# A FIRST COURSE

# IN

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# MAT4500 Notebook

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# Contents

# Acknowledgments

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CUHK(SZ)

# Notations and Conventions

| | |
|---|---|
| $\mathbb{R}^n$ | $n$-dimensional real space |
| $\mathbb{C}^n$ | $n$-dimensional complex space |
| $\mathbb{R}^{m\times n}$ | set of all $m \times n$ real-valued matrices |
| $\mathbb{C}^{m\times n}$ | set of all $m \times n$ complex-valued matrices |
| $x_i$ | $i$th entry of column vector $\boldsymbol{x}$ |
| $a_{ij}$ | $(i, j)$th entry of matrix $\boldsymbol{A}$ |
| $\boldsymbol{a}_i$ | $i$th column of matrix $\boldsymbol{A}$ |
| $\boldsymbol{a}_i^{\mathrm{T}}$ | $i$th row of matrix $\boldsymbol{A}$ |
| $\mathbb{S}^n$ | set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all $i, j$ |
| $\mathbb{H}^n$ | set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all $i, j$ |
| $\boldsymbol{A}^{\mathrm{T}}$ | transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all $i, j$ |
| $\boldsymbol{A}^{\mathrm{H}}$ | Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all $i, j$ |
| $\mathrm{trace}(\boldsymbol{A})$ | sum of diagonal entries of square matrix $\boldsymbol{A}$ |
| $\mathbf{1}$ | A vector with all 1 entries |
| $\mathbf{0}$ | either a vector of all zeros, or a matrix of all zeros |
| $\boldsymbol{e}_i$ | a unit vector with the nonzero element at the $i$th entry |
| $\mathcal{C}(\boldsymbol{A})$ | the column space of $\boldsymbol{A}$ |
| $\mathcal{R}(\boldsymbol{A})$ | the row space of $\boldsymbol{A}$ |
| $\mathcal{N}(\boldsymbol{A})$ | the null space of $\boldsymbol{A}$ |
| $\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$ | the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$ |

# Chapter 1

# Week1

## 1.1. Tuesday

### 1.1.1. Analogs of deterministic differential equations

**Problem 1.** Consider the first order homogeneous ODE

$$\begin{cases} \dfrac{\mathrm{d}N(t)}{\mathrm{d}t} = a(t)N(t) \\ N(0) = N_0 \end{cases}$$

$N(t)$ is described as the **size** of population at time $t$; $a(t)$ is the given (deterministic) function describing the **rate** of growth of population at time $t$; $N_0$ is a given constant.

The question raises: What if $a(t)$ is no longer deterministic, instead $a(t)$ is subject to some random effect, e.g.,

$$a(t) = r(t) \cdot \text{noise}, \text{ or } r(t) + \text{noise},$$

where $r(t)$ is deterministic, and the "noise" term is something random. Then how to solve the corresponding differential equation?

**Problem 2.** Suppose $Q(t)$ describes the charge at time $t$ in an electricity circuit.

$$\begin{cases} LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = F(t), \\ Q(0) = Q_0, \quad Q'(0) = Q_0' \end{cases}$$

$L$ is described as the **inductance**, $R$ is the **resistance**, $C$ is the **capacity**, and $F(t)$ is the **potential source**.

The Question raises: what if $F(t)$ involves some randomness? e.g.,

$$F(t) = G(t) + \text{noise}$$

where $G(t)$ is deterministic. How to solve the problem?

R The differential equations with some coefficients involved randomness are called the stochastic differential equations. Clearly, solutions to SDEs should also involved "randomness".

### 1.1.2. Optimal Stopping

**Problem 3.** Suppose someone holds an asset (e.g., stock, house, etc.) He plans to sell it at some future time. Denote $X(t)$ to be the price of the asset at time $t$, satisfying

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = rX(t) + \alpha X(t) \cdot \text{noise}$$

where $r, \alpha$ are given constants. Our goal is to choose time $\tau$ to solve

$$\max_{\tau \geq 0} \mathbb{E}X(\tau)$$

where the optimal solution $\tau^*$ is the optimal stopping time.

### 1.1.3. Stochastic Control

**Problem 4 (Portfolio Selection).** Suppose a person wants to invest into i) a riskless/safe asset (e.g., bond); or ii) a risky asset (e.g, stock).

The price of the saft asset $X_0(t)$ satisfies

$$\frac{\mathrm{d}X_0(t)}{\mathrm{d}t} = \rho X_0(t),$$

where $\rho > 0$ is a given constant. Therefore, $X(t)$ is exponentially growing function.

The price of risky assrt $X_1(t)$ satisfies

$$\frac{\mathrm{d}X_1(t)}{\mathrm{d}t} = \mu X_1(t) + \sigma X_1(t) \cdot \text{noise}$$

where $\mu, \sigma > 0$ are the given constants.

Suppose $u(t)$ is the fraction of his wealth to be invested into the risky asset; the remaining $1 - u(t)$ part to be invested into the saft asset. The wealth at time $t$ is denoted to be $v(t)$. Suppose the person has the utility function $U(\cdot)$. The terminal time is $T$. The objective function is

$$\max_{u(t), 0 \leq t \leq T} \mathbb{E}[U(v^u(T))]$$

If we impose no-shot selling constant, we further require

$$0 \leq u(t) \leq 1, \forall t \in [0, T]$$

**Problem 5 (Option Pricing).** Suppose at time 0, a person in the long position in an European call option has the right to buy the asset at a specified price $K$ at some future time $t$. How much the person should pay to the short position for the option? We can model this problem by Black-Sholes Formula.