A FIRST COURSE

IN

NUMERICAL ANALYSIS

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MAT4001 Notebook

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Contents

Acknowledgments

This book is from the MAT4001 in fall semester, 2018.

CUHK(SZ)

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Wednesday

1.1.1. Introduction to Imaginary System

Definition 1.1 [Complex Number] A complex number z is a pair of real numbers:

$$z = (x, y),$$

where x is the **real** part and y is the **imaginary part** of z, denoted as

$$\text{Re}z = x \quad \text{Im}z = y$$

R Note that the complex multiplication does not correspond to any standard vector operation. However, $(\mathbb{C}, +)$ and $(\mathbb{C} \setminus \{0\}, \cdot)$ forms a field:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

$$z_1 + z_2 = z_2 + z_1$$

$$z + 0 = 0 + z = z$$

$$z + (-z) = (-z) + z = 0$$

There is no other Eucliean space that can form a field.

Proposition 1.1 $zz' = 0$ if and only if $z = 0$ or $z' = 0$.

Proof. Rewrite the product as a linear system

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and discuss the determinant of the coefficient matrix. ■

Solving quadratic equation with one unknown. We can apply the imaginary number to solve the quadratic equations. For example, to solve $z^2 - 2z + 2 = 0$, the first method is to substitute z with $x + iy$; the second method is to simplify it into standard form to solve it.

Definition 1.2 If $z \neq 0$, then z^{-1} is the complex number satisfying $z \cdot z^{-1} = 1$.

Suppose $z = (x, y)$ and $z^{-1} = (u, v)$. After simplification, we derive

$$\begin{cases} xu - yv = 1 \\ xv + yu = 0 \end{cases} \implies \begin{cases} u = \dfrac{x}{x^2 + y^2} \\ v = \dfrac{-y}{x^2 + y^2} \end{cases}$$

Definition 1.3 [Division] The division between complex numbers is defined as:

$$\tfrac{z_1}{z_2} = z_1 \cdot z_2^{-1}, \quad \text{when } z_2 \neq 0$$

■ **Example 1.1**

$$\frac{3 - 4i}{1 + i} = (3 - 4i)\left(\frac{1}{2} - \frac{1}{2}i\right) = -\frac{1}{2} - \frac{7}{2}i$$

$$\frac{10}{(1+i)(2+i)(3+i)} = \frac{10}{(1+3i)(3+i)} = \frac{10}{10i} = \frac{1}{i} = -i$$

Definition 1.4 [Complex Conjugate] The complex number $x - iy$ is called the **complex conjugate** of $z = x + iy$, which is denoted by $\bar{z}$.

The following properties hold for complex conjugate:

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2. \quad \overline{\frac{z_1}{z_2}} = \frac{\bar{z}_1}{\bar{z}_2}$$

$$\mathrm{Re} z = \frac{z+\bar{z}}{2}, \quad \mathrm{Im} z = \frac{z-\bar{z}}{2i}$$

1.1.2. Algebraic and geometric properties

Definition 1.5 [Algebraic Region]

1. The complex plane: the z-plane, i.e., $\mathbb{C}$
2. Vector in $\mathbb{R}^2$: $(x, y) = x + iy = z \in \mathbb{C}$
3. Modulus of z:

$$|z| = \sqrt{x^2 + y^2} \quad \text{distance to the origin}$$

Note that

$$|z| = 0 \Longleftrightarrow z = 0, \quad |z_1 - z_2| = 0 \Longleftrightarrow z_1 = z_2$$

Definition 1.6 [Circle in plane] A circle with center z_0 and radius R is defined as follows in $\mathbb{C}$:

$$\{z \in \mathbb{C} \mid |z - z_0| = R\}$$

Proposition 1.2 Complex roots of polynomials with real coefficients appear in conjugate pairs.

Proof. Given $P(z_0) = 0$, we derive

$$P(z_0) = \overline{P(z_0)} = 0.$$

■

Note that a polynomial with real coefficients of degree 3 must have at least one real root.

Conjugate Product. Note that the conjugate product leads to the square of modulus:

$$z \cdot \bar{z} = |z|^2 \iff (x + iy)(x - iy) = x^2 + y^2$$

Such a property can be used to simplify quotient of two complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2} = \frac{x_1 x_2 + y_1 y_2 + (y_1 x_2 - x_1 y_2)i}{x_2^2 + y_2^2}$$

■ **Example 1.2**

$$\frac{-1+3i}{2-i} = \frac{(-1+3i)(2+i)}{(2-i)(2+i)} = \frac{-5+5i}{5} = -1 + i$$

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

■

We can use conjugate to show the **triangle inequality**:

Proposition 1.3 — **Triangle Inequality.** $|z_1 + z_2| \le |z_1| + |z_2|$.

