MA4266 Notes

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Contents

1 Fundamental Groups

1.1 Homotopy and Path Homotopy

Let $I = [0, 1]$ be the interval.

Definition 1.1 (Homotopy) *If f and f' are continuous maps of the topological space X into the topological space Y, we say that f is **homotopic** to f' if there is a continuous map $F : X \times I \to Y$ such that*

$$F(x, 0) = f(x) \text{ and } F(x, 1) = f'(x)$$

*for each x. The map F is called a **homotopy** between f and f' and we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is **nulhomotopic**.*

Example:
Let A be any convex subspace of $\mathbb{R}^n$. f and g are two continuous maps from X to A. Then f and g are homotopic, the map

$$F(x, t) = (1 - t)f(x) + tg(x)$$

is a homotopy between them.

Recall a path in X from x_0 to x_1 is a continuous map $f : [0, 1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. We say that x_0 is the initial point, and x_1 the final point of the path f.

Lemma 1.2 *Suppose $f : [0, 1] \to X$ is a path, then it is homotopic to some constant map.*

Remark 1.2.1 *In fact, the constant map can be chosen to be the map that maps $[0, 1]$ to x, where x is some point in $f([0, 1])$.*

Proof: Consider the map $F : I \times I \to X$ given by

$$F(s, t) = f((1 - t)s)$$

Then F is continuous and $F(s, 0) = f(s)$, $F(s, 1) = f(0)$, so F is a desired homotopy. □

Lemma 1.3 *If X is path connected. Then for any path f and g on X, f is homotopic to g.*

Proof: Let σ_i be a path between $f(i)$ to $g(i)$. Then the map

$$F(s, t) = \sigma_s(t) \tag{1.1}$$

is a homotopy between f and g. □

Lemma 1.4 *If $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.*

Proof: Let F be a homotopy between h and h'; G be a homotopy between k and k'. Then one can check that the map

$$H(x,t) = \begin{cases} k(F(x,2t)) & 0 \leq t \leq \frac{1}{2} \\ G(h'(x), 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}. \tag{1.2}$$

is a homotopy between $k \circ h$ and $k' \circ h'$. □

Definition 1.5 (contractible) *A space X is said to be* ***contractible*** *if the identity map $i_X : X \to X$ is nulhomotopic.*

Lemma 1.6 *Every contractible space X is path connected.*

Proof: Let $c_y : X \to X$, $x \mapsto y$ be a constant map such that i_X is homotopic to, and F be the homotopy between i_X and c_y. We show that X is path connected by showing that every $x \in X$, there is a path between x and y. Since F is a homotopy between i_x and c_y, then $F(x,0) = x$, $F(x,1) = y$. Define $g : [0,1] \to X$ by

$$g(s) = F(x,s).$$

Note $g(0) = x$ and $g(1) = y$, and g is the composition of F and $s \mapsto (x,s)$, which is continuous. Hence g is a path.
□

Definition 1.7 (Path Homotopy) *Two paths f and f', mapping the interval $I = [0,1]$ into X, are said to be* ***path homotopic*** *if they have the same initial point x_0 and the same final point x_1, and if there is a continuous map $F : I \times I \to X$ such that*

$$F(s,0) = f(s) \text{ and } F(s,1) = f'(s)$$
$$F(0,t) = x_0 \text{ and } F(1,t) = x_1,$$

for each $s \in I$ and each $t \in I$. We call F a ***path homotopy*** *between f and f' and write $f \simeq_p f'$.*

Lemma 1.8 *The relations $\simeq$ and $\simeq_p$ are equivalence relations.*

Remark 1.8.1 *If f is a path, we shall denote its path homotopy equivalence class by $[f]$.*

Proof: Reflexive: given f, it is clear that $f \simeq f$. Note $F(x,t) = f(x)$ is the required homotopy. If f is a path, then F is a path homotopy.
Symmetric: if $f \simeq f'$, then let F be a homotopy between f and f'. Define $G(x,t) = F(x, 1-t)$, it is clear that G is a homotopy between f' and f. If f is a path homotopy, so is G.
Transitive: suppose that $f \simeq f'$ and $f' \simeq f''$. We show $f \simeq f''$. Let F be a homotopy between f and f', and let F' be a homotopy between f' and f''. Define $G : X \times I \to Y$ by

$$G(x,t) = \begin{cases} F(x,2t), & \text{for } x \in \left[0, \frac{1}{2}\right], \\ F'(x, 2t-1), & \text{for } x \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

We can check that G is well-defined, continuous using the Pasting Lemma, and is indeed a homotopy between f and f''. Lastly, if F and F' are path homotopies, then so is G. $\square$

Notation: given space X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y.

Proposition 1.9

1. *If Y is contractible, then for any X, the set $[X, Y]$ has a single element.*
2. *If X is contractible, and Y is path connected, then $[X, Y]$ has a single element.*

Proof:

1. Let $f : X \to Y$ be arbitrary, and suppose i_Y is nulhomotopic to the map $c_y : Y \to Y$ given by $q \mapsto y$, where y is a fixed element in Y. We show f is homotopic to the constant map $c : X \to Y$, $p \mapsto y$.

   Let F be the homotopy between i_Y and c_y. Then $F(q, 0) = q$ and $F(q, 1) = y$. Then consider the map $G : X \times I \to Y$ given by
   $$G(p, t) = F(f(p), t).$$
   Note G is continuous, and $G(p, 0) = f(p)$, $G(p, 1) = y$. Thus $G(p, t)$ is a homotopy between f and the constant map c.

2. First show any map from X to Y is nulhomotopic, next showing the constant maps are homotopic.

$\square$

1.2 Fundamental Group

Definition 1.10 (Product) *If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2. We define the **product** $f * g$ of f and g to be the path h given by the equations:*

$$h(s) = \begin{cases} f(2s), & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(2s-1), & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Remark 1.10.1 *The function h is well-defined and continuous, by the pasting lemma; it is a path in X from x_0 to x_2. We think of h as the path whose first half is the path f and whose second half is the path g.*

Proposition 1.11 *The product operation on paths induces a well-defined operation on path-homotopy class (which still denoted by $*$), defined by the equation*
$$[f] * [g] = [f * g].$$

Proof: We show the operation is well-defined. Let $f \simeq_p f'$ and $g \simeq_p g'$, we show $f * g \simeq_p f' * g'$.
Let F be a path homotopy between f and f', G be the path homotopy between g and g'. Then the map H, defined by

$$H(x,t) = \begin{cases} F(2x,t), & 0 \le x \le \frac{1}{2},\, t \in [0,1] \\ G(2x-1,t), & \frac{1}{2} \le x \le 1,\, t \in [0,1] \end{cases}.$$

Then one can check that H is a path homotopy between $f * g$ and $f' * g'$. □

Definition 1.12 (Constant Path) *Given $x \in X$, let e_x denote* ***the constant path at*** *x, i.e., the path $e_x : I \to X$ carrying all of I to the point x. Then $[e_x]$ is the path-homotopy equivalence class of e_x.*

Definition 1.13 (Reverse) *Given the path f in X from x_0 to x_1, let $\bar{f}$ be the path defined by*

$$\bar{f}(s) = f(1-s).$$

it is called the ***reverse*** *of f.*

Proposition 1.14 *The reverse operation on paths induces a well-defined operation (inverse) on path-homotopy classes, defined by the equation*

$$[f]^{-1} = [\bar{f}].$$

Proof: Let $f \simeq_p g$, we show that $\bar{f} \simeq_p \bar{g}$. Let F be a path homotopy between f and g, then the map G defined by

$$G(x,t) = F(1-x,t)$$

is a path homotopy between f' and g'. □

Definition 1.15 (Fundamental Group) *Let X be a topological space, let x_0 be a point of X, i.e., (X, x_0) is a pointed topological space. A path in X that begins and ends at x_0 is called a* ***loop*** *based at x_0. The set of path homotopy class of loops based at x_0 with the operation $*$, is called the* ***fundamental group*** *of X relative to the base point x_0, which is denoted by $\pi_1(X, x_0)$.*

Remark 1.15.1 *The elements of $\pi_1(X, x_0)$ are the path-homotopy equivalence classes. Group operation is defined by $[f] * [g] = [f * g]$, the identity element is $[e_{x_0}]$ and the inverse of $[f]$ is $[\bar{f}]$.*

Remark 1.15.2 *Let R^n denote Euclidean n-space. Then $\pi_1(\mathbb{R}^n, x_0)$ is the trivial group. More, generally, if X is any convex subset of $\mathbb{R}^n$, then $\pi_1(X, x_0)$ is the trivial group. In particular, the unit ball $B^n = \{x \mid x_1^2 + \cdots + x_n^2 \le 1\}$ has trivial fundamental group. This is because, for if f is a loop based at x_0, then the straight-line homotopy is a path homotopy between f and the constant path at x_0.*

Remark 1.15.3 *$\pi_1(X, x_0)$ depends on only the path component of X containing x_0. For this reason, it is usual to deal with only path-connected spaces when studying the fundamental group.*

By constructing the path homotopy explicitly, we can show the following theorem:

Theorem 1.16

- *(Associativity) If f, g and h are path for which $f(1) = g(0)$, $g(1) = h(0)$, then*

$$[f] * ([g] * [h]) = ([f] * [g]) * [h].$$

- *(Right and left identities) If f is a path in X from x_0 to x_1, then*

$$[f] * [e_{x_1}] = [f] \quad \textit{and} \quad [e_{x_0}] * [f] = [f].$$

- *(Inverse) If f is a path in X from x_0 to x_1, then*

$$[f] * [\bar{f}] = [e_{x_0}] \quad \textit{and} \quad [\bar{f}] * [f] = [e_{x_1}].$$

Definition 1.17 *Let α be a path in X from x_0 to x_1. We define*

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1), \quad \textit{by} \quad \hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha].$$

Remark 1.17.1 *$\hat{\alpha}(\gamma) = [\alpha]^{-1} * \gamma * [\alpha]$, is well-defined. And the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ depends only on the path-homotopy class of α.*

Theorem 1.18 *$\hat{\alpha}$ is a group isomorphism.*

Proof: It is clear that $\hat{\alpha}$ is a group homomorphism. Next, the map $\hat{\bar{\alpha}}$ is the inverse of $\hat{\alpha}$, hence $\hat{\alpha}$ is an isomorphism. □

Corollary 1.18.1 *If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$. In particular, if X is path connected, all the groups $\pi_1(X, x)$ are isomorphic, for $x \in X$.*

Proposition 1.19 *Let x_0 and x_1 be points of the path-connected space X. Then $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.*

Proof: $\Rightarrow$: Suppose $\pi_1(X, x_0)$ is abelian. Then for any α, β, we know that $\alpha * \bar{\beta}$ is a loop at x_0. Then we have $[\beta * \bar{\alpha}] * [\alpha * \bar{\beta}] = [\alpha * \bar{\beta}][\beta * \bar{\alpha}] = e$, this shows that $\hat{\alpha}$ and $\hat{\bar{\beta}}$ are inverse of each other, hence $\hat{\alpha} = \hat{\beta}$.

$\Leftarrow$: Let $[f], [g] \in \pi_1(X, x_0)$, then for any path α from x_0 to x_1, we have $[\bar{\alpha}] * [f] * [\alpha] = [\bar{\alpha}] * [\bar{g}] * [f] * [g] * [\alpha]$, so $[f] = [g]^{-1} * [f] * [g]$ as desired. □

Definition 1.20 (Simply Connected) *A space X is said to be **simply connected** if and only if it is a path-connected space and the **fundamental group** $\pi_1(X, x_0)$ is trivial for any / every $x_0 \in X$.*

Example: Any convex subset of $\mathbb{R}^n$ is simply connected. A connected open subset U of $\mathbb{C} = \mathbb{R}^2$ is simply connected iff its complement $\bar{C} \setminus U$ in $\bar{C} = \mathbb{C} \cup \{\infty\}$ is connected.

Lemma 1.21 *In a simply connected space X, any two paths having the same initial and final points are path homotopic.*

Proof: Let α and β be two paths from x_0 to x_1. Then $\alpha * \bar{\beta}$ is defined and is a loop on X based at x_0. Since X is simply connected, this loop is path homotopic to the constant loop at x_0. Then

$$[a * \tilde{\beta}] * [\beta] = [e_{x_0}] * [\beta].$$

It follows that $[\alpha] = [\beta]$. □

1.3 Functorial Property

The following two lemmas are easy to check:

Lemma 1.22 *If $k : X \to Y$ is a continuous map, and if F is a path homotopy in X between the path f and f'. Then $k \circ F$ is a path homotopy in Y between the paths $k \circ f$ and $k \circ f'$.*

Lemma 1.23 *If $k : X \to Y$ is a continuous map, and if f and g are paths in X with $f(1) = g(0)$, then $k \circ (f * g) = (k \circ f) * (k \circ g)$.*

Definition 1.24 (h_*) *Suppose $h : X \to Y$ is a continuous map that carries the point x_0 of X to the point y_0 of Y. We denote this fact by writing $h : (X, x_0) \to (Y, y_0)$. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*([f]) = [h \circ f]$.*

Remark 1.24.1 *Note $h_*(\gamma)$ is well-defined: if F is a path homotopy between the paths f and f', then $f \circ F$ is a path homotopy between the paths $h \circ f$ and $h \circ f'$.*

Remark 1.24.2 *The homomorphism h_* depends not only on the map $h : X \to Y$ but also on the choice of the base point x_0. If x_0 and x_1 are two different points of X, we cannot use the same symbol h_* to stand for two different homomorphism, one having domain $\pi_1(X, x_0)$ and the other having domain $\pi_1(X, x_1)$. Even if X is path connected, so these groups are isomorphic, they are still not the same group. In such a case, we shall use the notation*

$$(h_{x_0})_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

for the first homomorphism and $(h_{x_1})_*$ for the second.

Lemma 1.25 *$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group homomorphism.*

Proof: For loops f, g in X based at x_0, we have

$$(h \circ f) * (h \circ g) = h \circ (f * g).$$

So $h_*([f] \circ [g]) = h_*([f]) * h_*([g])$. □

Theorem 1.26 (Functorial property of π_1)

1. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous then $(k \circ h)_* = k_* \circ h_*$.*
2. *If $i : (X, x_0) \to (X, x_0)$ is the identity map. Then i_* is the identity homomorphism.*

Proof: By definition,

$$(k \circ h)_*([f]) = [(k \circ h) \circ f]$$
$$(k_* \circ h_*)([f]) = k_*(h_*([f])) = k_*([h \circ f]) = [k \circ (h \circ f)].$$

Similarly, $i_*([f]) = [i \circ f] = [f]$. $\square$

Corollary 1.26.1 *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X and with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$. I.e., $\pi_1(X, x_0)$ is a topological invariant of (X, x_0).*

Proof: Let $k : (Y, y_0) \to (X, x_0)$ be the inverse of h. Then $k_* \circ h_* = (k \circ h)_* = i_*$, where i is the identity map of (X, x_0); and $h_* \circ k_* = j_*$, where j is the identity map of (Y, y_0). Since i_* and j_* are the identity homomorphism of groups, $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, then k_* is the inverse of h_*. $\square$

Theorem 1.27 *Let $h : X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$, then*

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}.$$

I.e., the following diagram of maps commutes:

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "(h_{x_0})_*"] \arrow[d, "\hat{\alpha}"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\beta}"] \\
\pi_1(X, x_1) \arrow[r, "(h_{x_1})_*"'] & \pi_1(Y, y_1)
\end{tikzcd}$$

Proof: Let $[f] \in \pi_1(X, x_0)$, then we need to show

$$[h \tilde{\circ} \alpha] * [h \circ f] * [h \circ \alpha] = [h \circ (\tilde{\alpha} * f * \alpha)].$$

It suffices to show $[h \tilde{\circ} \alpha] = [h \circ \tilde{\alpha}]$, but in fact $h \tilde{\circ} \alpha = h \circ \tilde{\alpha}$ (as one may check), so we are done. $\square$

1.4 Covering Space

Let B be a topological space - base space of interest.

Definition 1.28 (B-space) *A B-**space** or a space over B consists of:*

- *a topological space E, and*
- *a continuous map $p : E \to B$ (Structural Map).*

*Suppose $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are B-spaces. A B-**space map** from E_1 to E_2 is a continuous map*

$f : E_1 \to E_2$ such that $p_2 \circ f = p_1$, i.e., such that the diagram

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

commutes.

A B-space map $f : E_1 \to E_2$ is an ***isomorphism*** *of B-spaces if there exists a B-space map $g : E_2 \to E_1$ such that*

$$g \circ f = id_{E_1} \quad \text{and} \quad f \circ g = id_{E_2}.$$

Examples:

- For any topological space F, the projection map $p : B \times F \to B$, $(b, \alpha) \mapsto b$ makes $B \times F$ into a B-space.
- If $p : E \to B$ is a B-space. For any subset $B' \subseteq B$, get $E' := p^{-1}(B') \subset E$, and the map $p|_{E'} : E' \to B'$. Then E' becomes a space over B', called the **restriction** of p (or E) over B'
- If $p : E \to B$ is a B-spaces, the identity map $id_E : E \to E$ is a B-space map.
- If $f : E_1 \to E_2$ and $g : E_2 \to E_3$ are B-spaces maps, the composite map $g \circ f : E_1 \to E_3$ is a B-space map.

Definition 1.29 *Let $p : E \to B$ be a B-space. An open set U of B is* ***evenly covered*** *by p (or p is trivial as a covering space over U) if there exists a non-empty discrete space F such that the restriction of p over U is isomorphic to $U \times F$ as U-spaces. I.e., the inverse image $p^{-1}(U) \subseteq E$ can be written as a non-empty disjoint union $\bigsqcup_{\alpha \in F} V_\alpha$ of open sets in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into* ***slices****.*

Remark 1.29.1 *Note that if U is evenly covered by p and W is an open set contained in U, then W is also evenly covered by p.*

Definition 1.30 (Covering Space) *A* ***covering space of*** *B is a B-space $p : E \to B$ which is locally trivial (in which case p is a covering map), i.e., for every point b of B, there exists an open neighbourhood U of b that is evenly covered by p (such that p is trivial as a covering map over U, or $p^{-1}(U)$ is trivial as a covering space over U). If this is the case, then p is called a* ***covering map****, and E is said to be a covering space of B.*

Remark 1.30.1 *If $p : E \to B$ is a covering map, then*

1. *for each $b \in B$, the fiber $p^{-1}(b) \subset E$ given with the subspace topology from E has the discrete topology.*
2. *$p : E \to B$ is an open map and surjective.*
3. *p is a local homeomorphism of E with B, i.e., for every point e of E, there exists an open neighborhood V of e such that $p|_V : V \to p(V)$ is a homeomorphism (Note that $p(V)$ is open in B).*

Example:

- The identity map $id_B : B \to B$ is a covering map.

