EE2026
Digital Design

NUMBER SYSTEMS

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Outline

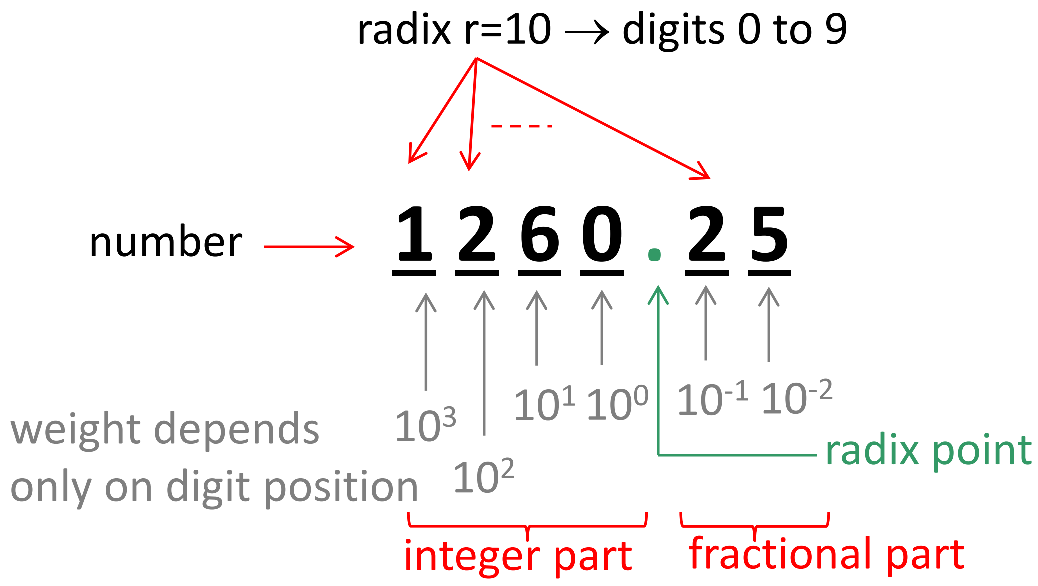
- Positional number systems
- Radix conversion
- Binary arithmetic
- Binary signed representation
- Binary-coded decimal (BCD)

Positional Number Systems

- Decimal number system: radix r=10

Terminology

- Radix (or base)
- Radix point
- Numerals made of digits (0 → r-1)
- Place value (or weight) is in the power of the base (positive on the left and negative on the right side of the radix point)

