A FIRST COURSE

IN

ANALYSIS

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MAT2006 Notebook

Lecturer

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Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 3

Week3

3.1. Tuesday

3.1.1. Application of Heine-Borel Theorem

Theorem 3.1 Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ which converges in $|x| < 1$. If for every $x \in [0,1)$, there exists $n(= n(x))$ such that $\sum_{n+1}^{\infty} a_k x^k = 0$, then f is a polynomial, i.e., n does not depend on x.

The idea is to construct a sequence of points $\{x_n\}$ satisfying $f(x_k) = a_0 + \cdots + a_m x_k^m$, i.e., infinite points coincide $f(x)$ with a polynomial, which implies f is a polynomial.

Proof. Construct $E_N := \{x \in [0, \frac{1}{2}] \mid \sum_{k=N+1}^{\infty} a_k x^k = 0\}$. It follows that

$$[0, \frac{1}{2}] = \bigcup_{N=1}^{\infty} E_N,$$

which implies that at least one E_N is uncountable, say, E_m is uncountable. In particular, E_m is infinite

By Bolzano-Weierstrass Theorem, there exists a sequence $\{x_k\} \subset E_m$ with limit x_0 in E_m as E_m is closed. Hence, $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ holds for the sequence $\{x_m\}$. Intuitively we conclude the power series and the analytics function coincide each other for every point $x \in (-1,1)$.

$$f(x) \equiv a_0 + a_1 x + \cdots + a_m x^m$$

■

However, the proof above does not show why a sequence coincide $f(x)$ with a polynomial could imply f is a polynomial for every point. We summarize this induction as the proposition(3.1) and give a proof below. Before that we formulate what we want to prove precisely:

> Let f be analytic, i.e., $f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$ on $(-1, 1)$; and $f(x_k) = \sum_{i=1}^{m} a_i x_k^i$ for all $k \geq 1$, where $\{x_k\}$ is a sequence with limit x_0. Then $f(x) = \sum_{i=1}^{m} a_i x^i$ on $(-1, 1)$.

To show this statement, we construct

$$g(x) = f(x) - \sum_{i=1}^{m} a_i x^i \implies g(x_k) = 0, \forall k \geq 1$$

It suffices to show $g \equiv 0$ on $(-1, 1)$. Moreover, if we construct $y_k := x_k - x_0$, and set $f(x) = a_0 + a_1(x - x_0) + \cdots$, then it suffices to prove the proposition given below:

Proposition 3.1 Let g be analytic, i.e., $g(x) = b_0 + b_1 x + \cdots + b_n x^n + \cdots$ on $(-1, 1)$; and $g(x_k) = 0$ for all $k \geq 1$, where $\{x_k\} \to 0$. Then $g \equiv 0$ on $(-1, 1)$ (i.e., $b_0 = b_1 = \cdots = 0$)

Proof.
- Note that $g(0) = 0$ due to continuity property. Also, $g(0) = b_0 = 0$, which follows that

$$g(x) = x(b_1 + b_2 x + \cdots + b_n x^{n-1} + \cdots \tag{3.1}$$

- Substituting x with x_k in Eq.(3.1), we derive

$$0 = g(x_k) = x_k(b_1 + b_2 x_k + \cdots + b_n x_k^{n-1} + \cdots \tag{3.2}$$

  Taking limit both sides for (3.2), we derive $b_1 = 0$.

- By applying the same trick, we conclude $b_0 = b_1 = \cdots = 0$ (the rigorous proof requires induction).

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Now we talk about some advanced topics in Analysis.

3.1.2. Set Structure Analysis

Definition 3.1 [Nowhere Dense] A set B is said to be **nowhere dense** if its closure $\overline{B}$ contains no non-empty open set. ■

For example,

$$B = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, ..\} \implies \overline{B} = B \bigcup \{0\},$$

which contains no non-empty open set.

Definition 3.2 [1st category] A set of B is said to be of 1st category if it can be written as the **union** of **finitely** many or **countably** many **nowhere** dense sets. ■

Definition 3.3 [2rd category] A set is said to be of 2rd category if it is **not** of 1st category ■

Theorem 3.2 — **Baire-Category Theorem.**
- $\mathbb{R}$ is of 2rd category, i.e.,
- $\mathbb{R}$ cannot be written as the union of countably many nowhere dense sets, i.e.,
- if $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$, then at least one A_n whose closure contains a non-empty open set.

Proof.
- Assume $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$ such that all A_n's are nowhere dense. It follows that

$$\mathbb{R} \setminus \overline{A_1} \text{ is open,}$$

since $\overline{A_1}$ is closed and its complement is open.

