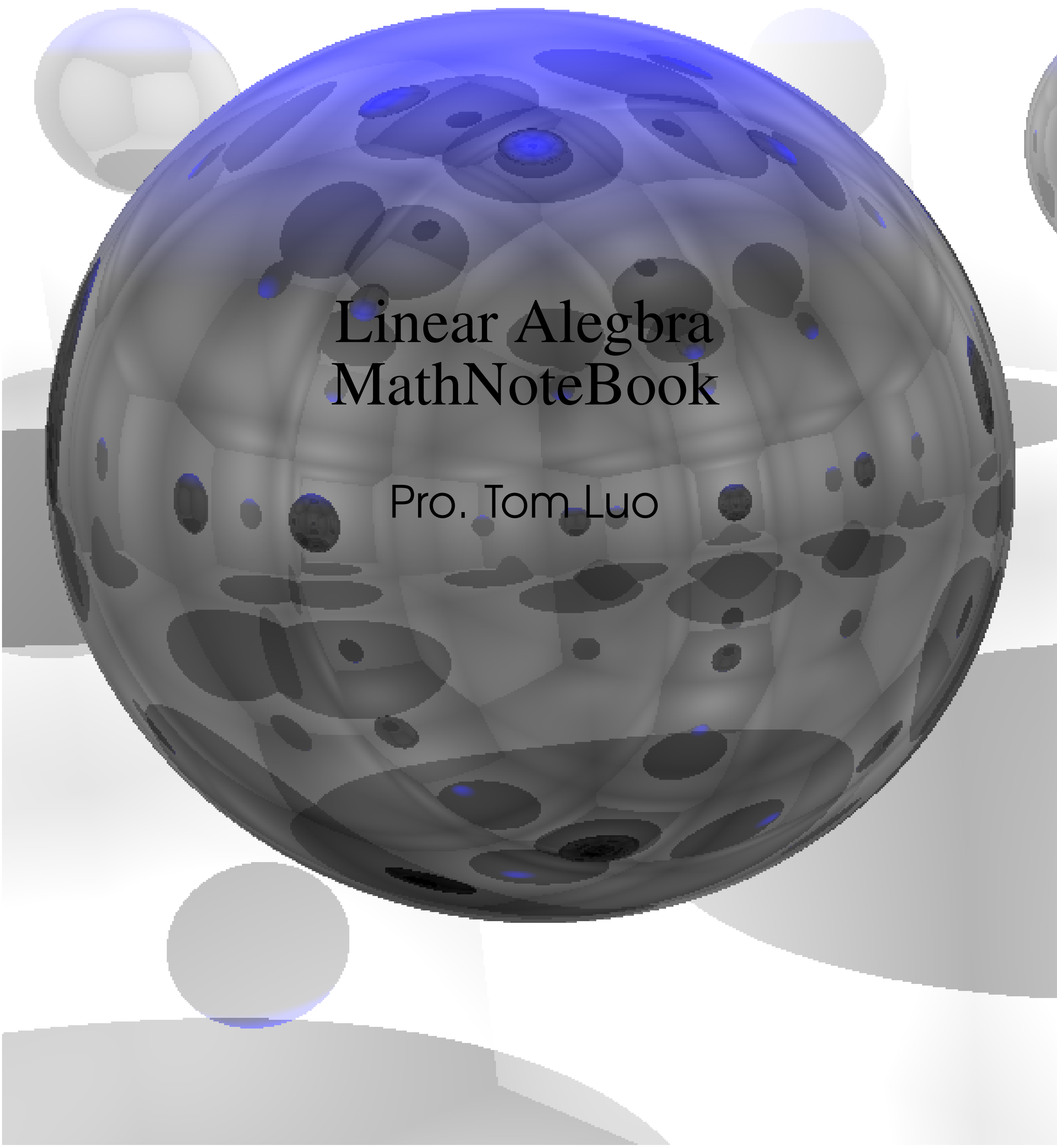

Linear Alegbra MathNoteBook

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7 — Week3

7.1 Tuesday

7.1.1 Introduction

Why do we learn Linear Algebra?

So, we raise the question again, why do we learn LA?

- Baisis of AI/ML/SP/etc.
  In information age, *artificial intelligence, machine learning, structured programming*, and otherwise gains great popularity among researchers. LA is the basis of them, so in order to explore science in modern age, you should learn LA well.
- Solving linear system of equations.
  How to solve linear system of equations efficiently and correctly is the **key** question for mathematicians.
- Internal grace.
  LA is very beautiful, hope you enjoy the beauty of math.
- Interview questions.
  LA is often used for interview questions for phd. Because the upper bound of difficulty for LA is **infinity**, interviewer often choose LA to question phd.