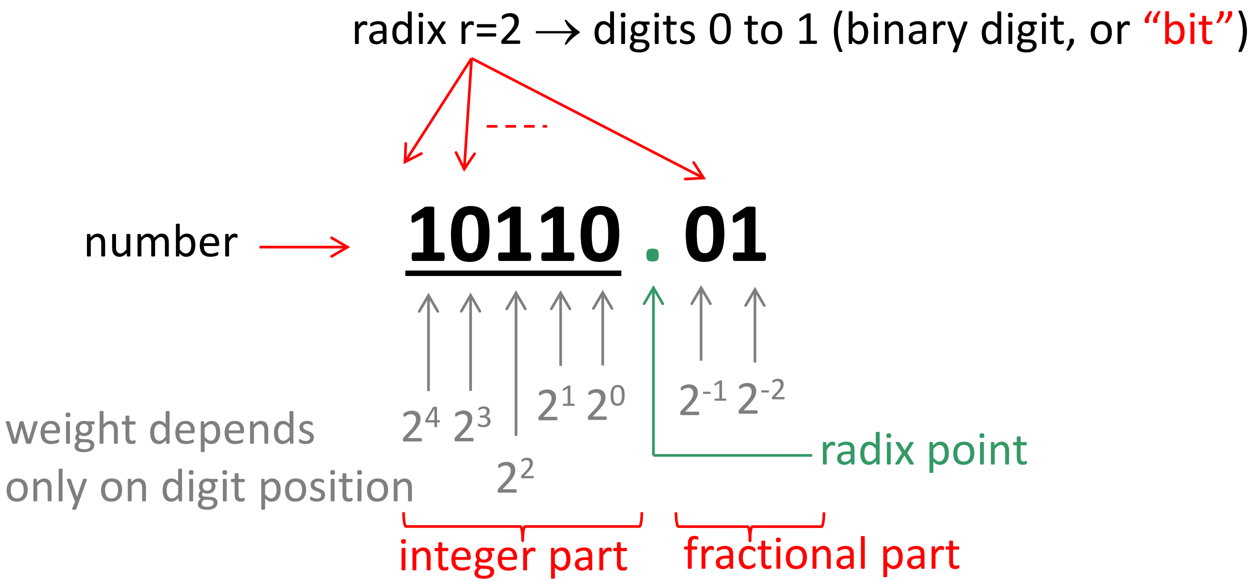


$$N = 1 \times 10^3 + 2 \times 10^2 + 6 \times 10^1 + 0 \times 10^0 + 2 \times 10^{-1} + 5 \times 10^{-2} = 1260.25$$

weighted sum of each digit (each digit is weighted by its place value)

Binary Number System

- Binary number system: radix r=2



- Decimal equivalent

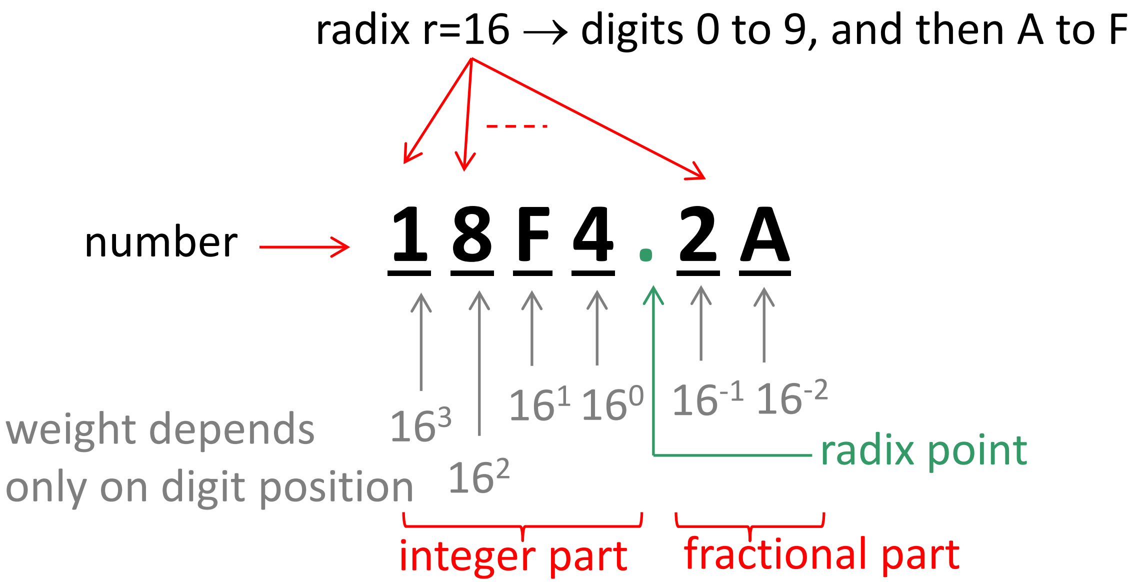
$$(N)_2 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}$$

$$= 16 + 0 + 4 + 2 + 0 + 0 + \frac{1}{4}$$

$$(N)_{10} = 22.25$$

Hexadecimal Number System

- Hexadecimal number system: radix r=16



hex	dec
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9
A	10
B	11
C	12
D	13
E	14
F	15

