35. $\displaystyle\int \frac{7\,dx}{(x-1)\sqrt{x^2-2x-48}}$

36. $\displaystyle\int \frac{dx}{(2x+1)\sqrt{4x+4x^2}}$

37. $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

38. $\displaystyle\int \frac{d\theta}{\cos\theta-1}$

39. $\displaystyle\int \frac{dx}{1+e^x}$
Hint: Use long division.

40. $\displaystyle\int \frac{\sqrt{x}}{1+x^3}\,dx$
Hint: Let $u = x^{3/2}$.

41. $\displaystyle\int \frac{e^{3x}}{e^x+1}\,dx$

42. $\displaystyle\int \frac{2^x-1}{3^x}\,dx$

43. $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\,dx$

44. $\displaystyle\int \frac{\tan\theta+3}{\sin\theta}\,d\theta$

Theory and Examples

45. **Area** Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

46. **Volume** Find the volume of the solid generated by revolving the region in Exercise 45 about the x-axis.

47. **Arc length** Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

48. **Arc length** Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

49. **Centroid** Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

50. **Centroid** Find the centroid of the region bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

51. The functions $y = e^{x^3}$ and $y = x^3e^{x^3}$ do not have elementary antiderivatives, but $y = (1+3x^3)e^{x^3}$ does. Evaluate
$$\int (1+3x^3)\,e^{x^3}\,dx.$$

52. Use the substitution $u = \tan x$ to evaluate the integral
$$\int \frac{dx}{1+\sin^2 x}.$$

53. Use the substitution $u = x^4+1$ to evaluate the integral
$$\int x^7\sqrt{x^4+1}\,dx.$$

54. **Using different substitutions** Show that the integral
$$\int \left((x^2-1)(x+1)\right)^{-2/3}\,dx$$
can be evaluated with any of the following substitutions.
a. $u = 1/(x+1)$
b. $u = ((x-1)/(x+1))^k$ for $k = 1,\ 1/2,\ 1/3,\ -1/3,\ -2/3,$ and -1
c. $u = \arctan x$
d. $u = \tan^{-1}\sqrt{x}$
e. $u = \tan^{-1}((x-1)/2)$
f. $u = \arccos x$
g. $u = \cosh^{-1} x$
What is the value of the integral?

8.2 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form
$$\int u(x)\,v'(x)\,dx.$$
It is useful when u can be differentiated repeatedly and v' can be integrated repeatedly without difficulty. The integrals
$$\int x\cos x\,dx \qquad \text{and} \qquad \int x^2e^x\,dx$$
are such integrals because $u(x) = x$ or $u(x) = x^2$ can be differentiated repeatedly, and $v'(x) = \cos x$ or $v'(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like
$$\int \ln x\,dx \qquad \text{and} \qquad \int e^x\cos x\,dx.$$
In the first case, the integrand $\ln x$ can be rewritten as $(\ln x)(1)$, and $u(x) = \ln x$ is easy to differentiate while $v'(x) = 1$ easily integrates to x. In the second case, each part of the integrand appears again after repeated differentiation or integration.

Product Rule in Integral Form

If u and v are differentiable functions of x, the Product Rule says that
$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[u(x)v(x)]\,dx = \int [u'(x)v(x) + u(x)v'(x)]\,dx$$

or

$$\int \frac{d}{dx}[u(x)v(x)]\,dx = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int u(x)v'(x)\,dx = \int \frac{d}{dx}[u(x)v(x)]\,dx - \int v(x)u'(x)\,dx,$$

leading to the following **integration by parts** formula.

Integration by Parts Formula

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx \tag{1}$$

This formula allows us to exchange the problem of computing the integral $\int u(x)v'(x)\,dx$ for the problem of computing a different integral, $\int v(x)u'(x)\,dx$. In many cases, we can choose the functions u and v so that the second integral is easier to compute than the first. There can be many choices for u and v, and it is not always clear which choice works best, so sometimes we need to try several.

The formula is often given in differential form. With $v'(x)\,dx = dv$ and $u'(x)\,dx = du$, the integration by parts formula becomes

Integration by Parts Formula—Differential Version

$$\int u\,dv = uv - \int v\,du \tag{2}$$

The next examples illustrate the technique.