- For any non-empty discrete space F, the projection map $p : B \times F \to B$, $(b, \alpha) \mapsto b$ is a covering map (in fact in the sense of globally trivial).
- If $p : E \to B$ and $p' : E' \to E$ are finite covering maps, then $p \circ p' : E' \to B$ is a finite covering map.

Theorem 1.31 *The exponential map $p : \mathbb{R} \to S^1 \subset \mathbb{C} \setminus \{0\}$ given by the equation*

$$p(x) = (\cos 2\pi x, \sin 2\pi x) \text{ or equivalently } p(x) = e^{2\pi i x}$$

is a covering map.

Proof: It is clear that the map is continuous. Next, by drawing the pictures, we can see that the map is locally trivial. □

Remark 1.31.1 *The map $p : \mathbb{R}_+ \to S^1$ given by the equation $x \mapsto e^{2\pi i x}$ is surjective and a local homeomorphism, but it is not a covering map. The point $b = (1, 0)$ has no neighborhood U that is evenly covered by p.*

Examples of Covering Maps:

- The map $p : S^1 \to S^1$ given in equation by $p(z) = z^2$ is a covering map.
- More generally, for any $n \in \mathbb{Z} \setminus \{0\}$, the n^{th} power map $S^1 \to S^1$, $z \mapsto z^n$ is a covering map.

Proposition 1.32 *Let $p : E \to B$ a covering map. If B_0 is a subspace of B, and $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ obtained by restricting p is a covering map.*

Remark 1.32.1

$$p^{-1}(B_0) = \quad \begin{tikzcd} E_0 \arrow[r, hook] \arrow[d, "p|_{E_0}"'] & E \arrow[d, "p"] \\ B_0 \arrow[r, hook] & B \end{tikzcd}$$

Proof: Given $b_0 \in B_0$, let U be an open set in B containing b_0 that is evenly coverer by p; let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Then $U \cap B_0$ is a neighborhood of b_0 in B_0, and the sets $V_\alpha \cap E_0$ are disjoint open sets in E_0 whose union in $p^{-1}(U \cap B_0)$, and each is mapped homeomorphically onto $U \cap B_0$ by p. □

Proposition 1.33 *If $p : E \to B$ and $p' : E' \to B'$ are covering maps, then $p \times p' : E \times E' \to B \times B'$ is a covering map.*

Remark 1.33.1

$$\begin{tikzcd} & E \times E' \arrow[dl] \arrow[dr] \arrow[dd] & \\ E \arrow[dd, "p"'] & & E' \arrow[dd, "p'"] \\ & B \times B' \arrow[dl] \arrow[dr] & \\ B & & B' \end{tikzcd}$$

Proof: Given $b \in B$ and $b' \in B'$, let U and U' be neighborhoods of b and b' respectively, that are evenly covered by p and p'. Let $\{V_\alpha\}$ and $\{V'_\beta\}$ be partitions of $p^{-1}(U)$ and $(p')^{-1}(U')$, into slices. Then the inverse image under $p \times p'$ of the open set $U \times U'$ is the union of all the sets $V_\alpha \times V'_\beta$. These are disjoint open sets of $E \times E'$, and each is mapped homeomorphically onto $U \times U'$ by $p \times p'$. $\square$

Proposition 1.34 *Let $q : X \to Y$ and $r : Y \to Z$ be covering maps. Let $p = r \circ q$, if $r^{-1}(z)$ is finite for each $z \in Z$, p is a covering map.*

Proof: Let $z \in Z$ be arbitrary. Then exists O_z, such that $r^{-1}(O_z) = \bigsqcup_{i=1}^{k} V_k$, since $r^{-1}(z)$ is finite. Then for each V_k, there exists y_k such that $r(y_k) = z$. Then for each such y_k, exists $W_k \subset X$, that is trivially covered by q. Then take the intersection of $r(W_k)$'s we get an open set around z that is trivially covered by p. $\square$

Example:

- Consider the space $T = S^1 \times S^1$ (the torus). The product map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is a covering of the torus by the plane $\mathbb{R}^2$.
- Let b_0 denote the point $p(0)$ of S^1 let B_0 denote the figure-eight space given by

  $$B_0 = (S^1 \times \{b_0\}) \cup (\{b_0\} \times S^1)$$

  of $S^1 \times S^1$ union of two circles that have a point in common. The space $E_0 = p^{-1}(B)$ is the "infinite grid"

  $$E_0 = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$$

  The map $p_0 : E_0 \to B_0$ obtained by restricting $p \times p$ is a covering map.
- Consider the covering map of the punctured plane by the open upper half-plane:

  $$\mathbb{R} \times \mathbb{R}_+ \to S^1 \times \mathbb{R}_+ \to \mathbb{R}^2 - \{0\} = \mathbb{C}^\times$$

  given by $(x, r) \mapsto (e^{2\pi i x}, r) \mapsto re^{2\pi i x}$.

Proposition 1.35 *Let $p : E \to B$ be a covering map. Suppose U is a open set of B that is evenly covered by p. Then if U is connected, then the partition of $p^{-1}(U)$ into slices is unique.*

Proof: Let $\{V_\alpha\}$ and $\{W_\beta\}$ be partitions of $p^{-1}(U)$. Fix an arbitrary α and let $e \in V_\alpha$. Then there is a unique β such that $e \in W_\beta$. We show that $V_\alpha = W_\beta$.

Suppose by way of contradiction that $V_\alpha \neq W_\beta$. Then $V_\alpha \setminus W_\beta \neq \emptyset$ (by homeomorphism) and $V_\alpha \cap W_\beta \neq \emptyset$. $V_\alpha \cap W_\beta$ is open, and $V_\alpha = (V_\alpha \cap W_\beta) \cup (V_\alpha \setminus W_\beta)$. We show that $V_\alpha \setminus W_\beta$ is open to get a contradiction, as by homeomorphism, V_α should be connected.
Let $x \in V_\alpha \setminus W_\beta$, there exists $\beta_x \neq \beta$ such that $x \in W_{\beta_x}$. Notice that $V_\alpha \setminus W_\beta = V_\alpha \cap \bigcup_{x \in V_\alpha \setminus W_\beta} W_{\beta_x}$ which is open. Hence the result follows. $\square$

Lemma 1.36 *Let $p : E \to B$ be a covering map; let B be connected. Then if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$. In such a case, E is called a **k-fold covering of** B.*

Proof: Suffices to show that the set of elements b, such that $|p^{-1}(b)| = |p^{-1}(b_0)|$ is both open and closed. This is clear by the local triviality property. $\square$

Proposition 1.37 *Let $p : E \to B$ be a covering map. Then if B is Hausdorff, E is Hausdorff. The same applies if B is regular, completely regular, or locally compact Hausdorff.*
In addition if B is compact, and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

1.5 Unique Path / Homotopy Lifting Property of Covering Maps

Definition 1.38 (Lifting) *Let $p : E \to B$ be a B-space; let $f : X \to B$ be a continuous map. A **lifting** of f to E over p is a continuous map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Example:

- Consider the covering map $p : \mathbb{R} \to S^1$. The path $f(s) = (\cos \pi s, \sin \pi s)$ lifts to the path $\tilde{f}(s) = \frac{s}{2}$ beginning at 0 and ending at $\frac{1}{2}$. The path $g(s) = (\cos \pi s, -\sin \pi s)$ lifts to the path $\tilde{g}(s) = -\frac{s}{2}$ beginning at 0 and ending at $-\frac{1}{2}$. The path $h(s) = (\cos 4\pi s, \sin 4\pi s)$ lifts to the path $\tilde{h}(s) = 2s$ beginning at 0 and ending at 2.

Lemma 1.39 *Let $p : E \to B$ be a covering map, let $f : X \to B$ be a continuous map. Suppose $\tilde{f} : X \to E$ and $\tilde{f}' : X \to E$ are lifting of f to E. Let $\Delta(\tilde{f}, \tilde{f}') := \{x \in X : \tilde{f}(x) = \tilde{f}'(x)\}$ be their equality locus. If X is locally connected, then $\Delta(\tilde{f}, \tilde{f}')$ is open and closed in X.*

Proof: Let $x \in X$ be any point. Since $p : E \to B$ is a covering map, there exists open neighborhood $U \subseteq B$ containing $f(x)$ such that $p^{-1}(U) = \bigsqcup_{\alpha \in F} V_\alpha$ is non-empty disjoint union and for each α, $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

Now $W := f^{-1}(U) \subseteq X$ is an open neighborhood of x. Since X is locally connected by hypothesis, we may shrink W if necessary and assume that W is connected.

Suppose $x \in \Delta(\tilde{f}, \tilde{f}')$. Then $\exists! \alpha_0 \in F$ such that $\tilde{f}(x) = \tilde{f}'(x) \in V_{\alpha_0}$. Then both $\tilde{f}$ and $\tilde{f}'$ must map W into V_{α_0} because V_α's are disjoint and W is connected. Since $p|_{V_{\alpha_0}} : V_{\alpha_0} \to U$ is a homeomorphism, and $\tilde{f}, \tilde{f}'$ are liftings of f over p, it follows that $\forall w \in W$, we must have $\tilde{f}(w) = \tilde{f}'(w)$ in V_{α_0}, namely, both equal to $(p|_{V_{\alpha_0}})^{-1}(f(w))$. Hence $W \subseteq \Delta(\tilde{f}, \tilde{f}')$. This shows $\Delta(\tilde{f}, \tilde{f}')$ is open in X.

Now suppose $x \notin \Delta(\tilde{f}, \tilde{f}')$. Then $\exists! \alpha \neq \beta \in F$ such that $\tilde{f}(x) \in V_\beta$, $\tilde{f}'(x) \in V_\beta$. Now $\tilde{f}$ and $\tilde{f}'$ maps W into V_α and V_β respectively. It follows that $\forall w \in W$, we must have $\tilde{f}(w) \neq \tilde{f}'(w)$ in E. Hence $\Delta(\tilde{f}, \tilde{f}')$ is closed in X. $\square$

Lemma 1.40 *Let $p : E \to B$ be a covering map, let $f : X \to B$ be a continuous map. Suppose*

- *There exists open $U \subseteq B$ with $f(X) \subseteq U$ such that U is evenly covered by p;*
- *There exists a connected subset $X_0 \subseteq X$ such that there exists a lifting $\tilde{f}_0 : X_0 \to E$ of $f|_{X_0}$ to E.*

Then there exists a lifting $\tilde{f} : X \to E$ of f to E such that $\tilde{f}|_{X_0} = \tilde{f}_0$.

Proof: Since p is trivial as a covering map over U, we have $p^{-1}(U) = \bigsqcup_{\alpha \in F} V_\alpha$ is a non-empty disjoint union and for each α, $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

Consider the lifting $\tilde{f}_0 : X_0 \to E$ of $f|_{X_0}$ to E. Since $f(X) \subseteq U$ by hypothesis, we have $f(X_0) \subseteq U$ as well and so $\tilde{f}_0(X_0) \subseteq p^{-1}(U) = \bigsqcup_{\alpha \in F} V_\alpha$. Since $X_0 \subseteq X$ is connected, $\exists! \beta \in F$ such that $\tilde{f}_0(X_0) \subseteq V_\beta$. Since $p|_{V_\beta} : V_\beta \to U$ is a homeomorphism, it follows that $(p|_{V_\beta}) \circ \tilde{f}_0 = f|_{X_0}$. Hence $\tilde{f} := (p|_{V_\beta})^{-1} \circ f : X \to E$ is a lifting of f to E, and $\tilde{f}|_{X_0} = \tilde{f}_0$. $\square$

Theorem 1.41 (Unique Path Lifting Property) *Let $p : E \to B$ be a covering map, let $f : [0,1] \to B$ be any path in B, beginning at $b_0 = f(0)$. For any choice of $e_0 \in p^{-1}(b_0)$ in the fiber over b_0, there exists a unique lifting $\tilde{f} : [0,1] \to E$ to a path E beginning at e_0.*

$$\begin{tikzcd}
& & E \arrow[d, "p"] \\
\{0\} \arrow[r, hook] \arrow[urr] & {[0,1]} \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & B
\end{tikzcd}$$

Proof: Cover B by open sets U each of which is evenly covered by p. Apply the Lebesgue number lemma to choose subdivision of I, $s_0 < s_1 < \cdots < s_n$ fine enough such that for each i, $f([s_i, s_{i+1}])$ lies in such an open set U. This is possible since I is compact.

Apply Lemma 1.40 to

$$\begin{tikzcd}
& & E \arrow[d, "p"] \\
\{0\} \arrow[r, hook] \arrow[urr] & {[s_0 = 0, s_1]} \arrow[r, "f"'] & B
\end{tikzcd}$$

to get $\tilde{f} : [s_0, s_1] \to E$ lifting $f|_{[s_0, s_1]}$. Next, inductively apply Lemme 1.40 to

$$\begin{tikzcd}
& & E \arrow[d, "p"] \\
\{s_i\} \arrow[r, hook] \arrow[urr] & {[s_i, s_{i+1}]} \arrow[r, "f"'] & B
\end{tikzcd}$$

to get $\tilde{f} : [s_i, s_{i+1}] \to E$ lifting $f|_{[s_i, s_{i+1}]}$. In particular, $\tilde{f}$ is continuous by the Pasting Lemma.

Uniqueness follows from Lemma 1.39, since $[0,1]$ is locally connected and connected. $\square$

Theorem 1.42 (Unique Path-homotopy Lifting Property) *Let $p: E \to B$ a covering map. Let $F: I \times I \to B$ be any continuous map with $F(0,0) = b_0$. For any choice of $e_0 \in p^{-1}(b_0)$ in the fiber over b_0. There exists a unique lifting $\tilde{F}: I \times I \to E$ with $\tilde{F}(0,0) = e_0$. Moreover, if F is a path homotopy, then $\tilde{F}$ is a path homotopy.*

Proof: The proof of uniqueness and existence is very similar to the proof of Theorem 1.41, by applying the inductive step block by block.

Now suppose F is a path homotopy, we show that $\tilde{F}$ is a path homotopy. The map F carries the entire left edge $0 \times I$ of I^2 into a single point b_0 of B. Because $\tilde{F}$ is a lifting of F, it carries this edge into the set $p^{-1}(b_0)$. But this set has the discrete topology as a subspace of E. Since $0 \times I$ is connected and $\tilde{F}$ is continuous, then $\tilde{F}(0 \times I)$ is connected and thus must equal to a one-point set. Similarly, $\tilde{F}(1 \times I)$ must be a one-point set. Thus $\tilde{F}$ is a path homotopy. $\square$

Proposition 1.43 *Let $p: E \to B$ be a covering map. Let f and g be two paths in B from b_0 to b_1. Let $e_0 \in p^{-1}(b_0)$ be a point in the fiber over b_0. Let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in E beginning at e_0. If f and g are path homotopic in B, then $\tilde{f}$ and $\tilde{g}$ end at the same point $e_1 \in p^{-1}(b_1)$ and are path homotopic.*

Proof: Let $F: I \times I \to B$ be the path homotopy between f and g. Then $F(0,0) = b_0$. Let $\tilde{F}: I \times I \to E$ be the lifting of F to E such that $\tilde{F}(0,0) = e_0$. Then $\tilde{F}$ is a path homotopy, so that $\tilde{F}(0 \times I) = \{e_0\}$ and $\tilde{F}(1 \times I)$ is one-point set $\{e_1\}$.
The restriction $\tilde{F}|_{I \times 0}$ of $\tilde{F}$ is a path on E beginning at e_0 that is a lifting of $F|_{I \times 0}$. By uniqueness of path lifting, we must have $\tilde{F}(s,0) = \tilde{f}(s)$. Similarly, we can show $\tilde{F}(s,1) = \tilde{g}(s)$. Therefore, $\tilde{F}$ is a path homotopy between $\tilde{f}$ and $\tilde{g}$. $\square$

Lemma 1.44 *Let $p: E \to B$ be a covering map. Let α and β be paths in B with $\alpha(1) = \beta(0)$; let $\tilde{\alpha}$ and $\tilde{\beta}$ be liftings of them such that $\tilde{\alpha}(1) = \tilde{\beta}(0)$. Show that $\tilde{\alpha} * \tilde{\beta}$ is a lifting of $\alpha * \beta$.*

Proof: Direct verification from definition, one can show that $p \circ (\tilde{\alpha} * \tilde{\beta}) = \alpha * \beta$. $\square$

Definition 1.45 (The Lifting Correspondence) *Let $p: E \to B$ be a covering map. Let $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. For any path-homotopy class α of paths in B beginning at b_0, there exists a unique path-homotopy class $\tilde{\alpha}_{e_0}$ of paths in E beginning at e_0 lifting α, I.e., given path-homotopy class α, to define $\tilde{\alpha}_{e_0}$, do the following:*

- *Choose path $f \in \alpha$ in B beginning at b_0.*
- *Apply Unique path lifting property to get the lifting $\tilde{f}$ of f to a path in E beginning at e_0.*
- *Set $\tilde{\alpha}_{e_0}$ to be the path-homotopy equivalence class of $\tilde{f}$.*

By Proposition 1.43, every path $\tilde{f}$ in $\tilde{\alpha}_{e_0}$ has the same end point which we denote by $e_0 * \alpha$ or $\phi(\alpha) \in E$. i.e.,

$e_0 * \alpha = \phi(\alpha) = \tilde{f}(1) = \tilde{\alpha}_{e_0}(1) \in p^{-1}(f(1))$ *for any / every lifting* $\tilde{f}$ *of* $f \in \alpha$ *beginning at* e_0. *Thus, if* α *is a path-homotopy class of paths in* B *from* b_0 *to* b_1. *We have a well-defined map* $- * \alpha : p^{-1}(b_0) \to p^{-1}(b_1)$, $e_0 \mapsto e_0 * \alpha$ *called the* ***lifting correspondence induced by*** α.

The following proposition is easy to see:

Proposition 1.46

1. *If* 1_{b_0} *is the path-homotopy class of the constant path at* b_0 *in* B, *then* $- * 1_{b_0} : p^{-1}(b_0) \to p^{-1}(b_0)$ *is the identity map on* $p^{-1}(b_0)$.