- Decimal equivalent

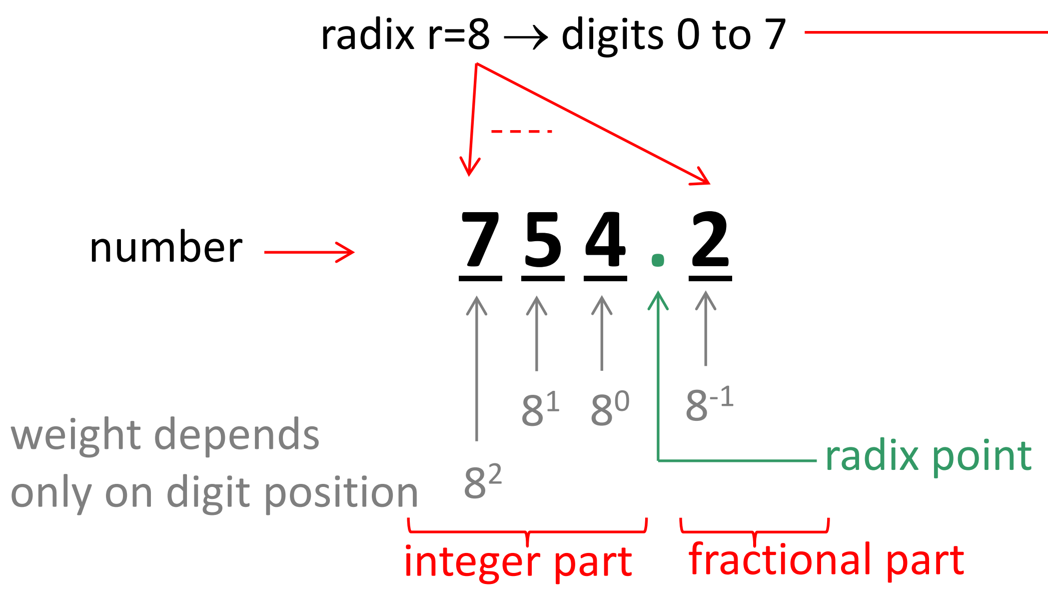
$$(N)_{16} = 1 \times 16^3 + 8 \times 16^2 + F \times 16^1 + 4 \times 16^0 + 2 \times 16^{-1} + 10 \times 16^{-2}$$

$$= 4096 + 2048 + 240 + 4 + \frac{2}{16} + \frac{10}{256}$$

$$(N)_{10} = 6388 + \frac{21}{128} \approx 6388.16$$

Octal Number System

- Octal number system: radix r=8



Oct	Dec
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
?	8
?	9
?	10

- Decimal equivalent

$$(N)_8 = 7 \times 8^2 + 5 \times 8^1 + 4 \times 8^0 + 2 \times 8^{-1}$$

$$(N)_{10} = 448 + 40 + 4 + \frac{2}{8} \approx 492.25$$

Generalization to Any Positional Number System

- Generic number system with radix r

radix r → digits 0 to r-1

$$(\boldsymbol{A})_{\boldsymbol{r}} = (\boldsymbol{a_n}\ \boldsymbol{a_{n-1}} \dots \boldsymbol{a_o}\,.\,\boldsymbol{a_{-1}} \dots \boldsymbol{a_{-m}})_{\boldsymbol{r}}$$

r^n r^{n-1} ... r^0 r^{-1} ... r^{-m}

radix point

- Decimal equivalent is weighted sum of all digits

$$(A)_r = (a_n a_{n-1} \dots a_o . a_{-1} \dots a_{-m})_r$$

$$(A)_{10} = a_n \times r^n + a_{n-1} \times r^{n-1} + \dots a_o \times r^0 + a_{-1} \times r^{-1} + \dots a_{-m} \times r^{-m}$$

$$= \sum_{i=-m}^{n} a_i r^i$$

Radix Conversion

- Three types of conversions
  - Radix r (r≠10) $\rightarrow$ decimal
  - Decimal $\rightarrow$ radix r (r≠10)
  - Conversion among binary, octal and hex numbers

- Radix r (r≠10) $\rightarrow$ decimal is trivial (decimal equivalent) $(A)_{10} = \sum_{i=-m}^{n} a_i r^i$
  - Binary $\rightarrow$ decimal

$$(10110.01)_2 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}$$
$$= 16 + 0 + 4 + 2 + 0 + 0 + \frac{1}{4} = (22.25)_{10}$$

  - Hex $\rightarrow$ decimal

$$(18F4.2A)_{16} = 1 \times 16^3 + 8 \times 16^2 + F \times 16^1 + 4 \times 16^0 + 2 \times 16^{-1} + 10 \times 16^{-2}$$
$$= 6388 + \frac{21}{128} \approx (6388.16)_{10}$$

