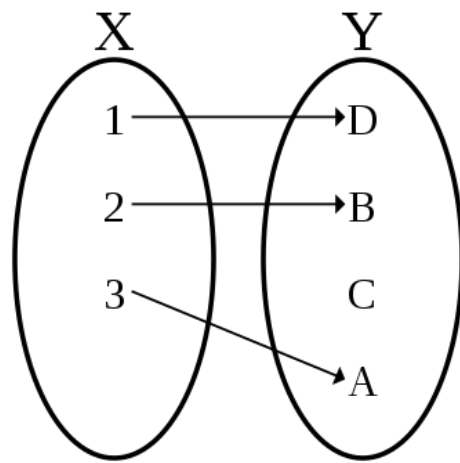
(a) A one-to-one but not onto mapping
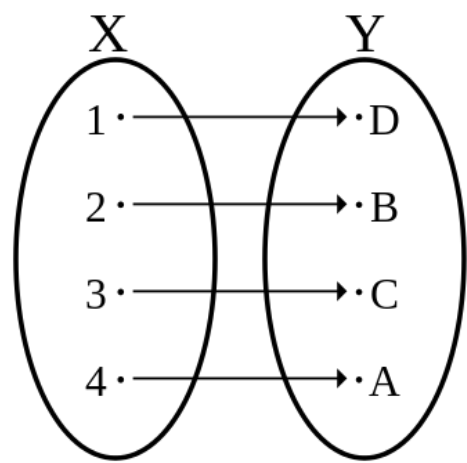
(b) A one-to-one onto mapping
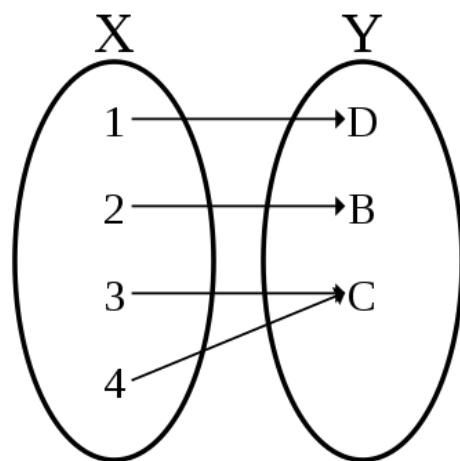
(c) A onto but not one-to-one mapping
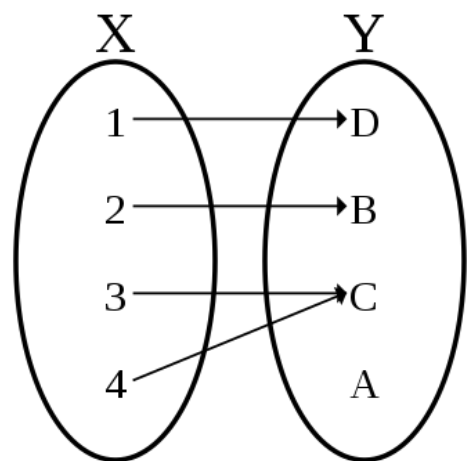
(d) Neither a one-to-one nor onto mapping

Figure 1.1: Illustrations of one-to-one/onto mappings
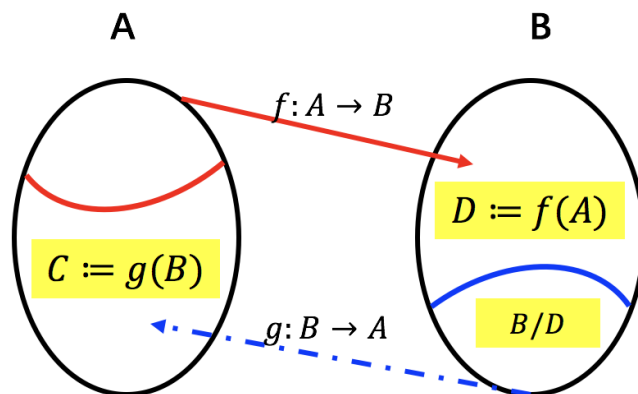
Figure 1.2: Illustration of Schroder-Bernstein Theorem

the subset $gf[g(B \setminus D)]$ (which is also a *trouble-maker*)... so on and so forth. Therefore, we should study the *union of these trouble makers*, i.e., we define

$$A_1 := g(B \setminus D), \quad A_2 := gf(A_1), \quad \cdots, \quad A_n := gf(A_{n-1}),$$

Then we study the union of infinite sets

$$S := A_1 \bigcup A_2 \bigcup \cdots \bigcup A_n \bigcup \cdots$$

Define

$$F(a) = \begin{cases} f(a), & a \in A \setminus S \\ g^{-1}(a), & a \in S \end{cases}$$

We claim that $F : A \mapsto B$ is one-to-one onto mapping.

F is onto mapping. Given any element $b \in B$, it follows two cases:

1. $g(b) \in S$. It implies $F(g(b)) = g^{-1}(g(b)) = b$.

2. $g(b) \notin S$. It implies $b \in D$, since otherwise $b \in B \setminus D \implies g(b) \in g(B \setminus D) \subseteq S$, which is a contradiction. $b \in D$ implies that $\exists a \in A$ s.t. $f(a) = b$.

   Then we study the relationship between $gf(S)$ and S. Verify by yourself that

   $$S = g(B \setminus D) \bigcup gf(S)$$

   With this relationship, we claim $a \notin S$, since otherwise $a \in S \implies gf(a) \in S$, but $gf(a) = g(b) \notin S$, which is a contradiction.

   Hence, $F(a) = f(a) = b$.

Hence, for any element $b \in B$, we can find a element from A such that the mapping for which is equal to b, i.e., F is onto mapping.

F is one-to-one mapping. Assume not, verify by yourself that the only possibility is that $\exists a_1 \in A \setminus S$ and $a_2 \in S$ such that $F(a_1) = F(a_2)$, i.e., $f(a_1) = g^{-1}(a_2)$, which follows

$$gf(a_1) = a_2 \in S = A_1 \bigcup A_2 \bigcup \cdots \tag{1.1}$$

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We claim that Eq.(1.1) is false. Note that $gf(a_1) \notin A_1 := g(B \setminus D)$, since otherwise $f(a_1) \in B \setminus D$, which is a contradiction; note that $gf(a_1) \notin A_2$, since otherwise $gf(a_1) \in gfg(B \setminus D) \implies a_1 \in g(B \setminus D) = A_1 \subseteq S$, which is a contradiction.

Applying the similar trick, we wil show that $gf(a_1) \notin A_k$ for $k \geq 1$. Hence, Eq.(1.1) is false, the proof is complete.

> ■ **Example 1.1** Given two sets $A := (0,1]$ and $B := [0,1)$. Now we apply the idea in the proof above to construct a one-to-one onto mapping from A to B:
>
> - Firstly we construct two one-to-one mappings:
>
> $$f : A \mapsto B \qquad g : B \mapsto A$$
> $$f(x) = \frac{1}{2}x \qquad g(x) = x$$
>
> - It follows that $B \setminus D = (\frac{1}{2}, 1)$, $gf(B \setminus D) = (\frac{1}{4}, 1)$, so on and so forth.
>
> $$S = (\frac{1}{2}, 1) \bigcup (\frac{1}{4}, 1) \bigcup \cdots$$
>
> - Hence, the one-to-one onto mapping we construct is
>
> $$F(x) = \begin{cases} \dfrac{1}{2}x, & x \in A \setminus S \\ x, & x \in S \end{cases}$$
>
> - Conversely, to construct the inverse mapping, we define
>
> $$f(x) = x \quad g(x) = \tfrac{1}{2}x$$
>
> - It follows that $D = (0,1)$, $B \setminus D = \{1\}$. Then
>
> $$S = \left\{ \frac{1}{2} \right\} \bigcup \cdots = \left\{ \frac{1}{2}, \frac{1}{4}, \cdots \right\}$$

- Hence, the function we construct for inverse mapping is

$$F(x) = \begin{cases} x, & x \neq \dfrac{1}{2^m} \\ 2x, & x = \dfrac{1}{2^m} \end{cases} \qquad (m = 1,2,3,\ldots)$$

1.3.2. Connectedness of Real Numbers

There are two approaches to construct real numbers. Let's take $\sqrt{2}$ as an example.

1. The first way is to use **Dedekind Cut**, i.e., every non-empty subset has a least upper bound. Therefore, $\sqrt{2}$ is actually the least upper bound of a non-empty subset

$$\{x \in \mathbb{Q} \mid x^2 < 2\}.$$

2. Another way is to use **Cauchy Sequence**, i.e., every Cauchy sequence is convergent. Therefore, $\sqrt{2}$ is actually the limit of the given sequence of decimal approximations below:

$$\{1, 1.4, 1.41, 1.414, 1.4142, \ldots\}$$

We will use the second approach to define real numbers. Every real number r essentially represents a collection of cauchy sequences with limit r, i.e.,

$$r \in \mathbb{R} \implies \left\{ \{x_n\}_{n=1}^{\infty} \,\middle|\, \lim_{n \to \infty} x_n = r \right\}$$

Let's give a formal definition for cauchy sequence and a formal definition for real number.

Definition 1.5 [Cauchy Sequence]

- Any sequence of rational numbers $\{x_1, x_2, \cdots\}$ is said to be a **cauchy sequence** if for every $\epsilon > 0$, $\exists N$ s.t. $|x_n - x_m| < \epsilon$, $\forall m, n \geq N$

- Two cauchy sequences $\{x_1, x_2, \dots\}$ and $\{y_1, y_2, \dots\}$ are said to be **equivalent** if for every $\epsilon > 0$, there $\exists N$ s.t. $|x_n - y_n| < \epsilon$ for $\forall n \geq N$.

- A real number is a **collection** of **equivalent** cauchy sequences. It can be represented by a cauchy sequence:

$$x \in \mathbb{R} \sim \{x_1, x_2, \dots, x_n, \dots\},$$

where x_j is a rational number.

  R   Let $\xi_\mathbb{Q}$ denote a collection of any cauchy sequences. Then once we have equivalence relation, the whole collection $\xi_\mathbb{Q}$ is partitioned into several disjoint subsets, i.e., equivalence classes. Hence, the real number space $\mathbb{R}$ are the equivalence classes of $\xi_\mathbb{Q}$.

The real numbers are well-defined, i.e., given two real numbers $x \sim \{x_1, x_2, \dots\}$ $y \sim \{y_1, y_2, \dots\}$, we can define add and multiplication operator.

$$x + y \sim \{x_1 + y_1, x_2 + y_2, \dots\}$$
$$x \cdot y \sim \{x_1 \cdot y_1, x_2 \cdot y_2, \dots\}$$

We will show how to define $x > 0$ in next lecture, this construction essentially leads to the lemma below:

Proposition 1.2 $\mathbb{Q}$ are dense in $\mathbb{R}$.

In the next lecture we will also show the completeness of $\mathbb{R}$:

Theorem 1.2 $\mathbb{R}$ is complete, i.e., every cauchy sequence of real numbers converges.

Recommended Reading:

Prof. Katrin Wehrheim, MIT Open Course, Fall 2010, Analysis I Course Notes, Online avaiable:

https://ocw.mit.edu/courses/mathematics

/18-100b-analysis-i-fall-2010/readings-notes/MIT18_100BF10_Const_of_R.pdf

Chapter 2

Week2

2.1. Wednesday

2.1.1. Review and Announcement

The quiz results will not be posted.

In this lecture we study the number theories.

The office hour is 2 - 4pm, TC606 on Wednesday

2.1.2. Irrational Number Analysis

Definition 2.1 [Algebraic Number] A number $x \in \mathbb{R}$ is said to be an **algebraic number** if it satisfies the following equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{2.1}$$

where $a_n, a_{n-1}, \ldots, a_0$ are integers and not all zero. We say x is **of degree** n if $a_n \neq 0$ and x is not the root of any polynomial with lower degree. ∎

Definition 2.2 A number $x \in \mathbb{R}$ is **transcendental** if it is not an algebraic number. ∎

The first example is that all rational numbers are algebraic, since rational number $\frac{p}{q}$ satisfies $qx - p = 0$. Also, $\sqrt{2}$ is algebraic. We leave an exericse: show that e and π are all transcendental. In history Joseph Liouville (1844) have constructed the first transcendental number. Let's look at the insights of his construction in this lecture:

Proposition 2.1 The set of all algebraic numbers is countable.

Proof. 1. Let $\mathcal{P}_n$ denote the set of all polynomials of degree n (Here we assert polynomials have all integer coefficients by default.), i.e.,

$$\mathcal{P}_n = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_j \in \mathbb{Z}\}$$

The set $\mathcal{P}_n$ have the one-to-one onto mapping to the set $\{(a_n, a_{n-1}, \ldots, a_0) \mid a_j \in \mathbb{Z}\} \subseteq \mathbb{Z}^{n+1}$, which implies $\mathcal{P}_n$ is **countable**.

2. Let $\mathcal{R}_n$ denote the set of all real roots of polynomials in $\mathcal{P}_n$. Since each polynomial of degree n has at most n real roots, the set $\mathcal{R}_n$ is a countable union of finite sets, which is at most countable. It is easy to show $\mathcal{R}_n$ is infinite, and thus countable.

3. Hence, we construct the set of all algebraic numbers $\bigcup_{n=1}^{\infty} \mathcal{R}_n$, which is countable since countably union of countable sets is also countable.

$\blacksquare$

How fast to approximate rational numbers using rational numbers? How fast to approximate irrational numbers using rational numbers? How fast to approximate transcendental numbers using rational numbers? We need the definition for the rate of approximation first to answer these questions.

> **Definition 2.3** A real number ξ is **approximable** by **rational numbers to order** n if $\exists$ a constant $K = K(\xi)$ such that the inequality
>
> $$\left| \frac{p}{q} - \xi \right| \leq \frac{K}{q^n}$$
>
> has **infinitely many** solutions $\frac{p}{q} \in \mathbb{Q}$ with $q > 0$ and p, q are integers without any common divisors. $\blacksquare$

Intuitively, a rational number is approximable by rational numbers. Now we study its rate of approximation by applying this definition.

■ Example 2.1 Suppose a rational number is approximable to oder α (which is a parameter). To calculate the value of α, it suffices to choose (p_k, q_k) such that

$$\left| \frac{p_k}{q_k} - \frac{p}{q} \right| \leq \frac{K}{q^\alpha}$$

Note that

$$\left| \frac{p_k}{q_k} - \frac{p}{q} \right| = \left| \frac{p_k q - p q_k}{q_k q} \right| \geq \frac{1}{q q_k} = \frac{1/q}{q_k},$$

- $\frac{p}{q}$ is approximable by rational numbers to order 1:

  If we construct $(p_k, q_k) = (kp - 1, kq)$, it follows that

  $$\left| \frac{p_k}{q_k} - \frac{p}{q} \right| = \frac{1}{kq} = \frac{1}{q_k^1}$$

- $\frac{p}{q}$ is approximable by rational numbers to order no higher than 1:

  Otherwise suppose it is approximable to order $n > 1$. The inequality holds for infinitely many (p_k, q_k):
  $$\frac{1/q}{q_k} \leq \left| \frac{p_k}{q_k} - \frac{p}{q} \right| \leq \frac{K}{q_k^n} \implies \frac{1}{q} q_k^{n-1} \leq K \tag{2.2}$$

  Since infinite (p_k, q_k) satisfy the inequality (2.2), we can choose a solution such that q_k is arbitrarily large, which falsify (2.2).

In summary, any rational number $\frac{p}{q}$ is approximable by rational numbers to order 1 and no higher than 1. ■

Liouville had shown that the transcendental number has the higher approxiamtion rate than rational and algebraic numbers, which is counter-intuitive. Let's review his process of proof:

Theorem 2.1 — Liouville, 1844. A real algebraic number ξ of degree n is not approximable by rational numbers to any order greater than n.

We can show some numbers is not algebraic, i.e., transcendental by applying this

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theorem:

> **■ Example 2.2** [1st Constructed Transcendental Number] Given a number
>
> $$\xi := \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots,$$
>
> we aim to show it is transcendental. Assume that it is an algebraic number of order n, then we construct the first n tails of ξ:
>
> $$\xi_n = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{n!}}$$
>
> It follows that
>
> $$\begin{aligned}
|\xi_n - \xi| &= \frac{1}{10^{(n+1)!}} + \frac{1}{10^{(n+2)!}} + \cdots \\
&= \frac{1}{10^{(n+1)!}} \left[1 + \frac{1}{10^{n+2}} + \frac{1}{10^{(n+2)(n+3)}} + \cdots \right] \\
&\leq \frac{1}{10^{(n+1)!}} \cdot 2 = \frac{2}{(10^{n!})^{n+1}} = \frac{2}{q^{n+1}}
\end{aligned}$$
>
> which implies $|\xi - \frac{p}{q}| \leq \frac{K}{q^{n+1}}$ has one solution ξ_n.
>
> We can construct infinitely many solutions from this solution:
>
> $$\xi_{n,1} = \xi_n + \frac{1}{10^{n+2}}, \quad \xi_{n,2} = \xi_n + \frac{1}{10^{(n+2)(n+3)}}, \quad \cdots,$$
>
> Hence, this number is approximable by rational numbers to order $n+1$, which contradicts the fact that it is an algebraic number of degree n. ■

Proof. Given an algebraic number ξ of degree n, there exists a polynomial whose roots contain ξ:

$$f(x) \equiv a_n x^n + \cdots + a_1 x + a_0 = 0.$$

We fix an interval around ξ, i.e, $I_\lambda = [\xi - \lambda, \xi + \lambda]$ (with $\lambda = \lambda(\xi) \in (0,1)$) such that I_λ contains no other root of f except ξ.

Hence, the value of f at any rational number $\frac{p}{q}$ inside I_λ is given by:

$$\left| f\left(\frac{p}{q}\right) \right| = \left| a_n \frac{p^n}{q^n} + \cdots + a_1 \frac{p}{q} + a_0 \right| = \left| \frac{a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_0 q^n}{q^n} \right| \neq 0$$

Hence, $\left| f\left(\frac{p}{q}\right) \right| \geq \frac{1}{q^n}$, which implies that

$$\frac{1}{q^n} \leq \left| f\left(\frac{p}{q}\right) \right| \tag{2.3a}$$

$$= \left| f\left(\frac{p}{q}\right) - f(\xi) \right| \tag{2.3b}$$

$$\leq |f'(\eta)| \left| \xi - \frac{p}{q} \right| \tag{2.3c}$$

$$\leq M \left| \xi - \frac{p}{q} \right| \tag{2.3d}$$

with $M := \max_{\eta \in I_\lambda} f(\eta)$. Note that from (2.3b) to (2.3c) is due to mean value theorem.

Or equivalently, $|\xi - \frac{p}{q}| \geq \frac{1/M}{q^n}$ applies for any rational number $\frac{p}{q}$ inside the interval I_λ.

- Verify by yourself that ξ is not approximable by rational numbers inside the interval I_λ to any order greater than n.

- For any rational number $\frac{p}{q} \notin I_\lambda$, we have

$$\left| \frac{p}{q} - \xi \right| \geq \lambda(\xi) \geq \frac{\lambda(\xi)}{q^n}$$

for $q \geq 1, n \geq 1$. It is obvious that ξ is not approximable by rational numbers outside the interval I_λ to any order greater than n.

The two cases above complete the proof. ∎

It's hard to determine which order the transcendental number is approximable by rational numbers. However, we can assrt that there is a "fast" approximation to transcendental numbers by applying countinued faction expansion.

Continued Fraction Expansion. Let x be irrational, then intuitively x could be represented as an infinite continued fraction as below:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{2.4}$$

We denote the continued fraction (2.4) as $[a_0; a_1, a_2, \ldots]$. Let's define the rigorous process of continued fraction expansion, i.e., how to find a_i:

- We set $a_0 = \lfloor x \rfloor$, which implies that

$$x := a_0 + \xi_0 = a_0 + \cfrac{1}{\frac{1}{\xi_0}} \qquad \text{for } 0 < \xi_0 < 1.$$

- We set $a_1 = \lfloor \frac{1}{\xi_0} \rfloor$, which implies that

$$x := a_0 + \cfrac{1}{a_1 + \xi_1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\frac{1}{\xi_1}}}$$

- After $n+1$ steps we obtain the continued fraction of x:

$$[a_0; a_1, a_2, \ldots, a_n + \xi_n]$$

We continue this process iteratively with

$$\frac{1}{\xi_n} = a_{n+1} + \xi_{n+1}$$

with $\xi_{n+1} \in (0, 1)$.

Such a process will continue without end as x is irrational. After $n+1$ steps alternatively, we write

$$x = [a_0; a_1, \ldots, a_n, a'_{n+1}] \quad \text{with } a'_{n+1} := a_{n+1} + \xi_{n+1}.$$

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Observations from continued fraction expansion.

1. For $x = [a_0; a_1, \ldots] \notin \mathbb{Q}$, consider its nth convergent term

$$\frac{p_n}{q_n} := [a_0; a_1, \ldots, a_n], \quad n \geq 0$$

   note that p_n and q_n can be computed iteratively:

$$p_0 = a \qquad\qquad q_0 = 1$$
$$p_1 = a_1 a_0 + 1 \qquad\qquad q_1 = a_1$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$p_n = a_n p_{n-1} + p_{n-2} \qquad\qquad q_n = a_n q_{n-1} + q_{n-2}$$

   Note that (p_n, q_n) have no common divisors. (exercise)

   Corollary 2.1 $q_n \geq n$ for $\forall n$.

   Proof. Note that $q_{n-1} \leq q_n$ for $\forall n \geq 1$; and that $q_{n-1} < q_n$ for $\forall n > 1$. ∎

2. From the first observation, $x := [a_0; a_1, \ldots, a'_{n+1}]$ can be written as

$$x = \frac{p_{n+1}}{q_{n+1}} = \frac{a'_{n+1} p_n + p_{n-1}}{a'_{n+1} q_n + q_{n-1}}$$

   Corollary 2.2 If $\frac{p_n}{q_n} (n \geq 0)$ is the nth convergent term of x, then

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$$

   Proof. First note that for $k \geq 2$,

$$p_{k-1} q_k - p_k q_{k-1} = p_{k-1}(a_k q_{k-1} + q_{k-2}) - (a_k p_{k-1} + p_{k-2}) q_{k-1}$$
$$= -(p_{k-2} q_{k-1} - p_{k-1} q_{k-2})$$

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After computation,

$$
\begin{aligned}
\left| x - \frac{p_n}{q_n} \right| &= \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \\
&= \left| \frac{a'_{n+1}p_n + p_{n-1}}{a'_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n} \right| = \left| \frac{p_{n-1}q_n - p_n q_{n-1}}{q_n(a'_{n+1}q_n + q_{n-1})} \right| \\
&= \left| \frac{(-1)^n p_1 q_0 - p_0 q_1}{q_n(a'_{n+1}q_n + q_{n-1})} \right| = \frac{1}{q_n(a'_{n+1}q_n + q_{n-1})} \\
&< \frac{1}{q_n q_{n+1}}
\end{aligned}
$$

∎

Corollary 2.3 Furthermore, for the convergent term $\frac{p_n}{q_n}(n \geq 0)$ of x, we have

$$
\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}
$$

3. The sequence $\{[a_0, a_1, \ldots, a_n]\}$ is a Cauchy sequence. (Exercise)

btw, π is approximable by rational number of order 42.

2.2. Friday

2.2.1. Set Analysis

This lecture will discuss different kinds of sets. Now recall our common sense:

Definition 2.4 [Interval]

- Open interval:

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$$

- Closed interval:

$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

- Half open intervals:

$$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

$$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

∎

Definition 2.5 [Open sets] A set A is open if $\forall x \in A$, there exists $(a,b) \subseteq A$ such that $x \in (a,b)$. ∎

Theorem 2.2   1. An open set in $\mathbb{R}$ is a **disjoint** union of finitely many or countably many open intervals.

2. The union of any collection of open sets is open.

3. The intersection of **finitely** many open sets is open.

The proof is omitted, check Rudin's book for reference.

Ⓡ Note that the intersection of **countably** many open sets may not be nessarily open.

$$\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0,1]$$

Definition 2.6 [Neighborhood] A **neighborhood** N of a point $a \in \mathbb{R}$ is an open interval containing a. ∎

Definition 2.7 [Limit Point] x is a **limit point** of the set A if for any neighborhood N of x, N contanins a point $a \in A$ such that $a \neq x$. ∎

Definition 2.8 [Closed Set] A set A is **closed** if A contains all of its limit points. ∎

Proposition 2.2 A is **closed** of and only if $\mathbb{R} \setminus A$ is open.

2.2.2. Set Analysis Meets Sequence

Definition 2.9 [Limit Point of sequence] Given a sequence $\{a_n\}$, i.e.,

$$a_1, a_2, a_3, \ldots,$$

a point x is said to be the **limit point** of $\{a_n\}$ if there exists a subsequence $\{x_{n_1}, x_{n_2}, \ldots\}$ converging to x. ∎

Does there exist a sequence of rational numbers such that every irrational number is a limit point? Yes, and we use an example as illustration.

■ **Example 2.3** $\{q_1, q_2, \ldots\}$ is a sequence of all rational numbers. For example, to construct a subsequence with limit $\sqrt{2}$, we pick:

$$q_{m_1} \in (\sqrt{2} - 1, \sqrt{2} + 1) \setminus (\sqrt{2} - \frac{1}{2}, \sqrt{2} + \frac{1}{2})$$
$$q_{m_2} \in (\sqrt{2} - \frac{1}{2}, \sqrt{2} + \frac{1}{2}) \setminus (\sqrt{2} - \frac{1}{3}, \sqrt{2} + \frac{1}{3})$$
$$\ldots$$
$$q_{m_k} \in (\sqrt{2} - \frac{1}{k}, \sqrt{2} + \frac{1}{k}) \setminus (\sqrt{2} - \frac{1}{k+1}, \sqrt{2} + \frac{1}{k+1})$$

> The same argument works for all irrational numbers, also for all rational numbers. ∎

2.2.3. Completeness of Real Numbers

Now we use Cauchy sequence to construct the completeness of real numbers. First let's give a proof of three important theorems. Note that the proof and applications of these theorems are mandatory.

> **Theorem 2.3** — **Bolzano-Weierstrass.** Every bounded sequence has a convergent subsequence.

> **Theorem 2.4** — **Cantor's Nested Interval Lemma.** A sequence of nested closed bounded intervals $I_1 \supseteq I_2 \supseteq \cdots$ has a non-empty intersection, i.e., $\bigcap_{k=1}^{\infty} I_k \neq \varnothing$.

> **Theorem 2.5** — **Heine-Borel.** Any open cover $\{\mathcal{U}\}$ of a bounded closed set E consists of a finite sub-cover, i.e, $E \subseteq$ the union of $\{\mathcal{U}\}$.

Proof for Bolzano-Weierstrass Theorem.

- Suppose $\{a_1, a_2, \dots\}$ is a bounded sequence, w.l.o.g., $\{a_1, a_2, \dots\} \subseteq [-M, M]$. We pick $a_{n_1} = a_1$.