2. *If* α *is a path-homotopy class of paths in* B *from* b_0 *to* b_1 *and* β *is a path-homotopy class of paths in* B *from* b_1 *to* b_2 *(so* $\alpha * \beta$ *is a path-homotopy class of paths in* B *from* b_0 *to* b_2*). Then the following diagram commutes:*

$$\begin{tikzcd} p^{-1}(b_0) \arrow[r, "-*\alpha"] \arrow[dr, "-*(\alpha*\beta)"'] & p^{-1}(b_0) \arrow[d, "-*\beta"] \\ & p^{-1}(b_2) \end{tikzcd}$$

   I.e., for any $e_0 \in p^{-1}(b_0)$,

$$(e_0 * \alpha) * \beta = e_0 * (\alpha * \beta).$$

Definition 1.47 (The Monodromy Action) *Let* $p : E \to B$ *be a covering map. Let* $b_0 \in B$. *For any element* α *of* $\pi_1(B, b_0)$, *the lifting correspondence induced by* α *is then a map from* $p^{-1}(b_0)$ *to itself:*

$$- * \alpha : p^{-1}(b_0) \to p^{-1}(b_0), \;\; e_0 \mapsto e_0 * \alpha$$

The ***monodromy action*** *of* $\pi_1(B, b_0)$ *on* $p^{-1}(b_0)$ *is the map*

$$* : p^{-1}(b_0) \times \pi_1(B, b_0) \to p^{-1}(b_0), \;\; (e_0, \alpha) \mapsto e_0 * \alpha$$

This is a well-defined right action of $\pi_1(B, b_0)$ *on* $p^{-1}(b_0)$ *by Proposition 1.46.*

Theorem 1.48 *Let* $p : E \to B$ *be a covering map. Let* $b_0 \in B$. *Consider the monodromy action of* $\pi_1(B, b_0)$ *on* $p^{-1}(b_0)$. *If* E *is path connected, the action is transitive. If* E *is simply connected, the action is simply transitive.*

Proof: If E is path connected, for any $e_0, e_1 \in p^{-1}(b_0)$, there exists a path $\tilde{f}$ in E from e_0 to e_1. Then $f = p \circ \tilde{f}$ is a loop in B at b_0 (with $\tilde{f}$ as lifting beginning at e_0). So if we take $\alpha := [f] \in \pi_1(B, b_0)$, then $e_0 * \alpha = \tilde{f}(1) = e_1$. Hence the action is transitive.

If E is simply connected, then it is path-connected. Suppose $\alpha \in \pi_1(B, b_0)$ and $e_0 \in p^{-1}(b_0)$ such that $e_0 * \alpha = e_0$. Choose a loop $f \in \alpha$ in B based at b_0 representing α. Apply Unique path lifting property to get the lifting $\tilde{f}$ of f to a path in E beginning at e_0. Since $e_0 * \alpha = e_0$ by hypothesis, we must have $\tilde{f}(1) = e_0$ as well. So $\tilde{f}$ is in fact a loop in E based at e_0. Since E is simply connected, there exists a path homotopy $\tilde{F}$ in E between $\tilde{f}$ and 1_{e_0}. Then $p \circ \tilde{F}$ is a path homotopy in B between f and 1_{b_0}. So $\alpha = [f] = [1_{b_0}] = 1$ in $\pi_1(B, b_0)$. Hence the action is simply transitive. $\square$

Theorem 1.49 *Let $p : E \to B$ be a covering map. Let $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. Then*

1. *The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism (an injective homomorphism).*
2. *Let f be a loop in B based at b_0. Then $[f] \in p_*(\pi_1(E, e_0))$ if and only if f lifts to a loop in E based at e_0.*
3. *Let $H = p_*(\pi_1(E, e_0))$. The monodromy action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$ induces a well-defined injective map $\Phi : H \setminus \pi_1(B, b_0) \to p^{-1}(b_0)$, $\;H\alpha \mapsto e_0 * \alpha$. If E is a path connected, Φ is bijective.*

Proof:

1. Suppose $\tilde{h}$ is a loop in E at e_0 representing an element of $\ker(p_*)$. Then $p \circ \tilde{h}$ is a loop in B based at b_0 representing $1_{b_0} \in \pi_1(B, b_0)$. Let F be a path homotopy between $p \circ \tilde{h}$ and the constant loop at b_0. Apply Unique Path-homotopy lifting property, we get the lifting $\tilde{F}$ of F such that $\tilde{F}(0, 0) = e_0$. Then $\tilde{F}$ is a path homotopy in E between $\tilde{h}$ and the constant loop at e_0. Hence $[\tilde{h}] = 1_{e_0}$ in $\pi_1(E, e_0)$.
2. If f lifts to a loop $\tilde{f}$ in E based at e_0.Then $[f] = [p \circ \tilde{f}] = p_*([\tilde{f}])$ lies in $p_*(\pi_1(E, e_0)) \subseteq \pi_1(B, b_0)$.

   Conversely, suppose $[f] \in p_*(\pi_1(E, e_0))$, so there exists a loop $\tilde{h}$ in E based at e_0 such that $[f] = [p \circ \tilde{h}]$ in $\pi_1(B, b_0)$. Let F be a path homotopy between f and $p \circ \tilde{h}$ in B. Apply Uniqueness path-homotopy lifting property to get the lifting $\tilde{F}$ of F such that $\tilde{F}(0, 0) = e_0$. Then $\tilde{F}$ is a path homotopy in E between the lifting path $\tilde{f}$ of f beginning at e_0 and the loop $\tilde{h}$ based at e_0. Hence $\tilde{f}$ is also a loop based at e_0.
3. Suffices to show that under the monodromy action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$, the stabilizer subgroup of $e_0 \in p^{-1}(b_0)$ is $H = p_*(\pi_1(E, e_0))$. Suppose $[f] \in H$, by part (2), f lifts to a loop $\tilde{f}$ in E based at e_0. So $e_0 * [f] = \tilde{f}(1) = e_0$, so $H \subseteq \mathrm{Stab}_{\pi_1(B,b_0)}(e_0)$. Conversely, suppose $\alpha \in \mathrm{Stab}_{\pi_1(B,b_0)}(e_0)$, so $e_0 * \alpha = e_0$. Choose a loop $f \in \alpha$ in B based at b_0 representing α. Apply Uniqueness path lifting property to get the lifting $\tilde{f}$ of f to a path in E beginning at e_0. Since $e_0 * \alpha = e_0$ by hypothesis, we must have $\tilde{f}(1) = e_0$ as well. So $\tilde{f}$ is in fact a loop in E based at e_0. Then $\tilde{f} \in H$.

$\square$

Corollary 1.49.1 *Let $p : E \to B$ be a covering map, with E path connected. If B is simply connected, then p is a homeomorphism.*

Proof: Since E is path connected, the $\Phi : H \setminus \pi_1(B, b_0) \to p^{-1}(b_0)$ is bijective for every $b_0 \in B$. Since B is simply connected, then $\pi_1(B, b_0)$ is trivial, so $p^{-1}(b_0)$ contains exactly one element. So p is a bijection. Lastly, p is open, hence p^{-1} is also continuous. So p is a homeomorphism. $\square$

Theorem 1.50 *The fundamental group of S^1 is isomorphic to $\mathbb{Z}$. More precisely, consider the exponential covering map*

$$p : \mathbb{R} \to S^1, \quad p(x) = (\cos 2\pi x, \sin 2\pi x)$$

Since $\mathbb{R}$ is simply connected, the monodromy action of $\pi_1(S^1, 1)$ on $p^{-1}(1) = \mathbb{Z}$ is simply transitive. So we have a bijective map $\phi : \pi_1(S^1, 1) \to \mathbb{Z}$, $\alpha \mapsto 0 * \alpha$. ϕ is a homomorphism, hence an isomorphism of groups.

Proof: Suppose $\alpha, \beta \in \pi_1(S^1, 1)$. Choose loops $f \in \alpha$, $g \in \beta$ in S^1 based at 1 representing α, β. Apply Unique path lifting property to get the liftings $\tilde{f}$ of f and $\tilde{g}$ of g to paths in $\mathbb{R}$ beginning at 0. Let $n = \tilde{f}(1)$ and $m = \tilde{g}(1)$ in $p^{-1}(1) = \mathbb{Z}$. Thus $0 * \alpha = n$ and $0 * \beta = m$ by definition. Let $\hat{g}$ be the path $\hat{g}(s) = n + \tilde{g}(s)$. By the properties of the exponential map, $p : \mathbb{R} \to S^1$, it follows that $\hat{g}$ is the unique lifting of g beginning at n. Then the product path $\tilde{f} * \hat{g}$ is defined, and it is the unique lifting of $f * g$ beginning at 0, and the end point of this paths is $\hat{g}(1) = n + m$. Since $f * g$ represents $\alpha * \beta$, it follows that $0 * (\alpha * \beta) = n + m$. Hence ϕ is a homomorphism. □

Remark 1.50.1 *Let $p_0 : I \to S^1$ be the restriction $p : \mathbb{R} \to S^1$ to the unit interval. Then p_0 is a loop in S^1 whose lift to $\mathbb{R}$ begins at 0 and ends at 1, so $0 * [p_0] = 1$ Thus under the isomorphism $\phi : \pi_1(S^1, 1) \to \mathbb{Z}$, $\alpha \mapsto 0 * \alpha$, $[p_0]$ is mapped to 1, and generates $\pi_1(S^1, b_0)$.*

Corollary 1.50.1 *S^1 and I (or any convex set) are not homeomorphic.*

Theorem 1.51 *The fundamental group of T^2 (the 2-dimensional torus) is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.*

Proof: Firstly note that T^2 is path connected, hence we just need to show that $p_1(T^2, \theta)$, $\theta = (1, 1)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Define the map $p \times p : \mathbb{R} \times \mathbb{R} \to T^2 = S^1 \times S^1$ given by

$$p \times p(x, y) = (p(x), p(y)),$$

where p is the exponential map from $\mathbb{R}$ to S^1. Then one can check that $p \times p$ is a covering map.

Now suppose $\alpha, \beta \in \pi_1(T^2, \theta)$, then choose loops $f \in \alpha$ and $g \in \beta$ in T^2 based at 1. Apply the Uniqueness path lifting property to get the lift $\tilde{f}$ of f and $\tilde{g}$ of g to paths in $\mathbb{R} \times \mathbb{R}$ beginning at $(0, 0)$. Let $(n, m) = \tilde{f}(1)$ and $(n', m') = \tilde{g}(1)$ in $(p \times p)^{-1}(1, 1)$. Then $0 * \alpha = (n, m)$ by definition. Let $\hat{g}$ be the path $\hat{g}(x, y) = (n, m) + \tilde{g}(x, y)$. Then it follows that $\hat{g}$ is the unique lifting of g beginning at (n, m). Then the product path $\tilde{f} * \hat{g}$ is defined, and is the unique lifting of $f * g$ beginning at 0. The end point of the paths is $\hat{g}(1) = (n + n', m + m')$. So we have that $0 * (\alpha * \beta) = (n + n', m + m')$. Hence the map $\phi : \pi_1(T^2, (1, 1)) \to \mathbb{Z} \times \mathbb{Z}$, $\alpha \mapsto 0 * \alpha$ is an homomorphism. Since $\mathbb{R} \times \mathbb{R}$ is simply connected, then this is an isomorphism. □

1.6 Retraction and Fixed Points

Definition 1.52 (Retraction) *Let X be a topological space, and let $A \subset X$ be a subspace. We have the canonical continuous inclusion map $j : A \to X$. A **retraction** of X onto A is a continuous map $r : X \to A$ such that $r \circ j = id_A$, i.e., $r|_A$ is the identity map of A. We say that A is a **retract** of X if there exists a retraction of X onto A.*

Lemma 1.53 *The retract of a contractible space is contractible.*

Proof: Let $r : X \to A$ be a retract, and H is a homotopy between the identity map on X and a constant map. Then

$$F(x, s) = r(H(x, s)).$$

is a homotopy between the identity map on A and a constant map. Hence A is contractible. $\square$

Lemma 1.54 *If A is a retract of X, then $j_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective.*

Proof: By functoriality, $r_* \circ j_* = (r \circ j)_* = (id_A)_* = id_{\pi_1(A,a)}$. Thus j_* must be injective. $\square$

Theorem 1.55 (No-rectraction Theorem) *There is no retraction of B^2 onto S^1.*

Proof: If S^1 were a retract of B^2, then $j_* : \pi_1(S^1, 1) \to \pi_1(B^2, 1)$ is injective. But $\pi_1(S^1, 1) \cong \mathbb{Z}$, whereas $\pi_1(B^2, 1) = 1$. $\square$

Recall if $f : X \to Y$ is a continuous map. We say f is **nulhomotopic in** Y iff f is homotopic to a constant map in Y.

Corollary 1.55.1 *The identity map $i : S^1 \to S^1$ is not nulhomotopic. In S^1, the inclusion map $j : S^1 \to \mathbb{R}^2 - 0$ is not nullhomotopic in $\mathbb{R}^2$.*

Proof: By functoriality, i_* must be $id_{\pi_1(S^1,1)}$, not trivial. By Lemma 1.54, $j_* : \pi_1(S^1, 1) \to \pi_1(\mathbb{R}^2 - 0, 1)$ is injective, so it is not trivial. Then by the following Lemma 1.56, we know j is not nulhomotopic. $\square$

Lemma 1.56 *Let $h : S^1 \to X$ be a continuous map from S^1. The following are equivalent:*

1. *h is nulhomotopic in X.*
2. *h extends to a continuous map $k : B^2 \to X$, i.e., such that $k \circ j = h$ where $j : S^1 \hookrightarrow B^2$.*
3. *h_* is the trivial homomorphism of fundamental groups.*

Proof: (1) $\Rightarrow$ (2): Let $H : S^1 \times I \to X$ be a homotopy between h and a constant map. Let $\pi : S^1 \times I \to B^2$ be the map $\pi(x, t) = (1 - t)x$. Then π is a quotient map; it collapses $S^1 \times 1$ to the point 0 and is otherwise injective. Because H is constant on $S^1 \times 1$, it induces via the quotient map a continuous map $k : B^2 \to X$, that is an extension of h.

(2) $\Rightarrow$ (3): If $j : S^1 \to B^2$ is the inclusion map, then h equals the composite $k \circ j$. But $j_* : \pi_1(S^1, b_0) \to \pi_1(B^2, b_0) = 1$ is trivial. Hence $h_* = k_* \circ j_*$ is trivial.

(3) $\Rightarrow$ (1): Let $p_0 : I \to S^1$ be a loop in S^1 such that $[p_0]$ generates $\pi_1(S^1, 1)$. Because h_* is trivial, the loop $f = h \circ p_0$ in X represents 1 in $\pi_1(X, x_0)$, where $x_0 = h(1)$. Therefore, there is a path homotopy F in X between f and the constant path at x_0. The map $p_0 \times id : I \times I \to S^1 \times I$ is a quotient map for each $t \in I$. It maps $0 \times t$ and $1 \times t$ to $b_0 \times t$ but is otherwise injective. The path homotopy F maps $0 \times I$ and $1 \times I$ and $I \times 1$ to the point x_0 of X, so it induces a continuous map $H : S^1 \times I \to X$ that is a homotopy between h and a constant map. $\square$

Definition 1.57 (Vector Field) *A* ***vector field*** *on* B^2 *is a continuous map* $v : B^2 \to \mathbb{R}^2$. v *is* ***nonvanishing*** *if* $v(x) \neq 0$ *for every* x*, in such a case* v *is a continuous map* $v : B^2 \to R^2 - 0$.

Theorem 1.58 *Given a nonvanishing vector field on* B^2, $b : B^2 \to R^2 - 0$*, there exists a point of* S^1 *where the vector field points directly inward and a point of* S^1 *where it points directly outward. I.e., there exists* $p \in S^1$ *such that* $v(p) \in -\mathbb{R}_{>0} \cdot p$ *and there exists* $q \in S^1$ *such that* $v(q) \in \mathbb{R}_{>0} \cdot q$.

Proof: Suppose for any $p \in S^1$, one has $v(p) \notin -\mathbb{R}_{>0} \cdot p$ (for the other case, consider $-v(p)$). Let $w : S^1 \to \mathbb{R}^2 - 0$ be the restriction of v to S^1. Because w extends to a map of B^2 into $\mathbb{R}^2 - 0$, it is nulhomotopic in $\mathbb{R}^2 - 0$. On the other hand, $F : S^1 \times I \to \mathbb{R}^2$ given by $F(x, t) = tx + (1 - t)w(x)$ is homotopy in $\mathbb{R}^2$ from the inclusion map $j : S^1 \to \mathbb{R}^2 - 0$ to the map $w : S^1 \to \mathbb{R}^2 - 0$.

We claim that for each $t \in I$, $x \in S^1$, we have $F(x, t) \neq 0$. Clear that for $t = 0$ and $t = 1$, $F(x, t) \neq 0$. If $F(x, t) = 0$ for some t with $0 < t < 1$, then $tx + (1 - t)w(x) = 0$, so $v(x) = w(x) \in -\mathbb{R}_{>0} \cdot x$ which is a contradiction. So F is a homotopy in $\mathbb{R}^2 - 0$ from $j : S^1 \to R^2 - 0$ to $w : S^1 \to \mathbb{R}^2 - 0$. So j is nulhomotopic in $\mathbb{R}^2 - 0$, which is a contradiction to Corollary 1.55.1. Thus there must exist $v(p) \in -\mathbb{R}_{>0} \cdot p$. □

Theorem 1.59 (Brouwer Fixed-Point Theorem for the Disc) *If* $f : B^2 \to B^2$ *is continuous, there exists a point* $x \in B^2$ *such that* $f(x) = x$.

Proof: Suppose that $f(x) \neq x$ for every $x \in B^2$. Then $v : B^2 \to R^2 - 0$ given by $v(x) = f(x) - x$ is a nonvanishing vector field on B^2. Then by Theorem 1.58, there exists $x \in S^1$ such that $v(x) \in \mathbb{R}_{>0} \cdot x$ points directly outward. So $f(x) - x = ax$ for some positive real number $a \in \mathbb{R}_{>0}$. So $f(x) = (1 + a)x$ would lie outside B^2, which is a contradiction. □

Remark 1.59.1 *Brouwer fixed-point theorem for* $B^n \subseteq \mathbb{R}^n$ *holds analogously: for any continuous map* $f : B^n \to B^n$*, there exists a point* $x \in B^n$ *such that* $f(x) = x$.

Corollary 1.59.1 *If* A *is a retract of* B^2*, then every continuous map* $f : A \to A$ *has a fixed point.*

Proof: Let $r : B^2 \to A$ be a retraction map, and $j : A \to B^2$ be the canonical inclusion. Then consider the map $j \circ f \circ r : B^2 \to B^2$, which is continuous. By the Brouwer Fixed Point theorem, $\exists x \in B^2$ such that $(j \circ f \circ r)(x) = x$, next checking the range and domain of the function, we must see that $x \in A$, and $f(x) = x$. □

Corollary 1.59.2 *If* $h : S^1 \to S^1$ *is nulhomotopic, then* h *has a fixed point and* h *maps some point* x *to its antipode* $-x$.

Proof: Since h is nulhomotopic, then $-h$ is also nulhomotopic, and they extends to continuous maps k and $-k$ from B^2 to S^1 such that $k|_{S^1} = h$ and $-k|_{S^1} = -h$. Hence by the Brouwer fixed point theorem, we have the desired result. □

Corollary 1.59.3 (Perron-Frobenius Theorem for 3 by 3 matrix) *Let* A *be a 3 by 3 matrix of positive real numbers. Then* A *has a positive real eigenvalue.*

Proof: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation of multiplication by A. Let B be $S^2 \cap \{(x_1, x_2, x_3) \,|\, x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0\}$ homeomorphic to the ball B^2. If $x = (x_1, x_2, x_3)$ is in B, then $T(x)$ is a vector, all of whose components are positive (since all entries of A are positive and at least one component of x is positive). So the map $x \to \frac{T(x)}{\|T(x)\|}$ is a continuous map of B to itself, by Brouwer fixed-point theorem for $B \approx B^2$, it has a fixed point x_0. Then $T(x_0) = \|T(x_0)\|x_0$, so T has the positive real eigenvalue $\|T(x_0)\|$. □

We will prove that there is no retraction $r : B^{n+1} \to S^n$, using this, the following can be proved in the same way we proved for the case $n = 1$.