Decimal → Radix r (r≠10): Integer Part

- Must find a systematic way
  - Decimal → binary $(102)_{10} = (A)_2 = (a_n a_{n-1} \ldots a_o . a_{-1} \ldots a_{-m})_r = ?$
  - Start simple: integer number

$$(102)_{10} = (A)_2 = a_n \times 2^n + a_{n-1} \times 2^{n-1} + \ldots + a_1 \times 2^1 + a_o$$

$$\underbrace{\underbrace{= (a_n \times 2^n + a_{n-1} \times 2^{n-1} + \ldots + a_1 \times 2^1)}_{\text{integer multiple of 2}} + a_o}_{\text{integer multiple of 2 if } a_o=0, \text{ (integer multiple of 2)+1 if } a_o=1}$$

integer multiple of 2 if a_o=0,
(integer multiple of 2)+1 if a_o=1 → **a_o is remainder of division by 2**

continue dividing quotient by 2

quotient $a_n \times 2^{n-1} + a_{n-1} \times 2^{n-2} + \ldots + a_1$

$$2 \overline{\left) a_n \times 2^n + a_{n-1} \times 2^{n-1} + \ldots + a_1 \times 2 + a_o \right.}$$
$$\underline{a_n \times 2^n + a_{n-1} \times 2^{n-1} + \ldots + a_1 \times 2}$$
$$a_o$$

remainder is a_0

$$a_n \times 2^{n-2} + a_{n-1} \times 2^{n-3} + \ldots + a_1$$
$$2 \overline{\left) a_n \times 2^{n-1} + a_{n-1} \times 2^{n-2} + \ldots + a_1 \right.}$$
$$\underline{a_n \times 2^{n-1} + a_{n-1} \times 2^{n-2} + \ldots}$$
$$a_1$$

remainder is a_1

- Numerical example of repeated division by 2: $(102)_{10}=(A)_2=(?)_2$

division by 2	quotient	remainder
102/2	51	$0 \rightarrow a_0$
51/2	25	$1 \rightarrow a_1$
25/2	12	$1 \rightarrow a_2$
12/2	6	$0 \rightarrow a_3$
6/2	3	$0 \rightarrow a_4$
3/2	1	$1 \rightarrow a_5$
1/2	0	$1 \rightarrow a_6$

stop divisions when quotient=0
(subsequent divisions are all zero)

$$(A)_2 = \dots 00001100110$$

Check (inverse calculation)

$$\begin{aligned}(A)_{10} &= a_6 \times 2^6 + a_5 \times 2^5 + a_4 \times 2^4 + a_3 \times 2^3 \\ &\quad + a_2 \times 2^2 + a_1 \times 2^1 + a_0 \times 2^0 \\ &= 1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 \\ &\quad + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \\ &= 64 + 32 + 0 + 0 + 4 + 2 + 0 \\ &= 102 \ \checkmark\end{aligned}$$

Decimal → Radix r (r≠10): Fractional Part

- Must find a systematic way for the fractional part too
  - Then, sum up with integer part for arbitrary numbers with integer and fractional part
  - Fractional number

$$(0.58)_{10} = (A)_2 = (0.a_{-1}a_{-2}\dots a_{-m+1}a_{-m})_2 =$$

$$= a_{-1}\times 2^{-1} + a_{-2}\times 2^{-2} + \dots + a_{-m+1}\times 2^{-m+1} + a_{-m}\times 2^{-m}$$

  - Interesting interpretation of a_{-1} after multiplying the number by 2

$$2\times(a_{-1}\times 2^{-1} + a_{-2}\times 2^{-2} + \dots + a_{-m+1}\times 2^{-m+1} + a_{-m}\times 2^{-m}) =$$

$$= \underbrace{a_{-1}}_{\text{integer part of } 2\times(A)_{10}} + \underbrace{a_{-2}\times 2^{-1} + \dots + a_{-m+1}\times 2^{-m+2} + a_{-m}\times 2^{-m+1}}_{\text{fractional part of } 2\times(A)_{10}}$$

a_{-1} can be simply found as the integer part of $2\times(A)_{10}$, then repeat for a_{-2}, a_{-3}...

- Numerical example of repeated multiplications by 2: $(0.58)_{10}=(A)_2=(?)_2$

multiply by 2	**product**	**integer part**
0.58x2	1.16	$1 \rightarrow a_{-1}$
0.16x2	0.32	$0 \rightarrow a_{-2}$
0.32x2	0.64	$0 \rightarrow a_{-3}$
0.64x2	1.28	$1 \rightarrow a_{-4}$
0.28x2	0.56	$0 \rightarrow a_{-5}$
0.56x2	1.12	$1 \rightarrow a_{-6}$
0.12x2	0.24	$0 \rightarrow a_{-7}$
0.24x2	0.48	$0 \rightarrow a_{-8}$