What is LA?

The main part of Mathematics is given below:

$$\text{mathematics}\begin{cases}\text{Analysis+Calculus}\\ \text{Algebra:foucs on structure}\\ \text{Geometry}\end{cases}$$

All parts of math are based on **axiom systems**. And **LA** is the significant part of *Algebra*, which focus on the linear structure.

7.1.2 Review of 2 weeks

Motivating question: How to solve linear system equations?
The basic method is **Gaussian Elimination** (To make equations *simpler*. The main idea is *induction*)

Given one equation $ax = b$, you can easily sovle it:

$$\Longrightarrow \text{“if } a > 0\text{, no solution.” or } x = \frac{b}{a}$$

By induction, *if you can sovle $n \times n$ systems, can you solve $(n+1) \times (n+1)$ systems?*
In this process, math notations is needed:

- matrix multiplication
- matrix inverse
- transpose, symmetric matrices

So in first two weeks, we just learn two things:

- *linear system could be solved **almost** by G.E.*
- *Furthermore, Gaussian Elimination is (almost) LU decomposition.*

But there is a question remained to be solved:

For **singular** system, How to solve it?

- When will it has no solution, when it has infinitely many solutions? (Note that singular system don’y has unique solution.)
- If it has infinitely many solutions, how to find and express these solutions?

If we express system as matrix, we only to answer the question: **How to solve rectangular?**

7.1.3 Examples of solving equations

- For square case, we often convert the system into $\boldsymbol{Rx} = \boldsymbol{c}$, where $\boldsymbol{R}$ is of *row echelon form*.
- But for rectangular case, *row echelon form*(ref) is not enough, we must convert it into **reduced row echelon form**(rref):

$$\boldsymbol{U}(\text{ref}) = \begin{bmatrix} \boxed{1} & 0 & \times & \times & \times & 0 & \times \\ 0 & \boxed{1} & \times & \times & \times & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \boldsymbol{R}(\text{rref}) = \begin{bmatrix} \boxed{1} & 0 & \times & \times & \times & 0 & \times \\ 0 & \boxed{1} & \times & \times & \times & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

■ **Example 7.1** We discuss how to solve square matrix of **rref**: If all rows have nonzero entry, we have:

$$\begin{bmatrix} 1 & & \text{0} & \\ & 1 & & \\ & & 1 & \\ \text{0} & & & 1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{c} \Longrightarrow \boldsymbol{x} = \boldsymbol{c}$$

We already sovled this system, but note that *the last row could be all zero*:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \boldsymbol{x} = \boldsymbol{c} \Longrightarrow \begin{cases} x_1 = c_1 \\ x_2 = c_2 \\ x_3 = c_3 \\ 0 = c_4 \end{cases}$$

So the result has two cases:

- If $c_4 \neq 0$, we have no solution of this system.
- If $c_4 = 0$, we have infinitely many solutions, which can be expressed as:

$$x_{\text{complete}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

  where x_4 could be arbitarary number.

Hence for the $n \times n$ systems, does Gaussian Elimination work?

Answer: Almost, except "pivot=0"case.

- All pivots $\neq 0 \implies$ system has unique solution.
- Some pivots $= 0$ (The matrix is singular)
  1. No solution.
  2. Infinitely many solution.

■

What is G.E. doing? (Nonsingular case.)

Abstrastion: We use matrix to represent system of equations (Chinese mathematicians fail to do this.):

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{23}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{m3}x_n = b_m \end{cases} \implies \boldsymbol{Ax} = \boldsymbol{b}$$

By postmultiplying $\boldsymbol{E}_{ij}$ and $\boldsymbol{P}_{ij}$ we do one step of elimination:

$$\boldsymbol{E}_{ij}\boldsymbol{Ax} = \boldsymbol{b} \qquad \boldsymbol{P}_{ij}\boldsymbol{Ax} = \boldsymbol{b}$$

By several steps of elimination, we obtain the final result:

$$\hat{\boldsymbol{L}}\boldsymbol{PAx} = \hat{\boldsymbol{L}}\boldsymbol{Pb}$$

where $\hat{\boldsymbol{L}}\boldsymbol{PA}$ represents a upper triangular matrix $\boldsymbol{U}$, $\hat{L}$ is the lower triangular matrix.

$$\implies \hat{\boldsymbol{L}}\boldsymbol{PA} = \boldsymbol{U} \implies \boldsymbol{PA} = \hat{\boldsymbol{L}}^{-1}\boldsymbol{U} \triangleq \boldsymbol{LU}$$

So Gaussian Elimination is almost the LU decomposition.

