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PURE MATHEMATICS

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MAT3006 & 3040 & 4002 Notebook

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Contents

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CUHK(SZ)

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i, j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i, j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i, j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Monday for MAT3040

1.1.1. Introduction to Advanced Linear Algebra

Advanced Linear Algebra is one of the most important course in MATH major, with pre-request MAT2040. This course will offer the really linear algebra knowledge.

What the content will be covered?.

- In MAT2040 we have studied the space $\mathbb{R}^n$; while in MAT3040 we will study the general vector space V.
- In MAT2040 we have studied the *linear transformation* between Euclidean spaces, i.e., $T : \mathbb{R}^n \to \mathbb{R}^m$; while in MAT3040 we will study the linear transformation from vector spaces to vector spaces: $T : V \to W$
- In MAT2040 we have studied the eigenvalues of $n \times n$ matrix $\boldsymbol{A}$; while in MAT3040 we will study the eigenvalues of a **linear operator** $T : V \to V$.
- In MAT2040 we have studied the dot product $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$; while in MAT3040 we will study the **inner product** $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle$.

Why do we do the generalization?. We are studying many other spaces, e.g., $\mathcal{C}(\mathbb{R})$ is called the space of all functions on $\mathbb{R}$, $\mathcal{C}^\infty(\mathbb{R})$ is called the space of all infinitely differentiable functions on $\mathbb{R}$, $\mathbb{R}[x]$ is the space of polynomials of one-variable.

■ **Example 1.1** 1. Consider the Laplace equation $\Delta f = 0$ with linear operator Δ:

$$\Delta : \mathcal{C}^{\infty}(\mathbb{R}^3) \to \mathcal{C}^{\infty}(\mathbb{R}^3) \quad f \mapsto \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

The solution to the PDE $\Delta f = 0$ is the 0-eigenspace of Δ.

2. Consider the Schrödinger equation $\hat{H} f = E f$ with the linear operator

$$\hat{H} : \mathcal{C}^{\infty}(\mathbb{R}^3) \to \mathbb{R}^3, \quad f \to \left[\frac{-\hbar^2}{2\mu}\nabla^2 + V(x,y,z)\right] f$$

Solving the equation $\hat{H} f = E f$ is equivalent to finding the eigenvectors of $\hat{H}$. In fact, the eigenvalues of $\hat{H}$ are **discrete**.

1.1.2. Vector Spaces

Definition 1.1 [Vector Space] A **vector space** over a field $\mathbb{F}$ (in particular, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is a set of objects V equipped with vector addiction and scalar multiplication such that

1. the vector addiction $+$ is closed with the rules:
   (a) **Commutativity**: $\forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, $\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}_2 + \boldsymbol{v}_1$.
   (b) **Associativity**: $\boldsymbol{v}_1 + (\boldsymbol{v}_2 + \boldsymbol{v}_3) = (\boldsymbol{v}_1 + \boldsymbol{v}_2) + \boldsymbol{v}_3$.
   (c) **Addictive Identity**: $\exists \mathbf{0} \in V$ such that $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$, $\forall \boldsymbol{v} \in V$.
2. the **scalar multiplication** is closed with the rules:
   (a) **Distributive**: $\alpha(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \alpha\boldsymbol{v}_1 + \alpha\boldsymbol{v}_2, \forall \alpha \in \mathbb{F}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$
   (b) **Distributive**: $(\alpha_1 + \alpha_2)\boldsymbol{v} = \alpha_1\boldsymbol{v} + \alpha_2\boldsymbol{v}$
   (c) **Compatibility**: $a(b\boldsymbol{v}) = (ab)\boldsymbol{v}$ for $\forall a, b \in \mathbb{F}$ and $\boldsymbol{b} \in V$.
   (d) $0\boldsymbol{v} = \mathbf{0}$, $1\boldsymbol{v} = \boldsymbol{v}$.

Here we study several examples of vector spaces:

■ **Example 1.2** For $V = \mathbb{F}^n$, we can define

1. Addictive Identity:
$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$
2. Scalar Multiplication:
$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$
3. Vector Addiction:
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

■

■ **Example 1.3** 1. It is clear that the set $V = M_{n\times n}(\mathbb{F})$ (the set of all $m \times n$ matrices) is a vector space as well.

