A JOURNEY

IN

PURE MATHEMATICS

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MAT3006 & 3040 & 4002 Notebook

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Contents

Acknowledgments

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CUHK(SZ)

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Monday for MAT3040

1.1.1. Introduction to Advanced Linear Algebra

Advanced Linear Algebra is one of the most important course in MATH major, with pre-request MAT2040. This course will offer the really linear algebra knowledge.

What the content will be covered?.

- In MAT2040 we have studied the space $\mathbb{R}^n$; while in MAT3040 we will study the general vector space V.

- In MAT2040 we have studied the *linear transformation* between Euclidean spaces, i.e., $T:\mathbb{R}^n \to \mathbb{R}^m$; while in MAT3040 we will study the linear transformation from vector spaces to vector spaces: $T:V\to W$

- In MAT2040 we have studied the eigenvalues of $n\times n$ matrix $\boldsymbol{A}$; while in MAT3040 we will study the eigenvalues of a **linear operator** $T:V\to V$.

- In MAT2040 we have studied the dot product $\boldsymbol{x}\cdot\boldsymbol{y}=\sum_{i=1}^n x_iy_i$; while in MAT3040 we will study the **inner product** $\langle \boldsymbol{v}_1,\boldsymbol{v}_2\rangle$.

Why do we do the generalization?. We are studying many other spaces, e.g., $\mathcal{C}(\mathbb{R})$ is called the space of all functions on $\mathbb{R}$, $\mathcal{C}^\infty(\mathbb{R})$ is called the space of all infinitely differentiable functions on $\mathbb{R}$, $\mathbb{R}[x]$ is the space of polynomials of one-variable.

■ **Example 1.1** 1. Consider the Laplace equation $\Delta f = 0$ with linear operator Δ:

$$\Delta : \mathcal{C}^\infty(\mathbb{R}^3) \to \mathcal{C}^\infty(\mathbb{R}^3) \quad f \mapsto (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})f$$

The solution to the PDE $\Delta f = 0$ is the 0-eigenspace of Δ.

2. Consider the Schrödinger equation $\hat{H}f = Ef$ with the linear operator

$$\hat{H} : \mathcal{C}^\infty(\mathbb{R}^3) \to \mathbb{R}^3, \quad f \to \left[\frac{-\hbar^2}{2\mu}\nabla^2 + V(x,y,z)\right]f$$

Solving the equation $\hat{H}f = Ef$ is equivalent to finding the eigenvectors of $\hat{H}$. In fact, the eigenvalues of $\hat{H}$ are **discrete**.

■

1.1.2. Vector Spaces

Definition 1.1 [Vector Space] A **vector space** over a field $\mathbb{F}$ (in particular, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is a set of objects V equipped with vector addiction and scalar multiplication such that

1. the vector addiction $+$ is closed with the rules:
   (a) **Commutativity**: $\forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, $\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}_2 + \boldsymbol{v}_1$.
   (b) **Associativity**: $\boldsymbol{v}_1 + (\boldsymbol{v}_2 + \boldsymbol{v}_3) = (\boldsymbol{v}_1 + \boldsymbol{v}_2) + \boldsymbol{v}_3$.
   (c) **Addictive Identity**: $\exists \mathbf{0} \in V$ such that $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$, $\forall \boldsymbol{v} \in V$.
2. the **scalar multiplication** is closed with the rules:
   (a) **Distributive**: $\alpha(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \alpha\boldsymbol{v}_1 + \alpha\boldsymbol{v}_2, \forall \alpha \in \mathbb{F}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$
   (b) **Distributive**: $(\alpha_1 + \alpha_2)\boldsymbol{v} = \alpha_1\boldsymbol{v} + \alpha_2\boldsymbol{v}$
   (c) **Compatibility**: $a(b\boldsymbol{v}) = (ab)\boldsymbol{v}$ for $\forall a, b \in \mathbb{F}$ and $\boldsymbol{b} \in V$.
   (d) $0\boldsymbol{v} = \mathbf{0}$, $1\boldsymbol{v} = \boldsymbol{v}$.

■

Here we study several examples of vector spaces:

■ **Example 1.2** For $V = \mathbb{F}^n$, we can define

1. Addictive Identity:
$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$
2. Scalar Multiplication:
$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$
3. Vector Addiction:
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

■

■ **Example 1.3** 1. It is clear that the set $V = M_{n\times n}(\mathbb{F})$ (the set of all $m \times n$ matrices) is a vector space as well.

2. The set $V = \mathcal{C}(\mathbb{R})$ is a vector space:
   (a) Vector Addiction:
$$(f+g)(x) = f(x) + g(x), \forall f, g \in V$$
   (b) Scalar Multiplication:
$$(\alpha f)(x) = \alpha f(x), \forall \alpha \in \mathbb{R}, f \in V$$
   (c) Addictive Identity is a zero function, i.e., $\mathbf{0}(x) = 0$ for all $x \in \mathbb{R}$.

■

Definition 1.2 A sub-collection $W \subseteq V$ of a vector space V is called a **vector subspace of** V if W itself forms a vector space, denoted by $W \leq V$.

■ **Example 1.4**

1. For $V = \mathbb{R}^3$, we claim that $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \leq V$
2. $W = \{(x, y, 1) \mid x, y \in \mathbb{R}\}$ is not the vector subspace of V.

Proposition 1.1 $W \subseteq V$ is a **vector subspace** of V iff for $\forall \boldsymbol{w}_1, \boldsymbol{w}_2 \in W$, we have $\alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2 \in W$, for $\forall \alpha, \beta \in \mathbb{F}$.

■ **Example 1.5**

1. For $V = M_{n \times n}(\mathbb{F})$, the subspace $W = \{A \in V \mid \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}\} \leq V$
2. For $V = \mathcal{C}^{\infty}(\mathbb{R})$, define $W = \{f \in V \mid \frac{\mathrm{d}^2}{\mathrm{d}x^2} f + f = 0\} \leq V$. For $f, g \in W$, we have

$$(\alpha f + \beta g)'' = \alpha f'' + \beta g'' = \alpha(-f) + \beta(-g) = -(\alpha f + \beta g),$$

which implies $(\alpha f + \beta g)'' + (\alpha f + \beta g) = 0$.

1.2. Monday for MAT3006

1.2.1. Overview on uniform convergence

Definition 1.3 [Convergence] Let $f_n(x)$ be a sequence of functions on an interval $I = [a, b]$. Then $f_n(x)$ converges **pointwise** to $f(x)$ (i.e., $f_n(x_0) \to f(x_0)$) for $\forall x_0 \in I$, if

$$\forall \varepsilon > 0, \exists N_{x_0,\varepsilon} \text{ such that } |f_n(x_0) - f(x_0)| < \varepsilon, \forall n \geq N_{x_0,\varepsilon}$$

We say $f_n(x)$ converges uniformly to $f(x)$, (i.e., $f_n(x) \rightrightarrows f(x)$) for $\forall x_0 \in I$, if

$$\forall \varepsilon > 0, \exists N_{\varepsilon} \text{ such that } |f_n(x_0) - f(x_0)| < \varepsilon, \forall n \geq N_{\varepsilon}$$

■

■ **Example 1.6** It is clear that the function $f_n(x) = \frac{n}{1+nx}$ converges pointwise into $f(x) = \frac{1}{x}$ on $[0, \infty)$, and it is uniformly convergent on $[1, \infty)$. ■

Proposition 1.2 If $\{f_n\}$ is a sequence of continuous functions on I, and $f_n(x) \rightrightarrows f(x)$, then the following results hold:

1. $f(x)$ is continuous on I.
2. f is (Riemann) integrable with $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.
3. Suppose furthermore that $f_n(x)$ is **continuously differentiable**, and $f_n'(x) \rightrightarrows g(x)$, then $f(x)$ is differentiable, with $f_n'(x) \to f'(x)$.

We can put the discussions above into the content of series, i.e., $f_n(x) = \sum_{k=1}^{n} S_k(x)$.

Proposition 1.3 If $S_k(x)$ is continuous for $\forall k$, and $\sum_{k=1}^{n} S_k \rightrightarrows \sum_{k=1}^{\infty} S_k$, then

1. $\sum_{k=1}^{\infty} S_k(x)$ is continuous,
2. The series $\sum_{k=1}^{\infty} S_k$ is (Riemann) integrable, with $\sum_{k=1}^{\infty} \int_a^b S_k(x)\,dx = \int_a^b \sum_{k=1}^{\infty} S_k(x)\,dx$
3. If $\sum_{k=1}^{n} S_k$ is continuously differentiable, and the derivative of which is uniform

convergent, then the series $\sum_{k=1}^{\infty} S_k$ is differentiable, with

$$\left(\sum_{k=1}^{\infty} S_k(x)\right)' = \sum_{k=1}^{\infty} S_k'(x)$$

Then we can discuss the properties for a special kind of series, say power series.

Proposition 1.4 Suppose the power series $f(x) = \sum_{k=1}^{\infty} a_k x^k$ has radius of convergence R, then

1. $\sum_{k=1}^{n} a_k x^k \rightrightarrows f(x)$ for any $[-L, L]$ with $L < R$.
2. The function $f(x)$ is continuous on $(-R, R)$, and moreover, is differentiable and (Riemann) integrable on $[-L, L]$ with $L < R$:

$$\int_0^x f(t)\,\mathrm{d}t = \sum_{k=1}^{\infty} \frac{a_k}{k+1} x^{k+1}$$

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

1.2.2. Introduction to MAT3006

What are we going to do.

1. (a) Generalize our study of (sequence, series, functions) on $\mathbb{R}^n$ into a metric space.

   (b) We will study spaces outside $\mathbb{R}^n$.

   Remark:

   - For (a), different metric may yield different kind of convergence of sequences. For (b), one important example we will study is $X = \mathcal{C}[a, b]$ (all continuous functions defined on $[a, b]$.) We will generalize X into $\mathcal{C}_b(E)$, which means the set of bounded continuous functions defined on $E \subseteq \mathbb{R}^n$.
   - The insights of analysis is to find a **unified** theory to study sequences/series on a metric space X, e.g., $X = \mathbb{R}^n, \mathcal{C}[a, b]$. In particular, for $\mathcal{C}[a, b]$, we will see that
     - most functions in $\mathcal{C}[a, b]$ are nowhere differentiable. (repeat part of

content in MAT2006)

- We will prove the existence and uniqueness of ODEs.
- the set poly$[a,b]$ (the set of polynomials on $[a,b]$) is dense in $\mathcal{C}[a,b]$. (analogy: $\mathbb{Q} \subseteq \mathbb{R}$ is dense)

2. Introduction to the Lebesgue Integration.

   For convergence of integration $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)$, we need the pre-conditions (a) $f_n(x)$ is continuous, and (b) $f_n(x) \rightrightarrows f(x)$. The natural question is that can we relax these conditions to

   (a) $f_n(x)$ is integrable?
   (b) $f_n(x) \to f(x)$ pointwisely?

   The answer is yes, by using the tool of Lebesgue integration. If $f_n(x) \to f(x)$ and $f_n(x)$ is Lebesgue integrable, then $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$, which is so called the **dominated convergence**.

1.2.3. Metric Spaces

We will study the **length** of an element, or the **distance** between two elements in an arbitrary set X. First let's discuss the length defined on a well-structured set, say vector space.

Definition 1.4 [Normed Space] Let X be a vector space. A **norm** on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that

1. $\|\boldsymbol{x}\| \ge 0$ for $\forall \boldsymbol{x} \in X$, with equality iff $\boldsymbol{x} = \boldsymbol{0}$
2. $\|\alpha\boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\|$, for $\forall \alpha \in \mathbb{R}$ and $\boldsymbol{x} \in X$.
3. $\|\boldsymbol{x}+\boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ (triangular inequality)

Any vector space equipped with $\|\cdot\|$ is called a **normed space**.

■ **Example 1.7** 1. For $X = \mathbb{R}^n$, define

$$\|\boldsymbol{x}\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2} \quad \text{(Euclidean Norm)}$$

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \quad (p\text{-norm})$$

2. For $X = \mathcal{C}[a,b]$, define

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, \mathrm{d}x\right)^{1/p}$$

Exercise: check the norm defined above are well-defined. ■

Here we can define the distance in an arbitrary set:

Definition 1.5 A set X is a **metric space** with metric (X,d) if there exists a (distance) function $d : X \times X \to \mathbb{R}$ such that

1. $d(\boldsymbol{x},\boldsymbol{y}) \geq 0$ for $\forall \boldsymbol{x},\boldsymbol{y} \in X$, with equality iff $\boldsymbol{x} = \boldsymbol{y}$.
2. $d(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{y},\boldsymbol{x})$.
3. $d(\boldsymbol{x},\boldsymbol{z}) \leq d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z})$.

■

■ **Example 1.8** 1. If X is a normed space, then define $d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$, which is so called the metric induced from the norm $\|\cdot\|$.

2. Let X be any (non-empty) set with $\boldsymbol{x},\boldsymbol{y} \in X$, the discrete metric is given by:

$$d(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Exercise: check the metric space defined above are well-defined. ■

R Adopting the infinite norm discussed in Example (1.7), we can define a metric on $\mathcal{C}[a,b]$ by

$$d_\infty(f,g) = \|f-g\|_\infty := \max_{x\in[a,b]} |f(x) - g(x)|$$

which is the correct metric to study the uniform convergence for $\{f_n\} \subseteq \mathcal{C}[a,b]$.