Example for solving rectangular system of rref

Recall the definition for rref:

Definition 7.1 — reduced row echelon form. Suppose a matrix has r *nonzero* rows, each row has leading 1 as pivots. If all columns with pivots (call it pivot column) are all zero entries apart from the pivot in this column, then this matrix is said to be **reduced row echelon form**(**rref**). ■

Next we want to show an example for how to solve non-square system of rref, note that in last lecture we know the solution is given by:

$$\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_{\text{special}}$$

■ **Example 7.2** We try to solve the system $\begin{bmatrix}1 & 3 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}\boldsymbol{x}=\boldsymbol{c}$.

- **step 1:**Find null space. Thus we only need to solve $\boldsymbol{R}\boldsymbol{x}=\boldsymbol{0}$.

$$\Longrightarrow \begin{bmatrix}1 & 3 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Longrightarrow \begin{cases}x_1+3x_2+0x_3-x_4=0\\ x_3+x_4=0\end{cases}$$

  What should we do next? We want to express the **pivot variable** as the form of **free variable.**
  Note that the pivot columns in $\boldsymbol{R}$ are column 1 and 3, so the pivot variable is x_1 and x_3. The free variable is the remaining variable, say, x_2 and x_4.
  Hence the expression for x_1 and x_3 is given by:

$$\begin{cases}x_1=-3x_2+x_4\\ x_2=-x_4\end{cases}$$

  Hence all solutions to $\boldsymbol{R}\boldsymbol{x}=\boldsymbol{0}$ are

$$\boldsymbol{x}_{\text{special}}=\begin{bmatrix}-3x_2+x_4\\x_2\\-x_4\\x_4\end{bmatrix}=x_2\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$$

  where x_2 and x_4 can be taken arbitararily.
- **step 2:**find particular solution to $\boldsymbol{R}\boldsymbol{x}=\boldsymbol{c}$.
  The trick for this step is to set $x_2=x_4=0$. (*set free variable to be zero and then derive the pivot variable.*)

$$\Longrightarrow \begin{bmatrix}1 & 3 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\0\\x_3\\0\end{bmatrix}=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} \Longrightarrow \begin{cases}x_1=c_1\\ x_3=c_2\\ 0=c_3\end{cases}$$

  $\Longrightarrow$
  - if $c_3=0$, then exists particular solution $\boldsymbol{x}_p=\begin{bmatrix}c_1\\0\\c_2\\0\end{bmatrix}$;
  - if $c_3\neq 0$, then $\boldsymbol{R}\boldsymbol{x}=\boldsymbol{c}$ has no solution.
- **Final solution:** Assume $c_3=0$, then all solution to $\boldsymbol{R}\boldsymbol{x}=\boldsymbol{c}$ is given by:

$$\boldsymbol{x}_{complete}=\boldsymbol{x}_p+\boldsymbol{x}_{\text{special}}=\begin{bmatrix}c_1\\0\\c_2\\0\end{bmatrix}+x_2\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$$

■

Next we show how to solve a general rectangular:

7.1.4 How to solve a general rectangular

For linear system $\boldsymbol{Ax}=\boldsymbol{b}$, where $\boldsymbol{A}$ is rectangular, we can solve this system as follows:

- **step1:** Gaussian Elimination.
With proper rows permutaion (postmultiply $\boldsymbol{P}_{ij}$) and row transformation (postmultiply $\boldsymbol{E}_{ij}$) we convert $\boldsymbol{A}$ into $\boldsymbol{R}$(rref), then we only need to solve $\boldsymbol{Rx}=\boldsymbol{c}$.

Example 7.3 The first example is a 3×4 matrix with two pivots:

$$\boldsymbol{A}=\begin{bmatrix}1&1&2&3\\2&2&8&10\\3&3&10&13\end{bmatrix}$$

clearly $a_{11}=1$ is the first pivot, clear row 2 and row 3 of this matrix:

$$\boldsymbol{A}\xRightarrow[\boldsymbol{E}_{31}=\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}]{\boldsymbol{E}_{21}=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}\begin{bmatrix}1&1&2&3\\0&0&4&4\\0&0&4&4\end{bmatrix}\xRightarrow[\boldsymbol{E}_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}]{\boldsymbol{E}_{12}=\begin{bmatrix}1&-\frac{1}{2}&0\\0&1&0\\0&0&1\end{bmatrix}}\begin{bmatrix}1&1&0&1\\0&0&4&4\\0&0&0&0\end{bmatrix}$$

$$\Longrightarrow\begin{bmatrix}1&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

If we want to solve $\boldsymbol{Ax}=\boldsymbol{b}$, firstly we should convert $\boldsymbol{A}$ into $\begin{bmatrix}1&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$ (rref). ■

