8.3 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\, dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we must find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form

$$\int \sin^m x \cos^n x\, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If ***m* is odd** in $\int \sin^m x \cos^n x\, dx$, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we substitute $u = \cos x$ and $du = -\sin x\, dx$.

Case 2 If ***n* is odd** in $\int \sin^m x \cos^n x\, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then substitute $u = \sin x$ and $du = \cos x\, dx$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

EXAMPLE 1 Evaluate $\int \sin^3 x \cos^2 x\, dx$.

Solution This is an example of Case 1.

$$\begin{aligned}
\int \sin^3 x \cos^2 x\, dx &= \int \sin^2 x \cos^2 x \sin x\, dx && m \text{ is odd.} \\
&= \int (1 - \cos^2 x)(\cos^2 x) \sin x\, dx && \sin^2 x = 1 - \cos^2 x \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x,\ du = -\sin x\, dx \\
&= \int (u^4 - u^2)\, du && \text{Distribute.} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C
\end{aligned}$$

EXAMPLE 2 Evaluate

$$\int \cos^5 x\, dx.$$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\begin{aligned}
\int \cos^5 x\, dx &= \int \cos^4 x \cos x\, dx \\
&= \int (1 - \sin^2 x)^2 \cos x\, dx \\
&= \int (1 - u^2)^2\, du && u = \sin x,\ du = \cos x\, dx \\
&= \int (1 - 2u^2 + u^4)\, du && \text{Square } 1 - u^2. \\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C
\end{aligned}$$

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x\, dx.$$

Solution This is an example of Case 3.

$$\begin{aligned}
\int \sin^2 x \cos^4 x\, dx &= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx && m \text{ and } n \text{ both even} \\
&= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx \\
&= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx \\
&= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right]
\end{aligned}$$

For the term involving $\cos^2 2x$, we use

$$\begin{aligned}
\int \cos^2 2x\, dx &= \frac{1}{2}\int (1 + \cos 4x)\, dx && \text{Use the identity } \cos^2\theta = (1 + \cos 2\theta)/2, \text{ with } \theta = 2x. \\
&= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + C_1.
\end{aligned}$$

For the $\cos^3 2x$ term, we have

$$\begin{aligned}
\int \cos^3 2x\, dx &= \int (1 - \sin^2 2x)\cos 2x\, dx && u = \sin 2x,\ du = 2\cos 2x\, dx \\
&= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right) + C_2.
\end{aligned}$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x\, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

Eliminating Square Roots

In the next example, we use the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx &= \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx \\
&= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \qquad \cos 2x \geq 0 \text{ on } [0, \pi/4] \\
&= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}.
\end{aligned}$$

Integrals of Powers of tan x and sec x

We know how to integrate the tangent and secant functions and their squares. To integrate higher powers, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 5 Evaluate

$$\int \tan^4 x\, dx.$$

Solution

$$\begin{aligned}
\int \tan^4 x\, dx &= \int \tan^2 x \cdot \tan^2 x\, dx \\
&= \int \tan^2 x \cdot (\sec^2 x - 1)\, dx \qquad \tan^2 x = \sec^2 x - 1 \\
&= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx \\
&= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx \qquad \tan^2 x = \sec^2 x - 1 \\
&= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx
\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\,du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C. \qquad \blacksquare$$

EXAMPLE 6 Evaluate

$$\int \sec^3 x\,dx.$$

Solution We integrate by parts using

$$u = \sec x, \qquad dv = \sec^2 x\,dx, \qquad v = \tan x, \qquad du = \sec x \tan x\,dx.$$

Then

$$\begin{aligned}
\int \sec^3 x\,dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x)\,dx && \text{Integrate by parts.}\\
&= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx && \tan^2 x = \sec^2 x - 1\\
&= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx
\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

and therefore

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C. \qquad \blacksquare$$

EXAMPLE 7 Evaluate

$$\int \tan^4 x \sec^4 x\,dx.$$

Solution

$$\begin{aligned}
\int (\tan^4 x)(\sec^4 x)\,dx &= \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x)\,dx && \sec^2 x = 1 + \tan^2 x\\
&= \int (\tan^4 x + \tan^6 x)(\sec^2 x)\,dx && \text{Distribute.}\\
&= \int (u^4 + u^6)\,du = \frac{u^5}{5} + \frac{u^7}{7} + C && u = \tan x,\ du = \sec^2 x\,dx\\
&= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C && \blacksquare
\end{aligned}$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\, dx, \qquad \int \sin mx \cos nx\, dx, \qquad \text{and} \qquad \int \cos mx \cos nx\, dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the following identities.

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x] \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x] \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x] \tag{5}$$

These identities come from the angle sum formulas for the sine and cosine functions (Section 1.3). They give functions whose antiderivatives are easily found.

