
A FIRST COURSE IN ANALYSIS

MAT2006 Notebook

Lecturer

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Notations and Conventions

\mathbb{R}^n	n -dimensional real space
\mathbb{C}^n	n -dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
x_i	i th entry of column vector \mathbf{x}
a_{ij}	(i, j) th entry of matrix \mathbf{A}
\mathbf{a}_i	i th column of matrix \mathbf{A}
\mathbf{a}_i^T	i th row of matrix \mathbf{A}
\mathbb{S}^n	set of all $n \times n$ real symmetric matrices, i.e., $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all i, j
\mathbb{H}^n	set of all $n \times n$ complex Hermitian matrices, i.e., $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
\mathbf{A}^T	transpose of \mathbf{A} , i.e, $\mathbf{B} = \mathbf{A}^T$ means $b_{ji} = a_{ij}$ for all i, j
\mathbf{A}^H	Hermitian transpose of \mathbf{A} , i.e, $\mathbf{B} = \mathbf{A}^H$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\text{trace}(\mathbf{A})$	sum of diagonal entries of square matrix \mathbf{A}
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
\mathbf{e}_i	a unit vector with the nonzero element at the i th entry
$\mathcal{C}(\mathbf{A})$	the column space of \mathbf{A}
$\mathcal{R}(\mathbf{A})$	the row space of \mathbf{A}
$\mathcal{N}(\mathbf{A})$	the null space of \mathbf{A}
$\text{Proj}_{\mathcal{M}}(\mathbf{A})$	the projection of \mathbf{A} onto the set \mathcal{M}

Chapter 2

Week2

2.1. Wednesday

2.1.1. Review and Announcement

The quiz results will not be posted.

In this lecture we study the number theories.

The office hour is 2 - 4pm, TC606 on Wednesday

2.1.2. Irrational Number Analysis

Definition 2.1 [Algebraic Number] A number $x \in \mathbb{R}$ is said to be an **algebraic number** if it satisfies the following equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad (2.1)$$

where a_n, a_{n-1}, \dots, a_0 are integers and not all zero. We say x is **of degree n** if $a_n \neq 0$ and x is not the root of any polynomial with lower degree. ■

Definition 2.2 A number $x \in \mathbb{R}$ is **transcendental** if it is not an algebraic number. ■

The first example is that all rational numbers are algebraic, since rational number $\frac{p}{q}$ satisfies $qx - p = 0$. Also, $\sqrt{2}$ is algebraic. We leave an exercise: show that e and π are all transcendental. In history Joseph Liouville (1844) have constructed the first transcendental number. Let's look at the insights of his construction in this lecture:

Proposition 2.1 The set of all algebraic numbers is countable.

Proof. 1. Let \mathcal{P}_n denote the set of all polynomials of degree n (Here we assert polynomials have all integer coefficients by default.), i.e.,

$$\mathcal{P}_n = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_j \in \mathbb{Z}\}$$

The set \mathcal{P}_n have the one-to-one onto mapping to the set $\{(a_n, a_{n-1}, \dots, a_0) \mid a_j \in \mathbb{Z}\} \subseteq \mathbb{Z}^{n+1}$, which implies \mathcal{P}_n is **countable**.

2. Let \mathcal{R}_n denote the set of all real roots of polynomials in \mathcal{P}_n . Since each polynomial of degree n has at most n real roots, the set \mathcal{R}_n is a countable union of finite sets, which is at most countable. It is easy to show \mathcal{R}_n is infinite, and thus countable.
3. Hence, we construct the set of all algebraic numbers $\bigcup_{n=1}^{\infty} \mathcal{R}_n$, which is countable since countably union of countable sets is also countable.

■

How fast to approximate rational numbers using rational numbers? How fast to approximate irrational numbers using rational numbers? How fast to approximate transcendental numbers using rational numbers? We need the definition for the rate of approximation first to answer these questions.

Definition 2.3 A real number ξ is **approximable by rational numbers to order n** if \exists a constant $K = K(\xi)$ such that the inequality

$$\left| \frac{p}{q} - \xi \right| \leq \frac{K}{q^n}$$

has **infinitely many** solutions $\frac{p}{q} \in \mathbb{Q}$ with $q > 0$ and p, q are integers without any common divisors.

■

Intuitively, a rational number is approximable by rational numbers. Now we study its rate of approximation by applying this definition.

■ **Example 2.1** Suppose a rational number is apprixmable to oder α (which is a parameter).

