

**A JOURNEY
IN
PURE MATHEMATICS**

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PURE MATHEMATICS
MAT3006 & 3040 & 4002 Notebook

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Acknowledgments

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Notations and Conventions

\mathbb{R}^n	n -dimensional real space
\mathbb{C}^n	n -dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
x_i	i th entry of column vector \mathbf{x}
a_{ij}	(i, j) th entry of matrix \mathbf{A}
\mathbf{a}_i	i th column of matrix \mathbf{A}
\mathbf{a}_i^T	i th row of matrix \mathbf{A}
\mathbb{S}^n	set of all $n \times n$ real symmetric matrices, i.e., $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all i, j
\mathbb{H}^n	set of all $n \times n$ complex Hermitian matrices, i.e., $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
\mathbf{A}^T	transpose of \mathbf{A} , i.e, $\mathbf{B} = \mathbf{A}^T$ means $b_{ji} = a_{ij}$ for all i, j
\mathbf{A}^H	Hermitian transpose of \mathbf{A} , i.e, $\mathbf{B} = \mathbf{A}^H$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\text{trace}(\mathbf{A})$	sum of diagonal entries of square matrix \mathbf{A}
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
\mathbf{e}_i	a unit vector with the nonzero element at the i th entry
$\mathcal{C}(\mathbf{A})$	the column space of \mathbf{A}
$\mathcal{R}(\mathbf{A})$	the row space of \mathbf{A}
$\mathcal{N}(\mathbf{A})$	the null space of \mathbf{A}
$\text{Proj}_{\mathcal{M}}(\mathbf{A})$	the projection of \mathbf{A} onto the set \mathcal{M}

4.4. Wednesday for MAT3040

Reviewing.

- Quotient Space:

$$V/W = \{\mathbf{v} + W \mid \mathbf{v} \in V\}$$

The elements in V/W are cosets. Note that V/W does not mean a subset of V .

- Define the canonical projection mapping

$$\pi_W : V \rightarrow V/W,$$

$$\text{with } \mathbf{v} \mapsto \mathbf{v} + W,$$

then we imply π_W is a surjective linear transformation with $\ker(\pi_W) = W$.

If $\dim(V) < \infty$, then by Rank-Nullity Theorem (2.3), we imply that

$$\dim(V) = \dim(W) + \dim(V/W),$$

i.e., $\dim(V/W) = \dim(V) - \dim(W)$.

- **(Universal Property I)** Every linear transformation $T : V \rightarrow W$ with $V' \leq \ker(T)$ can be descended to the composition of the canonical projection mapping $\pi_{V'}$ and the mapping

$$\tilde{T} : V/V' \rightarrow W$$

$$\text{with } \mathbf{v} + V' \mapsto T(\mathbf{v}).$$

In other words, the diagram (2.1) commutes:

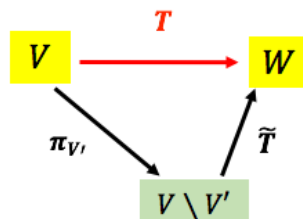


Diagram (2.1)

In other words, the mapping starting from either the black or red line gives the same result, i.e., $T(\mathbf{v}) = \tilde{T} \circ \pi_{V'}(\mathbf{v}) = \tilde{T}(\mathbf{v} + V')$ for any $\mathbf{v} \in V$.

- **(First Isomorphism Theorem)** Under the setting of Universal Property I (UPI), if T is a surjective linear transformation with $V' = \ker(T)$, then the \tilde{T} is an isomorphism.

