Solution to Assignment 7

I will appreciate it if you could give me some advice on my assignment!

December 26, 2018

1. The Lagrangian function and its derivatives are given by:

$$L(x,\mu) = \frac{1}{2}\left[-0.1(x_1-4)^2 + x_2^2 + \mu(1-x_1^2-x_2^2)\right] \tag{1a}$$

$$\nabla_x L(x,\mu) = \begin{pmatrix} -0.1(x_1-4) \\ x_2 \end{pmatrix} - \mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1b}$$

$$\nabla_{xx}^2 L(x,\mu) = \begin{pmatrix} -(0.1+\mu) & 0 \\ 0 & 1-\mu \end{pmatrix} \tag{1c}$$

The KKT conditions are given by:

$$\begin{cases} \nabla_x L(x,\mu) = 0 \\ \mu(1-x_1^2-x_2^2) = 0 \\ \mu \geq 0 \\ x_1^2+x_2^2 \geq 1 \end{cases} \iff \begin{cases} x_1 = \dfrac{4}{1+10\mu} \\ (1-\mu)x_2 = 0 \\ \mu(1-x_1^2-x_2^2) = 0 \\ \mu \geq 0 \\ x_1^2+x_2^2 \geq 1 \end{cases}$$

(a) When $\mu = 0$, we imply the KKT-point-Lagrangian-multiplier pair

$$(\boldsymbol{x}*, \mu^*) = (4, 0, 0).$$

Since the Hessian at this point is

$$\boldsymbol{H} = \begin{pmatrix} -0.1 & 0 \\ 0 & 1 \end{pmatrix},$$

Since $\boldsymbol{x}^*$ is inactive, the active set $V(\boldsymbol{x}^*)$ is the whole space $\mathbb{R}^2$. Also note that this Hessian matrix is *indefinite* on $\mathbb{R}^2$, we imply this KKT point is a saddle point.

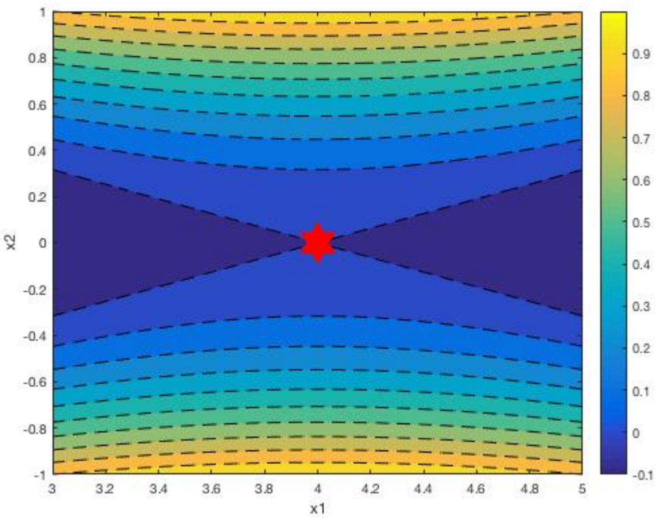

Figure 1: Contour plot near KKT point $(4, 0)$

Interpretation from Fig (1) We can see that the KKT point $(4, 0)$ is strictly inside the constraint, and we can pick a feasible direction that has more/less than 90 degree angle between the gradient direction, which implies that in some feasible direction $(4, 0)$ is local maximum, and in some feasible direction $(4, 0)$ is a local minimum. Thus $(4, 0)$ is a *saddle point*.

(b) When $\mu = 1$, we imply two KKT-point-Lagrangian-multiplier pairs:

$$(\boldsymbol{x}^*, \mu^*) = \left(\frac{4}{11}, \pm\sqrt{1 - (\frac{4}{11})^2}, 1 \right)$$

The Hessian at these two points are negative semi-definite:

$$\nabla^2_{xx} L(x, 1) = \begin{pmatrix} -1.1 & 0 \\ 0 & 0 \end{pmatrix},$$

and it can be psd only in subspace $\mathcal{O} = \{\boldsymbol{p} \mid \boldsymbol{p} = (0, \alpha), \alpha \in \mathbb{R}\}$. Since in our case it is clear that $V(x^*) \bigcap \mathcal{O}^c \neq \emptyset$, we conclude that these two KKT points cannot be local minimum.