- w.l.o.g., assume that $[0, M] \cap \{a_1, a_2, \dots\}$ is infinite (otherwise $[-M, 0] \cap \{a_1, a_2, \dots\}$ is infinite), then we pick $a_{n_2} \neq a_{n_1}$ such that $a_{n_2} \in [0, M]$.

- w.l.o.g., assume that $[0, \frac{M}{2}] \cap \{a_1, a_2, \dots\}$ is infinite, then we pick $a_{n_3} \neq a_{n_1}, a_{n_2}$ such that $a_{n_3} \in [0, \frac{M}{2}]$.

In this case, $\{a_{n_1}, a_{n_2}, \dots\}$ is Cauchy (by showing $|a_{n_k} - a_{n_l}| < \epsilon$ for large k, l), hence converges. ∎

Proof for Cantor's Nested Interval Lemma.

1. Pick $a_k \in I_k$ for $k = 1, 2, \dots$, thus the sequence $\{a_1, \dots, a_k, \dots\}$ is bounded. By Theorem (2.3), there exists a convergent sub-sequence $\{a_{k_l}\}$ (with limit a). It suffices to show $a \in \bigcup_{m=1}^{\infty} I_k$.

2. For fiexed m, there exists index j such that $a_{k_l} \in I_m$ for all $l \geq j$. Since I_m is closed, it must contain a_{k_l}'s limit point, i.e., $a \in I_m$.

3. Our choice is arbitrary m and hence a belongs to the intersection of all nested closed intervals. The proof is complete.

■

Before the proof of third theorem, let's have a review for open cover definitions:

Definition 2.10   [Open Cover] Let E be a subset of a metric space X. An open cover $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ of E is a collection of open sets in X whose union contains E, i.e., $E \subseteq \bigcup_{\alpha \in A} \mathcal{U}_\alpha$. A finite **subcover** of $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ is a **finite** sub-collection of $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ whose union still contains E. ∎

For example, consider $E := [\frac{1}{2}, 1)$ in metric space $\mathbb{R}$. Then the collection

$$\{I_n\}_{n=3}^\infty, \qquad \text{where } I_n := (\frac{1}{n}, 1 - \frac{1}{n})$$

is a open cover of E. Note that the finite subcover may not necessarily exist. In this example, the finite subcover of $\{I_n\}_{n=3}^\infty$ does not exist.

Proof for Heine-Borel Theorem.

Suppose $E := [0, M]$ is a bounded closed interval with an open cover $\{\mathcal{U}\}$. The trick of this proof is to construct a sequence of nested closed bounded intervals.

- **Base case** We choose $I_1 = E = [0, M]$

- **Inductive step** For example, Assume that E cannot be covered by finitely many open sets from $\{\mathcal{U}\}$, then at least one sub-interval $[0, \frac{M}{2}]$ or $[\frac{M}{2}, M]$ cannot be covered. Let I_2 be one of these sub-intervals that cannot be covered by finitely many elements of $\{\mathcal{U}\}$.

Repeating this process, we attain a nested bouned closed intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq$, which implies $\bigcap_{k=1}^\infty I_k \neq \varnothing$ (suppose $a \in \bigcap_{k=1}^\infty$), and $|I_k| = \frac{M}{2^k} \to 0$.

Note that $a \in E$ implies that there exists an open set ξ in $\{\mathcal{U}\}$ such that $a \in \xi$. Thus $(a - \epsilon, a + \epsilon) \in \xi$ for small ϵ. Note that there exists sufficiently large k such that $\frac{M}{2^k} < 2\epsilon$, and $a \in I_k$, which implies $I_k \subseteq \xi$, which is a contradiction. ∎

These theorems have simple applications:

Proposition 2.3 Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with the series convergent for $|x| < 1$. If for $\forall x \in [0, 1)$, there exists $n := n(x)$ such that $\sum_{k=n}^{\infty} a_k x^k = 0$, then f is a polynomial (that is independent from x, i.e., n does not depend on x.)

In next lecture we will continue to study the completeness of real numbers and will speed up.

Chapter 3

Week3

3.1. Tuesday

3.1.1. Application of Heine-Borel Theorem

> **Theorem 3.1**   Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ which converges in $|x| < 1$. If for every $x \in [0,1)$, there exists $n(= n(x))$ such that $\sum_{n+1}^{\infty} a_k x^k = 0$, then f is a polynomial, i.e., n does not depend on x.

The idea is to construct a sequence of points $\{x_n\}$ satisfying $f(x_k) = a_0 + \cdots + a_m x_k^m$, i.e., infinite points coincide $f(x)$ with a polynomial, which implies f is a polynomial.

Proof. Construct $E_N := \{x \in [0, \frac{1}{2}] \mid \sum_{k=N+1}^{\infty} a_k x^k = 0\}$. It follows that

$$[0, \frac{1}{2}] = \bigcup_{N=1}^{\infty} E_N,$$

which implies that at least one E_N is uncountable, say, E_m is uncountable. In particular, E_m is infinite

By Bolzano-Weierstrass Theorem, there exists a sequence $\{x_k\} \subset E_m$ with limit x_0 in E_m as E_m is closed. Hence, $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ holds for the sequence $\{x_m\}$. Intuitively we conclude the power series and the analytics function coincide each other for every point $x \in (-1,1)$.

$$f(x) \equiv a_0 + a_1 x + \cdots + a_m x^m$$

■

However, the proof above does not show why a sequence coincide $f(x)$ with a polynomial could imply f is a polynomial for every point. We summarize this induction as the proposition(3.1) and give a proof below. Before that we formulate what we want to prove precisely:

Let f be analytic, i.e., $f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$ on $(-1,1)$; and $f(x_k) = \sum_{i=1}^{m} a_i x_k^i$ for all $k \geq 1$, where $\{x_k\}$ is a sequence with limit x_0. Then $f(x) = \sum_{i=1}^{m} a_i x^i$ on $(-1,1)$.

To show this statement, we construct

$$g(x) = f(x) - \sum_{i=1}^{m} a_i x^i \implies g(x_k) = 0, \forall k \geq 1$$

It suffices to show $g \equiv 0$ on $(-1,1)$. Moreover, if we construct $y_k := x_k - x_0$, and set $f(x) = a_0 + a_1(x - x_0) + \cdots$, then it suffices to prove the proposition given below:

Proposition 3.1 Let g be analytic, i.e., $g(x) = b_0 + b_1 x + \cdots + b_n x^n + \cdots$ on $(-1,1)$; and $g(x_k) = 0$ for all $k \geq 1$, where $\{x_k\} \to 0$. Then $g \equiv 0$ on $(-1,1)$ (i.e., $b_0 = b_1 = \cdots = 0$)

Proof.   • Note that $g(0) = 0$ due to continuity property. Also, $g(0) = b_0 = 0$, which follows that

$$g(x) = x(b_1 + b_2 x + \cdots + b_n x^{n-1} + \cdots \tag{3.1}$$

• Substituting x with x_k in Eq.(3.1), we derive

$$0 = g(x_k) = x_k(b_1 + b_2 x_k + \cdots + b_n x_k^{n-1} + \cdots \tag{3.2}$$

Taking limit both sides for (3.2), we derive $b_1 = 0$.

• By applying the same trick, we conclude $b_0 = b_1 = \cdots = 0$ (the rigorous proof requires induction).

■

Now we talk about some advanced topics in Analysis.

3.1.2. Set Structure Analysis

> **Definition 3.1** [Nowhere Dense] A set B is said to be **nowhere dense** if its closure $\overline{B}$ contains no non-empty open set. ∎

For example,

$$B = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, ..\} \implies \overline{B} = B\bigcup\{0\},$$

which contains no non-empty open set.

> **Definition 3.2** [1st category] A set of B is said to be of 1st category if it can be written as the **union** of **finitely** many or **countably** many **nowhere** dense sets. ∎

> **Definition 3.3** [2rd category] A set is said to be of 2rd category if it is **not** of 1st category
>
> ∎

> **Theorem 3.2** — **Baire-Category Theorem.** • $\mathbb{R}$ is of 2rd category, i.e.,
>
> • $\mathbb{R}$ cannot be written as the union of countably many nowhere dense sets, i.e.,
>
> • if $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$, then at least one A_n whose closure contains a non-empty open set.

Proof. • Assume $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$ such that all A_n's are nowhere dense. It follows that

$$\mathbb{R} \setminus \overline{A_1} \text{ is open,}$$

since $\overline{A_1}$ is closed and its complement is open.

• We construct an open set N_1 such that $\overline{N_1} \subseteq \mathbb{R} \setminus \overline{A_1}$. (e.g., there exists ε and $x \in \mathbb{R} \setminus \overline{A_1}$ such that $N_1 := B(x,\varepsilon) \subseteq \overline{N_1} \subseteq \mathbb{R} \setminus \overline{A_1}$.)

• Since A_2 is nowhere dense, we imply $\overline{A_2}$ does not contain N_1, i.e., $N_1 \setminus \overline{A_2}$ is open.

- By applying similar trick, we obtain a sequence of nested sets

$$\overline{N_1} \supseteq N_1 \supset \overline{N_2} \supset N_2 \cdots$$

The cantor's theorem implies that $\bigcap_{k=1}^{\infty} \overline{N_k} \neq \emptyset$.

- On the other hand, $\bigcap_{k=1}^{\infty} \overline{N_k} \subseteq \mathbb{R} \setminus \bigcup_{n=1}^{m} A_n$ for any finite m.

- Therefore, $\emptyset \neq \bigcup_{k=1}^{\infty} \overline{N_k} \subseteq \mathbb{R} \setminus \bigcup_{n=1}^{\infty} A_n = \emptyset$, which is a contradiction.

∎

Ⓡ $\mathbb{R}$ is of 2nd category, i.e., if $\mathbb{R} = \bigcup_{n=1}^{\infty} A_m$, then at least A_n whose closure contains a **non-empty** open sets; The theorem also holds if we replace $\mathbb{R}$ by a **complete** metric space (essentially the same proof).

Most proof for $\mathbb{R}$ can be generalized into metric space, the proof for which is essentially the same. Now let's introduce the metric space informally.

Metric Space. A metric space is an ordered pair (M, d), where M is a set and d is a metric on M, i.e., d is a distance function defined for two points on M. Here we list several examples:

The Real Line. For $\mathbb{R}$, $d(x, y) = |x - y|$. Note that $(\mathbb{Q}, d)$ and $(\mathbb{R} \setminus \mathbb{Q}, d)$ are also metric spaces, but not complete.

n-**Cell Real Space.** $\mathbb{R}^n$, with $d(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}|$ is a metric space.

Bounded Sequences. The set of all bounded sequences on $\mathbb{R}$ is a metric space, with d defined as:

$$d(\{x_n\}, \{y_n\}) = \sup\{|x_i - y_i| \mid i = 1, 2, \dots\}$$

Bounded Functions. Similarly, the set of all bounded continuous functions on $\mathbb{R}$ (different domains), with

$$d_1(f, g) = \sup\{|f(x) - g(x)| \mid x \in \mathbb{R}\},$$

or

$$d_2(f,g) = \left(\int_0^1 |f(x) - g(x)|^2 \, dx \right)^{1.2}$$

is a metric space. Note that $(\mathcal{C}[0,1], d_1)$ is complete, and $(\mathcal{C}[0,1], d_2)$ is not complete. (exercise)

(R)

Different distance definition corresponds to different metric spaces.

Recall that a metric space is complete if all Cauchy sequence of which converge.

3.1.3. Reviewing

Definition 3.4 [Sequence] A sequence is defined as a kind of function $f : \mathbb{N} \to \mathbb{R}$, denoted as $\{f(0), f(1), \dots\}$. Conventionally we denote it as $x_1, x_2, \dots$ ∎

Definition 3.5 [Limit] A number α is the limit of $\{x_1, x_2, \dots\}$ if $\forall \epsilon > 0$, there $\exists N = N(\epsilon)$ such that $|x_k - \alpha| < \epsilon$ for $\forall k \geq N$, denoted by $\alpha_n \to \alpha$ ∎

Definition 3.6 [liminf & limsup]

$$\liminf_{k \to \infty} x_k := \lim_{n \to \infty} \inf_{k \geq n} x_k$$

which is the smallest limit point of the sequence

$$\limsup_{k \to \infty} x_k := \lim_{n \to \infty} \sup_{k \geq n} x_k$$

which is the largest limit point of the sequence. ∎

A sequence always has liminf and limsup.

Definition 3.7 [Partial Sum] Given the sequence $\{a_n\}$, its n-th partial sum are defined as:

$$s_n = a_1 + \cdots + a_n,$$

the series $\sum_i a_i$ is defined as the limit of the partial sum, ∎

Next lecture we will show that most continuous function is nowhere differentiable, by applying the Baire Category Theorem on $(\mathcal{C}[0,1], d_1)$

3.2. Friday

"Our first quiz is at 1:30-2:20pm on September 30th. That is next Sunday. There will be around 5 questions." the Grandpa said breezily,

3.2.1. Review

This lecture will review the continuty of function. Let's start with some easy examples:

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ -x, & x \notin \mathbb{Q} \end{cases}$$

This function is continuous nowhere expect for $x = 0$.

> **Definition 3.8** [Continuous] Given a function $f : D \mapsto \mathbb{R}$,
>
> - we say f is **continuous** at $x_0 \in D$ if for $\forall \varepsilon > 0$, $\exists \delta := \delta(\varepsilon, x_0) > 0$ s.t. $|f(y) - f(x_0)| < \varepsilon$ for $\forall |y - x_0| < \delta$

- f is continuous on D if it is continuous at every point in D.

- If $\delta := \delta(\varepsilon)$, i.e., δ is independent of $x_0 \in D$, then f is said to be **uniformly continuous** on D.

■

(R) The following statements are equivalent, you should show by yourself.

1. f is continuous
2. If $\{x_n\} \to x_0$ as $n \to \infty$, then $\{f(x_n)\} \to f(x_0)$ as $n \to \infty$.
3. $f^{-1}(A)$ is open/closed if the set A is open/closed.

Definition 3.9   [Compact] A set K is **compact** (cpt) if for evey open cover of K, there exists a **finite** sub-cover.   ■

(R) The compactness has an important connection with continuity, e.g., the continuous function f maps compact sets to compact sets.

There is a useful way to determine whether a point is continuous at f, which will be discussed in this lecture.

3.2.2. Continuity Analysis

Let's raise some examples first. From these examples we can see that the proof of continuousness is non-trival.

■ **Example 3.1**   1. Given a funciton

$$f(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

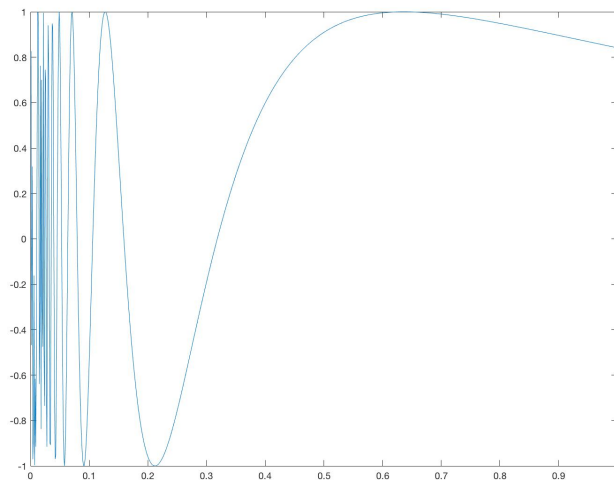From the graph we can see that f oscillates heavily near zero point.

Figure 3.1: Graph for f

It is easy to show that $\sup_{x,y \in N_\delta(0)} |f(x) - f(y)| = 2$ however small δ is.

2. For another function

$$g(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Conversely, it oscillates weakly near the zero point.
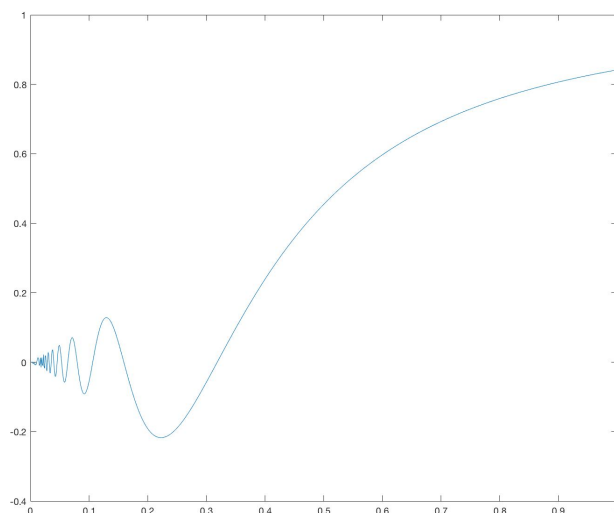
Figure 3.2: Graph for g

It is easy to show that $\sup_{x,y \in N_\delta(0)} |g(x) - g(y)| = 0$ as $\delta \to 0$.

∎

Definition 3.10 [oscillation]

- The oscillation of a function f on E is defined as

$$\omega(f;E) := \sup_{x,y \in E} |f(x) - f(y)|$$

- The oscillation of f at a single point x_0 is defined as

$$\lim_{\delta \to 0} \omega(f, N_\delta(x_0)) := w(f;x_0)$$

∎

R

- Here we abuse the notation to denote the oscillation at x_0 with $\omega(f;x_0)$, but note that $\omega(f;x_0) \neq \omega(f;\{x_0\})$.

- The well-definedness of $w(f;x_0)$ is because $\omega(f, N_\delta(x_0))$ is non-incresing as δ decreases and has a lower bound 0.

- A function f is continuous at x_0 iff $w(f,x_0) = 0$. (verify by yourself)

An classical example that illustrates a function can have continuous points in $\mathbb{R} \setminus \mathbb{Q}$ is shown below. We have faced this example in the diagnostic quiz:

■ **Example 3.2** The Dirichlet function is defined for $\mathbb{R} \setminus \{0\}$:

$$f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ \dfrac{1}{q}, & x = \dfrac{p}{q}, q > 0, (p,q) = 1 \end{cases}$$

The function f is continuous at x iff $x \notin \mathbb{Q}$. The set of all discontinuous points of f is $\mathbb{Q}$.

∎

Now the question turns out:

> *Does there exists a function g of which the set of all discontinuous points of g*
>
> *is* $\mathbb{R} \setminus \mathbb{Q}$*?*

Applying Baire-Category Theorem, we will show the answer to this question is no.

Proposition 3.2 Suppose f is continuous on a dense set in $\mathbb{R}$. Then the set of all discontinous points of f, denoted as T, must form a set of first category, i.e., (a countably union of nowhere dense sets)

$\textcircled{R}$ $\mathbb{R} \setminus \mathbb{Q}$ is of second category, otherwise assume

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{i=1}^{\infty} I_i$$

for nowhere dense sets I_i, which implies $\mathbb{R} = [\cup_{i=1}^{\infty} I_i] \cup [\cup_{q \in \mathbb{Q}} \{q\}]$ is a countably union of nowhere dense sets. Thus by applying proposition(3.2), the irrational number space cannot be the set of discontinuities.

The idea of the proof is to express T as countably union of sets, and argue that at least one of which must be nowhere dense.

Proof. We construct $D_n = \{x \in \mathbb{R} \mid w(f;x) \geq \frac{1}{n}\}$, which follows that

$$T = \bigcup_{n=1}^{\infty} D_n.$$

It suffices to show that D_n is nowhere dense for every n by contradiction.

Assume for some fixed n, D_n is not nowhere dense, i.e., $\overline{D_n}$ contains an **open** interval I. Note that the set of continuous points is dense, we conclude that there exists a point a inside the interval I such that f is continuous at a. (why?) Also, there exists a sequence $\{b_k\} \subseteq D_n$ with limit a. (since you can verify D_n is closed)

Since f is continuous at a, there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \frac{1}{4n} \text{for } |x - a| < \delta. \tag{3.3a}$$

At the same time $\{b_k\} \subseteq (a - \delta, a + \delta)$ for all large k, i.e., $\omega(f;b_k) \geq \frac{1}{n}$. Hence, there exists a sequence $\{c_{kl}\}$ with limit b_k, such that the difference $|f(c_{kl}) - f(b_k)|$ is at least greater than $\frac{1}{2n}$ (why not $\frac{1}{n}$?), i.e.,

$$|f(c_{kl} - f(b_k)| \geq \frac{1}{2n} \tag{3.3b}$$

Meanwhile, for large l, note that c_{kl} is close to a, i.e., from (3.3a) we have

$$|f(c_{kl}) - f(a)| < \frac{1}{4n} \tag{3.3c}$$

Also, note that b_k is close to a for large k, i.e., from (3.3a) we have

$$|f(b_k) - f(a)| < \frac{1}{4n} \tag{3.3d}$$

Three inequalities (3.3b) to (3.3d) show a contradiction:

$$|f(c_{kl}) - f(b_k)| \leq |f(c_{kl}) - f(a)| + |f(b_k) - f(a)| < \frac{1}{4n} + \frac{1}{4n} = \frac{1}{2n}$$

$\blacksquare$

> **Theorem 3.3** Let f be the **pointwise** limit of a sequence of continuous functions $\{f_n\}$, i.e., $f(x) = \lim_{n \to \infty} f_n(x)$. Then the set if all discontinuous points of f must be a set of first category.

(R) Review the uniform limit version of this theorem;

Proof. We claim $D_\varepsilon = \{x \in \mathbb{R} \mid \omega(f, x) \geq \varepsilon\}$ is nowhere dense for any $\varepsilon > 0$. (fixed ε).

Assume $\overline{D_\varepsilon}$ contains an open set, or equivalently, D_ε contains an open set $\mathcal{U}$ (since D_ε is closed, i.e., $\overline{D_\varepsilon} = D_\varepsilon$). Define

$$A_{mn} = \left\{ x \in U \mid |f_m(x) - f_n(x)| \leq \frac{\varepsilon}{4} \right\}.$$

Proposition 3.3 The set A_{mn} is closed.

The proof of proposition is moved in the end.

Set $A_m = \bigcap_{n \geq m} A_{mn}$, which is also closed, and

$$A_m \subseteq \left\{ x \in U \mid |f_m(x) - f(x)| \leq \frac{\varepsilon}{4} \right\}$$

For every $x \in U$, as $\lim_{n \to \infty} f_n(x) = f(x)$, we have $x \in \bigcup_{m=1}^{\infty} A_m$, which implies $U \subseteq \bigcup_{m=11}^{\infty} A_m$. Applying the Baire Category Theorem, there exists one A_m containing an open set W.

For $x_0 \in W \subseteq \mathcal{U} \subseteq D_\varepsilon$, pick $\{x_n\} \subseteq W$ with limit x_0 such that

$$|f(x_n) - f(x_0)| \geq \frac{3}{4}\varepsilon \tag{3.4a}$$

At the same time, since $x_n, x_0 \in W$, it follows that

$$|f_m(x_n) - f(x_n)| \leq \frac{\varepsilon}{4} \tag{3.4b}$$

$$|f_m(x_0) - f(x_0)| \leq \frac{\varepsilon}{4} \tag{3.4c}$$

From (3.4a) to (3.4c), we conclude that

$$\begin{aligned}
\frac{3}{4}\varepsilon &\leq |f(x_n) - f(x_0)| \\
&\leq |f_m(x_n) - f(x_n)| + |f_m(x_n) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\
&\leq \frac{1}{2}\varepsilon + |f_m(x_n) - f_m(x_0)|
\end{aligned}$$

Or equivalently,

$$|f_m(x_n) - f_m(x_0)| \geq \frac{\varepsilon}{4}, \forall n \tag{3.4d}$$

which implies $\omega(f_m; x_0) \geq \frac{\varepsilon}{4}$, which implies f_m is discontinuous at x_0, which is a contradiction. ∎

Ⓡ

- The proof for A_{mn} is closed is easy: pick any sequence $\{x_k\} \subseteq A_{mn}$ with

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limit x, it suffices to show $x \in A_{mn}$, i.e.,

$$\lim_{k \to \infty} |f_m(x_k) - f_m(x_k)| \leq \frac{\varepsilon}{4}$$

- The applications of Baire Category Theorem give us an estimation of how large and how small a set is. In next lecture we will see how large is the set of continuous but nowhere differential functions.

Chapter 4

Week4

4.1. Wednesday

This lecture will talk about the applications of Baire-Category Theorem and continuity analysis.

4.1.1. Function Analysis

In last lecture we have studied that given a analytic function f, if f can be expressed as a partial sum of its series for each x, then f is a polynomial:

Proposition 4.1   Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ in $|x| < 1$. If for every $x \in (-1,1)$, there exists $n(= n(x))$ such that $\sum_{k=n+1}^{\infty} a_k x^k = 0$, then f is a polynomial, i.e., n is independent of x.

The idea of the proof is to construct a sequence of points such that f coincide with a polynomial over these points, which implies f is indeed a polynomial.

Now we study its stronger version, i.e., f may not be analytic, it only needs to be infinitely differentiable:

Proposition 4.2   Suppose $f \in \mathcal{C}^{\infty}[-1,1]$. If for every $|x| \leq 1$, there exists $n(= n(x))$ such that $f^{(n)}(x) = 0$, then f is a polynomial.

R Note that an analytic function (i.e., can be expressed as power series) is always infinitely differentiable, but the reverse direction is necessarily not.

For example, recall we have learnt a function

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x^2}\right), & x \neq 0 \\ \\ 0, & x = 0 \end{cases},$$

such that it is infinitely differentiablem but $f^{(n)} = 0$ for $n = 1, 2, 3, \ldots$. Hence, this function is not analytic at $x = 0$.

Proof. Construct a sequence of set

$$E_n = \{x \in [-1,1] \mid f^{(n)}(x) = 0\} \implies [-1,1] = \bigcup_{n=1}^{\infty} E_n,$$

with E_n closed (Exercise #1). Applying Baire-Category Theorem to $[-1,1]$, at least one E_{N_1} contains a non-empty open interval, say I_1 (Exercise #2).

1. On I_1, $f^{(N_1)} \equiv 0$, which implies f is a polynomial of degree $N_1 - 1$(Exercise #3).

2. If $I_1 = (-1,1)$, the proof is complete.

3. Otherwise, $[-1,1] \setminus I_1 \neq \varnothing$. Applying Baire-Category Theorem on the set $[-1,1] \setminus I_1 := \bigcup_{n=1}^{\infty} E_n \setminus I_1$, we conclude that at least one $E_{N_2} \setminus I_1$ contains a non-empty open interval, say I_2, on which f is a polymial of degree $N_2 - 1$.

4. Each time applying the same trick to construct $I_1, I_2, \ldots$, and make sure these are the **maximal** intervals with the desired properties. Finally, we reach the stage that:

   $f|_{x \in I_j}$ is a polynomial of order $N_j - 1$ for $j = 1, \ldots, \infty$ and $\bigcup_{j=1}^{\infty} I_j$ is dense on $[-1,1]$ (Exercise #4).

