A FIRST COURSE

IN

ANALYSIS

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MAT2006 Notebook

Lecturer

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Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

2.2. Friday

2.2.1. Set Analysis

This lecture will discuss different kinds of sets. Now recall our common sense:

Definition 2.4 [Interval]

- Open interval:

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$$

- Closed interval:

$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

- Half open intervals:

$$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

$$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

■

Definition 2.5 [Open sets] A set $\boldsymbol{A}$ is open if $\forall x \in \boldsymbol{A}$, there exists $(a,b) \subseteq \boldsymbol{A}$ such that $x \in (a,b)$. ■

Theorem 2.2
1. An open set in $\mathbb{R}$ is a **disjoint** union of finitely many or countably many open intervals.
2. The union of any collection of open sets is open.
3. The intersection of **finitely** many open sets is open.

The proof is omitted, check Rudin's book for reference.

R Note that the intersection of **countably** many open sets may be open.

$$\bigcup_{n=1}^{\infty}\left(-\frac{1}{n}, 1+\frac{1}{n}\right) = [0,1]$$

Definition 2.6 [Neighborhood] A **neighborhood** N of a point $a \in \mathbb{R}$ is an open set containing a. ■

Definition 2.7 [Limit Point] x is a **limit point** of the set $\boldsymbol{A}$ if for any neighborhood N of x, N contains a point $a \in A$ such that $a \neq x$. ■

Definition 2.8 [Closed Set] A set $\boldsymbol{A}$ is **closed** if $\boldsymbol{A}$ contains all of its limit points. ■

Proposition 2.2 $\boldsymbol{A}$ is **closed** of and only if $\mathbb{R} \setminus \boldsymbol{A}$ is open.

2.2.2. Set Analysis Meets Sequence

Definition 2.9 [Limit Point of sequence] Given a sequence $\{a_n\}$, i.e.,

$$a_1, a_2, a_3, \ldots,$$

a point x is said to be the **limit point** of $\{a_n\}$ if there exists a subsequence $\{x_{n_1}, x_{n_2}, \ldots\}$ converging to x. ■

Does there exist a sequence of rational numbers such that every irrational number is a limit point? Yes, and we use an example as illustration.

■ **Example 2.3** $\{q_1, q_2, \ldots\}$ is a sequence of all rational numbers. For example, to construct a subsequence with limit $\sqrt{2}$, we pick:

$$\begin{aligned}
&q_{m_1} \in (\sqrt{2} - 1, \sqrt{2} + 1) \setminus (\sqrt{2} - \frac{1}{2}, \sqrt{2} + \frac{1}{2}) \\
&q_{m_2} \in (\sqrt{2} - \frac{1}{2}, \sqrt{2} + \frac{1}{2}) \setminus (\sqrt{2} - \frac{1}{3}, \sqrt{2} + \frac{1}{3}) \\
&\ldots \\
&q_{m_k} \in (\sqrt{2} - \frac{1}{k}, \sqrt{2} + \frac{1}{k}) \setminus (\sqrt{2} - \frac{1}{k+1}, \sqrt{2} + \frac{1}{k+1})
\end{aligned}$$

The same argument works for all irrational numbers, also for all rational numbers. ■

2.2.3. Completeness of Real Numbers

Now we use Cauchy sequence to construct the completeness of real numbers. First let's give a proof of three important theorems. Note that the proof and applications of these theorems are mandatory.

Theorem 2.3 — **Bolzano-Weierstrass.** Every bounded sequence has a convergent subsequence.

Theorem 2.4 — **Cantor's Nested Interval Lemma.** A sequence of nested closed bounded intervals $I_1 \supseteq I_2 \supseteq \cdots$ has a non-empty intersection, i.e., $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$.

Theorem 2.5 — **Heine-Borel.** Any open cover $\{\mathcal{U}\}$ of a bounded closed set $\boldsymbol{E}$ consists of a finite sub-cover, i.e, $\boldsymbol{E} \subseteq$ the union of $\{\mathcal{U}\}$.

Proof for Bolzano-Weierstrass Theorem.

