EE2026
Digital Design

BOOLEAN ALGEBRA

Massimo ALIOTO
Dept of Electrical and Computer Engineering
Email: massimo.alioto@nus.edu.sg

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Outline

- Definitions and postulates
- Theorems
- Boolean functions and truth table
- Boolean function simplification using algebra manipulation

Boolean Algebra

- Developed as formal algebraic system in 1854 by George Boole (English mathematician, philosopher, and logician)
  - Huntington formulated the postulates in1904 as formal definition
  - Boolean Algebra is the mathematical foundation for digital system design, including computers
  - First applied by C.E Shannon (MIT) to the practical problem of analyzing networks of relays (switches)
  - Useful for control and logic systems

- Boolean algebra is ultimately described by
  - a set of elements B={0,1} (binary symbols)
  - operator $\overline{\ldots}$ (NOT) on single operand
  - operators · (AND), + (OR) on multiple operands

binary constants or variables (as in common algebra)

· has precedence over +

Boolean Algebra

- Huntington postulates as foundation of Boolean algebra
  - General algebraic structure based on axioms (rules), leading to more practical set-theoretic definition (operations are a consequence of postulates)

1. Closure
   - $\forall x, y \in B, x + y \in B$ (outcome of operation is still in B, obviously)
   - $\forall x, y \in B, x \cdot y \in B$ (outcome of operation is still in B, obviously)

2. Neutral elements of $+$ and $\cdot$
   - There exists a 0 and 1 element in B, such that
     - $x + 0 = x$ ← 0 is neutral elements for +
     - $x \cdot 1 = x$ ← 1 is neutral elements for ·

3. Commutative Law
   - $x + y = y + x$
   - $x \cdot y = y \cdot x$

Boolean Algebra

4. Distributive Law

- $x \cdot (y + z) = x \cdot y + x \cdot z$ ($\cdot$ over $+$, easy to remember)
- $x + (y \cdot z) = (x + y) \cdot (x + z)$ ($+$ over $\cdot$ not intuitive, still true)

5. Complement

- $\forall x \in B$, there exists an element $\bar{x} \in B$ (complement of x) such that
  - $x + \bar{x} = 1$
  - $x \cdot \bar{x} = 0$

6. There exist at least two distinct elements in the set B (obvious, it was introduced for more general algebras with more than two symbols)

Operators in Boolean Algebra

- Derived from axioms:

truth table (simplest expression of a function)

OR: $A + B$

A	B	$A + B$
0	0	0
0	1	1
1	0	1
1	1	1

$$0 + 0 = 0$$
$$0 + 1 = 1$$
$$1 + 0 = 1$$
$$1 + 1 = 1$$

AND: $A \cdot B$

A	B	$A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1

$$0 \cdot 0 = 0$$
$$0 \cdot 1 = 0$$
$$1 \cdot 0 = 0$$
$$1 \cdot 1 = 1$$

NOT: $\overline{A}$

A	$\overline{A}$
0	1
1	0

$$\mathrm{A} = 0 \rightarrow \overline{A} = 1$$
$$\mathrm{A} = 1 \rightarrow \overline{A} = 0$$

priority: NOT has highest precedence, followed by AND and OR

$\rightarrow$ NOT(A · B + C) = NOT((A · B) + C)

Boolean vs. Elementary Algebra

Boolean algebra	Ordinary algebra
No associative law. But it can be derived from the other postulates	Associative law is included: $a + (b + c) = (a + b) + c$
Distributive law: $\mathrm{x} + (y \cdot z) = (x + y) \cdot (x + z)$ valid	Not valid
No additive or multiplicative inverses, therefore there are no subtraction and division operation	Subtraction and division operations exist
Complement operation available	No complement operation
Boolean algebra: Undefined set of elements; Switching algebra: a two-valued Boolean algebra, whose element set only has two elements, 0 and 1	Dealing with real numbers and constituting an infinite set of elements