- We construct an open set N_1 such that $\overline{N_1} \subseteq \mathbb{R} \setminus \overline{A_1}$. (e.g., there exists ε and $x \in \mathbb{R} \setminus \overline{A_1}$ such that $N_1 := B(x, \varepsilon) \subseteq \overline{N_1} \subseteq \mathbb{R} \setminus \overline{A_1}$.)
- Since A_2 is nowhere dense, we imply $\overline{A_2}$ does not contain N_1, i.e., $N_1 \setminus \overline{A_2}$ is open.

- By applying similar trick, we obtain a sequence of nested sets

$$\overline{N_1} \supseteq N_1 \supset \overline{N_2} \supset N_2 \cdots$$

  The cantor's theorem implies that $\bigcap_{k=1}^{\infty} \overline{N_k} \neq \emptyset$.

- On the other hand, $\bigcap_{k=1}^{\infty} \overline{N_k} \subseteq \mathbb{R} \setminus \bigcup_{n=1}^{m} A_n$ for any finite m.

- Therefore, $\emptyset \neq \bigcup_{k=1}^{\infty} \overline{N_k} \subseteq \mathbb{R} \setminus \bigcup_{n=1}^{\infty} A_n = \emptyset$, which is a contradiction.

■

R $\mathbb{R}$ is of 2nd category, i.e., if $\mathbb{R} = \bigcup_{n=1}^{\infty} A_m$, then at least A_n whose closure contains a **non-empty** open sets; The theorem also holds if we replace $\mathbb{R}$ by a **complete** metric space (essentially the same proof).

Most proof for $\mathbb{R}$ can be generalized into metric space, the proof for which is essentially the same. Now let's introduce the metric space informally.

Metric Space. A metric space is an ordered pair (M,d), where M is a set and d is a metric on M, i.e., d is a distance function defined for two points on M. Here we list several examples:

The Real Line. For $\mathbb{R}$, $d(x,y) = |x-y|$. Note that $(\mathbb{Q},d)$ and $(\mathbb{R}\setminus\mathbb{Q},d)$ are also metric spaces, but not complete.

n-Cell Real Space. $\mathbb{R}^n$, with $d(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}|$ is a metric space.

Bounded Sequences. The set of all bounded sequences on $\mathbb{R}$ is a metric space, with d defined as:

$$d(\{x_n\},\{y_n\}) = \sup\{|x_i - y_i| \mid i = 1,2,\ldots\}$$

Bounded Functions. Similarly, the set of all bounded continuous functions on $\mathbb{R}$ (different domains), with

$$d_1(f,g) = \sup\{|f(x) - g(x)| \mid x \in \mathbb{R}\},$$

or

$$d_2(f,g) = (\int_0^1 |f(x) - g(x)|^2 \, dx)^{1.2}$$

is a metric space. Note that $(\xi[0,1], d_1)$ is complete, and $(\xi[0,1], d_2)$ is not complete. (exercise)

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Different distance definition corresponds to different metric spaces.

Recall that a metric space is complete if all Cauchy sequence of which converge.

3.1.3. Reviewing

Definition 3.4 [Sequence] A sequence is defined as a kind of function $f : \mathbb{N} \to \mathbb{R}$, denoted as $\{f(0), f(1), \dots\}$. Conventionally we denote it as $x_1, x_2, \dots$

Definition 3.5 [Limit] A number α is the limit of $\{x_1, x_2, \dots\}$ if $\forall \epsilon > 0$, there $\exists N = N(\epsilon)$ such that $|x_k - \alpha| < \epsilon$ for $\forall k \geq N$, denoted by $\alpha_n \to \alpha$

Definition 3.6 [liminf & limsup]

$$\liminf_{k \to \infty} x_k := \lim_{n \to \infty} \inf_{k \geq n} x_k$$

which is the smallest limit point of the sequence

$$\limsup_{k \to \infty} x_k := \lim_{n \to \infty} \sup_{k \geq n} x_k$$

which is the largest limit point of the sequence.

A sequence always has liminf and limsup.

Definition 3.7 [Partial Sum] Given the sequence $\{a_n\}$, its n-th partial sum are defined as:

$$s_n = a_1 + \cdots + a_n,$$

the series $\sum_i a_i$ is defined as the limit of the partial sum,

Next lecture we will show that most continuous function is nowhere differentiable, by applying the Baire Category Theorem on $(\xi[0,1], d_1)$