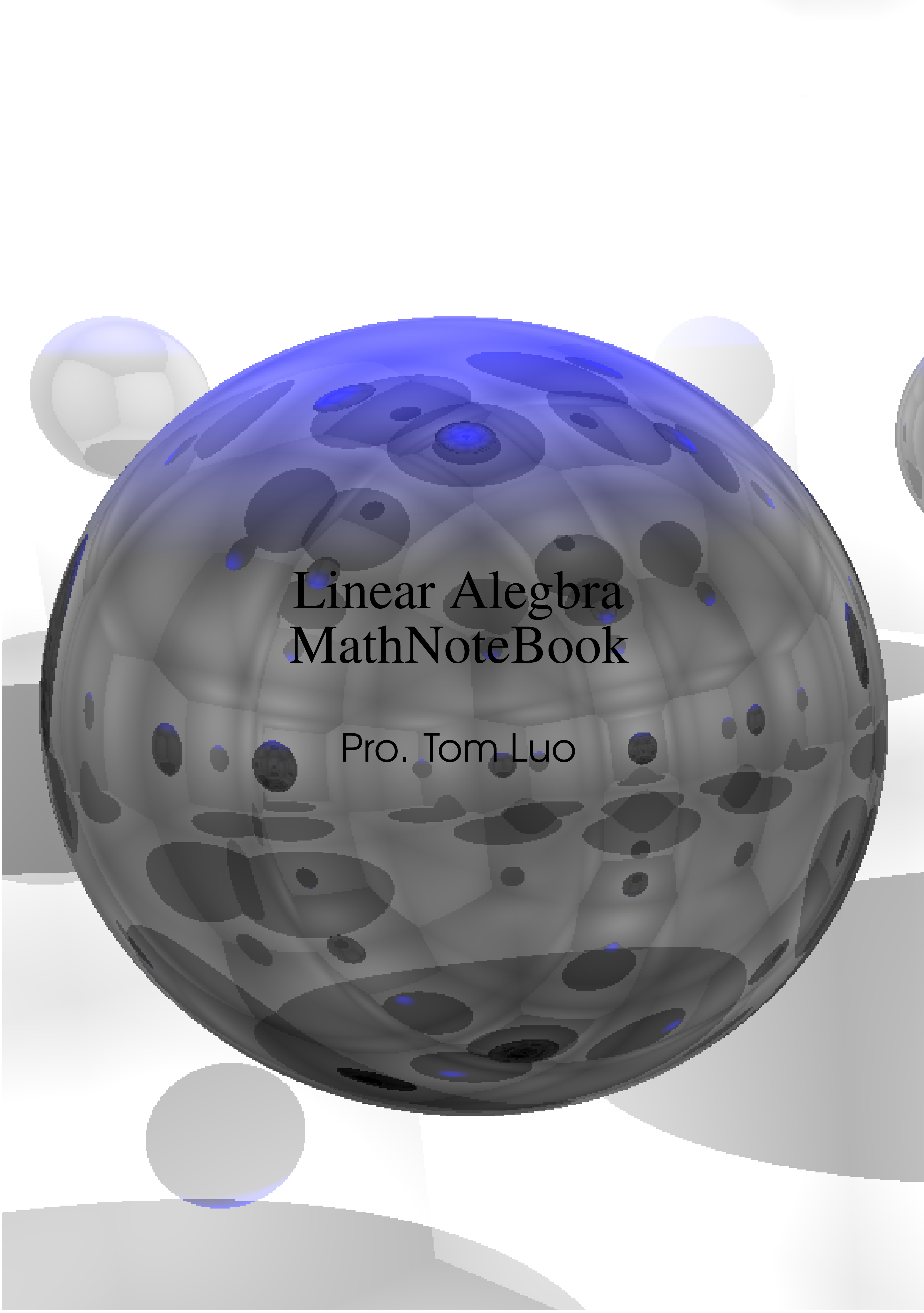

Linear Alegbra MathNoteBook

Pro. Tom Luo

Tuesday
- Introduction
- Gaussian Elimination
- Complexity Analysis

Friday
- Matrix Multiplication
- Elementary Matrix
- Properties of Matrix
- Permutation Matrix
- LU decomposition
- LDU decomposition
- LU Decomposition with row exchanges

Tuesday
- Review
- Special matrix multiplication case
- Inverse

Tuesday
- Introduction
- Review of 2 weeks
- Examples of solving equations
- How to solve a general rectangular

Thursday
- Review
- Remarks on solving linear system equations
- Linearly dependence
- Basis and dimension

8 — Week3

8.1 Thursday

8.1.1 Review

Last time you may be confused about how to compute $N(\boldsymbol{A})$ or $y_1, y_2, \dots, y_{n-r}$ (step2). Now let's review the whole steps for solving rectangular bu using block matrix:

- Firstly, we convert our rref into the form $\begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$ by switching columns.