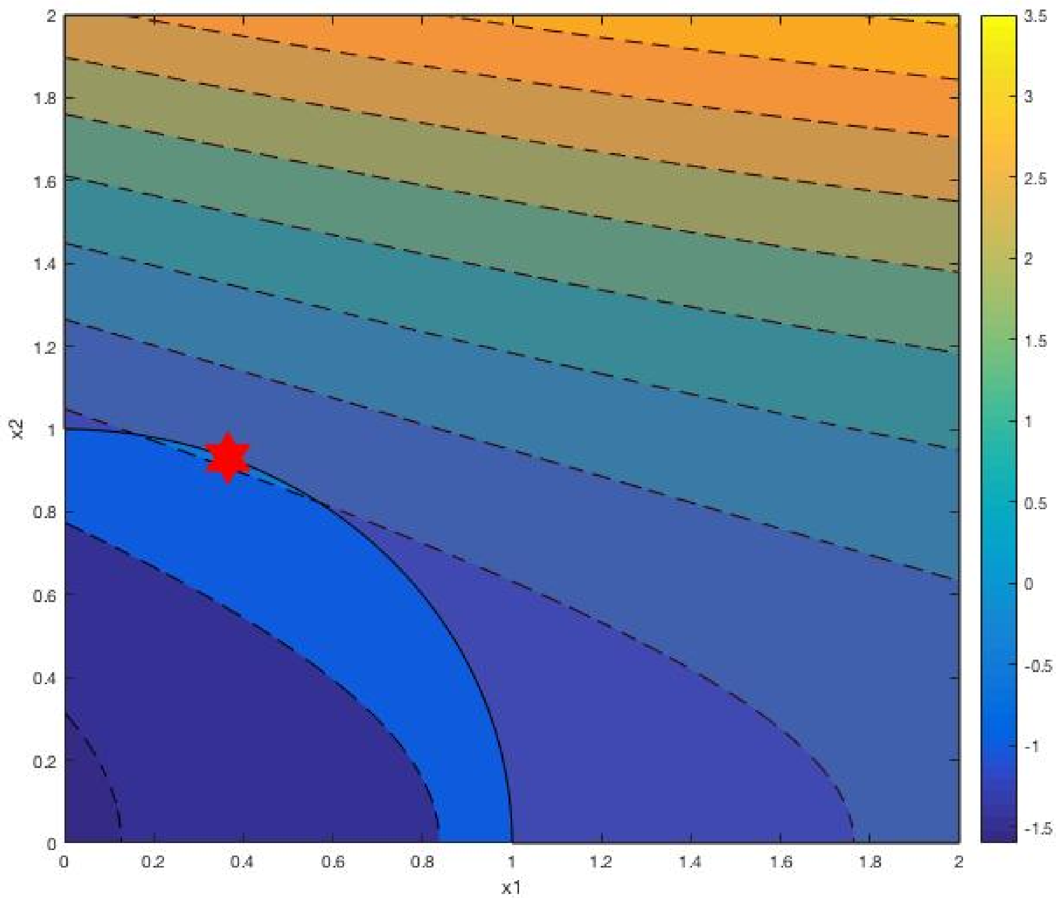

Figure 2: Contour plot near KKT point $(4/11, \sqrt{1-(4/11)^2})$

Interpretation from Fig (2) We can see that the KKT point $(4/11, \sqrt{1-(4/11)^2})$ is on the boundary of the constraint, and we can pick a feasible direction that has more/less than 90 degree angle between the gradient direction, which implies that in some feasible direction $(4/11, \sqrt{1-(4/11)^2})$ is local maximum, and in some feasible direction $(4/11, \sqrt{1-(4/11)^2})$ is a local minimum. Thus $(4/11, \sqrt{1-(4/11)^2})$ is a *saddle point*.