Definition 1.6 Let (X,d) be a metric space. An **open ball** centered at $\boldsymbol{x} \in X$ of radius r is the set

$$B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in X \mid d(\boldsymbol{x},\boldsymbol{y}) < r\}.$$

■ **Example 1.9** 1. For $X = \mathbb{R}^2$, we can draw the $B_1(\boldsymbol{0})$ with respect to the metrics d_1, d_2:

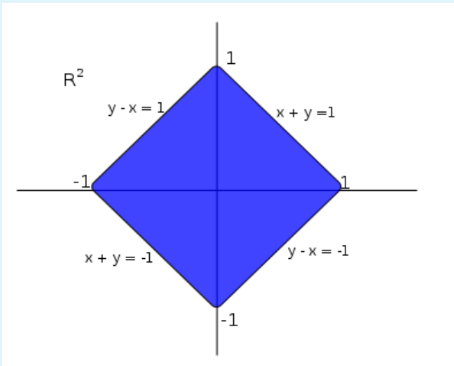


Figure 1.1: $B_1(\boldsymbol{0})$ w.r.t. the metric d_1

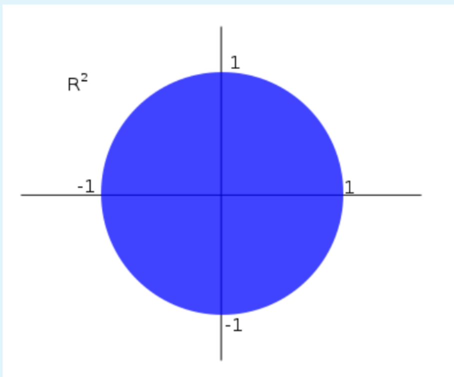
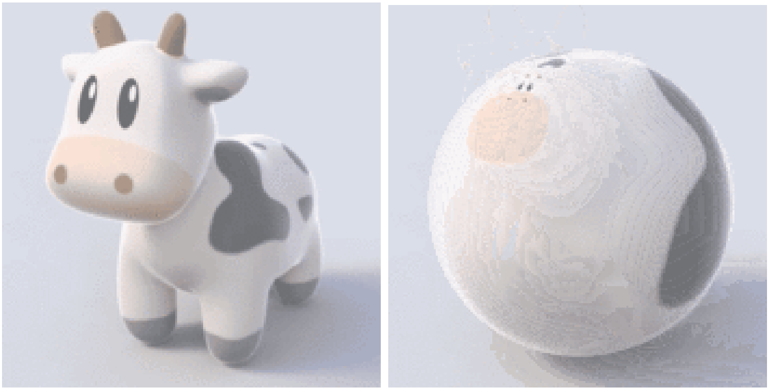


Figure 1.2: $B_1(\boldsymbol{0})$ w.r.t. the metric d_2

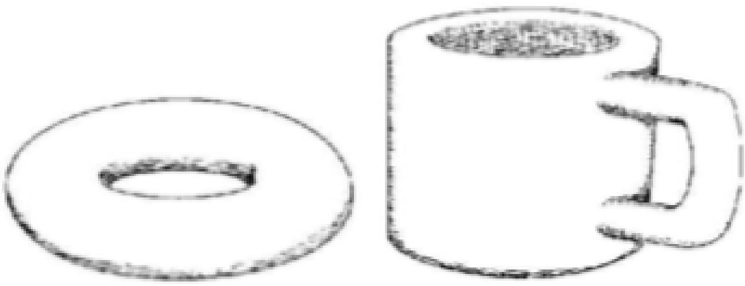
1.3. Monday for MAT4002

1.3.1. Introduction to Topology

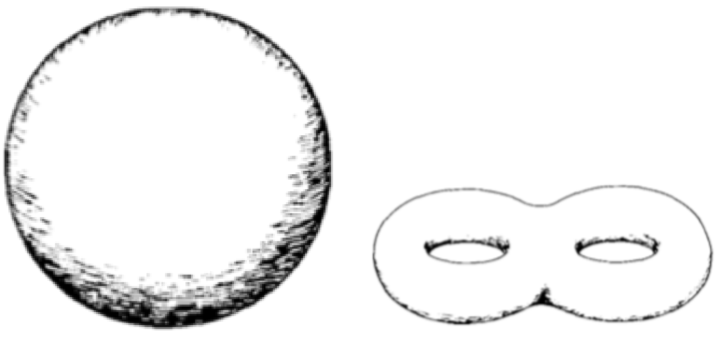
We will study global properties of a geometric object, i.e., *the distrance between 2 points in an object is totally ignored*. For example, the objects shown below are essentially invariant under a certain kind of transformation:



Another example is that the coffee cup and the donut have the same topology:



However, the two objects below have the intrinsically different topologies:



In this course, we will study the phenomenon described above mathematically.

1.3.2. Metric Spaces

In order to ingnore about the distances, we need to learn about distances first.

Definition 1.7 [Metric Space] Metric space is a set X where one can measure distance between any two objects in X.

Specifically speaking, a metric space X is a non-empty set endowed with a function (distance function) $d : X \times X \to \mathbb{R}$ such that

1. $d(\boldsymbol{x},\boldsymbol{y}) \geq 0$ for $\forall \boldsymbol{x},\boldsymbol{y} \in X$ with equality iff $\boldsymbol{x} = \boldsymbol{y}$
2. $d(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{y},\boldsymbol{x})$
3. $d(\boldsymbol{x},\boldsymbol{z}) \leq d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z})$ (triangular inequality)

■ **Example 1.10**

1. Let $X = \mathbb{R}^n$, with

$$d_2(\boldsymbol{x},\boldsymbol{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

$$d_\infty(\boldsymbol{x},\boldsymbol{y}) = \max_{i=1,\dots,n} |x_i - y_i|$$

2. Let X be any set, and define the discrete metric

$$d(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Homework: Show that (1) and (2) defines a metric.

Definition 1.8 [Open Ball] An **open ball** of radius r centered at $\boldsymbol{x} \in X$ is the set

$$B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in X \mid d(\boldsymbol{x},\boldsymbol{y}) < r\}$$

■ **Example 1.11** 1. The set $B_1(0,0)$ defines an open ball under the metric $(X = \mathbb{R}^2, d_2)$, or the metric $(X = \mathbb{R}^2, d_\infty)$. The corresponding diagram is shown below:

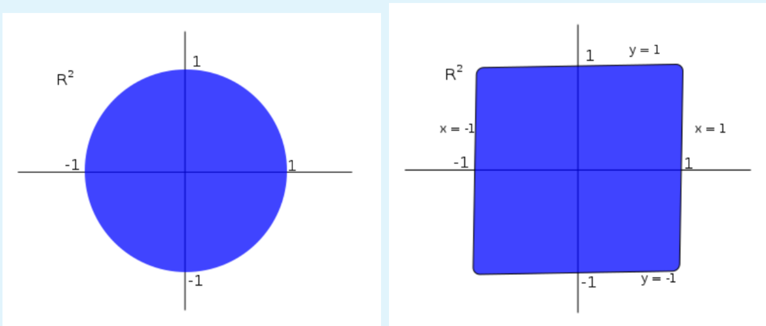


Figure 1.3: Left: under the metric $(X = \mathbb{R}^2, d_2)$; Right: under the metric $(X = \mathbb{R}^2, d_\infty)$

2. Under the metric $(X = \mathbb{R}^2, \text{discrete metric})$, the set $B_1(0,0)$ is one single point, also defines an open ball.

■

Definition 1.9 [Open Set] Let X be a metric space, $U \subseteq X$ is an open set in X if $\forall u \in U$, there exists $\epsilon_u > 0$ such that $B_{\epsilon_u}(u) \subseteq U$. ■

Definition 1.10 The **topology** induced from (X, d) is the collection of all open sets in (X, d), denoted as the symbol $\mathcal{T}$. ■

Proposition 1.5 All open balls $B_r(\boldsymbol{x})$ are open in (X, d).

Proof. Consider the example $X = \mathbb{R}$ with metric d_2. Therefore $B_r(x) = (x - r, x + r)$. Take $\boldsymbol{y} \in B_r(\boldsymbol{x})$ such that $d(\boldsymbol{x}, \boldsymbol{y}) = q < r$ and consider $B_{(r-q)/2}(\boldsymbol{y})$: for all $z \in B_{(r-q)/2}(\boldsymbol{y})$, we have

$$d(\boldsymbol{x}, \boldsymbol{z}) \le d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z}) < q + \frac{r-q}{2} < r,$$

which implies $\boldsymbol{z} \in B_r(x)$. ■

Proposition 1.6 Let $(X, \boldsymbol{d})$ be a metric space, and $\mathcal{T}$ is the topology induced from (X, d), then

1. let the set $\{G_\alpha \mid \alpha \in \mathcal{A}\}$ be a collection of (uncountable) open sets, i.e., $G_\alpha \in \mathcal{T}$,

then $\bigcup_{\alpha \in \mathcal{A}} G_\alpha \in \mathcal{T}$.

2. let $G_1, \ldots, G_n \in \mathcal{T}$, then $\bigcap_{i=1}^n G_i \in \mathcal{T}$. The finite intersection of open sets is open.

Proof. 1. Take $x \in \bigcup_{\alpha \in \mathcal{A}} G_\alpha$, then $x \in G_\beta$ for some $\beta \in \mathcal{A}$. Since G_β is open, there exists $\epsilon_x > 0$ s.t.

$$B_{\epsilon_x}(x) \subseteq G_\beta \subseteq \bigcup_{\alpha \in \mathcal{A}} G_\alpha$$

2. Take $x \in \bigcap_{i=1}^n G_i$, i.e., $x \in G_i$ for $i = 1, \ldots, n$, i.e., there exists $\epsilon_i > 0$ such that $B_{\epsilon_i}(x) \subseteq G_i$ for $i = 1, \ldots, n$. Take $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$, which implies

$$B_\epsilon(x) \subseteq B_{\epsilon_i}(x) \subseteq G_i, \forall i$$

which implies $B_\epsilon(x) \subseteq \bigcap_{i=1}^n G_i$

■

Exercise.

1. let $\mathcal{T}_2, \mathcal{T}_\infty$ be topologies induced from the metrices d_2, d_∞ in $\mathbb{R}^2$. Show that $J_2 = J_\infty$, i.e., every open set in $(\mathbb{R}^2, d_2)$ is open in $(\mathbb{R}^2, d_\infty)$, and every open set in $(\mathbb{R}^2, d_\infty)$ is open in $(\mathbb{R}_2, d_2)$.
2. Let $\mathcal{T}$ be the topology induced from the discrete metric (X, d_{discrete}). What is $\mathcal{T}$?

1.4. Wednesday for MAT3040

1.4.1. Review

1. Vector Space: e.g., $\mathbb{R}, M_{n\times n}(\mathbb{R}), \mathcal{C}(\mathbb{R}^n), \mathbb{R}[x]$.
2. Vector Subspace: $W \le V$, e.g.,
   (a) $V = \mathbb{R}^2$, the set $W := \mathbb{R}^2_+$ is not a vector subspace since W is not closed under scalar multiplication;
   (b) the set $W = \mathbb{R}^2_+ \bigcup \mathbb{R}^2_-$ is not a vector subspace since it is not closed under addition.
   (c) For $V = \mathbb{M}_{3\times 3}(\mathbb{R})$, the set of invertible 3×3 matrices is not a vector subspace, since we cannot define zero vector inside.
   (d) Exercise: How about the set of all singular matrices? Answer: it is not a vector subspace since the vector addition does not necessarily hold.

1.4.2. Spanning Set

Definition 1.11 [Span] Let V be a vector space over $\mathbb{F}$:

1. A linear combination of a subset S in V is of the form

$$\sum_{i=1}^{n}\alpha_i \boldsymbol{s}_i, \quad \alpha_i \in \mathbb{F}, \boldsymbol{s}_i \in S$$

   Note that the summation should be finite.

2. The **span** of a subset $S \subseteq V$ is

$$\text{span}(S) = \left\{ \sum_{i=1}^{n}\alpha_i \boldsymbol{s}_i \,\middle|\, \alpha_i \in \mathbb{F}, \boldsymbol{s}_i \in S \right\}$$

3. S is a spanning set of V, or say S spans V, if

$$\text{span}(S) = V.$$

■ **Example 1.12** For $V = \mathbb{R}[x]$, define the set

$$S = \{1, x^2, x^4, \dots, x^6\},$$

then $2 + x^4 + \pi x^{106} \in \text{span}(S)$, while the series $1 + x^2 + x^4 + \cdots \notin \text{span}(S)$.

It is clear that $\text{span}(S) \neq V$, but S is the spanning set of $W = \{p \in V \mid p(x) = p(-x)\}$.

■ **Example 1.13** For $V = M_{3\times 3}(\mathbb{R})$, let $W_1 = \{\boldsymbol{A} \in V \mid \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}\}$ and $W_2 = \{\boldsymbol{B} \in V \mid \boldsymbol{B}^{\mathrm{T}} = -\boldsymbol{B}\}$ (the set of skew-symmetric matrices) be two vector subspaces. Define the set

$$\boldsymbol{S} := W_1 \bigcup W_2$$

Exercise: $\boldsymbol{S}$ spans V.

Proposition 1.7 Let S be a subset in a vector space V.

1. $S \subseteq \text{span}(S)$
2. $\text{span}(S) = \text{span}(\text{span}(S))$
3. If $\boldsymbol{w} \in \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \setminus \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$, then

$$\boldsymbol{v}_1 \in \text{span}\{\boldsymbol{w}, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\} \setminus \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$$

Proof. 1. For each $\boldsymbol{s} \in S$, we have

$$\boldsymbol{s} = 1 \cdot \boldsymbol{s} \in \text{span}(S)$$

2. From (1), it's clear that $\text{span}(S) \subseteq \text{span}(\text{span}(S))$, and therefore suffices to show $\text{span}(\text{span}(S)) \subseteq \text{span}(S)$:

Pick $\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i \in \operatorname{span}(\operatorname{span}(S))$, where $\boldsymbol{v}_i \in \operatorname{span}(S)$. Rewrite

$$\boldsymbol{v}_i = \sum_{j=1}^{n_i} \beta_{ij} \boldsymbol{s}_j, \quad \boldsymbol{s}_j \in S,$$

which implies

$$\begin{aligned}\boldsymbol{v} &= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n_i} \beta_{ij} \boldsymbol{s}_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n_i} (\alpha_i \beta_{ij}) \boldsymbol{s}_j,\end{aligned}$$

i.e., $\boldsymbol{v}$ is the finite combination of elements in S, whcih implies $\boldsymbol{v} \in \operatorname{span}(S)$.

3. By hypothesis, $\boldsymbol{w} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n$ with $\alpha_1 \neq 0$, which implies

$$\boldsymbol{v}_1 = -\frac{\alpha_2}{\alpha_1} \boldsymbol{v}_2 + \cdots + \left(-\frac{1}{\alpha_1} \boldsymbol{w}\right)$$

which implies $\boldsymbol{v}_1 \in \operatorname{span}\{\boldsymbol{w}, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$. It suffices to show $\boldsymbol{v}_1 \notin \operatorname{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$. Suppose on the contrary that $\boldsymbol{v}_1 \in \operatorname{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$. It's clear that $\operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} = \operatorname{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$. (left as exercise). Therefore,

$$\emptyset = \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \setminus \operatorname{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\},$$

which is a contradiction.

■

1.4.3. Linear Independence and Basis

Definition 1.12 [Linear Independence] Let S be a (not necessarily finite) subset of V. Then S is **linearly independent** (l.i.) on V if for any finite subset $\{\boldsymbol{s}_1, \dots, \boldsymbol{s}_k\}$ in S,

$$\sum_{i=1}^{k} \alpha_i \boldsymbol{s}_i = 0 \Longleftrightarrow \alpha_i = 0, \forall i$$

■ **Example 1.14** For $V = \mathcal{C}(\mathbb{R})$,

1. let $S_1 = \{\sin x, \cos x\}$, which is l.i., since

$$\alpha \sin x + \beta \cos x = \mathbf{0}(\text{means zero function})$$

   Taking $x = 0$ both sides leads to $\beta = 0$; taking $x = \frac{\pi}{2}$ both sides leads to $\alpha = 0$.