EXAMPLE 8 Evaluate

$$\int \sin 3x \cos 5x\, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned} \int \sin 3x \cos 5x\, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C. \end{aligned}$$

EXERCISES 8.3

Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–22.

1. $\int \cos 2x\, dx$

2. $\int_0^{\pi} 3 \sin \frac{x}{3}\, dx$

3. $\int \cos^3 x \sin x\, dx$

4. $\int \sin^4 2x \cos 2x\, dx$

5. $\int \sin^3 x\, dx$

6. $\int \cos^3 4x\, dx$

7. $\int \sin^5 x\, dx$

8. $\int_0^{\pi} \sin^5 \frac{x}{2}\, dx$

9. $\int \cos^3 x\, dx$

10. $\int_0^{\pi/6} 3 \cos^5 3x\, dx$

11. $\int \sin^3 x \cos^3 x\, dx$

12. $\int \cos^3 2x \sin^5 2x\, dx$

13. $\int \cos^2 x\, dx$

14. $\int_0^{\pi/2} \sin^2 x\, dx$

15. $\int_0^{\pi/2} \sin^7 y\, dy$

16. $\int 7 \cos^7 t\, dt$

17. $\int_0^{\pi} 8 \sin^4 x\, dx$

18. $\int 8 \cos^4 2\pi x\, dx$

19. $\int 16 \sin^2 x \cos^2 x\, dx$

20. $\int_0^{\pi} 8 \sin^4 y \cos^2 y\, dy$

21. $\int 8 \cos^3 2\theta \sin 2\theta\, d\theta$

22. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta\, d\theta$

Integrating Square Roots

Evaluate the integrals in Exercises 23–32.

23. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx$

24. $\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$

25. $\int_0^{\pi} \sqrt{1-\sin^2 t}\,dt$

26. $\int_0^{\pi} \sqrt{1-\cos^2\theta}\,d\theta$

27. $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}}\,dx$

28. $\int_0^{\pi/6} \sqrt{1+\sin x}\,dx$

$\left(\textit{Hint}\text{: Multiply by } \sqrt{\frac{1-\sin x}{1-\sin x}}.\right)$

29. $\int_{5\pi/6}^{\pi} \frac{\cos^4 x}{\sqrt{1-\sin x}}\,dx$

30. $\int_{\pi/2}^{3\pi/4} \sqrt{1-\sin 2x}\,dx$

31. $\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\,d\theta$

32. $\int_{-\pi}^{\pi} (1-\cos^2 t)^{3/2}\,dt$

Powers of Tangents and Secants

Evaluate the integrals in Exercises 33–52.

33. $\int \sec^2 x \tan x\,dx$

34. $\int \sec x \tan^2 x\,dx$

35. $\int \sec^3 x \tan x\,dx$

36. $\int \sec^3 x \tan^3 x\,dx$

37. $\int \sec^2 x \tan^2 x\,dx$

38. $\int \sec^4 x \tan^2 x\,dx$

39. $\int_{-\pi/3}^{0} 2\sec^3 x\,dx$

40. $\int e^x \sec^3 e^x\,dx$

41. $\int \sec^4\theta\,d\theta$

42. $\int \tan^4 x \sec^3 x\,dx$

43. $\int_{\pi/4}^{\pi/2} \csc^4\theta\,d\theta$

44. $\int \sec^6 x\,dx$

45. $\int 4\tan^3 x\,dx$

46. $\int_{-\pi/4}^{\pi/4} 6\tan^4 x\,dx$

47. $\int \tan^5 x\,dx$

48. $\int \cot^6 2x\,dx$

49. $\int_{\pi/6}^{\pi/3} \cot^3 x\,dx$

50. $\int 8\cot^4 t\,dt$

51. $\int_{\pi/4}^{\pi/3} \tan^5\theta \sec^4\theta\,d\theta$

52. $\int \cot^3 t \csc^4 t\,dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 53–58.

53. $\int \sin 3x \cos 2x\,dx$

54. $\int \sin 2x \cos 3x\,dx$

55. $\int_{-\pi}^{\pi} \sin 3x \sin 3x\,dx$

56. $\int_0^{\pi/2} \sin x \cos x\,dx$

57. $\int \cos 3x \cos 4x\,dx$

58. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x\,dx$

Exercises 59–64 require the use of various trigonometric identities before you evaluate the integrals.

59. $\int \sin^2\theta \cos 3\theta\,d\theta$

60. $\int \cos^2 2\theta \sin\theta\,d\theta$

61. $\int \cos^3\theta \sin 2\theta\,d\theta$

62. $\int \sin^3\theta \cos 2\theta\,d\theta$

63. $\int \sin\theta \cos\theta \cos 3\theta\,d\theta$

64. $\int \sin\theta \sin 2\theta \sin 3\theta\,d\theta$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 65–70.