- Suppose $\{a_1, a_2, \dots\}$ is a bounded sequence, w.l.o.g., $\{a_1, a_2, \dots\} \subseteq [-M, M]$. We pick $a_{n_1} = a_1$.
- w.l.o.g., assume that $[0, M] \bigcap \{a_1, a_2, \dots\}$ is infinite (otherwise $[-M, 0] \bigcap \{a_1, a_2, \dots\}$ is infinite), then we pick $a_{n_2} \neq a_{n_1}$ such that $a_{n_2} \in [0, M]$.
- w.l.o.g., assume that $[0, \frac{M}{2}] \bigcap \{a_1, a_2, \dots\}$ is infinite, then we pick $a_{n_3} \neq a_{n_1}, a_{n_2}$ such that $a_{n_3} \in [0, \frac{M}{2}]$.

In this case, $\{a_{n_1}, a_{n_2}, \dots\}$ is Cauchy (by showing $|a_{n_k} - a_{n_l}| < \epsilon$ for large k, l), hence converges. ■

Proof for Cantor's Nested Interval Lemma.

1. Pick $a_k \in I_k$ for $k = 1, 2, \dots$, thus the sequence $\{a_1, \dots, a_k, \dots\}$ is bounded. By Theorem (2.3), there exists a convergent sub-sequence $\{a_{k_l}\}$ (with limit a). It suffices to show $a \in \bigcup_{m=1}^{\infty} I_k$.

2. For fiexed m, there exists index j such that $a_{k_l} \in I_m$ for all $l \geq m$. Since I_m is closed, it must contain a_{k_l}'s limit point, i.e., $a \in I_m$.
3. Our choice is arbitrary m and hence a belongs to the intersection of all nested closed intervals. The proof is complete.

■

Before the proof of third theorem, let's have a review for open cover definitions:

Definition 2.10 [Open Cover] Let $\boldsymbol{E}$ be a subset of a metric space X. An open cover $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ of $\boldsymbol{E}$ is a collection of open sets in X whose union contains $\boldsymbol{E}$, i.e., $\boldsymbol{E} \subseteq \bigcup_{\alpha \in A} \mathcal{U}_\alpha$. A finite **subcover** of $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ is a **finite** sub-collection of $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ whose union still contains $\boldsymbol{E}$.

For example, consider $\boldsymbol{E} := [\frac{1}{2}, 1)$ in metric space $\mathbb{R}$. Then the collection

$$\{I_n\}_{n=3}^{\infty}, \qquad \text{where } I_n := (\frac{1}{n}, 1 - \frac{1}{n})$$

is a open cover of $\boldsymbol{E}$. Note that the finite subcover may not necessarily exist. In this example, the finite subcover of $\{I_n\}_{n=3}^{\infty}$ does not exist.

Proof for Heine-Borel Theorem.

Suppose $\boldsymbol{E} := [0, M]$ is a bounded closed interval with an open cover $\{\mathcal{U}\}$. The trick of this proof is to construct a sequence of nested closed bounded intervals.

- **Base case** We choose $I_1 = \boldsymbol{E} = [0, M]$
- **Inductive step** For example, Assume that $\boldsymbol{E}$ cannot be covered by finitely many open sets from $\{\mathcal{U}\}$, then at least one sub-interval $[0, \frac{M}{2}]$ or $[\frac{M}{2}]$ cannot be covered. Let I_2 be one of these sub-intervals that cannot be covered by finitely many elements of $\{\mathcal{U}\}$.

Repeating this process, we attain a nested bouned closed intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq$, which implies $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$ (suppose $a \in \bigcap_{k=1}^{\infty}$), and $|I_k| = \frac{M}{2^k} \to 0$.

Note that $a \in \boldsymbol{E}$ implies that there exists an open set ξ in $\{\mathcal{U}\}$ such that $a \in \xi$. Thus $(a-\epsilon, a+\epsilon) \in \xi$ for small ϵ. Note that there exists sufficiently large k such that $\frac{M}{2^k} < 2\epsilon$, and $a \in I_k$, which implies $I_k \subseteq \xi$, which is a contradiction. ■

These theorems have simple applications:

Proposition 2.3 Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with the series convergent for $|x| < 1$. If for $\forall x \in [0,1)$, there exists $n := n(x)$ such that $\sum_{k=n}^{\infty} a_k x^k = 0$, then f is a polynomial (that is independent from x, i.e., n does not depend on x.)

In next lecture we will continue to study the completeness of real numbers and will speed up.