

**A JOURNEY
IN
PURE MATHEMATICS**

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MAT3006 & 3040 & 4002 Notebook

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Notations and Conventions

\mathbb{R}^n	n -dimensional real space
\mathbb{C}^n	n -dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
x_i	i th entry of column vector \mathbf{x}
a_{ij}	(i, j) th entry of matrix \mathbf{A}
\mathbf{a}_i	i th column of matrix \mathbf{A}
\mathbf{a}_i^T	i th row of matrix \mathbf{A}
\mathbb{S}^n	set of all $n \times n$ real symmetric matrices, i.e., $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all i, j
\mathbb{H}^n	set of all $n \times n$ complex Hermitian matrices, i.e., $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
\mathbf{A}^T	transpose of \mathbf{A} , i.e, $\mathbf{B} = \mathbf{A}^T$ means $b_{ji} = a_{ij}$ for all i, j
\mathbf{A}^H	Hermitian transpose of \mathbf{A} , i.e, $\mathbf{B} = \mathbf{A}^H$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\text{trace}(\mathbf{A})$	sum of diagonal entries of square matrix \mathbf{A}
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
\mathbf{e}_i	a unit vector with the nonzero element at the i th entry
$\mathcal{C}(\mathbf{A})$	the column space of \mathbf{A}
$\mathcal{R}(\mathbf{A})$	the row space of \mathbf{A}
$\mathcal{N}(\mathbf{A})$	the null space of \mathbf{A}
$\text{Proj}_{\mathcal{M}}(\mathbf{A})$	the projection of \mathbf{A} onto the set \mathcal{M}

Chapter 3

Week3

3.1. Monday for MAT3040

Reviewing.


1. Complementation. Suppose $\dim(V) = n < \infty$, then $W \leq V$ implies $\exists W'$ such that

$$W \oplus W' = V.$$

2. Given the linear transformation $T : V \rightarrow W$, define the set $\ker(T)$ and $\text{Im}(T)$
3. Isomorphism of vector spaces:

$$T : V \cong W$$

4. Rank-Nullity Theorem

 On isomorphism T on vector spaces,

1. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent in V if and only if $\{T\mathbf{v}_1, \dots, T\mathbf{v}_k\}$ is linearly independent.

Question 8 in Homework 1: if $T : V \rightarrow W$ is injective, then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent in V implies $\{T\mathbf{v}_1, \dots, T\mathbf{v}_k\}$ is linearly independent.

2. The same goes if we replace the linearly independence by spans.
3. If $\dim(V) = n$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ forms a basis of V if and only if $\{T\mathbf{v}_1, \dots, T\mathbf{v}_n\}$ forms a basis of W . In particular, $\dim(V) = \dim(W)$.

Actually, if $\dim(V) = \dim(W) = n$, then $V \cong W$.

3.1.1. Change of Basis and Matrix Representation

Definition 3.1 Let V be a finite dimensional vector space and $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ an ordered basis of V . The **coordinate vector** of a vector $\mathbf{v} \in V$ is given by:

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n \implies \mapsto [\mathbf{v}]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Note that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \neq \{\mathbf{v}_2, \mathbf{v}_1, \dots, \mathbf{v}_n\}$ w.r.t. ordered basis.

■ **Example 3.1** Given $V = M_{2 \times 2}(\mathbb{F})$ and

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

any matrix has the coordinate vector w.r.t. B :

$$\left[\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \right]_B = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 3 \end{pmatrix}$$

However, given another ordered basis

$$B = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

any matrix has the coordinate vector w.r.t. B_1 :

$$\left[\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \right]_{B_1} = \begin{pmatrix} 4 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

Theorem 3.1 The mapping

$$[\]_B : V \mapsto \mathbb{F}^n$$

is an isomorphism of vector spaces.

Proof. 1. The operator $[\]_B$ is well-defined: suppose

$$[\mathbf{v}]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \quad [\mathbf{v}]_B = \begin{pmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{pmatrix},$$

then

$$\begin{aligned} \mathbf{v} &= \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n \\ &= \alpha'_1 \mathbf{v}_1 + \cdots + \alpha'_n \mathbf{v}_n \end{aligned}$$

By the uniqueness of coordinates, we imply $\alpha_i = \alpha'_i$ for $i = 1, \dots, n$.

2. The operator $[\]_B$ is a linear transformation, i.e.,

$$[p\mathbf{v} + q\mathbf{w}]_B = p[\mathbf{v}]_B + q[\mathbf{w}]_B, \quad p, q \in \mathbb{F}$$

3. The operator $[\]_B$ is surjective: suppose

$$[\mathbf{v}]_B = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{F}^n,$$

then $\mathbf{v} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n = \mathbf{0}$.

4. The injective is clear.

Therefore, $[\]_B$ is an isomorphism. ■

Exercise: if $\dim(V) = \dim(W)$, and $T : V \rightarrow W$ is injective, then $V \cong W$.

■ **Example 3.2** For $V = P_3[x]$ and its basis $B = \{1, x, x^2, x^3\}$.

To check if the set $\{1 + x^2, 3 - x^3, x - x^3\}$ is linearly independent, it suffices to check the corresponding coordinate vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

is linearly independent (Do Gaussian Elimination and check the number of pivots). ■

■ **Example 3.3** Questions: if B_1, B_2 form two basis of V , then how are $[\mathbf{v}]_{B_1}, [\mathbf{v}]_{B_2}$ related to each other.