Corollary 1.59.4

1. *The identity map $i : S^n \to S^n$ is not nulhomotopic.*
2. *The inclusion map $j : S^n \to R^{n+1} - 0$ is not nulhomotopic.*
3. *Every nonvanishing vector field on B^{n+1} points directly outward at some point of S^n and directly inward at some point of S^n.*
4. *Every continuous map $f : B^{n+1} \to B^{n+1}$ has a fixed point.*
5. *Every $n+1$ by $n+1$ matrix with positive real entries has a positive eigenvalue.*
6. *If $h : S^n \to S^n$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.*

1.7 Deformation Rectracts and Homotopy Type

Definition 1.60 (Deformation Retract) *A **deformation retract** of X onto A is a continuous map $H : X \times I \to X$ such that*

- *for all $x \in X$, one has $H(x, 0) = x$;*
- *for all $x \in X$, one has $H(x, 1) \in A$;*
- *for all $t \in I$, for all $a \in A$, one has $H(a, t) = a$.*

Equivalently, such that the map $r : X \to A$ defined by $r(x) = H(x, 1)$ is a retraction of X onto A and H is a homotopy between the identity map of X and the map $j \circ r$ such that each point of A remains fixed during the homotopy.

*We say that A is a **deformation retract** of X if there exists a deformation retraction of X onto A.*

Lemma 1.61 *If A is a deformation detract of X and B is a deformation retract of A, then B is a deformation retract of X.*

Proof: Let F be a deformation retraction of X onto A and G be a deformation retraction of A onto B, then

$$H(x,t) = \begin{cases} F(x, 2s) & 0 \leq s \leq \frac{1}{2} \\ G(F(x,1), 2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

Then H is a deformation retraction of X onto B. $\square$

Recall that if A is a retract of X, then $j_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective.

Lemma 1.62 *Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps. If h and k are homotopic and the image of the base point x_0 of X remains fixed at y_0 during the homotopy, then the homomorphism $h_*, k_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $[f] \mapsto [h \circ f]$, $[f] \mapsto [k \circ f]$ are equal.*

Proof: By assumption, there is a homotopy $H : X \times I \to Y$ between h and k such that $H(x_0, t) = y_0$ for all t. If f is a loop in X based at x_0, the composite

$$I \times I \xrightarrow{f \times id} X \times I \xrightarrow{H} Y$$

is a path homotopy between $h \circ f$ and $k \circ f$ because H maps $x_0 \times I$ to y_0. $\square$

Theorem 1.63 *Let A be a deformation retract of X; let $x_0 \in A$. Then the inclusion map $j : (A, x_0) \to (X, x_0)$ induces an isomorphism $j_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ of fundamental groups.*

Proof: Since $r : X \to A$ defined by $r(x) = H(x, 1)$ is a retraction of X onto A, then we already have

$$r_* \circ j_* = (r \circ j)_* = (id_A)_* = id_{\pi_1(A, x_0)}.$$

Since H is a homotopy between id_X and the map $j \circ r$ fixing A, then by Lemma 1.62, we also have

$$j_* \circ r_* = (j \circ r)_* = (id_X)_* = id_{\pi_1(X, x_0)}.$$

$\square$

Remark 1.63.1 *In particular, this also shows that r_* is an isomorphism.*

Corollary 1.63.1 *The inclusion map $j : S^n \to \mathbb{R}^{n+1} - 0$ induces an isomorphism $j_* : \pi_1(S^n, b_0) \to \pi_1(\mathbb{R}^{n+1} - 0, b_0)$ of fundamental groups, $b_0 = (1, 0, \cdots, 0)$.*

Proof: S^n is a deformation retraction of $R^{n+1} \setminus 0$ given by

$$(x, t) \longmapsto (1 - t)x + \frac{tx}{\|x\|}.$$

$\square$

Definition 1.64 (Homotopy Type)

- *A **homotopy inverse** of a continuous map $f : X \to Y$ is a continuous map $g : Y \to X$ such that $g \circ f : X \to X$ is homotopic to the identity map of X, and $f \circ g : Y \to Y$ is homotopic to the identity map of Y (Then f is a homotopy inverse of g).*
- *A **homotopy equivalence** from X to Y is a continuous map $f : X \to Y$ for which there exists a homotpy*

inverse $g : Y \to X$.

- *X and Y are* ***homotopy equivalent*** *(of the same homotopy type) if and only if there exists a homotopy equivalence from X to Y.*

Remark 1.64.1 *Homotopy inverse may not be unique. Homotopy equivalence is an equivalence relation among topological spaces. An equivalence class for this equivalence relation is a* ***homotopy type****.*

Example:

- A homeomorphism is a homotopy equivalence.
- Let X be a topological space and let $A \subset X$ be a deformation retract of X. Then the canonical inclusion $j : A \to X$ is a homotopy equivalence from A to X.
  If $H : X \times I \to X$ is a deformation retraction of X onto A, then the map $r : X \to A$ defined by $r(x) = H(x, 1)$ is a retraction of x onto A, i.e., $r \circ j = id_A$. And H is a homotopy between the identity map of X and the map $j \circ r$ such that each point of A remains fixed during the homotopy. So $j : A \to X$ and $r : X \to A$ are homotopy inverses of each other.

Lemma 1.65 *Let $h, k : X \to Y$ be continuous maps, $h(x_0) = y_0$, $k(x_0) = y_1$. If h and k are homotopic, then there exists a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$. More precisely, if $H : X \times I \to Y$ is a homotopy from h to k, then $\alpha : I \to Y$ defined by $\alpha(t) = H(x_0, t)$ is such a path.*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] & {[g]} \arrow[d] \\
& \pi_1(Y, y_1) & {[\tilde{\alpha} * g * \alpha]}
\end{tikzcd}$$

Proof: Let $f : I \to X$ be a loop in X based at x_0. We must show that

$$k_*([f]) = \hat{\alpha}(h_*([f])).$$

This equation states that $[k \circ f] = [\tilde{\alpha}] * [h \circ f] * [\alpha]$, or equivalently, that

$$[\alpha] * [k \circ f] = [h \circ f] * [\alpha].$$

To begin, consider the paths f_0 and f_1 in the space $X \times I$ given by the equations

$$f_0(s) = (f(s), 0) \quad \text{and} \quad f_1(s) = (f(s), 1).$$

Consider also the path c in $X \times I$ given by the equation $c(t) = (x_0, t)$. Then $H \circ f_0 = h \circ f$, $H \circ f_1 = k \circ f$, while $H \circ c$ equals the path α. Let $F : I \times I \to X \times I$ be the map $F(s, t) = (f(s), t)$. Consider the following paths in $I \times I$, $\beta_0(s) = (s, 0)$ and $\beta_1(s) = (s, 1)$, $\gamma_0(t) = (0, t)$ and $\gamma_1(t) = (1, t)$. Then $F \circ \beta_0 = f_0$, $F \circ \beta_1 = f_1$, while $F \circ \gamma_0 = F \circ \gamma_1 = c$. Then $\beta_0 * \gamma_1$, $\gamma_0 * \beta_1$ are paths in $I \times I$ from $(0, 0)$ to $(1, 1)$. Since $I \times I$ is convex, there is a path homotopy G between them. Then $F \circ G$ is a path homotopy in $X \times I$ between $f_0 * c$ and $c * f_1$. $H \circ (F \circ G)$ is a path homotopy in Y between $(H \circ f_0) * (H \circ c) = (h \circ f) * \alpha$ and $(H \circ f) * (H \circ f_1) = \alpha * (k \circ f)$. □

Corollary 1.65.1 *Let $h, k : X \to Y$ be homotopic continuous maps, if h_* is injective or surjective or trivial, then so is k_*.*

Proof: Since $\hat{\alpha}$ is an isomorphism of fundamental groups. $\square$

Corollary 1.65.2 *If $h : X \to Y$ is nullhomotopic, then h_* is the trivial homomorphism.*

Theorem 1.66 *Let $f : X \to Y$ be a homotopy equivalence, with $f(x_0) = y_0$. Then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism, i.e., $\pi_1(X, x_0)$ is a homotopy type invariant of (X, x_0).*

Proof: Let $g : Y \to X$ be a homotopy inverse for f. Consider the maps

$$\begin{tikzcd} (X, x_0) \arrow[r, "f"] & (Y, y_0) \arrow[dl, "g"'] \\ (X, x_1) \arrow[r, "f"] & (Y, y_1) \end{tikzcd} \qquad f(x_0) = y_0$$

and the induced homomorphism

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "(f_{x_0})_*"] & \pi_1(Y, y_0) \arrow[dl, "g_*"'] \\ \pi_1(X, x_1) \arrow[r, "(f_{x_1})_*"] & \pi_1(Y, y_1) \end{tikzcd}$$

By hypothesis, $g \circ f : (X, x_0) \to (X, x_1)$ is homotopic to $id_X : (X, x_0) \to (X, x_0)$. So by Lemma 1.65, there is a path α in X such that $(g \circ f)_* = \hat{\alpha} \circ (id_X)_* = \hat{\alpha}$. So $g \circ (f_{x_0})_* = (g \circ f)_*$ is an isomorphism. Similarly, $(f_{x_1})_* \circ g_* = (f \circ g)_*$ is an isomorphism. Therefore, g_* is an isomorphism and so is $(f_{x_0})_* = (g_*)^{-1} \circ \hat{\alpha}$. $\square$

Corollary 1.66.1 *A space X is contractible if and only if X has the homotopy type of a one-point space.*

Proof: If X is contractible, then the identity map is nulhomotopic, so $\exists$ homotopy $H : X \times I \to X$ between id_X and e_x for some $x \in X$. Consider the canonical inclusion map $j : \{x\} \to X$. Then $e_x \circ j = id_{\{x\}}$, and $j \circ e_x$ is homotopic to id_X by H, hence X is homotopic to the one-point space $\{x\}$.

Conversely, if X has a homotopy type of a one-point space, then there exists a map $f : X \to \{p\}$ that has a homotopy inverse $g : \{p\} \to X$. Since $g \circ f$ is homotopic to id_X, and clearly $g \circ f$ is the constant map, then X is contractible. $\square$

Remark 1.66.1 *Facts by Martin Fuchs: two spaces X and Y have the same homotopy type if and only if there exists a single space Z such that X and Y are homeomorphic to deformation retracts of Z.*

1.8 Fundamental Group of a Union

Theorem 1.67 *Suppose $X = U \cup V$ where U and V are open, path connected sets of X with $U \cap V$ non-empty and path connected. Let $x_0 \in U \cap V$, let $i : U \to X$, $j : V \to X$ be the inclusion mappings. Then $\pi_1(X, x_0)$ is generated by the images of $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$. I.e., for any loop f in X based at x_0 there exists loops g_i in U or in V based at x_0 such that f is path homotopic in X to $(g_1 * (g_2 * (\cdots * g_n)))$.*

Proof: Let $f : [0,1] \to X$ be a loop in X based at x_0. Since $\{U, V\}$ is an open covering of X, apply the Lebesgue number lemma to choose finite subdivision of $[0,1]$, $0 = a_0 < a_1 < \cdots < a_n = 1$ fine enough such that for each i, $f([a_{i-1}, a_i])$ is contained in either U or V.

If for any index i, we have $f(a_i) \notin U \cap V$, then $i \neq 0$ and $i \neq n$, because $f(a_0) = f(a_n) = x_0$ is in $U \cap V$. So both $f([a_{i-1}, a_i])$ and $f([a_i, a_{i+1}])$ is contained in either U or V. So $f(a_i) \in U \setminus V$ implies $f([a_{i-1}, a_{i+1}])$ is contained in U, the analogous statement holds for $f(a_i) \in V \setminus U$, so we may delete a_i.

WLOG, we may assume that $f(a_i) \in U \cap V$ for each i. For each index $i \in \{1, \cdots, n\}$, define f_i to be the path in X given by the positive linear map of $[0,1]$ onto $[a_{i-1}, a_i]$ followed by f and choose a path a_i in $U \cap V$ from x_0 to $f(a_i)$ with a_0 and a_n being the constant path at x_0. Set $g_i = (a_{i-1} * f_i) * \bar{\alpha}_i$. Then g_i is a loop in U or in V based at x_0 and

$$[g_1] * [g_2] * \cdots * [g_n] = [f_1] * [f_2] * \cdots * [f_n] = [f] \quad \text{in } \pi_1(X, x_0).$$

$\square$

Corollary 1.67.1 *Suppose $X = U \cup V$ where U and V are open sets of X with $U \cap V$ non-empty and path connected. If U and V are simply connected, then X is simply connected.*

Proof: As its fundamental group is generated by trivial elements. $\square$

Theorem 1.68 *If $n \geq 2$, the n-sphere S^n is simply connected.*

Proof: Let $p = (0, \cdots, 0, 1)$, $q = (0, \cdots, 0, -1)$ in S^n. Let $U = S^n - p$ and $V = S^n - q$. Then by the stereographical projection $f : (S^n - p) \to \mathbb{R}^n$, $f(x) = f(x_1, \cdots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \cdots, x_n$ is a homeomorphism with inverse givenby $g : \mathbb{R}^n \to (S^n - p)$, $g(y) = g(y_1, \cdots, y_n) = (t(y)y_1, \cdots, t(y)y_n)$, where $t(y) = \frac{2}{1 + \|y\|^2}$. Thus U and V are homeomorphic to $\mathbb{R}^n$, simply connected. Now $U \cap V = S^n - p - q$ is nonempty, homeomorphic to $\mathbb{R}^n - 0$ under stereographic projection, so $U \cap V$ is path connected. $\square$

1.9 Fundamental Groups of Some Surfaces

Theorem 1.69 *$\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic with $\pi_1(X, x_0) \times \pi_1(Y, y_0)$. More precisely, let $p : X \times Y \to X$, $q : X \times Y \to Y$ be the projection map. The homomorphisms $p_* : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0)$, $q_* : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(Y, y_0)$ induce the homomorphism $\Phi : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$, given by*

$$[f] \longmapsto \Phi([f]) = p_*([f]) \times q_*([f]) = [p \circ f] \times [q \circ f].$$

Then Φ is an isomorphism.

Proof: Φ is surjective: Let $g : I \to X$ be a loop in X based at x_0, $h : I \to Y$ be a loop in Y based at y_0. Then $f : I \to X \times Y$ given by $f(s) = g(s) \times h(s)$, then f is a loop in $X \times Y$ based at $x_0 \times y_0$ and $\Phi([f]) = [p \circ f] \times [q \circ f]$ is $[g] \times [h]$ by construction.

Φ is injective: suppose $f : I \to X \times Y$ is a loop in $X \times Y$ based at $x_0 \times y_0$ such that $\Phi([f]) = [p \circ f] \times [q \circ f]$ is the identity element of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$. Then there exists path homotopies G in X from $p \circ f$ to e_{x_0} and H in Y from $q \circ f$ to e_{y_0}. Then $F : I \times I \to X \times Y$ given by $F(s,t) = G(s,t) \times H(s,t)$ is a path homotopy in $X \times Y$ from f to the constant loop based at $x_0 \times y_0$. So $[f]$ is the identity element of $\pi_1(X \times Y, x_0 \times y_0)$. $\square$

Corollary 1.69.1 *The fundamental group of the torus $T = S^1 \times S^1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Or more generally,*

$$\pi_1(\underbrace{S^1 \times \cdots \times S^1}_{n\ times}, p_0) \cong \mathbb{Z}^{\oplus n}.$$

Definition 1.70 (Real Projective n-space) *The **real projective** n**-space** P^n is the quotient space obtained from S^n by identifying each point $x \in S^n$ with its antipode $-x \in S^n$:*

$$P^n = S^n/\{\pm 1\} = (R^{n+1} - 0)/\mathbb{R}^\times.$$

Definition 1.71 (Topological n-manifold) *A **toplogical** n**-manifold** is a topological space X such that*

- *X is locally homeomorphic to $\mathbb{R}^n$, i.e., for any $x \in X$, there exists an open neighbourhood $U \subseteq X$ of x such that U is homeomorphic to $\mathbb{R}^n$ (or $B_{\mathbb{R}^n}(1)$).*
- *X is Hausdorff.*
- *X is second-countable.*

*Then n is called the **dimension** of the manifold.*

*A **surface** is a connected topological 2-manifold.*

Theorem 1.72 *P^n is a compact topological n-manifold. The quotient map $p : S^n \to P^n$ is a 2-fold covering map.*

Corollary 1.72.1 *For $n \geq 2$, $\pi_1(P^n, y)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In particular, $\pi_1(P^2, y) \cong \mathbb{Z}/2\mathbb{Z}$. For $n = 1$, then $\pi_1(P^1, y) = \mathbb{Z}$.*

Lemma 1.73 *The fundamental group of the figure eight is not abelian.*

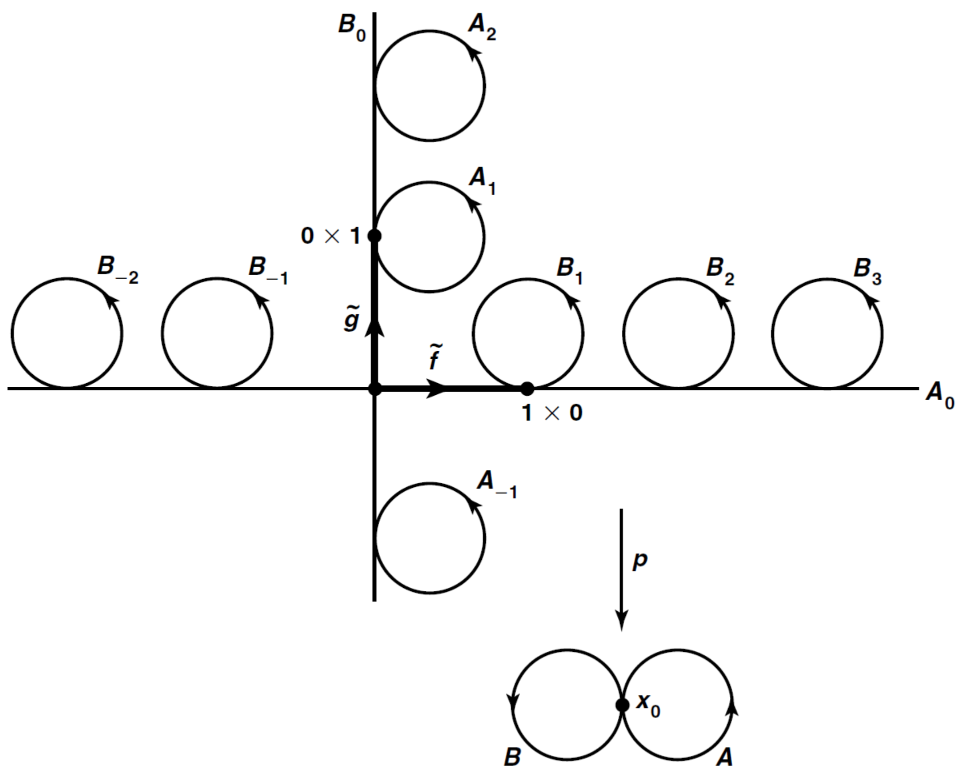
Proof: Let X be the union of two circles A and B in $\mathbb{R}^2$ whose intersection consists of the single point x_0. We describe a certain covering space E of X.