5. Construct and claim that

$$H = [-1,1] \setminus \bigcup_{j=1}^{\infty} I_j = \{-1,1\}.(\text{Exercise \#5})$$

6. Combining (4) and (5), we derive f satisfies the condition in Proposition(4.2), and therefore is a polynomial. (Exercise #6)

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■

Verification. Here we give some hints for the exercises above:

1. Since the inverse image of $\{0\}$ is closed for continuous functions, and $f^{(n)}(\cdot)$ is continuous, we derive E_n's are closed.

2. The Baire-Category Theorem asserts that for a non-empty complete metric space X, or any subsets of X with **non-empty** interior, if it is the countably union of **closed** sets, then one of these **closed** sets has non-empty interior.

3. By integrating $f^{(N_1)}$ for N_1 times, e.g.,

$$f^{(N_1)} = 0 \implies f^{(N_1-1)} = \int f^{(N_1)}\,\mathrm{d}x = a_0 \implies \cdots \implies f = a_{N_1-1}x^{N_1-1} + \cdots + a_0$$

4. Let $I \subseteq [-1,1]$ be any open interval. Its clousure can be expressed as:

$$\bar{I} = \bigcup_{n=1}^{\infty} \bar{I} \bigcap E_n$$

Applying Baire category theorem to $\bar{I}$, at least one $\bar{I} \bigcap E_{n'}$ contains an open interval I'. Thus, $I' \subseteq I$ and $I' \subseteq E_{n'}$, which implies $I' \in \bigcup_{j=1}^{\infty} I_j$ (recall that I_j's are picked maximally). This means that $\bigcup_{j=1}^{\infty} I_j \bigcap I$ is non-empty for arbitrary open interval I, which implies $\bigcup_{j=1}^{\infty} I_j$ is dense.

5. We have seen that $\bigcup_{j=1}^{\infty} I_j$ is an open, dense **proper** subset of $[0,1]$, which means H is **non-empty**, closed, and nowhere dense in $[0,1]$. In order to show $H = \{-1,1\}$, it suffices to show H does not contain open intervals. Otherwise applying Baire-Category Theorem to $H = \bigcup_{n=1}^{\infty} E_n \bigcap H$ again, for some fixed n^*, $E_{n^*} \cap H$ contains an open interval I^*. Thus, $I^* \subseteq E_{n^*}$ and $I^* \subseteq H \implies I^* \subseteq (\bigcup_{j=1}^{\infty} I_j)^c$, which leads to a contradiction as I_j's are picked maximally.

6. Hence, $\bigcup_{j=1}^{\infty} I_j = (-1,1)$, i.e., $f(x) = \sum_{k=0}^{\infty} a_k x^k$ in $|x| < 1$.

A simpler and more clear proof is presented in the website

https://mathoverflow.net/questions/34059/

if-f-is-infinitely-differentiable-then-f-coincides-with-a-polynomial

We have seen some examples of nowhere differentiable functions. Now we show that almost functions are nowhere differentiable.

Notations. We denote $\mathcal{C}[0,1]$ as the set of all continuous functions on $[0,1]$. One corresponding metric is defined as:

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|, \quad \forall f, g \in \mathcal{C}[0,1].$$

Remember that $(\mathcal{C}[0,1], d)$ is complete.

> **Theorem 4.1** The set of all nowhere differentiable functions in $(\mathcal{C}[0,1], d)$ is dense, i.e., forms a 2nd Category.

The trick is to show the complement of the set of nowhere differentiable functions, i.e., the set of functions that have a **finite** derivative at some point, forms a 1st Category.

Proof. Construct

$$E_n = \left\{ f \in \mathcal{C}[0,1] \,\middle|\, \begin{array}{l} \forall 0 < h < 1 - x, \left| \dfrac{f(x+h) - f(x)}{h} \right| \leq n, \\ \text{for some } 0 \leq x \leq 1 - \frac{1}{n} \end{array} \right\}$$

Thus the union of all E_n will contain all functions having a finite **right hand derivative** at some point in $[0,1)$.

Proposition 4.3 E_n is closed, i.e., for a sequence of function $\{f_m\} \subseteq E_n$ such that $f_m \to f$, we have $f \in E_n$.

Proposition 4.4 E_n is nowhere dense, i.e., $(\mathcal{C} \setminus E_n$ is dense):

After showing these two propositions, we conclude that the set of functions, with a right derivatives at some point, is a set of the first category. Similarly, we can repeat these steps for left derivatives. In summary, the set of functions with a well-defined derivatives forms a 1st Category. The proof is complete. ∎

Proof of Proposition(4.3). Since $\{f_m\} \subseteq E_n$, there exists a sequence of $\{x_m\}$ such that for

each m,

$$0 \leq x_m \leq 1 - \frac{1}{n}$$

$$|f_m(x_m + h) - f_m(x_m)| \leq hn,$$

for $\forall 0 < h < 1 - x_m$. As $\{x_m\}$ is bounded, there exists a subsequence $\{x_{m,k}\}$ of $\{x_m\}$ with limit $x \in [0, 1 - \frac{1}{n}]$.

For $\forall 0 < h < 1 - x$, we have that $0 < h < 1 - x_{m,k}$ for large k. Applying triangle inequality, we obtain:

$$
\begin{aligned}
|f(x + h) - f(x)| &\leq |f(x + h) - f(x_{m,k} + h)| + |f(x_{m,k} + h) - f_m(x_{m,k} + h)| \\
&\quad + |f_m(x_{m,k} + h) - f_m(x_{m,k})| + |f_m(x_{m,k}) - f(x_{m,k})| + |f(x_{m,k}) - f(x)| \\
&\leq |f(x + h) - f(x_{m,k} + h)| + d(f, f_k) + nh + d(f_k, f) + |f(x_k) - f(x)|.
\end{aligned}
$$

Taking $k \to \infty$, we find all terms in RHS goes to zero except nh:

$$|f(x + h) - f(x)| \leq nh \implies f \in E_n.$$

$\blacksquare$

Proof of Proposition(4.4). In order to show E_n is nowhere dense, by using the fact that E_n is closed, it suffices to show that an arbitrary open neighborhood $B(f, \varepsilon)$ will contain elements from the set $C[0,1] \setminus E_n$, i.e., it suffices to create a function in $B(f, \varepsilon)$ that cannot be in E_n for fixed ε.

- Construct a piecewise linear function $\phi_N(x)$ on $[0,1]$ first:

$$
\phi_N(x) = \begin{cases}
N(x - \frac{k}{N}), & \frac{k}{N} \leq x \leq \frac{k+1}{N}, k = 0, 2, \ldots, N \\
-N(x + \frac{k+1}{N}), & \frac{k}{N} \leq x \leq \frac{k+1}{N}, k = 1, 3, \ldots, N-1
\end{cases}
$$

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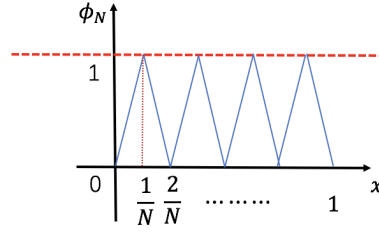
Figure 4.1: Plot of function $\phi_N(x)$

As we can see, N is the maximum slope of the piecewise linear function ϕ_N.

- Let M be the maximum slope of the piecewise linear function f, and pick a positive even integer m such that

$$\frac{1}{2}mN\varepsilon > M + n.$$

Then we construct function

$$g(x) = f(x) + \frac{1}{2}\varepsilon\phi_{mN}(x)$$

As we can see, $d(f,g) = \frac{1}{2}\varepsilon < \varepsilon$, thus $g \in B(f,\varepsilon)$. Also note that

$$\left|\frac{g(x+h) - g(x)}{h}\right| \geq \left|\frac{1/2\varepsilon(\phi_{mN}(x+h) - \phi_{mN}(x))}{h}\right| - \left|\frac{f(x+h) - f(x)}{h}\right|$$

$$\geq \frac{1}{2}\varepsilon\left|\frac{(\phi_{mN}(x+h) - \phi_{mN}(x))}{h}\right| - M$$

$$= \frac{1}{2}mN\varepsilon - M > n$$

for x in $(0, 1 - \frac{1}{mN})$ and some $h \in (0, 1 - x)$. Hence, $g \notin E_n$. The proof is complete.

∎

4.1.2. Continuity Analysis

Recall the definition for continuity:

- A function f is said to be continuous at $x_0 \in I$ if $\forall \varepsilon > 0$, there exists $\delta > 0$ (δ depends on x_0 and ε) such that

$$|f(x) - f(x_0)| < \varepsilon, \quad \forall |x - x_0| < \delta$$

- A function f is continuous on I if it is continuous at every point in I.

Definition 4.1 [Uniform] We say f is **uniformly continuous** on I if $\forall \varepsilon > 0$, there exists δ (depend only on ε, but independent of $x \in I$) such that

$$|f(y) - f(x)| < \varepsilon, \text{ if } |x - y| < \delta$$

R It is useful to note that the uniform continuity places a upper bound on the growth of the function at every point, i.e., the function cannot grow too fast.

■ **Example 4.1** Given a function $f(x) = x^2$,

1. Is it uniformly continuous on $[0,1]$?

   Yes, intuitively the growth of x^2 is limited within bounded interval.

2. Is it uniformly continuous on $\mathbb{R}$?

   No, intuitively the growth of x^2 tends to infinite as $x \to \infty$.

   Proof: For fixed x, if $|y - x| < \delta$, if we choose $|x| \geq \frac{\varepsilon}{2\delta} + \frac{\delta}{2}$, then

   $$\underbrace{|f(y) - f(x)|}_{\varepsilon} = |y^2 - x^2| = |y + x| \underbrace{|y - x|}_{\delta}$$

   $$\geq (|2x| - |x - y|)|y - x| \geq (\frac{\varepsilon}{\delta} + \delta - \delta)\delta = \varepsilon$$

   which is a contradiction.
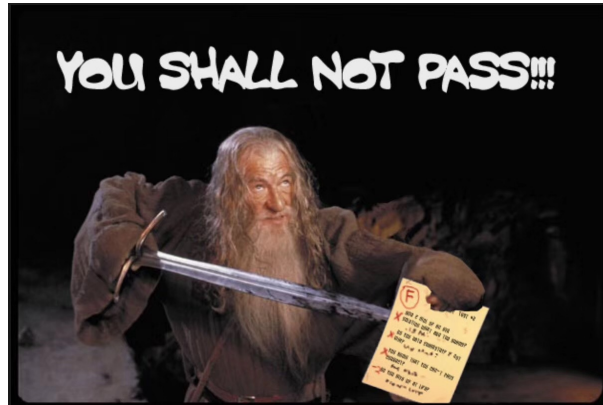
Figure 4.2: The proof and application for the Theorem(4.2) is Mandatory. If you don't know how to do it in the exam, Prof.Ni will fail you without hesitation.

Theorem 4.2 Suppose that f is continuous on a compact set D. Then f is uniformly continuous on D.

Proof. For given $\varepsilon > 0$, since f is continuous at x, there exists $\delta_x > 0$ s.t.

$$|f(y) - f(x)| < \tfrac{\varepsilon}{2}, \quad \text{if } |y - x| < \delta_x.$$

Construct an open cover $\{B_{\delta_x}(x) \mid x \in D\}$ of D with

$$B_{\delta_x}(x) = \{y \in D \mid |y - x| < \frac{1}{2}\delta_x\}.$$

The set D is compact implies there exists a finite subcover:

$$D \subseteq B_{\delta_{x_1}}(x_1) \bigcup B_{\delta_{x_2}}(x_2) \bigcup \cdots \bigcup B_{\delta_{x_k}}(x_k). \tag{4.1}$$

Construct $\delta > 0$ such that $B_\delta(x)$ must be contained entirely in one of the ball, say $B_{\delta_{x_j}}(x_j)$ (Exercise #7)

Therefore given $|y - x| < \delta$ we imply $x, y \in B_{\delta_{x_j}}(x_j)$ for some j, which follows that

$$|f(y) - f(x)| \leq |f(y) - f(x_j)| + |f(x_j) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

■

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Verification of Exercise. Such a δ is constructed as

$$\delta = \frac{1}{2}\min\{\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_k}\}.$$

Thus for any x, y with $|y - x| < \delta$, by(4.1), there exists j such that $x \in B_{\delta_{x_j}}(x_j)$, and hence

$$|x - x_j| < \frac{1}{2}\delta_{x_j} \tag{4.2}$$

Also, we have

$$|y - x_j| \le |y - x| + |x - x_j| \le \delta + \frac{1}{2}\delta_{x_j} \le \delta, \tag{4.3}$$

i.e., y is also in $B_{\delta_{x_j}}(x_j)$.

Definition 4.2 [Convex] A real-valued function f defined in (a, b) is said to be convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

whenever $a < x < b, a < y < b, 0 < t < 1$. ∎

Check Rudin's book for the proof that a convex function is always continuous.

4.2. Friday

This lecture will finish the topic for continuity, and we will have a very simple, easy quiz on Sunday.

4.2.1. Continuity Analysis

> **Definition 4.3** [Lipschitz Continuity] A function f is **Lipschitz continuous** at x_0 if there exists a constant M (depend on x_0) such that
>
> $$|f(x) - f(x_0)| \leq M|x - x_0|,$$
>
> for $\forall |x - x_0|$ small. ∎

(R)

- Note that Lipschitzness places a upper bound on the growth of the funciton that is linear in the perturbation, i.e., $|x - x_0|$.

- Also notice that Lipschitz functions need not be differentiable, e.g., $f(x) = |x|$ is Lipschitz continuous at $x = 0$.

- However, differentiable functions with bounded derivative are always Lipschitz.

- Lipschitz continuous functions are always continuous. (choose $\delta = \varepsilon/M$)

A property similar to Lipschitzness is that of Holder continuity.

> **Definition 4.4** [Holder] A function f is said to be **Holder continuous** of order α at x_0 if there exists a constant M such that
>
> $$|f(x) - f(x_0)| \leq M|x - x_0|^{\alpha}$$
>
> for $\forall |x - x_0|$ small, where $0 < \alpha < 1$. ∎

- $f(x) = \sqrt{|x|}$ is Holder continuous of order $1/2$ at $x = 0$.

- Holder continuous functions are always continuous (choose $\delta = (\varepsilon/M)^{1/\alpha}$)

Holder Continuity for Differentiable Equation. Solving differentiable equations is a core topic in pure math. When solving the ODE $u'' = f(x)$ with f continuous, we can say $u \in \mathcal{C}^2$. However, when talking about the PDE $u_{xx} + u_{yy} = f(x,y)$ with continuous f, u is not necessarily twice continuously differentiable. Instead, u is almost $\mathcal{C}^2$. However, if given the extra condition $f \in \mathcal{C}^\alpha_{\text{holder}}$, then we imply $u \in \mathcal{C}^{2+\alpha}$. Holder continuous is vital important for future mathematics study.

Definition 4.5 [Convex] A funciton f is said to be **convex** in (a,b) if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for $\forall x,y \in (a,b)$ and $\forall t \in [0,1]$ ∎

(R) The geometrically meaning for convexity is that the function evaluated in the line segment is lower than secant line between x and y, i.e., a convex function f lies below secant line.
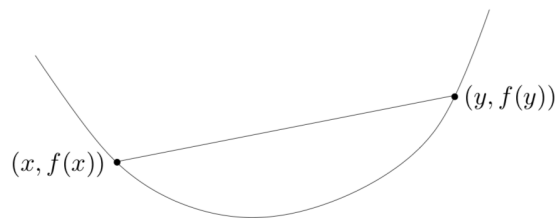
Figure 4.3: Graph of a convex function. The line segment between any two points on the graph lies above the graph.

An intuitive statement is that a convex function is always continuous. This statement can be shown pictorially.

Proposition 4.5 A convex function must be continuous.
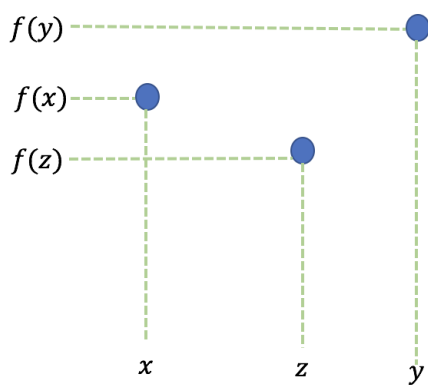
(a) Pick x and y at either side of z
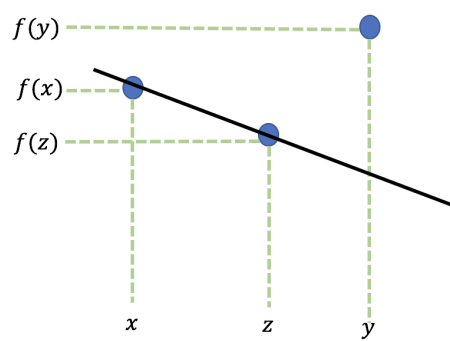
(b) The point y is above the secant line
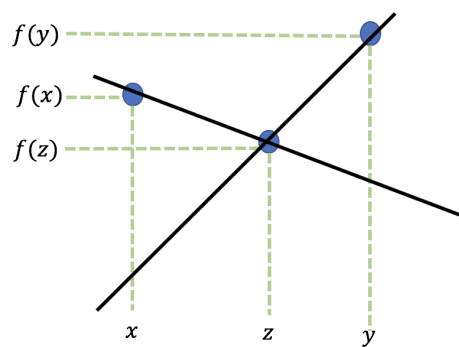
(c) The point x is below the secant line
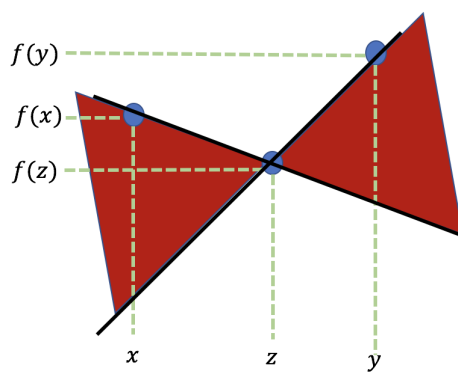
(d) The function is inside the red region

Figure 4.4: Graphic Proof for Proposition(4.5)

Proof. The proof is given in Fig:(4.4):

- To show the continuity for z, pick x and y at either side.

- By convexity, the y lies above the secant line between x and z, otherwise draw a secant line between x and y, then z lines above the secant line (contradiction).

- Again, x lines below the secant line between z and y.

- Hence, the function lies inside the red region shown in Fig:(4.4(d)).

- Pick a sequence $\{z_n\} \to z$, the function $f(z_n)$ must converge to $f(z)$.

The proof is complete. ∎

The following two statements are left as exercise:

Proposition 4.6 Assume that f is continuous real function defined in (a,b) such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in (a,b)$. Then f is convex.

Ⓡ We can pick an example to show that dropping out the continuity condition will make this statement false.

4.2.2. Monotone Analysis

Definition 4.6 [Monotone] The function f is said to be **monotone increasing (decreasing)** if $f(x) \leq f(y)$ $(f(x) \geq f(y))$ whenever $x < y$. ∎

Ⓡ

- For monotone functions,

$$\lim_{x \to x_0+} f(x), \quad \lim_{x \to x_0-} f(x),$$

always exist.

- The difference

$$\lim_{x \to x_0+} f(x) - \lim_{x \to x_0-} f(x)$$

is called the **jump discontinuity** at $x = x_0$.

Proposition 4.7 A monotone function f on (a,b) is continuous except at possibly a countable number of points.

We show the statement is true for interval $[a + \frac{1}{n}, b - \frac{1}{n}]$ first. That's because the function inside a closed intercal is finite.

Proof. • Let S_k denote the set of all points in $[a + \frac{1}{n}, b - \frac{1}{n}]$ on which f has a jump discontinuity no less than $\frac{1}{k}$.

- Then the set S_k must be finite for every k, otherwise the total jump of S_k will be infinite, which contradicts the range of f.

- The set of all discontinuous points of f on $[a + \frac{1}{n}, b - \frac{1}{n}]$ is the union $\bigcup_{k=1}^{\infty} S_k$, therefore is at most countable.

- Note that

$$(a,b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}],$$

the countably union of at most countable sets is at most countable. Tshe proof is complete.

∎

Exercise. Given the Riemann function

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ \dfrac{1}{q}, & x = \dfrac{p}{q}, (p,q) = 1. \end{cases}$$

Is f Lipschitz continuous, Holder continuous, or whatever at the given point $x_0 \notin \mathbb{Q}$?

4.2.3. Cantor Set

Now we describe a complicated subset of $\mathbb{R}$ which is uncountable. This set is called the Cantor set.

Geometric description. We start with the unit interval

$$F_0 = [0,1]$$

Now define a new set

$$F_1 = F_0 \setminus (\frac{1}{3}, \frac{2}{3}) = [0,1/3] \bigcup [2/3,1],$$

i.e., we obtain F_1 by deleting the open middle third of F_0.

Next we obtain a new set F_2 by deleting the open middle thirds of each of the intervals making up F_1:

$$F_2 = [0,1/9] \bigcup [2/9,1/3] \bigcup [2/3,7/9] \bigcup [8/9,1]$$

Continue in this way to obtain sets F_n, $n \geq 0$, where F_n consists of 2^n disjoint closed intervals of length 3^{-n}, formed by deleting the middle thirds of the intervals making up F_{n-1}.

The **Cantor set** is defined to be the intersection of these sets:

$$F = \bigcup_{n=1}^{\infty} F_n$$

Arithmetic description. We also have

$$F = \left\{ x \in [0,1] : x = \sum_{n=1}^{\infty} a_n 3^{-n}, a_n \in \{0,2\}, n \geq 1 \right\}$$

Here we can also describe F as the set of reals with a tenary expansion

$$0.a_1 a_2 \ldots a_n \ldots \qquad a_n \in \{0,2\}.$$

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For example, given $x = 3/4$, how to find the tenary expansion $x \sim 0.a_1a_2\ldots$? Clearly $3/4 \in [\frac{2}{3}, 1]$, i.e., $a_1 = 1$. The next interval is determined by which interval among $[\frac{2}{3}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$ that x belongs. It is clearly that it belongs to the first interval, and so $a_2 = 0$. Thus we can find a_n recursively via this way to get the tenary expansion.

Proposition 4.8 There is a one-to-one correspondence between F and F_0.

Proof. For any $x \in F$, we write $x = 0.a_1a_2 \cdots a_j$ with $a_j = 0$ or 2. Also, we can construct our $y = 0.b_1b_2 \cdots$ with $b_j = \frac{a_j}{2}$, i.e., y is any number with binary expansion. Therefore, we establishes a one to one correspondence between F and the set of points in $(0, 1)$. ∎

Also, there is a one-to-one correspondence between F and F_0 implies F is **uncountable**.

Proposition 4.9 The measure of F is 0.

Proof. The F_1 is constructed by taking away an interval with length $\frac{1}{3}$. The F_2 is constructed by taking away intervals with total length $\frac{2}{9}$, etc. Therefore, the toal length of taking-away intervals is given by:

$$
\begin{aligned}
\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots \\
&= \frac{1}{3} + \frac{2}{3^2} + \frac{4}{3^3} + \cdots \\
&= \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) \\
&= 1
\end{aligned}
$$

∎

Proposition 4.10 F is closed and nowhere dense.

Proof. F is closed since it is the complement of the union of open intervals. The Cantor set is nowhere dense as its closure has empty interior. ∎

The proof for proposition(4.11) is left as exercise.

Proposition 4.11 Every point in F is a limit point of F

Definition 4.7 [Perfect] A closed set S is **perfect** if every point in S is a limit point of S. ∎

The proof for proposition(4.12) is left as exercise.

Proposition 4.12 Any perfect set is uncountable.

Chapter 5

Week5

5.1. Wednesdays

Today we will discuss topics about differentiation. Note that the topics in Friday will be more difficult.

5.1.1. Differentiation

Notations and Conventions. Given two functions ϕ and ξ, then we denote $\phi(x) = O(\xi(x))$ near x_0 if there exists a constant C such that

$$\left|\frac{\phi(x)}{\xi(x)}\right| \leq C \text{ for } x \text{ near } x_0,$$

i.e., $|\phi(x)| \leq C|\xi(x)|$; also, $\phi(x) = o(\xi(x))$ near x_0 if

$$\left|\frac{\phi(x)}{\xi(x)}\right| \to 0 \text{ as } x \to x_0,$$

i.e., $\forall \varepsilon > 0$, there exists $\delta > 0$ s.t. $|\phi(x)| \leq \varepsilon|\xi(x)|$ if $|x - x_0| \leq \delta$.

Ⓡ

1. In particular, if $f(x) \to 0$ as $x \to x_0$, we write $f(x) = o(1)$ near x_0.
2. $|\frac{\phi(x)}{h}| = o(1) \Longleftrightarrow |\phi(x)| = o(h)$.

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Definition 5.1 [Derivative] Given a function f, if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists,}$$

we say that f is **differentiable** at x_0 and the limit is called the **derivative** of f at x_0, denoted as $f'(x_0)$. ∎

Geometric Interpretation. Derivative has its geometric meaning:
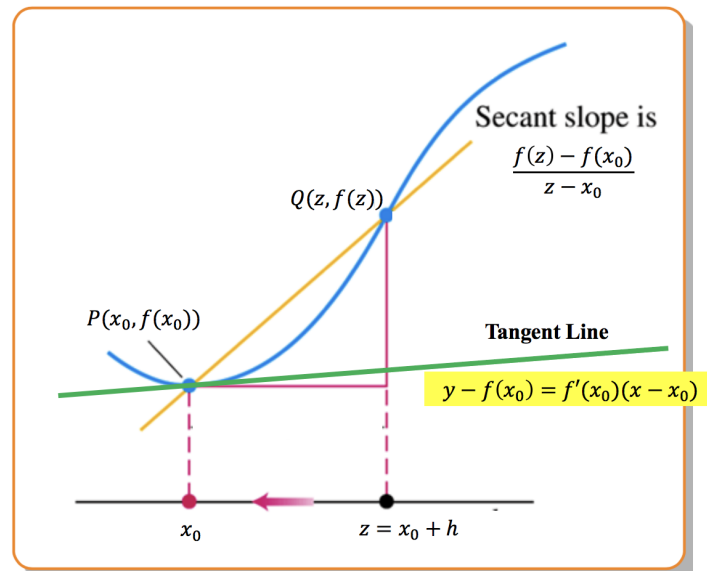
Figure 5.1: Interpretations of Derivative

Here the derivative is essentially the **slope** of the tangent line to the curve $y = f(x)$ at $x = x_0$, where the tangent line is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Here the tangent line is defined as the limit of secant line, i.e., taking limit $z \to x_0$, the secant line between x_0 and z becomes the tangent line.