2. let $S_2 = \{\sin^2 x, \cos^2 x, 1\}$, which is linearly dependent, since

$$1 \cdot \sin^2 x + 1 \cdot \cos^2 x + (-1) \cdot 1 = 0, \forall x$$

3. Exercise: For $V = \mathbb{R}[x]$, let $S = \{1, x, x^2, x^3, \dots,\}$, which is l.i.:

   Pick $x^{k_1}, \dots, x^{k_n} \in S$ with $k_1 < \cdots < k_n$. Consider that the euqation

$$\alpha_1 x^{k_1} + \cdots + \alpha_n x^{k_n} = \mathbf{0}$$

   holds for all x, and try to solve for $\alpha_1, \dots, \alpha_n$ (one way is differentation.)

■

Definition 1.13 [Basis] A subset S is a **basis** of V if

(a) S spans V;

(b) S is l.i.

■

■ **Example 1.15**

1. For $V = \mathbb{R}^n$, $S = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is a basis of V
2. For $V = \mathbb{R}[x]$, $S = \{1, x, x^2, \dots\}$ is a basis of V
3. For $V = M_{2\times 2}(\mathbb{R})$,

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis of V

R Note that there can be many basis for a vector space V.

Proposition 1.8 Let $V = \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$, then there exists a subset of $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$, which is a basis of V.

Proof. If $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ is l.i., the proof is complete.

Suppose not, then $\alpha_1\boldsymbol{v}_1 + \cdots + \alpha_m\boldsymbol{v}_m = \mathbf{0}$ has a non-trivial solution. w.l.o.g., $\alpha_1 \neq 0$, which implies

$$\boldsymbol{v}_1 = -\frac{\alpha_2}{\alpha_1}\boldsymbol{v}_2 + \cdots + \left(\frac{\alpha_m}{\alpha_1}\right)\boldsymbol{v}_m \implies \boldsymbol{v}_1 \in \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_m\}$$

By the proof in (c), Proposition (1.7),

$$\text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\} = \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_m\},$$

which implies $V = \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_m\}$.

Continuse this argument finitely many times to guarantee that $\{\boldsymbol{v}_i, \boldsymbol{v}_{i+1}, \dots, \boldsymbol{v}_m\}$ is l.i., and spans V. The proof is complete. ∎

Corollary 1.1 If $V = \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ (i.e., V is finitely generated), then V has a basis. (The same holds for non-finitely generated V).

Proposition 1.9 If $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ is a basis of V, then every $\boldsymbol{v} \in V$ can be expressed uniquely as

$$\boldsymbol{v} = \alpha_1\boldsymbol{v}_1 + \cdots + \alpha_n\boldsymbol{v}_n$$

Proof. Since $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ spans V, so $\boldsymbol{v} \in V$ can be written as

$$\boldsymbol{v} = \alpha_1\boldsymbol{v}_1 + \cdots + \alpha_n\boldsymbol{v}_n \tag{1.1}$$

Suppose further that

$$\boldsymbol{v} = \beta_1 \boldsymbol{v}_1 + \cdots + \beta_n \boldsymbol{v}_n, \tag{1.2}$$

it suffices to show that $\alpha_i = \beta_i$ for $\forall i$:

Subtracting (1.1) into (1.2) leads to

$$(\alpha_1 - \beta_1)\boldsymbol{v}_1 + \cdots + (\alpha_n - \beta_n)\boldsymbol{v}_n = 0.$$

By the hypothesis of linear independence, we have $\alpha_i - \beta_i = 0$ for $\forall i$, i.e., $\alpha_i = \beta_i$. ■

1.5. Wednesday for MAT3006

Reviewing.

- Normed Space: a norm on a vector space
- Metric Space
- Open Ball

1.5.1. Convergence of Sequences

Since $\mathbb{R}^n$ and $C[a,b]$ are both metric spaces, we can study the convergence in $\mathbb{R}^n$ and the functions defined on $[a,b]$ at the same time.

Definition 1.14 [Convergence] Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is **convergent** to x if $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n,x) < \varepsilon, \forall n \geq N.$$

We can denote the convergence by

$$x_n \to x, \quad \text{or} \quad \lim_{n\to\infty} x_n = x, \quad \text{or} \quad \lim_{n\to\infty} d(x_n,x) = 0$$

Proposition 1.10 If the limit of $\{x_n\}$ exists, then it is unique.

R Note that the proposition above does not necessarily hold for topology spaces.

Proof. Suppose $x_n \to x$ and $x_n \to y$, which implies

$$0 \leq d(x,y) \leq d(x,x_n) + d(x_n,y), \forall n$$

Taking the limit $n \to \infty$ both sides, we imply $d(x,y) = 0$, i.e., $x = y$. ■

■ **Example 1.16** 1. Consider the metric space $(\mathbb{R}^k, d_\infty)$ and study the convergence

$$\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x} \Longleftrightarrow \lim_{n\to\infty}\left(\max_{i=1\dots,k} |x_{n_i} - x_i|\right) = 0$$
$$\Longleftrightarrow \lim_{n\to\infty} |x_{n_i} - x_i| = 0, \forall i = 1, \dots, k$$
$$\Longleftrightarrow \lim_{n\to\infty} x_{n_i} = x_i,$$

i.e., the convergence defined in d_∞ is the same as the convergence defined in d_2.

2. Consider the convergence in the metric space $(\mathcal{C}[a,b], d_\infty)$:

$$\lim_{n\to\infty} f_n = f \Longleftrightarrow \lim_{n\to\infty}\left(\max_{[a,b]} |f_n(x) - f(x)|\right) = 0$$
$$\Longleftrightarrow \forall \varepsilon > 0, \forall x \in [a,b], \exists N_\varepsilon \text{ such that } |f_n(x) - f(x)| < \varepsilon, \forall n \geq N_\varepsilon$$

which is equivalent to the uniform convergence of functions, i.e., the convergence defined in d_2.

■

Definition 1.15 [Equivalent metrics] Let d and ρ be metrics on X.

1. We say ρ is **stronger** than d (or d is **weaker** than ρ) if

$$\exists K > 0 \text{ such that } d(x,y) \leq K\rho(x,y), \forall x,y \in X$$

2. The metrics d and ρ are equivalent if there exists $K_1, K_2 > 0$ such that

$$d(x,y) \leq K_1\rho(x,y) \leq K_2 d(x,y)$$

■

R The strongerness of ρ than d is depicted in the graph below:

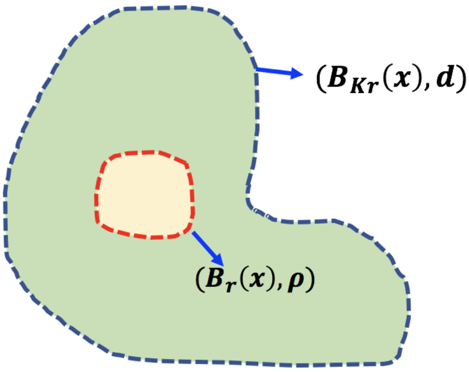


Figure 1.4: The open ball $(B_r(x), \rho)$ is contained by the open ball $(B_{Kr}(x), d)$

For each $x \in X$, consider the open ball $(B_r(x), \rho)$ and the open ball $(B_{Kr}(x), d)$:

$$B_r(x) = \{y \mid \rho(x,y) < r\}, \quad B_{Kr}(x) = \{z \mid d(x,z) < Kr\}.$$

For $y \in (B_r(x), \rho)$, we have $d(x,y) < K\rho(x,y) < Kr$, which implies $y \in (B_{Kr}(x), d)$, i.e, $(B_r(x), \rho) \subseteq (B_{Kr}(x), d)$ for any $x \in X$ and $r > 0$.

■ **Example 1.17** 1. d_1, d_2, d_∞ in $\mathbb{R}^n$ are equivalent

$$d_1(\boldsymbol{x},\boldsymbol{y}) \leq d_\infty(\boldsymbol{x},\boldsymbol{y}) \leq n d_1(\boldsymbol{x},\boldsymbol{y})$$
$$d_2(\boldsymbol{x},\boldsymbol{y}) \leq d_\infty(\boldsymbol{x},\boldsymbol{y}) \leq \sqrt{n} d_2(\boldsymbol{x},\boldsymbol{y})$$

We use two relation depiected in the figure below to explain these two inequalities:

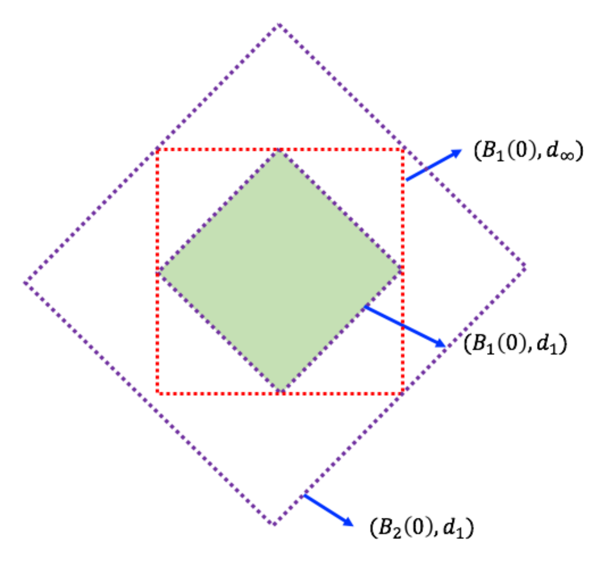


Figure 1.5: The diagram for the relation $(B_1(x), d_1) \subseteq (B_\infty(x), d_\infty) \subseteq (B_2(x), d_1)$ on $\mathbb{R}^2$

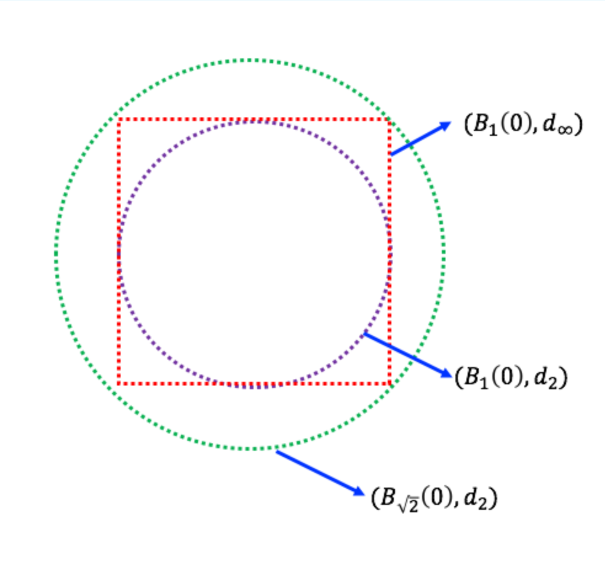


Figure 1.6: The diagram for the relation $(B_1(x), d_2) \subseteq (B_\infty(x), d_\infty) \subseteq (B_{\sqrt{2}}(x), d_2)$ on $\mathbb{R}^2$

It's easy to conclude the simple generalization for example (1.16):

Proposition 1.11 If d and ρ are equivalent, then

$$\lim_{n\to\infty} d(x_n, x) = 0 \Longleftrightarrow \lim_{n\to\infty} \rho(x_n, x) = 0$$

Note that this does not necessarily hold for topology spaces.

2. Consider d_1, d_∞ in $\mathcal{C}[a,b]$:

$$d_1(f,g) := \int_a^b |f-g|\,\mathrm{d}x \le \int_a^b \sup_{[a,b]} |f-g|\,\mathrm{d}x = (b-a)d_\infty(f,g),$$

i.e., d_∞ is stronger than d_1. Question: Are they equivalent? **No**.

Justification. Consider $f_n(x) = n^2 x^n (1-x)$. Check that

$$\lim_{n\to\infty} d_1(f_n(x), 1) = 0, \quad \text{but } d_\infty(f_n(x), 1) \to \infty$$

The peak of f_n may go to infinite, while the integration converges to zero. Therefore d_1 and d_∞ have different limits. We will discuss this topic at Lebsegue integration again. ■

1.5.2. Continuity

Definition 1.16 [Continuity] Let $f : (X,d) \to (Y,d)$ be a function and $x_0 \in X$. Then f is continuous at x_0 if $\forall \varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x,x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon$$

The function f is continuous in X if f is continuous for all $x_0 \in X$.

Proposition 1.12 The function f is continuous at x if and only if for all $\{x_n\} \to x$ under d, $f(x_n) \to f(x)$ under ρ.

Proof. *Necessity:* Given $\varepsilon > 0$, by continuity,

$$d(x,x') < \delta \implies \rho(f(x'), f(x)) < \varepsilon. \tag{1.3}$$

Consider the sequence $\{x_n\} \to x$, then there exists N such that $d(x_n, x) < \delta$ for $\forall n \ge N$. By applying (1.3), $\rho(f(x_n), f(x)) < \varepsilon$ for $\forall n \ge N$, i.e., $f(x_n) \to f(x)$.

Sufficiency: Assume that f is not continuous at x, then there exists ε_0 such that for $\delta_n = \frac{1}{n}$, there exists x_n such that

$$d(x_n, x) < \delta_n, \text{ but } \rho(f(x_n), f(x)) > \varepsilon_0.$$

Then $\{x_n\} \to x$ by our construction, while $\{f(x_n)\}$ does not converge to $f(x)$, which is a contradiction. ■

Corollary 1.2 If the function $f : (X, d) \to (Y, \rho)$ is continuous at x, the function $g : (Y, \rho) \to (Z, m)$ is continuous at $f(x)$, then $g \circ f : (X, d) \to (Z, m)$ is continuous at x.

Proof. Note that

$$\{x_n\} \to x \overset{(a)}{\Longrightarrow} \{f(x_n)\} \to f(x) \overset{(b)}{\Longrightarrow} \{g(f(x_n))\} \to g(f(x)) \overset{(c)}{\Longrightarrow} g \circ f \text{ is continuous at } x.$$

where $(a), (b), (c)$ are all by proposition (1.12). ■

1.5.3. Open and Closed Sets

We have open/closed intervals in $\mathbb{R}$, and they are important in some theorems (e.g, continuous functions bring closed intervals to closed intervals).

Definition 1.17 [Open] Let (X, d) be a metric space. A set $U \subseteq X$ is open if for each $x \in U$, there exists $\rho_x > 0$ such that $B_{\rho_x}(x) \subseteq U$. The empty set $\emptyset$ is defined to be open. ■

■ **Example 1.18** Let $(\mathbb{R}, d_2 \text{ or } d_\infty)$ be a metric space. The set $U = (a, b)$ is open. ■

Proposition 1.13
1. Let (X, d) be a metric space. Then all open balls $B_r(x)$ are open
2. All open sets in X can be written as a union of open balls.