2. The set $V = \mathcal{C}(\mathbb{R})$ is a vector space:
   (a) Vector Addiction:
   $$(f+g)(x) = f(x) + g(x), \forall f, g \in V$$
   (b) Scalar Multiplication:
   $$(\alpha f)(x) = \alpha f(x), \forall \alpha \in \mathbb{R}, f \in V$$
   (c) Addictive Identity is a zero function, i.e., $\mathbf{0}(x) = 0$ for all $x \in \mathbb{R}$.

■

Definition 1.2 A sub-collection $W \subseteq V$ of a vector space V is called a **vector subspace of** V if W itself forms a vector space, denoted by $W \leq V$.

Example 1.4 1. For $V = \mathbb{R}^3$, we claim that $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \leq V$

2. $W = \{(x, y, 1) \mid x, y \in \mathbb{R}\}$ is not the vector subspace of V.

Proposition 1.1 $W \subseteq V$ is a **vector subspace** of V iff for $\forall \boldsymbol{w}_1 + \boldsymbol{w}_2 \in W$, we have $\alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2 \in W$, for $\forall \alpha, \beta \in \mathbb{F}$.

Example 1.5 1. For $V = M_{n \times n}(\mathbb{F})$, the subspace $W = \{A \in V \mid \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}\} \leq V$

2. For $V = \mathcal{C}^{\infty}(\mathbb{R})$, define $W = \{f \in V \mid \frac{\mathrm{d}^2}{\mathrm{d}x^2} f + f = 0\} \leq V$. For $f, g \in W$, we have

$$(\alpha f + \beta g)'' = \alpha f'' + \beta g'' = \alpha(-f) + \beta(-g) = -(\alpha f + \beta g),$$

which implies $(\alpha f + \beta g)'' + (\alpha f + \beta g) = 0$.

1.2. Monday for MAT3006

1.2.1. Overview on uniform convergence

Definition 1.3 [Convergence] Let $f_n(x)$ be a sequence of functions on an interval $I = [a, b]$. Then $f_n(x)$ converges **pointwise** to $f(x)$ (i.e., $f_n(x_0) \to f(x_0)$) for $\forall x_0 \in I$, if

$$\forall \varepsilon > 0, \exists N_{x_0,\varepsilon} \text{ such that } |f_n(x_0) - f(x_0)| < \varepsilon, \forall n \geq N_{x_0,\varepsilon}$$

We say $f_n(x)$ converges uniformly to $f(x)$, (i.e., $f_n(x) \rightrightarrows f(x)$) for $\forall x_0 \in I$, if

$$\forall \varepsilon > 0, \exists N_{\varepsilon} \text{ such that } |f_n(x_0) - f(x_0)| < \varepsilon, \forall n \geq N_{\varepsilon}$$

■ **Example 1.6** It is clear that the function $f_n(x) = \frac{n}{1+nx}$ converges pointwise into $f(x) = \frac{1}{x}$ on $[0, \infty)$, and it is uniformly convergent on $[1, \infty)$.

Proposition 1.2 If $\{f_n\}$ is a sequence of continuous functions on I, and $f_n(x) \rightrightarrows f(x)$, then the following results hold:

1. $f(x)$ is continuous on I.
2. f is (Riemann) integrable with $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.
3. Suppose furthermore that $f_n(x)$ is **continuously differentiable**, and $f_n'(x) \rightrightarrows g(x)$, then $f(x)$ is differentiable, with $f_n'(x) \to f'(x)$.

We can put the discussions above into the content of series, i.e., $f_n(x) = \sum_{k=1}^{n} S_k(x)$.

Proposition 1.3 If $S_k(x)$ is continuous for $\forall k$, and $\sum_{k=1}^{n} S_k \rightrightarrows \sum_{k=1}^{\infty} S_k$, then

1. $\sum_{k=1}^{\infty} S_k(x)$ is continuous,
2. The series $\sum_{k=1}^{\infty} S_k$ is (Riemann) integrable, with $\sum_{k=1}^{\infty} \int_a^b S_k(x)\,dx = \int_a^b \sum_{k=1}^{\infty} S_k(x)\,dx$
3. If $\sum_{k=1}^{n} S_k$ is continuously differentiable, and the derivative of which is uniform

convergent, then the series $\sum_{k=1}^{\infty} S_k$ is differentiable, with

$$\left(\sum_{k=1}^{\infty} S_k(x)\right)' = \sum_{k=1}^{\infty} S_k'(x)$$

Then we can discuss the properties for a special kind of series, say power series.