Theorems of Boolean Algebra

#	Theorem		
1	$A + A = A$	$A \cdot A = A$	Tautology Law
2	$A + 1 = 1$	$A \cdot 0 = 0$	Union Law
3	$\overline{(\overline{A})} = A$		Involution Law
4	$A + (B + C)$ $= (A + B) + C$	$A \cdot (B \cdot C) = (A \cdot B) \cdot C$	Associative Law
5	$\overline{A + B} = \bar{A} \cdot \bar{B}$	$\overline{A \cdot B} = \bar{A} + \bar{B}$	De Morgan's Law
6	$A + A \cdot B = A$	$A \cdot (A + B) = A$	Absorption Law
7	$A + \bar{A} \cdot B = A + B$	$A \cdot (\bar{A} + B) = A \cdot B$	
8	$AB + A\bar{B} = A$	$(A + B)(A + \bar{B}) = A$	Logical adjacency
9	$AB + \bar{A}C + BC$ $= AB + \bar{A}C$	$(A + B)(\bar{A} + C)(B + C)$ $= (A + B)(\bar{A} + C)$	Consensus Law

duality (swap OR and AND, swap 0 and 1)

Analytical Expressions and Truth Tables

- Analytical expressions for Boolean functions use logical relationship between binary variables via operators
  - Evaluated by determining the binary value of the expression for all possible values of the variables
  - Examples:

$$F_1 = A + B \qquad F_3 = A + BC$$

$$F_2 = A \cdot B \qquad F_4 = \bar{A}\bar{B}C + AB\bar{C}$$

- Truth table lists all possible combinations of input values and corresponding output
  - Examples:
  - Repeat for F_2 and F_4

$\boldsymbol{F_1 = A + B}$

A	B	F_1
0	0	0
0	1	1
1	0	1
1	1	1

$\boldsymbol{F_3 = A + BC}$

A	B	C	F_3
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Using Truth Tables to Prove Theorems

- Exhaustively check validity for all possible input values
  - Example: prove De Morgan's laws

$$\overline{A+B} = \bar{A}\cdot\bar{B}$$

A	B	$\overline{\boldsymbol{A}+\boldsymbol{B}}$	$\bar{\boldsymbol{A}}\cdot\bar{\boldsymbol{B}}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

equality true for all input values
$\rightarrow$ proved

$$\overline{A\cdot B} = \bar{A}+\bar{B}$$

A	B	$\overline{\boldsymbol{A}\cdot\boldsymbol{B}}$	$\bar{\boldsymbol{A}}+\bar{\boldsymbol{B}}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

  - Example: prove other laws

$$A+\bar{A}\cdot B = A+B$$

A	B	$\boldsymbol{A}+\bar{\boldsymbol{A}}\cdot\boldsymbol{B}$	$\boldsymbol{A}+\boldsymbol{B}$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

$$A\cdot(A+B) = A$$

A	B	$\boldsymbol{A}\cdot(\boldsymbol{A}+\boldsymbol{B})$	$\boldsymbol{A}$
0	0	0	0
0	1	0	0
1	0	1	1
1	1	1	1

Special Forms of Boolean Expressions: SOP, POS

- A Boolean expression can be rewritten in many formally different (but equivalent) forms
  - Sum of products (SOP)
    - Logic sum of product terms
    - Example:

$$F_1(A,B,C) = \overbrace{\underbrace{\bar{A}\bar{B}C}_{\text{products}} + \underbrace{\bar{A}B\bar{C}}_{\text{products}} + \underbrace{A\bar{B}C}_{\text{products}}}^{\text{sum}}$$

  - Product of sums (POS)
    - Logic product of sum terms
    - Example:

$$F_2(A,B,C) = \overbrace{\underbrace{(A+B+C)}_{\text{sums}} \cdot \underbrace{(A+\bar{B}+\bar{C})}_{\text{sums}} \cdot \underbrace{(\bar{A}+B+C)}_{\text{sums}} \cdot \underbrace{(\bar{A}+\bar{B}+C)}_{\text{sums}} \cdot \underbrace{(\bar{A}+\bar{B}+\bar{C})}_{\text{sums}}}^{\text{product}}$$

Special Forms of Boolean Expressions: Canonical

- Definition of minterms and maxterms
  - Minterm: product term that contains **all** variables of the function
    - examples in Boolean Function $Z=f(A,B,C)$
    - $ABC,\ A\bar{B}C,\ \bar{A}\bar{B}C$ are minterms (contain all variables)
    - $AC,\ A\bar{B},\ \bar{A}$ are not minterms (some variable is missing)

"·" often omitted

  - Maxterm: product term that contains **all** variables of the function
    - examples in Boolean Function $Z=f(A,B,C)$
    - $A+B+C,\ A+\bar{B}+\bar{C},\ \bar{A}+B+C$ are maxterms (contain all variables)
    - $A+B,\ \bar{B}+\bar{C},\ B$ are not maxterms (some variable is missing)