Suppose $V = \mathbb{R}^n$ and $B_1 = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. For any $\mathbf{v} \in V$,

$$\mathbf{v} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1 \mathbf{e}_1 + \cdots + \alpha_n \mathbf{e}_n \implies [\mathbf{v}]_{B_1} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Also, it is clear that the B_2 forms a basis as well:

$$B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

which implies

$$[\mathbf{v}]_{B_2} = \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_2 - \alpha_3 \\ \vdots \\ \alpha_{n-1} - \alpha_n \\ \alpha_n \end{pmatrix}$$

Proposition 3.1 — Change of Basis. Let $A = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $A' = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be two basis of a vector space V . Suppose $\mathbf{v}_j = \sum_{i=1}^n \alpha_{ij} \mathbf{w}_i$ for $j = 1, \dots, n$. Then the **change of basis matrix**

$$\mathcal{C}_{A',A} = \left(\alpha_{ij} \right)_{i,j=1,\dots,n}$$

satisfies the following:

$$\mathcal{C}_{A',A} [\mathbf{v}]_A = [\mathbf{v}]_{A'} \quad (3.1)$$

Also, the matrix $\mathcal{C}_{A',A}$ is invertible with the inverse

$$(\mathcal{C}_{A',A})^{-1} = \mathcal{C}_{A,A'}$$

where $\mathcal{C}_{A,A'} = (\beta_{ij})$, with β_{ij} satisfying

$$\mathbf{w}_j = \sum_{i=1}^n \beta_{ij} \mathbf{v}_i$$

Proof. Consider $\mathbf{v} = \mathbf{v}_j$, then LHS of (3.1) is

$$(\alpha_{ij})\mathbf{e}_j = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix}$$

the RHS of (3.1) is

$$[\mathbf{v}_j]_{A'} = \left[\sum_{i=1}^n \alpha_i \mathbf{w}_i \right]_{A'} = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix} = \text{LHS}$$

Therefore, $\mathcal{C}_{A',A}[\mathbf{v}_j]_A = [\mathbf{v}_j]_{A'}$ for $\forall j = 1, \dots, n$.

Then for all $\mathbf{v} = r_1 \mathbf{v}_1 + \dots + r_n \mathbf{v}_n$,

$$\begin{aligned} \mathcal{C}_{A',A}[\mathbf{v}]_A &= \mathcal{C}_{A',A}[r_1 \mathbf{v}_1 + \dots + r_n \mathbf{v}_n]_A \\ &= \mathcal{C}_{A',A}[r_1 [\mathbf{v}_1]_A + \dots + r_n [\mathbf{v}_n]_A] \\ &= \sum_{j=1}^n r_j \mathcal{C}_{A',A}[\mathbf{v}_j]_A \\ &= \sum_{j=1}^n r_j [\mathbf{v}_j]_{A'} \\ &= \left[\sum_{j=1}^n r_j \mathbf{v}_j \right]_{A'} \\ &= (\mathbf{v})_{A'} \end{aligned}$$

Now, suppose

$$\begin{aligned} \mathbf{v}_j &= \sum_{i=1}^n \alpha_{ij} \mathbf{w}_i \\ &= \sum_{i=1}^n \alpha_{ij} \sum_{k=1}^n \beta_{ki} \mathbf{v}_k \\ &= \sum_{k=1}^n \left(\sum_{i=1}^n \beta_{ki} \alpha_{ij} \right) \mathbf{v}_i \end{aligned}$$

By the uniqueness of coordinates, we imply

$$\left(\sum_{i=1}^n \beta_{ki} \alpha_{ij} \right) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

where

$$\left(\sum_{i=1}^n \beta_{ki} \alpha_{ij} \right) = (\mathcal{C}_{AA'} \mathcal{C}_{A'A})$$

Therefore, $(\mathcal{C}_{AA'} \mathcal{C}_{A'A}) = \mathbf{I}_n$. ■

■ **Example 3.4** Back to Example (3.3), suppose

$$B_1 = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}, \quad B_2 = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$$

and suppose $\mathbf{w}_i = \mathbf{e}_1 + \dots + \mathbf{e}_i$. Therefore,

$$\mathcal{C}_{B_1, B_2} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$\mathcal{C}_{B_1, B_2}[\mathbf{v}]_{B_2} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 - \alpha_2 \\ \vdots \\ \alpha_{n-1} - \alpha_n \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = [\mathbf{v}]_{B_2}$$

Definition 3.2 Let $T : V \rightarrow W$ be a linear transformation, and

$$\mathcal{A} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}, \quad \mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$$

be bases of V and W , respectively. The **matrix representation** of T with respect to

(w.r.t.) \mathcal{A} and \mathcal{B} is given by:

$$T(\mathbf{v}_j) = \sum_{i=1}^m \alpha_{ij} \mathbf{w}_i \implies (T)_{\mathcal{B}\mathcal{A}} = (\alpha_{ij})_{i,j=1,\dots,m},$$

where $(T)_{\mathcal{B}\mathcal{A}} \in M_{m \times m}(\mathbb{F})$. ■