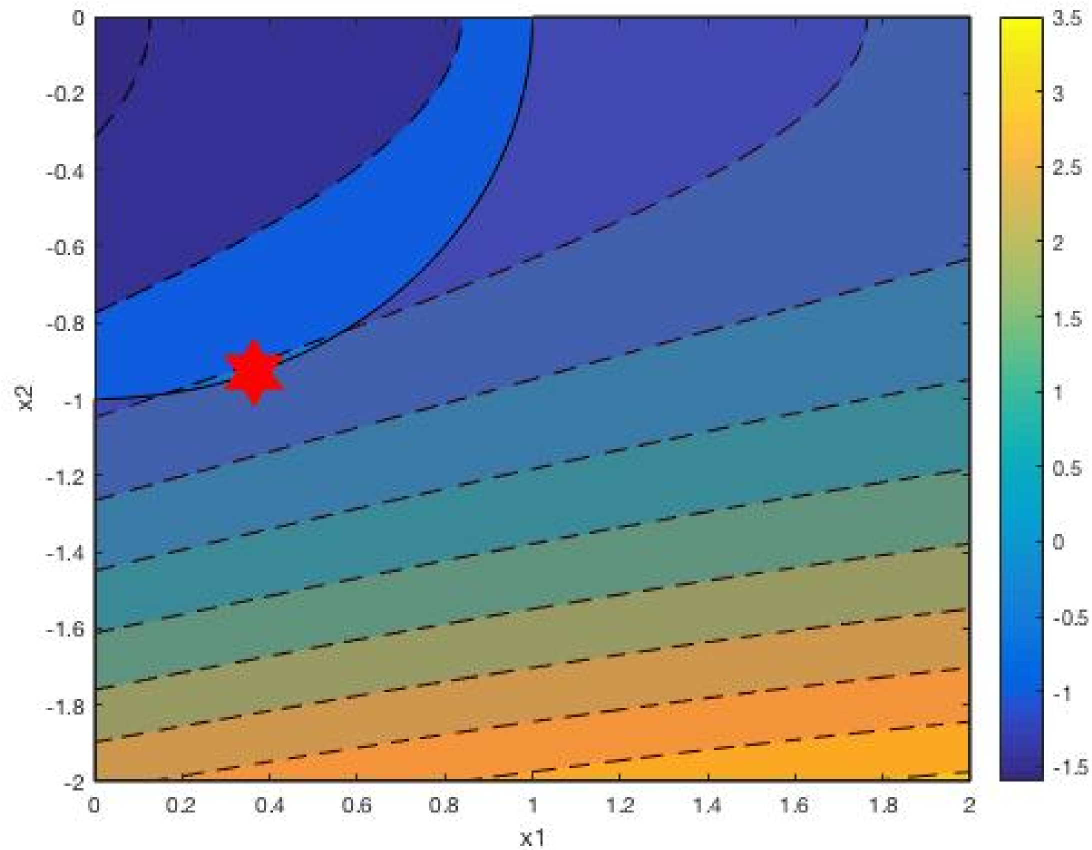

Figure 3: Contour plot near KKT point $(4/11, -\sqrt{1-(4/11)^2})$

Interpretation from Fig (3) We can see that the KKT point $(4/11, -\sqrt{1-(4/11)^2})$ is on the boundary of the constraint, and we can pick a feasible direction that has more/less than 90 degree angle between the gradient direction, which implies that in some feasible direction $(4/11, -\sqrt{1-(4/11)^2})$ is local maximum, and in some feasible direction $(4/11, -\sqrt{1-(4/11)^2})$ is a local minimum. Thus $(4/11, -\sqrt{1-(4/11)^2})$ is a *saddle point*.

(c) When $\mu \neq 1$ and $\mu > 0$, we imply the KKT-point-Lagrangian-multiplier pair

$$(\boldsymbol{x}*, \mu^*) = (1, 0, 0.3).$$

The Hessian at this point is given by

$$\nabla^2_{xx} L(x, 1) = \begin{pmatrix} -0.4 & 0 \\ 0 & 0.7 \end{pmatrix}.$$

The set of points at which $\langle \nabla h(\boldsymbol{x}^*), \boldsymbol{y} \rangle = 0$ is given by $V(\boldsymbol{x}^*) = \{\boldsymbol{y} = (0, y_1) \mid y_1 \in \mathbb{R}\}$. By Lagrangian multiplier theorem, since $\nabla^2_{xx} L(x, 1)$ is positive definite at $V(x^*)$, we imply that this point must be a local minimum.