The space E is the subspace of the plane consisting of the x-axis and the y-axis, along with tiny circles tangent to these axes, one circle tangent to the x-axis at each nonzero integer point and one circle tangent to the y-axis at each nonzero integer point.

The projection map $p : E \to X$ wraps the x-axis around the circle A and wraps the y-axis around the other circle B; in each case the integer points are mapped by p into the base point x_0. Each circle tangent to an integer point on the x-axis is mapped homeomorphically by p onto B, while each circle tangent to an integer point on the y-axis is mapped homeomorphically onto A; in each case the point of tangency is mapped onto the point x_0. We leave it to you to check mentally that the map p is indeed a covering map.

We could write this description down in equations if we wished, but the informal description seems to us easier to follow.

Now let $\tilde{f} : I \to E$ be the path $\tilde{f}(s) = s \times 0$, going along the x-axis from the origin to the point 1×0. Let $\tilde{g} : I \to E$ be the path $\tilde{g}(s) = 0 \times s$, going along the y-axis from the origin to the point 0×1. Let $f = p \circ \tilde{f}$ and $g = p \circ \tilde{g}$; then f and g are loops in the figure eight based at x_0, going around the circles A and B, respectively.



We assert that $f * g$ and $g * f$ are not path homotopic, so that the fundamental group of the figure eight is not abelian.

To prove this assertion, let us lift each of these to a path in E beginning at the origin. The path $f * g$ lifts to a path that goes along the x-axis from the origin to 1×0 and then goes once around the circle tangent to the x-axis at 1×0. On the other hand, the path $g * f$ lifts to a path in E that goes along the y-axis from the origin to 0×1, and then goes once around the circle tangent to the y-axis at 0×1. Since the lifted paths do not end at the same point, $f * g$ and $g * f$ cannot be path homotopic.
□

Definition 1.74 (Double Torus) *The* ***double torus*** $T\#T$ *is the surface obtained by taking two copies of torus deleting a small open disc from each of them and pasting the remaining pieces together along their edges.*

Theorem 1.75 *The Fundamental group of the double torus is non-abelian.*

Proof: The figure eight X is a retract of $T\#T$. $\square$

1.10 The Borsuk-Ulam Theorem

Definition 1.76 *A map $h : S^n \to S^n$ between spheres is* ***antipode-preserving*** *(or odd) if $h(-x) = -h(x)$ for all $x \in S^n$.*

Theorem 1.77 *For any $n \in \mathbb{N}$, if $h : S^n \to S^n$ is continuous and antipode-preserving, then h is of odd degree and hence not nulhomotopic.*

Theorem 1.78 *If $h : S^1 \to S^1$ is continuous and antipode-preserving, then $h_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is nontrivial, and hence h is not nulhomotopic.*

Proof: Regard S^1 as a subgroup of $\mathbb{C}^\times$.
Step 0: we may assume $h(1) = 1$,

- Let $\rho : S^1 \to S^1$ be any rotation of S^1 sending $h(1) \in S^1$ to $1 \in S^1$;
- ρ is antipode-preserving, hence $\rho \circ h$ is also antipode-preserving, and $\rho \circ h$ fixes $1 \in S^1$.
- if H is a homotopy between h and constant map, then $\rho \circ H$ is a homotopy between $\rho \circ h$ and a constant map. Hence $\rho \circ H$ not nulhomotopic implies h not nulhomotpic.

Step 1: Let $q : S^1 \to S^1$ be the "squaring map", $q(z) = z^2$, then

- q is a covering map (2-fold).
- q is a quotient map, with $q^{-1} = \{\pm\sqrt{z}\}$ (q identifies $S^1/\{\pm 1\} = \mathbb{RP}^1$ with S^1).
- Since h is continuous and antipode-preserving, it induces a continuous map $k : S^1 \to S^1$ such that $k \circ q = q \circ h$.

Step 2: $k_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is nontrivial.

- If $\tilde{f}$ is any path in S^1 from 1 to -1, then $f = q \circ \tilde{f}$ is a loop in S^1 based at 1.
- $h \circ \tilde{f}$ is also a path in S^1 from 1 to -1, because $h(1) = 1$ and h is antipode-preserving.
- $[f]$ is a nontrivial element of $\pi_1(S^1, 1)$ because $\tilde{f}$ is a lifting of f over the covering map q that begins at 1 but does not end at 1.
- Thus $k_*[f] = [k \circ (q \circ \tilde{f})] = [q \circ (h \circ \tilde{f})]$ is a nontrivial element of $\pi_1(S^1, 1)$ because $h \circ \tilde{f}$ is a lifting of $q \circ (h \circ \tilde{f})$ over the covering map q that beings at 1 but does not end at 1.

Step 3: $h_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is nontrivial.

- By functorality, the square

$$\begin{tikzcd}
\pi_1(S^1,1) \arrow[r, "h_*"] \arrow[d, "q_*"'] & \pi_1(S^1,1) \arrow[d, "q_*"] \\
\pi_1(S^1,1) \arrow[r, "k_*"'] & \pi_1(S^1,1)
\end{tikzcd}$$

  commutes

- k_* is injective, it is a nontrivial homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ (as k_* is a nontrivial homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$);
- q_* is injective because q is a covering map.
- Therefore h_* is injective, hence nontrivial.

$\square$

Theorem 1.79 *For any $m, n \in \mathbb{N}$ with $m > n > 0$, there does not exist a continuous and antipode-preserving map $g : S^m \to S^n$ from m-sphere to n-sphere.*

Theorem 1.80 *There does not exist a continuous and antipode-preserving map $g : S^2 \to S^1$ from* 2*-sphere to* 1*-sphere.*

Proof: Suppose $g : S^2 \to S^1$ is a continuous and antipode-preserving map. Let $j : S^1 \to S^2$ be the inclusion map of the equator, also continuous and antipode-preserving. Then $h = g \circ j$ is a continuous and antipode-preserving map $S^1 \to S^1$. By Theorem 1.78, h is not nulhomotopic. Let E be the upper hemisphere of S^2, then $g|_E$ is a continuous extension of h to E which is isomorphic to the ball B^2, so h is nulhomotopic (By Lemma 1.56), which is a contradiction. $\square$

Theorem 1.81 (Borsuk-Ulam Theorem) *For any $n \in \mathbb{N}_{\geq 1}$ and any continuous map $f : S^{n+1} \to \mathbb{R}^{n+1}$ there exists $x \in S^n$ such that $f(x) = f(-x)$ in $\mathbb{R}^{n+1}$.*

Corollary 1.81.1 *Any continuous map $f : S^{n+1} \to \mathbb{R}^{n+1}$ is not injective.*

Theorem 1.82 (Borsuk-Ulam Theorem for S^2) *For any continuous map $f : S^2 \to \mathbb{R}^2$, there exists $x \in S^n$ such that $f(x) = f(-x)$ in $\mathbb{R}^2$.*

Proof: Suppose $f : S^2 \to \mathbb{R}^2$ is a continuous map such that $f(x) \neq f(-x)$ for all $x \in S^2$. Then $g : S^2 \to S^1$ given by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

is a continuous and antipode-preserving map which contradicts Theorem 1.80. $\square$

Theorem 1.83 (Lusternik-Schnirelmann Theorem) *Let $n \geq 1$, let S^n be covered by $n+1$ many closed subsets, then at least one of them contains a pair of antipodal points.*

Theorem 1.84 *Suppose $A_0, A_1, \cdots, A_n \subseteq S^n$ are $n+1$ many closed subsets and $A_0 \cup A_1 \cup \cdots \cup A_n = S^n$. Consider the map $f : S^n \to \mathbb{R}^n$ given by*

$$f(x) = (d(x_1, A_1), \cdots, d(x, A_n)).$$

This map is continuous and $\mathbb{R}$-valued function. And $d(x, A_i) = 0$ iff $x \in A_i$ as A_i is closed. By Borsuk-Ulam Theorem, $\exists \pm x \in S^n$ such that $f(x) = f(-x)$ in $\mathbb{R}^n$. I.e., $\forall i \in \{1, \cdots, n\}$, $d(x, A_i) = d(-x, A_i)$. If $\exists i \in \{1, \cdots, n\}$, such that $d(x, A_i) = d(-x, A_i) = 0$ then $x, -x \in A_i$. Suppose not, then $x, -x \in A_0$.

Theorem 1.85 (Bisection Theorem in $\mathbb{R}^n$) *For any $n \in \mathbb{N}_{>0}$ and n many bounded Jordan-measurable sets (boundary has Lebesgue Measure 0) in $\mathbb{R}^n$, there exists a hyper-plane (of dimension $n-1$) in $\mathbb{R}^n$ that bisects them all.*

Theorem 1.86 (Bisection Theorem in $\mathbb{R}^2$) *For any two bounded polygonal regions in $\mathbb{R}^2$, there exists a line in $\mathbb{R}^2$ that bisects each of them (with 2-dimensional area).*

Proof: We take two bounded polygonal regions A_1 and A_2 in the plane $\mathbb{R}^2 \times 1$ in $\mathbb{R}^3$, and show there is a line L in this plane that bisects each of them.

Given a point u of S^2, let us consider the plane P in $\mathbb{R}^3$ passing through the origin that has u as its unit normal vector . This plane divides $\mathbb{R}^3$ into two half-planes; let $f_i(u)$ equal to the area of that portion of A_i that lies on the same side of P as does the vector u.

If u is the unit vector $\mathbf{k}$, then $f_i(u)$ is the area of A_i; and if $u = -k$, then $f_i(u) = 0$. Otherwise, the plane P intersects the plane $\mathbb{R}^2 \times 1$ in a line L that splits $\mathbb{R}^2 \times 1$ into two half-planes, and $f_i(u)$ is the area of that part of A_i that lies on one side of this line. Replacing u by $-u$ gives us the same plane P, but the other half-space, so that $f_i(-u)$ is the area of that part of A_i that lies on the other side of P. It follows that

$$f_i(u) + f_i(-u) = \text{Area } A_i.$$

Now consider the map $F : S^2 \to \mathbb{R}^2$ given by $F(u) = (f_1(u), f_2(u))$. The Borsuk-Ulam theorem gives us a point u of S^2 for which $F(u) = F(-u)$. Then $f_i(u) = f_i(-u)$ for $i = 1, 2$, that $f_i(u) = \frac{1}{2}\text{Area } A_i$, as desired. $\square$

2 Covering Spaces and Fundamental Group

Throughout this section, unless otherwise stated, all topological space are summed to be path connected and locally path connected.

2.1 Equivalence of Covering Spaces

Definition 2.1 (Equivalence Of Covering Maps) *Let $p : E \to B$ and $p' : E' \to B$ be covering spaces of B. An isomorphism of B-spaces $h : E \to E'$ is called an* ***equivalence of covering maps****. If there exists an equivalence, then p, p' or E, E' are said to be* ***equivalent*** *as covering spaces of B.*

Lemma 2.2 (The General Lifting Lemma) *Let $p : E \to B$ be a covering map, and let $b_0 \in B$. Let $e_0 \in p^{-1}(b_0)$ be in the fiber over B_0. Let Y be path connected and locally path connected, $f : Y \to B$ be a continuous map and let $y_0 \in Y$ be such that $f(y_0) = b_0$. Then there exists a lifting $\tilde{f} : Y \to E$ of f with $\tilde{f}(y_0) = e_0$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0))$ as subgroups of $\pi_1(B, b_0)$. If such lifting $\tilde{f}$ exists, it is unique.*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

Proof: Uniqueness: Suppose $\tilde{f} : Y \to E$ and $\tilde{f}' : Y \to E$ are liftings of f to E. Let $\Delta(\tilde{f}, \tilde{f}') = \{y \in Y : \tilde{f}(y) = \tilde{f}'(y)\}$ be their equality locus. Then as Y is locally connected, then $\Delta(\tilde{f}, \tilde{f}')$ is both open and closed. Since Y is path connected, and $\Delta(\tilde{f}, \tilde{f}')$ is non-empty, we conclude that $\tilde{f} = \tilde{f}'$.

$\Rightarrow$: Suppose there exists a lifting $\tilde{f} : Y \to E$ of f with $\tilde{f}(y_0) = e_0$. Then

$$f_*(\pi_1(Y, y_0)) = (p \circ \tilde{f})_*(\pi_1(Y, y_0)) = p_*(\tilde{f}_*(\pi_1(Y, y_0)))$$

which is a subset of $p_*(\pi_1(E, e_0))$.

$\Leftarrow$: suppose $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0))$. Define the map $\tilde{f} : Y \to E$ as follows:

- For any $y_1 \in Y$, choose a path α in Y from y_0 to y_1 (possible since Y is path connected).
- Apply the unique path lifting property to lift the path $f \circ \alpha$ in B to the path γ in E beginning at $e_0 \in E$.
- Set $\tilde{f}(y_1) := \gamma(1) \in E$ the end point of γ.

We will check that the map $\tilde{f} : Y \to E$ is well-defined and continuous. Then for any $y_1 \in Y$ we have

$$(p \circ \tilde{f})(y_1) = p(\gamma(1)) = (f \circ \alpha)(1) = f(y_1)$$

so $\tilde{f}$ is a lifting of f with $\tilde{f}(y_0) = e_0$.

Well-definedness: suppose α and β are paths in Y from y_0 to y_1. The reverse path $\bar{\beta}$ in Y is from y_1 to y_0. So $a * \bar{\beta}$ is a loop in Y at y_0 and $f \circ (\alpha * \bar{\beta})$ is a loop in B at b_0. Apply the unique path lifting property to lift the path $f \circ \alpha$ and $f \circ \bar{\beta}$ in B to the path γ and δ in E beginning at $e_0 \in E$ and $\gamma(1) \in E$ respectively. Then $\gamma * \delta$ is a path in E beginning at $e_0 \in E$, lifting the loop $f \circ (\alpha * \bar{\beta})$ in B starting at b_0 and ending in $p^{-1}(b_0)$. Since $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0))$ by hypothesis, it follows that $[f \circ (\alpha * \bar{\beta})] \in \pi_1(B, b_0)$ belongs to $p_*(\pi_1(E, e_0))$. So by theorem 1.49, the lifting $\gamma * \delta$ of $f \circ (\alpha * \bar{\beta})$ is a loop in E at e_0. Hence the unique lifting of $f \circ \beta$ to a path in E beginning at $e_0 \in E$ must be the reverse path $\bar{\delta}$, and so must end at $\gamma(1)$ as well. Hence $\tilde{f}(\gamma_1) := \gamma(1) \in E$ is well-defined, independent of the choice of path α.

Continuity: let N be an open neighborhood of $\tilde{f}(y_1)$ in E. WTS there exists an open neighbourhood W of y_1 in Y such that $\tilde{f}(W) \subseteq N$.

- Choose a path connected open neighbourhood U of $f(y_1)$ in B which is evenly covered by p (possible since B is locally path connected).
- Let $V_0 \subseteq p^{-1}(U)$ denote the unique slice such that $\tilde{f}(y_1) \in V_0$. Thus the restriction $p_0 : V_0 \to U$ of p is a homeomorphism. We may shrink U and V_0 if necessary and assume $V_0 \subseteq N$.
- Since f is continuous, $W = f^{-1}(U)$ is an open neighborhood of y_1 in Y such that $f(W) \subseteq U$. We may shrink W if necessary and assume W is path-connected.
- Let γ be the unique lifting in E beginning at $e_0 \in E$ of the path $f \circ \alpha$ in B.
- Given $y \in W$, choose a path β in W from y_1 to y (possible since W is path connected).
- Then $f \circ \beta$ is a path in $U \subseteq B$ from $f(y_1)$ to $f(y)$ so $\delta = p_0^{-1} \circ f \circ \beta$ is a path in $V_0 \subseteq E$ beginning at $\tilde{f}(y_1) = \gamma(1) \in E$ lifting the path $f \circ \beta$ in B.
- Hence $\gamma * \delta$ must be the unique lifting in E beginning at $e_0 \in E$ of the path $f \circ (\alpha * \beta)$ in B from b_0 to $f(y)$.
- Since $\tilde{f}(y)$ is well-defined, it follows that $\tilde{f}(y) = (y * \delta)(1) = \delta(1)$ lies in $V_0 \subseteq N$
- Hence $\tilde{f}(W) \subseteq N$.

□

Corollary 2.2.1 *For $n > 1$, every continuous map $f : S^n \to S^1$ is nulhomotopic and every continuous map $g : P^n \to S^1$ is nulhomotopic.*

Proof: Let the base point be implicit, as all spaces under consideration is path connected. Since for $n > 1$, $\pi_1(S^n) = 1$, $\pi_1(P^n) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(S^1) = \mathbb{Z}$. Then as f_* and g_* are homomorphisms, it can only be the trivial homomorphism. Hence there is a lifting of S^n and P^n to the covering space $\mathbb{R}$ which is contractible, and hence would induce a nulhomotopy. □

Theorem 2.3 *Let $p : E \to B$ and $p' : E' \to B$ covering spaces of B and let $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$, $e_0' \in (p')^{-1}(b_0)$ be in the fiber over b_0. Then there exists an equivalence $h : E \to E'$ of covering spaces with $h(e_0) = e_0'$ if and only if*

$$p_*(\pi_1(E, e_0)) \quad \text{and} \quad p'_*(\pi_1(E', e_0'))$$

are equal as subgroups of $\pi_1(B, b_0)$. If such an equivalence h exists, it is unique.

Proof: Uniqueness of h follows from the fact that it is a lifting of $p : E \to B$ over $p' : E' \to B$.

$\Rightarrow$: Suppose there exists an equivalence $h : E \to E'$ of covering spaces with $h(e_0) = e_0'$. Then h is a homeomorphism and so $h_* : \pi_1(E, e_0) \to \pi_1(E', e_0')$ is an isomorphism. Hence

$$p_*(\pi_1(E_0)) = (p' \circ h)_*(\pi_1(E, e_0)) = p'_*(h_*(\pi_1(E, e_0))) = p'_*(\pi_1(E', e_0')).$$

$\Leftarrow$: Suppose $p_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e_0'))$ in $\pi_1(B, b_0)$.

