SECURITY (COMP0141): MATH MEETS CRYPTOGRAPHY

MATH MEETS CRYPTOGRAPHY

Two interesting settings to consider from a cryptographic perspective:

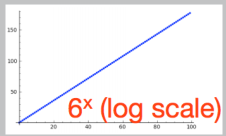
The finite field F_p for a very large prime p (1024 bits or more)

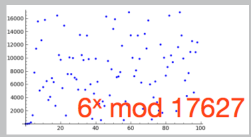
The ring $(Z/NZ)^*$ for N = pq for very large primes p,q (1024 bits or more)

DISCRETE LOGARITHM

Discrete logarithm problem: for a fixed prime p, given g and y, find x such that $g^x = y \bmod p$

Example: $6^x = 10000 \bmod 17627$





This problem seems to be very difficult to solve (like for modern computers and large enough p, until the heat death of the sun)

Exponentiation behaves very regularly over the integers but very irregularly (almost randomly) when taken modulo a prime

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THE RING (Z/NZ)*

Euler totient function φ is $\varphi(N) = |\{x \text{ in } \{0,\ldots,N-1\} \mid \gcd(x,N) = 1\}|$

Euler's theorem: $x^{\varphi(N)} = 1 \bmod N$ for $x \in (Z/NZ)^*$

Now let $N = pq$ for p and q two different odd primes

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RSA

RSA problem: given an integer N = pq, find p and q

Example: the RSA-1024 challenge is to find p and q for N =
135066410865995223349603216278805969938881475605667027524485143851526510604859533833940287150571909441798207282164471551373680419703964191743046496589274256239341020864383202110372958725762358509643110564073501508187510676594629205563685529475213500852879416377328533906109750544334999811150056977236890927563

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We saw before that any number can be uniquely factored into prime powers, but finding these factors for large numbers is considered to be very hard

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$$\begin{aligned}\varphi(N) = \varphi(pq) &= pq - \left|\{x : \gcd(x,pq) \neq 1\}\right| \\ &= pq - \left|\{x : p \mid x\}\right| - \left|\{x : q \mid x\}\right| + \left|\{0\}\right| \\ &= pq - q - p + 1 = (p-1)(q-1)\end{aligned}$$

This means that $(Z/NZ)^*$ has $\varphi(N) = (p-1)(q-1)$ elements

ONE-WAY FUNCTIONS

More generally, discrete log and RSA are examples of something called a **one-way function**

This is a function f( ) such that

- (1) it is easy to compute f(x) for all x, but
- (2) it is assumed to be very difficult to compute x given f(x), or in fact to compute any y such that f(y) = f(x)

Discrete log: $f(x) = g^x \bmod p$

RSA: $f(p,q) = pq$

Getting into some cryptography already, both discrete log and RSA exemplify a cryptographic primitive called a one-way function, which we're going to rely on a lot to provide both confidentiality and integrity