■ **Example 4.2** Suppose that $U, W \leq V$ with $U \cap W = \{\mathbf{0}\}$, then define the mapping

$$\begin{aligned} \phi : U \oplus W &\rightarrow U \\ \text{with } \phi(\mathbf{u} + \mathbf{w}) &= \mathbf{u} \end{aligned}$$

Ⓡ Exercise: if $U, W \leq V$ but $U \cap W \neq \{\mathbf{0}\}$, then the mapping

$$\begin{aligned} \phi : U + W &\rightarrow U \\ \text{with } \mathbf{u} + \mathbf{w} &\mapsto \mathbf{u} \end{aligned} \quad \text{is not well-defined:}$$

Suppose that $\mathbf{0} \neq \mathbf{v} \in U \cap W$ and for any $\mathbf{u} \in U, \mathbf{w} \in W$, we construct

$$\mathbf{u}' = \mathbf{u} - \mathbf{v} \in U, \quad \mathbf{w}' = \mathbf{w} + \mathbf{v} \in W \implies \phi(\mathbf{u}' + \mathbf{w}') = \mathbf{u} - \mathbf{v}$$

Therefore we get $\mathbf{u} + \mathbf{w} = \mathbf{u}' + \mathbf{w}'$ but $\phi(\mathbf{u} + \mathbf{w}) \neq \phi(\mathbf{u}' + \mathbf{w}')$.

Back to the situation $U \cap W = \{\mathbf{0}\}$, then it's clear that $\phi : U \oplus W \rightarrow U$ is surjective linear transformation with $\ker(\phi) = W$. Therefore, construct the new mapping

$$\begin{aligned} \tilde{\phi} : U \oplus W / W &\rightarrow U \\ \text{with } \mathbf{u} + \mathbf{w} + W &\mapsto \phi(\mathbf{u} + \mathbf{w}) \end{aligned}$$

We imply $\tilde{\phi}$ is an isomorphism by First Isomorphism Theorem. ■

Now we study the generalized quotients, which is defined to satisfy the generalized version of universal property I.

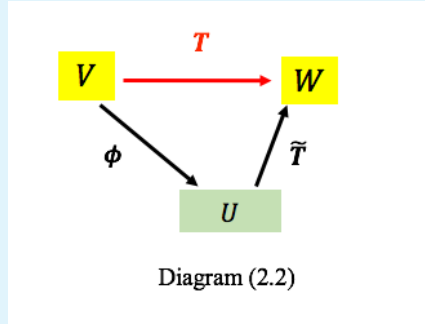
Definition 4.7 [Universal Property for Quotients] Let V be a vector space and $V' \leq V$. Consider the collection of linear transformations

$$\text{Obj} = \left\{ T : V \rightarrow W \left| \begin{array}{l} T \text{ is a linear transformation} \\ V' \leq \ker(T) \end{array} \right. \right\}$$

(For example, $\pi_{V'} : V \rightarrow V/V'$ is an element from the set Obj .)

An element $(\phi : V \rightarrow U) \in \text{Obj}$ is said to satisfy the **universal property** if it satisfies the following:

Given any element $(T : V \rightarrow W) \in \text{Obj}$, we can extend the transformation ϕ with a uniquely existing $\tilde{T} : U \rightarrow W$ so that the diagram (2.2) commutes:



Or equivalently, for given $(T : V \rightarrow W) \in \text{Obj}$, there exists the unique mapping $\tilde{T} : U \rightarrow W$ such that $T = \tilde{T} \circ \phi$.

Theorem 4.3 — Universal Property II.

1. The mapping $(\pi_{V'} : V \rightarrow V/V') \in \text{Obj}$ is a universal object, i.e., it satisfies the universal property.
2. If $(\phi : V \rightarrow U)$ is a universal object, then $U \cong V/V'$, i.e., there is intrinsically “one” element in the set of universal objects.

Proof. 1. Consider any linear transformation $T : V \rightarrow W$ such that $V' \leq \ker(T)$, then define (construct) the same $\tilde{T} : V/V' \rightarrow W$ as that in UPI. Therefore, for given T , applying the result of UPI, we imply $T = \tilde{T} \circ \pi_{V'}$, i.e., $\pi_{V'}$ satisfies the

diagram (2.2).