Arithmetic Insights of derivative. The limit

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

can be equivalently rewritten as:

- $\left|\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0)\right| \to 0 (= o(1))$ as $h \to 0$,

- $|f(x_0 + h) - f(x_0) - f'(x_0)h| = o(h)$ as $h \to 0$,

- $f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)$ as $h \to 0$. (useful equivalent definition for derivative)

Substituting $h = x - x_0$ into the above equation, we derive

$$|f(x) - [f(x_0) + f'(x_0)(x - x_0)]| = o(x - x_0), \quad \textbf{Important Formula}$$

which essentially means that *the differential provides the best linear approximation to the function in a neightborhood of a point.*

5.1.2. Basic Rules of Differentiation

Given two functions g, f, we study the derivative evaluated for the composite function $g \circ f$:

Proposition 5.1 $(g \circ f)' = g'(f(x)) \cdot f'(x)$

Let's see how the engineer gives a proof: (recall MAT1001 slides, they exactly done this)

Engineers' proof. Recall the definition

$$(g \circ f)'(x) := \lim_{h \to 0} \frac{(g \circ f)(x_0 + h) - (g \circ f)(x)}{h}.$$

Note that

$$\frac{(g \circ f)(x_0 + h) - (g \circ f)(x)}{h} = \frac{(g \circ f)(x_0 + h) - (g \circ f)(x)}{f(x + h) - f(x)} \frac{f(x + h) - f(x)}{h}.$$

Taking the limit $h \to 0$, the first term approaches $g'(f(x))$, and the second approaches $f'(x)$.   ∎

Comments. Note that $f(x+h) - f(x)$ may not be necessarily non-zero, and there is meaningless to pick a zero term in denominator.

Mathematicians' Proof. Recall the definition

$$(g \circ f)(x+h) - (g \circ f)(x) := g(f(x) + f'(x)h + o(h)) - g(f(x))$$

The differentiablity of g gives us $g(y) = g(y_0) + g'(y_0)(y - y_0) + o(y - y_0)$, and therefore

$$
\begin{aligned}
(g \circ f)(x+h) - (g \circ f)(x) &= g(f(x) + f'(x)h + o(h)) - g(f(x)) \\
&= \{g(f(x)) + g'(f(x))\left[(f'(x)h) + o(h)\right]\} - g(f(x)) \\
&= g'(f(x))f'(x)h + \underbrace{g'(f(x))}_{\text{fixed as } x \text{ is fixed}} o(h) = g'(f(x))f'(x)h + o(h),
\end{aligned}
$$

$\blacksquare$

and therefore $(g \circ f)'(x) = g'(f(x))f'(x)$.

5.1.3. Analysis on Differential Calculus

Differentiablity implies continuity.

Proposition 5.2 If f is differentiable at x_0, then it is continuous at x_0.

Proof. By definition of differentiablity,

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + o(x - x_0).$$

Taking the limit $x \to x_0$, the RHS approaches 0, i.e., $\lim_{x \to x_0} f(x) = f(x_0)$. $\blacksquare$

> **Theorem 5.1 — Rolle's Theorem.** Suppose f is differentiable on $[a, b]$, then $f'(x_0) = 0$ if $x_0 \in (a, b)$ is a local maximum or local minimum.

Proof. Note that

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Suppose that x_0 is a local maximum, then

- $x > x_0$ and x close to x_0 implies $f(x) - f(x_0) \leq 0$, i.e., $\lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$

- Similarly, $x < x_0$ and x close to x_0 implies $\lim_{x \to x-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$.

Therefore

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x-} \frac{f(x) - f(x_0)}{x - x_0} = 0.$$

∎

(R) Geometrically this theorem is obvious. It asserts that at an extreme point of a differentiable function, the tangent line to its graph is horizontal.

Estimation on finite increments. The following two proposition are the most frequently used and important methods of studying numerical-valued functions.

Theorem 5.2 — **Mean-Value Theorem.** Suppose f is differentiable on $[a,b]$, then there exists a point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

The proof relies on Rolle's theorem, i.e., we need to construct a function h on $[a,b]$ that has a interior local maximum or local minimum. This is one useful trick.

Proof. We set $g(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$, i.e., g is a secant line between $(a, f(a))$ and $(b, f(b))$. Then we consider the auxiliary function $h(x) = f(x) - g(x)$, which implies $h(a) = h(b) = 0$.

- If $h \equiv 0$, then $g \equiv f$, and therefore $g' \equiv f'$, i.e.,

$$g'(x) = \frac{f(b) - f(a)}{b - a} = f'(x), \qquad \forall x \in (a, b)$$

- Otherwise, h is positive or negative somewhere in (a, b). w.l.o.g., h is positive somewhere in (a, b). Thus h assumes its maximum in (a, b), say c (Exercise # 1). By rolle's theorem $h'(c) = 0$, which impleis $f'(c) = g'(c) = (f(b) - f(a))/(b - a)$.

■

Applying Mean-Value Theorem, we give a more useful verison of Rolle's Theorem:

Corollary 5.1 [Rolle's Theorem verison 2] Suppose f is differentiable on $[a, b]$ with $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. By Mean-Value Theorem,

$$f(b) - f(a) = 0 = f'(c)(b - a) \implies f'(c) = 0.$$

■

Exercise Verification. Recall the proposition we have learnt:

Proposition 5.3 The range of a continuous function over a compact set is also compact.

As a result, $\sup_{x \in [a,b]} h = \max_{x \in [a,b]} h$, i.e., h assumes its maximum in $[a, b]$.

(R) Mean Value Theorem is correct, but not precise, i.e., we obtain an estimate of a function using affine. Now we are interested in approximations of a function by a polynomial $P_n(b) = P_n(b; a) = c_0 + c_1(b - a) + \cdots + c_n(b - a)^n$. The Taylor's theorem gives the answer using integrals.

Theorem 5.3 — Taylor's Theorem. Let f be n times differentiable on the open interval

with $f^{(n-1)}$ continuous on the closed interval between a and x, then

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(n-1)}(a)\frac{(x-a)^{n-1}}{(n-1)!} + R_n(x),$$

where the remainder is given by:

$$R_n(x) = \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}\,\mathrm{d}t$$

Proof. All techniques in this proof is **integration by parts**:

$$f(x) = f(a) + \int_a^x f'(t)\,\mathrm{d}t = f(a) - \int_a^x f'(t)\,\mathrm{d}(x-t)$$
$$= f(a) - f'(t)(x-t)|_b^a + \int_a^x f''(t)\,\mathrm{d}t = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)\,\mathrm{d}t$$

Applying the similar trick gives the result. ∎

Motivation of Continuosly differentiable.

Ⓡ If $f'(x_0) > 0$, then f is increasing at x_0. However, it does not mean that f is increasing in a neighborhoold of x_0. That's because f' may not be continuous.

Some mathematicians give an example of a function that is everywhere differentiable but nowhere monotone:

> ■ **Example 5.1** There exists a function haing the following properties:
>
> 1. f is differentiable with $|f'| \leq 1$ on $\mathbb{R}$
> 2. Both $\{x \in \mathbb{R} \mid f'(x) > 0\}$ and $\{x \in \mathbb{R} \mid f'(x) < 0\}$ are dense in $\mathbb{R}$
> 3. $\{x \in \mathbb{R} \mid f'(x) = 0\}$ is also dense in $\mathbb{R}$
> 4. f is the difference of 2 monotone functions
> 5. f' is **not** Riemann integrable.
>
> Reference: `http://citeseerx.ist.psu.edu/viewdoc/download?`

In the future when handling such topic, we will assume f is **continuously differentiable**, denoted as $f \in \mathcal{C}^1$. Then the example above will not appear.

5.2. Friday

We plan to have a make-up lecture tomorrow, which will cover problems in Quiz 1.

5.2.1. Analysis on Derivative

Not all functions could be the derivative of some functions. Let's give sufficient or necessary conditions for that.

Criteria 1: The image of the derivative of any function should be continuous.

> **Theorem 5.4** — **Intermediate-Value Property for derivative.** Let f be differentiable on $[a,b]$, and let y_0 be a number between $f'(a)$ and $f'(b)$. Then there exists $c \in (a,b)$ such that $f'(c) = y_0$.

Proof. Consider an auxiliary function

$$h(x) = f(x) - y_0 x,$$

which follows that $h'(a) = f'(a) - y_0$ and $h'(b) = f'(b) - y_0$. w.l.o.g., $f'(a) < f'(b)$, and therefore $h'(b) < 0 < h'(a)$, i.e.,

$$\begin{cases} \lim_{x \to b+} \frac{h(x)-h(b)}{x-b} < 0 \\ \lim_{x \to a-} \frac{h(x)-h(a)}{x-a} > 0 \end{cases} \implies \begin{cases} h(x) < h(a) \text{ for some } x > a \\ h(x) < h(b) \text{ for some } x > b \end{cases},$$

i.e., h has a minimum on $[a,b]$, say at c. By Rolle's theorem, $h'(c) = 0$, i.e., $f'(c) = y_0$. ∎

> R This theorem tells us that the image for the derivative of any function (on $[a,b]$) contains the whole interval between $f'(a)$ and $f'(b)$. Hence, the step function cannot be the derivative of other functions, since its image does not contain the interval $[0,1]$ but only the endpoints.

However, it does not asset that the derivative of any function should be continuous (the image should be). A simple example of this is the function

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \neq 0 \\ \\ 0, & x = 0 \end{cases}$$

the derivative $f'(x)$ is not continuous at $x = 0$.

The question turns out how continuity should be for the derivative of any function?

Criteria 2: The derivative of any function should be continuous on a dense set. For the derivative of any function f, if exists, can be expressed as the pointwise limit:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{n \to \infty} \underbrace{n \left[f(x + \frac{1}{n}) - f(x) \right]}_{f_n(x)} \\ &:= \lim_{n \to \infty} f_n(x), \end{aligned}$$

with $f_n(x) = n \left[f(x + \frac{1}{n} - f(x)) \right]$. Thus the set of all discontinuous points of f' is a set of first categroy (Recall Theorem(3.3)). In particular, f' must be continuous on a dense set.

5.2.2. Analysis on Mean-Value Theorem

The following proposition is a useful generalization of the standrad mean-value theorem, and is also based on the Rolle's theorem. From this theorem we can also imply the L-Hopital's Rule:

Theorem 5.5 — **Cauchy's Mean-Value Theorem.** Let f and g be two differentiable

function on $[a,b]$. Then there exists a point $c \in (a,b)$ such that

$$g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)] \tag{5.1}$$

Ⓡ

1. When $g(x) := x$, Cauchy's Mean-Value Theorem becomes the Mean-Value Theorem.

2. If in addiction $g'(x) \neq 0$ for each $x \in (a,b)$, then $g(b) \neq g(a)$ and we have the equality version of (5.1):

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

The idea of proof is to construct an auxiliary function satisfying the hypotheses of Rolle's theorem version 2 (Corollary (5.1)), which is the same trick used in Theorem(5.2) and (5.4).

Proof. We construct a function

$$h(x) = g(x)[f(b) - f(a)] - f(x)[g(b) - g(a)],$$

which follows that $h(a) = h(b) = g(a)f(b) - f(a)g(b)$. By Rolle's theorem, there exists $c \in (a,b)$ s.t. $h'(c) = 0$, i.e., $g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)]$. ∎

Now we pause to discuss a special but very useful technique for findind the limit of a ratio of functions, known as L-Hopital's Rule[1].

Theorem 5.6 — L-hospital's Rule. Suppose f and g are both differentiable on (a,b), and $g'(x) \neq 0$ in (a,b). Suppose that

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = l,$$

---

[1]G.F.de l'Hopital, a French mathematician, a capable student of Johann Bernoulli. The L-Hopital's Rule is really due to Johann Bernoulli, but l'Hopital is so rich so that there is a deal on the table, and thus the rule was published in slightly altered altered form by "l'Hopital".

then

1. $\lim_{x \to a+} f(x) = 0 = \lim_{x \to a+} g(x)$ implies $\lim_{x \to a+} \frac{f(x)}{g(x)} = l$.

2. $\lim_{x \to a+} f(x) = +\infty, \lim_{x \to a+} g(x) = \infty$ implies $\lim_{x \to a+} \frac{f(x)}{g(x)} = l$.

We discuss the proof for the first case, and the second case is left as exercise.

Proof. Note that $g(x) \neq g(y)$ for $\forall x, y \in (a, b)$, otherwise there will be some $c \in (a, b)$ such that $g'(c) = 0$. By Cauchy's Mean-Value Theorem, for $x, y \in (a, b)$, there exists $z \in (y, x)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)}$$

Or equivalently,

$$\frac{f(x)}{g(x)} = \frac{f(y)}{g(x)} + \frac{f'(z)}{g'(z)} \left[1 - \frac{g(y)}{g(x)} \right], \forall x, y \in (a, b). \tag{5.2}$$

For fixed x, pick $y < x$ which is close to $a+$ such that $\frac{f(y)}{g(x)}$ and $\frac{g(y)}{g(x)}$ both small (the reason we can do that is because $\lim_{y \to a+} f(y) = 0 = \lim_{y \to a+} g(y)$). Thus taking the limit $x \to a+$, by (5.2), we derive

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(z)}{g'(z)} = l.$$

∎

We can also derive the L-hopital's rule by Taylor expansion, but keep note that we should add one moer condition that $f, g \in C^1$ (so that we can apply taylor expansion)

Proof. We expand $f(x)$ and $g(x)$ in a small neighborhood of a with $x \neq a$:

$$\begin{aligned}
\frac{f(x)}{g(x)} &= \frac{f(a) + f'(a)(x - a) + o(x - a)}{g(a) + g'(a)(x - a) + o(x - a)} \\
&= \frac{(x - a)[f'(a) + o(1)]}{(x - a)[g'(a) + o(1)]} \\
&= \frac{f'(a) + o(1)}{g'(a) + o(1)} \to \frac{f'(a)}{g'(a)}, \qquad x \to a.
\end{aligned}$$

∎

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Note that L-hopital's rule is a technique that should be cleverly used, otherwise the limit can be messy to handle. Prof.Yeye Ni does not like this technique. Now we give an example:

■ **Example 5.2**

$$\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/(1-\cos x)} = \lim_{x \to 0} \exp \left[\frac{\ln(\frac{\sin x}{x})}{1 - \cos x} \right]$$

$$= \exp \left[\lim_{x \to 0} \frac{\ln(\frac{\sin x}{x})}{1 - \cos x} \right]$$

$$= \exp \left[\lim_{x \to 0} \frac{\cos x - \sin x / x}{\sin^2 x} \right] \leftarrow \text{L-hopital's Rule}$$

$$= \exp \left[\lim_{x \to 0} \frac{x \cos x - \sin x}{x \sin^2 x} \right]$$

$$= \exp \left[\lim_{x \to 0} \frac{\cos x - x \sin x - \cos x}{\sin^2 x + 2x \sin x \cos x} \right] \leftarrow \text{L-hopital's Rule}$$

$$= \exp \left[\lim_{x \to 0} \frac{-x}{\sin x + 2x \cos x} \right]$$

$$= \exp \left[\lim_{x \to 0} \frac{-1}{\cos x + 2 \cos x - 2x \sin x} \right] \leftarrow \text{L-hopital's Rule}$$

$$= e^{-1/3}$$

An alternative way is to apply Taylor expansion. Recall that

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\cos x = 1 - \frac{1}{2} x^2 + o(x^2)$$

$$\ln(1 + z) = z - z^2 + o(z^2).$$

Therefore we have

$$\ln(\frac{\sin x}{x}) = \ln(1 - \frac{1}{6}x^2 + o(x^2)) = \ln(1 + x^2(-\frac{1}{6} + o(1)))$$

Finally as $x \to 0$,

$$\frac{\ln(\frac{\sin x}{x})}{1 - \cos x} = \frac{x^2(-\frac{1}{6} + o(1))}{x^2(\frac{1}{2} + o(1))} = -\frac{1}{3}$$

Plan for next week. Next week, with Taylor theorem, we will discuss taylor's polynomial, and finally taylor series (let $n \to \infty$).

Also, given a function $f \in \mathcal{C}^\infty$, we will discuss topics about:

1. do we always have the taylor series converge in some neighborhood?
2. Suppose it does, does it necessarily converge to $f(x)$?

   One counter-example is the typical function

   $$f(x) = \begin{cases} \exp\left[-\dfrac{1}{x^2}\right], & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

   The taylor expansion at $x = 0$ is always zero, and therefore does not converge to $f(x)$.

Hence what conditon could guarantee the correctness of convergence?

5.3. Saturday: Comments on Quiz 1

The performance in Quiz 1 is not satisfying in general. The quiz one will not be counted, but the next exam will. It is designed for a rigorous mathematical course.

5.3.1. First Question

The first is from Schroder-Bernstein Theorem:

$A = (-1,1)$ and $B = [-1,1]$. We are given two one-to-one mapping $f(x) = x : A \mapsto B$ and $g(x) = \frac{x}{2} : B \mapsto A$. We are required to construct a one-to-one onto mapping from A to B.

Solution.

$$B \setminus D = \{-1,1\}$$
$$g(B \setminus D) = \{-\frac{1}{2}, \frac{1}{2}\}$$
$$gfg(B \setminus D) = \{-\frac{1}{4}, \frac{1}{4}\}$$

$$\ldots\ldots\ldots\ldots$$

Hence $S = \{\ldots, -\frac{1}{4}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \ldots\}$, and

$$F(x) = \begin{cases} f(x) = x, x \in A \setminus S \\ g^{-1}(x) = 2x, x \in S \end{cases}$$

∎

5.3.2. Second Question

The first question appears on assginement 1, while the second question appears in diagnostic quiz. **Always keep in mind that the same question or similar question may appear again in future exam.**

1. Compute limit $\lim_{x \to 0+} x^x$: as $x \to 0+$,

$$x^x = \exp(\ln(x^x)) = \exp(x \ln x)$$

$$= \exp(\frac{\ln x}{1/x}) = \exp(\frac{(\ln x)'}{(1/x)'}) = \exp(\frac{1/x}{-1/x^2})$$

$$= \exp(-x) \to 1$$

5.3.3. Third Question

Also appears in assignment 1.

Suppose $a_n \geq 0, \forall n$, and the series $\sum_{n=1}^{\infty} a_n$ converges. The new sequence $\{A_n\}$ is given by

$$A_n = \sqrt{\sum_{k=n}^{\infty} a_k} - \sqrt{\sum_{k=n+1}^{\infty} a_k}.$$

Show that $\sum_{n=1}^{\infty} A_n$ also convetges and $a_n = o(A_n)$ as $n \to \infty$.

Solution. The partial sum gives

$$A_1 + A_2 + \cdots + A_l$$

$$= (\sqrt{\sum_{k=1}^{\infty} a_k} - \sqrt{\sum_{k=2}^{\infty} a_k}) + (\sqrt{\sum_{k=2}^{\infty} a_k} - \sqrt{\sum_{k=3}^{\infty} a_k}) + \cdots + (\sqrt{\sum_{k=l}^{\infty} a_k} - \sqrt{\sum_{k=l+1}^{\infty} a_k})$$

$$= \sqrt{\sum_{k=1}^{\infty} a_k} - \sqrt{\sum_{k=l+1}^{\infty} a_k} \to \sqrt{\sum_{k=1}^{\infty} a_k} \qquad \text{as } l \to \infty$$

Also,

$$\frac{a_n}{A_n} = \frac{a_n}{\sqrt{\sum_{k=n}^{\infty} a_k} - \sqrt{\sum_{k=n+1}^{\infty} a_k}}$$

$$= \frac{a_n(\sqrt{\sum_{k=n}^{\infty} a_k} + \sqrt{\sum_{k=n+1}^{\infty} a_k})}{a_n} = \sqrt{\sum_{k=n}^{\infty} a_k} + \sqrt{\sum_{k=n+1}^{\infty} a_k}$$

$$= o(1)$$

∎

5.3.4. Fourth Question

This question is about Baire-Category Theorem:

> The set of rational numbers $\mathbb{Q}$ is not a countably intersection of open
> sets.

Proof. Assume it is. Suppose

$$\mathbb{Q} = \bigcap_{n=1}^{\infty} O_n,$$

where O_n's are open set containing $\mathbb{Q}$, and therefore dense.

Consider the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} (\mathbb{R} \setminus O_n)$. $\mathbb{R} \setminus O_n$ is closed $\forall n$, containing no open set, since otherwise open set $\mathbb{R} \setminus O_n$ will contain rational numbers. Therefore $\mathbb{R} \setminus O_n$ is nowhere dense, and therefore $\mathbb{R} \setminus \mathbb{Q}$ is first Category, i.e.,

$$\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \bigcup \mathbb{Q}$$

is of first Category, which is a contradiction. ∎

5.3.5. Fifth Question

The growth of the uniform continuous function.It is essentially the test on mathematical maturity. You are required to show that

> if $f : \mathbb{R} \mapsto \mathbb{R}$ is **uniformly continuous**, then f can grow at ∞ at most
> linearly, i.e., $\frac{f(x)}{x^2} \to 0$, $x \to \infty$, i.e., $|f(x)| \leq Cx$ for large x.

Proof. Recall the definition for uniform continuous: $\forall \varepsilon > 0, \exists \delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ if $|x - y| < \delta$. Take $\varepsilon = 1$. Look at the postive axis: for $(n-1)\delta < x \leq n\delta$, we have

$$|f(x) - f(0)| \leq n \leq \frac{x}{\delta} + 1$$

and therefore

$$|f(x)| \leq |f(x) - f(0)| + |f(0)| \leq \frac{x}{\delta} + 1 + |f(0)| \implies \frac{f(x)}{x^2} \to 0.$$

■

5.3.6. Grading policy

We will not follow the partial grading policy in this course.

The next exam will be similar.

Furthermore, the **final** exam will be more **comprehensive**.

Chapter 6

Week6

6.1. Wednesday

Given a function $f : C^\infty$, i.e., $f^{(n)}(x)$ exists for all $n \in \mathbb{N}$, we have learnt the Taylor's polynomial (of order n), whose derivatives up to order n are the same as the corresponding derivatives of $f(x)$ at the expanding point.

Now the Taylor's series is such that its derivatives at any order are the same as the corresponding derivatives of $f(x)$ at the expanding point. This lecture will mainly discuss the follows two questions:

1. Does Taylor series always converge? Not necessarily.
2. Suppose it does, does it necessarily converge to f? Also not.

Before the discussion, let's review some preliminaries:

6.1.1. Reviewing

Taylor's Formula.

> **Theorem 6.1** — **Taylor's Theorem.** Let $f(x)$ be infinitely differentiable, then the Taylor's formula for $f(x)$ is given by:
>
> $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R_n(x;a),$$
>
> with
>
> $$R_n(x;a) = \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}\,\mathrm{d}t$$

To discuss the convergence of taylor's series, we set

$$c_n = \frac{f^{(n)}(a)}{n!},$$

thus it suffices to discuss the convergence of the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$.

6.1.2. Convergence Analysis

Power Series. Here we present several tests for the convergence of general series.

> **Theorem 6.2** — **Root Test.** Given $\sum_{n=0}^{\infty} a_n$, let $\alpha = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$, then
>
> 1. If $\alpha < 1$, $\sum a_n$ converges **absolutely**;
> 2. If $\alpha > 1$, $\sum a_n$ diverges;
> 3. If $\alpha = 1$, the test gives no information.

The condition (2) and (3) is clear. Here we only give a proof for (1):

Proof. Since $\limsup_{n \to \infty} \sqrt[n]{|a_n|} < 1$, we choose a positive number $\beta < 1$ such that

$$\sqrt[n]{|a_n|} < \beta, \quad \forall n \geq N,$$

i.e., $|a_n| < \beta^n$ for $n \geq N$. Since $\sum \beta^n$ converges, the same hold for $\sum |a_n|$. ∎

We can apply root test to derive the convergence domain for the power series $\sum c_n x^n$:

Proposition 6.1 Given $\sum c_n x^n$, let $R = \frac{1}{\limsup_{n \to \infty} |c_n|^{1/n}}$, then

1. $\sum c_n x^n$ converges absolutely for all $|x| < R$
2. $\sum c_n x^n$ diverges for $|x| > R$
3. The test gives no information if $|x| = R$.

The proof is left as exercise.

Proposition 6.2   Given the power series $f(x) = \sum c_n x^n$ with radius of convergence R, the radius of convergence for the derivatives $f^{(n)}(x)$ is also R, $n \geq 1$.

Proof. We show this statement for $f'(x)$. Since $f(x)$ converges absolutely for $|x| < R$, $f'(x)$ is given by:

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1},$$

and

$$\limsup_{n \to \infty} (n c_n)^{1/(n-1)} = \limsup_{n \to \infty} |c_n|^{1/n} |c_n|^{1/n},$$

by Root Test, the power series $\sum_{n=1}^{\infty} n c_n x^{n-1}$ converges absolutely for $|x| < R$.

The statement for higher order derivative follows the similar proof.   ∎

In the proof above, we assert but without proof that the power series converge absolutely implies its derivative can be taken term by term. The proof for this assrtion is given below:

Proposition 6.3   Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be a power series with convergence radius R, then $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$.

Proof. Fix $|x| < R$, and pick $r > 0$ s.t. $|x| < r < R$. For any $|y| < r$, consider the term $\frac{f(y)-f(x)}{y-x}$. Construct a function $g(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$, which has the same convergence radius as shown in the proof above, it suffices to show that $\frac{f(y)-f(x)}{y-x}$ is close enough to $g(x)$:

$$\frac{f(y) - f(x)}{y - x} = \sum_{n=0}^{\infty} c_n \frac{y^n - x^n}{y - x} \tag{6.1a}$$

$$= \sum_{n=0}^{\infty} c_n (y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}) \tag{6.1b}$$

where (6.1a) is because that the convergence of series implies addition and multiplication term by term; (6.1b) is because the expansion on $\frac{y^n - x^n}{y - x}$.

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Therefore we can give a bound on $|\frac{f(y)-f(x)}{y-x} - g(x)|$:

$$\left| \frac{f(y)-f(x)}{y-x} - g(x) \right| = \left| \sum_{n=0}^{N} c_n \left(\frac{y^n - x^n}{y-x} - nx^{n-1} \right) \right. \tag{6.2a}$$

$$\left. + \sum_{n=N+1}^{\infty} c_n (y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}) - \sum_{N+1}^{\infty} nc_n x^{n-1} \right| \tag{6.2b}$$

$$\leq \left| \sum_{n=0}^{N} c_n \left(\frac{y^n - x^n}{y-x} - nx^{n-1} \right) \right| \tag{6.2c}$$

$$+ \left| \sum_{n=N+1}^{\infty} c_n (y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}) \right| + \left| \sum_{N+1}^{\infty} nc_n x^{n-1} \right| \tag{6.2d}$$

$$\leq \left| \sum_{n=0}^{N} c_n \left(\frac{y^n - x^n}{y-x} - nx^{n-1} \right) \right| + 2 \sum_{n=N+1}^{\infty} n|c_n| r^{n-1}, \tag{6.2e}$$

where (6.2e) is because that $|x| < r$ and $|y| < r$.