Then we should identify **pivot variables** and **free variables**. we can follow the direction to derive these:
pivot $\Longrightarrow$ pivot columns $\Longrightarrow$ pivot columns $\Longrightarrow$ pivot variable

Example 7.4 we want to identify **pivot variables** and **free variables** of $\boldsymbol{R}$:

$$\boldsymbol{R}=\begin{bmatrix}\boxed{1}&0&\times&\times&\times&0&\times\\0&\boxed{1}&\times&\times&\times&0&\times\\0&0&0&0&0&\boxed{1}&\times\\0&0&0&0&0&0&0\end{bmatrix}$$

The pivot are r_{11},r_{22},r_{36}. So the pivot columns are column $1,2,6$. So the *pivot variables* are x_1,x_2,x_6; the *free variables* are x_3,x_4,x_5,x_7. ■

- **step2:** Compute null space $N(\boldsymbol{A})$. In order to find $N(\boldsymbol{A})$, we only need to compute $N(\boldsymbol{R})$.
  - For each of $(n-r)$ free variables,
    set value of **it** to be 1.
    set other **free variables** to be 0.
    Then solve $\boldsymbol{Rx}=\boldsymbol{0}$ to get special solution y_j for $j=1,2,\dots,n-r$.

■ **Example 7.5** continue with 3×4 matrix example:

$$\boldsymbol{R} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We want to find special solutions to $\boldsymbol{Rx} = \boldsymbol{0}$:

1. Set $x_2 = 1$ and $x_4 = 0$. Solve $\boldsymbol{Rx} = \boldsymbol{0}$, then $x_1 = -1$ and $x_3 = 0$.

   Hence one special solution is $y_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.
2. Set $x_2 = 0$ and $x_4 = 1$. Solve $\boldsymbol{Rx} = \boldsymbol{0}$, then $x_1 = -1$ and $x_3 = -1$.

   Then another special solution is $y_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

■

– When we get $(n-r)$ special solutions to $\boldsymbol{Rx} = \boldsymbol{0}$: $y_1, y_2, \dots, y_{n-r}$. Then $N(\boldsymbol{A}) = \operatorname{span}(y_1, y_2, \dots, y_{n-r})$.

■ **Example 7.6** We continue the example above, when we get all special solutions $y_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, y_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, **the null space contains all linear combinations of the special solutions.**

$$\boldsymbol{x}_{\text{special}} = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right) = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

where x_2, x_4 here could be arbitarary. ■

- **step3:** Compute a particular solution $\boldsymbol{x}_p$.

  The easiest way is to "read" from $\boldsymbol{Rx} = \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$:

  Suppose $\boldsymbol{R} \in \mathbb{R}^{m \times n}$ has $r (\leq m)$ pivot variables, then it has $(m-r)$ zero rows and (n-r) free variables. In order to have solution, we must have $c_{r+1} = \cdots = c_n = 0$. In other words, **For a solution to exist, zero rows in $\boldsymbol{R}$ must also be zero in $\boldsymbol{c}$.**

■ **Example 7.7** If $\boldsymbol{Rx} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{x} = \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, then in order to have a solution, we must let $c_3 \neq 0$. ■

So we have to discuss the particular solution case by case:

- **case1:** one of $c_{r+1}, \dots, c_n$ is nonzero, then the system has **no** solution.
- **case2:** $c_{r+1} = \cdots = c_n$, then a particular solution exists:

$$\boldsymbol{x}_p = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

We set all **free variables** to be zero, and pivot variables are from $\boldsymbol{c}$. More specifically, the first entry in $\boldsymbol{c}$ is exactly the value for the first pivot variable;the second entry in $\boldsymbol{c}$ is exactly the value for the second pivot variable......

■ **Example 7.8** If $\boldsymbol{Rx} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{x} = \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}$, we want to compute particular solution

$$\boldsymbol{x}_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Then we know x_2, x_4 are free variable, so $x_2 = x_4 = 0$; x_1, x_3 are pivot variable, so we have $\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

Hence the solution for $\boldsymbol{Rx} = \boldsymbol{c}$ is $\begin{bmatrix} c_1 \\ 0 \\ c_2 \\ 0 \end{bmatrix}$. ■

- **Final step:** All solution of $\boldsymbol{Ax} = \boldsymbol{b}$ are $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_{\text{special}}$, where $x_{\text{special}} \in N(\boldsymbol{A})$. $\boldsymbol{x}_p$ is defined in step3, $\boldsymbol{x}_{\text{special}}$ is defined in step2.

However, where does the number r come? r denotes the **rank** of a matrix, which will be discussed next lecture.