SECURITY (COMP0141): MATHEMATICAL BACKGROUND

BASIC MATHEMATICAL NOTATION

Sets: $A = \{1, 2\}$, $B = \{a, b, c\}$

Cardinality: $|A| = 2$, $|B| = 3$

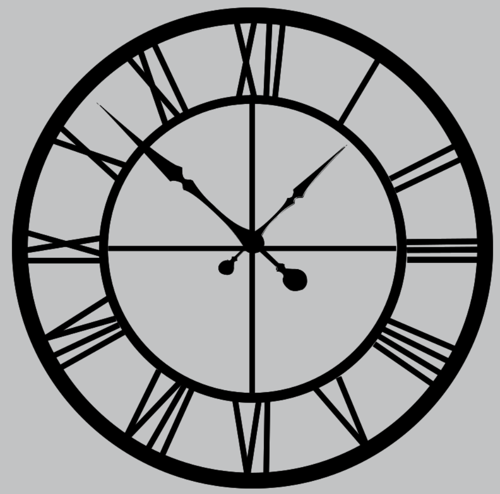
Set inclusion: $x \in A$ means x belongs to A ($1 \in A$, $3 \notin A$)

Integers: whole numbers (positive or negative)

Division: $x \mid y$ means x (evenly) divides y ($y = ax$ for some integer a)

Set conditions: $\{x : 2 \mid x\}$ means all x that are even

MODULAR (“CLOCK”) ARITHMETIC



6 = 6 mod 12

12 = 0 mod 12

14 = 2 mod 12

We technically should use ≡ to denote equivalence but will use = instead

True or false?

14 = 2 mod 12

37 = 26 mod 7

5 = -10 mod 3

MODULAR ARITHMETIC

Given integers $x > 0$, y, z we write $z \equiv y \bmod x$ when $x \mid (z-y)$

More examples: $5 = 1 \bmod 2$, $18 = 3 \bmod 5$

Given $x > 0$ and z, we can find unique a and $y \in \{0,1,\ldots,x-1\}$ such that

$$z = ax + y$$

remainder

quotient

More examples: $5 = 2*2 + 1$, $18 = 3*5 + 3$

If $z = ax + y$ then $z = y \bmod x$ (because $z - y = ax$ and $x \mid ax$)

GREATEST COMMON DIVISOR

Greatest common divisor of a and b is the largest number that divides both a and b; i.e., $\gcd(a, b) = \max\{d : d \mid a \text{ and } d \mid b\}$

Bézout's identity: for all a,b there are r,s such that gcd(a,b) = ra + sb

PROOF OF BEZOUT'S IDENTITY

Bézout's identity: for all a,b there are r,s such that gcd(a,b) = ra + sb

Proof:

(1) Define $d = \min\{ra + sb \mid ra + sb > 0\}$

(2) Want to show that gcd(a,b) = d, which we can do by showing that (a) $\gcd(a,b) \leq d$ and that (b) $d \leq \gcd(a,b)$

(2a) Write d = ra + sb. Since $\gcd(a,b) \mid a$ and $\gcd(a,b) \mid b$ we have $\gcd(a,b) \mid ra + sb$ so $\gcd(a,b) \mid d$

(2b) If $d \mid a$ and $d \mid b$ then $d \leq \gcd(a,b)$, so we show this instead. Assume $a = kd + t$ for $0 < t < d$ (meaning $d \nmid a$). Then $t = a - kd = a - k(ra + sb) = (1 - kr)a - ksb$. But this implies that $0 < t = Ra + Sb < d$ (for R = 1-kr and S = ks) which contradicts the fact that d is minimal

EUCLIDEAN ALGORITHM

Euclidean algorithm: used to efficiently compute gcd(a,b) for a > b

-start by finding r_0 such that $a = q_0 b + r_0$ (so $r_0 = b \bmod a$)

-then r_1 such that $b = q_1 r_0 + r_1$ (so $r_1 = b \bmod r_0$)

-eventually, get to $r_{n-2} = q_n r_{n-1} + 0$

-this final non-zero remainder r_{n-1} is the gcd of a and b

EUCLIDEAN ALGORITHM

Euclidean algorithm: used to efficiently compute gcd(a,b) for a > b

Example: find gcd(270,192)

- 270 = 1*192 + 78
- 192 = 2*78 + 36
- 78 = 2*36 + 6
- 36 = 6*6 + 0
- so gcd(270,192) = 6

EUCLIDEAN ALGORITHM

Why does the algorithm work?

-There are finitely many numbers between a and 0 so it terminates

-Bottom up (r_{n-1} divides a and b):

$r_{n-2} = q_n r_{n-1} + 0 \Rightarrow r_{n-1} = 0 \bmod r_{n-2}$

$r_{n-3} = q_{n-1} r_{n-2} + r_{n-1} \Rightarrow r_{n-1} = 0 \bmod r_{n-3}$

… $\Rightarrow r_{n-1} = 0 \bmod a$ and $0 \bmod b$ (so r_{n-1} is a common divisor)

-Top down (any other divisor divides r_{n-1}):

$a = q_0 b + r_0 \Rightarrow c$ dividing a and b divides r_0

$b = q_1 r_0 + r_1 \Rightarrow c$ divides r_1

… $\Rightarrow c$ divides r_{n-1} (so r_{n-1} is the greatest common divisor)

EXTENDED EUCLIDEAN ALGORITHM

Extended Euclidean algorithm: used to calculate r,s such that gcd(a,b) = ra + sb

In the Euclidean algorithm, focused just on remainders and ignored quotients: equations of the form $r_{i-2} = q_{i-1}r_{i-1} + r_i$, or $r_i = r_{i-2} - q_{i-1}r_{i-1}$

