A JOURNEY

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MAT3006 & 3040 & 4002 Notebook

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Contents

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Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Monday for MAT3040

1.1.1. Introduction to Advanced Linear Algebra

Advanced Linear Algebra is one of the most important course in MATH major, with pre-request MAT2040. This course will offer the really linear algebra knowledge.

What the content will be covered?.

- In MAT2040 we have studied the space $\mathbb{R}^n$; while in MAT3040 we will study the vector space V.
- In MAT2040 we have studied the *linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$, i.e., left-multiplying some matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$; while in MAT3040 we will study the linear transformation from vector spaces to vector spaces: $T : V \to W$
- In MAT2040 we have studied the eigenvalues of $n \times n$ matrix $\boldsymbol{A}$; while in MAT3040 we will study the eigenvalues of a **linear operator** $T : V \to V$.
- In MAT2040 we have studied the dot product $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$; while in MAT3040 we will study the **inner product** $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle$.

Why do we do the generalization?. We are studying many other spaces, e.g., $\mathcal{C}(\mathbb{R})$ is called the space of all functions on $\mathbb{R}$, $\mathcal{C}^\infty(\mathbb{R})$ is called the space of all infinitely differentiable functions on $\mathbb{R}$, $\mathbb{R}[x]$ is the space of polynomials of one-variable.

For example, the Laplace equation $\nabla^2 f = 0$:

$$\nabla^2 : C^\infty(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3), \qquad f \mapsto (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})f$$

The solution of $\nabla^2 f = 0$ is the 0-eigenspace of ∇^2.

Consider the Schriodiageis equation

$$\hat{H} : C^\infty(\mathbb{R}^3) \to \mathbb{R}^3, \quad f \mapsto \left(-\tfrac{\hbar^2}{2\mu}\nabla + V(x,y,z)\right) f$$

we aim to solve the equation $\hat{H}f = Ef$, where E denotes the energy. It suffices to find the eigenvectors of $\hat{H}$.

In fact, the eigenvalues of $\hat{H}$ are **discrete**.

1.1.2. Vector Spaces

Definition 1.1 [Vector Space] A **vector space** over a field $\mathbb{F}$ (in particular, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is a set of objects V such that

1. they can be added subject to the rules:
   (a) **Commutativity**: $\forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, $\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}_2 + \boldsymbol{v}_1$.
   (b) **Associativity**: $\boldsymbol{v}_1 + (\boldsymbol{v}_2 + \boldsymbol{v}_3) = (\boldsymbol{v}_1 + \boldsymbol{v}_2) + \boldsymbol{v}_3$.
   (c) **Addictive Identity**: $\exists \mathbf{0} \in V$ such that $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$, $\forall \boldsymbol{v} \in V$.
2. **scalar multiplication** satisfying
   (a) **Distributive**: $\alpha(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \alpha\boldsymbol{v}_1 + \alpha\boldsymbol{v}_2, \forall \alpha \in \mathbb{F}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$
   (b) **Distributive**: $(\alpha_1 + \alpha_2)\boldsymbol{v} = \alpha_1\boldsymbol{v} + \alpha_2\boldsymbol{v}$
   (c) $0\boldsymbol{v} = \mathbf{0}$, $1\boldsymbol{v} = \boldsymbol{v}$.

■ **Example 1.1** For $V = \mathbb{F}^n$, we can define

- $$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

- $$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

- $$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

■

■ Example 1.2 1. It is clear that the set $V = M_{n\times n}(\mathbb{F})$ (the space of all $m \times n$ matrices) is a vector space aswell.

2. The set $V = \mathcal{C}(\mathbb{R})$ is a vector space:
   - For $\forall f, g \in V$, $f + g$ is defined by $(f+g)(x) = f(x) + g(x)$
   - For $\alpha \in \mathbb{R}$, αf is defined by $(\alpha f)(x) = \alpha f(x)$. In this case, $\mathbf{0}$ is a zero function, i.e., $\mathbf{0}(x) = 0$ for all $\boldsymbol{x} \in \mathbb{R}$.