EXAMPLE 1 Find

$$\int x \cos x\,dx.$$

Solution There is no obvious antiderivative of $x \cos x$, so we use the integration by parts formula

$$\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int v(x)\,u'(x)\,dx$$

to change this expression to one that is easier to integrate. We first decide how to choose the functions $u(x)$ and $v(x)$. There is more than one way to do this, but here we choose to factor the expression $x \cos x$ into

$$u(x) = x \quad \text{and} \quad v'(x) = \cos x.$$

Next we differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 1 \quad \text{and} \quad v(x) = \sin x.$$

When finding an antiderivative for $v'(x)$, we have a choice of how to pick a constant of integration C. We choose the constant $C = 0$, since that makes this antiderivative as simple as possible. We now apply the integration by parts formula:

$$\int \underset{u(x)}{x} \underset{v'(x)}{\cos x}\, dx = \underset{u(x)}{x} \underset{v(x)}{\sin x} - \int \underset{v(x)}{\sin x} \underset{u'(x)}{(1)}\, dx \qquad \text{Integration by parts formula}$$

$$= x \sin x + \cos x + C \qquad \text{Integrate and simplify.}$$

and we have found the integral of the original function. ■

There are at least four apparent choices available for $u(x)$ and $v'(x)$ in Example 1:

1. Let $u(x) = 1$ and $v'(x) = x \cos x$.
2. Let $u(x) = x$ and $v'(x) = \cos x$.
3. Let $u(x) = x \cos x$ and $v'(x) = 1$.
4. Let $u(x) = \cos x$ and $v'(x) = x$.

We used choice 2 in Example 1. The other three choices lead to integrals that we do not know how to evaluate. For instance, Choice 3, with $u'(x) = \cos x - x \sin x$, leads to the integral

$$\int (x \cos x - x^2 \sin x)\, dx.$$

The goal of integration by parts is to go from an integral $\int u(x)v'(x)\, dx$ that we don't see how to evaluate to an integral $\int v(x)u'(x)\, dx$ that we can evaluate. Generally, we choose $v'(x)$ first to be as much of the integrand as we can readily integrate; then we let $u(x)$ be the leftover part. When finding $v(x)$ from $v'(x)$, any antiderivative will work, and we usually pick the simplest one. In particular, no arbitrary constant of integration is needed in $v(x)$ because it would simply cancel out of the right-hand side of Equation (2).

EXAMPLE 2 Find $\int \ln x\, dx$.

Solution We have not yet seen how to find an antiderivative for $\ln x$. If we set $u(x) = \ln x$, then $u'(x)$ is the simpler function $1/x$. It may not appear that a second function $v'(x)$ is multiplying $u(x) = \ln x$, but we can choose $v'(x)$ to be the constant function $v'(x) = 1$. We use the integration by parts formula given in Equation (1), with

$$u(x) = \ln x \quad \text{and} \quad v'(x) = 1.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = \frac{1}{x} \quad \text{and} \quad v(x) = x.$$

Then

$$\int \underset{u(x)}{\ln x} \cdot \underset{v'(x)}{1}\, dx = \underset{u(x)}{(\ln x)}\, \underset{v(x)}{x} - \int \underset{v(x)}{x} \underset{u'(x)}{\frac{1}{x}}\, dx \qquad \text{Integration by parts formula}$$

$$= x \ln x - \int 1\, dx$$

$$= x \ln x - x + C \qquad \text{Simplify and integrate.}$$ ■

In the following examples we use the differential form to indicate the process of integration by parts. The computations are the same, with du and dv providing shorter expressions for $u'(x)\, dx$ and $v'(x)\, dx$.

Sometimes we have to use integration by parts more than once, as in the next example.

EXAMPLE 3 Evaluate

$$\int x^2 e^x\, dx.$$

Solution We use the integration by parts formula given in Equation (1), with

$$u(x) = x^2 \quad \text{and} \quad v'(x) = e^x.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 2x \quad \text{and} \quad v(x) = e^x.$$