To show the uniqueness of \tilde{T} , suppose there exists $\tilde{S} : V/V' \rightarrow W$ such that the diagram (2.3) commutes.

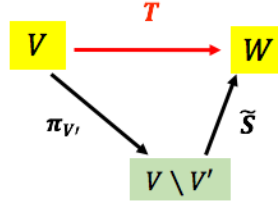


Diagram (2.3)

It suffices to show the mapping $\tilde{S} = \tilde{T}$: for any $\mathbf{v} + V' \in V/V'$, we have

$$\tilde{S}(\mathbf{v} + V') := \tilde{S} \circ \pi_{V'}(\mathbf{v}) = T(\mathbf{v}),$$

where the first equality is due to the surjectivity of $\pi_{V'}$. By the result of UPI, $T(\mathbf{v}) = \tilde{T}(\mathbf{v} + V')$. Therefore $\tilde{T}(\mathbf{v} + V') = \tilde{S}(\mathbf{v} + V')$ for all $\mathbf{v} + V' \in V/V'$. The proof is complete.

2. Suppose that $(\phi : V \rightarrow U)$ satisfies the universal property. In particular, the following two diagrams hold:

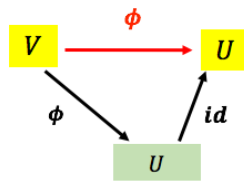


Diagram (2.4)

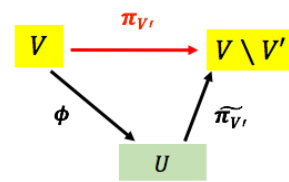


Diagram (2.5)

Since $(\pi_{V'})$ satisfies the universal property, in particular, the following two diagrams hold:

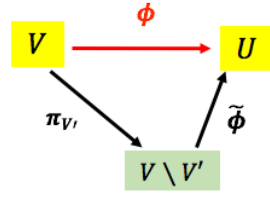


Diagram (2.6)

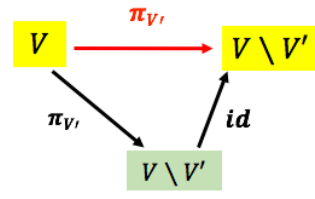


Diagram (2.7)

Then we claim that: Combining Diagram (2.5) and (2.6), we imply the diagram (2.8):

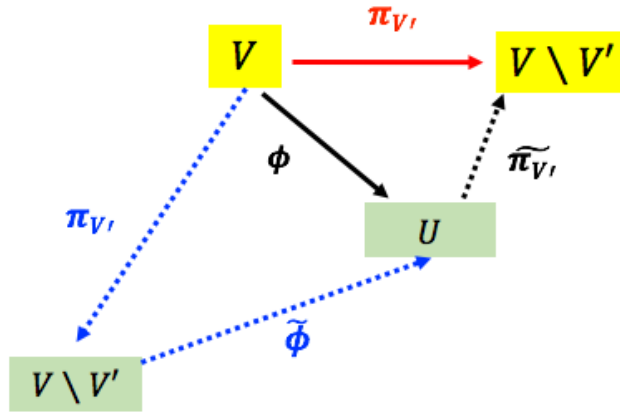


Diagram (2.8)

Graph Description: Note that this diagram commutes, i.e., the mapping starting from either the red line or the dash line gives the same result, i.e., $\pi_{V'} = \tilde{\pi}_{V'} \circ \tilde{\phi} \circ \pi_{V'}$, i.e., $\tilde{\pi}_{V'} \circ \tilde{\phi} = id$

Therefore, $\tilde{\pi}_{V'} \circ \tilde{\phi} = id$ implies $\tilde{\pi}_{V'}$ is surjective and $\tilde{\phi}$ is injective.