Note that the absolute convergence of the power series $\sum nc_n x^{n-1}$ for $|x| < R$ implies the convergence of $\sum n|c_n| r^{n-1}$, i.e., $\exists N$ such that

$$\sum_{n=N+1}^{\infty} n|c_n| r^{n-1} < \frac{\varepsilon}{3}.$$

As $y \to x$, $\sum_{n=0}^{N} c_n \left(\frac{y^n - x^n}{y-x} - nx^{n-1} \right) \to 0$. Therefore,

$$\limsup_{y \to x} \left| \frac{f(y) - f(x)}{y-x} - g(x) \right| \leq 0 + \frac{2}{3}\varepsilon < \varepsilon,$$

the proof is complete. ∎

Definition 6.1 [Analytic] We say that f is **analytic** at c if f can be represented by a power series in a neighborhood of a, i.e.,

$$R_n(x) = \frac{1}{(n-1)!} \int_0^x f^{(n)}(t)(x-t)^{n-1} \, dt \to 0, \quad \text{as } n \to \infty$$

Note that f is analytic implies the existence of all its derivatives, but the converse is not true in general (why)? Adding one more condition can the converse become true.

> **Theorem 6.3** — **Bernstein's Theorem.** If f and all of its derivatives are non-negative in an interval $I = [a,b]$, then f is analytic on $(a,b)/$

Proof. It suffices to show $R_n(x) \to 0$. Note that

$$R_n(x) = \frac{1}{(n-1)!} \int_0^x f^{(n)}(t)(x-t)^{n-1}\, dt \tag{6.3a}$$

$$= \frac{1}{(n-1)!} \int_0^x f^{(n)}(t)(\frac{x-t}{b-t})^{n-1}(b-t)^{n-1}\, dt \tag{6.3b}$$

$$\leq \frac{1}{(n-1)!} \int_0^x f^{(n)}(t)(b-t)^{n-1}\, dt (\frac{x-a}{b-a})^{n-1} \tag{6.3c}$$

$$= R_n(b)(\frac{x-a}{b-a})^{n-1} \tag{6.3d}$$

$$\leq f(b)(\frac{x-a}{b-a})^{n-1} \to 0, \qquad \text{as } n \to \infty \tag{6.3e}$$

where (6.3b) is by re-arrangement terms; (6.3c) is because $(\frac{x-t}{b-t})^{n-1} \leq (\frac{x-a}{b-a})^{n-1}$ and the non-negativity of $f^{(n)}$; (6.3e) is because of the non-negativity of f and all its derivatives.

The proof is complete. ∎

> ■ **Example 6.1**　　1. $f(x) = e^x$ admits Taylor expansion at $x = 0$:
>
> $$e^x = 1 + x + \cdots + \frac{1}{n!}x^n + \cdots$$
>
> The radius of convergence for e^x is infinity, i.e., its taylor series converges on $\mathbb{R}$.
>
> 2. Also, $f(x) = e^x$ admits Taylor expansion for domain $\mathbb{C}$:
>
> $$e^z = 1 + z + \cdots + \frac{1}{n!}z^n + \cdots$$
>
> and therefore
>
> $$e^{iy} = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots + i\left(y - \frac{y^3}{3!} + \cdots\right)$$

3.
$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots$$

4.
$$\sin y = y - \frac{y^3}{3!} + \cdots$$

5.
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^{n-1}}{n}x^n + \cdots$$

6.
$$(1+x)^\alpha = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + o(x^3)$$

Finally, let's discuss the convergence for series $\sum \ln \cos \frac{1}{n^\alpha}$:

$$\cos \frac{1}{n^\alpha} = 1 - \frac{1}{2!}\frac{1}{n^{2\alpha}} + O(\frac{1}{n^{4\alpha}})$$
$$\ln \cos \frac{1}{n^\alpha} = \ln \left[1 - \frac{1}{2!}\frac{1}{n^{2\alpha}} + O(\frac{1}{n^{4\alpha}}) \right]$$
$$= 1 - \frac{1}{2!}\frac{1}{n^{2\alpha}} + O(\frac{1}{n^{4\alpha}})$$
$$\sum \ln \cos \frac{1}{n^\alpha} = -\frac{1}{2} \sum \left[\frac{1}{n^{2\alpha} + O(\frac{1}{n^{4\alpha}})} \right]$$
$$= -\frac{1}{2} \sum \frac{1}{n^{2\alpha}} \left[1 + O(\frac{1}{n^{2\alpha}}) \right],$$

which is convergent since $\sum \frac{1}{n^{2\alpha}}$ converges.

6.2. Friday

6.2.1. Recap

Announcement. As promised, the taylor series and uniform continuity will definitely show up in the mid-term. (remember the Theorem(4.2)?)

Summary about Taylor series. This lecture will mainly discuss the integration, but first let's review what we have learnt from last lecture.

The Taylor series has its connection with complex numbers:

> ■ **Example 6.2** Given a function $f(x) = \frac{1}{1+x^2}$, which is infinitely differentiable. However, when expanding its Taylor series at point $x = 0$, we have
>
> $$f(x) = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \cdots, \quad \text{holds for } x^2 < 1$$
>
> Why would the function f have taylor series convergent only hold for $x^2 < 1$, but it is infinitely differentiable on the whole real line?
>
> **Proposition 6.4** The function $f(z)$ for $z \in \mathbb{C}$ is infinitely differentiable on domain D iff it has Taylor series convergent on D. Note that real function does not necessarily have such a property.
>
> If extending the domain into complex plane, the function $\frac{1}{1+z^2}$ have poles $\pm i$, and thus have no chance to have taylor expansion beyond $|z| < 1$. Then projecting the domain $\{z : |z| < 1\}$ into the real line, we derive the function f have Taylor series convergent for $|x| < 1$. ■

Exercise. Exercise: find the taylor series of $\frac{1}{1+x^2}$ at $x = 1$ and determine its radius of convergence. Answer: $R = \sqrt{2}$

6.2.2. Riemann Integration

Set Up. Given a **bounded** function f on the closed (finite) interval $[a,b]$. A partition $\mathcal{P}$ is a set of points $\{x_i\}_{i=0}^n$:

$$a_1 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$$

where the **mesh** of $\mathcal{P}$ is defined to be $\lambda(\mathcal{P}) = \max_{1 \leq i \leq n} |\Delta x_i|$.

On each interval $[x_{i-1}, x_i]$, define

$$m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x), \quad M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$$

The lower sum and upper low sum associated with partition $\mathcal{P}$ is defined as:

$$L(\mathcal{P}, f) = \sum_{i=1}^n m_i (x_i - x_{i-1}) = \sum_{i=1}^n m_i \Delta x_i$$

$$U(\mathcal{P}, f) = \sum_{i=1}^n M_i (x_i - x_{i-1}) = \sum_{i=1}^n M_i \Delta x_i$$

Now we define the lower and upper Riemann intergral as:

$$\underline{\int_a^b} f(x) \, dx = \sup_{\mathcal{P}} L(\mathcal{P}, f)$$

$$\overline{\int_a^b} f(x) = \inf_{\mathcal{P}} U(\mathcal{P}, f)$$

These definitions are well-defined.

Definition 6.2   [integrable] We say that f is (Riemann) integrable if

$$\underline{\int_a^b} f(x) \, dx = \overline{\int_a^b} f(x) \, dx,$$

and denote the common values of which as $\int_a^b f(x) \, dx$. The set of all **Riemann integrable functions** on $[a,b]$ is denoted as $\mathcal{R}[a,b]$. ∎

■ **Example 6.3**   Check if these functions are integrable or not:

1. $f(x) \equiv 1$ on $[0,1]$; then $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x) = 1$.

2. Dirichlet function:
$$D(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$$

   Verify that this function always has lower sum 0 and upper sum 1, and therefore not integrable.

3. Riemann function on $[0,1]$:
$$R(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ \dfrac{1}{q}, & x = \dfrac{p}{q}, q > 0, (p,q) = 1 \end{cases}$$

   We will show that it is integrable later. First keep in mind that the function with countably many discontinuites on $[a,b]$ is integrable.

4. The function defined on $[0,1]$:
$$f(x) = \begin{cases} 0, & x = 0 \\ \sin\dfrac{1}{x}, & x \neq 0 \end{cases}$$

   It is integrable.

5. Evaluate the limit from definition
$$\lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right].$$

   Construct
$$x_n = \sum_{k=1}^{n} \frac{1}{n+k} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+k/n}$$

   Note that x_n is just the lower sum associated with the partitition
$$\mathcal{P} = \{x_0 := 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_n := 1\}.$$

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As $n \to \infty$, inituitively $\mathcal{L}(\mathcal{P}, f) \to \mathcal{U}(\mathcal{P}, f)$. Therefore

$$x_n \to \int_0^1 \frac{dx}{1+x} = \log 2$$

∎

Definition 6.3 [Refinement] Given a partition $\mathcal{P}$, we say $\mathcal{P}^*$ is a refinement of $\mathcal{P}$ if $\mathcal{P}^*$ contains all the sub-division points of $\mathcal{P}$ ∎

Proposition 6.5 Let $f : [a,b] \mapsto \mathbb{R}$ with $m \le f(x) \le M$ on $[a,b]$, then

1. $L(\mathcal{P}, f) \le L(\mathcal{P}^*, f)$ and $U(\mathcal{P}^*, f) \le U(\mathcal{P}, f)$ holds for any refinement $\mathcal{P}^*$ of $\mathcal{P}$

2. $L(\mathcal{P}_1, f) \le U(\mathcal{P}_2, f)$ for any refinements $\mathcal{P}_1, \mathcal{P}_2$.

3.
$$m(b-a) \le \underline{\int_a^b} f(x)\, dx \le \overline{\int_a^b} f(x)\, dx \le M(b-a)$$

4. f is **Riemann integrable** iff $\forall \varepsilon$, there exists $\mathcal{P}$ s.t. $U(\mathcal{P}, f) - L(\mathcal{P}, f) \le \varepsilon$.

Proof. 1. Check Theorem(6.4) in Rudin's book for detail

2. Take the $\mathcal{P}^*$ as common refinement for $\mathcal{P}_1, \mathcal{P}_2$, and show that

$$L(\mathcal{P}_1, f) \le L(\mathcal{P}^*, f) \le U(\mathcal{P}^*, f) \le U(\mathcal{P}_2, f)$$

3. For every $\mathcal{P}$,
$$m(b-a) \le L(\mathcal{P}, f) \le U(\mathcal{P}, f) \le M(b-a).$$

4. Check Theorem(6.6) in Rudin's book for detail.

∎

Theorem 6.4 If f is continuous on $[a,b]$, then f is Riemann integrable on $[a,b]$.

Proof. The function f is continuous on $[a,b]$ implies f is uniform continuous, i.e., $\forall \varepsilon > 0, \exists \delta > 0$ s.t. for $|x - y| < \delta$,
$$|f(x) - f(y)| < \varepsilon.$$

Pick a partition $\mathcal{P} = \{x_0 := a, x_1 := a+h, x_2 := a+2h, \ldots, x_n := a+nh := b\}$ with $h = \frac{b-a}{n} < \delta$. It follows that on interval $[x_{i-1}, x_i]$, we have

$$M_i - m_i < \varepsilon \implies U(\mathcal{P}, f) - L(\mathcal{P}, f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i \leq \varepsilon \sum_{i=1}^{n}\Delta x_i = \varepsilon(b-a)$$

∎

We aeert a proposition but without proof first:

Corollary 6.1 If f is continuous expect for finitely many points on $[a,b]$, then f is Riemann integrable.

We can apply this theorem to show that $f(x) = \sin\frac{1}{x}$ is integrable, since f has only one discontinuity on $[0,1]$.

Alternatively, we can separate the interval $[0,1]$ as $[\epsilon, 1]$ and $[0, \epsilon]$. Note that on $[\epsilon, 1]$ f is continuous, thus Riemann integrable, and then we can bound the difference of lower and upper sum by taking $\epsilon \to 0$.

We can also apply this corollary to show the Riemann function is integrable, but we need another useful tool: uniform convergence.

Theorem 6.5 Let $\{f_n\}$ be a sequence of **Riemann integrable** functions on $[a,b]$, and f_n converges uniformly to f. Then f is Riemann integrable and

$$\int_a^b f(x) = \lim_{n\to\infty}\int_a^b f_n$$

(R) Note that this theorem essentially says that a sequence of uniformly convergent function $\{f_n\}$ has the property

$$\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx$$

which may not necessarily true if we ignore the uniform convergence condition.

■ **Example 6.4** [counter-example for pointwise convergent function sequence] For the function sequence $\{f_n\}$ with

$$f_n(x) = \begin{cases} n, & x \in [0, \frac{1}{n}) \\ 0, & x \notin (0, \frac{1}{n}) \end{cases}$$

we find that

$$f = \lim_{n \to \infty} f_n = 0 \implies \int_a^b f \, dx = 0,$$

while $\int_0^1 f_n(x) \, dx = 1$. ■

Let's review the definition for uniform convergence:

Definition 6.4 [Uniform Convergence] Let f be the pointwise limit of f_n, then f_n is said to converge uniformly to f if

$$\sup_{a \le x \le b} |f_n(x) - f(x)| \to 0, \text{ as } n \to \infty.$$

■

(R) The value $\sup_{a \le x \le b} |f_n(x) - f(x)|$ tends to zero for pointwise convergent function f_n, e.g., $f_n(x) = x^n$ for $x \in [0, 1]$.

Corollary 6.2 Riemann function is Riemann integrable.

Proof. Construct the function

$$f_N(x) = \begin{cases} \dfrac{1}{q}, & x = \dfrac{p}{q}, 0 < q < N, (p, q) = 1, \\ 0, & \text{otherwise} \end{cases}$$

Thus f_N is Riemann integrable since it contains finitely many discontinuities. We can also show it is uniformly convergent with limit $R(x)$. Therefore the Riemann function is Riemann integrable. ■

Chapter 7

Week7

7.1. Wednesday

Announcement. Our mid-term is on next Wednesday in Liwen Building, from 8:00am to 10:00am. We will cover everything until this Friday.

7.1.1. Integrable Analysis

Recap. Given a sequence of functions $\{f_n\}$ with pointwise limit f, we are curious about whether the equation holds:

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \left[\lim_{n\to\infty} f_n(x)\right] \mathrm{d}x$$

Let's give a counter-example to show this equaiton may not necessarily true.

> ■ **Example 7.1**   Let $\{f_n\}$ defined on $[0,1]$ with
>
> $$f_n(x) = \begin{cases} n, & \text{if } x \in (0,\frac{1}{n}) \\ 0, & \text{otherwise} \end{cases}$$
>
> We find that $\int_0^1 f_n\,\mathrm{d}x = 1$, and $f_n \to 0$ as $n \to \infty$. Thus
>
> $$\int_0^1 \left[\lim_{n\to\infty} f_n(x)\right] \mathrm{d}x = 0 \neq \lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x$$

There is a **sufficient** condition that guarantees the equation holds:

> **Theorem 7.1** Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a,b]$. If f_n converges to f uniformly as $n \to \infty$, then f is also **Riemann integrable**, and
>
> $$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

> **Definition 7.1** We say that f_n converges to f uniformly as $n \to \infty$ on $[a,b]$ if for every $\varepsilon > 0$, there exists N such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [a,b]$ and for all $n \geq N$. ∎

Proof.
- **Step 1:** First we need to show that both $\int_a^b f_n(x)\,dx$ and $\int_a^b f(x)\,dx$ is well-defined, i.e., f and f_n is **uniformly bounded**, i.e., there exists $M, M' > 0$ such that $|f(x)| \leq M$ and $|f_n(x)| \leq M', \forall n$. First show that $\{f_n\}$ is uniformly bounded:

$$|f_n(x) - f_k(x)| = |f_n(x) - f(x) + f(x) - f_k(x)| \tag{7.1a}$$

$$\leq |f_n(x) - f(x)| + |f(x) - f_k(x)| \tag{7.1b}$$

Due to the uniform convergence of $\{f_n\}$, we choose $\varepsilon := 1$, then there exists $N > 0$ s.t.

$$|f_m(x) - f(x)| < 1, \qquad \forall m \geq N. \tag{7.1c}$$

Therefore, we give a bound on (7.1a):

$$|f_n(x) - f_k(x)| < 2, \quad \forall n, k \geq N \tag{7.1d}$$

In particular, take $k = N$, thus

$$|f_n(x) - f_N(x)| < 2 \implies |f_n(x)| < |f_N(x)| + 2, \qquad \forall n \geq N, \tag{7.1e}$$

i.e., every f_n for $n \geq N$ is bounded from $|f_N(x)|$ as 2. Therefore, we have $\{f_n\}_{n=1}^\infty$ is uniformly bounded by M. (just set $M = \max\{|f_1(x)|, \ldots, |f_{N-1}(x)|, |f_N| + 2\}$.)

Another application of (7.1c) gives the uniform boundness of f:

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + |f_N(x)|.$$

- **Step 2:** Argue the Riemann integrability of f. Define $\varepsilon_n = \sup_{a \leq x \leq b} |f_n(x) - f(x)|$, and $\varepsilon_n \to 0$ as $n \to \infty$. Therefore, we give bounds on f:

$$- \varepsilon_n + f_n(x) \leq f(x) \leq \varepsilon_n + f_n(x) \tag{7.2a}$$

So that the lower and upper integrals of f satisfy:

$$\underline{\int_a^b} [-\varepsilon_n + f_n(x)] \, dx \leq \underline{\int_a^b} f(x) \, dx \leq \overline{\int_a^b} f(x) \, dx \leq \int_a^b [\varepsilon_n + f_n(x)] \, dx \tag{7.2b}$$

Note that f_n is integrable, so we can remove the upper and lower integral symbols of $f_n \pm \varepsilon_n$:

$$\int_a^b f_n(x) - \varepsilon_n \, dx \leq \underline{\int_a^b} f(x) \, dx \leq \overline{\int_a^b} f(x) \, dx \leq \int_a^b f_n(x) - \varepsilon_n \, dx \tag{7.2c}$$

Hence we give a bound on the difference of upper and lower integrals of f:

$$0 \leq \overline{\int_a^b} f(x) \, dx - \underline{\int_a^b} f(x) \, dx \leq 2(b-a)\varepsilon_n, \tag{7.2d}$$

Since $\varepsilon_n \to 0$ as $n \to \infty$, the upper and lower integrals of f are equal. Thus $f \in \mathcal{R}[a,b]$.

- Another application of (7.2c) now yields

$$\left| \int_a^b f(x) - f_n(x) \, dx \right| \leq \int_a^b \varepsilon_n \, dx = \varepsilon_n(b-a), \tag{7.3}$$

which implies $\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx$.

$\blacksquare$

(R) The sequence of functions also remains the question that:

Would the equation (7.4) holds?

$$\lim_{n \to \infty} f'_n(x) = \left[\lim_{n \to \infty} f_n(x) \right]' \tag{7.4}$$

Equation (7.4) holds also depends on the uniform convergence of $\{f_n\}$.

7.1.2. Elementary Calculus Analysis

Theorem 7.2 — **Fundamental Theorem of Calculus.** If $f : [a,b] \mapsto \mathbb{R}$ is continuous, then the function $F(x) = \int_a^x f(t) \, dt$ is **differentiable** with $F' = f$.

Proof. The proof is simply by definition, keep in mind that difference quotient is useful in proofs related to differentiation.

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h} \left[\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt \right] - f(x) \tag{7.5a}$$

$$= \frac{1}{h} \int_x^{x+h} f(t) \, dt - f(x) \tag{7.5b}$$

$$= \frac{1}{h} \int_x^{x+h} f(t) \, dt - \frac{1}{h} \left[\int_x^{x+h} 1 \, dt \right] f(x) \tag{7.5c}$$

$$= \frac{1}{h} \int_x^{x+h} f(t) \, dt - \frac{1}{h} \int_x^{x+h} f(x) \, dt \tag{7.5d}$$

$$= \frac{1}{h} \int_x^{x+h} [f(t) - f(x)] \, dt, \tag{7.5e}$$

which implies that

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| \, dt,$$

Then apply continuity condition to give a bound on $|f(t) - f(x)|$:

Since f is continuous at x, for $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ if $|y - x| < \delta$. Therefore,

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| \, dt \leq \frac{1}{h} \int_x^{x+h} \varepsilon \, dt = \varepsilon,$$

If $h < \delta$, we imply

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

$\blacksquare$

The integraiton by parts is an important part from Calculus, the core idea is from the product rule for differentiation.

> **Theorem 7.3** — **Integration by Parts.** Given two functions $f, g \in C^1[a,b]$, (similar to $(fg)' = f'g + fg'$), we have
>
> $$\int_a^b (fg)' \, dx = \int_a^b f'g \, dx + \int_a^b fg' \, dx,$$
>
> or equivalently,
> $$(fg)(a) - (fg)(b) = \int_a^b f'g \, dx + \int_a^b fg' \, dx,$$
>
> i.e.,
>
> $$\int_a^b fg' \, dx = (fg)|_a^b - \int_a^b f'g \, dx$$

There are two versions of change of variables in Calculus. We will discuss the difference of these.

Proposition 7.1 — **Change of variables,version 1.** Let $\phi : [\alpha, \beta] \mapsto [a, b]$ be a continuously differentiable function such that

$$\phi(\alpha) = a, \quad \phi(\beta) = b.$$

Then for every continuous function $f : [a,b] \mapsto \mathbb{R}$, we have

$$\int_a^b f(x) \, dx = \int_\alpha^\beta f(\phi(t))\phi'(t) \, dt$$

Proof. Define $F(x) = \int_a^x f(t) \, dt$, which implies

$$\frac{dF(x)}{dx} = f(x), \qquad \int_a^b f(x) \, dx = F(b).$$

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Observe that
$$\frac{\mathrm{d}F(\phi(t))}{\mathrm{d}t} = \frac{\mathrm{d}F(\phi(t))}{\phi(t)}\frac{\phi(t)}{\mathrm{d}t} = f(\phi(t))\phi'(t)$$

Or equivalently,
$$\frac{\mathrm{d}}{\mathrm{d}t}(F \circ \phi)(t) = f(\phi(t))\phi'(t)$$

Therefore,

$$\int_\alpha^\beta (F \circ \phi)'(t)\,\mathrm{d}t = \int_\alpha^\beta f(\phi(t))\phi'(t)\,\mathrm{d}t \tag{7.6}$$

$$= (F \circ \phi)(\beta) - (F \circ \phi)(\alpha) = F(\phi(\beta)) - F(\phi(\alpha)) \tag{7.7}$$

$$= F(b) - F(a) = F(b) \tag{7.8}$$

$$= \int_a^b f(x)\,\mathrm{d}x \tag{7.9}$$

■

Proposition 7.2 — Change of variables,version 2. Let $\phi : [\alpha,\beta] \mapsto [a,b]$ be continuously differentiable and **strictly monotone**. Then for any $f \in \mathcal{R}[a,b]$, we have

1. $f(\phi(t))\phi'(t) \in \mathcal{R}[\alpha,\beta]$

2.
$$\int_\alpha^\beta f(\phi(t))\phi'(t) = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,\mathrm{d}x$$

Ⓡ

- Comparing proposition(7.2) to (7.1), note that we **relax** f from being continuously differentiable to being Riemann integrable; but **restrict** ϕ to be **strictly monotone**.

- The proof for proposition(7.2) is messy. For most time functions we have faced is *not continuous*, but we can break into finite sub-intervals and apply proposition(7.1). Thus the benifit for proposition(7.2) is not such huge. In practice, proposition(7.1) is enough.

Last, let's discuss a initutive fact of Riemann sum, i.e., as the mesh goes to zero, Riemann sums always converges to their corresponding integration

Theorem 7.4 Let $f \in \mathcal{R}[a,b]$. Then a Riemann sum $S(\mathcal{P}, f)$ converges to $\int_a^b f(x)\, dx$ as the mesh $\lambda(\mathcal{P}) \to 0$, i.e.,

$$\sum_{i=1}^{n} f(t_i)\Delta x_i \to \int_a^b f(x)\, dx, \qquad \text{as } \max_{1 \le i \le n} \Delta x_i \to 0,$$

where $t_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$.

We apply this theorem to evaluate some limits:

■ **Example 7.2** 1. Evaluate the limit

$$\lim_{n \to \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right].$$

$$
\begin{aligned}
x_n &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \\
&= \frac{1}{n}\left[\frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{n}{2n} \right] \\
&= \frac{1}{n}\left[\frac{1}{1+1/n} + \frac{1}{1+2/n} + \cdots + \frac{1}{1+n/n} \right] \\
&= \Delta x_i \left[f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n}{n}) \right]
\end{aligned}
$$

which is essentially the Riemann sum of function $f(x) = \frac{1}{1+x}$ over interval $[0,1]$.

Therefore, as $n \to \infty$,

$$x_n \to \int_0^1 \frac{1}{1+x}\, dx$$

2. Evaluate the limit

$$\lim_{n \to \infty} \frac{1^\alpha + \cdots + n^\alpha}{n^\alpha}$$

$$
\begin{aligned}
x_n &= \frac{1}{n} \frac{1^\alpha + \cdots + n^\alpha}{n^\alpha} \\
&= \frac{1}{n}\left[(\frac{1}{n})^\alpha + (\frac{2}{n})^\alpha + \cdots + (\frac{n}{n})^\alpha \right] \\
&= \Delta x_i \left[f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n}{n}) \right]
\end{aligned}
$$

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As $n \to \infty$,

$$x_n \to \int_0^1 x^\alpha \, dx = \frac{1}{\alpha+1} x^{\alpha+1} \Big|_0^1 = \frac{1}{\alpha+1}$$

∎

7.2. Friday

Announcement. Our final exam is on 2018.12.16 (Sunday), one day before the final exam. The exact time depends on the room schedule. Today we are going to talk about the improper integral, which is the last topic in the chapter Riemann integration.

7.2.1. Improper Intergrals

Motivation. So far, to define the integration, we pre-assume f and the interval are bounded. In practice, however, we are faced two situations:

1. Interval may be unbounded, i.e., $(-\infty, b], [a, \infty), (-\infty, +\infty)$.

   For example, how to define the integration of f over $[a, \infty)$? A reasonable definition is

   $$\int_a^\infty f(x)\,\mathrm{d}x := \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x$$

2. f is unbounded. Specially speaking, f may be defined on $[a, b)$ with limit to b as infinity.

   In such a situation, similarly, a reasonable definition is

   $$\int_a^b f(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0+} \int_a^{b-\varepsilon} f(x)\,\mathrm{d}x$$

■ **Example 7.3**   1. Given the integral

$$\int_0^1 \frac{1}{x^\alpha}\,\mathrm{d}x = \lim_{\varepsilon\to 0+} \int_\varepsilon^1 \frac{1}{x^\alpha}$$

$$= \begin{cases} \lim_{\varepsilon\to 0+} \left[\frac{1}{(1-\alpha)x^{\alpha-1}}\big|_\varepsilon^1 \right], & \alpha \neq 1 \\ \lim_{\varepsilon\to 0+} \left[\ln x\big|_\varepsilon^1 \right], & \alpha = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{1-\alpha}, & \alpha < 1 \\ \infty, & \alpha \geq 1 \end{cases}$$

We have a function that is unbounded on $[0,1]$ but may have a bounded region (integration) over this interval. The interpretation is that whether the integration of $\frac{1}{x^\alpha}$ is bounded depends on the increasing rate of this function. (not **singular** too much)

2. Another example:

$$\int_1^\infty \frac{1}{x^\alpha} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^\alpha}$$

$$= \begin{cases} \lim_{b \to \infty} \left[\frac{1}{(1-\alpha)x^{\alpha-1}} \Big|_1^b \right], & \alpha \neq 1 \\ \lim_{b \to \infty} \left[\ln x \Big|_1^b \right], & \alpha = 1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{1-\alpha}, & \alpha > 1 \\ \infty, & \alpha \leq 1 \end{cases}$$

The interpretation, in this case, whether the integration of $\frac{1}{x^\alpha}$ is bounded depends on the decreasing rate of this function. (decreasing/converging fast enough)

We summary the examples above as a useful proposition:

Proposition 7.3 — **Convergence of Reciprocal Integration.**

1. The integral $\displaystyle\int_0^1 \frac{1}{x^\alpha} \, dx$ with $\alpha > 0$,

   • Converges if $\alpha < 1$;

   • Diverges if $\alpha \geq 1$

2. The integral $\displaystyle\int_1^\infty \frac{1}{x^\alpha} \, dx$ with $\alpha > 0$,

   • Converges if $\alpha > 1$;

   • Diverges if $\alpha \leq 1$

Notations. We define the general integrals as $\int_a^w f$, where w can be finite, i.e., normal integrals; or $w = \infty$, i.e., improper integrals.