> ■ **Example 8.1** Last time our rref is given by:
>
> $$\boldsymbol{R} = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
>
> We notice that column 3 is pivot column, so we can switch it into the second column. (By switching column 2 and column 3):
>
> $$\boldsymbol{R} \implies \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$
>
> ■

- Then our system equation is translated (We use 3×4 matrix to show the whole process.):

$$\boldsymbol{Rx} = \boldsymbol{c} \implies \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Because we have changed the columns, so here row 2 and row 3 is also switched respectively. And then x_1 and x_2 are pivot variables, x_3 and x_4 are free variables. Then we derive:

$$\Longrightarrow \begin{cases} \boldsymbol{I}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\boldsymbol{B}\begin{bmatrix}x_3\\x_4\end{bmatrix}=\begin{bmatrix}c_1\\c_2\end{bmatrix} \\ \qquad\qquad 0=c_3 \end{cases}$$

- If $c_3\neq 0$, then there is no solution; next, let's assume $c_3=0$. Then *pivot variables* could be expressed as the form of *free variables*:

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}c_1\\c_2\end{pmatrix}-\boldsymbol{B}\begin{pmatrix}x_3\\x_4\end{pmatrix}$$

Hence all solutions to $\boldsymbol{Rx}=\boldsymbol{c}$ is obtained:

$$\boldsymbol{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}c_1\\c_2\\0\\0\end{bmatrix}-\begin{bmatrix}\boldsymbol{B}\begin{pmatrix}x_3\\x_4\end{pmatrix}\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\x_3\\x_4\end{bmatrix}$$

Suppose $-\boldsymbol{B}=\begin{bmatrix}\hat{\boldsymbol{y}}_1 & \hat{\boldsymbol{y}}_2\end{bmatrix}$, then pivot variables is equivalent to

$$\Longrightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}c_1\\c_2\end{pmatrix}+x_3\hat{\boldsymbol{y}}_1+x_4\hat{\boldsymbol{y}}_2$$

- So the complete solution to the system is

$$\boldsymbol{x}=\begin{pmatrix}c_1\\c_2\\0\\0\end{pmatrix}+\begin{pmatrix}x_3\hat{\boldsymbol{y}}_1+x_4\hat{\boldsymbol{y}}_2\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\x_3\\x_4\end{pmatrix} \tag{8.1}$$

$$=\begin{pmatrix}c_1\\c_2\\0\\0\end{pmatrix}+x_3\begin{pmatrix}\hat{\boldsymbol{y}}_1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}\hat{\boldsymbol{y}}_2\\0\\0\end{pmatrix}+x_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+x_4\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{8.2}$$

$$=\underbrace{\begin{pmatrix}c_1\\c_2\\0\\0\end{pmatrix}}_{\boldsymbol{x}_p}+\underbrace{x_3\begin{pmatrix}\hat{\boldsymbol{y}}_1\\1\\0\end{pmatrix}+x_4\begin{pmatrix}\hat{\boldsymbol{y}}_2\\0\\1\end{pmatrix}}_{\boldsymbol{x}_{\text{special}}} \tag{8.3}$$

where x_3 and x_4 could be arbitarary.

- If we set $x_3=x_4=1$, then we check whether $\begin{pmatrix}\hat{\boldsymbol{y}}_1\\1\\0\end{pmatrix}+\begin{pmatrix}\hat{\boldsymbol{y}}_2\\0\\1\end{pmatrix}$ belongs to null space (sanity check):

$$\begin{pmatrix}\hat{\boldsymbol{y}}_1\\1\\0\end{pmatrix}+\begin{pmatrix}\hat{\boldsymbol{y}}_2\\0\\1\end{pmatrix}=\begin{pmatrix}-\boldsymbol{B}\\\boldsymbol{I}\end{pmatrix}\Longrightarrow\begin{pmatrix}\boldsymbol{I}&\boldsymbol{B}\\\boldsymbol{0}&\boldsymbol{0}\end{pmatrix}\begin{bmatrix}-\boldsymbol{B}\\\boldsymbol{I}\end{bmatrix}=\begin{bmatrix}-\boldsymbol{B}+\boldsymbol{B}\\\boldsymbol{0}\end{bmatrix}=\begin{bmatrix}\boldsymbol{0}\\\boldsymbol{0}\end{bmatrix}$$

If our rectangular matrix is $m \times n (m > n)$, how to solve it?
Answer: Also, we do G.E. to get rref, which will be of the form

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ 0 & \dots & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

8.1.2 Remarks on solving linear system equations

The two possibilities for linear equations depend on m and n:

Theorem 8.1 Let m denote number of equations, n denote number of variables. For number of solutions for $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, we obtain:

- $m < n$: either no solution or infinitely many solutions
- $m \geq n$: no solution;unique solution $(N(\boldsymbol{A}) = \boldsymbol{0})$; or infinitely many solutions.