Also, combining Diagram (2.6) and (2.5), we imply diagram (2.9):

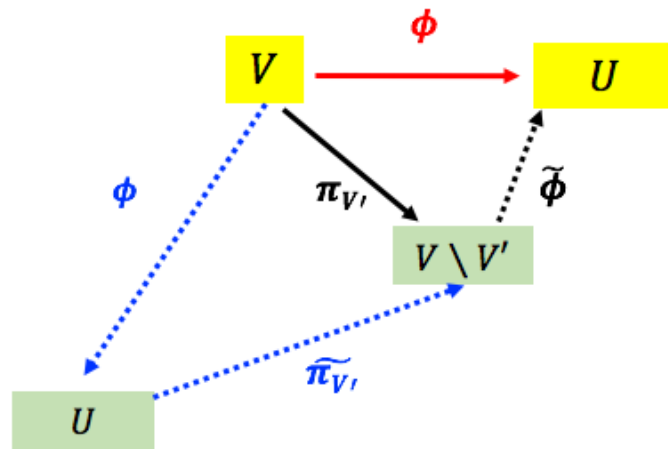


Diagram (2.9)

Graph Description: Note that this diagram commutes, i.e., the mapping starting from either the red line or the dash line gives the same result, i.e., $\phi = \tilde{\phi} \circ \tilde{\pi}_{V'} \circ \phi$, i.e., $\tilde{\phi} \circ \tilde{\pi}_{V'} = id$

Therefore, $\tilde{\phi} \circ \tilde{\pi}_{V'} = id$ implies $\tilde{\phi}$ is surjective and $\tilde{\pi}_{V'}$ is injective.

Therefore, both $\tilde{\phi} : U \rightarrow V/V'$ and $\tilde{\pi}_{V'} : V/V' \rightarrow U$ are bijective, i.e., $U \cong V/V'$.

The proof is complete. ■

4.4.1. Dual Space

Definition 4.8 Let V be a vector space over a field \mathbb{F} . The **dual vector space** V^* is defined as

$$\begin{aligned} V^* &= \text{Hom}_{\mathbb{F}}(V, \mathbb{F}) \\ &= \{f : V \rightarrow \mathbb{F} \mid f \text{ is a linear transformation}\} \end{aligned}$$

- **Example 4.3** 1. Consider $V = \mathbb{R}^n$ and define $\phi_i : V \rightarrow \mathbb{R}$ as the i -th component of input:

$$\phi_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i,$$

Then we imply $\phi_i \in V^*$. On the contrary, $\phi_i^2 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i^2$ is not in V^*

2. Consider $V = \mathbb{F}[x]$ and define $\phi : V \rightarrow \mathbb{F}$ as:

$$\phi(p(x)) = p(1),$$

It's clear that $\phi \in V^*$:

$$\begin{aligned} \phi(ap(x) + bq(x)) &= ap(1) + bq(1) \\ &= a\phi(p(x)) + b\phi(q(x)) \end{aligned}$$

3. Also, $\psi : V \rightarrow \mathbb{F}$ by $\psi(p(x)) = \int_0^1 p(x) dx$ is in V^* .
 4. Also, for $V = M_{n \times n}(\mathbb{F})$, the mapping $\text{tr} : V \rightarrow \mathbb{F}$ by $\text{tr}(M) = \sum_{i=1}^n M_{ii}$ is in V^* .
 However, the $\det : V \rightarrow \mathbb{F}$ is not in V^*

Definition 4.9 Let V be a vector space, with basis $B = \{v_i \mid i \in I\}$ (I can be finite or countable, or uncountable). Define

$$B^* = \{f_i : V \rightarrow \mathbb{F} \mid i \in I\},$$

where f_i 's are defined on the basis B :

$$f_i(v_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Then we extend f_i 's linearly, i.e., for $\sum_{j=1}^N \alpha_j v_j \in V$,

$$f_i\left(\sum_{j=1}^N \alpha_j v_j\right) = \sum_{j=1}^N \alpha_j f_i(v_j).$$

It's clear that $f_i \in V^*$ is well-defined. ■

Our question is that whether the B^* can be the basis of V^* ?