Proposition 1.4 Suppose the power series $f(x) = \sum_{k=1}^{\infty} a_k x^k$ has radius of convergence R, then

1. $\sum_{k=1}^{n} a_k x^k \rightrightarrows f(x)$ for any $[-L, L]$ with $L < R$.
2. The function $f(x)$ is continuous on $(-R, R)$, and moreover, is differentiable and (Riemann) integrable on $[-L, L]$ with $L < R$:

$$\int_0^x f(t)\,\mathrm{d}t = \sum_{k=1}^{\infty} \frac{a_k}{k+1} x^{k+1}$$
$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

1.2.2. Introduction to MAT3006

What are we going to do.

1. (a) Generalize our study of (sequence, series, functions) on $\mathbb{R}^n$ into a metric space.

   (b) We will study spaces outside $\mathbb{R}^n$.

   Remark:

   - For (a), different metric may yield different kind of convergence of sequences. For (b), one important example we will study is $X = \mathcal{C}[a,b]$ (all continuous functions defined on $[a,b]$.) We will generalize X into $\mathcal{C}_b(E)$, which means the set of bounded continuous functions defined on $E \subseteq \mathbb{R}^n$.
   - The insights of analysis is to find a **unified** theory to study sequences/series on a metric space X, e.g., $X = \mathbb{R}^n, \mathcal{C}[a,b]$. In particular, for $\mathcal{C}[a,b]$, we will see that
     - most functions in $\mathcal{C}[a,b]$ are nowhere differentiable. (repeat part of

content in MAT2006)

- We will prove the existence and uniqueness of ODEs.
- the set $\text{poly}[a,b]$ (the set of polynomials on $[a,b]$) is dense in $\mathcal{C}[a,b]$. (analogy: $\mathbb{Q} \subseteq \mathbb{R}$ is dense)

2. Introduction to the Lebesgue Integration.

   For convergence of integration $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)$, we need the pre-conditions (a) $f_n(x)$ is continuous, and (b) $f_n(x) \rightrightarrows f(x)$. The natural question is that can we relax these conditions to

   (a) $f_n(x)$ is integrable?

   (b) $f_n(x) \to f(x)$ pointwisely?

   The answer is yes, by using the tool of Lebesgue integration. If $f_n(x) \to f(x)$ and $f_n(x)$ is Lebesgue integrable, then $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$, which is so called the **dominated convergence**.

1.2.3. Metric Spaces

We will study the **length** of an element, or the **distance** between two elements in an arbitrary set X. First let's discuss the length defined on a well-structured set, say vector space.

Definition 1.4 [Normed Space] Let X be a vector space. A **norm** on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that

1. $\|\boldsymbol{x}\| \geq 0$ for $\forall \boldsymbol{x} \in X$, with equality iff $\boldsymbol{x} = \boldsymbol{0}$
2. $\|\alpha \boldsymbol{x}\| = |\alpha| \|\boldsymbol{x}\|$, for $\forall \alpha \in \mathbb{R}$ and $\boldsymbol{x} \in X$.
3. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ (triangular inequality)

Any vector space equipped with $\|\cdot\|$ is called a **normed space**.

■ **Example 1.7** 1. For $X=\mathbb{R}^n$, define

$$\|\boldsymbol{x}\|_2=\left(\sum_{i=1}^n x_i^2\right)^{1/2} \quad \text{(Euclidean Norm)}$$

$$\|\boldsymbol{x}\|_p=\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \quad (p\text{-norm})$$

2. For $X=\mathcal{C}[a,b]$, define

$$\|f\|_\infty=\max_{x\in[a,b]}|f(x)|$$

$$\|f\|_p=\left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p}$$

Exercise: check the norm defined above are well-defined. ■

Here we can define the distance in an arbitrary set:

Definition 1.5 A set X is a **metric space** with metric (X,d) if there exists a (distance) function $d: X\times X\to\mathbb{R}$ such that

1. $d(\boldsymbol{x},\boldsymbol{y})\ge 0$ for $\forall \boldsymbol{x},\boldsymbol{y}\in X$, with equality iff $\boldsymbol{x}=\boldsymbol{y}$.
2. $d(\boldsymbol{x},\boldsymbol{y})=d(\boldsymbol{y},\boldsymbol{x})$.
3. $d(\boldsymbol{x},\boldsymbol{z})\le d(\boldsymbol{x},\boldsymbol{y})+d(\boldsymbol{y},\boldsymbol{z})$.