■

Definition 1.2 A sub-collection $W \subseteq V$ of a vector space V is called a **vector subspace of** V if W itself forms a vector space. We use the notation $W \leq V$. ■

■ Example 1.3 1. For $V = \mathbb{R}^3$, we claim that $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \leq V$

2. $W = \{(x, y, 1) \mid x, y \in \mathbb{R}\}$ is not the vector subspace of V.

■

Proposition 1.1 $W \subseteq V$ is a **vector subspace** of V iff for $\forall \boldsymbol{w}_1 + \boldsymbol{w}_2 \in W$, we have

$\alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2 \in W$, for $\forall \alpha, \beta \in \mathbb{F}$.

■ **Example 1.4**

- For $V = M_{n\times n}(\mathbb{F})$, define $W = \{A \in V \mid \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}\} \leq V$
- For $V = \mathcal{C}^{\infty}(\mathbb{R})$, define $W = \{f \in V \mid \frac{\mathrm{d}^2}{\mathrm{d}x^2} f + f = 0\} \leq V$. For $f, g \in W$, we have

$$(\alpha f + \beta g)'' = \alpha f'' + \beta g'' = \alpha(-f) + \beta(-g) = -(\alpha f + \beta g),$$

which implies $(\alpha f + \beta g)'' + (\alpha f + \beta g) = 0$.

■

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 4 & 5 & 14 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right].$$

1.2. Monday for MAT3006

1.2.1. Overview on uniform convergence

Definition 1.3 [Convergence] Let $f_n(x)$ be a sequence of functions on an integral $I = [a,b]$. Then $f_n(x)$ converges **pointwise** to $f(x)$ (i.e., $f_n(x_0) \to f(x_0)$) for $\forall x_0 \in I$, if

$$\forall \varepsilon > 0, \exists N_{x_0,\varepsilon} \text{ such that } |f_n(x_0) - f(x_0)| < \varepsilon, \forall n \ge N_{x_0,\varepsilon}$$

We say $f_n(x)$ converges uniformly to $f(x)$, (i.e., $f_n(x) \to\to f(x)$) for $\forall x_0 \in I$, if

$$\forall \varepsilon > 0, \exists N_\varepsilon \text{ such that } |f_n(x_0) - f(x_0)| < \varepsilon, \forall n \ge N_\varepsilon$$

■

■ **Example 1.5** It is clear that the function $f_n(x) = \frac{n}{1+nx}$ converges pointwisely into $f(x) = \frac{1}{x}$ on $[0,\infty)$, and it is uniformly convergent on $[1,\infty)$ ■

Proposition 1.2 If $f_n(x)$ is continuous on I, $\forall n$, and $f_n(x) \to\to f(x)$. Then

1. $f(x)$ is continuous on I.
2. $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$.
3. Suppose furthermore that $f_n(x)$ is **continuously differentiable**, and $f_n'(x) \to\to g(x)$, then $f(x)$ is differentiable, and $f_n'(x) \to f'(x)$.

Proposition 1.3 Putting the discussions above into the content of series, i.e., $f_n(x) = \sum_{k=1}^n S_k(x)$. If $S_k(x)$ is continuous for $\forall k$, and $f_n(x) \to\to f(x) := \sum_{k=1}^\infty S_k(x)$, then

1. $f(x) = \sum_{k=1}^\infty S_k(x)$ is continuous,
2. $\sum_{k=1}^\infty \int_a^b S_k(x)\,\mathrm{d}x = \int_a^b \sum_{k=1}^\infty S_k(x)\,\mathrm{d}x$
3. blabla, then
$$\left(\sum_{k=1}^\infty S_k(x)\right)' = \sum_{k=1}^\infty S_k'(x)$$

Proposition 1.4 For power series, i.e., $S_k(x) = a_k x^k$. Suppose $f(x) = \sum_{k=1}^\infty a_k x^k$ has radius of convergence R, then

$$\sum_{k=1}^{n} a_k x^k \to\to f(x)$$

for any $[-L, L]$ with $L < R$. Then $f(x)$ is continuous, and

$$\int_0^x f(t)\,\mathrm{d}t = \sum_{k=1}^{\infty} \frac{a_k}{k+1} x^{k+1}$$

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

1.2.2. Introduction to MAT3006

What are we going to do.

1. (a) Generalize our study of (sequence, series, functions) on $\mathbb{R}^n$ into a metric space.
   (b) We will study spaces outside $\mathbb{R}^n$.

