A FIRST COURSE

IN

NUMERICAL ANALYSIS

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MATRIX4001 Notebook

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Contents

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Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all i, j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i, j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Wednesday

1.1.1. Introduction to Imaginary System

Definition 1.1 [Complex Number] A complex number z is a pair of real numbers:

$$z = (x, y),$$

where x is the **real** part and y is the **imaginary part** of z, denoted as

$$\text{Re}z = x \quad \text{Im}z = y$$

R Note that the complex multiplication does not correspond to any standard vector operation. However, $(\mathbb{C}, +)$ and $(\mathbb{C} \setminus \{0\}, \cdot)$ forms a field:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

$$z_1 + z_2 = z_2 + z_1$$

$$z + 0 = 0 + z = z$$

$$z + (-z) = (-z) + z = 0$$

There is no other Eucliean space that can form a field.

Proposition 1.1 $zz' = 0$ if and only if $z = 0$ or $z' = 0$.

Proof. Rewrite the product as a linear system

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and discuss the determinant of the coefficient matrix. ■

Solving quadratic equation with one unknown. We can apply the imaginary number to solve the quadratic equations. For example, to solve $z^2 - 2z + 2 = 0$, the first method is to substitute z with $x + iy$; the second method is to simplify it into standard form to solve it.

Definition 1.2 If $z \neq 0$, then z^{-1} is the complex number satisfying $z \cdot z^{-1} = 1$.

Suppose $z = (x,y)$ and $z^{-1} = (u,v)$. After simplification, we derive

$$\begin{cases} xu - yv = 1 \\ xv + yu = 0 \end{cases} \Longrightarrow \begin{cases} u = \dfrac{x}{x^2 + y^2} \\ v = \dfrac{-y}{x^2 + y^2} \end{cases}$$

Definition 1.3 [Division] The division between complex numbers is defined as:

$$\tfrac{z_1}{z_2} = z_1 \cdot z_2^{-1}, \quad \text{when } z_2 \neq 0$$

Example 1.1

$$\frac{3-4i}{1+i} = (3-4i)\left(\frac{1}{2} - \frac{1}{2}i\right) = -\frac{1}{2} - \frac{7}{2}i$$

$$\frac{10}{(1+i)(2+i)(3+i)} = \frac{10}{(1+3i)(3+i)} = \frac{10}{10i} = \frac{1}{i} = -i$$

Definition 1.4 [Complex Conjugate] The complex number $x - iy$ is called the **complex conjugate** of $z = x + iy$, which is denoted by $\bar{z}$.

The following properties hold for complex conjugate:

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2. \quad \overline{\frac{z_1}{z_2}} = \frac{\bar{z}_1}{\bar{z}_2}$$

$$\text{Re}z = \frac{z+\bar{z}}{2}, \quad \text{Im}z = \frac{z-\bar{z}}{2i}$$

1.1.2. Algebraic and geometric properties

Definition 1.5 [Algebraic Region]

1. The complex plane: the z-plane, i.e., $\mathbb{C}$
2. Vector in $\mathbb{R}^2$: $(x, y) = x + iy = z \in \mathbb{C}$
3. Modulus of z:

$$|z| = \sqrt{x^2 + y^2} \quad \text{distance to the origin}$$

Note that

$$|z| = 0 \Longleftrightarrow z = 0, \quad |z_1 - z_2| = 0 \Longleftrightarrow z_1 = z_2$$

Definition 1.6 [Circle in plane] A circle with center z_0 and radius R is defined as follows in $\mathbb{C}$:

$$\{z \in \mathbb{C} \mid |z - z_0| = R\}$$

Proposition 1.2 Complex roots of polynomials with real coefficients appear in conjugate pairs.

Proof. Given $P(z_0) = 0$, we derive

$$P(z_0) = \overline{P(z_0)} = 0.$$

■

Note that a polynomial with real coefficients of degree 3 must have at least one real root.