■

■ **Example 1.8** 1. If X is a normed space, then define $d(\boldsymbol{x},\boldsymbol{y})=\|\boldsymbol{x}-\boldsymbol{y}\|$, which is so called the metric induced from the norm $\|\cdot\|$.

2. Let X be any (non-empty) set with $\boldsymbol{x},\boldsymbol{y}\in X$, the discrete metric is given by:

$$d(\boldsymbol{x},\boldsymbol{y})=\begin{cases}0, & \text{if } x=y\\ 1, & \text{if } x\ne y\end{cases}$$

Exercise: check the metric space defined above are well-defined. ■

R Adopting the infinite norm discussed in Example (1.7), we can define a metric on $\mathcal{C}[a,b]$ by

$$d_\infty(f,g)=\|f-g\|_\infty:=\max_{x\in[a,b]}|f(x)-g(x)|$$

which is the correct metric to study the uniform convergence for $\{f_n\} \subseteq \mathcal{C}[a,b]$.

Definition 1.6 Let (X,d) be a metric space. An **open ball** centered at $\boldsymbol{x} \in X$ of radius r is the set

$$B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in X \mid d(\boldsymbol{x},\boldsymbol{y}) < r\}.$$

■ **Example 1.9** 1. For $X = \mathbb{R}^2$, we can draw the $B_1(\mathbf{0})$ with respect to the metrics d_1, d_2:

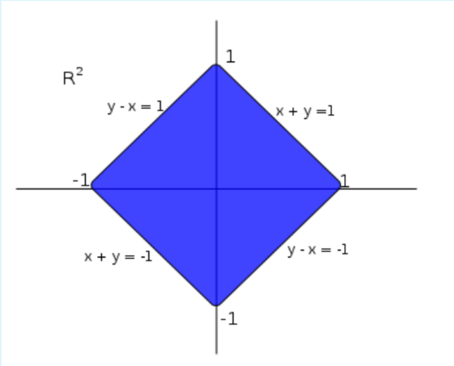


Figure 1.1: $B_1(\mathbf{0})$ w.r.t. the metric d_1

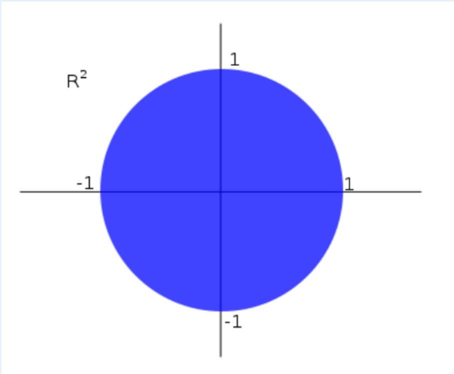
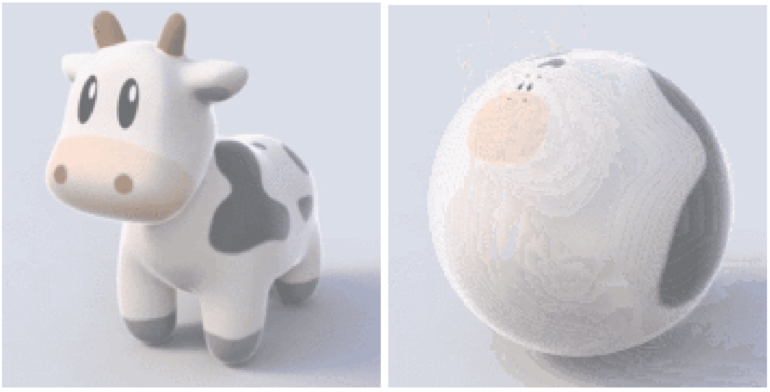