Proofoutline for $m < n$ case: Recall we can convert $\boldsymbol{Ax} = \boldsymbol{b}$ into $\boldsymbol{Rx} = \boldsymbol{c}$:

$$\begin{bmatrix} 1 & & & \times & \times \\ & \ddots & & \times & \times \\ & & 1 & \times & \times \\ 0 & 0 & 0 & 0 & 0 \\ \dots & & & & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_r \\ c_{r+1} \\ \vdots \\ c_n \end{bmatrix}$$

Note that $x_1.x_2....,x_r$ is pivot variables (This is because of column switching). Hence we have $(n-r)$ free variables, thus $N(\boldsymbol{A})$ is spanned by $(n-r)$ special vectors $y_1, y_2, \dots, y_{n-r}$.
Hence we only need to show $n > r$ *given the condition* $n > m$:
Obviously, $r \leq m$, and we have $n > m$, so we obtain $n > r$. ■

So we get the proposition immediately:

Proposition 8.1 For system $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m < n$,
it either has no solution or infinitely many solutions.

Corollary 8.1 For system $\boldsymbol{Ax} = \boldsymbol{0}$, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m < n$,
it always has infinitely many solutions.

What is r?

We ask the question again, what is r? Let's see some examples before answering this question.

■ **Example 8.2** If we want to solve system of equations of size 1000 as the following:

$$\begin{cases} x_1+x_2=3 \\ 2x_1+2x_2=6 \\ \cdots \\ 1000x_1+1000x_2=3000 \end{cases}$$

It seems very difficult when hearding it has 1000 equations, but the remaining 999 equations could be redundant (They actually don't exist):

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ \vdots & \vdots \\ 1000 & 1000 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

■

Here we see only one equation $x_1+x_2=3$ is true, the remaining part is not true. So we claim that r is the number of "true" equations. But what is the definition for "true" equations? Let's discuss the definition for *linear dependence* first.

8.1.3 *Linearly dependence*

Definition 8.1 — linearly dependence. The vectors $\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_n$ in linear space $\boldsymbol{V}$ are **linearly dependent** if there exists $c_1,c_2,\dots,c_n\in\mathbb{R}$ s.t.

$$c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\cdots+c_n\boldsymbol{v}_n=\boldsymbol{0}.$$

In other words, it means one of v_i could be expressed as linear combination of others. When assuming $c_n\neq 0$, we can express $\boldsymbol{v}_n$ as:

$$\boldsymbol{v}_n=-\frac{c_1}{c_n}\boldsymbol{v}_1-\frac{c_2}{c_n}\boldsymbol{v}_2-\cdots-\frac{c_{n-1}}{c_n}\boldsymbol{v}_{n-1}.$$

■

Definition 8.2 — linearly independence. The vectors $\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_n$ in linear space $\boldsymbol{V}$ are **linearly independent** if the two statements are equivalent:

- $c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\cdots+c_n\boldsymbol{v}_n=\boldsymbol{0}$
- All scalars $c_1=c_2=\cdots=c_n=0$.

In other words, if $\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_n$ are not **linearly dependent**, they must be **linearly independent**. ■

(R) Note that **only** in this course, if we say vectors are dependent, we mean they are **linearly** dependent. And we often express *dependent* as *dep.*; we also sometimes express *linearly dependent* as *lin. dep.*; express *linearly independent* as *lin. ind.*

Here we pick some examples to help you understand lin. dep. and lin. ind.:

Example 8.3

- $\boldsymbol{v}_1 = (1,1)$ and $\boldsymbol{v}_2 = (2,2)$ are **dep.** because
$$(-2) \times \boldsymbol{v}_1 + \boldsymbol{v}_2 = \mathbf{0}.$$
- The only one vector $\boldsymbol{v}_1 = 2$ is **ind.** because
$$c\boldsymbol{v}_1 = \mathbf{0} \Longleftrightarrow c = 0.$$
- The only one vector $\boldsymbol{v}_1 = 0$ is **dep.** because
$$2 \times \boldsymbol{v}_1 = \mathbf{0}$$
- $\boldsymbol{v}_1 = (1,2)$ and $\boldsymbol{v}_2 = (0,0)$ are **dep.** because
$$0 \times \boldsymbol{v}_1 + 1 \times \boldsymbol{v}_2 = \mathbf{0}.$$
- The upper triangular matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ has three column vectors:
$$\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$
$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are **ind.** because
$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0} \Longleftrightarrow c_1 = c_2 = c_3 = 0.(\text{Why?})$$