- Canonical form: SOP = sum of only minterms
  (POS = product of only maxterms)

SOP **POS**

$F_1(A,B,C) = \bar{A}B\bar{C} + A\bar{B}\bar{C} + A\bar{B}C + AB\bar{C}$ ← canonical form → $F_3(A,B,C) = (A+B+C)(A+B+\bar{C})(A+\bar{B}+\bar{C})$

$F_2(A,B,C) = \bar{A}B\bar{C} + A\bar{B} + \bar{B}C + AB\bar{C}$ ← non canonical form → $F_4(A,B,C) = (A+B+C)(A+\bar{C})(A+\bar{B}+\bar{C})$

non minterms

non maxterm

Minterm Association with Input Value

- In n-variable function, there are 2^n different minterms
  - Example with n=3

A	B	C	minterm
0	0	0	$\boxed{\bar{A}\cdot\bar{B}\cdot\bar{C}}$
0	0	1	$\bar{A}\cdot\bar{B}\cdot C$
0	1	0	$\bar{A}\cdot B\cdot\bar{C}$
0	1	1	$\boxed{\bar{A}\cdot B\cdot C}$
1	0	0	$A\cdot\bar{B}\cdot\bar{C}$
1	0	1	$A\cdot\bar{B}\cdot C$
1	1	0	$A\cdot B\cdot\bar{C}$
1	1	1	$\boxed{A\cdot B\cdot C}$

input value making minterm = 1

  - Input value such that $A\cdot B\cdot C = 1 \leftarrow 111$

    $\bar{A}\cdot\bar{B}\cdot\bar{C} = 1 \leftarrow 000$

    $\bar{A}\cdot B\cdot C = 1 \leftarrow 011$
  - There exists **at least** one input value such that given minterm = 1

    (rule: 1 if not complemented, 0 if complemented)
  - Value of the above minterms is 0 under any other input value (mostly 0, 1 only for specific input)
  - There exists **one and only** one input value such that given minterm = 1

- Direct (one-to-one) relation between minterms or values making them 1

Maxterm Association with Input Value

- In n-variable function, there are 2^n different maxterms
  - Example with n=3

A	B	C	maxterm
0	0	0	$\boxed{A+B+C}$
0	0	1	$A+B+\bar{C}$
0	1	0	$A+\bar{B}+C$
0	1	1	$A+\bar{B}+\bar{C}$
1	0	0	$\boxed{\bar{A}+B+C}$
1	0	1	$\bar{A}+B+\bar{C}$
1	1	0	$\bar{A}+\bar{B}+C$
1	1	1	$\boxed{\bar{A}+\bar{B}+\bar{C}}$

input value making maxterm = 0

  - Input value such that $A+B+C$ = 0 ← 000
    $\bar{A}+\bar{B}+\bar{C}$ = 0 ← 111
    $\bar{A}+B+C$ = 0 ← 100
  - There exists **at least** one input value such that given maxterm = 0 ← duality
    (rule: 0 if not complemented, 1 if complemented)
  - Value of the above maxterms is 1 under any other input value (mostly 1, 0 only for specific input) ← duality
  - There exists **one and only** one input value such that given maxterm = 0

- Direct (one-to-one) relation between maxterms or values making them 0

Convert Truth Table → Canonical Form: SOP (CSOP)

- In n-variable function, there are 2^n different minterms and maxterms
  - Example with n=3: sum of 2 products (CSOP)

A	B	C	F	minterm	maxterm
0	0	0	0	$\bar{A}\cdot\bar{B}\cdot\bar{C}$	$A+B+C$
0	0	1	0	$\bar{A}\cdot\bar{B}\cdot C$	$A+B+\bar{C}$
0	1	0	**1**	$\boxed{\bar{A}\cdot B\cdot\bar{C}}$	$A+\bar{B}+C$
0	1	1	0	$\bar{A}\cdot B\cdot C$	$A+\bar{B}+\bar{C}$
1	0	0	**1**	$\boxed{A\cdot\bar{B}\cdot\bar{C}}$	$\bar{A}+B+C$
1	0	1	0	$A\cdot\bar{B}\cdot C$	$\bar{A}+B+\bar{C}$
1	1	0	0	$A\cdot B\cdot\bar{C}$	$\bar{A}+\bar{B}+C$
1	1	1	0	$A\cdot B\cdot C$	$\bar{A}+\bar{B}+\bar{C}$