Proof.

$$\begin{aligned}
|z_1+z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)} \\
&= |z_1|^2+|z_2|^2+z_1\bar{z}_2+\overline{z_1\bar{z}_2} \\
&= |z_1|^2+|z_2|^2+2\mathrm{Re}(z_1\bar{z}_2) \\
&\le |z_1|^2+|z_2|^2+2|z_1\bar{z}_2| \\
&= |z_1|^2+|z_2|^2+2|z_1z_2| = (|z_1|+|z_2|)^2.
\end{aligned}$$

■

Corollary 1.1 1. $||z_1|-|z_2|| \le |z_1 \pm z_2|$.

2. If $|z| \le 1$, then $|z^2+z+1| \le 3$

Proof. 1. Note that

$$|z_1| = |z_1 \pm z_2 \mp z_2| \le |z_1 \pm z_2| + |z_2| \implies |z_1|-|z_2| \le |z_1 \pm z_2|$$

Similarly, $|z_2|-|z_1| \le |z_1 \pm z_2|$.

2.
$$|z^2+z+1| \le |z^2|+|z+1| \le |z|^2+|z|+1 \le 1+1+1 = 3.$$

■

Proposition 1.4 — **Cauchy-Schwarz inequality.** If $z_1,\ldots,z_n$ and $w_1,\ldots,w_n$ are complex numbers, then

$$\left[\sum_{k=1}^{n} z_k w_k\right]^2 \le \left[\sum_{k=1}^{n} |z_k|^2\right]\left[\sum_{k=1}^{n} |w_k|^2\right]$$

1.1.3. Polar and exponential forms

Definition 1.7 [Polar Form] The polar form of a nonzero complex number z is:

$$z = r(\cos\theta + i\sin\theta)$$

where (r,θ) is the polar coordinates of (x,y).

$$(r,\theta) \implies (x,y): \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$(x,y) \implies (r,\theta): \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

Note that θ is said to be the **argument** of z, i.e., $\theta = \arg z$. The augument is not unique, i.e.,

$$z = r(\cos\theta + i\sin\theta) r(\cos(\theta+2\pi) + i\sin(\theta+2\pi))$$

■

If given an argument of z, then we form the set of arguments of z:

$$\{\theta + 2n\pi \mid n \in \mathbb{Z}\}$$

Definition 1.8 [Principal Value] The principal value of $\arg z$, denoted by $\text{Arg}z$, is the unique value of $\arg z$ such that $-\pi < \arg z \leq \pi$ ■

■ **Example 1.3**

1. $\text{Arg}z = \pi$ implies $z = r(\cos\pi + i\sin\pi) = -r < 0$, which is a negative real number.
2. $\text{Arg}z = 0$ implies $z = r(\cos 0 + i\sin 0) = r > 0$m which is a positive real number.
3. $\text{Arg}z = -\frac{\pi}{2}$ implies $z = r(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})) = -ri$
4. $\text{Arg}z = \frac{\pi}{2}$ implies $z = ri$
5. Particularly, $\pm i = \cos(\pm\frac{\pi}{2}) + i\sin(\pm\frac{\pi}{2})$

■

Product in polar form. Given $z_i = r_i(\cos\theta_i + i\sin\theta_i)$ for $i = 1,2$, we can compute its product:

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\ &= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

Thus, $\arg(z_1z_2) = \arg z_1 + \arg z_2$.

R Note that $\operatorname{Arg}(z_1z_2) \neq \operatorname{Arg} z_1 + \operatorname{Arg} z_2$. ($\operatorname{Arg}(z_1z_2)$ should be restricted to be within the interval $(-\pi,\pi]$)

Inverse in Polar form. Given $z = r(\cos\theta + i\sin\theta)$, we aim to find the inverse such that $zz^{-1} = 1$. Hence, $z^{-1} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$.

If we obtain the inverse, we can compute the division $\frac{z_1}{z_2}$:

$$\frac{z_1}{z_2} = r_1(\cos\theta_1 + i\sin\theta_1)\frac{1}{r_2}(\cos(-\theta_2) + i\sin(-\theta_2)) = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

Thus, $\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$.

Euler Identity. The Euler Identity is given by:

$$e^{ix} = \cos x + i\sin x$$

The proof requires Taylor's expansion.