- Apply the existence claim of the general lifting Lemma to the covering map $p' : E' \to B$ and the continuous map $p : E \to B$. We get he existence of a continuous map $h : E \to E'$ with $h(e_0) = e_0'$ and $p' \circ h = p$.
- Apply the existence claim of the general lifting Lemma to the covering map $p : E \to B$ and the continuous map $p' : E' \to B$, we get the existence of a continuous map $k : E' \to E$ with $k(e_0') = e_0$ and $p \circ k = p'$.
- Both $k \circ h : E \to E$ and $id_E : E \to E$ satisfies $p \circ - = p$ and $-(e_0) = e_0$. Apply the uniqueness claim of the general lifting Lemma to the covering map $p : E \to B$ and the continuous map $p : E \to B$, we get $k \circ h = id_E$.
- Similarly, we get $h \circ k = id_{E'}$.

□

Lemma 2.4 *Let $p : E \to B$ be a covering space of B, and let $b_0 \in B$ let $e_0 \in p^{-1}(b_0)$ be in the fiber over b_0. Then*

1. *For any point $e_1 \in p^{-1}(b_0)$ in the fiber over b_0 and any path γ in E from e_0 to e_1, setting $\alpha = p \circ \gamma$ loop in B at b_0, we have*
$$[\alpha] * p_*(\pi_1(E, e_1))_*[\alpha]^{-1} = p_*(\pi_1(E, e_0))$$
as subgroups of $\pi_1(B, b_0)$.

2. *For any subgroup $H \subseteq \pi_1(B, b_0)$ conjugate to $p_*(\pi_1(E, e_0))$ in $\pi_1(B, b_0)$, there exists a point $e_1 \in p^{-1}(b_0)$ in the fiber over b_0 such that $H = p_*(\pi_1(E, e_1))$.*

Proof:

1. An element $[h] \in p_*(\pi_1(E, e_1))$ is represented by a loop in h in B at b_0 of the form $h = p \circ \beta$ where β is a loop in E at e_1. Then
$$[\alpha] * [h] * [\alpha]^{-1} = [p \circ \gamma] * [p \circ \beta] * [p \circ \bar{\gamma}] = p_*[\gamma * \beta * \bar{\gamma}]$$
lies in $p_*(\pi_1(E, e_0))$ since $\gamma * \beta * \bar{\gamma}$ is a loop in E at e_0. Thus $[\alpha] * p_*(\pi_1(E, e_1)) * [\alpha]^{-1} \subseteq p_*(\pi_1(E, e_0))$. Now $\bar{\gamma}$ is a path in E from e_1 to e_0 and $[\alpha]^{-1} = [p \circ \bar{\gamma}]$. So $[\alpha]^{-1} * p_*(\pi_1(E, e_0)) * [\alpha] \subseteq p_*(\pi_1(E, e_1))$ by the preceding result. Hence $[\alpha] * p_*(\pi_1(E, e_1)) * [\alpha]^{-1} = p_*(\pi_1(E, e_0))$.

2. Since H is conjugate to $p_*(\pi_1(E, e_0))$ in $\pi_1(B, b_0)$, there exists $[\alpha] \in \pi_1(B, b_0)$ such that $[\alpha] * H * [\alpha]^{-1} = p_*(\pi_1(E, e_0))$ and $[\alpha]$ is represented by a loop α in B at b_0. Let γ be the unique lifting of α to a path in E beginning at $e_0 \in E$. Then $e_1 = \gamma(1) \in p^{-1}(b_0)$ lies in the fiber over b_0. So $[\alpha] * p_*(\pi_1(E, e_1)) * [\alpha]^{-1} = p_*(\pi_1(E, e_0)) = [\alpha] * H * [\alpha]^{-1}$. So $H = p_*(\pi_1(E, e_1))$.

□

Theorem 2.5 *Let $p : E \to B$ and $p' : E' \to B$ covering spaces of B and let $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$, $e'_0 \in (p')^{-1}(b_0)$ be in the fiber over b_0. Then there exists an equivalence $h : E \to E'$ of covering spaces if and only if*

$$p_*(\pi_1(E, e_0)) \quad \textit{and} \quad p'_*(\pi_1(E', e'_0))$$

are conjugate as subgroups of $\pi_1(B, b_0)$.

Proof: $\Rightarrow$: Suppose there exists an equivalence $h : E \to E'$ of covering spaces. Then $e'_1 = h(e_0) \in (p')^{-1}(b_0)$ lies in the fiber over b_0. By Theorem 2.3, we have $p_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e'_1))$ and by Lemma 2.4, this is conjugate to $p'_*(\pi_1(E', e'_0))$ in $\pi_1(B, b_0)$ (Since E is path connected).

$\Leftarrow$: Suppose $p_*(\pi_1(E, e_0))$ and $p'_*(\pi_1(E', e'_0))$ are conjugate in $\pi_1(B, b_0)$. By Lemma 2.3, there exists $e'_1 \in (p')^{-1}(b_0)$ such that $p_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e'_1))$. Then by Theorem 2.3, there exists an equivalence $h : E \to E'$ of covering spaces with $h(e_0) = e'_1$. □

Corollary 2.5.1 *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Then $H = p_*(\pi_1(E, e_0))$ is a normal subgroup of $\pi_1(B, b_0)$ if and only if for every pair of points e_1, e_2 of $p^{-1}(b_0)$, there is an equivalence $h : E \to E$ with $h(e_1) = e_2$.*

Corollary 2.5.2 *The map from the set of equivalence classes of covering spaces of B to the set of conjugacy classes of subgroups of $\pi_1(B, b_0)$, where $[p : E \to B] \longmapsto [p_*(\pi_1(E, e_0))$ for any choice of $e_0 \in p^{-1}(b_0)]$ is well-defined and injective (but not necessarily surjective).*

Corollary 2.5.3 *The set of equivalence classes of covering spaces of S^1 is either the equivalence class of the exponential map or the equivalence class of the n^{th} power map, where $n \in \mathbb{Z}_{\neq 0}$.*

Note that if $\pi_1(B, b_0)$ is abelian, then the set of conjugacy classes of subgroups of $\pi_1(B, b_0)$ is equal to the set of subgroups of $\pi_1(B, b_0)$.

2.2 The Universal Covering Space

Definition 2.6 (Universal Covering Space) *A **universal covering space** of B is a covering space $p : E \to B$ such that E is simply connected.*

Remark 2.6.1 *Not all base space B has a universal covering space.*

Theorem 2.7 (Universal Property of A Universal Covering Space) *Let $p : E \to B$ be a universal covering space of B. Let $b_0 \in B$ and let $e_0 \in p^{-1}(b_0)$ in the fiber over b_0. Then for any covering space $p' : E' \to B$ and any $e'_0 \in (p')^{-1}(b_0)$ there exists a unique B-space map $h : E \to E'$ such that $h(e_0) = e'_0$.*

Remark 2.7.1 *As we will see next, that the map h is in fact a covering map.*

Proof: Since E is simply connected, $p_*(\pi_1(E, e_0)) = 1$ is always a subset of $p'_*(\pi_1(E', e'_0))$ as subgroups of $\pi_1(B, b_0)$. Hence by the general lifting lemma applied to the covering map $p' : E' \to B$ and the continuous map $p : E \to B$ gives the universal property. $\square$

Lemma 2.8 *Let $p : E \to B$ and $p' : E' \to B$ be a covering map to B. Then any B-space map $h : E \to E'$ is also a covering map*

Proof: By hypothesis, we have $p' \circ h = p$, i.e., the diagram

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

commutes. We have to show that for every point of E', there exists an open neighorhood which is evenly covered by h.

We claim that h is surjective. Fix $e_0 \in E$, let $e'_0 := h(e_0) \in E'$. For any $e'_1 \in E'$, choose a path β in E' from e'_0 to e'_1 (Possible since E is path connected). Then $p' \circ \beta$ is a path in B from $p'(e'_0) = p(e_0)$ to $p'(e'_1)$. Apply the Unique path lifting property to lift the path $p' \circ \beta$ in B to a path γ in E beginning at $e_0 \in E$. Then $h \circ \gamma$ is a path in E' beginning at e'_0, and $p' \circ h \circ \gamma = p \circ \gamma = p' \circ \beta$. So $h \circ \gamma$ and β are both liftings in E' beginning at e'_0 of the path $p' \circ \beta$ in B. By the uniqueness of path liftings, we have $\beta = h \circ \gamma$, so $e'_1 = \beta(1) = h(\gamma(1))$ lies in $h(E)$.

Next we show that E' is locally evenly covered by h. For any $e'_0 \in E'$, let $b_0 := p'(e'_0) \in B$. Since p and p' are covering maps to B, there exists an open neighborhood U of b_0 in B which is evenly covered by both p and p'. We may shrink U if necessary and assume U is path-connected. Every slice of $p^{-1}(U)$, $(p')^{-1}(U)$ is homeomorphic to U and hence path-connected. For any slice $W \subseteq p^{-1}(U)$, we have $h(W) \subseteq (p')^{-1}(U)$, so h must map W into a single slice of $(p')^{-1}(U)$. Let $V \subseteq (p')^{-1}(U)$ denote the unique slice such that $e'_0 \in V$. Then $h^{-1}(V)$ is the nonempty disjoint union of those slices of $p^{-1}(U)$ mapped by h into V (nonempty because h is surjective). If $W \subseteq h^{-1}(V)$ is one such slice, then

$$\begin{tikzcd} W \arrow[rr, "h|_W"] \arrow[dr, "p|_W"'] & & V \arrow[dl, "p'|_V"] \\ & U & \end{tikzcd}$$

commutes and since $p|_W$ and $p'|_V$ are homeomorphisms, so must $h|_W$ also be a homeomorphism. $\square$

Lemma 2.9 *Let B be path connected and locally path connected. Let $p : E \to B$ be a covering map (where E does not have to be path connected). If E_0 is a path component of E, then the map $p_0 : E_0 \to B$ obtained by restricting p is a covering map.*

Proof: Clear. $\square$

Lemma 2.10 *Let $p : E \to B$ and $p' : E' \to B$ be B-spaces, and $h : E \to E'$ is a B-space map. If p and h are covering maps, then p' is a covering map.*

Proof: Suppose p and h are covering maps, because $p = p' \circ h$ is surjective, then p' is also surjective.

Given $b_0 \in B$, let U be a path-connected neighbourhood of b_0 that is envenly covered by p. We show that U is also evenly covered by p'. Let $\{V_\beta\}$ be the collection of path components of $(p')^{-1}(U)$; these sets are disjoint and open in E'. We show for each β, the map p' carries V_β homeomorphically on U.

Let $\{U_\alpha\}$ be the collection of slices of $p^{-1}(U)$; they are disjoint, open and path connected, so they are path components of $p^{-1}(U)$. Now h maps each U_α into the set $(p')^{-1}(U)$; because U_α is connected, it must be mapped by h into one of the sets V_β. Therefore $h^{-1}(V_\beta)$ equals the union of a subcollection of the collection $\{U_\alpha\}$. If U_{α_0} is any one of the path components of $h^{-1}(V_\beta)$ then the map $h_0 : U_{\alpha_0} \to V_\beta$ obtained by restricting q is a covering map. In particular h_0 is surjective. Hence h_0 is a homeomorphism, being continuous, open, and injective as well. Consider the maps

$$\begin{tikzcd}
U_{\alpha_0} \arrow[rr, "h_0"] \arrow[dr, "p_0"'] & & V_\beta \arrow[dl, "p'_0"] \\
& U &
\end{tikzcd}$$

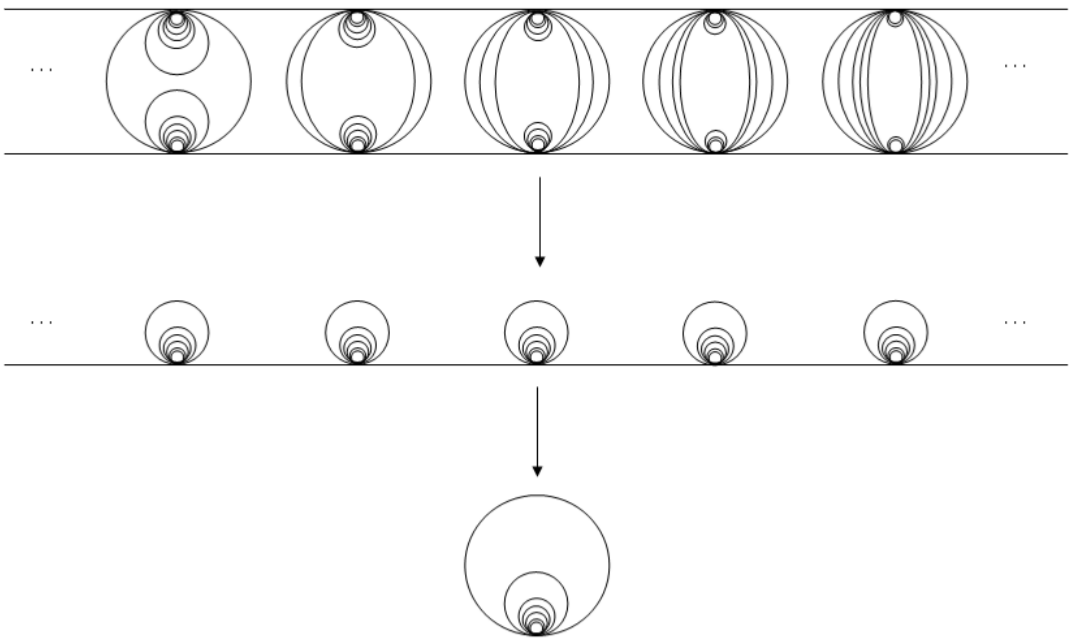
obtained by restricting p, p' and h. Because p_0, h_0 are homeomorphisms, so is p'_0. □

Lemma 2.11 *Let $p : E \to B$ and $p' : E' \to B$ be B-spaces, and $h : E \to E'$ is a B-space map. Suppose p' and h are covering maps, and B admits a universal covering space, then p is a covering map.*

Proof: Let $r : \tilde{B} \to B$ be a universal covering. Since p' is a covering map for B, then by Theorem 2.7, there is a B-space map $\tilde{r} : \tilde{B} \to E'$, and this map is a covering map by Lemma 2.8. So $\tilde{r} : \tilde{B} \to E'$ is a universal covering map, then again by Theorem 2.7, there is a covering map $\tilde{h} : \tilde{B} \to E$. Lastly, note that $p' \circ h \circ \tilde{h} = r$, then as r and $\tilde{h}$ are covering maps, $p' \circ h = p$ is also a covering map. □

Example: (Composition of covering maps may not be a covering map)
Let H be the Infinite Earing (Hawaiian Earing). Then for any $n \geq 1$, find a covering map $p_n : E_n \to H$ such that the restriction $p^{-1}(C_n) \to C_n$ is not a trivial covering map. Define $E = \bigsqcup_{n \geq 1} E_n$ and $p : E \to H$ the map induced by all the $p_n' s$, then p is not a covering map. However, there is a trivial covering map $t : T \to H$ and a (non trivial) covering map $q : E \to T$ such that $p = t \circ q$. An example can be seen below:

The top map is a 2-fold covering but the bottom map has infinite fibers. The composition isn't locally trivial and is therefore not a covering map. However, it is still a semicovering map in the sense that it is a local homeomorphism which has the unique path lifting property.



Lemma 2.12 *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. If E is simply connected, then b_0 has a neighborhood U such that the inclusion $i : U \to B$ induces the trivial homomorphism $i_* : \pi_1(U, b_0) \to \pi_1(B, b_0)$.*

Proof: Let U be a neighborhood of b_0 that is evenly covered by p; break $p^{-1}(U)$ into slices; let U_α be the slice containing e_0. Let f be a loop in U based at e_0. Because p defines a homeomorphism of U_α with U, the loop f lifts to a loop $\tilde{f}$ in U_α based at e_0. Since E is simply connected, there is a path homotopy $\tilde{F}$ between $\tilde{f}$ and a constant loop. Then $p \circ \tilde{F}$ is a path homotopy in B between f and a constant loop. □

2.3 Existence of Covering Spaces

Definition 2.13 (Semilocally Simply Connected) *A space B is* ***semilocally simply connected*** *iff for any $b \in B$, there exists an open neighborhood U of B in B such that the homomorphism $i_* : \pi_1(U, b) \to \pi_1(B, b)$ is trivial, where i is the inclusion map. That is, any loop in U at b is nulhomotopic in B.*

Remark 2.13.1 *Note if U satisfies this condition, then so does any smaller neighborhood of b_0, i.e., we can "shrink" U if necessary.*

Example:

- If B is simply connected, then it is semilocally simply connected.
- If B is locally simply connected, i.e., $\forall b_0 \in B$, $\forall$ open neighbourhood $V \subseteq B$ of b_0, $\exists$ open neighbourhood $U \subseteq B$ of b_0, such that U is simply connected, then it is semilocally connected.

- There exists semilocally simply connected space that is not locally simply connected.

Theorem 2.14 *Let B be a path connected and locally path connected space, $b_0 \in B$. Then B the following are equivalent:*

1. *B is semilocally simply connected.*
2. *There exists a universal covering space of B, i.e. for the trivial subgroup $H = 1$ of $\pi_1(B, b_0)$, there exists a covering space $p : E \to B$ and $e_0 \in p^{-1}(b_0)$ such that $H = p_*(\pi_1(E, e_0))$ as subgroups of $\pi_1(B, b_0)$ (Recall p_* is injective).*
3. *For any subgroup $H \subseteq \pi_1(B, b_0)$, there exists a covering space $p : E \to B$ and $e_0 \in p^{-1}(b_0)$ such that as subgroups of $\pi_1(B, b_0)$.*

i.e., if and only if the map from the set of equivalence classes of path connected covering spaces of B to the set of conjugacy classes of subgroups of $\pi_1(B, b_0)$, where $[p : E \to B]$ is mapped to the conjugacy classes of $p_*(\pi_1(E, e_0))$ for any choice of $e_0 \in p^{-1}(b_0)$ is also surjective.

Proof: It is clear that (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) : Let $p : E \to B$ be a universal covering space of B. Let $b_0 \in B$ and let $e_0 \in p^{-1}(b_0)$ be in the fiber over b_0. Let U be an open neighborhood of b_0 in B which is evenly covered by p. Let $V \subseteq p^{-1}(U)$ denote the unique slice such that $e_0 \in V$ so $p|_V : V \to U$ is a homeomorphism and $(p|_V)(e_0) = b_0$. Now consider any loop f in U at b_0. Then $\tilde{f} := (p|_V)^{-1} \circ f$ is a lifting in E beginning at e_0 of the loop f in U and $\tilde{f}$ is also a loop. Since E is simply connected, there exists a path homotopy $\tilde{F}$ in E between $\tilde{f}$ and a constant loop. Then $p \circ \tilde{F}$ is a path homotopy in B between f and a constant loop, i.e., $i_* : \pi_1(U, b) \to \pi_1(B, b_0)$ maps $[f]$ to 1.