In the extended algorithm, add in two extra variables with the same quotient: $s_i = s_{i-2} - q_{i-1}s_{i-1}$ and $t_i = t_{i-2} - q_{i-1}t_{i-1}$, start with $s_0 = 1$, $s_1 = 0$, $t_0 = 0$, and $t_1 = 1$ (because a = 1*a + 0*b and b = 0*a + 1*b)

EXTENDED EUCLIDEAN ALGORITHM

Example: find s and t such that gcd(18,13) = s*18 + t*13

18 mod 13

-(r) 5 = 18 - 1*13, (s) 1 - 0 = 1, (t) 0 - 1 = -1

13 mod 5

-(r) 3 = 13 - 2*5, (s) 0 - 2*1 = -2, (t) 1 - 2*(-1) = 3

5 mod 3

-(r) 2 = 5 - 1*3, (s) 1 - 1*(-2) = 3, (t) -1 - 1*(3) = -4

3 mod 2

-(r) 1 = 3 - 1*2, (s) -2 - 1*(3) = -5, (t) 3 - 1*(-4) = 7

2 mod 1

-(r) 0 = 2 - 2*1

So 1 = -5*18 + 7*13

MODULAR ARITHMETIC

Many of the usual laws of the integers also apply when computing modulo N

Associativity (both for + and *):

-(a + b mod N) + c mod N

= a + (b + c mod N) mod N

= a + b + c mod N

-(ab mod N)c mod N

= a(bc mod N) mod N

= abc mod N

MODULAR EXPONENTIATION

We often use modular exponentiations g^x mod p ($\underbrace{g*g*\ldots*g}_{\text{x times}}$ mod p)

Again, usual exponentiation rules apply

- $g^x g^y = g^{x+y}$ mod p
- $(g^x)^y = g^{xy}$ mod p

COMMUTATIVE RING Z/NZ

Associative: $(a + b) + c = a + (b + c) \bmod N$

$(ab)c = a(bc) \bmod N$

Distributive: $a(b+c) = ab + ac \bmod N$

$(a+b)c = ac + bc \bmod N$

Commutative: $a + b = b + a \bmod N$

$ab = ba \bmod N$

Additive identity: $a + 0 = 0 + a = a \bmod N$

Multiplicative identity: $1a = a1 = a \bmod N$

Additive inverse: $a + (-a) + (-a) + a = 0 \bmod N$

Multiplicative inverse?

MULTIPLICATIVE INVERSES

In general, some numbers have inverses but not all

Example: 3 mod 10

-3 * 7 = 21 = 1 + 20 = 1 + 2*10 = 1 mod 10

-so 3 = 7^{-1} mod 10

Example: 2 mod 10

-there is no number b such that 2b = 1 mod 10

We define (Z/NZ)* to be all invertible elements in Z/NZ

When does a number have an inverse?

MULTIPLICATIVE INVERSES

Element a has multiplicative inverse mod N if and only if gcd(a,N) = 1

Proof:

-if gcd(a,N) = 1 we can write ra + sN = 1 (by Bézout's identity) so ra = 1 mod N, meaning r = a^{-1}

-if a has an inverse r, then ra = 1 mod N, which means ra + sN = 1 for some s, which means gcd(a,N) = 1

The inverse a^{-1} is unique modulo N and can be efficiently computed using the Extended Euclidean algorithm

EXERCISE

What is Z/12Z?

What is (Z/12Z)*?

For each element in (Z/12Z)*, find its inverse.

ANSWERS

What is Z/12Z? {0,1,2,3,…,11}

What is (Z/12Z)*? {1,5,7,11}

For each element in (Z/12Z)*, find its inverse.

- $1^{-1} = 1$
- $5^{-1} = 5$ (5*5 = 25 = 2*12 + 1 = 1 mod 12)
- $7^{-1} = 7$ (7*7 = 49 = 4*12 + 1 = 1 mod 12)
- $11^{-1} = 11$ (11*11 = 121 = 10*12 + 1 = 1 mod 12)

MULTIPLICATIVE INVERSES

Element a has multiplicative inverse mod N if and only if gcd(a,N) = 1

When could we guarantee this to be the case?

PRIME NUMBERS

A natural number N is prime if its only divisors are 1 and N

Examples: 2, 3, 5, 7, 11, 13, 17, 19, …

If p is a prime and $p \mid ab$ then $p \mid a$ or $p \mid b$

Any natural number N has unique factorisation $N = p_1^{r1} p_2^{r2} \ldots p_s^{rs}$

This means $\gcd(a, p) \in \{1, p\}$ when p is prime

Corollary of Bézout's identity: if p is prime then for any $a \in \{1,\ldots,p-1\}$ there is an inverse r such that $1 = ra \bmod p$

FINITE FIELDS

What is $(Z/pZ)^*$ when p is a prime?

The elements that have $\gcd(x,p) = 1$, but for a prime this is all x not divisible by p, so $F_p^* = \{1,2,3,\ldots,p-1\}$

A field is a commutative ring where all non-zero elements are (multiplicatively) invertible

F_p is a field with p elements (or **order** p) so is called a finite field

PRIME-ORDER FINITE FIELDS

Associative: $(a + b) + c = a + (b + c) \bmod N$

$(ab)c = a(bc) \bmod N$

Distributive: $a(b+c) = ab + ac \bmod N$

$(a+b)c = ac + bc \bmod N$

Commutative: $a + b = b + a \bmod N$

$ab = ba \bmod N$

Additive identity: $a + 0 = 0 + a = a \bmod N$

Multiplicative identity: $1a = a1 = a \bmod N$

Additive inverse: $a + (-a) + (-a) + a = 0 \bmod N$

Multiplicative inverse: $a * a^{-1} = 1 \bmod N$