Definition 7.2   The improper integral $\int_a^w f$ is said to be

1. **Absolute convergent**, if $\int_a^w |f|$ converges;

2. **Conditionally convergent**, if $\int_a^w f$ converges while $\int_a^w |f|$ diverges.

∎

Proposition 7.4 — **Comparison Test.** If $0 \le f(x) \le g(x)$ on $[a,w)$, then

1. $\int_a^w g$ converges implies that $\int_a^w f$ converges;

2. $\int_a^w g$ diverges implies that $\int_a^w f$ diverges;

(R) When determining the convergence of improper integral $\int_a^w f \, dx$ using comparison test, we usually compares it with the reciprocal integral $\int_a^w \frac{1}{x^\alpha} \, dx$, with the parameter α appropriately chosen.

Proposition 7.5 — **Integral & Series Test.** Let $f : [1,\infty) \mapsto \mathbb{R}$ be a non-negative, non-increasing, integrable function on every closed interval $[1,b] \subseteq [1,\infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ and the integral $\int_1^\infty f(x) \, dx$ either both converge or both diverge.

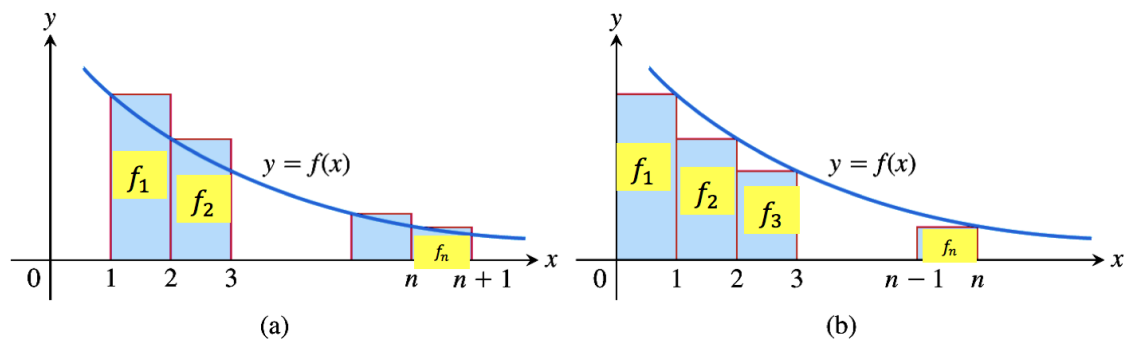
Figure 7.1: Graphic Interpretation of Proposition(7.5)

Graphic Interpretation of Proposition(7.5). Clearly, from Fig.(7.1a), we derive

$$\int_1^\infty f(x) \, dx \le f(1) + f(2) + \cdots + f(n) + \cdots \tag{7.10}$$

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Conversely, from Fig.(7.1b), we derive

$$f(2) + f(3) + \cdots + f(n) + \cdots \leq \int_1^\infty f(x)\, dx \tag{7.11}$$

A rigorous proof. Let's reformulate (7.10) and (7.11) a little bit to make it more rigorous:

$$\sum_{k=1}^n f(k) \geq \sum_{k=1}^n \int_k^{k+1} f(x)\, dx = \int_1^{n+1} f(x)\, dx \tag{7.12a}$$

$$\sum_{k=2}^n f(k) \leq \sum_{k=2}^n \int_{k-1}^k f(x)\, dx = \int_1^n f(x)\, dx \tag{7.12b}$$

- From (7.12a), if $\int_1^\infty f(x)$ diverges, i.e., $\lim_{n\to\infty} \int_1^{n+1} f(x)\, dx = \infty$, then the sequence of partial sums heads off to infinity as well, i.e., the series diverges. The contrapositive is also true, i.e., the series converges implies that the integral converges.

- From (7.12b), if $\int_1^\infty f(x)$ converges, which bounds the sequene of partial sums, we imply that the series converges. The contrapositive is also true, i.e., if the series diverges, so does the integral.

∎

■ **Example 7.4**   Determine the convergence of the following integrals:

1.

$$\int_1^\infty \frac{\sqrt{x}}{\sqrt{1+x^4}} \tag{7.13a}$$

Note that $\frac{\sqrt{x}}{\sqrt{1+x^4}} \sim \frac{c}{x^{3/2}}$, and therefore (7.13a) converges

2.

$$\int_1^\infty \exp(-x^2)\, dx \tag{7.13b}$$

Note that $\exp(-x^2)$ converges much faster than $x^{-\alpha}, \alpha > 1$, and thus (7.13b) converges.

3.

$$\int_2^\infty \ln x \, dx \qquad (7.13c)$$

We can compute $\int_2^\infty \ln x \, dx$ directly:

$$\int_2^\infty \ln x \, dx = [x \ln x - x]_{x=2}^\infty = \infty.$$

4.

$$() \int_0^{\pi/2} \ln(\sin x) \, dx \qquad (7.13d)$$

Note that $\sin x \sim x \implies \ln(\sin x) \sim \ln x$. Also, $\ln x$ increases slower than $x^{-\alpha}, \alpha < 1$:

$$\frac{\ln x}{\sqrt{x}} = \frac{1/x}{\frac{1}{2}\frac{1}{(\sqrt{x})^3}} = 2x^{1/2} \to 0,$$

and therefore (7.13d) converges.

5.

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad 0 \le k^2 < 1 \qquad \text{[Elliptic Integral]} \qquad (7.13e)$$

We reformulate it a little bit:

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^1 \frac{dx}{\sqrt{(1+x)(1-k^2x^2)}} \frac{1}{\sqrt{1-x}} \sim \int_0^1 \frac{d\eta}{\sqrt{\eta}},$$

and therefore (7.13e) converges.

6.

$$\int_0^\phi \frac{d\theta}{\sqrt{\cos\theta - \cos\phi}}, \quad 0 < \phi < \frac{\pi}{2} \qquad (7.13f)$$

Note that

$$\cos\theta - \cos\phi = -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2} \sim \phi - \theta \implies \frac{1}{\sqrt{\cos\theta - \cos\phi}} \sim \frac{1}{\sqrt{x}},$$

and therefore (7.13f) converges.

7.

$$\int_{\pi/2}^\infty \frac{\sin x}{x} \, dx \qquad (7.13g)$$

We solve this problem via integration by parts:

$$\int_{\pi/2}^{b} \frac{\sin x}{x} \, \mathrm{d}x = -\frac{1}{b} \cos b - \int_{\pi/2}^{b} \frac{\cos x}{x^2} \, \mathrm{d}x$$

Note that $\int_{\pi/2}^{b} |\cos \frac{\cos x}{x^2}| \, \mathrm{d}x \leq \int_{\pi/2}^{b} \frac{1}{x^2}$, and thus this integral converges. However, this integral diverges, since the integration

$$\int_{\pi/2}^{\infty} |\frac{\sin x}{x}| \, \mathrm{d}x$$

diverges. In summary, $\int_{\pi/2}^{b} \frac{\sin x}{x} \, \mathrm{d}x$ is conditionally convergent.

proof for divergence of $\int_{\pi/2}^{\infty} |\frac{\sin x}{x}| \, \mathrm{d}x$.

$$\int_{\pi/2}^{\infty} |\frac{\sin x}{x}| \, \mathrm{d}x \geq \int_{\pi/2}^{\infty} \frac{\sin^2 x}{x} \, \mathrm{d}x = \int_{\pi/2}^{\infty} \frac{1}{2x} \, \mathrm{d}x - \frac{1}{2} \int_{\pi/2}^{\infty} \frac{\cos 2x}{2x} \, \mathrm{d}x, \qquad (7.13\text{h})$$

where the same trick shows that $\int_{\pi/2}^{\infty} \frac{\cos 2x}{2x} \, \mathrm{d}x$ converges, and hence the subtraction (7.13h) diverges. ∎

Alternative proof for divergence of $\int_{\pi/2}^{\infty} |\frac{\sin x}{x}| \, \mathrm{d}x$. Consider the graph of $|\sin x|$:
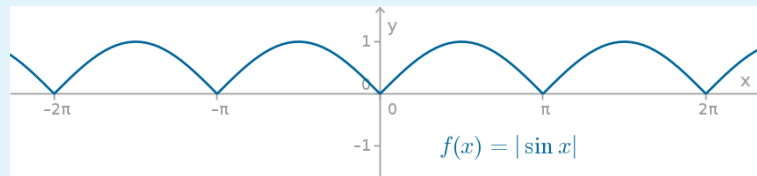
Figure 7.2: Graphic of $|\sin x|$

Thus the integrations of $|\sin x|$ over the interval $[2\pi n, (n+1)2\pi]$ are always the same:

$$\int_{2\pi n}^{2\pi(n+1)} |\sin x| \, \mathrm{d}x = \int_{0}^{2\pi} |\sin x| \, \mathrm{d}x = 4, \qquad n \in \mathbb{N}$$

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which follows that

$$\int_0^{2\pi N} \left| \frac{\sin x}{x} \right| dx = \sum_{n=0}^{N-1} \int_{2\pi n}^{2\pi(n+1)} \left| \frac{\sin x}{x} \right| dx \qquad (7.13i)$$

$$\geq \sum_{n=0}^{N-1} \frac{1}{2\pi(n+1)} \int_{2\pi n}^{2\pi(n+1)} |\sin x| \, dx \qquad (7.13j)$$

$$= \sum_{n=0}^{N-1} \frac{1}{2\pi(n+1)} \int_0^{2\pi} |\sin x| \, dx \qquad (7.13k)$$

$$= \sum_{n=0}^{N-1} \frac{2}{\pi(n+1)} \qquad (7.13l)$$

where in (7.13j) we lower bound the $\frac{1}{|x|}$ as $\frac{1}{n+1}$ for each n. The last sum (7.13l) diverges as $N \to \infty$, and so does the original integral $\int_0^\infty \left| \frac{\sin x}{x} \right| dx$. ∎

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Chapter 8

Week8

8.1. Friday

This lecture will discuss the multi-variable calculus.

8.1.1. Introduction to metric space

The multi-variable calculus aims to study the function $f : \mathbb{R}^m \mapsto \mathbb{R}^n$:

$$f(\underbrace{x_1, x_2, \ldots, x_m}_{\boldsymbol{x}}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$$

To begin with, let's assume $n = 1$, i.e., we study only one component in RHS first. The preliminaries for this concept are limit points or something else. Let's define them in high dimension first.

Generalization from $\mathbb{R}$ to $\mathbb{R}^2$. The distance between two points in $\mathbb{R}^2$ is usually defined as follows:

$$\begin{aligned} \boldsymbol{x} &= (x_1, x_2) \\ \boldsymbol{y} &= (y_1, y_2) \end{aligned} \implies d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \quad (L_2 \text{ norm})$$

- Sometimes another distance measrue is $d_1(\boldsymbol{x}, \boldsymbol{y}) = |x_1 - y_1| + |x_2 - y_2|$, which is called L_1 norm

- Or more generally, $d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max_{1 \leq i \leq n} |x_i - y_i|$, which is called L_∞ norm.

Those distance measures are essentially the same in order, i.e.,

$$\frac{1}{\sqrt{2}}d(\boldsymbol{x},\boldsymbol{y}) \leq d_1(\boldsymbol{x},\boldsymbol{y}) \leq \sqrt{2}d(\boldsymbol{x},\boldsymbol{y}), \qquad \forall x,y \in \mathbb{R}^2$$

$$d(\boldsymbol{x},\boldsymbol{y}) \leq d_\infty(\boldsymbol{x},\boldsymbol{y}) \leq \sqrt{2}d(\boldsymbol{x},\boldsymbol{y}), \qquad \forall x,y \in \mathbb{R}^2$$

(R)

- For those distance measures with the same order, the corresponding properties defined with those measures are also nearly the same. We always define the L_2 norm as our distance measure by default.

- However, one distance measure is different in order from those above:

$$\bar{d}(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{otherwise} \end{cases}$$

Definition 8.1 [Matric Space] The binary operation $d : \mathcal{H} \times \mathcal{H} \mapsto \mathbb{R}$ is called a matric if the following are satisfied:

1. $d(\boldsymbol{x},\boldsymbol{y}) \geq 0, \forall x,y \in \mathcal{H}$;
2. $d(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{y},\boldsymbol{x}), \forall x,y \in \mathcal{H}$;
3. $d(\boldsymbol{x},\boldsymbol{z}) \leq d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z}), \forall x,y,z \in \mathcal{H}$,

where $\mathcal{H}$ is called the **metric space**, e.g., $\mathbb{R}^m$ is a common metric space. ∎

The reason for defining matric is to describe **convergence** in high dimensions. Let's define the corresponding definitions related to convergence again:

Definition 8.2 [Open ball] The open ball is a set $B_r(\boldsymbol{a})$ such that

$$B_r(\boldsymbol{a}) := \{\boldsymbol{x} \in \mathcal{H} \mid d(\boldsymbol{x},\boldsymbol{a}) < r\} \tag{8.1}$$

∎

Some illustrations for $B_1(\boldsymbol{0})$ in the metric space $\mathbb{R}^2$ is shown as follows:
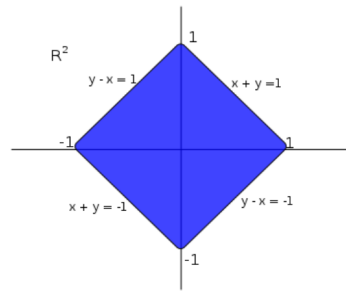
(a) The L_1 metric with unit ball $B_1(\mathbf{0})$
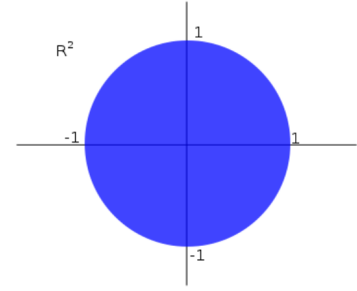
(b) The L_2 metric with unit ball $B_1(\mathbf{0})$
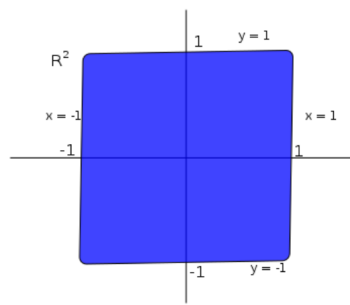
(c) The L_∞ metric with unit ball $B_1(\mathbf{0})$

Figure 8.1: illustrations for $B_1(\mathbf{0})$ in the metric space $\mathbb{R}^2$

Definition 8.3 [Convergence] The sequence defined on the metric space $(\mathcal{H}, d)$ is convergent to x_0 if $d(x_n, x_0) \to 0$ as $n \to \infty$, which is denoted as $\lim_{n \to \infty} x_n = x_0$, or simply $x_n \to x_0$ ∎

Check rudin's book in page 32 for the concepts about Openness, closedness, neighborhood, boundness, **compactness**, limit points, **connectness**. In particular, let's discuss something important.

Definition 8.4 [Compact] A set K is said to be compact ifs every open cover has a finite subcover. ∎

The proof of the proposition that K is compact if it is bounded and closed is left as exercise.

Definition 8.5 A set S is **pathwise** connected if for any two points $\boldsymbol{x}, \boldsymbol{y} \in S$, there exists a ''**path** '' Γ connecting $\boldsymbol{x}$ and $\boldsymbol{y}$, where Γ is a continuous function $[0,1] \mapsto S$ with the

property that $\Gamma(0) = \boldsymbol{x}, \Gamma(1) = \boldsymbol{y}$. ∎

Definition 8.6 [Domain] A domian is an open set which is **path-wise connected**. ∎

(R) An open set is not necessarily path-wise connected.

Definition 8.7 [Oscillation]

1. The **oscillation** of $f : X \subseteq \mathbb{R}^m \mapsto \mathbb{R}$ on a set $E \subseteq X$ is $\omega(f;E) = d(f(E))$, where d denote the diameter of the set $f(E)$, i.e.,

$$\omega(f;E) = d(f(E)) = \sup_{x,y \in E} d(f(x), f(y)).$$

2. The **oscillation** of f at a point $\boldsymbol{a}$ is

$$w(f;\boldsymbol{a}) = \lim_{r \to 0+} \omega(f;B_r(\boldsymbol{a}))$$

∎

Local Properties.

Proposition 8.1 — **Useful Properties.** Let f be a function mapping a metric space $\mathcal{H}$ to $\mathbb{R}^m$, then

1. f is continuous at the point $\boldsymbol{a}$ iff $\omega(f;\boldsymbol{a}) = 0$
2. If f is continuous at $\boldsymbol{a}$ with $f(\boldsymbol{a}) > 0$, then $f > 0$ in some neighborhood of $\boldsymbol{a}$
3. The linear combinations of continuous functions $(\alpha f + \beta g)$, component-wise products $(f \cdot g)$, or component-wise quotients $(\frac{f}{g}, g_i \neq 0)$ are also continuous functions

Proposition 8.2 — **Global Properties.** 1. Let $f : K \mapsto \mathbb{R}^n$ be a continuous function with K being compact, then we have

    (a) f is **uniformly continuous** on K

    (b) f is **bounded** on K

(c) f assumes its maximum and minimum on K

2. Intermediate Value property: If $f : E \mapsto \mathbb{R}$ is continuous, where E is **path-wise connected**, then $f(a) = A$ and $f(b) = B$ implies for all c between A and B, there exists $c \in E$ such that $f(c) = C$.

■ **Example 8.1**   1.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x,y) \neq (0,0) \\ \\ 0, & (x,y) = (0,0) \end{cases}$$

For any (x,y) on the line $y = mx$, we have

$$f(x,mx) = \frac{mx^2}{x^2 + m^2 x^2} = \frac{m}{1 + m^2},$$

thus $\lim_{(x,y) \to (0,0)} f(x,y)$ does not exist sicne different paths toward the origin may lead to a different limit. There is another interesting fact:

$$\begin{cases} \lim\limits_{x \to 0} \lim\limits_{y \to 0} f(x,y) = 0, \\ \\ \lim\limits_{y \to 0} \lim\limits_{x \to 0} f(x,y) = 0 \end{cases}$$

2.

$$f(x,y) = \begin{cases} x + y \sin \dfrac{1}{x}, & x \neq 0 \\ \\ 0, & x = 0 \end{cases}$$

Apply ε-δ language to verify that $\lim_{(x,y) \to (0,0)} f(x,y) = 0$, but

$$\begin{cases} \lim\limits_{x \to 0} \lim\limits_{y \to 0} f(x,y) = 0 \\ \\ \lim\limits_{y \to 0} \lim\limits_{x \to 0} f(x,y) \text{ does not exist} \end{cases}$$

Ⓡ   The continuity of a function at $x = a$ does not necessaily imply the interchangeability of limit processes. However, the uniform convergence can enable us to arrive at positive results.

3.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

For any (x,y) on the line $y = mx$, we have

$$f(x,mx) = \frac{mx^2 \cdot x}{x^4 + m^2 x^2} = \frac{mx}{x^2 + m^2} \to 0, \text{ as } x \to 0$$

However, for any (x,y) on the line $y = x^2$, we have

$$f(x,x^2) = \frac{x^2 \cdot x^2}{x^4 + x^4} = \frac{1}{2},$$

which means f is not continuous at $(0,0)$.

4.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^6 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

When $y = mx$, we have $f(x,mx) = \frac{mx}{x^4 + m^2} \to 0$.

When $y = x^3$, we have $f(x,x^3) = \frac{1}{2x} \to \infty$, i.e., f is unbounded near origin.

∎

Question. Is it possible that a sphere S^2 and a circle S^1 are situated such that the distance from any point on the sphere to any point on the circle is the same, i.e., is it possible for a function

$$d(x,y) : S^2 \times S^1 \mapsto \mathbb{R}$$

remains constant? When will this fact be possible in $\mathbb{R}^k$?

Chapter 9

Week9

9.1. Friday

We are now in the multi-variate differentiation part.

Comments on question in last lecture. The question left in the last lecture is that

> In dimension $\mathbb{R}^k$ with k to be determined, is it possible to find the smallest k such that a sphere S^2 and a circle S^1 have a way of putting to make each point from S^2 to S^1 have the same distance?

The answer is $k = 5$, Let's give an example. We define the sphere and the circle to be

$$S^2 = \{(x,y,z,0,0) \mid x^2 + y^2 + z^2 = 1\},$$
$$S^1 = \{(0,0,0,u,v) \mid u^2 + v^2 = 1\}$$

Therefore, the distance between any two points on the sphere and the circile, respectively, is

$$d = \sqrt{x^2 + y^2 + z^2 + u^+v^2} \equiv \sqrt{2}$$

Why $k \leq 4$ is not ok? Now we give a instructive proof:

Proof. For the case $k \leq 4$, let c_2 denote the center of sphere S^2 with radius r_2; c_3 deote the center of circle S^1 with radius r_1. Any point on S^2 can be written as $c_2 + x$ with $x \in M := \{x \in \mathbb{R}^3 : \|x\| = r_2\}$; and any point on S^1 can be written as $c_1 + y$ with $x \in N := \{y \in \mathbb{R}^2 : \|y\| = r_1\}$. It follows that the distance between any two points can

be expressed as:

$$d(\boldsymbol{c}_2 + \boldsymbol{x}, \boldsymbol{c}_1 + \boldsymbol{y}) = \|(\boldsymbol{c}_2 + \boldsymbol{x}) - (\boldsymbol{c}_1 + \boldsymbol{y})\| = \|\boldsymbol{x}\|^2 + \|\boldsymbol{c}_2 - \boldsymbol{y} - \boldsymbol{c}_1\|^2 + 2\langle \boldsymbol{x}, \boldsymbol{c}_2 - \boldsymbol{y} - \boldsymbol{c}_1 \rangle$$

Note that the distance between any point in S^2 and any point in S^1 is the same. In particular, $d(\boldsymbol{c}_2 + \boldsymbol{x}, \boldsymbol{c}_1 + \boldsymbol{y}) = d(\boldsymbol{c}_2 - \boldsymbol{x}, \boldsymbol{c}_1 + \boldsymbol{y})$, which implies $\langle \boldsymbol{x}, \boldsymbol{c}_2 - \boldsymbol{y} - \boldsymbol{c}_1 \rangle = 0, \forall \boldsymbol{x} \in M, \boldsymbol{y} \in N$, i.e.,

$$\boldsymbol{c}_2 - \boldsymbol{c}_1 + N \subseteq M^{\perp},$$

and therefore

$$\dim(\boldsymbol{c}_2 - \boldsymbol{c}_1 + N) = \dim(N) = 2$$
$$\leq \dim(M^{\perp}) = k - 3,$$

i.e., $k \leq 5$ is the sufficient condition of the problem.

■

9.1.1. Preliminaries

Notations. Here we use lower-case and bolded alphabet to denote a vector, e.g., $\boldsymbol{x}$; upper-case and bolded alphabet to denote a matrix, e.g., $\boldsymbol{A}$; unbolded alphabet to denote a scalar, e.g., x_1 or A_1

Define $\boldsymbol{x} := (x_1, \ldots, x_m)$ and $\boldsymbol{y} := (y_1, \ldots, y_m)$, we define the L_2 norm

$$d(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\|$$
$$= \left[(x_1 - y_1)^2 + \cdots + (x_m - y_m)^2\right]^{1/2}$$

and the L_1, L_S (sup) norm:

$$d_1(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\| := |x_1 - y_1| + \cdots + |x_m - y_m|$$
$$d_S(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\|_S := \max_{1 \leq i \leq m} (x_i - y_i)$$

> **Definition 9.1** [Vector Norm] A norm $\|\cdot\|$ is a function from a vector space X to $\mathbb{R}$ such that
>
> 1. $\|x\| \geq 0$, $\forall x \in X$; and $\|x\| = 0$ iff $x = 0$;
> 2. $\|\lambda x\| = |\lambda| \|x\|$, for $\forall x \in X, \lambda \in \mathbb{R}$;
> 3. $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$
>
> ■

(R) Any norm defines a metric: $d(x,y) = \|x - y\|$. From now on, we pre-assume the norm to be L_2 norm and the metric to be L_2 metric, unless specified.

The norm are used to masure the length of a vector, or the distance of two vectors. Correspondingly, the inner product can be used to measure the angle between two angles:

> **Definition 9.2** [Inner Product] An **inner product** is a binary operation function $\langle \cdot, \cdot \rangle :$ $X \times X \mapsto \mathbb{R}$ such that
>
> 1. $\langle x, x \rangle \geq 0$ for $\forall x \in X$ and $\langle x, x \rangle = 0$ iff $x = 0$
> 2. $\langle x, y \rangle = \langle y, x \rangle$, for $\forall x, y \in X$
> 3. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ for $\forall x, y \in X$ and $\forall \lambda \in \mathbb{R}$
> 4. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
>
> ■

(R)

1. Any inner product always defines a norm, i.e., $\|x\| = \sqrt{\langle x, x \rangle}$.
2. The angle θ between vector x, y is defined as:

$$\theta = \cos^{-1}\left(\frac{\langle x, y \rangle}{\|x\| \|y\|}\right)$$

In particular, x and y are orthogonal, i.e., $\langle x, y \rangle = 0$ if $\theta = \pm\frac{\pi}{2}$; x, y are parallel if $\theta = 0$, or $\theta = \pm\pi$.

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9.1.2. Differentiation

Review for One-dimension. Given a function $f : \mathbb{R} \mapsto \mathbb{R}$, the derivative is defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

or equivalently, the value $f'(x_0)$ is said to be the derivative of f if it satisfies the equality:

$$
\begin{aligned}
0 &= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right] \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \left| \frac{f(x) - [f(x_0) + f'(x_0)(x - x_0)]}{x - x_0} \right|
\end{aligned}
$$

The interpretation is that the affine (linear function) $f(x_0) + f'(x_0)(x - x_0)$ approximates f near x_0 in at least first order. We can use the similar way to define the high-dimension derivative.

High-Dimension Derivative.