To calculate the value of α , it suffices to choose (p_k, q_k) such that

$$\left| \frac{p_k}{q_k} - \frac{p}{q} \right| \leq \frac{K}{q^\alpha}$$

Note that

$$\left| \frac{p_k}{q_k} - \frac{p}{q} \right| = \left| \frac{p_k q - p q_k}{q_k q} \right| \geq \frac{1}{q q_k} = \frac{1/q}{q_k},$$

- $\frac{p}{q}$ is approximable by rational numbers to order 1:

If we construct $(p_k, q_k) = (kp - 1, kq)$, it follows that

$$\left| \frac{p_k}{q_k} - \frac{p}{q} \right| = \frac{1}{kq} = \frac{1}{q_k^1}$$

- $\frac{p}{q}$ is approximable by rational numbers to order no higher than 1:

Otherwise suppose it is approximable to order $n > 1$. The inequality holds for infinitely many (p_k, q_k) :

$$\frac{1/q}{q_k} \leq \left| \frac{p_k}{q_k} - \frac{p}{q} \right| \leq \frac{K}{q_k^n} \implies \frac{1}{q} q_k^{n-1} \leq K \quad (2.2)$$

Since infinite (p_k, q_k) satisfy the inequality (2.2), we can choose a solution such that q_k is arbitrarily large, which falsify (2.2).

In summary, any rational number $\frac{p}{q}$ is approximable by rational numbers to order 1 and no higher than 1. ■

Liouville had shown that the transcendental number has the higher approxiamtion rate than rational and algebraic numbers, which is counter-intuitive. Let's review his process of proof:

Theorem 2.1 — Liouville, 1844. A real algebraic number ξ of degree n is not approximable by rational numbers to any order greater than n .

We can show some numbers is not algebraic, i.e., transcendental by applying this

theorem:

■ **Example 2.2** [1st Constructed Transcendental Number] Given a number

$$\xi := \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots,$$

we aim to show it is transcendental. Assume that it is an algebraic number of order n , then we construct the first n tails of ξ :

$$\xi_n = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{n!}}$$

It follows that

$$\begin{aligned} |\xi_n - \xi| &= \frac{1}{10^{(n+1)!}} + \frac{1}{10^{(n+2)!}} + \cdots \\ &= \frac{1}{10^{(n+1)!}} \left[1 + \frac{1}{10^{n+2}} + \frac{1}{10^{(n+2)(n+3)}} + \cdots \right] \\ &\leq \frac{1}{10^{(n+1)!}} \cdot 2 = \frac{2}{(10^{n!})^{n+1}} = \frac{2}{q^{n+1}} \end{aligned}$$

which implies $|\xi - \frac{p}{q}| \leq \frac{K}{q^{n+1}}$ has one solution ξ_n .

We can construct infinitely many solutions from this solution:

$$\xi_{n,1} = \xi_n + \frac{1}{10^{n+2}}, \quad \xi_{n,2} = \xi_n + \frac{1}{10^{(n+2)(n+3)}}, \quad \cdots,$$

Hence, this number is approximable by rational numbers to order $n + 1$, which contradicts the fact that it is an algebraic number of degree n . ■

Proof. Given an algebraic number ξ of degree n , there exists a polynomial whose roots contain ξ :

$$f(x) \equiv a_n x^n + \cdots + a_1 x + a_0 = 0.$$

We fix an interval around ξ , i.e, $I_\lambda = [\xi - \lambda, \xi + \lambda]$ (with $\lambda = \lambda(\xi) \in (0, 1)$) such that I_λ contains no other root of f except ξ .

Hence, the value of f at any rational number $\frac{p}{q}$ inside I_λ is given by:

$$\left| f\left(\frac{p}{q}\right) \right| = \left| a_n \frac{p^n}{q^n} + \cdots + a_1 \frac{p}{q} + a_0 \right| = \left| \frac{a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_0 q^n}{q^n} \right| \neq 0$$

Hence, $\left| f\left(\frac{p}{q}\right) \right| \geq \frac{1}{q^n}$, which implies that

$$\frac{1}{q^n} \leq \left| f\left(\frac{p}{q}\right) \right| \tag{2.3a}$$

$$= \left| f\left(\frac{p}{q}\right) - f(\xi) \right| \tag{2.3b}$$

$$\leq |f'(\eta)| \left| \xi - \frac{p}{q} \right| \tag{2.3c}$$

$$\leq M \left| \xi - \frac{p}{q} \right| \tag{2.3d}$$

with $M := \max_{\eta \in I_\lambda} f'(\eta)$. Note that from (2.3b) to (2.3c) is due to mean value theorem.

Or equivalently, $\left| \xi - \frac{p}{q} \right| \geq \frac{1/M}{q^n}$ applies for any rational number $\frac{p}{q}$ inside the interval I_λ .