We summarize this choice by setting $du = u'(x)\,dx$ and $dv = v'(x)\,dx$, so

$$du = 2x\,dx \quad \text{and} \quad dv = e^x\,dx.$$

We then have

$$\int \underset{u}{x^2}\, \underbrace{e^x\,dx}_{dv} = \underset{u}{x^2}\,\underset{v}{e^x} - \int \underset{v}{e^x}\, \underbrace{2x\,dx}_{du} = x^2 e^x - 2\int x e^x\,dx \qquad \text{Integration by parts formula}$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int \underset{u}{x}\, \underbrace{e^x\,dx}_{dv} = \underset{u}{x}\,\underset{v}{e^x} - \int \underset{v}{e^x}\, \underbrace{dx}_{du} = xe^x - e^x + C. \qquad \begin{array}{l}\text{Integration by parts Equation (2)}\\ u = x,\ \ dv = e^x\,dx\\ v = e^x,\ du = dx\end{array}$$

Using this last evaluation, we then obtain

$$\begin{aligned}\int x^2 e^x\,dx &= x^2 e^x - 2\int x e^x\,dx\\ &= x^2 e^x - 2xe^x + 2e^x + C,\end{aligned}$$

where the constant of integration is renamed after substituting for the integral on the right. ■

The technique of Example 3 works for any integral $\int x^n e^x\,dx$ in which n is a positive integer, because differentiating x^n will eventually lead to a constant, and repeatedly integrating e^x is easy.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 4 Evaluate

$$\int e^x \cos x\,dx.$$

Solution Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x\,dx, \quad v = -\cos x, \quad du = e^x\,dx.$$

Then

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\,dx)\right)\\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation, but with opposite signs. Adding the integral to both sides and adding the constant of integration gives

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration then gives

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x\, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\, dx,$$

so that

$$du = (n-1)(\cos^{n-2} x)(-\sin x\, dx) \qquad \text{and} \qquad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned}\int \cos^n x\, dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx.\end{aligned}$$

If we add

$$(n-1)\int \cos^n x\, dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned}\int \cos^3 x\, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\, dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.\end{aligned}$$

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both u' and v' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b u(x)\, v'(x)\, dx = u(x)\, v(x)\Big]_a^b - \int_a^b v(x)\, u'(x)\, dx \tag{3}$$

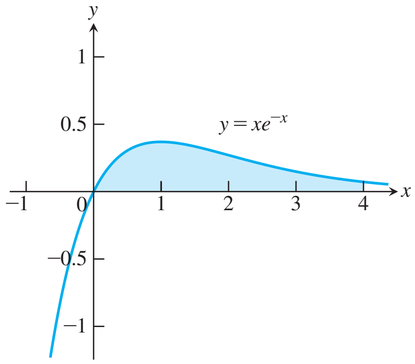


FIGURE 8.1 The region in Example 6.

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\,dx.$$

Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx && \text{Integration by parts Formula (3)}\\ &= [-4e^{-4} - (-0e^{-0})] + \int_0^4 e^{-x}\,dx\\ &= -4e^{-4} - e^{-x}\Big]_0^4\\ &= -4e^{-4} - (e^{-4} - e^{-0}) = 1 - 5e^{-4} \approx 0.91.\end{aligned}$$

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\,dx$
2. $\int \theta \cos \pi\theta\,d\theta$
3. $\int t^2 \cos t\,dt$
4. $\int x^2 \sin x\,dx$
5. $\int_1^2 x \ln x\,dx$
6. $\int_1^e x^3 \ln x\,dx$
7. $\int xe^x\,dx$
8. $\int xe^{3x}\,dx$
9. $\int x^2 e^{-x}\,dx$
10. $\int (x^2 - 2x + 1)e^{2x}\,dx$
11. $\int \tan^{-1} y\,dy$
12. $\int \arcsin y\,dy$
13. $\int x \sec^2 x\,dx$
14. $\int 4x \sec^2 2x\,dx$
15. $\int x^3 e^x\,dx$
16. $\int p^4 e^{-p}\,dp$
17. $\int (x^2 - 5x)e^x\,dx$
18. $\int (r^2 + r + 1)e^r\,dr$
19. $\int x^5 e^x\,dx$
20. $\int t^2 e^{4t}\,dt$
21. $\int e^\theta \sin \theta\,d\theta$
22. $\int e^{-y} \cos y\,dy$
23. $\int e^{2x} \cos 3x\,dx$
24. $\int e^{-2x} \sin 2x\,dx$

Using Substitution

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\,ds$
26. $\int_0^1 x\sqrt{1 - x}\,dx$
27. $\int_0^{\pi/3} x \tan^2 x\,dx$
28. $\int \ln(x + x^2)\,dx$
29. $\int \sin(\ln x)\,dx$
30. $\int z(\ln z)^2\,dz$

Evaluating Integrals

Evaluate the integrals in Exercises 31–56. Some integrals do not require integration by parts.