Proof. 1. Let $y \in B_r(x)$, i.e., $d(x, y) := q < r$. Consider the open ball $B_{(r-q)/2}(y)$. It suffices to show $B_{(r-q)/2}(y) \subseteq B_r(x)$. For any $z \in B_{(r-q)/2}(y)$,

$$d(x,z) \leq d(x,y) + d(y,z) < q + \frac{r-q}{2} = \frac{r+q}{2} < r.$$

The proof is complete.

2. Let $U \subseteq X$ be open, i.e., for $\forall x \in U$, there exists $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subseteq U$. Therefore

$$\{x\} \subseteq B_{\varepsilon_x}(x) \subseteq U, \forall x \in U$$

which implies

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_{\varepsilon_x}(x) \subseteq U,$$

i.e., $U = \bigcup_{x \in U} B_{\varepsilon_x}(x)$.

■

1.6. Wednesday for MAT4002

Reviewing.

- Metric Space (X,d)
- Open balls and open sets (note that the emoty set $\emptyset$ is open)
- Define the collection of open sets in X, say $\mathcal{T}$ is the topology.

Exercise.

1. Show that the $\mathcal{T}_2$ under $(X=\mathbb{R}^2,d_2)$ and $\mathcal{T}_\infty$ under $(X=\mathbb{R}^2,d_\infty)$ are the same.

   Ideas. Follow the procedure below:

   An open ball in d_2-metric is open in d_∞;

   Any open set in d_2-metric is open in d_∞;

   Switch d_2 and d_∞. ■

2. Describe the topology $\mathcal{T}_{\text{discrete}}$ under the metric space $(X=\mathbb{R}^2,d_{\text{discrete}})$.

   Outlines. Note that $\{x\}=B_{1/2}(x)$ is an open set.

   For any subset $W\subseteq\mathbb{R}^2$, $W=\bigcup_{w\in W}\{w\}$ is open.

   Therefore $\mathcal{T}_{\text{discrete}}$ is all subsets of $\mathbb{R}^2$. ■

1.6.1. Forget about metric

Next, we will try to define closedness, compactness, etc., without using the tool of metric:

Definition 1.18 [closed] A subset $V\subseteq X$ is **closed** if $X\setminus V$ is **open**.

Example 1.19 Under the metric space $(\mathbb{R},d_1)$,

$$\mathbb{R}\setminus[b,a]=(a,\infty)\bigcup(-\infty,b)\text{ is open}\implies[b,a]\text{ is closed}$$

Proposition 1.14 Let X be a metric space.

1. $\emptyset, X$ is closed in X
2. If F_α is closed in X, so is $\bigcap_{\alpha \in A} F_\alpha$.
3. If $F_1, \dots, F_k$ is closed, so is $\bigcup_{i=1}^{k} F_i$.

Proof. 1. Note that X is open in X, which implies $\emptyset = X \setminus X$ is closed in X; Similarly, $\emptyset$ is open in X, which implies $X = X \setminus \emptyset$ is closed in X;

2. The set F_α is closed implies there exists open $U_\alpha \subseteq X$ such that $F_\alpha = X \setminus U_\alpha$. By De Morgan's Law,

$$\bigcap_{\alpha \in A} F_\alpha = \bigcap_{\alpha \in A} (X \setminus U_\alpha) = X \setminus (\bigcup_{\alpha \in A} U_\alpha).$$

By part (a) in proposition (1.6), the set $\bigcup_{\alpha \in A} U_\alpha$ is openm which implies $\bigcap_{\alpha \in A} F_\alpha$ is closed.

3. The result follows from part (b) in proposition (1.6) by taking complements.

■

We illustrate examples where open set is used to define convergence and continuity.

1. Convergence of sequences:

Definition 1.19 [Convergence] Let (X, d) be a metric space, then $\{x_n\} \to x$ means

$$\forall \varepsilon > 0, \exists N \text{ such that } d(x_n, x) < \varepsilon, \forall n \geq N.$$

We will study the convergence by using open sets instead of metric.

Proposition 1.15 Let X be a metric space, then $\{x_n\} \to x$ if and only if for $\forall$ open set $U \ni x$, there exists N such that $x_n \in U$ for $\forall n \geq N$.

Proof. *Necessity*: Since $U \ni x$ is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. Since $\{x_n\} \to x$, there exists N such that $d(x_n, x) < \varepsilon$, i.e., $x_n \in B_\varepsilon(x) \subseteq U$ for $\forall n \geq N$.

Sufficiency: Let $\varepsilon > 0$ be given. Take the open set $U = B_\varepsilon(x) \ni x$, then there exists N such that $x_n \in U = B_\varepsilon(x)$ for $\forall n \geq N$, i.e., $d(x_n, x) < \varepsilon$, $\forall n \geq N$.

■

2. Continuity:

Definition 1.20 [Continuity] Let (X, d) and (Y, ρ) be given metric spaces. Then $f : X \to Y$ is continuous at $x_0 \in X$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon.$$

The function f is continuous on X if f is continous for all $x_0 \in X$.

We can get rid of metrics to study continuity:

Proposition 1.16 (a) The function f is continuous at x if and only if for all open $U \ni f(x)$, there exists $\delta > 0$ such that the set $B(x, \delta) \subseteq f^{-1}(U)$.

(b) The function f is continuous on X if and only if $f^{-1}(U)$ is open in X for each open set $U \subseteq Y$.

During the proof we will apply a small lemma:

Proposition 1.17 f is continuous at x if and only if for all $\{x_n\} \to x$, we have $\{f(x_n)\} \to f(x)$.

Proof. (a) *Necessity*:

Due to the openness of $U \ni f(x)$, there exists a ball $B(f(x), \varepsilon) \subseteq U$.

Due to the continuity of f at x, there exists $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \varepsilon$, which implies

$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \subseteq U,$$

which implies $B(x, \delta) \subseteq f^{-1}(U)$.

Sufficiency:

Let $\{x_n\} \to x$. It suffices to show $\{f(x_n)\} \to f(x)$. For each open $U \ni f(x)$, by hypothesis, there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(U)$.

Since $\{x_n\} \to x$, there exists N such that

$$x_n \in B_\delta(x) \subseteq f^{-1}(U), \forall n \geq N \implies f(x_n) \in U, \forall n \geq N$$

Let $\varepsilon > 0$ be given, and then construct the $U = B_\varepsilon(f(x))$. The argument above shows that $f(x_n) \in B_\varepsilon(f(x))$ for $\forall n \geq N$, which implies $\rho(f(x_n), f(x)) < \varepsilon$, i.e., $\{f(x_n)\} \to f(x)$.

(b) For the forward direction, it suffices to show that each point x of $f^{-1}(U)$ is an interior point of $f^{-1}(U)$, which is shown by part (a); the converse follows trivially by applying (a).

■

R As illustracted above, convergence, continuity, (and compactness) can be defined by using open sets $\mathcal{T}$ only.

1.6.2. Topological Spaces

Definition 1.21 A **topological space** $(X, \mathcal{T})$ consists of a (non-empty) set X, and a family of subsets of X ("open sets" $\mathcal{T}$) such that

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T}$ implies $U \bigcap V \in \mathcal{T}$
3. If $U_\alpha \in \mathcal{T}$ for all $\alpha \in \mathcal{A}$, then $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T}$.

The elements in $\mathcal{T}$ are called **open subsets** of X. The $\mathcal{T}$ is called a **topology** on X. ■

■ **Example 1.20** 1. Let (X, d) be any metric space, and

$$\mathcal{T} = \{\text{all open subsets of } X\}$$

It's clear that $\mathcal{T}$ is a topology on X.

2. Define the discrete topology

$$\mathcal{T}_{\text{dis}} = \{\text{all subsets of } X\}$$

It's clear that $\mathcal{T}_{\text{dis}}$ is a topology on X, (which also comes from the discrete metric (X, d_{discrete})).

R We say $(X, \mathcal{T})$ is induced from a metric (X, d) (or it is **metrizable**) if $\mathcal{T}$ is the faimly of open subsets in (X, d).

3. Consider the indiscrete topology $(X, \mathcal{T}_{\text{indis}})$, where X contains more than one element:

$$\mathcal{T}_{\text{indis}} = \{\emptyset, X\}.$$

Question: is $(X, \mathcal{T}_{\text{indis}})$ metrizable? No. For any metric d defined on X, let x, y be distinct points in X, and then $\varepsilon := d(x, y) > 0$, hence $B_{\frac{1}{2}\varepsilon}(x)$ is a open set belonging to the corresponding induced topology. Since $x \in B_{\frac{1}{2}\varepsilon}(x)$ and $y \notin B_{\frac{1}{2}\varepsilon}(y)$, we conclude that $B_{\frac{1}{2}\varepsilon}(x)$ is neither $\emptyset$ nor X, i.e., the topology induced by any metric d is not the indiscrete topology.

4. Consider the cofinite topology $(X, \mathcal{T}_{\text{cofin}})$:

$$\mathcal{T}_{\text{cofin}} = \{U \mid X \setminus U \text{ is a finite set}\} \bigcup \{\emptyset\}$$

Question: is $(X, \mathcal{T}_{\text{cofin}})$ metrizable?

Definition 1.22 [Equivalence] Two metric spaces are **topologically equivalent** if they give rise to the same topology.

Example 1.21 Metrics d_1, d_2, d_∞ in $\mathbb{R}^n$ are topologically equivalent.

1.6.3. Closed Subsets

Definition 1.23 [Closed] Let $(X,\mathcal{T})$ be a topology space. Then $V \subseteq X$ is **closed** if $X \setminus V \in J$

■ **Example 1.22** Under the topology space $(\mathbb{R},\mathcal{T}_{\text{usual}})$, $(b,\infty)\bigcup(-\infty,a) \in \mathcal{T}$. Therefore,

$$[a,b] = \mathbb{R} \setminus \Big((b,\infty)\bigcup(-\infty,a)\Big)$$

is closed in $\mathbb{R}$ under usual topology.

R It is important to say that V is **closed in** X**.** You need to specify the underlying the space X.

Chapter 2

Week2

2.1. Monday for MAT3040

Reviewing.

1. Linear Combination and Span
2. Linear Independence
3. Basis: a set of vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is called a **basis** for V if $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is linearly independent, and $V = \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$.

   Lemma: Given $V = \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$, we can find a basis for this set. Here V is said to be **finitely generated**.
4. Lemma: The vector $\boldsymbol{w} \in \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \setminus \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ implies that

$$\boldsymbol{v}_1 \in \text{span}\{\boldsymbol{w}, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\} \setminus \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$$

2.1.1. Basis and Dimension

Theorem 2.1 Let V be a finitely generated vector space. Suppose $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ and $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ are two basis of V. Then $m = n$. (where m is called the **dimension**)

Proof. Suppose on the contrary that $m \neq n$. Without loss of generality (w.l.o.g.), assume that $m < n$. Let $\boldsymbol{v}_1 = \alpha_1 \boldsymbol{w}_1 + \cdots + \alpha_n \boldsymbol{w}_n$, with some $\alpha_i \neq 0$. w.l.o.g., assume $\alpha_1 \neq 0$. Therefore,

$$\boldsymbol{v}_1 \in \text{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \setminus \text{span}\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \tag{2.1}$$

which implies that $\boldsymbol{w}_1 \in \text{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \setminus \text{span}\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$.

Then we claim that $\{\boldsymbol{v}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$ is a basis of V:

1. Note that $\{\boldsymbol{v}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$ is a spannning set:

$$\begin{aligned}\boldsymbol{w}_1 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\} &\implies \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\} \\ &\implies \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\} \subseteq \operatorname{span}\{\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}\end{aligned}$$

   Since $V = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$, we have $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\} = V$.

2. Then we show the linear independence of $\{\boldsymbol{v}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$. Consider the equation

$$\beta_1 \boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2 + \cdots + \beta_n \boldsymbol{w}_n = \mathbf{0}$$

   (a) When $\beta_1 \neq 0$, we imply

$$\boldsymbol{v}_1 = \left(-\frac{\beta_2}{\beta_1}\right)\boldsymbol{w}_2 + \cdots + \left(-\frac{\beta_n}{\beta_1}\right)\boldsymbol{w}_n \in \operatorname{span}\{\boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\},$$

   which contradicts (2.1).

   (b) When $\beta_1 = 0$, then $\beta_2 \boldsymbol{w}_2 + \cdots + \beta_n \boldsymbol{w}_n = \mathbf{0}$, which implies $\beta_2 = \cdots = \beta_n = 0$, due to the independence of $\{\boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$.

Therefore, $\boldsymbol{v}_2 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$, i.e.,

$$\boldsymbol{v}_2 = \gamma_1 \boldsymbol{v}_1 + \cdots + \gamma_n \boldsymbol{v}_n,$$

where $\gamma_2, \ldots, \gamma_n$ cannot be all zeros, since otherwise $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ are linearly dependent, i.e., $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ cannot form a basis. w.l.o.g., assume $\gamma_2 \neq 0$, which implies

$$\boldsymbol{w}_2 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_3, \ldots, \boldsymbol{w}_n\} \setminus \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_3, \ldots, \boldsymbol{w}_n\}.$$

Following the simlar argument above, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_3, \ldots, \boldsymbol{w}_n\}$ forms a basis of V.

Continuing the argument above, we imply $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m, \boldsymbol{w}_{m+1}, \ldots, \boldsymbol{w}_n\}$ is a basis of V.

Since $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\}$ is a basis as well, we imply

$$\boldsymbol{w}_{m+1} = \delta_1\boldsymbol{v}_1 + \cdots + \delta_m\boldsymbol{v}_m$$

for some $\delta_i \in \mathbb{F}$, i.e., $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m,\boldsymbol{w}_{m+1}\}$ is linearly dependent, which is a contradction.

■

■ **Example 2.1** A vector space may have more than one basis.