Exponential Form. The exponential form of z in polar form is given by:

$$z = re^{i\theta}$$

Then it is convenient to define produt, inverse, and division:

$$
\begin{aligned}
(r_1e^{i\theta_1}) \cdot (r_2e^{i\theta_2}) &= r_1r_2e^{i(\theta_1+\theta_2)} \\
\frac{1}{re^{i\theta}} &= \frac{1}{r}e^{i(-\theta)} \\
\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} &= \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}
\end{aligned}
$$

Nonuniqueness. $z = re^{i\theta} = re^{i(\theta+2n\pi)}$

Equality. Two complex numbers are equal means that:

$$
r_1e^{i\theta_1} = r_2e^{i\theta_2} \Longleftrightarrow \begin{cases} r_1 = r_2 \\ \theta_1 = \theta_2 + 2k\pi, k \in \mathbb{Z} \end{cases}
$$

Circle. The circle centered at the origin with radius R can be described as:

$$
|z| = R \Longleftrightarrow z = Re^{i\theta}, \quad 0 \le \theta < 2\pi
$$

The circle centered at z_0 with radius R can be described as:

$$
|z - z_0| = R \Longleftrightarrow z = z_0 + Re^{i\theta}, \quad 0 \le \theta < 2\pi
$$

Neighborhoold. The ϵ-neighborhood of the point z_0 is given by:

$$
|z - z_0| < \epsilon
$$

If delete the center, it is given by:

$$
0 < |z - z_0| < \epsilon
$$

1.2. Powers and Roots

Powers. The powers of $z = re^{i\theta}$ is given by:

$$z^n = r^n e^{in\theta}$$

$$z^{-n} = r^{-n} e^{i(-n)\theta}$$

Thus we derive the **De Moiver's Formula**:

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = \cos n\theta + i\sin n\theta.$$

It is useful for computing powers tha contains complex number. For example,

$$(1+i)^n = (\sqrt{2}e^{i\frac{\pi}{4}})^n = 2^{n/2}e^{\frac{in\pi}{4}}$$

Proposition 1.5

$$\sin(n\theta) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k+1} (-1)^k \cos^{n-2k-1}\theta \sin^{2k+1}\theta,$$

where $\lfloor x \rfloor$ denotes the largest integer that not exceeds x.

Solving high order equations. The powers of complex can also be used to solve high order equations.

■ **Example 1.4** To sovle the equation $z^n = 1$, we express $z = re^{i\theta}$. It follows that

$$(re^{i\theta})^n = 1e^{i0} \implies \begin{cases} r^n = 1 \\ n\theta = 2k\pi \end{cases} \implies \begin{cases} r = 1 \\ \theta = \dfrac{2k\pi}{n} \end{cases}$$

Thus, the distinct n-th roots(of unity) are given by:

$$\exp(i\frac{2k\pi}{n}) = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \qquad k = 0, 1, 2, \ldots, n-1.$$

If we denote $w_n = \exp(i\frac{2\pi}{n})$, we derive the roots:

$$1, w_n, w_n^2, \ldots, w_n^{n-1}.$$

Roots of high order equations. Suppose $z_0 = r_0 e^{i\theta_0}$, we aim to sovle $z^n = z_0$:

$$r^n e^{in\theta} = r_0 e^{i\theta_0} \implies \begin{cases} r = r_0^{1/n} \\ \theta = \dfrac{\theta_0 + 2k\pi}{n} \end{cases}$$

Thus the distinct nth roots are given by:

$$r_0^{1/n} \exp(i\frac{\theta_0 + 2k\pi}{n}), \qquad k = 0, 1, 2, \ldots, n-1.$$

Actually, $(\mathbb{C}, +)$ forms a group:

Also, $(\mathbb{C} \setminus \{0\}, \cdot)$ forms a group.

The product for imaginary numbers is different from vector product:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$$

Also, we can define the crossover product $\vec{v} \times \vec{w}$.

modulus:

$$|z| = \sqrt{x^2 + y^2}$$

direction angle (Argument):

$$\tan\theta = \frac{y}{x}$$

Using the polar coordination, we find $z = x + iy$ can be transformed into

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= r\left[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)\right] \end{aligned}$$

Principal argument:

$$-\pi < \mathrm{Arg} z \leq \pi$$

Conjugate form of imaginary number:

$$\bar{z} = x - iy$$

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$z \cdot \bar{z} = |z|^2$$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

Proposition 1.6 $|z + w| \leq |z| + |w|$

Proposition 1.7 $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2.$

Proposition 1.8

$$\Re z = \frac{z + \bar{z}}{2}, \quad \Im z = \tfrac{z - \bar{z}}{2i}$$