31. $\int x \sec x^2\,dx$
32. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$
33. $\int x(\ln x)^2\,dx$
34. $\int \frac{1}{x(\ln x)^2}\,dx$
35. $\int \frac{\ln x}{x^2}\,dx$
36. $\int \frac{(\ln x)^3}{x}\,dx$
37. $\int x^3 e^{x^4}\,dx$
38. $\int x^5 e^{x^3}\,dx$
39. $\int x^3\sqrt{x^2 + 1}\,dx$
40. $\int x^2 \sin x^3\,dx$
41. $\int \sin 3x \cos 2x\,dx$
42. $\int \sin 2x \cos 4x\,dx$
43. $\int \sqrt{x} \ln x\,dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
45. $\int \cos\sqrt{x}\,dx$
46. $\int \sqrt{x}\,e^{\sqrt{x}}\,dx$
47. $\int_0^{\pi/2} \theta^2 \sin 2\theta\,d\theta$
48. $\int_0^{\pi/2} x^3 \cos 2x\,dx$
49. $\int_{2/\sqrt{3}}^2 t \sec^{-1} t\,dt$
50. $\int_0^{1/\sqrt{2}} 2x \arcsin(x^2)\,dx$
51. $\int x \arctan x\,dx$
52. $\int x^2 \tan^{-1}\frac{x}{2}\,dx$
53. $\int (1 + 2x^2)e^{x^2}\,dx$
54. $\int \frac{xe^x}{(x + 1)^2}\,dx$
55. $\int \sqrt{x}(\arcsin\sqrt{x})\,dx$
56. $\int \frac{(\sin^{-1} x)^2}{\sqrt{1 - x^2}}\,dx$

Theory and Examples

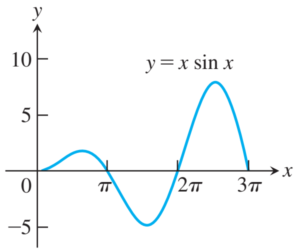
57. Finding area Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis (see the accompanying figure) for

a. $0 \le x \le \pi$.

b. $\pi \le x \le 2\pi$.

c. $2\pi \le x \le 3\pi$.

d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



58. Finding area Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the accompanying figure) for

a. $\pi/2 \le x \le 3\pi/2$.

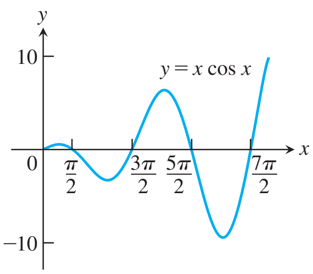
b. $3\pi/2 \le x \le 5\pi/2$.

c. $5\pi/2 \le x \le 7\pi/2$.

d. What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



59. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

60. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

a. about the y-axis.

b. about the line $x = 1$.

61. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about

a. the y-axis.

b. the line $x = \pi/2$.

62. Finding volume Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about

a. the y-axis.

b. the line $x = \pi$.

(See Exercise 57 for a graph.)

63. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.

a. Find the area of the region.

b. Find the volume of the solid formed by revolving this region about the x-axis.

c. Find the volume of the solid formed by revolving this region about the line $x = -2$.

d. Find the centroid of the region.

64. Consider the region bounded by the graphs of $y = \arctan x$, $y = 0$, and $x = 1$.

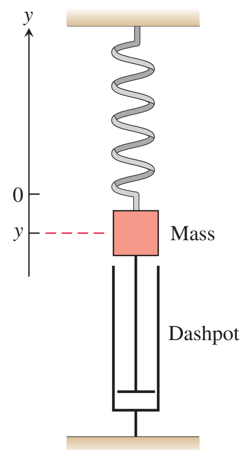
a. Find the area of the region.

b. Find the volume of the solid formed by revolving this region about the y-axis.

65. Average value A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t}\cos t, \quad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



66. Average value In a mass-spring-dashpot system like the one in Exercise 65, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 67–73, use integration by parts to establish the reduction formula.