Suppose $V = \mathbb{F}^n$, it is clear that $\dim(V) = n$, and

$$\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\} \text{ is a basis of } V, \text{ where } \boldsymbol{e}_i \text{ denotes a unit vector.}$$

There could be other basis of V, such as

$$\left\{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\begin{pmatrix}1\\1\\\vdots\\0\end{pmatrix},\dots,\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix},\right\}$$

Actually, the columns of any invertible $n\times n$ matrix forms a basis of V. ■

■ **Example 2.2** Suppose $V = M_{m\times n}(\mathbb{R})$, we claim that $\dim(V) = mn$:

$$\left\{E_{ij}\middle|\begin{matrix}1\le i\le m\\1\le j\le n\end{matrix}\right\} \text{ is a basis of } V,$$

where E_{ij} is $m\times n$ matrix with 1 at (i,j)-th entry, and 0s at the remaining entries. ■

■ **Example 2.3** Suppose $V = \{\text{all polynomials of degree} \le \text{n}\}$, then $\dim(V) = n+1$. ■

■ **Example 2.4** Supppose $V = \{\boldsymbol{A}\in M_{n\times n}(\mathbb{R}) \mid \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}\}$, then $\dim(V) = \frac{n(n+1)}{2}$. ■

■ **Example 2.5** Let $W = \{\boldsymbol{B} \in M_{n\times n}(\mathbb{R}) \mid \boldsymbol{B}^{\mathrm{T}} = -\boldsymbol{B}\}$, then $\dim(V) = \frac{n(n-1)}{2}$. ■

(R) Sometimes it should be classified the field $\mathbb{F}$ for the scalar multiplication to define a vector space. Conside the example below:

1. Let $V = \mathbb{C}$, then $\dim(\mathbb{C}) = 1$ for the scalar multiplication defined under the field $\mathbb{C}$.
2. Let $V = \text{span}\{1, i\} = \mathbb{C}$, then $\dim(\mathbb{C}) = 2$ for the scalar multiplication defined under the field $\mathbb{R}$, since all $z \in V$ can be written as $z = a + bi$, $\forall a, b \in \mathbb{R}$.
3. Therefore, to aviod confusion, it is safe to write

$$\dim_{\mathbb{C}}(\mathbb{C}) = 1, \quad \dim_{\mathbb{R}}(\mathbb{C}) = 2.$$

2.1.2. Operations on a vector space

Note that the basis for a vector space is characterized as the **maximal linearly independent set**.

Theorem 2.2 — Basis Extension. Let V be a finite dimensional vector space, and $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be a linearly independent set on V, Then we can extend it to the basis $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_n\}$ of V.

Proof.

- Suppose $\dim(V) = n > k$, and $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ is a basis of V. Consider the set $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\} \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$, which is linearly dependent, i.e.,

$$\alpha_1 \boldsymbol{w}_1 + \cdots + \alpha_n \boldsymbol{w}_n + \beta_1 \boldsymbol{v}_1 + \cdots + \beta_k \boldsymbol{v}_k = \boldsymbol{0},$$

  with some $\alpha_i \neq 0$, since otherwise this equation will only have trivial solution. w.l.o.g., assume $\alpha_1 \neq 0$.

- Therefore, consider the set $\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$. We keep removing elements from $\{\boldsymbol{w}_2,\dots,\boldsymbol{w}_n\}$ until we first get the set

$$S\bigcup\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\},$$

with $S\subseteq\{\boldsymbol{w}_1,\boldsymbol{w}_2,\dots,\boldsymbol{w}_n\}$ and $S\bigcup\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}$ is linearly independent, i.e., S is a maximal subset of $\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_n\}$ such that $S\bigcup\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}$ is linearly independent.

- Rewrite $S=\{\boldsymbol{v}_{k+1},\dots,\boldsymbol{v}_m\}$ and therefore $S'=\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k,\boldsymbol{v}_{k+1},\dots,\boldsymbol{v}_m\}$ are linearly independent. It suffices to show S' spans V.
  - Indeed, for all $\boldsymbol{w}_i\in\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_n\}$, $\boldsymbol{w}_i\in\operatorname{span}(S')$, since otherwise the equation

    $$\alpha\boldsymbol{w}_i+\beta_1\boldsymbol{v}_1+\cdots+\beta_m\boldsymbol{v}_m=\mathbf{0}\implies\alpha=0,$$

    which implies that $\beta_1\boldsymbol{v}_1+\cdots+\beta_m\boldsymbol{v}_m=\mathbf{0}$ admits only trivial solution, i.e.,

    $$\{\boldsymbol{w}_i\}\bigcup S'=\{\boldsymbol{w}_i\}\bigcup S\bigcup\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_k\}\text{ is linearly independent,}$$

    which violetes the maximality of S.

  Therefore, all $\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_n\}\subseteq\operatorname{span}(S')$, which implies $\operatorname{span}(S')=V$.

  Therefore, S' is a basis of V.

■

R Start with a spanning set, we keep removing something to form a basis; start with independent set, we keep adding something to form a basis.

In other words, the basis is both the minimal spanning set, and the maximal linearly independent set.

Definition 2.1 [Direct Sum] Let W_1,W_2 be two vector subspaces of V, then

1. $W_1\bigcap W_2:=\{\boldsymbol{w}\in V\mid\boldsymbol{w}\in W_1,\text{ and }\boldsymbol{w}\in W_2\}$
2. $W_1+W_2:=\{\boldsymbol{w}_1+\boldsymbol{w}_2\mid\boldsymbol{w}_i\in W_i\}$

3. If furthermore that $W_1 \bigcap W_2 = \{\mathbf{0}\}$, then $W_1 + W_2$ is denoted as $W_1 \oplus W_2$, which is called **direct sum**.

Proposition 2.1 $W_1 \bigcap W_2$ and $W_1 + W_2$ are vector subspaces of V.

2.2. Monday for MAT3006

Reviewing.

1. Equivalent Metric:
$$d_1(\boldsymbol{x},\boldsymbol{y}) \le Kd_2(\boldsymbol{x},\boldsymbol{y}) \le K'd_1(\boldsymbol{x},\boldsymbol{y})$$
In $\mathcal{C}[0,1]$, the metric d_1 and d_∞ are not equivalent:
For $f_n(x) = x^n n^2(1-x)$, $d_1(f_n,0) \to 1$ and $d_\infty(f_n,0) \to \infty$. Suppose on contrary that
$$d_1(f_n,0) \le Kd_\infty(\boldsymbol{x},\boldsymbol{y}) \le K'd_1(\boldsymbol{x},\boldsymbol{y}).$$
Taking limit both sides, we imply the immediate term goes to infinite, which is a contradiction.
2. Continuous functions: the function f is continuous is equivalent to say for $\forall x_n \to x$, we have $f(x_n) \to f(x)$.
3. Open sets: Let (X,d) be a metric space. A set $U \subseteq X$ is open if for each $x \in U$, there exists $\rho_x > 0$ such that $B_{\rho_x}(x) \subseteq U$.

R Unless stated otherwise, we assume that
$$\mathcal{C}[a,b] \longleftrightarrow (\mathcal{C}[a,b],d_\infty)$$
$$\mathbb{R}^n \longleftrightarrow (\mathbb{R}^n,d_2)$$

2.2.1. Remark on Open and Closed Set

■ **Example 2.6** Let $X = \mathcal{C}[a,b]$, show that the set
$$U := \{f \in X \mid f(x) > 0, \forall x \in [a,b]\} \quad \text{is open.}$$

Take a point $f \in U$, then

$$\inf_{[a,b]} f(x) = m > 0.$$

Consider the ball $B_{m/2}(f)$, and for $\forall g \in B_{m/2}(f)$,

$$\begin{aligned}
|g(x)| &\geq |f(x)| - |f(x) - g(x)| \\
&\geq \inf_{[a,b]} |f(x)| - \sup_{[a,b]} |f(x) - g(x)| \\
&\geq m - \frac{m}{2} \\
&= \frac{m}{2} > 0, \ \forall x \in [a,b]
\end{aligned}$$

Therefore, we imply $g \in U$, i.e., $B_{m/2}(f) \subseteq U$, i.e., U is open in X. ▪

Proposition 2.2 Let (X,d) be a metric space. Then

1. $\emptyset, X$ are open in X
2. If $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ are open in X, then $\bigcup_{\alpha \in \mathcal{A}}$ is also open in X
3. If $U_1, \dots, U_n$ are open in X, then $\bigcap_{i=1}^{n} U_i$ are open in X

R Note that $\bigcap_{i=1}^{\infty} U_i$ is not necessarily open if all U_i's are all open:

$$\bigcap_{i=1}^{\infty} \left(-\frac{1}{i}, 1 + \frac{1}{i}\right) = [0,1]$$

Definition 2.2 [Closed] The closed set in metric space (X,d) are the complement of open sets in X, i.e., any closed set in X is of the form $V = X \setminus U$, where U is oepn. ▪

For example, in $\mathbb{R}$,

$$[a,b] = \mathbb{R} \setminus \{(-\infty, a) \bigcup (b, \infty)\}$$

Proposition 2.3 1. $\emptyset, X$ are closed in X

2. If $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ are closed subsets in X, then $\bigcap_{\alpha \in \mathcal{A}} V_\alpha$ is also closed in X
3. If $V_1, \dots, V_n$ are closed in X, then $\bigcup_{i=1}^{n} V_i$ is also closed in X.

R Whenever you say U is open or V is closed, you need to specify the underlying space, e.g.,

$$\textbf{Wrong}: U \text{ is open}$$

$$\textbf{Right}: U \text{ is open in } X$$

Proposition 2.4 The following two statements are equivalent:

1. The set V is closed in metric space (X,d).
2. If the sequence $\{v_n\}$ in V converges to x, then $x \in V$

Proof. Necessity.

Suppose on the contrary that $\{v_n\} \to x \notin V$. Since $X \setminus V \ni x$ is open, there exists an open ball $B_\varepsilon(x) \subseteq X \setminus V$.

Due to the convergence of sequence, there exists N such that $d(v_n, x) < \varepsilon$ for $\forall n \geq N$, i.e., $v_n \in B_\varepsilon(x)$, i.e., $v_n \notin V$, which contradicts to $\{v_n\} \subseteq V$.

Sufficiency.

Suppose on the contrary that V is not closed in X, i.e., $X \setminus V$ is not open, i.e., there exists $x \notin V$ such that for all open $U \ni x$, $U \bigcap V \neq \emptyset$. In particular, take

$$U_n = B_{1/n}(x), \implies \exists v_n \in B_{1/n}(x) \bigcap V,$$

i.e., $\{v_n\} \to x$ but $x \notin V$, which is a contradiction. ■

Proposition 2.5 Given two metric space (X,d) and (Y,ρ), the following statements are equivalent:

1. A function $f : (X,d) \to (Y,\rho)$ is continuous on X
2. For $\forall U \subseteq Y$ open in Y, $f^{-1}(U)$ is open in X.
3. For $\forall V \subseteq Y$ closed in Y, $f^{-1}(V)$ is closed in X.

■ **Example 2.7** The mapping $\Psi : \mathcal{C}[a,b] \to \mathbb{R}$ is defined as:

$$f \mapsto f(c)$$

where Ψ is called a **functional**.

Show that Ψ is continuous by using d_∞ metric on $\mathcal{C}[a,b]$:

1. Any open set in $\mathbb{R}$ can be written as countably union of open disjoint intervals, and therefore suffices to consider the pre-image $\Psi^{-1}(a,b) = \{f \mid f(c) \in (a,b)\}$. Following the similar idea in Example (2.6), it is clear that $\Psi^{-1}(a,b)$ is open in $(\mathcal{C}[a,b], d_\infty)$. Therefore, Ψ is continuous.
2. Another way is to apply definition.

We now study open sets in a subspace $(Y, d_Y) \subseteq (X, d_X)$, i.e.,

$$d_Y(y_1, y_2) := d_X(y_1, y_2).$$

Therefore, the open ball is defined as

$$\begin{aligned}
B_\varepsilon^Y(y) &= \{y' \in Y \mid d_Y(y, y') < \varepsilon\} \\
&= \{y' \in Y \mid d_X(y, y') < \varepsilon\} \\
&= \{y' \in X \mid d_X(y, y') < \varepsilon, y' \in Y\} \\
&= B_\varepsilon^X(y) \bigcap Y
\end{aligned}$$

Proposition 2.6 All open sets in the subspace $(Y, d_Y) \subseteq (X, d_X)$ are of the form $U \bigcap Y$, where U is open in X.

Corollary 2.1 For the subspace $(Y, d_Y) \subseteq (X, d_X)$, the mapping $i : (Y, d_Y) \to (X, d_X)$ with $i(y) = y, \forall y \in Y$ is continuous.

Proof. $i^{-1}(U) = U \bigcap Y$ for any subset $U \subseteq X$. The results follows from proposition (2.5). ∎

R It's important to specify the underlying space to describe an open set.

For example, the interval $[0, \frac{1}{2})$ is not open in $\mathbb{R}$, while $[0, \frac{1}{2})$ is open in $[0, 1]$,

since
$$[0,\frac{1}{2}) = (-\frac{1}{2},\frac{1}{2})\bigcap[0,1].$$

2.2.2. Boundary, Closure, and Interior

Definition 2.3 Let (X,d) be a metric space, then

1. A point x is a **boundary point** of $S \subseteq X$ (denoted as $x \in \partial S$) if for any open $U \ni x$, then both $U \bigcap S, U \setminus S$ are non-empty.
   (one can replace U by $B_{1/n}(x)$, with $n = 1,2,\dots$)
2. The **closure** of S is defined as $\overline{S} = S \bigcup \partial S$.
3. A point x is an **interior point** of S (denoted as $x \in S^\circ$) if there $\exists U \ni x$ open such that $U \subseteq S$. We use S° to denote the set of interior points.

Proposition 2.7

1. The closure of S can be equivalently defined as
$$\overline{S} = \bigcap\{C \in X \mid C \text{ is closed and } C \supseteq S\}$$
   Therefore, $\overline{S}$ is the smallest closed set containing S.
2. The interior set of S can be equivalently defined as
$$S^\circ = \bigcup\{U \subseteq X \mid U \text{ is open and } U \subseteq S\}$$
   Therefore, S° is the largest open set contained in S.

Example 2.8 For $S = [0,\frac{1}{2}] \subseteq X$, we have

1. $\partial S = \{0,\frac{1}{2}\}$
2. $\overline{S} = [0,\frac{1}{2}]$
3. $S^\circ = (0,\frac{1}{2})$

Proof. 1. (a) Firstly, we show that $\overline{S}$ is closed, i.e., $X \setminus \overline{S}$ is open.

- Take $x \notin \overline{S}$. Since $x \notin \partial S$, there $\exists B_r(x) \ni x$ such that

$$B_r(x) \bigcap S, \quad \text{or} \quad B_r(x) \setminus S \text{ is } \emptyset.$$

- Since $x \notin S$, the set $B_r(x) \setminus S$ is not empty. Therefore, $B_r(x) \bigcap S = \emptyset$.
- It's clear that $B_{r/2}(x) \bigcap S = \emptyset$. We claim that $B_{r/2}(x) \bigcap \overline{S}$ is empty. Suppose on the contrary that

$$y \in B_{r/2}(x) \bigcap \partial S,$$

which implies that $B_{r/2}(y) \bigcap S \neq \emptyset$. Therefore,

$$B_{r/2}(y) \subseteq B_r(x) \implies B_r(x) \bigcap S \supseteq B_{r/2}(y) \bigcap S \neq \emptyset,$$

which is a contradiction.