(1) $\Rightarrow$ (3) : we need to show that if B is semilocally connected, then for any subgroup $H \subseteq \pi_1(B, b_0)$, there exists a covering space $p : E \to B$ and $e_0 \in p^{-1}(b_0)$, such that $H = p_*(\pi_1(E, e_0))$ as subgroups of $\pi_1(B, b_0)$.

Let $H \subseteq \pi_1(B, b_0)$ be a given subgroup.

- Step 1. Construct the set E and the map $p : E \to B$.

  Let $\mathcal{P}$ be the set of all paths α in B beginning at $b_0 \in B$. Define the relation $\sim_H$ on $\mathcal{P}$ by setting $\alpha \sim_H \beta$ iff $\alpha(1) = \beta(1)$ and $[\alpha * \bar{\beta}] \in H$, i.e., $H * [\alpha] = H * [\beta]$ as path-homotopy equivalence classes of paths in B. We can check that $\sim_H$ is an equivalence relation on $\mathcal{P}$. Denote the $\sim_H$ equivalence class of $\alpha \in \mathcal{P}$ by $\alpha^\#$. Let $E := \mathcal{P}/\sim_H = \{\alpha^\# : \alpha \in \mathcal{P}\}$ be the set of $\sim_H$ equivalence classes. Define the map $p : E \to B$ by setting for all $\alpha^\#$, $p(\alpha^\#) = \alpha(1)$. This is well-defined and independent of choice of representatives. It is clear that p is surjective as B is path connected. Also note that: (i) if $[\alpha] = [\beta]$ as path-homotopy equivalence class of paths in B, then $\alpha^\# = \beta^\#$ in E; (ii) if $\alpha^\# = \beta^\#$, then for any path δ in B beginning at $\alpha(1) = \beta(1)$, one has $(\alpha * \delta)^\# = (\beta * \delta)^\#$ in E.

- Step 2: Define the topology on E.
  For any $\alpha \in \mathcal{P}$, and any path-connected neighborhood U of $\alpha(1)$ in B, define

$$B(U, \alpha) := \{(\alpha * \delta)^\# \in E : \delta \text{ a path in } U \subseteq B \text{ beginning at } \alpha(1)\}.$$

Then $\alpha^{\#} \in B(U, \alpha)$. Define the topology of E to be that generated by$\{B(U, \alpha)\}_{\alpha \in \mathcal{P}, U}$ (We will show this is indeed a basis).
If $\beta^{\#} \in B(U, \alpha)$, then we claim that $\alpha^{\#} \in B(U, \beta)$ and $B(U, \alpha) = B(U, \beta)$. By hypothesis there exists a path δ in U such that $\beta^{\#} = (\alpha * \delta)^{\#}$, then $\alpha^{\#} = ((\alpha * \delta) * \bar{\delta})^{\#}$. A general element of $B(U, \beta)$ is of the form $(\beta * \gamma)^{\#}$ where γ is a path in U beginning at $\beta(1)$. Then $(\beta * \gamma)^{\#} = ((\alpha * \delta) * \gamma)^{\#} = (\alpha * (\delta * \gamma))^{\#}$ lies in $B(U, \alpha)$. Hence $B(U, \beta) \subseteq B(U, \alpha)$ and similarly, we have the other direction.
Next, note for any $\beta^{\#} \in B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$, there exists a path-connected neighborhood V of $\beta(1)$ in B such that $V \subseteq U_1 \cap U_2$ (Recall B is locally path connected). Hence one has

$$\beta^{\#} \in B(V, \beta) \subseteq B(U_1, \beta) \cap B(U_2, \beta) = B(U_1, \alpha_1) \cap B(U_2, \alpha_2).$$

Thus $\{B(U, \alpha)\}_{\alpha \in \mathcal{P}, U}$ is a basis.

- Step 3: Verify the map $p : E \to B$, $\alpha^{\#} \mapsto \alpha(1)$ is continuous and open.
  p is maps open set to open set because $p(B(U, \alpha)) = U$ for all basis open set $B(U, \alpha)$ of E (This is clear).
  p is continuous: for any $\alpha^{\#} \in E$ and any open neighborhood W of $p(\alpha^{\#})$ in B, choose a path-connected neighborhood U of $p(a^{\#}) = \alpha(1)$ in B with $U \subseteq W$. Then $B(U, \alpha)$ is an open neighborhood of $\alpha^{\#}$ in E and one has $p(B(U, \alpha)) = U \subseteq W$ so p is continuous at $\alpha^{\#}$.

- Step 4: p is a covering map.
  For any $b_1 \in B$, choose a path-connected neighborhood U of b_1 in B such that the homomorphism $\pi_1(U, b_1) \to \pi_1(B, b_1)$ induced by inclusion is trivial (B is semilocally simply connected, locally path connected). Then we claim U is evenly covered by p.
  In fact $p^{-1}(U) = \bigsqcup_{\alpha} B(U, \alpha)$, disjoint union of the $B(U, \alpha)'s$ where α runs over all paths in B from b_0 to b_1. Note the $B(U, \alpha)$'s are disjoint for if $\beta^{\#}$ belongs to $B(U, \alpha) \cap B(U, \alpha_2)$, then $B(U, \alpha_1) = B(U, \beta) = B(U, \alpha_2)$.
  Next, we show that $p|_{B(U,\alpha)} : B(U, \alpha) \to U$ is a homeomorphism. It suffices to show that $p|_B(U, \alpha)$ is bijective, as it is continuous and open. We know p maps $B(U, \alpha)$ onto U, so we show injectivity. Suppose that

$$p((\alpha * \delta_1)^{\#}) = p((\alpha * \delta_2)^{\#}),$$

  where δ_1 and δ_2 are paths in U. Then $\delta_1(1) = \delta_2(1)$. Because the homomorphism $\pi_1(U, b_1) \to \pi_1(B, b_1)$ induced by inclusion is trivial, $\delta_1 * \bar{\delta_2}$ is a path homotopic in B to the constant loop, Then $[\alpha * \delta_1] = [\delta * \delta_2]$ in B, so that $(\alpha * \delta_1)^{\#} = (\alpha * \delta_2)^{\#}$, as desired. In conclusion, $p : E \to B$ is a covering map.

- Step 5. Lifting of paths in B.
  Let $e_0 \in E$ be the $\sim_H$ equivalence class of the constant path at b_0, so $p(e_0) = b_0$ in B. Given a path α in B beginning at b_0, we calculates its lift to a path in E beginning at e_0 and show that this lift ends at $\alpha^{\#}$.
  To begin, given $c \in [0, 1]$, let $\alpha_c : I \to B$ denote th path defined by equation

$$\alpha_c(t) = \alpha(Tc) \quad \text{for } 0 \leq t \leq 1.$$

  Then α_c is the "portion" of α that runs from $\alpha(0)$ to $\alpha(c)$. In particular, α_0 is the constant path at b_0, and

α_1 is the path α itself. We define $\tilde{\alpha} : I \to E$ by the equation

$$\tilde{\alpha}(c) = (\alpha_c)^{\#}$$

and show that $\tilde{\alpha}$ is continuous. Then $\tilde{\alpha}$ is a lifting of α that begins at e_0 and ends at $(\alpha_1)^{\#} = \alpha^{\#}$.
To verify continuity of $\tilde{\alpha}$ at the point c of $[0,1]$. Let W be a basis element in E about the point $\tilde{\alpha}(c)$. Then W equals $B(U, \alpha_c)$ for some path-connected neighborhood U of $\alpha(c)$. Choose $\epsilon > 0$ so that for $|c-t| < \epsilon$, the point $\alpha(t)$ lies in U. WE show that if d is a point of $[0,1]$ with $|c-d| < \epsilon$, then $\tilde{\alpha}(d) \in W$.
Suppose $|c-d| < \epsilon$. Consider first that when $d > c$, let δ be a path from $\alpha(c)$ to $\alpha(d)$ in U. Then since $[\alpha_d] = [\alpha_c * \delta]$, we have

$$\tilde{\alpha}(d) = (\alpha_d)^{\#} = (\alpha_c * \delta)^{\#}.$$

Since δ lies in U, we have $\tilde{\alpha}(d) \in B(U, \alpha_c)$, as desired. Similarly, we can show for the case where $d < c$, hence $\tilde{\alpha}$ is continuous, so is a lifting of α.

- Step 6. E is path connected. For if $\alpha^{\#}$ is any point of E, then the lift $\tilde{\alpha}$ of the path α is a path in E from e_0 to $\alpha^{\#}$.

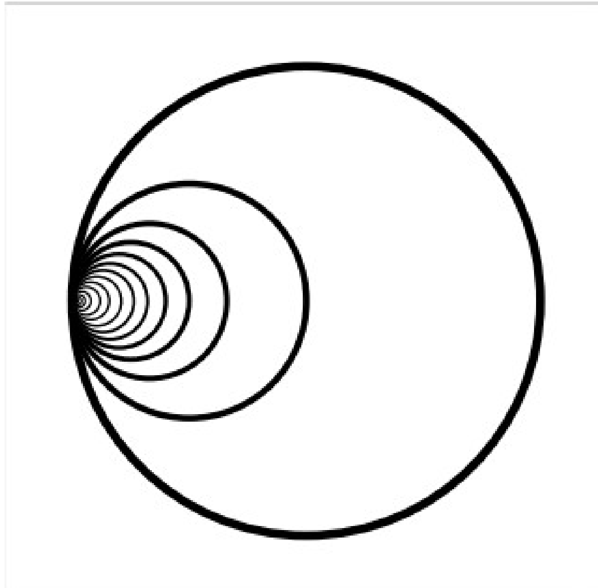
- Step 7. Verify $H = p_*(\pi_1(E, e_0))$.
  Let α be a loop in B at b_0. Let $\tilde{\alpha}$ be its lift to E beginning at $e_)$. Then $[\alpha] \in p_*(\pi_1(E, e_0))$ if and only if $\tilde{\alpha}$ is a loop in E. Now the final point of $\tilde{\alpha}$ is the point $\alpha^{\#}$, and $\alpha^{\#} = e_0$ if nad only if α is equivalent to the constant path at b_0, i.e., if and only if $[\alpha * \bar{e}_{b_0}] \in H$. This happens if and only if $[\alpha] \in H$.

$\square$

Theorem 2.15 *Let B be a path connected and locally path connected, semilocally simply connected space, and let $b_0 \in B$. Then the map from the set of equivalence classes of path connected covering spaces of B to the set of conjugacy classes of subgroups of $\pi_1(B, b_0)$, $[p : E \to B] \longmapsto [p_*(\pi_1(E, e_0))$ for any choice of $e_0 \in p^{-1}(b_0)]$ is a well-defined order reversing bijection.*

Corollary 2.15.1 *Let B be a path connected and locally path connected. Then B has a universal covering space if and only if B is locally simply connected.*

Example: (The Infinite Earing)
Let C_n be the circle of radius $1/n$ in $\mathbb{R}^2$ with center at the point $(1/n, 0)$. Let X be the union of these circles as a subspace of $\mathbb{R}^2$.



X is path connected and locally path connected , but not semilocally simply connected. So X does not have a universal covering space. More precisely, if U is any neighborhood of $b_0 = 0$, then $i_* : \pi_1(U, b_0) \to \pi_1(X, b_0)$ is not trivial.
Given n, there is a retraction $r : X \to C_n$ obtained by letting r map each circle C_i for $i \neq n$ to the point b_0. Choose n large enough that C_n lies in U. Then in the following diagram of homomorphisms induced by inclusion, j_* injective (maps $\mathbb{Z}$ to $\mathbb{Z}$); hence i_* cannot be trivial.

$$\begin{tikzcd}
\pi_1(C_n, b_0) \arrow[rr, "j_*"] \arrow[dr, "k_*"'] & & \pi_1(X, b_0) \\
& \pi_1(U, b_0) \arrow[ur, "i_*"'] &
\end{tikzcd}$$

2.4 Covering Transformation

Definition 2.16 (Covering Transformation) *Let $p : E \to B$ be a covering space of B. A* ***covering transformation of*** *p (of E over B) is an equivalence of $p : E \to B$ with itself, i.e., an isomorphism h of B-spaces:*

$$\begin{tikzcd}
E \arrow[rr, "h"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\
& B &
\end{tikzcd}$$

We denote the group of all covering transformations (automorphism) of p to be $\mathcal{C}(E, p, B)$ or $\mathrm{Aut}(E/B)$ or $\mathrm{Aut}(p)$. Let $b_0 \in B$. Then $\mathrm{Aut}(p)$ acts from the left on the fiber $p^{-1}(b_0)$ over b_0: $\mathrm{Aut}(p) \times p^{-1}(b_0) \to p^{-1}(b_0)$, $(h, e_0) \mapsto h(e_0)$, this is called the ***covering transformation action****.*

Remark 2.16.1 *If give $\mathrm{Aut}(p)$ the natural group operation (composition), then the covering transformation action is indeed an action.*

Lemma 2.17 *Let $p : E \to B$ be a covering map, let $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. The covering transformation action* $\mathrm{Aut}(p)$ *on $p^{-1}(b_0)$ is fixed-point free, i.e., the map $\Psi : \mathrm{Aut}(p) \to p^{-1}(b_0)$, $h \mapsto h(e_0)$ is injective. Let $H = p_*(\pi_1(E, e_0)) \subseteq \pi_1(B, b_0)$, let $N(H) = [g \,|\, gHg^{-1} = H]$ be the normalizer of H in $\pi_1(B, b_0)$. Then the Ψ-image of* $\mathrm{Aut}(p)$ *in $p^{-1}(b_0)$ is equal to the Φ-image (the monodromy action) of $H \setminus N(H)$ in $p^{-1}(b_0)$.*

Proof: Ψ is injective because $h \in \mathrm{Aut}(p)$ is a lifting of $p : E \to B$ over itself. So by the general lifting lemma, it is uniquely determined by $h(e_0)$.

$\subseteq$: Suppose $h \in \mathrm{Aut}(p)$ and $e_1 = h(e_0)$ in $p^{-1}(b_0)$, so by the General Lifting Lemma, one has $p_*(\pi_1(E, e_0)) = p_*(\pi_1(E, e_1))$. Choose a path γ in E from e_0 to e_1, let $\alpha := p \circ \gamma$. Then α is a loop in B at b_0, so by Lemma 2.4, we have

$$[\alpha] * p_*(\pi_1(E, e_1)) * [\alpha]^{-1} = p_*(\pi_1(E, e_0))$$

as subgroups of $\pi_1(B, b_0)$. Then $[\alpha] \in N(H)$ in $\pi_1(B, b_0)$, and so

$$\Psi(h) = h(e_0) = e_1 = \gamma(1) = \Phi(H \cdot [\alpha])$$

lies in the Φ-image of $H \setminus N(H)$.

$\supseteq$: Suppose $[a] \in N(H)$, represented by loop α in B at b_0. Let γ be the unique path lifting of α over p beginning at e_0. Let $e_1 := \gamma(1)$. Then by Lemma 2.4, we have

$$[a] * p_*(\pi_1(E, e_1)) * [\alpha]^{-1} = p_*(\pi_1(E, e_0))$$

So by the General Lifting Lemma, there exists a B-space map $h : E \to E$ with $h(e_0) = e_1$. Reversing roles, we see that $h \in \mathrm{Aut}(p)$. Then $\Phi(H \cdot [\alpha]) = \gamma(1) = e_1 = h(e_0) = \Psi(h)$ lies in the Ψ-image of $\mathrm{Aut}(p)$. □

Theorem 2.18 *Let $p : E \to B$ be a covering map, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$. Let $H = p_*(\pi_1(E, e_0)) \subseteq \pi_1(B, b_0)$. The composite map*

$$\Phi^{-1} \circ \Psi : \mathrm{Aut}(p) \xrightarrow{\Psi} (\Psi - image \,/\, \Phi - image\ in\ p^{-1}(b_0)) \xrightarrow{\Phi^{-1}} H \setminus N(H)$$

is a group homomorphism (hence isomorphism).

Proof: We know Ψ is bijective to its image, and Φ is bijective to its image (if the covering space is path connected), so $\Phi^{-1} \circ \Psi$ is well-defined and bijective. So we only need to show the map is a homomorphism.

Let $h, k \in \mathrm{Aut}(p)$, and $e_1 = h(e_0) = \Psi(h)$, $e_2 = k(e_0) = \Psi(k)$ in $p^{-1}(b_0)$. Choose paths γ, δ in E from e_0 to e_1 and e_2 respectively. Let $\alpha := p \circ \gamma$ and $\beta := p \circ \delta$, then α, β are loops in B at b_0. So $H \cdot [\alpha] = \Phi^{-1}(e_1)$ and $H \cdot [\beta] = \Phi^{-1}(e_2)$ lies in $H \setminus N(H)$ (By bijectivity).

Let $e_3 = h(e_2) = h(k(e_0))$, then we must show that $H \cdot [\alpha * \beta] = e_3$, since in order for the composite map to be a homomorphism, we need $\Phi^{-1} \circ \Psi(h \circ k)$ to map to $H \cdot [\alpha * \beta]$. The path $h \circ \delta$ in E is from $h(e_0) = e_1$ to $h(e_2) = h(k(e_0)) = e_3$ and is thus the unique path lifting of β over p beginning at e_1 (Note $p \circ h = p$). Hence $\gamma * (h \circ \delta)$ is the unique path lifting of $\alpha * \beta$ over p beginning at e_0. So

$$\Phi(H \cdot [\alpha * \beta]) = (\gamma * (h \circ \delta))(1) = e_3 = h(k(e_0)) = \psi(h \circ k).$$

□

Corollary 2.18.1 *Let $p : E \to B$ be a universal covering space of B. Then for any choice of $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, the map $\mathrm{Aut}(p) \to \pi_1(B, b_0)$, $h \mapsto [p \circ \gamma]$ for any γ in B from e_0 to $h(e_0)$ is an isomorphism.*

Proof: Since p is a universal covering map, then E is simply connected, so $H \setminus N(H) \cong \pi_1(B, b_0)$. And we can see from the proof of the theorem, that the map given in the corollary is indeed an isomorphism. □

Definition 2.19 (Galois) *A covering map $p : E \to B$ is called **regular or Galois** if E is path connected and $H = p_*(\pi_1(E, e_0))$ is a normal subgroup in $\pi_1(B, b_0)$ for any choice of $b_0 \in B$, $e_0 \in p^{-1}(b_0)$.*

Remark 2.19.1 *If $\pi_1(B, b_0)$ is abelian, then any covering map is Galois.*

Corollary 2.19.1 *The covering map is Galois if and only if one of the following holds: for any choice of $b_0 \in B$, $e_0 \in p^{-1}(b_0)$,*

1. *$N(H) = \pi_1(B, b_0)$;*
2. *the Ψ-image of $\mathrm{Aut}(p)$ in $p^{-1}(b_0)$ is $p^{-1}(b_0)$ (since the Monodromy action is transitive);*
3. *$\mathrm{Aut}(p)$ acts (simply) transitively on $p^{-1}(b_0)$;*
4. *for any $e_1, e_2 \in p^{-1}(b_0)$, there exists $h \in \mathrm{Aut}(p)$ such that $h(e_1) = e_2$.*

If this is the case, $\Phi^{-1} \circ \Psi : \mathrm{Aut}(p) \to H \setminus \pi_1(B, b_0)$ is an isomorphism.