Check (inverse calculation)

$$
\begin{aligned}
(A)_{10} &= 1 \times 2^{-1} + 1 \times 2^{-4} + 1 \times 2^{-6} \\
&= \frac{1}{2} + \frac{1}{16} + \frac{1}{64} \\
&= 0.578125 \\
&\approx 0.58 \checkmark
\end{aligned}
$$

$$(A)_2 = 10010100\ldots \longleftarrow$$

- **conversion certainly stops when product=0**
- **may never end (non-terminating representation, even if $(A)_{10}$ may terminate)**
- **in this case, stop at the required precision (system specification, no. of bits)**

Summary Table for Conversion of 4-bit Integers

Numbers with Different Radixes

Decimal (radix 10)	Binary (radix 2)	Octal (radix 8)	Hexadecimal (radix 16)
00	0000	00	0
01	0001	01	1
02	0010	02	2
03	0011	03	3
04	0100	04	4
05	0101	05	5
06	0110	06	6
07	0111	07	7
08	1000	10	8
09	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Conversion among Hex, Octal and Binary

- Hex ↔ binary
  - Each Hex digit → 4 bits
  - Or each group of 4 bits → 1 Hex digit (starting from radix point and add zeroes if necessary to have all groups of 4 bits)

- Octal ↔ binary
  - Each octal digit (0…7) → 3 bits (indeed 0…7)
  - Or each group of 3 bits → 1 octal digit (starting from radix point)

- Hex ↔ Octal
  - Use binary as an intermediate step
  - Hex → binary → octal
  - Octal → binary → Hex

Numbers with Different Radixes

Decimal (radix 10)	Binary (radix 2)	Octal (radix 8)	Hexadecimal (radix 16)
00	0000	00	0
01	0001	01	1
02	0010	02	2
03	0011	03	3
04	0100	04	4
05	0101	05	5
06	0110	06	6
07	0111	07	7
08	1000	10	8
09	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Examples (Hex, Octal, Binary Conversion)

Hex → bin:

$(A45F)_{16}$

$(\underline{1010}\ \underline{0100}\ \underline{0101}\ \underline{1111})_2$

Bin → Hex:

$00\,(\underline{11}\ \underline{1010}\ \underline{1101}\ \underline{0111})_2$

$(3\ A\ D\ 7)_{16}$

complete with zeroes on the left in integer numbers (same number), on the right in fractional numbers

Oct → bin:

$(475)_8$

$(\underline{100}\ \underline{111}\ \underline{101})_2$

Bin → oct:

$0\,(\underline{10}\ \underline{111}\ \underline{101}\ \underline{110})_2$

$(2\ 7\ 5\ 6)_8$

Other Examples (Hex, Octal, Binary Conversion)

Hex → oct:

$(\mathbf{A45F})_{16}$

$(\underline{1010}\ \underline{0100}\ \underline{0101}\ \underline{1111})_2$

$\mathbf{00}(\underline{1}\ \underline{010}\ \underline{010}\ \underline{001}\ \underline{011}\ \underline{111})_2$

regroup in 3-bit groups, add zeroes

⇩

convert each 3-bit group

$(\mathbf{122137})_8$

Oct → Hex:

$(\mathbf{653})_8$

$(\underline{110}\ \underline{101}\ \underline{011})_2$

$\mathbf{000}(\underline{1}\ \underline{1010}\ \underline{1011})_2$

⇩

$(\mathbf{1AB})_{16}$

- For the fractional part: same, just group digits by starting from the position after the radix point (same number when adding zeroes on the right

Hex → bin:

$(\mathbf{0\ .\ A45F})_{16}$

⇩

$(0\ .\ \underline{1010}\ \underline{0100}\ \underline{0101}\ \underline{1111})_2\ \mathbf{00...}$

add zeroes to the right (same number, not needed in this example)

Radix Conversion: Summary and Generalization

- Radix r ($r \neq 10$) $\rightarrow$ decimal
  - Just use definition of positional number system (weighted sum of all digits)

- Decimal $\rightarrow$ Radix r ($r \neq 10$)
  - Integer part $\rightarrow$ repeated division by r and take the remainder
  - Fractional part $\rightarrow$ repeated multiplication by r and take the integer part
  - Add integer and fractional part

- Conversion among binary, octal and Hex numbers
  - Grouping: 1 Hex digit = 4 bits, 1 octal digit = 3 bits
  - Conversion with binary as intermediate step + regroup: Hex $\rightarrow$ oct carried out as Hex $\rightarrow$ Binary $\rightarrow$ Octal, and vice versa