Therefore, $x \in X \setminus \overline{S}$ implies $B_{r/2}(x) \bigcap \overline{S} = \emptyset$, i.e., $X \setminus \overline{S}$ is open, i.e., $\overline{S}$ is closed.

(b) Secondly, we show that $\overline{S} \subseteq C$, for any closed $C \supseteq S$, i.e., suffices to show $\partial S \subseteq C$.

Take $x \in \partial S$, and construct a sequence

$$x_n \in B_{1/n}(x) \bigcap S.$$

Here $\{x_n\}$ is a sequence in $S \subseteq C$ converging to x, which implies $x \in C$, due to the closeness of C in X.

Combining (a) and (b), the result follows naturally. (Question: do we need to show the well-defineness?)

2. Exercise. Show that

$$S^\circ = S \setminus \partial S = X \setminus (\overline{X \setminus S}).$$

Then it's clear that S° is open, and contained in S.

∎

The next lecture we will talk about compactness and sequential compactness.

2.3. Monday for MAT4002

Reviewing.

1. Topological Space $(X,\mathcal{J})$: a special class of topological space is that induced from metric space (X,d):
$$(X,\mathcal{T}), \quad \text{with } \mathcal{T} = \{\text{all open sets in } (X,d)\}$$
2. Closed Sets $(X \setminus U)$ with U open.

Proposition 2.8 Let $(X,\mathcal{T})$ be a topological space,

1. $\emptyset, X$ are closed in X
2. V_1, V_2 closed in X implies that $V_1 \bigcup V_2$ closed in X
3. $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ closed in X implies that $\bigcap_{\alpha\in\mathcal{A}} V_\alpha$ closed in X

Proof. Applying the De Morgan's Law
$$(X \setminus \bigcup_{i\in I} U_i) = \bigcap_{i\in I}(X \setminus U_i)$$
■

2.3.1. Convergence in topological space

Definition 2.4 [Convergence] A sequence $\{x_n\}$ of a topological space $(X,\mathcal{T})$ converges to $x \in X$ if $\forall U \ni x$ is open, there $\exists N$ such that $x_n \in U, \forall n \geq N$.

Example 2.9 1. The topology for the space $(X = \mathbb{R}^n, d_2) \to (X,\mathcal{T})$ (i.e., a topological space induced from meric space $(X = \mathbb{R}^n, d_2)$) is called a **usual topology** on $\mathbb{R}^n$.

When I say $\mathbb{R}^n$ (or subset of $\mathbb{R}^n$) is a topological space, it is equipeed with usual topology.

Convergence of sequence in $(\mathbb{R}^n,\mathcal{T})$ is the usual convergence in analysis.

For $\mathbb{R}^n$ or metric space, the limit of sequence (if exists) is unique.

2. Consider the topological space $(X, \mathcal{T}_{\text{indiscrete}})$. Take any sequence $\{x_n\}$ in X, it is convergent to any $x \in X$. Indeed, for $\forall U \ni x$ open, $U = X$. Therefore,

$$x_n \in U(= X), \forall n \geq 1.$$

3. Consider the topological space $(X, \mathcal{T}_{\text{cofinite}})$, where X is infinite. Consider $\{x_n\}$ is a sequence satisfying $m \neq n$ implies $x_m \neq x_n$. Then $\{x_n\}$ is convergent to any $x \in X$. (Question: how to define openness for $\mathcal{T}_{\text{cofinite}}$ and $\mathcal{T}_{\text{indiscrete}}$)?
4. Consider the topological space $(X, \mathcal{T}_{\text{discrete}})$, the sequence $\{x_n\} \to x$ is equivalent to say $x_n = x$ for all sufficiently large n.

■

R The limit of sequences may not be unique. The reason is that "$\mathcal{T}$ is not big enough". We will give a criterion to make sure the limit is unique in the future. (Hausdorff)

Proposition 2.9 If $F \subseteq (X, \mathcal{T})$ is closed, then for any convergent sequence $\{x_n\}$ in F, the limit(s) are also in F.

Proof. Let $\{x_n\}$ be a sequence in F with limit $x \in X$. Suppose on the contrary that $x \notin F$ (i.e., $x \in X \setminus F$ that is open). There exists N such that

$$x_n \in X \setminus F, \forall n \geq N,$$

i.e., $x_n \notin F$, which is a contradiction. ■

R The converse may not be true. If the $(X, \mathcal{T})$ is metrizable, the converse holds. Counter-example: Consider the co-countable topological space $(X, \mathcal{T}_{\text{co-co}})$, where

$$\mathcal{T}_{\text{co-co}} = \{U \mid X \setminus U \text{ is a countable set}\} \bigcup \{\emptyset\},$$

and X is uncontable. Let $F \subsetneq$ be an un-countable set such that is closed under limits, e.g., $[0,1]$. It's clear that $X \setminus F \notin \mathcal{T}_{\text{co-co}}$, i.e., F is not closed.

2.3.2. Interior, Closure, Boundary

Definition 2.5 Let $(X,\mathcal{T})$ be a topological space, and $A \subseteq X$ a subset.

1. The **interior** of A is

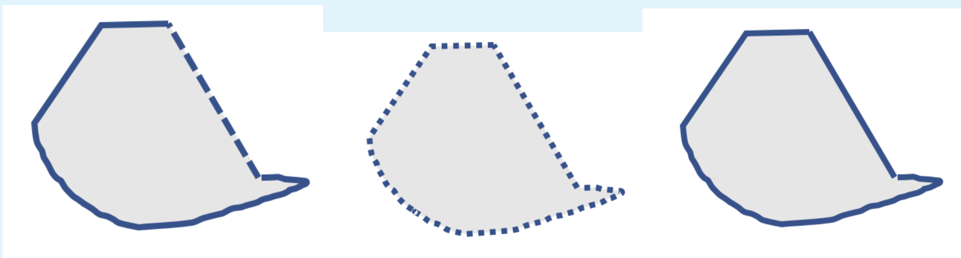
$$A^\circ = \bigcup_{U \subseteq A, U \text{ is open}} U$$

2. The **closure** of A is

$$\overline{A} = \bigcap_{A \subseteq V, V \text{ is closed}} V$$

If $\overline{A} = X$, we say that A is dense in X.

The graph illustration of the definition above is as follows:



(a) Illustration of A (b) Illustration of A° (c) Illustration of $\overline{A}$

Figure 2.1: Graph Illustrations

■ **Example 2.10**

1. For $[a,b) \subseteq \mathbb{R}$, we have:

$$[a,b)^\circ = (a,b), \quad \overline{[a,b)} = [a,b]$$

2. For $X = \mathbb{R}$, $\mathbb{Q}^\circ = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$.
3. Consider the discrete topology $(X, \mathcal{T}_{\text{discrete}})$, we have

$$S^\circ = S, \quad \overline{S} = S$$

The insights behind the definition (2.5) is as follows

Proposition 2.10

1. A° is the largest open subset of X contained in A; $\overline{A}$ is the smallest closed subset of X containing A.
2. If $A \subseteq B$, then $A^\circ \subseteq B$ and $\overline{A} \subseteq \overline{B}$
3. A is open in X is equivalent to say $A^\circ = A$; A is closed in X is equivalent to say $\overline{A} = A$.

■ **Example 2.11** Let (X,d) be a metric space. What's the closure of an open ball $B_r(x)$?

The direct intuition is to define the closed ball

$$\bar{B}_r(x) = \{y \in X \mid d(x,y) \leq r\}.$$

Question: is $\bar{B}_r(x) = \overline{B_r(x)}$?

1. Since $\bar{B}_r(x)$ is a closed subset of X, and $B_r(x) \subseteq \bar{B}_r(x)$, we imply that

$$\overline{B_r(x)} \subseteq \bar{B}_r(x)$$

2. However, we may find an example such that $\overline{B_r(x)}$ is a proper subset of $\bar{B}_r(x)$: Consider the discrete metric space (X, d_{discrete}) and for $\forall x \in X$,

$$B_1(x) = \{x\} \implies \overline{B_1(x)} = \{x\}, \quad \bar{B}_1(x) = X$$

The equality $\bar{B}_r(x) = \overline{B_r(x)}$ holds when (X,d) is a normed space.

Here is another characterization of $\overline{A}$:

Proposition 2.11

$$\overline{A} = \{x \in X \mid \forall \text{open } U \ni x, U \bigcap A \neq \varnothing\}$$

Proof. Define

$$S = \{x \in X \mid \forall \text{open } U \ni x, U \bigcap A \neq \varnothing\}$$

It suffices to show that $\overline{A} = S$.

1. First show that S is closed:

$$X \setminus S = \{x \in X \mid \exists U_x \ni x \text{ open s.t. } U_x \bigcap A = \emptyset\}$$

Take $x \in X \setminus S$, we imply there exists open $U_x \ni x$ such that $U_x \bigcap A = \emptyset$. We claim $U_x \subseteq X \setminus S$:

- For $\forall y \in U_x$, note that $U_x \ni y$ that is open, such that $U_x \bigcap A = \emptyset$. Therefore, $y \in X \setminus S$.

Therefore, we have $x \in U_x \subseteq X \setminus S$ for any $\forall x \in X \setminus S$.

Note that

$$X \setminus S = \bigcup_{x \in X \setminus S} \{x\} \subseteq \bigcup_{x \in X \setminus S} U_x \subseteq X \setminus S,$$

which implies $X \setminus S = \bigcup_{x \in X \setminus S} U_x$ is open, i.e., S is closed in X.

2. By definition, it is clear that $A \subseteq S$:

$$\forall a \in A, \forall \text{open } U \ni a, U \bigcap A \supseteq \{a\} \neq \emptyset \implies a \in S.$$

Therefore, $\overline{A} \subseteq \overline{S} = S$.

3. Suppose on the contrary that there exists $y \in S \setminus \overline{A}$.

Since $y \notin \overline{A}$, by definition, there exists $F \supseteq A$ closed such that $y \notin F$.

Therefore, $y \in X \setminus F$ that is open, and

$$(X \setminus F) \bigcap A \subseteq (X \setminus A) \bigcap A = \emptyset \implies y \notin S,$$

which is a contradiction. Therefore, $S = \overline{A}$.

■

Definition 2.6 [accumulation point] Let $A \subseteq X$ be a subset in a topological space. We call $x \in X$ are an **accumulation point (limit point)** of A if

$$\forall U \subseteq X \text{ open s.t. } U \ni x, (U \setminus \{x\}) \bigcap A \neq \emptyset.$$

The set of accumulation points of A is denoted as A' ■

Proposition 2.12 $\overline{A} = A \bigcup A'$.

2.4. Wednesday for MAT3040

Reviewing.

- Basis, Dimension
- Basis Extension
- $W_1 \bigcap W_2 = \emptyset$ implies $W_1 \oplus W_2 = W_1 + W_2$ (Direct Sum).

2.4.1. Remark on Direct Sum

Proposition 2.13 The set $W_1 + W_2 = W_1 \oplus W_2$ iff any $\boldsymbol{w} \in W_1 + W_2$ can be uniquely expressed as

$$\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2,$$

where $\boldsymbol{w}_i \in W_i$ for $i = 1, 2$.

R We can also define addiction among finite set of vector spaces $\{W_1, \dots, W_k\}$. If $\boldsymbol{w}_1 + \cdots + \boldsymbol{w}_k = \mathbf{0}$ implies $\boldsymbol{w}_i = 0, \forall i$, then we can write $W_1 + \cdots + W_k$ as

$$W_1 \oplus \cdots \oplus W_k$$

Proposition 2.14 — **Complementation.** Let $W \le V$ be a vector subspace of a fintie dimension vector space V. Then there exists $W' \le V$ such that

$$W \oplus W' = V.$$

Proof. It's clear that $\dim(W) := k \le n := \dim(V)$. Suppose $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is a basis of W.

By the basis extension proposition, we can extend it into $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_n\}$, which is a basis of V.

Therefore, we take $W' = \text{span}\{\boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_n\}$, which follows that

1. $W + W' = V$: $\forall \boldsymbol{v} \in V$ has the form

$$\boldsymbol{v} = (\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k) + (\alpha_{k+1} \boldsymbol{v}_{k+1} + \cdots + \alpha_n \boldsymbol{v}_n),$$

   where $\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k \in W$ and $\alpha_{k+1} \boldsymbol{v}_{k+1} + \cdots + \alpha_n \boldsymbol{v}_n \in W'$.

2. $W \bigcap W' = \{\mathbf{0}\}$: Suppose $\boldsymbol{v} \in W \bigcap W'$, i.e.,

$$\begin{aligned} \boldsymbol{v} &= (\beta_1 \boldsymbol{v}_1 + \cdots + \beta_k \boldsymbol{v}_k) + (0\boldsymbol{v}_{k+1} + \cdots + 0\boldsymbol{v}_n) \in W \\ &= (0\boldsymbol{v}_1 + \cdots + 0\boldsymbol{v}_k) + (\beta_{k+1} \boldsymbol{v}_{k+1} + \cdots + \beta_n \boldsymbol{v}_n) \in W'. \end{aligned}$$

   By the uniqueness of coordinates, we imply $\beta_1 = \cdots = \beta_n = 0$, i.e., $\boldsymbol{v} = \mathbf{0}$.

Therefore, we conclude that $W \oplus W' = V$. ■

2.4.2. Linear Transformation

Definition 2.7 [Linear Transformation] Let V, W be vector spaces. Then $T : V \to W$ is a **linear transformation** if

$$T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) = \alpha T(\boldsymbol{v}_1) + \beta T(\boldsymbol{v}_2),$$

for $\forall \alpha, \beta \in \mathbb{F}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$.

■ **Example 2.12**

1. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined as $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$ (where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$) is a linear transformation.
2. The transformation $T : \mathbb{R}[x] \to \mathbb{R}[x]$ defined as

$$p(x) \mapsto T(p(x)) = p'(x), \quad p(x) \mapsto T(p(x)) = \int_0^x p(t)\,\mathrm{d}t$$

   is a linear transformation

3. The transformation $T : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ defined as

$$\boldsymbol{A} \mapsto \text{trace}(\boldsymbol{A}) := \sum_{i=1}^{n} a_{ii}$$

is a linear transformation.

However, the transformation

$$\boldsymbol{A} \mapsto \det(\boldsymbol{A})$$

is not a linear transformation.

■

Definition 2.8 [Kernel/Image] Let $T : V \to W$ be a linear transfomation.