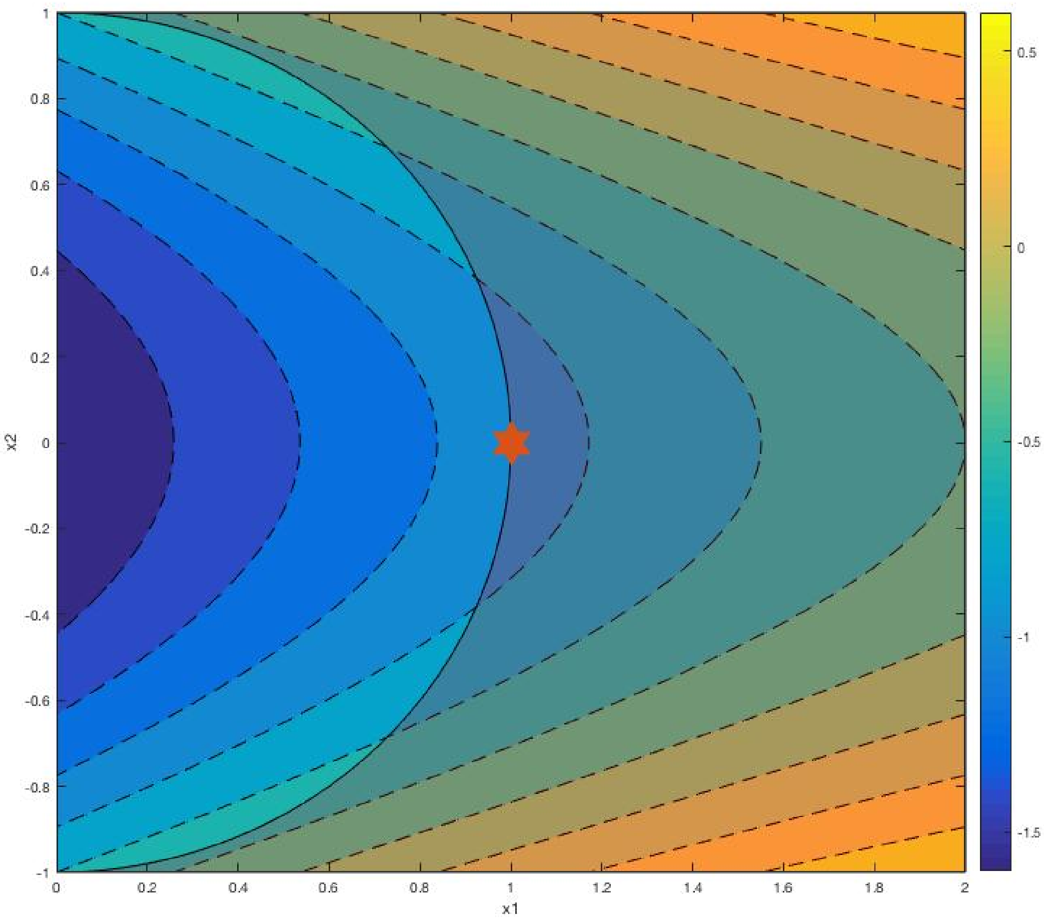

Figure 4: Contour plot near KKT point $(1, 0)$

Interpretation from Fig (4) We can see that the KKT point $(1, 0)$ is on the boundary of the constraint, and the *gradient* direction at this point is $(1, 0)$. From the figure we can also see that the feasible direction at this point should has the *positive x-axis component*, which implies that any feasible direction is a ascent direction, i.e., $(1, 0)$ is a *local minimum*

In summary, there are four KKT points with multipliers given below:

$$\begin{aligned} (\boldsymbol{x}_1, \mu_1) &= (4, 0, 0), & (\boldsymbol{x}_2, \mu_2) &= (\tfrac{4}{11}, \sqrt{1 - (\tfrac{4}{11})^2}, 1), \\ (\boldsymbol{x}_3, \mu_3) &= (\tfrac{4}{11}, -\sqrt{1 - (\tfrac{4}{11})^2}, 1), & (\boldsymbol{x}_4, \mu_4) &= (1, 0, 0.3) \end{aligned}$$

There is only one local minimum point, that is $(\boldsymbol{x}^*, \mu^*) = (1, 0, 0.3)$.