Figure 1.2: $B_1(\mathbf{0})$ w.r.t. the metric d_2

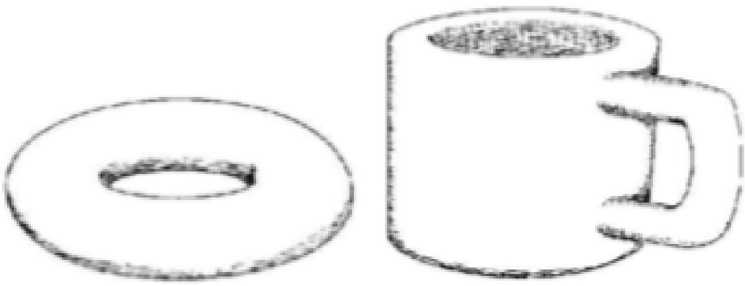
1.3. Monday for MAT4002

1.3.1. Introduction to Topology

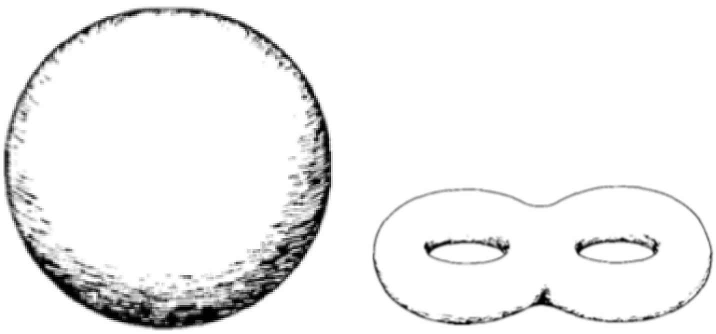
We will study global properties of a geometric object, i.e., *the distrance between 2 points in an object is totally ignored*. For example, the objects shown below are essentially invariant under a certain kind of transformation:



Another example is that the coffee cup and the donut have the same topology:



However, the two objects below have the intrinsically different topologies:



In this course, we will study the phenomenon described above mathematically.

1.3.2. Metric Spaces

In order to ingnore about the distances, we need to learn about distances first.

Definition 1.7 [Metric Space] Metric space is a set X where one can measure distance between any two objects in X.

Specifically speaking, a metric space X is a non-empty set endowed with a function (distance function) $d : X \times X \to \mathbb{R}$ such that

1. $d(\boldsymbol{x},\boldsymbol{y}) \geq 0$ for $\forall \boldsymbol{x},\boldsymbol{y} \in X$ with equality iff $\boldsymbol{x} = \boldsymbol{y}$
2. $d(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{y},\boldsymbol{x})$
3. $d(\boldsymbol{x},\boldsymbol{z}) \leq d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z})$ (triangular inequality)

■

■ **Example 1.10**

1. Let $X = \mathbb{R}^n$, with

$$d_2(\boldsymbol{x},\boldsymbol{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

$$d_\infty(\boldsymbol{x},\boldsymbol{y}) = \max_{i=1,\dots,n} |x_i - y_i|$$

2. Let X be any set, and define the discrete metric

$$d(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Homework: Show that (1) and (2) defines a metric. ■

Definition 1.8 [Open Ball] An **open ball** of radius r centered at $\boldsymbol{x} \in X$ is the set

$$B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in X \mid d(\boldsymbol{x},\boldsymbol{y}) < r\}$$

■

■ **Example 1.11** 1. The set $B_1(0,0)$ defines an open ball under the metric $(X = \mathbb{R}^2, d_2)$, or the metric $(X = \mathbb{R}^2, d_\infty)$. The corresponding diagram is shown below:

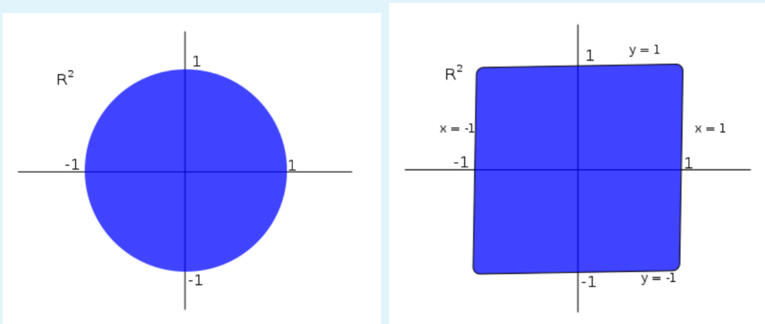


Figure 1.3: Left: under the metric $(X = \mathbb{R}^2, d_2)$; Right: under the metric $(X = \mathbb{R}^2, d_\infty)$

2. Under the metric $(X = \mathbb{R}^2, \text{discrete metric})$, the set $B_1(0,0)$ is one single point, also defines an open ball.

■

Definition 1.9 [Open Set] Let X be a metric space, $U \subseteq X$ is an open set in X if $\forall u \in U$, there exists $\epsilon_u > 0$ such that $B_{\epsilon_u}(u) \subseteq U$. ■

Definition 1.10 The **topology** induced from (X, d) is the collection of all open sets in (X, d), denoted as the symbol $\mathcal{T}$. ■

Proposition 1.5 All open balls $B_r(\boldsymbol{x})$ are open in (X, d).