1. The **kernel** of T is

$$\ker(T) = T^{-1}(\mathbf{0}) = \{\boldsymbol{v} \in V \mid T(\boldsymbol{v}) = \mathbf{0}\}$$

2. The **image** (or range) of T is

$$\text{Im}(T) = T(\boldsymbol{v}) = \{T(\boldsymbol{v}) \in W \mid \boldsymbol{v} \in V\}$$

■

■ **Example 2.13** 1. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$, then

$$\ker(T) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \mathbf{0}\} = \text{Null}(\boldsymbol{A}) \qquad \text{Null Space}$$

and

$$\text{Im}(T) = \{\boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^n\} = \text{Col}(\boldsymbol{A}) = \text{span}\{\text{columns of } \boldsymbol{A}\} \qquad \text{Column Space}$$

2. For $T(p(x)) = p'(x)$, $\ker(T) = \{\text{constant polynomials}\}$ and $\text{Im}(T) = \mathbb{R}[x]$.

■

Proposition 2.15 The kernel or image for a linear transformation $T : V \to W$ also forms a vector subspace:

$$\ker(T) \leq V, \quad \mathrm{Im}(T) \leq W$$

Proof. For $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \ker(T)$, we imply

$$T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) = \mathbf{0},$$

which implies $\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2 \in \ker(T)$.

The remaining proof follows similarly. ■

Definition 2.9 [Rank/Nullity] Let V, W be finite dimensional vector spaces and $T : V \to W$ a linear transformation. Then we define

$$\mathrm{rank}(T) = \dim(\mathrm{im}(T))$$

$$\mathrm{nullity}(T) = \dim(\ker(T))$$

R Let

$$\mathrm{Hom}_{\mathbb{F}}(V, W) = \{\text{all linear transformations } T : V \to W\},$$

and we can define the addiction and scalar multiplication to make it a vector space:

1. For $T, S \in \mathrm{Hom}_{\mathbb{F}}(V, W)$, define

$$(T + S)(\boldsymbol{v}) = T(\boldsymbol{v}) + S(\boldsymbol{v}),$$

   which implies $T + S \in \mathrm{Hom}_{\mathbb{F}}(V, W)$.

2. Also, define

$$(\gamma T)(\boldsymbol{v}) = \gamma T(\boldsymbol{v}), \qquad \text{for } \forall \gamma \in \mathbb{F},$$

   which implies $\gamma T \in \mathrm{Hom}_{\mathbb{F}}(V, W)$.

In particular, if $V=\mathbb{R}^n, W=\mathbb{R}^m$, then

$$\mathrm{Hom}_{\mathbb{F}}(V,W)=M_{m\times n}(\mathbb{R}).$$

Proposition 2.16 If $\dim(V)=n, \dim(W)=m$, then $\dim(\mathrm{Hom}_{\mathbb{F}}(V,W))=mn$.

Proposition 2.17 There are anternative characterizations for the injectivity and surjectivity of lienar transformation T:

1. The linear transformation T is injective if and only if

$$\ker(T)=0, \Longleftrightarrow \mathrm{nullity}(T)=0.$$

2. The linear transformation T is surjective if and only if

$$\mathrm{im}(T)=W, \Longleftrightarrow \mathrm{rank}(T)=\dim(W).$$

3. If T is bijective, then T^{-1} is a linear transformation.

Proof. 1. (a) For the forward direction of (1),

$$\boldsymbol{x}\in\ker(T) \implies T(\boldsymbol{x})=0=T(\boldsymbol{0}) \implies \boldsymbol{x}=\boldsymbol{0}$$

(b) For the reverse direction of (1),

$$T(\boldsymbol{x})=T(\boldsymbol{y}) \implies T(\boldsymbol{x}-\boldsymbol{y})=\boldsymbol{0} \implies \boldsymbol{x}-\boldsymbol{y}\in\ker(T)=\boldsymbol{0} \implies \boldsymbol{x}=\boldsymbol{y}$$

2. The proof follows similar idea in (1).
3. Let $T^{-1}: W\to V$. For all $\boldsymbol{w}_1,\boldsymbol{w}_2\in W$, there exists $\boldsymbol{v}_1,\boldsymbol{v}_2\in V$ such that $T(\boldsymbol{v}_i)=\boldsymbol{w}_i$, i.e., $T^{-1}(\boldsymbol{w}_i)=\boldsymbol{v}_i$ $i=1,2$.

Consider the mapping

$$\begin{aligned} T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) &= \alpha T(\boldsymbol{v}_1) + \beta T(\boldsymbol{v}_2) \\ &= \alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2, \end{aligned}$$

which implies $\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2 = T^{-1}(\alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2)$, i.e.,

$$\alpha T^{-1}(\boldsymbol{w}_1) + \beta T^{-1}(\boldsymbol{w}_2) = T^{-1}(\alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2).$$

■

Definition 2.10 [isomorphism]

We say the vector subspaces V and W are isomorphic if there exists a bijective linear transfomation $T: V \to W$. $(V \cong W)$

This mapping T is called an **isomorphism** from V to W.

(R) If $\dim(V) = \dim(W) = n < \infty$, then $V \cong W$:

Take $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}, \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ as basis of V and W, respectively. Then one can construct $T: V \to W$ satisfying $T(\boldsymbol{v}_i) = \boldsymbol{w}_i$ for $\forall i$ as follows:

$$T(\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n) = \alpha_n \boldsymbol{w}_1 + \cdots + \alpha_n \boldsymbol{w}_n \ \forall \alpha_i \in \mathbb{F}$$

It's clear that our constructed T is a linear transformation.

(R) $V \cong W$ doesn't imply any linear transformations $T: V \to W$ is an isomorphism. e.g., $T(\boldsymbol{v}) = \mathbf{0}$ is not an isomorphic if $W \neq \{\mathbf{0}\}$.

Theorem 2.3 — **Rank-Nullity Theorem.** Let $T: V \to W$ be a linear transformation with $\dim(V) < \infty$. Then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).$$

Proof. Since $\ker(T) \leq V$, by proposition (2.14), there exists $V_1 \leq V$ such that

$$V = \ker(T) \oplus V_1.$$

1. Consider the transformation $T\mid_{V_1}: V_1 \to T(V_1)$, which is an isomorphism, since:

   - Surjectivity is immediate
   - For $\boldsymbol{v} \in \ker(T\mid_{V_1})$,

     $$T(\boldsymbol{v}) = \mathbf{0} \implies \boldsymbol{v} \in \ker(T),$$

     which implies $\boldsymbol{v} = \mathbf{0}$ since $\boldsymbol{v} \in \ker(T) \cap V_1 = 0$, i.e., the injectivity follows.

   Therefore, $\dim(V_1) = \dim(T(V_1))$.

2. Secondly, given an isomorphism T from X to Y with $\dim(X) < \infty$, then $\dim(X) = \dim(T(X))$. The reason follows from assignment 1 questions (8-9):

   $$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \text{ is a basis of } X \implies \{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_k)\} \text{ is a basis of } Y$$

3. Note that $T(V_1) = T(V) = \mathrm{im}(T)$, since:

   - for $\forall \boldsymbol{v} \in V$, $\boldsymbol{v} = \boldsymbol{v}_k + \boldsymbol{v}_1$, where $\boldsymbol{v}_k \in \ker(T), \boldsymbol{v}_1 \in V_1$, which implies

     $$T(\boldsymbol{v}) = T(\boldsymbol{v}_k) + T(\boldsymbol{v}_1) = \mathbf{0} + T(\boldsymbol{v}_1),$$

     i.e., $T(V) \subseteq T(V_1) \subseteq T(V)$, i.e., $T(V) = T(V_1)$.

4. By the proof of complementation,

$$\begin{aligned}
\dim(V) &= \dim(\ker(T)) + \dim(V_1) \\
&= \mathrm{nullity}(T) + \dim(T(V_1)) \\
&= \mathrm{nullity}(T) + \dim(T(V)) \\
&= \mathrm{nullity}(T) + \dim(\mathrm{im}(T)) \\
&= \mathrm{nullity}(T) + \mathrm{rank}(T).
\end{aligned}$$

2.5. Wednesday for MAT3006

2.5.1. Compactness

This lecture will talk about the generalization of closeness and boundedness property in $\mathbb{R}^n$. First let's review some simple definitions:

Definition 2.11 [Compact] Let (X,d) be a metric space, and $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ a collection of open sets.

1. $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is called an **open cover** of $E\subseteq X$ if $E\subseteq\cup_{\alpha\in\mathcal{A}}U_\alpha$
2. A **finite subcover** of $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is a finite sub-collection $\{U_{\alpha_1},\dots,U_{\alpha_n}\}\subseteq\{U_\alpha\}$ covering E.
3. The set $E\subseteq X$ is **compact** if every open cover of E has a finite subcover.

■

A well-known result is talked in MAT2006:

Theorem 2.4 — **Heine-Borel Theorem.** The set $E\subseteq\mathbb{R}^n$ is **compact** if and only if E is closed and bounded.

However, there's a notion of sequentially compact, and we haven't identify its gap and relation with compactness.

Definition 2.12 [Sequentially Compact] Let (X,d) be a metric space. Then $E\subseteq X$ is **sequentially compact** if every sequence in E has a convergent subsequence with limit in E. ■

A well-known result is talked in MAT2006:

Theorem 2.5 — **Bolzano-Weierstrass Theorem.** The set $E\subseteq\mathbb{R}^n$ is closed and bounded if and only if E is sequentially compact.

Actually, the definitions of comapctness and the sequential compactness are equivalent under a metric space.

Theorem 2.6 Let (X,d) be a metric space, then $E \subseteq X$ is compact if and only if E is sequentially compact.

Proof. Necessity

Suppose $\{x_n\}$ is a sequence in E, it suffices to show it has a convergent subsequence. Consider the tail of $\{x_n\}$, say

$$F_n = \{x_k \mid k \geq n\} \implies F_1 \supseteq F_2 \supseteq \cdots.$$

- Note that $\cap_{i=1}^{\infty} F_i \neq \emptyset$. Assume not, then we imply $\cup_{i=1}^{\infty}(E \setminus F_i) = E$, i.e., $\{E \setminus F_i\}_{i=1}^{\infty}$ a open cover of E. By the compactness of E, we imply there exists a finite subcover of E:
$$E = \bigcup_{j=1}^{r}(E \setminus F_{i_j}) \implies \bigcap_{j=1}^{r} F_{i_j} = \emptyset \implies F_{i_j} = \emptyset, \forall j$$
which is a contradiction, and there must exist an element $x \in \cap_{n=1}^{\infty} F_i$.
- For any $n \geq 1$, the open ball $B_{1/n}(x)$ must intersect with the n-th tail of the sequence $\{x_n\}$:
$$B_{1/n}(x) \cap \{x_k \mid k \geq n\} \neq \emptyset$$
Pick the r-th intersection, say x_{n_r}, which implies that the subsequence $x_{n_r} \to x$ as $r \to \infty$. The proof for necessity is complete.

Sufficiency

Firstly, let's assume the claim below hold (which will be shown later):

Proposition 2.18 If $E \subseteq X$ is sequentially compact, then for any$\varepsilon > 0$, there exists finitely many open balls, say $\{B_\varepsilon(x_1), \ldots, B_\varepsilon(x_n)\}$, covering E.

Suppose on the contrary that there exists an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of E, that has no finite subcover.

- By proposition (2.18), for $n \geq 1$, there are finitely many balls of radius $1/n$ covering E. Due to our assumption, there exists a open ball $B_{1/n}(y_n)$ such that $B_{1/n}(y) \cap E$ cannot be covered by finitely many members in $\{U_\alpha\}_{\alpha \in \mathcal{A}}$.

- Pick $x_n \in B_{1/n}(y_n)$ to form a sequence. Due to the sequential compactness of E, there exists a subsequence $\{x_{n_j}\} \to x$ for some $x \in E$.

- Since $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers E, there exists a U_β containing x. Since U_β is open and the radius of $B_{1/n_j}(y_{n_j})$ tends to 0, we imply that, for sufficiently large n_j, the set $B_{1/n_j}(y_{n_j}) \cap E$ is contained in U_β.

  In oteher words, U_β forms a **single** subcover of $B_{1/n}(y) \cap E$, which contradicts to our choice of $B_{1/n_j}(y_{n_j}) \cap E$. The proof for sufficiency is complete.

■

Proof for proposition (2.18). Pick $B_\varepsilon(x_1)$ for some $x_1 \in E$. Suppose $E \setminus B_\varepsilon(x_1) \neq \emptyset$. We can find $x_2 \notin B_\varepsilon(x_1)$ such that $d(x_2, x_1) \geq \varepsilon$.

Suppose $E \setminus (B_\varepsilon(x_1) \bigcup B_\varepsilon(x_2))$ is non-empty, then we can find $x_3 \notin B_\varepsilon(x_1) \bigcup B_\varepsilon(x_2)$ so that $d(x_j, x_3) \geq \varepsilon$, $j = 1, 2$.

Keeping this procedure, we obtain a sequence $\{x_n\}$ in E such that

$$E \setminus \bigcup_{j=1}^{n} B_\varepsilon(x_j) \neq \emptyset, \qquad \text{and} \qquad d(x_j, x_n) \geq \varepsilon, j = 1, 2, \dots, n-1.$$

By the sequential compactness of E, there exists $\{x_{n_j}\}$ and $x \in E$ so that $x_{n_j} \to x$ as $j \to \infty$. But then $d(x_{n_j}, x_{n_k}) < d(x_{n_j}, x) + d(x_{n_k}, x) \to 0$, which contradicts that $d(x_j, x_n) \geq \varepsilon$ for $\forall j < n$.

Therefore, one must have $E \setminus \bigcup_{j=1}^{N} B_\varepsilon(x_j) = \emptyset$ for some finite N.

The proof is complete. ■

R

1. Given the condition metric space,

$$\text{Sequential Compactness} \Longleftrightarrow \text{Compactness}$$

2. Given the condition metric space, we will show that

$$\text{Compactness} \implies \text{Closed and Bounded}$$

However, the converse may not necessarily hold. Given the condition the metric space is $\mathbb{R}^n$, then

$$\text{Compactness} \Longleftrightarrow \text{Closed and Bounded}$$

Proposition 2.19 Let (X,d) be a metric space. Then $E \subseteq X$ is compact implies that E is closed and bounded.

Proof.
1. Let $\{x_n\}$ be a convergent sequence in E. By sequential compactness, $\{x_{n_j}\} \to x$ for some $x \in E$. By the uniqueness of limits, under metric space, $\{x_n\} \to x$ for $x \in E$. The closeness is shown
2. Take $x \in E$ and consider the open cover $\bigcup_{n=1}^{\infty} B_n(x)$ of E. By compactness,

$$E \subseteq \bigcup_{i=1}^{k} B_{n_i}(x) = B_{n_k}(x),$$

which implies that for any $y,z \in E$,

$$d(y,z) \le d(y,x) + d(x,z) \le n_k + n_k = 2n_k.$$

The boundness is shown.

■

Here we raise several examples to show that the coverse does not necessarily hold under a metric space.

Example 2.14 Given the metric space $\mathcal{C}[0,1]$ and a set $E = \{f \in \mathcal{C}[0,1] \mid 0 \le f(x) \le 1\}$. Notice that E is closed and bounded:

- $E = \cap_{x\in[0,1]} \Psi_x^{-1}([0,1])$, where $\Psi_x(f) = f(x)$, which implies that E is closed.
- Note that $E \subseteq B_2(\mathbf{0}) = \{f \mid |f| < 2\}$, i.e., E is bounded.

However, E may not be compact. Consider a sequence $\{f_n\}$ with

$$f_n(x) = \begin{cases} nx, & 0 \le x \le \frac{1}{n} \\ 1, & \frac{1}{n} \le x \le 1 \end{cases}$$

Suppose on the contrary that E is sequentially compact, therefore there exists a subsequence $\{f_{n_k}\} \to f$ under d_∞ metric, which implies, $\{f_{n_k}\}$ uniformly converges to f.

By the definition of $f_n(x)$, we imply

$$f(x) = \begin{cases} 0, & x = 0 \\ 1, & x \in (0,1] \end{cases}$$

However, since d_∞ indicates uniform convergence, the limit for $\{f_{n_k}\}$, say f, must be continuous, which is a contradiction.

Theorem 2.7 Let the set E be compact in (X,d) and the function $f : (X,d) \to (Y,\rho)$ is continuous. Then $f(E)$ is compact in Y.

Note that the technique to show compactness by using the sequential compactness is very useful. However, this technique only applies to the metric space, but fail in general topological spaces.

Proof. Let $\{y_n\} = \{f(x_n)\}$ be any sequence in $f(E)$.

- By the compactness of X, $\{x_n\}$ has a convergent subsequence $\{x_{n_r}\} \to x$ as $r \to \infty$.
- Therefore, $\{y_{n_r}\} := \{f(x_{n_r})\} \to f(x)$ by the continuity of f.
- Therefore, $f(E)$ is sequentially compact, i.e., compact.

■

R The Theorem (2.7) is a generalization of the statement that *a continuous function on $\mathbb{R}^n$ admits its minimum and maximum*. Note that such an extreme value property no longer holds for arbitrary closed, bounded sets in a general metric space, but it continues to hold when the sets are strengthened to compact ones.

Another characterization of compactness in $\mathcal{C}[a,b]$ is shown in the Ascoli-Arzela Theorem (see Theorem (14.1) in MAT2006 Notebook).

2.5.2. Completeness

Definition 2.13 [Complete] Let (X,d) be metric space.

1. A sequence $\{x_n\}$ in (X,d) is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists some N such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.
2. A subset $E \subseteq X$ is said to be **complete** if every Cauchy sequence in E is convergent.

■

■ **Example 2.15** The set $X = \mathcal{C}[a,b]$ is complete:

- Suppose $\{f_n\}$ is Cauchy in $\mathcal{C}[a,b]$, i.e., $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$ for $\forall x \in [a,b]$.
- By the compactness of $\mathbb{R}$, the sequence $f_n(x) \to f(x)$ for some $f(x) \in \mathbb{R}, \forall x \in [a,b]$. It suffices to show $f_n \to f$ uniformly:
  - For fixed $\varepsilon > 0$, there exists $N > 0$ such that

$$d_\infty(f_n, f_{n+k}) < \frac{\varepsilon}{2}, \qquad \forall n \geq N, k \in \mathbb{N}$$

which implies that for $\forall x \in [a,b]$, $\forall n \geq N, k \in \mathbb{N}$,

$$|f_n(x) - f_{n+k}(x)| < \frac{\varepsilon}{2} \implies \lim_{k\to\infty} |f_n(x) - f_{n+k}(x)| \leq \frac{\varepsilon}{2}$$

Therefore, we imply

$$|f_n(x) - f(x)| = \lim_{k\to\infty} |f_n(x) - f_{n+k}(x)| \le \frac{\varepsilon}{2} < \varepsilon, \qquad \forall n \ge N, x \in [a,b]$$

The proof is complete.

■

2.6. Wednesday for MAT4002

Reviewing.

1. Interior, Closure:
$$\overline{A} = \{x \mid \forall U \ni x \text{ open}, U \bigcap A \neq \emptyset\}$$

2. Accumulation points

2.6.1. Remark on Closure

Definition 2.14 [Sequential Closure] Let A_S be the set of limits of any convergent sequence in A, then A_S is called the **sequential closure** of A.

Definition 2.15 [Accumulation/Cluster Points] The set of accumulation (limit) points is defined as
$$A' = \{x \mid \forall U \ni x \text{ open }, (U \setminus \{x\}) \bigcap A \neq \emptyset\}$$

R

1. (a) There exists some point in A but not in A':
$$A = \{1, 2, 3, \dots, n, \dots\}$$
Then any point in A is not in A'

   (b) There also exists some point in A' but not in A:
$$A = \{\frac{1}{n} \mid n \geq 1\}$$
Then the point 0 is in A' but not in A.

2. The closure $\overline{A} = A \bigcup A'$.
3. The size of the sequencial closure A_S is between A and $\overline{A}$, i.e., $A \subseteq A_S \subseteq \overline{A}$:

It's clear that $A \subseteq A_S$, since the sequence $\{a_n := a\}$ is convergent to a for $\forall a \in A$.

For all $a \in A_S$, we have $\{a_n\} \to a$. Then for any open $U \ni a$, there exists N such that $\{a_N, a_{N+1}, \dots\} \subseteq U \bigcap A \neq \emptyset$. Therefore, $a \in \overline{A}$, i.e., $A_S \subseteq \overline{A}$.

Question: Is $A_S = \overline{A}$?

Proposition 2.20 Let (X,d) be a metric space, then $A_S = \bar{A}$.

Proof. Let $a \in \overline{A}$, then there exists $a_n \in B_{1/n}(a) \bigcap A$, which implies $\{a_n\} \to a$, i.e., $a \in A_S$. ■

R If $(X, \mathcal{T})$ is metrizable, then $A_S = \overline{A}$. The same goes for first countable topological spaces. However, A_S is a proper subset of A in general:

Let $A \subseteq X$ be the set of continuous functions, where $X = \mathbb{R}^{\mathbb{R}}$ denotes the set of all real-valued functions on $\mathbb{R}$, with the topology of pointwise convergence.

Then $A_S = B_1$, the set of all functions of first Baire-Category on $\mathbb{R}$; and $[A_S]_S = B_2$, the set of all functions of second Baire-Category on $\mathbb{R}$. Since $B_1 \neq B_2$, we have $[A_S]_S = A_S$. Note that $\overline{\overline{A}} = \overline{A}$. We conclude that A_S cannot equal to $\overline{A}$, since the sequential closure operator cannot be idemotenet.

Definition 2.16 [Boundary] The **boundary** of $\boldsymbol{A}$ is defined as

$$\partial \boldsymbol{A} = \overline{A} \setminus A^\circ$$

■

Proposition 2.21 Let $(X, \mathcal{T})$ be a topological space with $A, B \subseteq X$.

$$\overline{X \setminus A} = X \setminus A^\circ, \quad (X \setminus B)^\circ = X \setminus \overline{B} \quad \partial A = \overline{A} \bigcap (\overline{X \setminus A})$$

Proof.

$$X \setminus A^\circ = X \setminus \left(\bigcup_{U \text{ is open}, \, U \subseteq A} U \right) \tag{2.2a}$$

$$= \bigcap_{U \text{ is open}, \, U \subseteq A} (X \setminus U) \tag{2.2b}$$

$$= \bigcap_{V \text{ is closed}, \, F \supseteq X \setminus A} F \tag{2.2c}$$

$$= \overline{X \setminus A} \tag{2.2d}$$

Denoting $X \setminus A$ by B, we obtain:

$$(X \setminus B)^\circ = A^\circ \tag{2.3a}$$

$$= X \setminus (X \setminus A^\circ) \tag{2.3b}$$

$$= X \setminus \overline{X \setminus A} \tag{2.3c}$$

$$= X \setminus \overline{B} \tag{2.3d}$$

By definition of ∂A,

$$\partial A = \overline{A} \setminus A^\circ \tag{2.4a}$$

$$= \overline{A} \bigcap (X \setminus A^\circ) \tag{2.4b}$$

$$= \overline{A} \bigcap (\overline{X \setminus A}) \tag{2.4c}$$

■

2.6.2. Functions on Topological Space

Definition 2.17 [Continuous] Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map. Then the function f is continuous, if

$$U \in \mathcal{T}_Y \implies f^{-1}(U) \in \mathcal{T}_X$$

■ **Example 2.16**

1. The identity map $\mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T})$ defined as $x \mapsto x$ is continuous
2. The identity map $\mathrm{id} : (X, \mathcal{T}_{\text{discrete}}) \to (X, \mathcal{T}_{\text{indiscrete}})$ defined as $x \mapsto x$ is continuous. Since $\mathrm{id}^{-1}(\emptyset) = \emptyset$ and $\mathrm{id}^{-1}(X) = X$
3. The identity map $\mathrm{id} : (X, \mathcal{T}_{\text{indiscrete}}) \to (X, \mathcal{T}_{\text{discrete}})$ defined as $x \mapsto x$ is not continuous.

Proposition 2.22 If $f : X \to Y$, and $g : Y \to Z$ be continuous, then $g \circ f$ is continuous

Proof. For given $U \in \mathcal{T}_Z$, we imply

$$g^{-1}(U) \in \mathcal{T}_Y \implies f^{-1}(g^{-1}(U)) \in \mathcal{T}_X,$$

i.e., $(g \circ f)^{-1}(U) \in \mathcal{T}_X$ ■

Proposition 2.23 Suppose $f : X \to Y$ is continuous between two topological spaces. Then $\{x_n\} \to X$ implies $\{f(x_n)\} \to f(x)$.

Proof. Take open $U \ni f(x)$, which implies $f^{-1}(U) \ni x$. Since $f^{-1}(U)$ is open, we imply there exists N such that

$$\{x_n \mid n \geq N\} \subseteq f^{-1}(U),$$

i.e., $\{f(x_n) \mid n \geq N\} \subseteq U$ ■

We use the notion of Homeomorphism to describe the equivalence between two topological spaces.

Definition 2.18 [Homeomorphism] A **homeomorphism** between spaces topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a bijection

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y),$$

such that both f and f^{-1} are continuous.

2.6.3. Subspace Topology

Definition 2.19 Let $A \subseteq X$ be a non-empty set. The **subspace topology** of A is defined as:

1. $\mathcal{T}_A := \{U \bigcap A \mid U \in \mathcal{T}_A\}$
2. The **coarsest** topology on A such that the **inclusion map**

$$i : (A, \mathcal{T}_A) \to (X, \mathcal{T}_X), \quad i(x) = x$$

   is continuous.

   (We say the topology $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$, or $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ e.g., $\mathcal{T}_{\text{discrete}}$ is the finest topology, and $\mathcal{T}_{\text{indiscrete}}$ is coarsest topology.)

3. The (**unique**) topology such that for any $(Y, \mathcal{T}_Y)$,

$$f : (Y, \mathcal{T}_Y) \to (A, \mathcal{T}_A)$$

   is continuous iff $i \circ f : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ (where i is the inclusion map) is continuous.

Proposition 2.24 The definition (1) and (2) in (2.19) are equivalent.

Outline. The proof is by applying

$$i^{-1}(S) = S \bigcap A, \quad \forall S$$

■

■ **Example 2.17** Let all English and numerical letters be subset of $\mathbb{R}^2$:

P,6

The homeomorphism can be constrcuted between these two English letters.

Proposition 2.25 The definition (2) and (3) in (2.19) are equivalent.

Proof. Necessity.

- For $\forall U \in \mathcal{T}_X$, consider that

$$(i \circ f)^{-1}(U) = f^{-1}(i^{-1}(U)) = f^{-1}(U \bigcap A)$$

  since $U \bigcap A \in \mathcal{T}_A$ and f is continuous, we imply $(i \circ f)^{-1}(U) \in \mathcal{T}_Y$

- For $\forall U' \in \mathcal{T}_A$, we have $U' = U \bigcap A$ for some $U \in \mathcal{T}_X$. Therefore,

$$f^{-1}(U') = f^{-1}(U \bigcap A) = f^{-1}(i^{-1}(U)) = (i \circ f)^{-1}(U) \in \mathcal{T}_Y.$$

The sufficiency is left as exercise. ■

Proposition 2.26

1. The definition (1) in (2.19) does define a topology of A
2. Closed sets of A under subspace topology are of the form $V \bigcap A$, where V is closed in X

Proposition 2.27 Suppose $(A, \mathcal{T}_A) \subseteq (X, \mathcal{T}_X)$ is a subspace topology, and $B \subseteq A \subseteq X$. Then

1. $\bar{B}^A = \bar{B}^X \bigcap A$.
2. $B^{\circ A} \supseteq B^{\circ X}$

Proof. By proposition (2.26), $\bar{B}^X \bigcap A$ is closed in A, and $\bar{B}^X \bigcap A \supset B$, which implies

$$\bar{B}^A \subseteq \bar{B}^X \bigcap A$$

Note that $\bar{B}^A \supset B$ is closed in A, which implies $\bar{B}^A = V \bigcap A \subseteq V$, where V is closed in X. Therefore,

$$\bar{B}^X \subseteq V \implies \bar{B}^X \bigcap A \subseteq V \bigcap A = \bar{B}^A$$

Therefore, $\bar{B}^A = \bar{B}^X \subseteq V$

■

Can we have $B^{\circ X} = B^{\circ A}$?

2.6.4. Basis (Base) of a topology

Roughly speaking, a basis of a topology is a family of "generators" of the topology.

Definition 2.20 Let $(X, \mathcal{T})$ be a topological space. A family of subsets $\mathcal{B}$ in X is a **basis** for $\mathcal{T}$ if

1. $\mathcal{B} \subseteq \mathcal{T}$, i.e., everything in $\mathcal{B}$ is open
2. Every $U \in \mathcal{T}$ can be written as union of elements in $\mathcal{B}$.

■ **Example 2.18** 1. $\mathcal{B} = \mathcal{T}$ is a basis.

2. For $X = \mathbb{R}^n$,

$$\mathcal{B} = \{B_r(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{Q}^n, r \in \mathbb{Q} \bigcap (0, \infty)\}$$

Exercise: every $(a,b) = \bigcup_{i \in I} (p_i, q_i)$ for $p_i, q_i \in \mathbb{Q}$.

Therefore, $\mathcal{B}$ is countable.

Proposition 2.28 If $(X, \mathcal{T})$ has a countable basis, e.g., $\mathbb{R}^n$, then $(X, \mathcal{T})$ has a second-countable space.