The code is appendixed in the next page.

```
%% contour plot
clear
x1 = 4; x2 = 0;
%x1 = 4/11; x2 = sqrt(1-x1^2);
%x1 = 4/11; x2 = -sqrt(1-x1^2);
%x1 = 1; x2 = 0;
x = 3:0.001:5; y = -1:0.001:1;
%x = 0:0.001:2; y = 0:0.001:2;
%x = 0:0.001:2; y = 0:-0.001:-2;
%x = x1-1:0.001:x1+1; y = x2-1:0.001:x2+1;
[X,Y] = meshgrid(x,y);
Z = -0.1 * (X - 4).^2 + Y.^2;
figure
contourf(X,Y,Z,'--');
colorbar;
xlabel('x1')
ylabel('x2')
hold on

n = 1000;
t = linspace(0,pi/2,n);
%t = linspace(0,pi/2,n);
%t = linspace(-pi/2,0,n);
%t = linspace(-pi/2,pi/2,n);
xc1 = cos(t);
yc1 = sin(t);
fill([xc1 0 2 2],[yc1 2 2 0],[1 0 0],'facealpha',0.2)
%fill([xc1 0 2 2],[yc1 2 2 0],[1 0 0],'facealpha',0.2)
%fill([xc1 2 2 0],[yc1 0 -2 -2],[1 0 0],'facealpha',0.2)
%fill([xc1 2 2 0],[yc1 1 -1 -1],[1 0 0],'facealpha',0.2)
scatter(x1,x2,1000,'r','hexagram','field')
%scatter(x1,x2,1000,'r','hexagram','field')
%scatter(x1,x2,1000,'r','hexagram','field')
%scatter(1,0,1000,'hexagram','field')
xlim([3,5]) ylim([-1,1])
%xlim([0,2]) ylim([0,2])
%xlim([0,2]) ylim([-2,0])
%xlim([0,2]) ylim([-1,1])
```

2. The original problem is equivalent to

$$-\min \quad \begin{array}{l} -y^{\mathrm{T}}x \\ x^{\mathrm{T}}Qx \leq 1 \end{array} \tag{2}$$

The Lagrangian function for (2) is given by:

$$L(x,y,\mu) = -y^{\mathrm{T}}x + \mu(x^{\mathrm{T}}Qx - 1)$$

The KKT necessary condition is given by:

$$\begin{cases} \nabla L(x,y,\mu) = 0 \\ \mu \geq 0 \\ \mu(x^{\mathrm{T}}Qx - 1) = 0 \\ x^{\mathrm{T}}Qx \leq 1 \end{cases} \Longleftrightarrow \begin{cases} -y + 2\mu Qx = 0 \\ \mu \geq 0 \\ \mu(x^{\mathrm{T}}Qx - 1) = 0 \\ x^{\mathrm{T}}Qx \leq 1 \end{cases}$$

(a) When $\mu = 0$, we imply from the first condition that the only possibility is $y = 0$. Therefore the optimal value is clearly 0, which can be re-written as $\sqrt{y^{\mathrm{T}}Q^{-1}y}$.

(b) When $\mu \neq 0$, we imply from the first condition that

$$x = \frac{1}{2\mu}Q^{-1}y. \tag{3}$$

Substituting (3) into $x^{\mathrm{T}}Qx = 1$ (by complementarity condition), we derive

$$\mu^2 = \frac{1}{4}y^{\mathrm{T}}Q^{-1}y \implies \mu = \frac{1}{2}\sqrt{y^{\mathrm{T}}Q^{-1}y},$$

which is strictly positive because of the second condition and the positive definiteness of Q. Therefore the optimal solution should be

$$x = \frac{1}{\sqrt{y^{\mathrm{T}}Q^{-1}y}}Q^{-1}y,$$

and therefore the optimal value is given by:

$$y^{\mathrm{T}}x = \frac{y^{\mathrm{T}}Q^{-1}y}{\sqrt{y^{\mathrm{T}}Q^{-1}y}} = \sqrt{y^{\mathrm{T}}Q^{-1}y}$$

To derive the inequality, for the case $x = 0$, the inequality is clearly holds since $LHS = 0 = RHS$. For $x \neq 0$, define

$$s := \frac{x}{\sqrt{x^{\mathrm{T}}Qx}}$$

which follows that $s^{\mathrm{T}}Qs = \frac{x^{\mathrm{T}}Qx}{(\sqrt{x^{\mathrm{T}}Qx})^2} = 1$. Thus consider the maximization problem

$$\begin{array}{ll} \max & y^{\mathrm{T}}s \\ & s^{\mathrm{T}}Qs = 1 \end{array} \tag{4}$$

Note that (2) is a relaxation of problem (4), which implies the optimal value for (4) should be no more than that for (2), i.e.,

$$\max y^{\mathrm{T}}s \leq \sqrt{y^{\mathrm{T}}Q^{-1}y} \implies \frac{y^{\mathrm{T}}x}{\sqrt{x^{\mathrm{T}}Qx}} \leq \sqrt{y^{\mathrm{T}}Q^{-1}y}$$

By rearranging terms we derive:

$$(y^{\mathrm{T}}x)^2 \leq (y^{\mathrm{T}}Q^{-1}y) \cdot (x^{\mathrm{T}}Qx)$$