Relation between lin.ind. and equations

The following statements are equivalent:

- Vectors $a_1, a_2, \dots, a_n \in \mathbb{R}^m$ are dep.
- $\exists$ c_i not all zero s.t. $\sum_{i=1}^n c_i a_i = \mathbf{0}$.
- $\exists$ some $\boldsymbol{c} \neq \mathbf{0}$ s.t.
$$\boldsymbol{A}\boldsymbol{c} = \begin{bmatrix} a_1 \mid \dots \mid a_n \end{bmatrix} \boldsymbol{c} = \mathbf{0}$$

So what if $m < n$, when checking corollary (8.1), we immediately obtain:

Corollary 8.2 When vectors $a_1, a_2, \dots, a_n \in \mathbb{R}^m (m < n)$ are dependent, there exists infinitely solutions $c_1, c_2, \dots, c_n$ such that $\sum_{i=1}^n c_i a_i = \mathbf{0}$.

So we say the true equations are those linearly independent equations.

8.1.4 Basis and dimension

Definition 8.3 — Basis. The vectors $v_1, \dots, v_n$ form a **basis** for a vector space $\boldsymbol{V}$ if and only if:

1. $v_1, \dots, v_n$ are **linearly independent**.
2. $v_1, \dots, v_n$ span $\boldsymbol{V}$.

■

■ **Example 8.4** In $\mathbb{R}^3$, $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ form a basis.

$\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not a basis, since it doesn't span $\mathbb{R}^3$.

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}$ don't form a basis, since they don'y linearly independent.

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}$ form a basis. ■

We feel that the number of vectors for basis of $\mathbb{R}^3$ is always 3, is this a coincidence? The theorem below gives the answer.

Theorem 8.2 If $v_1, v_2, \dots, v_m$ is a basis; $w_1, w_2, \dots, w_n$ is a basis for the same vector space $\boldsymbol{V}$, then $n = m$.

In order to proof it, let's try simple case first:

proofoutline.

- Let's consider $\boldsymbol{V} = \mathbb{R}$ case first:
  For $\mathbb{R}$, 1 forms a basis.
  Given any two vectors x and y, they are not a basis for $\mathbb{R}$. It is because that
  - if $x = 0$ or $y = 0$, they are not ind.
  - otherwise, $y = \frac{y}{x} \times x \implies \frac{y}{x} \times x + (-1) \times y = 0$. So they are not ind.
- Then we consider $\boldsymbol{V} = \mathbb{R}^3$ case:
  For $\mathbb{R}^3$, $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ is a basis.
  We want to show if $v_1, v_2, \dots, v_m$ is a basis, then $m = 3$.
  - Let's proof $m = 4$ is impossible (4 vectors in $\mathbb{R}^3$ cannot be a basis.):
    We only need to show for $\forall a_1, a_2, a_3, a_4 \in \mathbb{R}^3$ they must be dep.
    $\iff \boldsymbol{Ax} = \boldsymbol{0}$ has nonzero solutions, where $\boldsymbol{A} = \begin{bmatrix} a_1 & a_2 & \dots & a_4 \end{bmatrix} \in \mathbb{R}^{3\times 4}$.
    By corollary (8.1), it is obviously true.
  - The same argument could show any basis for $\mathbb{R}^3$ satisfies $m \le 3$.
  - Then let's prove $m = 2$ is impossible (2 vectors in $\mathbb{R}^2$ cannot be a basis):
    We only need to show for $\forall a_1, a_2 \in \mathbb{R}^3$, they cannot span the whole space.
    If this is not true, then $\boldsymbol{Ax} = \boldsymbol{b}$ must have solution, where $\boldsymbol{A} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \in \mathbb{R}^{3\times 2}$.
    However, this kind matrix may have no solution, which forms a contradiction.
  - The same arugment could show any basis for $\mathbb{R}^3$ satisfies $m \ge 3$.

