A FIRST COURSE

IN

ANALYSIS

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MAT2006 Notebook

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Contents

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CUHK(SZ)

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m\times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i,j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $a_{ij} = a_{ji}$ for all i,j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i,j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i,j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i,j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Wednesday

Recommended Reading.

1. (Springer-Lehrbuch) V. A. Zorich, J. Schüle-Analysis I-Springer (2006).
2. (International series in pure and applied mathematics) Walter Rudin, Principles of Mathematical Analysis-McGraw-Hill (1976).
3. Terence Tao, Analysis I,II-Hindustan Book Agency (2006)
4. (Cornerstones) Anthony W. Knapp, Basic real analysis-Birkhäuser (2005)

1.1.1. Introduction to Set

For a set $\mathcal{A} = \{1,2,3\}$, we have $2^3 = 8$ subsets of $\mathcal{A}$. We are interested to study the collection of sets.

Definition 1.1 [Collection of Subsets] Given a set $\mathcal{A}$, the the collection of subsets of $\mathcal{A}$ is denoted as $2^{\mathcal{A}}$. ■

We use Candinal to describe the order of number of elements in a set.

Definition 1.2 Given two sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ and $\mathcal{B}$ are said to be **equivalent** (or have the same**candinal**) if there exists a 1-1 onto mapping from $\mathcal{A}$ to $\mathcal{B}$. ■

Definition 1.3 [Countability] The set $\mathcal{A}$ is said to be **countable** if $\mathcal{A} \sim \mathbb{N} = \{1,2,3,\dots\}$; an infinite set $\mathcal{A}$ is **uncountable** if it is not equivalent to $\mathbb{N}$. ■

R Note that the set of integers, i.e., $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ is also countable; the set of rational numbers, i.e., $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$ is countable.

We skip the process to define real numbers.

Proposition 1.1 The set of real numbers $\mathbb{R}$ is **uncountable**.

For example, $\sqrt{2} \notin \mathbb{Q}$. Some inrational numbers are the roots of some polynomials, such a number is called **algebraic** numbers. However, some inrational numbers are not, such a number is called **transcendental**. For example, π is **not** algebraic. We will show that the collection of algebraic numbers are countable in the future.

There are two steps for the proof for proposition(1.1):

Proof. 1. $2^{\mathbb{N}}$ is **uncountable**:

Assume $2^{\mathbb{N}}$ is countable, i.e.,

$$2^{\mathbb{N}} = \{A_1, A_2, \dots, A_k, \dots\}$$

Define $B := \{k \in \mathbb{N} \mid k \notin A_k\}$, it is a collection of subscripts such that the subscript k does not belong to the corresponding subsets A_k.

It follows that $B \in 2^{\mathbb{N}} \implies B = A_n$ for some n. Then it follows two cases:

- If $n \in A_n$, then $n \notin B = A_n$, which is a contradiction
- Otherwise, $n \in B = A_n$, which is also a contradiction.

The proof for the claim $2^{\mathbb{N}}$ is **uncountable** is complete.

2. $\mathbb{R} \sim 2^{\mathbb{N}}$:

Firstly we have $\mathbb{R} \sim (0,1)$. This can be shown by constructing a one-to-one mapping:

$$f : \mathbb{R} \mapsto (0,1) \quad f(x) = \frac{1}{\pi}\arctan x + \frac{1}{2}, \forall x \in \mathbb{R}$$

Secondly, we show that $2^{\mathbb{N}} \sim (0,1)$. We construct a mapping f such that

$$f : 2^{\mathbb{N}} \mapsto (0,1),$$

where for $\forall A \in 2^{\mathbb{N}}$,

$$f(A) = 0.a_1a_2a_3\ldots, \quad a_j = \begin{cases} 2, & \text{if } j \in A \\ 4, & \text{if } j \notin A \end{cases}$$

This function is only 1-1 mapping but not onto mapping.

Reversely, we construct a 1-1 mapping from $(0,1)$ to $2^{\mathbb{N}}$. We construct a mapping g such that

$$g : (0,1) \mapsto 2^{\mathbb{N}}$$

where for any real number from $(0,1)$, we can write it into binary expansion:

$$\text{binary form:} \quad 0.a_1a_2\ldots \quad \text{where } a_j = 0 \text{ or } 1.$$

Hence, we construct $g(0.a_1a_2\ldots) = \{j \in \mathbb{N} \mid a_j = 0\} \subseteq \mathbb{N}$, which implies $g(\cdot) \in 2^{\mathbb{N}}$.

R Our intuition is that two 1-1 mappings in the reverse direction will lead to a 1-1 **onto** mapping. If this is true, then we complete the proof. This intuition is the **Schroder-Berstein Theorem**.

■

Defining Binary Form. However, during this proof, we must be careful about the binary form of a real number from $(0,1)$. Now we give a clear definition of Binary Form:

For a real number a, to construct its binary form, we define

$$a_1 = \begin{cases} 0, & \text{if } a \in (0, \frac{1}{2}) \\ 1, & \text{if } a \in [\frac{1}{2}, 1). \end{cases}$$

After having chosen $a_1, a_2, \ldots, a_{j-1}$, we define a_j to be the largest integer such that

$$\frac{1}{2}a_1 + \frac{1}{2^2}a_2 + \cdots + \frac{a_j}{2^j} \leq a$$

Then the binary form of a is $a := 0.a_1 a_2 \ldots$.

Theorem 1.1 — Schroder-Berstein Theorem. If $f : A \mapsto B$ and $g : A \mapsto B$ are both 1-1 mapping, then there exists a 1-1 onto mapping from A to B.

Exercise: Show that $(0,1]$ and $[0,1)$ have 1-1 onto mapping without applying Schroder-Berstein Theorem.

The next lecture we will take a deeper study into the proof of Schroder-Berstein Theorem and the real number.

1.2. Quiz

1. Show that the sequence $\{x_n\}$ is convergent, where

$$x_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \cdots + \frac{\sin n}{2^n}.$$

2. Compute the following limits:

   (a)
$$\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/(1-\cos x)}$$

   (b)
$$\lim_{n \to \infty} \int_0^1 \frac{x^n}{1+\sqrt{x}} \, \mathrm{d}x$$

3. Justify that the natural number e is irrational, where

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

4. Every rational x can be written in the form $x = p/q$, where $q > 0$ and p and q are integers without any common divisors. When $x = 0$, we take $q = 1$. Consider the function f defined on $\mathbb{R}^1$ by

$$f(x) = \begin{cases} 0, & x \text{ is irrational} \\ \dfrac{1}{q}, & x = \frac{p}{q}. \end{cases}$$

   Find:

   (a) all continuities of $f(x)$;
   (b) all discontinuities of $f(x)$

   and prove your results.