- Verify by yourself that ξ is not approximable by rational numbers inside the interval I_λ to any order greater than n .
- For any rational number $\frac{p}{q} \notin I_\lambda$, we have

$$\left| \frac{p}{q} - \xi \right| \geq \lambda(\xi) \geq \frac{\lambda(\xi)}{q^n}$$

for $q \geq 1, n \geq 1$. It is obvious that ξ is not approximable by rational numbers outside the interval I_λ to any order greater than n .

The two cases above complete the proof. ■

It's hard to determine which order the transcendental number is approximable by rational numbers. However, we can assert that there is a "fast" approximation to transcendental numbers by applying continued fraction expansion.

Continued Fraction Expansion. Let x be irrational, then intuitively x could be represented as an infinite continued fraction as below:

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} \quad (2.4)$$

We denote the continued fraction (2.4) as $[a_0; a_1, a_2, \dots]$. Let's define the rigorous process of continued fraction expansion, i.e., how to find a_i :

- We set $a_0 = \lfloor x \rfloor$, which implies that

$$x := a_0 + \xi_0 = a_0 + \frac{1}{\frac{1}{\xi_0}} \quad \text{for } 0 < \xi_0 < 1.$$

- We set $a_1 = \lceil \frac{1}{\xi_0} \rceil$, which implies that

$$x := a_0 + \frac{1}{a_1 + \xi_1} = a_0 + \frac{1}{a_1 + \frac{1}{\frac{1}{\xi_1}}}$$

- After $n + 1$ steps we obtain the continued fraction of x :

$$[a_0; a_1, a_2, \dots, a_n + \xi_n]$$

We continue this process iteratively with

$$\frac{1}{\xi_n} = a_{n+1} + \xi_{n+1}$$

with $\xi_{n+1} \in (0, 1)$.

Such a process will continue without end as x is irrational. After $n + 1$ steps alternatively, we write

$$x = [a_0, a_1, \dots, a_n, a'_{n+1}] \quad \text{with } a'_{n+1} := a_{n+1} + \xi_{n+1}.$$

Observations from continued fraction expansion.

1. For $x = [a_0; a_1, \dots] \notin \mathbb{Q}$, consider its n th convergent term

$$\frac{p_n}{q_n} := [a_0, a_1, \dots, a_n], \quad n \geq 0$$

note that p_n and q_n can be computed iteratively:

$$\begin{array}{ll} p_0 = a & q_0 = 1 \\ p_1 = a_1 a_0 + 1 & q_1 = a_1 \\ \vdots & \vdots \\ p_n = a_n p_{n-1} + p_{n-2} & q_n = a_n q_{n-1} + q_{n-2} \end{array}$$

Note that (p_n, q_n) have no common divisors. (exercise)

Corollary 2.1 $q_n \geq n$ for $\forall n$.

Proof. Note that $q_{n-1} \leq q_n$ for $\forall n \geq 1$; and that $q_{n-1} < q_n$ for $\forall n > 1$. ■

2. From the first observation, $x := [a_0; a_1, \dots, a'_{n+1}]$ can be written as

$$x = \frac{p_{n+1}}{q_{n+1}} = \frac{a'_{n+1} p_n + p_{n-1}}{a'_{n+1} q_n + q_{n-1}}$$

Corollary 2.2 If $\frac{p_n}{q_n} (n \geq 0)$ is the n th convergent term of x , then

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$$

Proof. First note that for $k \geq 2$,

$$\begin{aligned} p_{k-1} q_k - p_k q_{k-1} &= p_{k-1} (a_k q_{k-1} + q_{k-2}) - (a_k p_{k-1} + p_{k-2}) q_{k-1} \\ &= -(p_{k-2} q_{k-1} - p_{k-1} q_{k-2}) \end{aligned}$$

After computation,

$$\begin{aligned}
\left| x - \frac{p_n}{q_n} \right| &= \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \\
&= \left| \frac{a'_{n+1}p_n + p_{n-1}}{a'_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n} \right| = \left| \frac{p_{n-1}q_n - p_nq_{n-1}}{q_n(a'_{n+1}q_n + q_{n-1})} \right| \\
&= \left| \frac{(-1)^n p_1 q_0 - p_0 q_1}{q_n(a'_{n+1}q_n + q_{n-1})} \right| = \frac{1}{q_n(a'_{n+1}q_n + q_{n-1})} \\
&< \frac{1}{q_n q_{n+1}}
\end{aligned}$$

■

Corollary 2.3 Furthermore, for the convergent term $\frac{p_n}{q_n} (n \geq 0)$ of x , we have

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}$$

3. The sequence $\{[a_0, a_1, \dots, a_n]\}$ is a Cauchy sequence. (Exercise)

In next lecture, we will apply Liouville Theorem to construct the first transcendental number and discuss the completeness of real numbers.

btw, π is apprixmable by rational number of order 42.