65. $\int \frac{\sec^3 x}{\tan x}\,dx$

66. $\int \frac{\sin^3 x}{\cos^4 x}\,dx$

67. $\int \frac{\tan^2 x}{\csc x}\,dx$

68. $\int \frac{\cot x}{\cos^2 x}\,dx$

69. $\int x\sin^2 x\,dx$

70. $\int x\cos^3 x\,dx$

Applications

71. Arc length Find the length of the curve

$$y = \ln(\sin x), \quad \frac{\pi}{6} \le x \le \frac{\pi}{2}$$

72. Center of gravity Find the center of gravity of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

73. Volume Find the volume generated by revolving one arch of the curve $y = \sin x$ about the x-axis.

74. Area Find the area between the x-axis and the curve $y = \sqrt{1+\cos 4x}$, $0 \le x \le \pi$.

75. Centroid Find the centroid of the region bounded by the graphs of $y = x + \cos x$ and $y = 0$ for $0 \le x \le 2\pi$.

76. Volume Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sin x + \sec x$, $y = 0$, $x = 0$, and $x = \pi/3$ about the x-axis.

77. Volume Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \arctan x$, $x = 0$, and $y = \pi/4$ about the y-axis.

78. Average Value Find the average value of the function $f(x) = \frac{1}{1-\sin\theta}$ on $[0, \pi/6]$.

8.4 Trigonometric Substitutions

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a\tan\theta$, $x = a\sin\theta$, and $x = a\sec\theta$. These substitutions are effective in transforming integrals involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$ into integrals with respect to θ, since they come from the reference right triangles in Figure 8.2.

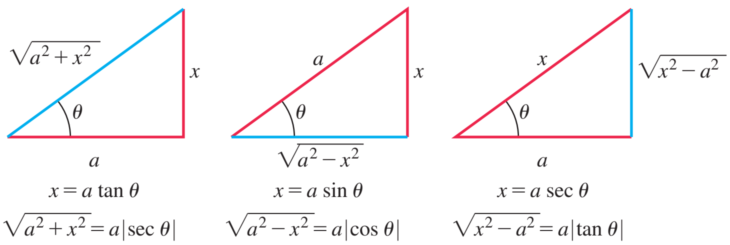

FIGURE 8.2 Reference triangles for the three basic substitutions, identifying the sides labeled x and a for each substitution.

With $x = a\tan\theta$,

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

With $x = a\sin\theta$,

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

With $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a\tan\theta$, we want to be able to set $\theta = \arctan(x/a)$ after the integration takes place. If $x = a\sin\theta$, we want to be able to set $\theta = \arcsin(x/a)$ when we're done, and similarly for $x = a\sec\theta$.

As we know from Section 7.6, the functions in these substitutions have inverses only for selected values of θ (Figure 8.3). For reversibility,

$$x = a\tan\theta \quad \text{requires} \quad \theta = \arctan\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \arcsin\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \operatorname{arcsec}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if} \quad \dfrac{x}{a} \ge 1, \\[2ex] \dfrac{\pi}{2} < \theta \le \pi & \text{if} \quad \dfrac{x}{a} \le -1. \end{cases}$$

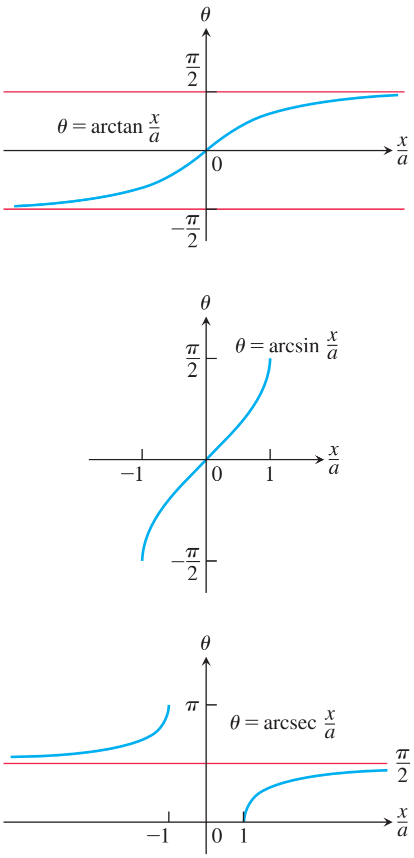

FIGURE 8.3 The arctangent, arcsine, and arcsecant of x/a, graphed as functions of x/a.

To simplify calculations with the substitution $x = a\sec\theta$, we will restrict its use to integrals in which $x/a \ge 1$. This will place θ in $[0, \pi/2)$ and make $\tan\theta \ge 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta$, free of absolute values, provided $a > 0$.