Proof. Consider the example $X = \mathbb{R}$ with metric d_2. Therefore $B_r(x) = (x - r, x + r)$. Take $\boldsymbol{y} \in B_r(\boldsymbol{x})$ such that $d(\boldsymbol{x}, \boldsymbol{y}) = q < r$ and consider $B_{(r-q)/2}(\boldsymbol{y})$: for all $z \in B_{(r-q)/2}(\boldsymbol{y})$, we have

$$d(\boldsymbol{x}, \boldsymbol{z}) \leq d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z}) < q + \frac{r - q}{2} < r,$$

which implies $\boldsymbol{z} \in B_r(x)$. ■

Proposition 1.6 Let $(X, \boldsymbol{d})$ be a metric space, and $\mathcal{T}$ is the topology induced from (X, d), then

1. let the set $\{G_\alpha \mid \alpha \in \mathcal{A}\}$ be a collection of (uncountable) open sets, i.e., $G_\alpha \in \mathcal{T}$,

then $\bigcup_{\alpha\in\mathcal{A}} G_\alpha \in \mathcal{T}$.

2. let $G_1,\dots,G_n \in \mathcal{T}$, then $\bigcap_{i=1}^n G_i \in \mathcal{T}$. The finite intersection of open sets is open.

Proof. 1. Take $x \in \bigcup_{\alpha\in\mathcal{A}} G_\alpha$, then $x \in G_\beta$ for some $\beta \in \mathcal{A}$. Since G_β is open, there exists $\epsilon_x > 0$ s.t.

$$B_{\epsilon_x}(x) \subseteq G_\beta \subseteq \bigcup_{\alpha\in\mathcal{A}} G_\alpha$$

2. Take $x \in \bigcap_{i=1}^n G_i$, i.e., $x \in G_i$ for $i = 1,\dots,n$, i.e., there exists $\epsilon_i > 0$ such that $B_{\epsilon_i}(x) \subseteq G_i$ for $i = 1,\dots,n$. Take $\epsilon = \min\{\epsilon_1,\dots,\epsilon_n\}$, which implies

$$B_\epsilon(x) \subseteq B_{\epsilon_i}(x) \subseteq G_i, \forall i$$

which implies $B_\epsilon(x) \subseteq \bigcap_{i=1}^n G_i$

■

Exercise.

1. let $\mathcal{T}_2, \mathcal{T}_\infty$ be topologies induced from the metrices d_2, d_∞ in $\mathbb{R}^2$. Show that $J_2 = J_\infty$, i.e., every open set in $(\mathbb{R}^2, d_2)$ is open in $(\mathbb{R}^2, d_\infty)$, and every open set in $(\mathbb{R}^2, d_\infty)$ is open in $(\mathbb{R}_2, d_2)$.

   Proof. The key is to show the equivalence of metric d_2 and d_∞.
   Take $\forall G \in \mathcal{T}_2$, and for each point $\boldsymbol{x} \in G$, there exists $\epsilon_{\boldsymbol{x}}$ such that with respect to d_2,

   $$B_{\epsilon_{\boldsymbol{x}}}(\boldsymbol{x}) \subseteq G \Longleftrightarrow d_2(\boldsymbol{x},\boldsymbol{y}) < \epsilon_{\boldsymbol{x}} \text{ implies } \boldsymbol{y} \in G.$$

   Note that $d_\infty(\boldsymbol{x},\boldsymbol{y}) < \sqrt{2}\epsilon_{\boldsymbol{x}}$ implies $d_2(\boldsymbol{x},\boldsymbol{y}) < \epsilon_{\boldsymbol{x}}$, i.e., $\boldsymbol{y} \in G$. In other words, G is open w.r.t. the metric d_∞. The converse follows similarly. ■

2. Let $\mathcal{T}$ be the topology induced from the discrete metric (X, d_{discrete}). What is $\mathcal{T}$?

   Proof. For any $\boldsymbol{x} \in X$, consider the open ball $B_r(\boldsymbol{x})$. For $r \le 1$, the open ball is the singleton $\{\boldsymbol{x}\}$; for $r > 1$, the open ball is the whole space. Note that any element A in $\mathcal{T}$ is represented as:

   $$A = \bigcup_{\boldsymbol{x}\in A} \{\boldsymbol{x}\}.$$

Therefore, the topology is generating set with generator signleton sets. ■