NATIONAL UNIVERSITY OF SINGAPORE
DEPARTMENT OF STATISTICS & DATA SCIENCE

ST2334 PROBABILITY AND STATISTICS

SEMESTER I, AY 2025/2026

Tutorial 04: Suggested Solutions

Exam-Like Questions

1. The answer is 0.5714.

   The sum of the column $x = 2$ is $0.05 + 0.10 + 0.2 = 0.35$, which is $P(X = 2)$. Therefore

   $$P(Y = 3|X = 2) = \tfrac{P(Y=3,X=2)}{P(X=2)} = \frac{0.20}{0.35} = 4/7 = 0.5714.$$

2. The answer is 2.6.

   The sum of the row $y = 2$ is $0.05 + 0.10 + 0.35 = 0.5$, which is $P(Y = 2)$. Thus the condition probability function $f_{X|Y}(x|2)$ is given as the values in the row of $y = 2$ divided by 0.5, i.e.,

   $$f_{X|Y}(1|2) = 0.05/0.5 = 0.1; \quad f_{X|Y}(2|2) = 0.2; \quad f_{X|Y}(3|2) = 0.7.$$

   As a consequence,

   $$E(X|Y = 2) = 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.7 = 2.6.$$

3. The answer is (b).

   $f_X(1) = 0.52$; so $E(X) = 1 f_X(1) = 0.52$;

   $f_Y(1) = 0.4$; so $E(Y) = 1 f_Y(1) = 0.4$.

   $E(3X + 2Y) = 3E(X) + 2E(Y) = 1.56 + 0.8 = 2.36$.

4. The answer is 0.619.

   The marginal probability density function of X is given by

   $$f_X(x) = \int_0^1 (x + y)\,\mathrm{d}y = \left[xy + \frac{y^2}{2}\right]_{y=0}^{1} = x + \frac{1}{2}, \quad 0 \le x \le 1.$$

   To compute $E(Y|X = 0.2)$, we need to obtain $f_{Y|X}(y|0.2)$ first.

   It is clear that $f_{Y|X}(y|0.2) = 0$ for $y \notin [0, 1]$. For $0 \le y \le 1$,

   $$f_{Y|X}(y|x = 0.2) = \frac{f(0.2, y)}{f_X(0.2)} = \frac{0.2 + y}{0.2 + 0.5} = \frac{2 + 10y}{7}.$$

   Therefore

   $$E(Y|X = 0.2) = \int_0^1 y\left(\frac{2 + 10y}{7}\right)\mathrm{d}y = \tfrac{1}{7}\left[y^2 + \frac{10y^3}{3}\right]_0^1 = \tfrac{13}{21} \approx 0.619.$$

5. The answer is (d).

   (a) Since $f_{X,Y}(x,y)$ is a joint probability density function,

$$1 = \int_3^5 \int_3^5 k(x^2+y^2)\,\mathrm{dy}\,\mathrm{dx} = k\int_3^5 \left[yx^2 + \frac{y^3}{3}\right]_{y=3}^5 \mathrm{dx} = \tfrac{2}{3}k\int_3^5 (3x^2+49)\,\mathrm{dx}$$
$$= \tfrac{2}{3}k\left[x^3+49x\right]_{x=3}^5 = {}^{392k}/_{3}.$$

   This implies $k = {}^{3}/_{392}$.

   (b)

$$P(3 \le X \le 4, 4 \le Y \le 5) = \tfrac{3}{392}\int_3^4 \int_4^5 (x^2+y^2)\,\mathrm{dy}\,\mathrm{dx} = \tfrac{3}{392}\int_3^4 \left[yx^2+\frac{y^3}{3}\right]_{y=4}^5 \mathrm{dx}$$
$$= \tfrac{3}{392}\int_3^4 \left(x^2+\frac{61}{3}\right)\mathrm{dx} = \tfrac{1}{392}\left[x^3+61x\right]_3^4 = \tfrac{1}{392}(98) = {}^{1}/_{4}.$$

   (c) It is clear that for $x \notin [3,5]$, $f(x) = 0$.
   For $3 \le x \le 5$,

$$f_X(x) = \tfrac{3}{392}\int_3^5 (x^2+y^2)\,\mathrm{dy} = \tfrac{3}{392}\left[x^2y+\frac{y^3}{3}\right]_{y=3}^5 = \tfrac{3}{392}\left(2x^2+\tfrac{98}{3}\right) = \tfrac{1}{196}(3x^2+49).$$

   (d) Using the previous part,

$$P(3.5 < X < 4) = \tfrac{1}{196}\int_{3.5}^4 (3x^2+49)\,\mathrm{dx} = \tfrac{1}{196}\left[x^3+49x\right]_{3.5}^4 = 0.2328.$$

Long Form Questions

1. (a) We first compute

$$E(X) = \sum x f_X(x) = 2(0.01)+\ldots+6(0.04) = 4.11;$$
$$E(X^2) = \sum x^2 f_X(x) = 2^2(0.01)+\ldots+6^2(0.04) = 17.63.$$

   Using the alternative formula:

$$V(X) = E(X^2) - [E(X)]^2 = 17.63 - 4.11^2 = 0.7379.$$

   (b) $E(Z) = E(3X-2) = 3E(X)-2 = 10.33$; $\quad V(Z) = V(3X-2) = 3^2V(X) = 6.6411$.