   Remark:

   - For (a), different metric may yield different kind of convergence of sequences. For (b), one important example we will study is $X = \mathcal{C}[a,b]$ (all continuous functions defined on $[a,b]$.) We will generalize X into $\mathcal{C}_b(E)$, which means the set of bounded continuous functions defined on $E \subseteq \mathbb{R}^n$.
   - The insights of analysis is to find a **unified** theory to study sequences/series on a metric space X, e.g., $X = \mathbb{R}^n, \mathcal{C}[a,b]$. In particular, for $\mathcal{C}[a,b]$, we will see that
     - most functions in $\mathcal{C}[a,b]$ are nowhere differentiable. (repeat part of content in MAT2006)
     - We will prove the existence and uniqueness of ODEs.
     - the set poly$[a,b]$ (the set of polynomials on $[a,b]$) is dense in $\mathcal{C}[a,b]$. (analogy: $\mathbb{Q} \subseteq \mathbb{R}$ is dense)
2. Introduction to the Lebesgue Integration.

For convergence of integration $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)$, we need the pre-conditions (a)$f_n(x)$ is continuous, and (b) $f_n(x) \to\to f(x)$. The natural question is that Can we relax the condition to

(a) $f_n(x)$ is integrable?

(b) $f_n(x) \to f(x)$ pointwisely?

The answer is yes, by using the tool of Lebesgue integration. If $f_n(x) \to f(x)$ and $f_n(x)$ is Lebesgue integrable, then $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$, which is so called the **dominated convergence**.

1.2.3. Metric Spaces

Normed Space. We will study the distance and length of elements in an arbitrary set X.

Definition 1.4 [Normed Space] Let X be a vector space. A **norm** on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that

1. $\|\boldsymbol{x}\| \geq 0$ for $\forall \boldsymbol{x} \in X$, with equality iff $\boldsymbol{x} = \boldsymbol{0}$
2. $\|\alpha \boldsymbol{x}\| = |\alpha| \|\boldsymbol{x}\|$, for $\forall \alpha \in \mathbb{R}$ and $\boldsymbol{x} \in X$.
3. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ (triangular inequality)

Any vector space equipped with $\|\cdot\|$ is called a **normed space**.

Example 1.6

1. For $X = \mathbb{R}^n$, define

$$\|\boldsymbol{x}\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2} \quad \text{(Euclidean Norm)}$$

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \quad (p\text{-norm})$$

2. For $X = \mathcal{C}[a,b]$, define

$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$$

$$\|f\|_p = \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p}$$

Exercise: check the norm defined above are well-defined.

Here raise a question: can we define the distance in an arbitrary set?

Definition 1.5 A set X is a **metric space** with metric (X,d) if there exists a (distance) function $d : X \times X \to \mathbb{R}$ such that

1. $d(\boldsymbol{x},\boldsymbol{y}) \geq 0$ for $\forall \boldsymbol{x},\boldsymbol{y} \in X$, with equality iff $\boldsymbol{x} = \boldsymbol{y}$.
2. $d(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{y},\boldsymbol{x})$.
3. $d(\boldsymbol{x},\boldsymbol{z}) \leq d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z})$.

Example 1.7

1. If X is a normed space, then define $d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$, which is so called the metric induced from the norm $\|\cdot\|$.
2. Let X be any (non-empty) set with $\boldsymbol{x},\boldsymbol{y} \in X$, the discrete metric is given by:

$$d(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Exercise: check the metric space defined above are well-defined.

So we have defined a metric on $\mathcal{C}[a,b]$ by

$$d_\infty(f,g) = \|f - g\|_\infty := \max_{x\in[a,b]} |f(x) - g(x)|$$

This is the correct metric to study the uniform convergence for $\{f_n\} \subseteq \mathcal{C}[a,b]$.

Definition 1.6 Let (X,d) be a metric space. An **open ball** centered at $\boldsymbol{x} \in X$ of radius r is

$$B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in X \mid d(\boldsymbol{x},\boldsymbol{y}) < r\}.$$

■ **Example 1.8** 1. $X = \mathbb{R}^2$, and $d_2(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_2$. Graph:

$d_1(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_1$. Graph: