Solution to Assignment 6

I will appreciate it if you could give me some advice on my assignment!

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1. The gradient of constraints at any feasible point (x, y) are given by:

$$\nabla h_1(x,y) = \begin{pmatrix} x \\ 0 \end{pmatrix}, \quad \nabla h_2(x,y) = \begin{pmatrix} 0 \\ y \end{pmatrix}, \quad \nabla h_3(x,y) = \begin{pmatrix} y \\ x \end{pmatrix}$$

   To show linear independent, for any $\alpha_1, \alpha_2, \alpha_3$ satisfying

$$\alpha_1 \nabla h_1(x,y) + \alpha_2 \nabla h_2(x,y) + \alpha_3 \nabla h_3(x,y) = 0, \tag{1}$$

   we equivalently have:

$$\begin{cases} \alpha_1 x + \alpha_3 y = 0 \\ \alpha_2 y + \alpha_3 x = 0 \end{cases}$$

   Left-multiplying x^{T} for both equalities in above system, by appling feasibility, we obtain:

$$\begin{cases} \alpha_1 x^{\mathrm{T}} x + \alpha_3 x^{\mathrm{T}} y = 0 \implies \alpha_1 \cdot 1 + \alpha_3 \cdot 0 = \alpha_1 = 0 \\ \alpha_2 x^{\mathrm{T}} y + \alpha_3 x^{\mathrm{T}} x = 0 \implies \alpha_2 \cdot 0 + \alpha_3 \cdot 1 = \alpha_3 = 0 \end{cases}$$

   Left-multiplying y^{T} for the second equality in above system, we obtain:

$$\alpha_2 y^{\mathrm{T}} y + \alpha_3 y^{\mathrm{T}} x = \alpha_2 \cdot 1 + \alpha_3 \cdot 0 = \alpha_2 = 0$$

   In other words, (1) implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$, which shows the linear independence for $\{\nabla h_1(x,y), \nabla h_2(x,y), \nabla h_3(x,y)\}$, i.e., the regularity holds for any feasible (x, y).

2. Since the regularity is satisfied, we apply the KKT necessary condition. Define the Lagrangian function as follows:

$$L(x, y, \lambda_1, \lambda_2, \lambda_3) = f(x,y) + \sum_{i=1}^{3} \lambda_i h_i(x,y) \tag{2}$$

   Applying the KKT condition for the local minimum-Lagrangian pair (x, y), we have:

$$\nabla_{x,y} L(x, y, \lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} Ax - b + \lambda_1 x + \lambda_3 y \\ Ay - c + \lambda_2 y + \lambda_3 x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   Or equivalently, the first-order necessary conditions for this QCQP are given as:

$$Ax - b + \lambda_1 x + \lambda_3 y = 0 \tag{3a}$$
$$Ay - c + \lambda_2 y + \lambda_3 x = 0 \tag{3b}$$

3. Left-multiplying x^{T} for (3a) and (3b), by applying feasibility, we obtain:

$$x^{\mathrm{T}}Ax - x^{\mathrm{T}}b + \lambda_1 x^{\mathrm{T}}x + \lambda_3 x^{\mathrm{T}}y = x^{\mathrm{T}}Ax - x^{\mathrm{T}}b + \lambda_1 = 0 \tag{4a}$$
$$x^{\mathrm{T}}Ay - x^{\mathrm{T}}c + \lambda_2 x^{\mathrm{T}}y + \lambda_3 x^{\mathrm{T}}x = x^{\mathrm{T}}Ay - x^{\mathrm{T}}c + \lambda_3 = 0 \tag{4b}$$

ALso, left-multiplying y^{T} for (3a) and (3b), by applying feasibility, we obtain:

$$y^{\mathrm{T}}Ax - y^{\mathrm{T}}b + \lambda_1 y^{\mathrm{T}}x + \lambda_3 y^{\mathrm{T}}y = y^{\mathrm{T}}Ax - y^{\mathrm{T}}b + \lambda_3 = 0 \tag{4c}$$
$$y^{\mathrm{T}}Ay - y^{\mathrm{T}}c + \lambda_2 y^{\mathrm{T}}y + \lambda_3 y^{\mathrm{T}}x = y^{\mathrm{T}}Ay - y^{\mathrm{T}}c + \lambda_2 = 0 \tag{4d}$$

Similifying (4a) to (4d), we derive:

$$\lambda_1 = -x^{\mathrm{T}}Ax + x^{\mathrm{T}}b \tag{5a}$$
$$\lambda_2 = -y^{\mathrm{T}}Ay + y^{\mathrm{T}}c \tag{5b}$$

and

$$\lambda_3 = -y^{\mathrm{T}}Ax + y^{\mathrm{T}}b, \qquad \text{or} \qquad \lambda_3 = -x^{\mathrm{T}}Ay + x^{\mathrm{T}}c \tag{5c}$$

4. The expression for $\nabla h(x,y)$ is given by:

$$\nabla h(x,y) = \begin{pmatrix} x & 0 & y \\ 0 & y & x \end{pmatrix}$$

Thus simplifying $\nabla^{\mathrm{T}} h(x,y) \cdot w = 0$ gives:

$$\begin{pmatrix} x^{\mathrm{T}} & 0 \\ 0 & y^{\mathrm{T}} \\ y^{\mathrm{T}} & x^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

Or equivalently,

$$\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = \langle \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = \langle \begin{pmatrix} y \\ x \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = 0, \tag{7a}$$

and

$$\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle + \langle \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = \langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = 0 \tag{7b}$$

Thus the system (7a) and (7b) shows that

$$\begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} u \\ v \end{pmatrix} \qquad \begin{pmatrix} y \\ x \end{pmatrix} \perp \begin{pmatrix} u \\ v \end{pmatrix}$$

Moreover, by the feasibility,

$$\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} y \\ x \end{pmatrix} \rangle = x^{\mathrm{T}}y + y^{\mathrm{T}}x = 2x^{\mathrm{T}}y = 0 \implies \begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} y \\ x \end{pmatrix}$$

Hence, we conclude that

$$\begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} u \\ v \end{pmatrix} \perp \begin{pmatrix} y \\ x \end{pmatrix}$$

5. From (5a) to (5c), we obtain that

$$
\begin{aligned}
2\lambda_3 - (\lambda_1 + \lambda_2) &= [-y^{\mathrm{T}}Ax + y^{\mathrm{T}}b] + [-x^{\mathrm{T}}Ay + x^{\mathrm{T}}c] && \text{(8a)}\\
&\qquad - [-x^{\mathrm{T}}Ax + x^{\mathrm{T}}b] - [-y^{\mathrm{T}}Ay + y^{\mathrm{T}}c] && \text{(8b)}\\
&= (x-y)^{\mathrm{T}}A(x-y) + (x-y)^{\mathrm{T}}(c-b) && \text{(8c)}\\
&= (x-y)^{\mathrm{T}}A(x-y) - \langle x-y, b-c\rangle && \text{(8d)}\\
&\geq -\langle x-y, b-c\rangle && \text{(8e)}\\
&= 0 && \text{(8f)}
\end{aligned}
$$

where (8c) and (8d) is by term arrangement; (8e) is due to the semi-definiteness of A; (8f) is because $(x-y) \perp (b-c)$.

Therefore, we conclude that

$$\lambda_3 \geq \frac{\lambda_1 + \lambda_2}{2}$$