- If F=1 only for input 010:
  $F=\bar{A}\cdot B\cdot\bar{C}$
- If F=1 only for inputs 010 and 100:
  $F=\bar{A}\cdot B\cdot\bar{C}+A\cdot\bar{B}\cdot\bar{C}$

  0 for input 100   0 for input 010

- Each minterm is 1 only for a specific input
  - no "interference" with other inputs → just add minterms independently

Convert Truth Table → Canonical Form: SOP (CSOP)

- In n-variable function, there are 2^n different minterms and maxterms
  - Example with n=3: sum of 4 products (CSOP)

A	B	C	F	minterm	maxterm
0	0	0	0	$\bar{A}\cdot\bar{B}\cdot\bar{C}$	$A+B+C$
0	0	1	0	$\bar{A}\cdot\bar{B}\cdot C$	$A+B+\bar{C}$
0	1	0	1	$\boxed{\bar{A}\cdot B\cdot\bar{C}}$	$A+\bar{B}+C$
0	1	1	0	$\bar{A}\cdot B\cdot C$	$A+\bar{B}+\bar{C}$
1	0	0	1	$\boxed{A\cdot\bar{B}\cdot\bar{C}}$	$\bar{A}+B+C$
1	0	1	1	$\boxed{A\cdot\bar{B}\cdot C}$	$\bar{A}+B+\bar{C}$
1	1	0	0	$A\cdot B\cdot\bar{C}$	$\bar{A}+\bar{B}+C$
1	1	1	1	$\boxed{A\cdot B\cdot C}$	$\bar{A}+\bar{B}+\bar{C}$

$$F=\bar{A}\cdot B\cdot\bar{C}+A\cdot\bar{B}\cdot\bar{C}+$$
$$+A\cdot\bar{B}\cdot C+A\cdot B\cdot C$$

- Truth table is expressed directly with canonical form by

1. Identifying minterms = 1 for the input values making F = 1
2. (Logically) add them up

Convert Truth Table → Canonical Form: POS (CPOS)

- Example with n=3: product of 3 sums (CPOS)

A	B	C	F	minterm	maxterm
0	0	0	0	$\bar{A}\cdot\bar{B}\cdot\bar{C}$	$\boxed{A+B+C}$
0	0	1	0	$\bar{A}\cdot\bar{B}\cdot C$	$\boxed{A+B+\bar{C}}$
0	1	0	1	$\bar{A}\cdot B\cdot\bar{C}$	$A+\bar{B}+C$
0	1	1	0	$\bar{A}\cdot B\cdot C$	$\boxed{A+\bar{B}+\bar{C}}$
1	0	0	1	$A\cdot\bar{B}\cdot\bar{C}$	$\bar{A}+B+C$
1	0	1	0	$A\cdot\bar{B}\cdot C$	$\boxed{\bar{A}+B+\bar{C}}$
1	1	0	0	$A\cdot B\cdot\bar{C}$	$\boxed{\bar{A}+\bar{B}+C}$
1	1	1	1	$A\cdot B\cdot C$	$\bar{A}+\bar{B}+\bar{C}$

$$F = (A+B+C)\cdot(A+B+\bar{C})\cdot (A+\bar{B}+\bar{C})\cdot(\bar{A}+B+\bar{C})\cdot(\bar{A}+\bar{B}+C)$$

each maxterm is 0 only for a specific input
(no "interference" with other inputs)
→ just multiply maxterms independently

- Truth table is expressed directly with canonical form by
  1. Identifying maxterms = 0 for the input values making F = 0
  2. (Logically) multiply them

Convert Canonical Form → Truth Table

- Just reverse the process: in SOP (POS), fill in the outputs = 1 (0) for the input values associated with the function minterms (maxterms)
  - Example: SOP

$$F_1(A,B,C) = \bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}C$$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

  - Example: POS

$$F_1(A,B,C) = (A+B+C) + (A+\bar{B}+\bar{C}) + (\bar{A}+B+C) + \\ +(\bar{A}+\bar{B}+C) + (\bar{A}+\bar{B}+\bar{C})$$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

- In this example, SOP and POS happen to express the very same function → SOP and POS just different forms

Convert SOP → POS and POS → SOP

- If starting from SOP, Boolean expression manipulations lead to complement of POS (not POS)
  - Example:

$$F_1(A,B,C) = \bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}C$$

complement needed → insert 2 to leave function unaltered

$$= \overline{\overline{\bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}C}}$$