   (c) The probability function Z is given by

x	2	3	4	5	6
$z = 3x-2$	4	7	10	13	16
$f_Z(z)$	0.01	0.25	0.40	0.30	0.04

   We have,

$$E(Z) = \sum z f_Z(z) = 4(0.01)+\ldots+16(0.04) = 10.33;$$
$$E(Z^2) = \sum z^2 f_Z(z) = 4^2(0.01)+\ldots+16^2(0.04) = 113.35.$$

   This gives $V(Z) = E(Z^2) - [E(Z)]^2 = 113.35 - 10.33^2 = 6.6411$.

2. (a)

$$P(0.6 < X < 1.2) = \int_{0.6}^{1.2} f(x)\,\mathrm{dx} = \int_{0.6}^{1} x\,\mathrm{dx} + \int_{1}^{1.2} (2-x)\,\mathrm{dx} = \left[\tfrac{x^2}{2}\right]_{0.6}^{1} + \left[2x - \tfrac{x^2}{2}\right]_{1}^{1.2} = 0.50.$$

(b) In this case,

$$\begin{aligned} E(X) &= \int_0^2 x f(x)\,\mathrm{dx} = \int_0^1 x^2\,\mathrm{dx} + \int_1^2 x(2-x)\,\mathrm{dx} \\ &= \left[x^3/3\right]_0^1 + \left[x^2 - x^3/3\right]_1^2 = 1/3 + 4 - 1 - 8/3 + 1/3 = 1, \end{aligned}$$

and

$$\begin{aligned} E(X^2) &= \int_0^2 x^2 f(x)\,\mathrm{dx} = \int_0^1 x^3 f(x)\,\mathrm{dx} + \int_1^2 x^2(2-x)\,\mathrm{dx} \\ &= \left[x^4/4\right]_0^1 + \left[2x^3/3 - x^4/4\right]_1^2 = 7/6. \end{aligned}$$

Thus,

$$V(X) = 7/6 - 1^2 = 1/6.$$

3. (a) X can only take values in $\{0,1,2,3\}$; and Y in $\{0,1,2\}$.
   - As only 4 pieces of fruit are selected, $x + y \leq 4$.
   - There are only 3 bananas, so at least 1 piece of the selected fruit must be an orange or an apple. Thus $x + y \geq 1$.

   Therefore we have

$$f(x,y) = \begin{cases} \dfrac{\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}}{\binom{8}{4}}, & x = 0,1,2,3,\ y = 0,1,2,\ 1 \leq x+y \leq 4; \\ 0, & \text{elsewhere.} \end{cases}$$

(b) $P(X=1, Y=1) = f(1,1) = \frac{\binom{3}{1}\binom{2}{1}\binom{3}{2}}{\binom{8}{4}} = 0.2571.$

(c) $P(X+Y \leq 2) = f(0,1) + f(0,2) + f(1,0) + f(1,1) + f(2,0) = 0.5.$

(d) Recall that the possible values of X are 0, 1, 2, 3. Since 4 pieces of fruit are selected, $(4-X)$ pieces of fruit must be selected from 5 pieces of apples and bananas. That is,

$$f_X(x) = \begin{cases} \dfrac{\binom{3}{x}\binom{5}{4-x}}{\binom{8}{4}}, & x = 0,1,2,3; \\ 0, & \text{otherwise.} \end{cases}$$

(e) For $x = 2$,

$$f_{Y|X}(y|2) = \begin{cases} \dfrac{\binom{2}{y}\binom{3}{4-2-y}}{\binom{5}{4-2}} = \dfrac{1}{10}\dbinom{2}{y}\dbinom{3}{2-y}, & y = 0,1,2; \\ 0, & \text{elsewhere.} \end{cases}$$

Finally, $P(Y=0|X=2) = \frac{1}{10}\binom{2}{0}\binom{3}{2} = 0.3.$

4. Note that for $x \notin [0,1]$, $f_X(x) = 0$, while for $x \in [0,1]$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,\mathrm{dy} = \int_1^2 \tfrac{12}{13} x(x+y)\,\mathrm{dy} \\ &= \tfrac{12}{13} x \left(x + \int_1^2 y\,\mathrm{dy} \right) = \tfrac{12}{13} x (x + 1.5)\,. \end{aligned}$$

For each $x \in [0,1]$, the conditional probability function $f_{Y|X}(y|x)$ for $y \in [1,2]$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{12}{13}x(x+y)}{\frac{12}{13}x(x+1.5)} = \frac{x+y}{x+1.5}.$$

We can compute

$$P(Y \leq 1.5|X=0.5) = \int_1^{1.5} \frac{0.5+y}{0.5+1.5}\,\mathrm{dy} = 0.4375.$$

Furthermore,

$$\begin{aligned} E(Y|X=0.5) &= \int_1^2 y\frac{0.5+y}{0.5+1.5}\,\mathrm{dy} \\ &= \tfrac{1}{2}\int_1^2 (0.5y+y^2)\,\mathrm{dy} \\ &= \tfrac{1}{2}\left(\tfrac{3}{4}+\tfrac{7}{3}\right) = 37/24. \end{aligned}$$