Conjugate Product. Note that the conjugate product leads to the square of modulus:

$$z \cdot \bar{z} = |z|^2 \iff (x + iy)(x - iy) = x^2 + y^2$$

Such a property can be used to simplify quotient of two complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2} = \frac{x_1 x_2 + y_1 y_2 + (y_1 x_2 - x_1 y_2)i}{x_2^2 + y_2^2}$$

■ **Example 1.2**

$$\frac{-1 + 3i}{2 - i} = \frac{(-1 + 3i)(2 + i)}{(2 - i)(2 + i)} = \frac{-5 + 5i}{5} = -1 + i$$

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

■

We can use conjugate to show the **triangle inequality**:

Proposition 1.3 — Triangle Inequality. $|z_1 + z_2| \le |z_1| + |z_2|$.

Proof.

$$
\begin{aligned}
|z_1+z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)} \\
&= |z_1|^2+|z_2|^2+z_1\bar{z}_2+\overline{z_1\bar{z}_2} \\
&= |z_1|^2+|z_2|^2+2\mathrm{Re}(z_1\bar{z}_2) \\
&\le |z_1|^2+|z_2|^2+2|z_1\bar{z}_2| \\
&= |z_1|^2+|z_2|^2+2|z_1z_2| = (|z_1|+|z_2|)^2.
\end{aligned}
$$

■

Corollary 1.1 1. $||z_1|-|z_2|| \le |z_1 \pm z_2|$.

2. If $|z| \le 1$, then $|z^2+z+1| \le 3$

Proof. 1. Note that

$$|z_1| = |z_1 \pm z_2 \mp z_2| \le |z_1 \pm z_2| + |z_2| \implies |z_1|-|z_2| \le |z_1 \pm z_2|$$

Similarly, $|z_2|-|z_1| \le |z_1 \pm z_2|$.

2.

$$|z^2+z+1| \le |z^2|+|z+1| \le |z|^2+|z|+1 \le 1+1+1 = 3.$$

■

Proposition 1.4 — **Cauchy-Schwarz inequality.** If $z_1,\ldots,z_n$ and $w_1,\ldots,w_n$ are complex numbers, then

$$\left[\sum_{k=1}^{n} z_k w_k\right]^2 \le \left[\sum_{k=1}^{n} |z_k|^2\right]\left[\sum_{k=1}^{n} |w_k|^2\right]$$

1.1.3. Polar and exponential forms

Definition 1.7 [Polar Form] The polar form of a nonzero complex number z is:

$$z = r(\cos\theta + i\sin\theta)$$

where (r,θ) is the polar coordinates of (x,y).

$$(r,\theta) \implies (x,y): \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$(x,y) \implies (r,\theta): \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

Note that θ is said to be the **argument** of z, i.e., $\theta = \arg z$. The augument is not unique, i.e.,

$$z = r(\cos\theta + i\sin\theta) r(\cos(\theta+2\pi) + i\sin(\theta+2\pi))$$

If given an argument of z, then we form the set of arguments of z:

$$\{\theta + 2n\pi \mid n \in \mathbb{Z}\}$$

Definition 1.8 [Principal Value] The principal value of $\arg z$, denoted by $\mathrm{Arg}z$, is the unique value of $\arg z$ such that $-\pi < \arg z \leq \pi$

Actually, $(\mathbb{C},+)$ forms a group:

Also, $(\mathbb{C}\setminus\{0\},\cdot)$ forms a group.

The product for imaginary numbers is different from vector product:

$$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2$$

Also, we can define the crossover product $\vec{v}\times\vec{w}$.

modulus:

$$|z| = \sqrt{x^2+y^2}$$

direction angle (Argument):

$$\tan\theta = \frac{y}{x}$$

Using the polar coordination, we find $z = x + iy$ can be transformed into

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= r\left[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)\right] \end{aligned}$$

Principal argument:

$$-\pi < \mathrm{Arg} z \leq \pi$$

Conjugate form of imaginary number:

$$\bar{z} = x - iy$$

$$\begin{aligned} \overline{z + w} &= \bar{z} + \bar{w} \\ z \cdot \bar{z} &= |z|^2 \\ \frac{1}{z} &= \frac{\bar{z}}{|z|^2} \end{aligned}$$

Proposition 1.5 $|z + w| \leq |z| + |w|$

Proposition 1.6 $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$.

Proposition 1.7

$$\Re z = \frac{z + \bar{z}}{2}, \quad \Im z = \tfrac{z - \bar{z}}{2i}$$