Proposition 7.6 Consider the problem (7.2), $x^* \in X$ is stationary point if and only if

$$x^* = [x^* - \alpha \nabla f(x^*)]^+. \quad \forall \alpha > 0$$

Proof.

- For the forward direction, the stationarity of x^* is equivalent to

$$\langle \nabla f(x^*), (x - x^*) \rangle \geq 0, \quad \forall x \in X \tag{7.5}$$

Let $z := x^* - \alpha \nabla f(x^*)$, and consider

$$\min_x \|x - z\|^2 = \|x - x^* + \alpha \nabla f(x^*)\|^2 \tag{7.6a}$$

$$= \|x - x^*\|^2 + 2\alpha \langle \nabla f(x^*), (x - x^*) \rangle + \alpha^2 \|\nabla f(x^*)\|^2 \tag{7.6b}$$

$$\geq \alpha^2 \|\nabla f(x^*)\|^2, \tag{7.6c}$$

which implies the minimum point for (7.6) is x^*, i.e., $x^* = [z]^+$.

- For the reverse direction, assume x^* is not the stationary point, i.e., $\exists x^0 \in X$ such that $\langle \nabla f(x^*), (x^0 - x^*) \rangle < 0$. We set $d^0 = x^0 - x^*$. For fixed $\alpha > 0$, we construct $x^1 \in X$ such that

$$d^1 := x^1 - x^* := \frac{-\alpha \langle \nabla f(x^*), d^0 \rangle}{\|d^0\|_2^2} d^0$$

Substituting x^1 into the problem(7.6), we have

$$\begin{aligned}
\|x^1 - z\|^2 &= \|d^1\|^2 + 2\alpha \langle \nabla f(x^*), d^1 \rangle + \alpha^2 \|\nabla f(x^*)\|^2 \\
&= \left[\frac{\alpha^2 \langle \nabla f(x^*), d^0 \rangle^2}{\|d^0\|^4} \right] \|d^0\|^2 + 2\alpha \cdot \frac{-\alpha \langle \nabla f(x^*), d^0 \rangle}{\|d^0\|_2^2} \langle \nabla f(x^*), d^0 \rangle + \alpha^2 \|\nabla f(x^*)\|^2 \\
&= -\frac{\alpha^2 \langle \nabla f(x^*), d^0 \rangle^2}{\|d^0\|^2} + \alpha^2 \|\nabla f(x^*)\|^2 \\
&< \alpha^2 \|\nabla f(x^*)\|^2 = \|x^* - z\|^2,
\end{aligned}$$

i.e., x^* cannot be the minimizer of problem(7.6), which contradicts to the fact that $x^* = [z]^+$.

■