> **Definition 9.3** [Differentiable] A map $f : U \mapsto \mathbb{R}^n$, where U is open in $\mathbb{R}^m$, is **differentiable** at $\boldsymbol{x}_0 \in U$ if
>
> $$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} \frac{\| f(\boldsymbol{x}) - f(\boldsymbol{x}_0) - \boldsymbol{L}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0) \|}{\| \boldsymbol{x} - \boldsymbol{x}_0 \|} = 0, \tag{9.1}$$
>
> where $\boldsymbol{L}(\boldsymbol{x}_0)$ is said to be the derivative of $f(\boldsymbol{x})$ at $\boldsymbol{x} = \boldsymbol{x}_0$, which is often denoted as $Df(\boldsymbol{x}_0)$, or $f'(\boldsymbol{x}_0)$. ∎

(R) Note that $f(\boldsymbol{x}), f(\boldsymbol{x}_0) \in \mathbb{R}^n$, but $(\boldsymbol{x} - \boldsymbol{x}_0) \in \mathbb{R}^m$, thus $L(\boldsymbol{x}_0)$ is a **linear transformation** from $\mathbb{R}^m$ to $\mathbb{R}^n$, i.e., a $n \times m$ matrix.

Interpretation. We re-write $f(\boldsymbol{x}) : \mathbb{R}^m \mapsto \mathbb{R}^n$ (a n-vector valued function of a m-vector argument) as:

$$f(\boldsymbol{x}) = \Big(f_1(\boldsymbol{x}) \quad \cdots \quad f_n(\boldsymbol{x}) \Big),$$

with each $f_i : \mathbb{R}^m \mapsto \mathbb{R}$ being a scalar-valued function of a m-vector argument. Let's study only one component with 2-argument function first, i.e., $n = 1, m = 2$.
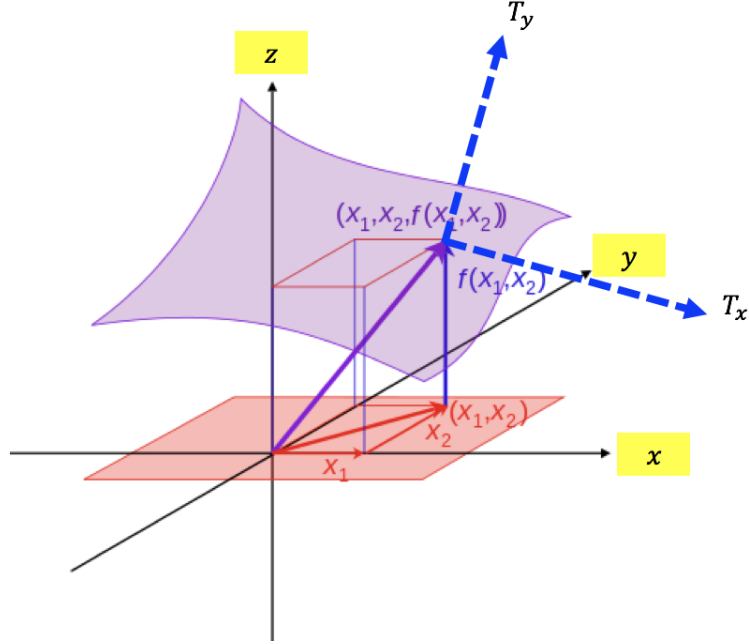
Figure 9.1: Diagram for partial derivatives

Given a surface $S : z = f(x_1, x_2)$, we study a point on the surface S, say $M_0(x_1, x_2, f(x_1, x_2))$. If the surface S intersects the plane $y = x_2$ with the point M_0, then the **partial derivative** $\frac{\partial f}{\partial x_1}(x_1, x_2)$ denotes the slope of the tangent line at M_0; the tangent line T_{x_1} can be denoted as a vector $T_{x_1} := (1, 0, \frac{\partial f}{\partial x_1}(x_1, x_2))$.

Similarly, if the surface S intersects the plane $x = x_1$ with the point M_0, then $\frac{\partial f}{\partial x_2}(x_1, x_2)$ denotes the slope of the tangent line at M_0; the tangent line T_{x_2} can be denoted as a vector $T_{x_2} := (0, 1, \frac{\partial f}{\partial x_2}(x_1, x_2))$.

Furthermore, the plane $\text{span}\{T_{x_1}, T_{x_2}\}$ denotes the tangent plane at point M_0.

Corollary 9.1 f is differentiable at x_0 implies all partial derivatives of f at x_0 exist.

The converse is not true. Let's raise a counter-example to explain that.

What guarntees f to be differntiable if all partial derivatives exist?

$$\frac{\|f(\boldsymbol{x}) - f(\boldsymbol{x}_0) - Df(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)\|}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} \to 0,$$

where $f : \mathbb{R}^m \mapsto \mathbb{R}^n$ and $Df(\boldsymbol{x}_0)$ be a $n \times m$ matrix. We re-write the formula for the derivative of f at $\boldsymbol{x} = \boldsymbol{x}_0$ first. Here we write $f(\boldsymbol{x}_0)$ in column vector form:

$$\begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} f_1(\boldsymbol{x}_0) \\ f_2(\boldsymbol{x}_0) \\ \vdots \\ f_n(\boldsymbol{x}_0) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \frac{\partial f_1}{\partial x_2}(x_0) & \cdots & \frac{\partial f_1}{\partial x_m}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) & \frac{\partial f_2}{\partial x_2}(x_0) & \cdots & \frac{\partial f_2}{\partial x_m}(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x_0) & \frac{\partial f_n}{\partial x_2}(x_0) & \cdots & \frac{\partial f_n}{\partial x_m}(x_0) \end{pmatrix} \begin{pmatrix} x_1 - x_{0,1} \\ x_2 - x_0 \\ \vdots \\ x_m - x_0 \end{pmatrix} + o(\|\boldsymbol{x} - \boldsymbol{x}_0\|),$$

$$\tag{9.2}$$

where $o(\|\boldsymbol{x} - \boldsymbol{x}_0\|)$ is a $m \times 1$ vector that has order less than $\|\boldsymbol{x} - \boldsymbol{x}_0\|$.

Each entry in LHS of (9.2) can be expressed as:

$$f_i(\boldsymbol{x}) = f_i(\boldsymbol{x}_0) + \nabla^{\mathrm{T}} f_i(\boldsymbol{x}_0) \cdot (\boldsymbol{x} - \boldsymbol{x}_0) + o(\|\boldsymbol{x} - \boldsymbol{x}_0\|),$$

with $\nabla^{\mathrm{T}} f_1(\boldsymbol{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{x}_0) & \cdots & \frac{\partial f_1}{\partial x_m}(\boldsymbol{x}_0) \end{pmatrix}$ to be a row vector, and $(\boldsymbol{x} - \boldsymbol{x}_0)$ to be a column vector.

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Sufficient Condition for differentiability. Recall that we have faced a function that is everywhere differentiable but nowhere monotone, which is very counter-inuitive. If adding the condition that such function is continuously differentiable, then such function is monotone. Similarly, the gap for corollary(9.1) is continuous differentiability.

> **Theorem 9.1** Let $f : U \mapsto \mathbb{R}^n$, where $U \subseteq \mathbb{R}^m$ is open. If all partial derivatives of f are **continuous** in U, then f is differentiable in U.

Proof for $n = 2, m = 1$ case. Consider $f : \mathbb{R}^2 \mapsto \mathbb{R}$, for $x_0 \in U$ with small h, k, we have:

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = [f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)] + [f(x_0 + h, y_0) - f(x_0, y_0)]$$
(9.3a)

$$\text{(by Mean Value Theorem, } \exists (c, d) \in [0, h] \times [0, k] \text{ such that)}$$
(9.3b)

$$= k \frac{\partial f}{\partial y}(x_0 + h, y_0 + c) + h \frac{\partial f}{\partial x}(x_0 + d, y_0) \tag{9.3c}$$

$$= k \frac{\partial f}{\partial y}(x_0, y_0 + c) + o(h) + h \frac{\partial f}{\partial x}(x_0, y_0) + o(h) \tag{9.3d}$$

$$= k \frac{\partial f}{\partial y}(x_0, y_0) + h \frac{\partial f}{\partial x}(x_0, y_0) + o(h) + o(k) \tag{9.3e}$$

Or we write it in compact matrix form:

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} + o(\|(h, k)\|),$$

i.e., the derivative of f at $x = x_0$ exists, say $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. Thus f is differentiable. ∎

The same proof can easily extend to $f : \mathbb{R}^n \mapsto \mathbb{R}$ by adding and subtracting $n - 1$ terms and then apply mean value theorem for n times; the extension to $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is clear by the same proof for each component.

Proof. Consider in general $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, and it suffices to show (9.2), and moreover, it

suffices to show each entry $f_i(\boldsymbol{x})$ in (9.2) can be expressed as

$$f_i(\boldsymbol{x}) = f_i(\boldsymbol{x}_0) + \nabla^{\mathrm{T}} f_i(\boldsymbol{x}_0) \cdot (\boldsymbol{x} - \boldsymbol{x}_0) + o(\|\boldsymbol{x} - \boldsymbol{x}_0\|).$$

Let's abuse notation to refer g as the i-th entry of f_i, i.e., $g : \mathbb{R}^n \mapsto \mathbb{R}$.

Let $\varepsilon > 0$, by continuity of partial derivatives, we can pick $\delta > 0$ such that for any $\|\boldsymbol{h}\| < \delta$, we have

$$\left\| \frac{\partial g}{\partial x_k}(\boldsymbol{x} + h) - \frac{\partial g}{\partial x_k}(\boldsymbol{x}) \right\| < \varepsilon, \qquad k = 1, 2, \ldots, n \tag{9.4}$$

Let $\boldsymbol{L} = (\frac{\partial g}{\partial x_1}(\boldsymbol{x}), \ldots, \frac{\partial g}{\partial x_n}(\boldsymbol{x}))$; $\phi_k(\boldsymbol{h}) = (h_1, h_2, \ldots, h_k, 0, \ldots, 0)$, $\phi_0(\boldsymbol{h}) = \boldsymbol{0}$, and suppose $\|\boldsymbol{h}\| < \delta$, , then we have:

$$|g(\boldsymbol{x} + \boldsymbol{h}) - g(\boldsymbol{x}) - \boldsymbol{L}\boldsymbol{h}| = \left| \sum_{k=1}^{n} [g(\boldsymbol{x} + \phi_k(\boldsymbol{h})) - g(\boldsymbol{x} + \phi_{k-1}(\boldsymbol{h})) - \frac{\partial g}{\partial x_k}(\boldsymbol{x})h_k] \right| \tag{9.5}$$

$$\leq \sum_{k=1}^{n} \left| g(\boldsymbol{x} + \phi_k(\boldsymbol{h})) - g(\boldsymbol{x} + \phi_{k-1}(\boldsymbol{h})) - \frac{\partial g}{\partial x_k}(\boldsymbol{x})h_k \right| \tag{9.6}$$

By the Mean Value Theorem, there exists $\boldsymbol{c}_k \in [\boldsymbol{x} + \phi_{k-1}(\boldsymbol{h}), \boldsymbol{x} + \phi_k(\boldsymbol{h})]$ such that $g(\boldsymbol{x} + \phi_k(\boldsymbol{h})) - g(\boldsymbol{x} + \phi_{k-1}(\boldsymbol{h})) = \frac{\partial g}{\partial x_k}(\boldsymbol{c}_k)h_k$, and therefore

$$|g(\boldsymbol{x} + \boldsymbol{h}) - g(\boldsymbol{x}) - \boldsymbol{L}\boldsymbol{h}| \leq \sum_{k=1}^{n} \left| \frac{\partial g}{\partial x_k}(\boldsymbol{c}_k)h_k - \frac{\partial g}{\partial x_k}(\boldsymbol{x})h_k \right| \tag{9.7}$$

$$= \sum_{k=1}^{n} \left| \frac{\partial g}{\partial x_k}(\boldsymbol{c}_k) - \frac{\partial g}{\partial x_k}(\boldsymbol{x}) \right| |h_k| \tag{9.8}$$

$$< \varepsilon \sum_{k=1}^{n} |h_k| \tag{9.9}$$

where (9.9) is due to the continuty of partial derivatives. To make life easier, we specify the norm to be L_1 norm, which follows that

$$|g(\boldsymbol{x} + \boldsymbol{h}) - g(\boldsymbol{x}) - \boldsymbol{L}\boldsymbol{h}| \leq \varepsilon \|\boldsymbol{h}\|_1,$$

which implies g is differentiable at $\boldsymbol{x}$. In paricular, if every component f_i is differentiable, then f is differentiable.

■

Planning for following weeks. The next lecture will talk about the chain rule, the inverse, the derivative of inverse, the directional derivative, the gradient. Next Friday will talk about Inverse function theorem and Implicit function theorem, which may last for two weeks.

Chapter 10

Week10

10.1. Wednesday

10.1.1. Preliminaries on Notations

The notations in this course are slightly different from that in textbook or other sources.

Let's make a justification, for example:

1. Derivative:

$$\frac{\partial F}{\partial x_i}, \quad F_{x_i}, \quad F_i, \quad \partial_{x_i} F, \quad \partial_i F, \quad D_{x_i} F, \quad D_i F$$

2. Gradient & Jacobian Matrix:

> **Definition 10.1** [Gradient] For $f : \mathbb{R}^n \to \mathbb{R}$, its **gradient** $\nabla f(\boldsymbol{x})$ is a $n \times 1$ **column vector**:
>
> $$[\nabla f(\boldsymbol{x})]_i = \frac{\partial f}{\partial x_i}(\boldsymbol{x})$$
>
> its **Jacobian** matrix $Df(\boldsymbol{x})$ is a $n \times 1$ **row vector**:
>
> $$Df(\boldsymbol{x}) = \nabla^{\mathrm{T}} f(\boldsymbol{x})$$
>
> ∎

> **Definition 10.2** [Jacobian matrix] For $f : \mathbb{R}^n \to \mathbb{R}^m$, suppose $f = (f_1, f_2, \ldots, f_m)$,

then the j-th row of **Jacobian matrix** of f is just the transpose of $\nabla f_j(\boldsymbol{x})$:

$$
Df(\boldsymbol{x}) = \begin{pmatrix} Df_1(\boldsymbol{x}) \\ Df_2(\boldsymbol{x}) \\ \vdots \\ Df_m(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \nabla^{\mathrm{T}} f_1(\boldsymbol{x}) \\ \nabla^{\mathrm{T}} f_2(\boldsymbol{x}) \\ \vdots \\ \nabla^{\mathrm{T}} f_m(\boldsymbol{x}) \end{pmatrix}
$$

This lecture will discuss some preliminaries and properties on multi-variate differentiation. Next lecture we will turn into implicit function theorem.

10.1.2. Analysis on multi-variate differentiation

First let's extend the chain rule to the multi-variate situation:

Definition 10.3 [Composition] Suppose $X \subseteq \mathbb{R}^n$ and $Y \in \mathbb{R}^m$. The function f maps X into Y, and the function g maps Y into $\mathbb{R}^k$. If f, g are differentiable at $\boldsymbol{x} = \boldsymbol{x}_0$, i.e., the Jacobian matrix $Df(x_0) \in \mathbb{R}^{m \times n}, Dg(x_0) \in \mathbb{R}^{k \times m}$ is well-defined, then the mapping $g \circ f$ is also differentiable at $\boldsymbol{x} = \boldsymbol{x}_0$, with the derivative

$$
D(g \circ f)(\boldsymbol{x}_0) = Dg(f(\boldsymbol{x}_0)) * Df(\boldsymbol{x}_0),
$$

or rewriting it into matrix form: $(\boldsymbol{y}_0 := f(\boldsymbol{x}_0))$

$$
D(g \circ f)(\boldsymbol{x}_0) := \begin{pmatrix} \frac{\partial}{\partial x_1}(g_1 \circ f) & \cdots & \frac{\partial}{\partial x_n}(g_1 \circ f) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}(g_k \circ f) & \cdots & \frac{\partial}{\partial x_n}(g_k \circ f) \end{pmatrix}_{k \times n}
$$

$$
= \begin{pmatrix} \frac{\partial g_1}{\partial y_1}(\boldsymbol{y}_0) & \cdots & \frac{\partial g_1}{\partial y_m}(\boldsymbol{y}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial y_1}(\boldsymbol{y}_0) & \cdots & \frac{\partial g_k}{\partial y_m}(\boldsymbol{y}_0) \end{pmatrix}_{k \times m} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\boldsymbol{x}_0) & \cdots & \frac{\partial f_n}{\partial x_n}(\boldsymbol{x}_0) \end{pmatrix}_{m \times n}
$$

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The proof is similar to the one-dimension case.

> **Definition 10.4** [Directional Derivative] Given a function $f : X(\subseteq \mathbb{R}^n) \mapsto \mathbb{R}^m$, for fixed
> $\boldsymbol{x}_0$ in E, suppose $\boldsymbol{v} \in \mathbb{R}^n$ is a unit vector, then the **directional derivative** of f at $\boldsymbol{x} = \boldsymbol{x}_0$
> in the direction of $\boldsymbol{v}$ is given by:
>
> $$D_{\boldsymbol{v}} f(\boldsymbol{x}_0) = \lim_{t \to 0} \frac{f(\boldsymbol{x}_0 + t\boldsymbol{v}) - f(\boldsymbol{x}_0)}{t}, \tag{10.1}$$
>
> where $t \in \mathbb{R}$. ∎

Proposition 10.1 The formula (10.1) can be re-written as:

$$D_{\boldsymbol{v}} f(\boldsymbol{x}_0) = Df(\boldsymbol{x}_0) * \boldsymbol{v}, \tag{10.2}$$

with the operator $*$ being the matrix multiplication.

Proof. For fixed $\boldsymbol{x}_0$, let $g(t) = f(\boldsymbol{x}_0 + t\boldsymbol{v})$ to be the one-argument scalar function, it follows that

$$\lim_{t \to 0} \frac{f(\boldsymbol{x}_0 + t\boldsymbol{v}) - f(\boldsymbol{x}_0)}{t} = \lim_{t \to 0} \frac{g(t) - g(0)}{t}$$
$$= g'(0)$$
$$= [Df(\boldsymbol{x}_0 + t\boldsymbol{v}) * \boldsymbol{v}]_{t=0}$$
$$= Df(\boldsymbol{x}_0) * \boldsymbol{v}$$

∎

> **Corollary 10.1** Given the n-argument scalar function $f : \mathbb{R}^n \to \mathbb{R}$, the **directional derivative** of f at $\boldsymbol{x} = \boldsymbol{x}_0$ in the direction of $\boldsymbol{v}$ can be expressed as:
>
> $$D_{\boldsymbol{v}} f(\boldsymbol{x}_0) = \langle \nabla f(\boldsymbol{x}_0), \boldsymbol{v} \rangle.$$

Thus we have defined the derivative for multi-variate function at any direction. Now we study in which direction does the function decrease/increase at the fastest rate.

■ **Example 10.1** The gradient actually tells us how the function behaves locally. Given a simple function $f(x,y) = x^2 + y^2$, its **level curve** is given as follows:
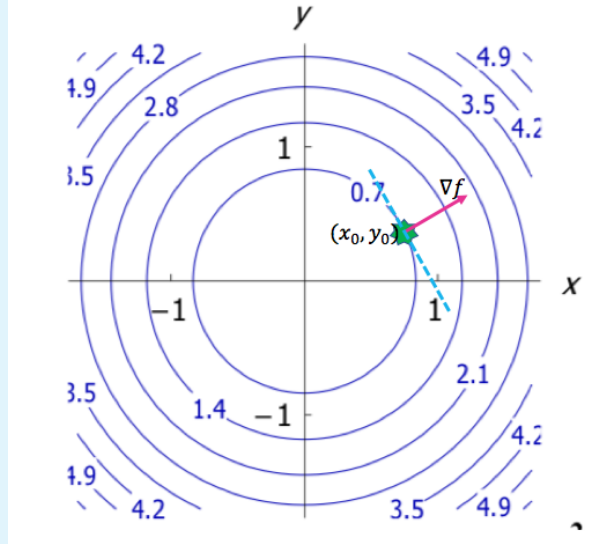
Figure 10.1: Contour plot for $f(x,y) = x^2 + y^2$

where each contour refers to the set $\{(x,y) \mid f(x,y) = \text{constant } C\}$.

The gradient at (x_0, y_0) is given by:

$$\nabla f(x_0, y_0) = (2x_0, 2y_0)^{\mathsf{T}}.$$

The tangent plane at $x^2 + y^2 = C$ can be denoted as a vector:

$$P = (-y, x)$$

Since $\langle \nabla f(x_0, y_0), P|_{(x_0,y_0)} \rangle = 0$, the gradient $\nabla f(x,y)$ is perpendicular to the level curves, with the direction pointing to where the function f increases the fastest. ■

Proposition 10.2 For $f : \mathbb{R}^n \to \mathbb{R}$, the direction $\frac{\nabla f(x)}{\|\nabla f(x)\|}$ gives the direction at which the function increases the fastest; the direction $-\frac{\nabla f(x)}{\|\nabla f(x)\|}$ gives the direction at which the function decreases the fastest.

The proof is simply by applying the Cauchy-Schawz inequality.

Interpretation in one-dimension. For $n = 1$, when $f' > 0$, f is increasing; when $f' < 0$, f is decreasing. Thus the sign of f' tells us at which direction the function increases/decreases.

Proposition 10.3 — ALgebra Rules for Gradients. Given two functions $f : X(\subseteq \mathbb{R}^n) \mapsto \mathbb{R}$ and $g : X(\subseteq \mathbb{R}^n) \mapsto \mathbb{R}$, if both f, g are differentiable, the product and division are also differentiable. In particular,

$$\nabla(gf) = g(\nabla f) + f(\nabla g)$$
$$\nabla(f/g) = \frac{g\nabla f - f\nabla g}{g^2}$$

Theorem 10.1 — Mean-Value Theorem. Let $f : X \subseteq \mathbb{R}^m \mapsto \mathbb{R}^n$ be differentiable in X, and $f = (f_1, f_2, \ldots, f_n)$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two points in X such that the line segment L connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ lies completely in X, then $\exists \boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n \in L$ such that

$$f_j(\boldsymbol{x}) - f_j(\boldsymbol{y}) = \langle \nabla f_j(c_j), \boldsymbol{x} - \boldsymbol{y} \rangle, j = 1, 2, \ldots, n$$

or equivalently,

$$f(\boldsymbol{x}) - f(\boldsymbol{y}) = Df(\boldsymbol{c}) * (\boldsymbol{x} - \boldsymbol{y}).$$

Proof. For fixed $\boldsymbol{x}, \boldsymbol{y}$, define a one-argument scalar function

$$\phi(t) = f(\boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x})), \quad 0 \leq t \leq 1$$

Applying MVT for one-variable, we derive

$$\phi(1) - \phi(0) = \phi'(t)(1 - 0)$$

Or equivalently,

$$f(\boldsymbol{y}) - f(\boldsymbol{x}) = Df(\underbrace{\boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x})}_{c}) * (\boldsymbol{y} - \boldsymbol{x}) = \langle \nabla f(\boldsymbol{c}), \boldsymbol{y} - \boldsymbol{x} \rangle$$

When can we change the order of differentiation?

■ **Example 10.2**

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ \\ 0, & x = y = 0 \end{cases}$$

After messy computation, we find

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right](0,0) = 1 \neq -1 = \frac{\partial^2 f}{\partial y \partial x}(0,0),$$

i.e., we cannot change the order of differentiation without a careful scrutiny. ■

Theorem 10.2   If $f : E(\subseteq \mathbb{R}^n) \mapsto \mathbb{R}^m$ has partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

at every point $x \in E$ where both partial derivatives are continuous.

Corollary 10.2   Generalizing it into higher orders

Proof. w.l.o.g., assume $i = 1, j = 2$. Define

$$F(h_1, h_2) = f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2) - f(x_1, x_2 + h_2) + f(x_1, x_2)$$

where the rectangle with vertices $(x_1, x_2), (x_1 + h_1, x_2), (x_1, x_2 + h_2), (x_1 + h_1, x_2 + h_2)$ are in E. We Define

$$\phi(t) = f(x_1 + th_1, x_2 + h_2) - f(x_1 + th_1, x_2),$$

then

$$F(h_1, h_2) = \phi(1) - \phi(0) = \phi'(\theta_1)$$

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Note that

$$\phi'(t) = \frac{\partial f}{\partial x_1}(x_1 + th_1, x_2 + h_2)h_1 - \frac{\partial f}{\partial x_1}(x_1 + th_1, x_2)h_1$$

which follows that

$$
\begin{aligned}
F(h_1, h_2) = \phi'(\theta_1) &= \left[\frac{\partial f}{\partial x_1}(x_1 + \theta_1 h_1, x_2 + h_2)h_1 - \frac{\partial f}{\partial x_1}(x_1 + \theta_1 h_1, x_2)\right]h_1 \\
&= \frac{\partial}{\partial x_2}\frac{\partial f}{\partial x_1}(x_1 + \theta_1 h_1, x_2 + \theta_2 h_2)h_2 h_1 \\
&= \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1 + \theta_1 h_1, x_2 + \theta_2 h_2)h_1 h_2
\end{aligned}
$$

Applying the same trick, we derive

$$\frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1 + \theta_1 h_1, x_2 + \theta_2 h_2)h_1 h_2 = F(h_1, h_2) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 + \theta_3 h_1, x_2 + \theta_4 h_2)h_1 h_2$$

or

$$\frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1 + \theta_1 h_1, x_2 + \theta_2 h_2) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 + \theta_3 h_1, x_2 + \theta_4 h_2)$$

for some $\theta_1, \theta_2, \theta_3, \theta_4$ between $(0,1)$. Taking $h_1, h_2 \to 0$, we conclude that

$$\frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1, x_2) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2)$$

∎

10.2. Friday

This lecture will talk about Taylor expansion and its applications, during which we will also have a brief review of linear algebra.

10.2.1. Multi-variate Taylor's Theorem

Recap for one-dimension. Given a function $f : \mathbb{R} \to \mathbb{R}$ with $f \in \mathcal{C}^n$, its Taylor's expansion is given by:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + R_n(x;x_0),$$

with

$$R_n(x;x_0) = \frac{1}{(n-1)} \int_{x_0}^x f^{(n)}(t)(x - t)^{n-1} \, \mathrm{d}t$$

Generalization into m-argument case.

> **Theorem 10.3 — Taylor's Theorem.** Given a function $f : \mathbb{R}^m \to \mathbb{R}$ with $f \in \mathcal{C}^n$, for the fixed point $\boldsymbol{x}$, and $\boldsymbol{h} = (h_1, h_2, \ldots, h_m)$, we have
>
> $$f(\boldsymbol{x} + \boldsymbol{h}) = f(\boldsymbol{x}) + \sum_{k=1}^{n-1} \frac{1}{k!} \left(h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} + \cdots + h_m \frac{\partial}{\partial x_m} \right)^k f(\boldsymbol{x}) + R_n(\boldsymbol{x};\boldsymbol{h}),$$
>
> $$\tag{10.3a}$$
>
> with
>
> $$R_n(\boldsymbol{x};\boldsymbol{h}) = \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!} \left(h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} + \cdots + h_m \frac{\partial}{\partial x_m} \right)^n f(\boldsymbol{x} + t\boldsymbol{h}) \, \mathrm{d}t \tag{10.3b}$$

Proof. We apply the Taylor's theorem in one-dimension case to finish he proof. Let

$$\phi(t) = f(\boldsymbol{x} + t\boldsymbol{h}),$$

which follows that

$$\phi(\tau) = \phi(0) + \phi'(0)\tau + \cdots + \frac{\phi^{(n-1)}(0)}{(n-1)!}\tau^{n-1} + \frac{1}{(n-1)!}\int_0^\tau \phi^{(n)}(t)(\tau-t)^{n-1}\,\mathrm{d}t$$

(10.4a)

$$\phi(\tau := 1) = \phi(0) + \phi'(0) + \cdots + \frac{1}{(n-1)!}\int_0^1 \phi^{(n)}(t)(1-t)^{n-1}\,\mathrm{d}t \qquad (10.4b)$$

Take $\phi'(0)$ as an example of computation:

$$\phi'(0) = \langle \nabla f(\boldsymbol{x}+t\boldsymbol{h}), \boldsymbol{h}\rangle|_{t=0}$$

$$= \langle \nabla f(\boldsymbol{x}), \boldsymbol{h}\rangle$$

$$= h_1 \frac{\partial f}{\partial x_1} + \cdots + h_m \frac{\partial f}{\partial x_m}$$

$$= \left(h_1 \frac{\partial}{\partial x_1} + \cdots + h_m \frac{\partial}{\partial x_m} \right) f$$

Taking such operator k times, we obtain

$$\phi^{(k)}(0) = \left(h_1 \frac{\partial}{\partial x_1} + \cdots + h_m \frac{\partial}{\partial x_m} \right)^k f \qquad (10.4c)$$

Substituting (10.4c) into (10.4b), we obtain the desired result.

∎

(R) Let's abuse the notation to let ∂_i denote $\frac{\partial}{\partial x_i}$; the term $\left(h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} + \cdots + h_m \frac{\partial}{\partial x_m} \right)^k$ denotes a kind of operator, e.g.,

$$(h_1\partial_1 + h_2\partial_2)^3 = h_1^3(\partial_1)^3 + 3h_1^2(\partial_1)^2 h_2\partial_2 + 3h_1\partial_1 h_2^2(\partial_2)^2 + h_2^3(\partial_2)^3$$

(R) We introduce the multi-index expression for Taylor's Theorem. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ with α_j's are non-negative integers, with the operators

$$|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_m \qquad (10.5a)$$

$$\boldsymbol{\alpha}! = \alpha_1! \alpha_2! \cdots \alpha_m! \qquad (10.5b)$$

Let $\boldsymbol{a} = (a_1, \ldots, a_m)$, with the operator **multiplication-combined component-wise** powers:

$$\boldsymbol{a}^{\boldsymbol{\alpha}} := a_1^{\alpha_1} \cdots a_m^{\alpha_m} \tag{10.5c}$$

We can re-write the binomial equation with the terms defined above:

$$
\begin{aligned}
(a_1 + a_2 + \cdots + a_m)^k &= \sum_{|\boldsymbol{\alpha}|=k} \frac{k!}{\alpha_1! \cdots \alpha_m!} a_1^{\alpha_1} \cdots a_m^{\alpha_m} \\
&= \sum_{|\boldsymbol{\alpha}|=k} \frac{k!}{\boldsymbol{\alpha}!} \boldsymbol{a}^{\boldsymbol{\alpha}}
\end{aligned}
\tag{10.5d}
$$

Define the combinatorial derivatives:

$$D^{\boldsymbol{\alpha}} f := \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_m} x_m} \tag{10.5e}$$

and therefore the k-th sub-term for the summation term in (10.3a) can be expressed as:

$$\sum_{i_1 + \cdots + i_m = k} \frac{\partial^k}{\partial^{i_1} x_1 \cdots \partial^{i_m} x_m} f(\boldsymbol{x}) h_1^{i_1} \cdots h_m^{i_m} = \sum_{|\boldsymbol{\alpha}|=k} \frac{k!}{\boldsymbol{\alpha}!} D^{\boldsymbol{\alpha}} f(\boldsymbol{x}) \boldsymbol{h}^{\boldsymbol{\alpha}} \tag{10.5f}$$

Substituting (10.5e) and (10.5f) into (10.3a), we can express the Taylor's formula in multi-index form:

$$f(\boldsymbol{x} + \boldsymbol{h}) = \sum_{|\boldsymbol{\alpha}|=0}^{n-1} \frac{1}{\boldsymbol{\alpha}!} D^{\boldsymbol{\alpha}} f(\boldsymbol{x}) \boldsymbol{h}^{\boldsymbol{\alpha}} + \sum_{|\boldsymbol{\alpha}|=n} n \int_0^1 \frac{(1-\boldsymbol{\alpha})^{n-1}}{\boldsymbol{\alpha}!} D^{\boldsymbol{\alpha}} f(\boldsymbol{x} + t\boldsymbol{h}) \boldsymbol{h}^{\boldsymbol{\alpha}} \, \mathrm{d}t \tag{10.6}$$

10.2.2. Application: Optimality Condition

Necessary optimal condition.

> **Theorem 10.4**   Given a function $f : U(\boldsymbol{x}_0) \subseteq \mathbb{R}^m \to \mathbb{R}^n$, where $U(\boldsymbol{x}_0)$ denotes a neighborhood of $\boldsymbol{x}_0$, suppose f has all the partial derivatives at $\boldsymbol{x}_0$. If f has a **local**

extremum (max or min) at $\boldsymbol{x} = \boldsymbol{x}_0$, then

$$\frac{\partial f}{\partial x_1}(\boldsymbol{x}_0) = \cdots = \frac{\partial f}{\partial x_m}(\boldsymbol{x}_0),$$

i.e., the local extremum of $\boldsymbol{x}_0$ implies the **critical point** of $\boldsymbol{x}_0$.

Proof. Let $\boldsymbol{x}_0 = (x_1^0, \ldots, x_m^0)$. To show $\partial_1 f(\boldsymbol{x}_0) = 0$, we fix all entries of $\boldsymbol{x}_0$ except for x_1^0, i.e., set $\phi(x_1) = f(x_1, x_2^0, \ldots, x_m^0)$. For this one-variable function ϕ, it has a local extremum at $x_1 = x_1^0$ implies $\phi'(x_1^0) = 0$, i.e.,

$$\phi'(x_1^0) = \frac{\partial f}{\partial x_1}(x_1^0, \ldots, x_m^0)$$

The similar results hold for $\partial_2 f(\boldsymbol{x}_0), \ldots, \partial_m f(\boldsymbol{x}_0)$. ∎

Theorem(10.4) is the necessary optimality condition. Before turning into the sufficient condition, let's have a brief review on linear algebra, or specifically, diagonalization.

Recap about Diagonalization.

Definition 10.5 [Positive Definite] A $m \times m$ symmetric matrix $\boldsymbol{A}$ is said to be

1. **positive definite** if $\boldsymbol{x}^{\mathsf{T}} \boldsymbol{A} \boldsymbol{x} > 0$ for $\forall \boldsymbol{x} \in \mathbb{R}^m \setminus \{\boldsymbol{0}\}$; or equivalently, all its eigenvalues are strictly positive

2. **negative definite** if $-\boldsymbol{A}$ is positive definite.

Theorem 10.5 — **Spectral Theorem.** Any **real symmetric** matrix $\boldsymbol{A} \in \mathbb{S}^m$ admits the eigen-decomposition

$$\boldsymbol{A} = \boldsymbol{Q} \boldsymbol{D} \boldsymbol{Q}^{\mathsf{T}},$$

with $\boldsymbol{D} \in \mathbb{R}^{m \times m}$ to be diagonal; $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ to be orthogonal.

Ⓡ Theorem (10.5) gives us the sufficient condition of diagonalization: real and symmetric. Since the Heessian matrix satisfies such a condition, we can diagonalize it without careful scrutiny.

Sufficient Optimality Condition.

> **Theorem 10.6**   Given a function $f : U(\boldsymbol{x}_0) \subseteq \mathbb{R}^m \to \mathbb{R}$ that is $\mathcal{C}^2$, suppose $\boldsymbol{x}_0$ is a critical point (i.e., $\nabla f(\boldsymbol{x}_0) = \boldsymbol{0}$).
>
> 1. If the matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j})_{m \times m}$ is positive definite, then $\boldsymbol{x}_0$ is a local minimum point
> 2. If the matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j})_{m \times m}$ is negative definite, then $\boldsymbol{x}_0$ is a local maximum point
> 3. Otherwise no further information.

Proof. We show the case 1 first.

Suppose $\boldsymbol{h} = (h_1, \ldots, h_m) \in \mathbb{R}^m$, then apply second order Taylor expansion at $\boldsymbol{x}_0$:

$$f(\boldsymbol{x}_0 + \boldsymbol{h}) = f(\boldsymbol{x}_0) + \frac{1}{2!} \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}_0) h_i h_j + o(\|\boldsymbol{h}\|^2) \tag{10.7a}$$

$$= f(\boldsymbol{x}_0) + \|\boldsymbol{h}\|^2 \left[\frac{1}{2!} \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{h_i}{\|\boldsymbol{h}\|} \frac{h_j}{\|\boldsymbol{h}\|} + o(1) \right] \tag{10.7b}$$

Define $a_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$, $q_i = \frac{h_i}{\|\boldsymbol{h}\|}$ with $\boldsymbol{q}$ to be a unit vector, we can rewrite the summation term in (10.7b) as the quadratic form:

$$\sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{h_i}{\|\boldsymbol{h}\|} \frac{h_j}{\|\boldsymbol{h}\|} = \boldsymbol{q}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{q} = \langle \boldsymbol{A}\boldsymbol{q}, \boldsymbol{q} \rangle \tag{10.7c}$$

Furthermore, we can derive a strictly positive lower bound on (10.7c) by applying spectral theorem:

$$\langle \boldsymbol{A}\boldsymbol{q}, \boldsymbol{q} \rangle = \langle \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{q}, \boldsymbol{q} \rangle \tag{10.7d}$$

$$= \langle \boldsymbol{D}\boldsymbol{Q}\boldsymbol{q}, \underbrace{\boldsymbol{Q}\boldsymbol{q}}_{\boldsymbol{y}} \rangle = \boldsymbol{y}^{\mathrm{T}} \boldsymbol{D} \boldsymbol{y} \tag{10.7e}$$

$$= \sum_{i=1}^m \lambda_i y_i^2, \tag{10.7f}$$

where $\lambda_i > 0$ are the diagonal entries of $\boldsymbol{D}$, i.e., eigenvalues of $\boldsymbol{A}$; y_i is the i-th entry of the unit vector $\boldsymbol{y}$. To give a bound on $\langle \boldsymbol{A}\boldsymbol{q}, \boldsymbol{q} \rangle$, we set $y_i = 0$ except for the smallest

eigenvalues of $\boldsymbol{A}$, i.e.,

$$\langle \boldsymbol{Aq}, \boldsymbol{q} \rangle \geq \min_{\|\boldsymbol{q}\|=1} \langle \boldsymbol{Aq}, \boldsymbol{q} \rangle = \min_{\|\boldsymbol{y}\|=1} \sum_{i=1}^{m} \lambda_i y_i^2 = \lambda_{\min}(\boldsymbol{A}) > 0 \qquad (10.7\text{g})$$

Substituting (10.7g) into (10.7b), we imply that

$$f(\boldsymbol{x}_0 + \boldsymbol{h}) \geq f(\boldsymbol{x}_0) + \|\boldsymbol{h}\|^2 \left[\lambda_{\min}(\boldsymbol{A}) + o(1) \right] \geq f(\boldsymbol{x}_0), \qquad (10.7\text{h})$$

which shows that $\boldsymbol{x}_0$ is the local minimum point. The case 2 can be shown similarly. ∎

Actually, we can simplify the messy steps from (10.7d) to (10.7g) with a simple fact in linear algebra:

> **Theorem 10.7** — **Courant-Fischer Formula.** Let $\boldsymbol{A} \in \mathbb{R}^m$ be a real symmetric matrix with $\lambda_{\max}, \lambda_{\min}$ to be the max. and min. eigenvalues of $\boldsymbol{A}$, respectively. Then $\lambda_{\max}, \lambda_{\min}$ can be characterized as:
>
> $$\lambda_{\max} = \max_{\boldsymbol{x} \in \mathbb{R}^m, \|\boldsymbol{x}\|_2 = 1} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x}, \quad \lambda_{\min} = \min_{\boldsymbol{x} \in \mathbb{R}^m, \|\boldsymbol{x}\|_2 = 1} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x}.$$

We will study the implicit function theorem in next lecture with a wonderful proof.

Chapter 11

Week11

11.1. Wednesday

Announcement. Our second quiz will happen next Friday (November 30), from 1:00 to 2:20 pm in the room Zhiren 205. The **emphasis** will be the content after the mid-term.

11.1.1. Recap

Given a function $f : \mathbb{R}^m \to \mathbb{R}$, suppose $\boldsymbol{x}_0 \in \mathbb{R}^m$. The point $\boldsymbol{x}_0$ is said to be a critical point if $Df(\boldsymbol{x}_0) = 0$, i.e., $\frac{\partial f}{\partial x_j}(\boldsymbol{x}_0) = 0$, $j = 1, 2, \ldots, m$.

> **Definition 11.1** [Critical Point] The point $\boldsymbol{x}_0 \in \mathbb{R}^m$ is said to be a critical point of $f : \mathbb{R}^m \to \mathbb{R}^n$ if the Jacobian matrix at $\boldsymbol{x} = \boldsymbol{x}_0$
>
> $$Df(\boldsymbol{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}_{n \times m}$$
>
> is **rank deficient**, i.e., $\mathrm{rank}(Df(\boldsymbol{x}_0)) < \min\{n, m\}$. ∎

Perhaps many classmates are confused with the size and notation of multi-variate formulas. Let's raise some rules to help you minimize errors:

1. Any vector is viewed as a column vector

2. The gradient ∇f of a scalar function of n-argument $f : \mathbb{R}^n \to \mathbb{R}$ is also viewed as

a column vector.

3. The Jacobian matrix Df of a vector function $f : \mathbb{R}^n \to \mathbb{R}^m$ with components $f_1, \ldots, f_m$ is the $m \times n$ matrix whose rows are the transposed vectors $\nabla^{\mathrm{T}} f_1, \ldots, \nabla^{\mathrm{T}} f_m$, i.e.,

$$Df = \begin{pmatrix} \nabla^{\mathrm{T}} f_1 \\ \vdots \\ \nabla^{\mathrm{T}} f_m \end{pmatrix}$$

Theorem 11.1 — **Sufficient Optimality Condition.** Given a function $f : U(\boldsymbol{x}_0)(\subseteq \mathbb{R}^m) \to \mathbb{R}$, suppose $\boldsymbol{x}_0$ is a critical point, then

1. If the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j} (\boldsymbol{x}_0) \right) \succ 0$, then $\boldsymbol{x}_0$ is the local minimum
2. If the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j} (\boldsymbol{x}_0) \right) \prec 0$, then $\boldsymbol{x}_0$ is the local maximum
3. Otherwise, the Hessian matrix gives no further information.

Proof. Taylor expansion $f(\boldsymbol{x}_0 + \boldsymbol{h})$ at point $\boldsymbol{x} = \boldsymbol{x}_0$:

$$f(\boldsymbol{x}_0 + \boldsymbol{h}) = f(\boldsymbol{x}_0) + \frac{1}{2!} \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j} (\boldsymbol{x}_0) h_i h_j + o(\|\boldsymbol{h}\|^2),$$

Then apply the definiteness of Hessian matrix to bound the second term on RHS. $\blacksquare$

■ **Example 11.1** Given the function $f(x, y) = x^4 + y^4 - 2x^2$, we have

$$\nabla f = \begin{pmatrix} 4x(x-1)(x+1) \\ 4y^3 \end{pmatrix},$$

which induces three critical points $(-1, 0), (0, 0), (1, 0)$, i.e., candidates for extreme points.

- The Hessian matrix is given by:

$$H = \begin{pmatrix} 12x^2 - 4 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

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The Hessian matrix evaluated at these critial points might give the further information on extreme points:

$$(-1,0) \quad \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{positive semi-definite}$$

Therefore, we cannot determine whether $(-1,0)$ is extreme point or not. Similarly, the Hessian matrix evaluated on the remaining two points are neither PD or ND, thus all these three points cannot be determined to be extreme point or not.

- However, by term arrangement,

$$f(x,y) = x^4 + y^4 - 2x^2 = (x^2 - 1)^2 + (y^4 - 1) \geq 0$$

Thus at $(\pm 1, 0)$, this function achieves its minimum value, i.e., $(\pm 1, 0)$ are (strictly) local minimum. At small neighborhood of $(0,0)$, as $h \to 0$, $f(h,0)$ is increasing; and $f(0,h)$ is decreasing.

11.1.2. Introduction to Implicit Function Theorem

The remaining lecture will discuss the proof of Implicit Function Theorem (IFT), which is an intuitive and elementary style.

Motivation. The purpose of this theorem is to understand the relation of some variables with others. Let's begin with a simple example.

■ **Example 11.2** Suppose we have the relation

$$F(x,y) := x^2 + y^2 - 1 = 0 \tag{11.1}$$

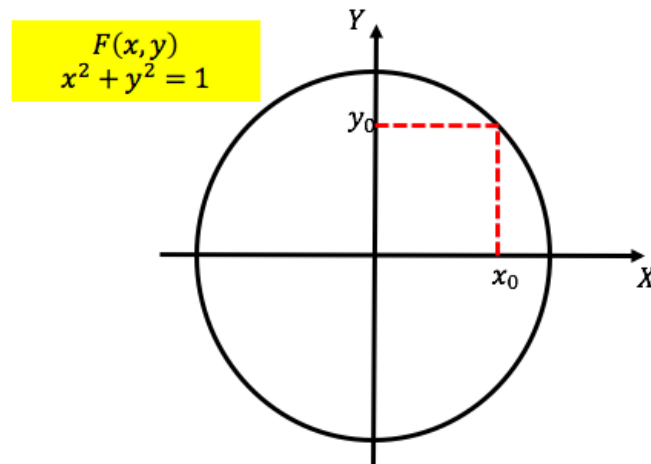This relation is just the unit circle, plotted in the figure below

Figure 11.1: The set of all points in $\mathbb{R}^2$ satisfying relation (11.1)

This relation shows that the location of y is dependent with x. We want to study under which condition can we have the one-to-one relation between x and y, say $y = f(x)$. For axample, the relation (11.1) pre-assumes the relation

$$y = \sqrt{1 - x^2}$$

near the small neighborhood of x_0. However, when $x_0 = \pm 1$, we can not obtain such a one-to-one relation. ∎

In other words, our interest is to answer the question

How is it possible to find out implicitly, the relation(11.1) can be represented as a one-to-one mapping $y = f(x)$?

The IFT gives sufficient condition for that. Moreover, after obtaining a form of function (one-to-one mapping), we could perform something fancy such as differentiation, integration, and otherwise. The IFT also gives the differential form of the function we have obtained.

First let's talk about IFT in two-variable case.

> **Theorem 11.2 — Elementary Version of IFT.** Let $F : U(x_0, y_0)(\subseteq \mathbb{R}^2) \to \mathbb{R}$ be a $\mathcal{C}^p$
> $(p \geq 1)$ function, with
>
> 1. $F(x_0, y_0) = 0$
> 2. $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$.
>
> Then there exists a neighborhood of (x_0, y_0), say $I_x \times I_y \subseteq U(x_0, y_0)$ with
>
> $$I_x = \{x \in \mathbb{R} \mid |x - x_0| < \alpha\}, \quad I_y = \{y \in \mathbb{R} \mid |y - y_0| < \beta\},$$
>
> and an **unique** function $f \in \mathcal{C}^p(I_x; I_y)$ satisfying
>
> $$F(x, y) = 0 \Longleftrightarrow y = f(x), \qquad \forall (x, y) \in I_x \times I_y \tag{11.2}$$
>
> $$f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}(x, f(x)) \qquad \left(= -\frac{F_x}{F_y}(x, f(x)) \right) \tag{11.3}$$

The proof are given below, and we use some diagrams to help you understand it.

Step 1: Applying the continuity of F. w.l.o.g., assume $F_y(x_0, y_0) > 0$. Due to the continuity, pick neighbborhood of (x_0, y_0), say $\tilde{U}(x_0, y_0)$ such that

$$F_y(x, y) > 0, \quad \forall (x, y) \in \tilde{U}(x_0, y_0) \tag{11.4}$$

Pick small $\beta > 0$, such that the line segment

$$l := \{(x_0, y) \mid |y - y_0| < \beta\} \subseteq \tilde{U}, \tag{11.5}$$

i.e., $F_y > 0$ on l, which implies $F(x_0, y - \beta) < 0$ and $F(x_0, y + \beta) > 0$.

By continuity of F, there $\exists$ small $\alpha > 0$ such that

$$F < 0 \text{ on } \{(x, y_0 - \beta) \mid |x - x_0| < \alpha\} \tag{11.6}$$

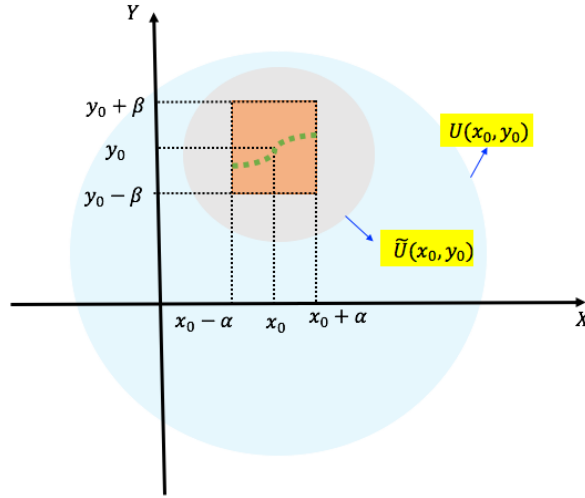$$F > 0 \text{ on } \{(x, y_0 + \beta) \mid |x - x_0| < \alpha\} \tag{11.7}$$

Figure 11.2: Interpretation of the constructive proof

By applying intermediate value theorem, there $\exists\, y = f(x)$ such that $F(x,y) = 0$. Note the the uniqueness of f is guarnteed since $F_y > 0$ on $\tilde{U}(x_0, y_0)$, i.e., the zero of F (on $\tilde{U}(x_0, y_0)$) is unique for fixed x.

Step 2: Show the continuity of f on I_x. Firstly, to show that f is continuous at x_0, it suffices to show that for fixed $\varepsilon > 0$,

$$\exists \delta > 0 \text{ such that } |f(x) - f(x_0)| < \varepsilon \text{ for } \forall\, x \text{ such that } |x - x_0| < \delta \tag{11.8}$$

Replace β with ε, and pick $\delta := \alpha$ in step1, the relation (11.8) holds clearly.

To show the continuity of f on the whole interval I_x, it suffices to repeat the arguments in step1 by replacing x_0 with any other points in I_x.

Step3: Show the differentiability of f. For any $x \in I_x$, pick small Δx such that $x + \Delta x \in I_x$. Define $y := f(x)$ and $y + \Delta y := f(x + \Delta x)$, which follows that

$$0 = F(x + \Delta x, y + \Delta y) - F(x, y) \tag{11.9}$$

$$= \langle \nabla F(x + \theta \Delta x, y + \theta \Delta y), \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \rangle \tag{11.10}$$

$$= F_x(x + \theta \Delta x, y + \theta \Delta y)\Delta x + F_y(x + \theta \Delta x, y + \theta \Delta y)\Delta y \tag{11.11}$$

where (11.9) is because $F(x + \Delta x, y + \Delta y) = F(x + \Delta x, f(x + \Delta x))$ and then applying (11.2); (11.10) is by applying mean-value theorem ($\theta \in [0,1]$); (11.11) is just term arrangement.

Therefore, solving (11.11) for $\Delta y / \Delta x$, we derive:

$$\frac{\Delta y}{\Delta x} = -\frac{F_x(x + \theta \Delta x, y + \theta \Delta y)}{F_y(x + \theta \Delta x, y + \theta \Delta y)} \to -\frac{F_x}{F_y}(x, f(x)), \tag{11.12}$$

as $\Delta x \to 0$. In other words, $f' = -\frac{F_x}{F_y}(x)$. The differentiability of f is shown thereby.

Alternative approach for evaluating the derivative of f. Once we have the differentiability condition, we could also evaluate the derivative directly. Note that $\phi(x) := F(x, f(x)) = 0$ for $\forall x \in I_x$. Differentiating $\phi(x) = 0$ both sides leads to

$$0 = \phi'(x) = F_x + F_y f' \qquad \text{in } I_x$$

and therefore $f' = -F_x/F_y$ in I_x.

Step4: Show that $f \in \mathcal{C}^p$. If $p = 2$, then the RHS of (11.3) is differentiable w.r.t. x, and we find

$$f'' = -\frac{[F_{xx} + F_{xy} \cdot f']F_y - F_x[F_{xy} + F_{yy} \cdot f']}{[F_y]^2}$$

The case for $p > 2$ can be shown by induction.

Theorem 11.3 — **Simple Generalization.** Let $F(x_1,\ldots,x_n;y) : U(\boldsymbol{x}_0,y_0)(\subseteq \mathbb{R}^{m+1}) \to \mathbb{R}$ be $\mathcal{C}^p$ function with $p \geq 1$. Suppose

1. $F(\boldsymbol{x}_0,y_0) = 0$
2. $\frac{\partial F}{\partial y}(\boldsymbol{x}_0,y_0) \neq 0$

Then there exists a neighborhood of (x_0,y_0), say $I_{\boldsymbol{x}} \times I_y \subseteq U(\boldsymbol{x}_0,y_0)$ with

$$I_{\boldsymbol{x}} = \{\boldsymbol{x} \in \mathbb{R}^m \mid \|\boldsymbol{x} - \boldsymbol{x}_0\| < \alpha\}, \quad I_y = \{y \in \mathbb{R} \mid |y - y_0| < \beta\},$$

and an **unique** function $f \in \mathcal{C}^p(I_{\boldsymbol{x}}, I_y)$ satisfying

$$F(x_1,\ldots,x_m,y) = 0 \iff F(x_1,\ldots,x_m,f(\boldsymbol{x})) = 0, \qquad \forall (\boldsymbol{x},y) \in I_{\boldsymbol{x}} \times I_y \qquad (11.13)$$

$$\frac{\partial f}{\partial x_j} = -\frac{\frac{\partial F}{\partial x_j}}{\frac{\partial F}{\partial y}} j = 1,\ldots,m \qquad (11.14)$$

Proof. The proof for the case $j = 1$ is just by fixing all variables in the functions $F(x_1,\ldots,x_m,y)$ except x_1 and y, and then following the same argument in Theorem(11.2) $\blacksquare$