A FIRST COURSE

IN

ANALYSIS

MAT2006 Notebook

Lecturer

Prof. Weiming Ni

The Chinese University of Hongkong, Shenzhen

Tex Written By

Mr. Jie Wang

The Chinese University of Hongkong, Shenzhen

香港中文大學（深圳）
The Chinese University of Hong Kong, Shenzhen

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Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 3

Week3

3.1. Tuesday

3.1.1. Application of Heine-Borel Theorem

Theorem 3.1 Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ which converges in $|x| < 1$. If for every $x \in [0,1)$, there exists $n(= n(x))$ such that $\sum_{n+1}^{\infty} a_k x^k = 0$, then f is a polynomial, i.e., n does not depend on x.

Proof. Let $E_N := \{x \in [0, \frac{1}{2}] \mid \sum_{k=N+1}^{\infty} a_k x^k = 0\}$. It follows that

$$[0, \frac{1}{2}] = \bigcup_{N=1}^{\infty} E_N,$$

which implies that at least one E_N is uncountable, say, E_m is uncountable. (In particular, E_m is infinite)

Therefore, (B-W) there $\exists$ a sequence $x_1, x_2, \dots, x_k, \dots \to x_0 \in E_m$ as E_m is closed.

Hence, $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ holds for the sequence $\{x_1, x_2, \dots\}$. Hence we conclude the power series and the analytics function coincide each other:

$$f(x) \equiv a_0 + a_1 x + \cdots + a_m x^m$$

■

Proposition 3.1 Let g be analytic, i.e., $g(x) = b_0 + b_1 x + \cdots + b_n x^n + \cdots$ on $(-1,1)$; and $g(x_k) = 0$ for all $k \geq 1$, where $\{x_k\} \to x_0$ (change 0 for simplicity). Then $g \equiv 0$ on

$(-1,1)$ (i.e., $b_0 = b_1 = \cdots = 0$)

First, observe that $g(0) = 0$ due to continuity property.

At the same time, $g(0) = b_0 = 0$. It follows that

$$g(x) = x(b_1 + b_2 x + \cdots + b_n x^{n-1} +)$$

Note that

$$0 = g(x_k) = x_k(b_1 + b_2 x_k + \cdots + b_n x_k^{n-1} +)$$

Taking limit both sides, we derive $b_1 = 0$.

Hence, $g(x) = x^2(b_2 + b_3 x + \cdots)$ Note $0 = g(x_k) = x_k^2(b_2 + b_3 x + \cdots) \implies b_2 = 0$.

We can show $b_k = 0$ for $\forall k$. The remaining proof requires induction.

Now we talk about something mature for understanding.

3.1.2. Set Structure Analysis

Definition 3.1 [Nowhere Dense] A set $\boldsymbol{B}$ is said to be **nowhere dense** if its closure $\overline{B}$ contain no non-empty open set. ▪

For example,

$$B = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, ..\} \implies \overline{B} = B \bigcup \{0\},$$

which contains no open set.

Definition 3.2 [1st category] A set of $\boldsymbol{B}$ is said to be of 1st category if it can be written as the **union** of **finitely** many or **countably** many **nowhere** dense sets. ▪

Definition 3.3 [2rd category] A set is said to be of 2rd category if it is **not** of 1st category ▪

Theorem 3.2 — **Baire-Category Theorem.** $\mathbb{R}$ is of 2rd category, i.e., $\mathbb{R}$ cannot be written as the union of countably many nowhere dense sets; or equivalently, if

$\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$, then at least one A_n whose closure contains a non-empty set.

Proof. Assume $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$ such that all A_n's are nowhere dense. It follows that

$$\mathbb{R} \setminus \overline{A_1} \text{ is open.}$$

We choose an open set N_1 such that $\overline{N_1} \subseteq \mathbb{R} \setminus \overline{A_1}$. Since A_2 is nowhere dense, we imply $\overline{A_2}$ does not contain N_1, i.e., $N_1 \setminus \overline{A_2}$ is open; choose an open set N_2 such that $\overline{N_2} \subseteq N_1 \setminus \overline{A_2}$.

A_3 is nowhere dense, i.e., $\overline{A_3}$ contains no open set. Thus $N_2 \setminus \overline{A_3}$ is non-empty open set; choose open set ...

Repeating this process, weobtain a sequence of nested sets $\overline{N_1} \supseteq N_1 \supset \overline{N_2} \supset N_2 \cdots$. The cantor's theorem implies that $\bigcap_{k=1}^{\infty} \overline{N_k} \neq \emptyset$.

On the other hand, $\bigcap_{k=1}^{\infty} \overline{N_k} \subseteq \mathbb{R} \setminus \bigcup_{n=1}^{m} A_n$ for any finite m.

Therefore, $\emptyset \neq \bigcup_{k=1}^{\infty} \overline{N_k} \subseteq \mathbb{R} \setminus \bigcup_{n=1}^{\infty} A_n = \emptyset$ ∎

most continuous function is nowhere differentiable. converge pointwise

review: sequence and series.

R $\mathbb{R}$ is of 2nd category, i.e., if $\mathbb{R} = \bigcup_{n=1}^{\infty} A_m$, then at least A_n whose closure contains a **non-empty** open sets; The theorem also holds if we replace $\mathbb{R}$ by a **complete** metric space (essentially the same proof).

For $\mathbb{R}$, $d(x,y) = |x-y|$, so it is a metric space; $\mathbb{Q}$ is also metric space.

The second example is $\mathbb{R}^n$, with $d(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}|$.

The set of all bounded sequences, with $d(\{x_n\},\{y_n\}) = \sup\{|x_i - y_i| \mid i = 1,2,\dots\}$

The set of all bounded continuous functions on $\mathbb{R}$ (different domains), with $d_1(f,g) = \sup\{|f(x)-g(x)| \mid x \in \mathbb{R}\}$, or $d_2(f,g) = (\int_0^1 |f(x)-g(x)|^2 \,\mathrm{d}x)^{1.2}$ Note that $(\xi[0,1],d_1)$ is complete, and $(\xi[0,1],d_2)$ is not complete. (exercise)

Different distance definition corresponds to different metric spaces.

Complete: all Cauchy sequence converge.

Definition 3.4 [Metric Space]

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To show that most continuous function is nowhere differentiable, we will apply the Baire Category Theorem on $(\xi[0,1], d_1)$

3.1.3. Reviewing

Definition 3.5 [Sequence] $f : \mathbb{N} \to \mathbb{R}$, denoted as $\{f(0), f(1), \dots\}$I conventionally we denote it as $x_1, x_2, \dots$

Definition 3.6 A number α is the limit of $\{x_1, x_2, \dots\}$ if $\forall \epsilon > 0$, there $\exists N = N(\epsilon)$ such that $|x_k - \alpha| < \epsilon$ for $\forall k \geq N$, denoted by $\alpha_n \to \alpha$

Definition 3.7

$$\lim_{k\to\infty} \inf x_k := \lim_{n\to\infty} \inf_{k\geq n} x_k$$

which is the smallest limit point of the sequence

$$\limsup_{k\to\infty} x_k := \lim_{n\to\infty} \sup_{k\geq n} x_k$$

which is the largest limit point of the sequence.

A sequence always have liminf and limsup.

Definition 3.8 [Partial Sum] The series $\sum_i a_i$, the partial sum are defined as:

$$s_n = a_1 + \cdots + a_n$$

the sum is defined as the limit of the partial sum,