67. $\displaystyle\int x^n \cos x\, dx = x^n \sin x - n\int x^{n-1} \sin x\, dx$

68. $\displaystyle\int x^n \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx$

69. $\int x^n e^{ax}\,dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$

70. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

71. $\int x^m(\ln x)^n\,dx$

$$= \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m(\ln x)^{n-1}\,dx,\ m \neq -1$$

72. $\int x^n\sqrt{x+1}\,dx$

$$= \frac{2x^n}{2n+3}(x+1)^{3/2} - \frac{2n}{2n+3}\int x^{n-1}\sqrt{x+1}\,dx$$

73. $\int \dfrac{x^n}{\sqrt{x+1}}\,dx$

$$= \frac{2x^n}{2n+1}\sqrt{x+1} - \frac{2n}{2n+1}\int \frac{x^{n-1}}{\sqrt{x+1}}\,dx$$

74. Use Example 5 to show that

$$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx$$

$$= \begin{cases} \left(\dfrac{\pi}{2}\right)\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, & n \text{ even} \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}, & n \text{ odd} \end{cases}$$

75. Show that

$$\int_a^b\left(\int_x^b f(t)\,dt\right)dx = \int_a^b (x-a)\,f(x)\,dx.$$

76. Use integration by parts to obtain the formula

$$\int\sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\,dx.$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned} \int f^{-1}(x)\,dx &= \int y f'(y)\,dy && y = f^{-1}(x),\quad x = f(y) \\ & && dx = f'(y)\,dy \\ &= y f(y) - \int f(y)\,dy && \text{Integration by parts with} \\ & && u = y,\ dv = f'(y)\,dy \\ &= x f^{-1}(x) - \int f(y)\,dy \end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned} \int \ln x\,dx &= \int y e^y\,dy && y = \ln x,\quad x = e^y \\ & && dx = e^y\,dy \\ &= ye^y - e^y + C \\ &= x\ln x - x + C. \end{aligned}$$

For the integral of $\arccos x$, we get

$$\begin{aligned} \int \arccos x\,dx &= x\arccos x - \int \cos y\,dy && y = \arccos x \\ &= x\arccos x - \sin y + C \\ &= x\arccos x - \sin(\arccos x) + C. \end{aligned}$$

Use the formula

$$\int f^{-1}(x)\,dx = x f^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \tag{4}$$

to evaluate the integrals in Exercises 77–80. Express your answers in terms of x.

77. $\int \operatorname{arcsec} x\,dx$

78. $\int \arctan x\,dx$

79. $\int \sec^{-1} x\,dx$

80. $\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\,dx = x f^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \tag{5}$$

Exercises 81 and 82 compare the results of using Equations (4) and (5).

81. Equations (4) and (5) give different formulas for the integral of $\arccos x$:

a. $\int \arccos x\,dx = x\arccos x - \sin(\arccos x) + C$ Eq. (4)

b. $\int \arccos x\,dx = x\arccos x - \sqrt{1-x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

82. Equations (4) and (5) lead to different formulas for the integral of $\arctan x$:

a. $\int \arctan x\,dx = x\arctan x - \ln\sec(\arctan x) + C$ Eq. (4)

b. $\int \arctan x\,dx = x\arctan x - \ln\sqrt{1+x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 83 and 84 with (**a**) Eq. (4) and (**b**) Eq. (5). In each case, check your work by differentiating your answer with respect to x.

83. $\int \sinh^{-1} x\,dx$

84. $\int \tanh^{-1} x\,dx$

8.3 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\, dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we must find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form

$$\int \sin^m x \cos^n x\, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If ***m* is odd** in $\int \sin^m x \cos^n x\, dx$, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we substitute $u = \cos x$ and $du = -\sin x\, dx$.

Case 2 If ***n* is odd** in $\int \sin^m x \cos^n x\, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then substitute $u = \sin x$ and $du = \cos x\, dx$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

EXAMPLE 1 Evaluate $\int \sin^3 x \cos^2 x\, dx$.

Solution This is an example of Case 1.

$$\begin{aligned}
\int \sin^3 x \cos^2 x\, dx &= \int \sin^2 x \cos^2 x \sin x\, dx && m \text{ is odd.} \\
&= \int (1 - \cos^2 x)(\cos^2 x) \sin x\, dx && \sin^2 x = 1 - \cos^2 x \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x,\ du = -\sin x\, dx \\
&= \int (u^4 - u^2)\, du && \text{Distribute.} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C
\end{aligned}$$