De Morgan's laws

$$= \overline{\overline{\bar{A}\bar{B}C} \cdot \overline{\bar{A}B\bar{C}} \cdot \overline{A\bar{B}C}}$$

not POS!

$$= \overline{(A + B + \bar{C}) \cdot (A + \bar{B} + C) \cdot (\bar{A} + B + \bar{C})}$$

  - To get POS, the above complement needs to be compensated upfront
    - start from **complemented** SOP
    - manipulate via De Morgan's laws and derive true POS
    - same when starting from POS to derive SOP

$$\overline{F_1(A,B,C)} = \overline{\bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}C}$$

$$= \overline{\bar{A}\bar{B}C} \cdot \overline{\bar{A}B\bar{C}} \cdot \overline{A\bar{B}C}$$

$$= (A + B + \bar{C}) \cdot (A + \bar{B} + C) \cdot (\bar{A} + B + \bar{C})$$

Convert Non-Canonical → Canonical via Truth Table

- First method: non-canonical form → truth table → canonical form
  - Example:

F = 1 for any input such that $B\bar{C}$ = 1

$$F_1(A,B,C) = \bar{A} + B\bar{C}$$

F = 1 for any input such that $\bar{A}$ = 1

A	B	C	F
0	0	0	**1**
0	0	1	**1**
0	1	0	**1**
0	1	1	**1**
1	0	0	0
1	0	1	0
1	1	0	**1**
1	1	1	0

  - Resulting SOP canonical form (from entries with F = 1)

$$F_1(A,B,C) = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}B\bar{C} + \bar{A}BC + AB\bar{C}$$

  - Resulting POS canonical form (from entries with F = 0)

$$F_1(A,B,C) = (\bar{A} + B + C)(\bar{A} + B + \bar{C})(\bar{A} + \bar{B} + \bar{C})$$

Non-Canonical → Canonical via Boolean Algebra

- Use Boolean properties to manipulate the non-canonical expression to let missing literals appear for canonical form
  - Example:

SOP → CSOP

$$F(x,y,z) = \bar{x}y + xz$$
$$= \bar{x}y \cdot 1 + x \cdot 1 \cdot z$$
$$= \bar{x}y(z+\bar{z}) + x(y+\bar{y})z$$
$$= \bar{x}yz + \bar{x}y\bar{z} + xyz + x\bar{y}z$$

For missing literals, complete minterms through postulates: $A \cdot 1 = A \quad and \quad A + \bar{A} = 1$

SOP → CPOS

1) express SOP as POS
   a) complement twice
   b) apply De Morgan's law
   c) expand
   d) re-apply De Morgan's law

$$F(x,y,z) \overset{1)}{=} \bar{x}y + xz$$
$$= \overline{\overline{\bar{x}y + xz}} \quad \text{a}$$
$$= \overline{(x+\bar{y})(\bar{x}+\bar{z})} \quad \text{b}$$
$$= \overline{x\bar{x} + x\bar{z} + \bar{x}\bar{y} + \bar{y}\bar{z}} \quad \text{c}$$
$$= (x+y)(\bar{x}+z)(y+z) \quad \text{d}$$
$$\overset{2)}{=} (x+y+z)\cdot(x+y+\bar{z})\cdot(\bar{x}+y+z)\cdot(\bar{x}+\bar{y}+z)\cdot$$
$$\cdot(x+y+z)\cdot(\bar{x}+y+z)$$

2) for missing literals, complete maxterms through distribution postulate

$A + (\boldsymbol{B}\boldsymbol{C}) = (A+\boldsymbol{B})(A+\boldsymbol{C})$
(A = incomplete sum, $\boldsymbol{C}$=NOT($\boldsymbol{B}$)=missing literal)

$x+y = (x+y) + \boldsymbol{z}\bar{\boldsymbol{z}}$
$= (x+y+\boldsymbol{z})(x+y+\bar{\boldsymbol{z}})$

Summary

- Postulates and theorems of Boolean algebra
- Boolean operators: AND, OR and NOT
- Boolean functions and truth tables
- Boolean function expressed in SOP and POS form
- Obtain SOP or POS from truth table
- Minterm, maxterm and canonical form (CSOP, CPOS)
- Convert truth table $\rightarrow$ CSOP and CPOS
- Convert CSOP and CPOS $\rightarrow$ truth table
- Convert CSOP $\rightarrow$ CPOS and vice versa
- Non-canonical $\rightarrow$ canonical form (via truth table, Boolean algebra)