- The same arugment could show any basis for $\mathbb{R}^n$ satisfies $m=n$.
- Next, let's consider general vector space:
  We assume $n<m$ (contradiction).
  We have known $v_1,\dots,v_n$ is a basis, our goal is to show $w_1,\dots,w_m$ cannot form a basis.
  $\Leftarrow \exists\, \boldsymbol{c}=\begin{bmatrix}c_1 & c_2 & \dots & c_m\end{bmatrix}^{\mathrm{T}} \neq \mathbf{0}$ s.t.

$$c_1w_1+c_2w_2+\cdots+c_mw_m=0. \tag{8.4}$$

Moreover, we can express $w_1,\dots,w_m$ in form of $v_1,\dots,v_n$:

$$\begin{cases} w_1 = a_{11}v_1+\cdots+a_{1n}v_n \\ \dots \\ w_m = a_{m1}v_1+\cdots+a_{mn}v_n \end{cases} \tag{8.5}$$

By (8.5), we can write (8.4) as:

$$\begin{aligned} 0 &= \sum_{j=1}^{m} c_jw_j \\ &= \sum_{j=1}^{m} c_j(\sum_{i=1}^{n} a_{ji}v_i) \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} c_ja_{ji}v_i \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} c_ja_{ji}v_i \\ &= \sum_{i=1}^{n} v_i\times(\sum_{j=1}^{m} c_ja_{ji}) \\ &= v_1\times(\sum_{j=1}^{m} c_ja_{j1})+v_2\times(\sum_{j=1}^{m} c_ja_{j2})+\cdots+v_n\times(\sum_{j=1}^{m} c_ja_{jn}) \end{aligned}$$

So, in order to let LHS=0, we only need to let each of RHS=0, more specifically, we only need to let $\sum_{j=1}^{m} c_ja_{j1}=\sum_{j=1}^{m} c_ja_{j2}=\cdots=\sum_{j=1}^{m} c_ja_{jn}=0$.
To write it into matrix form, we only need to let system $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}=\mathbf{0}$ has solution.
where $\boldsymbol{A}=\begin{bmatrix}a_{ij}\end{bmatrix}_{1\le m;1\le j\le n}$ and $\boldsymbol{c}=\begin{bmatrix}c_1 & c_2 & \dots & c_m\end{bmatrix}^{\mathrm{T}}$.
By corollary (8.1), since $\boldsymbol{A}^{\mathrm{T}}$ is $n\times m$ matrix where $n<m$, it has infinitely nonzero solution.
■

During the proof, we face two difficulties:

1. For arbitarily $\boldsymbol{V}$, we write a concrete form to express $w_1,w_2,\dots,w_m$.
2. We write matrix form to express $\sum_{j=1}^{m} c_ja_{j1}=\sum_{j=1}^{m} c_ja_{j2}=\cdots=\sum_{j=1}^{m} c_ja_{jn}=0$.

Next since all basis contains the same number of vectors, we can define the number of vectors to be dimension:

Definition 8.4 — Dimension. The **dimension** for a vector space is the number of vectors in a basis for it.

R Remember that vector space $\{0\}$ has dimension 0.
In order to denote the dimension for a given vector space $\boldsymbol{V}$, we often write it as $\dim(\boldsymbol{V})$.

■ **Example 8.5**
- $\mathbb{R}^n$ has dimension n.
- $\{$All $m \times n$ matrix$\}$ has dimension $m \cdot n$.
- $\{$All $n \times n$ symmetric matrix$\}$ has dimension $\frac{n(n+1)}{2}$.
- Let $\boldsymbol{P}$ denote the vector space of all polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$.
  dim($\boldsymbol{P}$)$\neq$ 3 since $1, x, x^2, x^3$ are ind.
  The same argument can show dim($\boldsymbol{P}$) is not equal to any real number, so dim($\boldsymbol{P}$)=∞

■

Human beings often ask a question: for a line and a plane, which is bigger?

1. Does plane has more point than a line?
   No, Cantor syas they have the same "number" of points by constructing a one-to-one mapping.
   Furthermore, $\mathbb{R}, \mathbb{R}^2, \dots, \mathbb{R}^n$ has the same number of points.
2. However, the plane has bigger dimension than a line. So from this point of view, a plane is bigger than a line.

You should know some common knowledge for dimension:

1. Programmer lives in **2** dimension world. (They only live with binary.)
2. Engineer lives in **3** dimension world. (They only live with enign.)
3. Physician lives in **4** dimension world. (They discuss time.)
4. String theories states that our world is **1**1 or **2**6 dimension, which has been proved by Qingshi Zhu.
5. For 3-body, they can perform dimension attack on you.

What is rank?

Finally let's answer the question: What is rank?
rank=dimension of row space of a matrix.
We will discuss it next lecture.