Definition 2.20 *Let X be a set, let G be a subgroup of $\mathrm{Aut}(X) = \mathrm{Perm}(X)$. The **orbit space** X/G is the quotient set of X modulo the equivalence relation $\sim$ of G-action: $x_1 \sim x_2$ iff there exists $g \in G$ such that $g(x_1) = x_2$. We have the quotient map $\pi : X \to X/G$, $x \mapsto G \cdot x =$ the G orbit of x. If X is a topological space and G is a subgroup of $\mathrm{Aut}_{cts}(X) = \mathrm{Homeo}(X)$, then X/G is given with the quotient topology.*

Definition 2.21 *Let X be a topological space, and G be a subgroup of $\mathrm{Aut}_{cts}(X) = \mathrm{Homeo}(X)$ of homeomorphisms of X. We say that G-action on X is **fixed-point free** iff for any $x \in X$, for any $g \in G \setminus \{e\}$, one has $g(x) \neq x$, i.e., no element of G other than the identity e has a fixed point. We say that G-action on X is properly discontinuous, iff for any $x \in X$, there exists a neighborhood U of x in X such that for any $g \in G \setminus \{e\}$, one has $g(U)$ disjoint from U, (then $g_0(U)$ disjoint from $g_1(U)$ whenever $g_0 \neq g_1$).*

Remark 2.21.1 *It is clear that properly discontinuous implies fixed-point free. When G is finite and X is Hausdorff, then the converse holds.*

Example:

- $\mathbb{Z}$ acting on $\mathbb{R}$ by translation is properly discontinuous.
- $\mathbb{Q}$ acting on $\mathbb{R}$ by translation is fixed-point free but not properly discontinuous.

Lemma 2.22 *Let $p : E \to B$ be a covering map. Then the action of $\mathrm{Aut}(p) \subseteq \mathrm{Homeo}(E)$ on E is properly discontinuous.*

Proof: Given $e_0 \in E$, let $b_0 = p(e_0) \in B$. By hypothesis, there exists a neighbourhood V of b_0 in B evenly covered by p. Let $U \subseteq p^{-1}(V)$ denote the unique slice such that $e_0 \in U$. The action of $\mathrm{Aut}(p)$ on each fiber is fixed-point free. Hence for any $x \in U$, for any $g \in \mathrm{Aut}(p) \setminus \{e\}$, one has $g(x) \neq x$ and so $g(x)$ lies in a slice of $p^{-1}(V)$ different from U. Hence $g(U)$ is disjoint from U. $\square$

Theorem 2.23 *Let X be path connected and locally path connected, $G \subseteq \mathrm{Homeo}(X)$. Then the quotient map $\pi : X \to X \setminus G$ is a covering map if and only if the G-action on X is properly discontinuous. In this case, $\pi : X \to X \setminus G$ is a Galois covering map and $\mathrm{Aut}(\pi) = G$ as subgroups of $\mathrm{Homeo}(X)$.*

Proof: $\Rightarrow$: Follows from Lemma 2.22.
$\Leftarrow$: Suppose the G-action on X is properly discontinuous. For any open $U \subseteq X$, one has $\pi^{-1}\pi(U) = \bigcup\limits_{g\in G} g(U)$ open in X, so $\pi(U)$ is open in X/G (π is the quotient map). Hence π is an open map (continuous as π is the quotient map).

Next, given $G \cdot x \in X/G$, represented by $x \in X$. There exists an open neighborhood U of x in X such that for any $g_0 \neq g_1$ in G, one has $g_0(U)$ disjoint from $g_1(U)$. Then $\pi^{-1}\pi(U) = \bigsqcup g(U)$ is disjoint union of the $g(U)$'s. For each $g \in G$, the map $\pi : g(U) \to \pi(U)$ is bijective, continuous, open, hence a homeomorphism. Therefore $\pi : X \to X/G$ is a covering map.

Now we show that $G = \mathrm{Aut}(\pi)$. For any $x \in X$, for any $g \in G$, one has $G \cdot g(x) = G \cdot x$ in X/G (same G-orbit), so $\pi \circ g = \pi$. Hence $G \subseteq \mathrm{Aut}(\pi)$. On the other hand, pick any $x \in X$, for any $h \in \mathrm{Aut}(\pi)$, one has $\pi \circ h = \pi$, so $G \cdot h(x) = G \cdot x$ in X/G. So there exists $g \in G$ such that $g(x) = h(x)$ in X. The action of $\mathrm{Aut}(\pi)$ on each fiber is fixed-point free. So $h = g$ lies in G, which shows that $\mathrm{Aut}(\pi) = G$.

Lastly, we verify the covering map is Galois. Since $\mathrm{Aut}(\pi) = G$ acts simply transitively on $\pi^{-1}\pi(x)$. Thus $\pi : X \to X/G$ is Galois by Corollary 2.19.1. $\square$

Theorem 2.24 *Let $p : X \to B$ be a Galois covering map. Let $G := \mathrm{Aut}(p)$. Then there exists a unique continuous map $k : X/G \to B$ such that*

$$\begin{tikzcd} X \arrow[dr, "\pi"] \arrow[dd, "p"'] & \\ & X/G \arrow[dl, "k"] \\ B & \end{tikzcd}$$

commutes, where $\pi : X \to X/G$ is the quotient map. Moreover, k is a homeomophism.

Proof: For any $g \in G$, one has $p \circ g = p$, so p is constant on each G-orbit. Then by the Universal property of quotient map, there exists a unique continuous map $k : X/G \to B$, such that $k \circ \pi = p$.

Now k is surjective, because p is surjective; and k is injective, because p is Galois (so $G = \mathrm{Aut}(p)$ acts simply transitively on each fiber of p). k is an open map, because X/G is given with the quotient topology from $\pi : X \to X/G$ and p is an open map. Hence k is a homeomorphism. $\square$

Definition 2.25 *Let $p_0 : X \to B$ be a given Galois covering map. Let $G := \mathrm{Aut}(p_0)$. A covering space of B* ***dominated*** *by X consists of:*

- *a covering space $p : E \to B$;*
- *a B-space map $\pi : X \to E$, $(p \circ \pi = p_0)$.*

Let

$$\begin{array}{c} X \\ {\scriptstyle\pi}\big\downarrow \\ E \\ {\scriptstyle p}\big\downarrow \\ B \end{array}$$

and

$$\begin{array}{c} X \\ {\scriptstyle\pi'}\big\downarrow \\ E' \\ {\scriptstyle p'}\big\downarrow \\ B \end{array}$$

be covering spaces of B dominated by X. A ***morphism*** *from E to E' is a B-space map $k : E \to E'$ such that $k \circ \pi = \pi'$, $p' \circ k = p$.*

Connection to Galois Correspondence:

Theorem 2.26 *Let $p_0 : X \to B$ be a Galois covering map. Let $G := \mathrm{Aut}(p_0)$. Then the map between the set of isomorphism classes of path connected covering spaces of B dominated by X and the set of subgroups of G,*

$$\begin{pmatrix} X \\ \pi \downarrow \\ E \\ p \downarrow \\ B \end{pmatrix} \longmapsto \mathrm{Aut}(\pi)$$

is a well-defined order-reversing bijection. The inverse of the map is given by mapping any subgroup H of G to the isomorphism class of

$$\begin{array}{c} X \\ {\scriptstyle H}\big\downarrow \\ X/H \\ {\scriptstyle p}\big\downarrow \\ B \end{array}$$

Also H (resp. $\mathrm{Aut}(\pi)$) is normal in $G : \mathrm{Aut}(p_0)$, iff $X/H \to B$ (resp $E \xrightarrow{p} B$) is Galois as covering map.

Remark 2.26.1 *If $p_0 : X \to B$ is the universal covering map, then $G := \mathrm{Aut}(p_0)$ is identified with $\pi_1(B, b_0)$.*

3 Seifert-van Kampen Theorem

3.1 Seifert-van Kampen Theorem

Recall the following Theorem:

Theorem 3.1 *Suppose $X = U \cup V$ where U and V are open, path connected sets of X with $U \cap V$ non-empty and path connected. Let $x_0 \in U \cap V$, let $i : U \to X$, $j : V \to X$ be the inclusion mappings. Then $\pi_1(X, x_0)$ is generated by the images of $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$. I.e., for any loop f in X based at x_0 there exists loops g_i in U or in V based at x_0 such that f is path homotopic in X to $(g_1 * (g_2 * (\cdots * g_n)))$.*

We can reformulate this theorem as below:

Theorem 3.2 *The canonical homomorphism $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ induced by i_* and j_* is surjective.*

Theorem 3.3 (Seifert-van Kampen Theorem, Classical Version) *The* ***kernel*** *of the canonical homomorphism $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(U \cup V, x_0)$ induced by i_* and j_* is the normal subgroup of $\pi_1(U, x_0) * \pi_1(V, x_0)$ generated by all elements represented by words of the form $(i_1(g)^{-1}, i_2(g))$, $g \in \pi_1(U \cap V, x_0)$, where*

$$i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0), \quad i_2 : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$$

are induced by the inclusion mappings. Thus $\pi_1(X, x_0)$ is the amalgamated product of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ over the image of $\pi_1(U \cap V, x_0)$, denoted as $\pi_1(X, x_0) = \frac{\pi_1(U,x_0)*\pi_1(V,x_0)}{\pi_1(U\cap V,x_0)}$.

Corollary 3.3.1 *If $U \cap V$ is simply connected, then the canonical homomorphism $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is an isomorphism.*

Corollary 3.3.2 *If V is simply connected, then $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ induces an isomorphism $\pi_1(U, x_0)/N \to \pi_1(X, x_0)$ where N is the normal subgroup of $\pi_1(U, x_0)$ generated by $(i_1)_*(\pi_1(U \cap V, x_0))$.*

Theorem 3.4 (Seifert-van Kampen Theorem, Modern Formulation) *Let $X = U \cup V$ where U and V are open sets of X, $x_0 \in U \cap V$. Suppose $U, V, U \cap V$ are path connected. Let i_1, i_2, j_1, j_2 be the homomorphism induced by inclusions:*

$$\begin{tikzcd}
& \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, "\Phi"] & H \\
& \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

For any group H and any homomorphism $\phi_1 : \pi_1(U, x_0) \to H$, $\phi_2 : \pi_1(V, x_0) \to H$ with $\phi_1 \circ i_1 = \phi_2 \circ i_2$, there exists a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$, $\Phi \circ j_2 = \phi_2$.

Remark 3.4.1 *The theorem is equivalent to the classic version of the Seifert-van Kampen Theorem. This is because the modern formulation asserts that $\pi_1(X, x_0)$ has the same universal property as $\frac{\pi_1(U,x_0)*\pi_1(V,x_0)}{\pi_1(U\cap V,x_0)}$. So we will only prove this version of the theorem.*

Notation: For any path f in X, we will use $[f]$ to denote the path-homotopy class of f in X, and $[f]_K$ denotes the class of f in $K \subset X$ if f lies in that subset.

Proof: Uniqueness: by Theorem 1.67 $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is surjective, so any element $[f] \in \pi_1(X, x_0)$ is a finite product of element of the form $j_1(g_1)$ with $g_1 \in \pi_1(U, x_0)$ and $j_2(g_2)$ with $g_2 \in \pi_1(V, x_0)$. So $\Phi([f])$ must be the corresponding finite product in H of the elements $\phi_1(g_1)$, $\phi_2(g_2)$, i.e., Φ is completely determined by ϕ_1, ϕ_2.

Existence: Let $x_0 \in U \cap V$. For each $x \in X$, choose a path α_x from x_0 to x such that α_x is the constant path at x_0 if $x = x_0$; α_x lies in $U \cap V$ if $x \in U \cap V$; α_x lies in U if $x \in U \setminus (U \cap V)$; α_x lies in V if $x \in V \setminus (U \cap V)$.

For each path f in U (resp. V), say from x to y. Define $L(f) := a_x * (f * \bar{a}_y)$, which is a loop in U (resp. V) based at x_0. So $[L(f)] = j_1([L(f)]_U)$ (resp. $j_2([L(f)]_V)$) as elements of $\pi_1(X, x_0)$.

For each path f in $X = U \cup V$. Choose a subdivision $0 = s_1 < \cdots < s_n = 1$ of $[0,1]$ such that f maps each subinterval $[s_{i-1}, s_i]$ into U or V. For each i, let $f_i : I \to [s_{i-1}, s_i] \xrightarrow{f} X$ be the composite path so f_i is a path in U or V. And

$$[f] = [f_1] * \cdots * [f_n]$$

as path-homotopy classes of paths in X. In particular, if f is a loop in X based at x_0, then

$$[f] = [L(f)] = [L(f_1)] * \cdots * [L(f_n)] = j_1([L(f_1)]_U) * \cdots * j_2([L(f_n)]_V)$$

as elements of $\pi_1(X, x_0)$.

Define the map $\Phi : \pi_1(X, x_0) \to H$ as follows: for any $\alpha \in \pi_1(X, x_0)$,

- Choose a loop f in X based at x_0 representing α.
- Choose a subdivision $[f] = [f_1] * \cdots * [f_n]$ with f_i path in U or V.
- For each i, choose U or V such that f_i is a path in it (if f_i is in $U \cap V$, always choose U).
- Set $\Phi(\alpha) = \phi_1([L(f_1)]_U) \cdots \phi_2([L(f_n)]_V)$ in H.

We need to show that $\Phi(\alpha)$ is well-defined, independent of choice of

1. U of V such that f_i is a path in it.
2. the subdivision $f \sim f_1 * \cdots * f_n$ of f
3. the loop f in X representing α.

Proof:

1. If f_i is a path in $U \cap V$, from x to y, then $x, y \in U \cap V$, and a_x, a_y lie in $U \cap V$. So $L(f_i)$ is a loop in $U \cap V$ based at x_0. With $[L(f_i)]_U = i_1([L(f_i)]_{U \cap V})$ as element of $\pi_1(U, x_0)$, $[L(f_i)]_V = i_2([L(f_i)]_{U \cap V})$ as elements of $\pi_1(V, x_0)$, and so

$$\phi_1([L(f_i)]_U) = (\phi_1 \circ i_1)([L(f_i)]_{U \cap V}) = (\phi_2 \circ i_2)([L(f_i)]_{U \cap V}) = \phi_2([L(f_i)]_V)$$

since $\phi_1 \circ i_1 = \phi_2 \circ i_2$.

2. For any two subdivisions of $I = [0, 1]$, we can do a common refinement. So suffices to show that $\Phi([f])$ remains unchanged when we adjoin a single additional point p to the subdivision $0 = s_0 < \cdots < s_n = 1$. Let i be the index such that $p \in [s_{i-1,s_i}]$.WLOG, assume f_i is a path in U. Then in the formula for $\Phi([f])$, the factor $\phi_1([L(f_i)]_U)$ is replaced by $\phi_1([L(f_i')]_U) \cdot \phi_1([L(f_i'')]_U)$ in H. Note $f_i \sim f_i' * f_i''$ path homotopic in U, so

$$[L(f_i)]_U = [L(f_i')]_U * [L(f_i'')]_U \text{ in } \pi_1(U, x_0).$$

Then $\phi_1([L(f_i)]_U) = \phi_1([L(f_i')]_U) \cdot \phi_1([L(f_i'')]_U)$ in H, as ϕ_1 is a homomorphism. Hence $\Phi([f]$ remains unchanged.

3. Suppose f, g are paths in X, path homotopic in X. Let $F : I \times I \to X$ be a path homotopy from f to g. Apply the Lebesgue number lemma to choose subdivisions of I, $s_0 < s_1 < \cdots < s_n$, $t_0 < t_1 < \cdots < t_m$ fine enough such that each rectangle $I_i \times J_j = [s_{i-1} \times s_i] \times [t_{j-1}, t_j]$ is mapped by F into U or V. For each j, let $f_j : I \to X$ be the paths in X given by $f_j(s) = F(s, t_j)$. Then $f_0 = f$, $f_m = g$, and for each j, the paths f_{j-1}, f_j are path homotopic in X, where the path homotopy satisfies the following Property (P): there exists a subdivision $s_0 < \cdots < s_n$ such that each rectangle $R_i = [s_{i-1}, s_i] \times I$ is mapped into U or V (WLOG assume it is in V). So we are reduced to showing the special case when the paths f, g in X are path homotopic in X, where the path homotopy $F : I \times I \to X$ satisfies (P).

   For each i, let $f_i : I \to [s_{i-1}, s_i] \xrightarrow{f} X$, $g_i : I \to [s_{i-1}, s_i] \xrightarrow{g} X$ be the composite path. Then the restriction of F to R_i is a homotopy from f_i to g_i in U or V but not a path homotopy (the end points may vary). Suppose F maps R_i into V, let $\beta_i : I \to X$ be the path in V given by $\beta_i(t) = F(s_i, t)$ from $f(s_i)$ to $g(s_i)$. Then β_0, β_n are constant paths at $f(0) = g(0)$, $f(1) = g(1)$ respectively. For each i, $f_i * \beta_i \sim \beta_{i-1} * g_i$ are path homotopic in V, so $f_i \sim \beta_{i-1} * g_i * \bar{\beta}_i$. Then

$$[L(f_i)]v = [L(\beta_{i-1})]_V * [L(g_i)]_V * [L(\bar{\beta}_i)]_V \quad \text{in } \pi_1(V, x_0).$$

   So

$$\begin{aligned}\Phi_2([L(f_i)]_V) &= \phi_2([L(\beta_{i-1})]_V) * \phi_2([L(g_i)]_V) * \phi_2([L(\bar{\beta}_i)]_V) \\ &= \phi_2([L(\beta_{i-1})]_V) * \phi_2([L(g_i)]_V) * \phi_2([L(\beta_i)]_V)^{-1}\end{aligned}$$

   Analogous results holds for if F maps R_i into U. Next, notice that $[L(\beta_0)]_U = [L(\beta_n)]_U = 1$ in $\pi_1(U, x_0)$ and $[L(\beta_0)]_V = [L(\beta_n)]_V = 1$ in $\pi_1(V, x_0)$. Also if $\beta_i \in U \cap V$, then

$$\begin{aligned}\phi_1([L(\beta_i)]_U) &= (\phi_1 \circ i_1)([L(\beta_i)]_{U\cap V}) \\ &= (\phi_2 \circ i_2)([L(\beta_i)]_{U\cap V}) \\ &= \phi_2([L(\beta_i)]_V) \text{ in } H.\end{aligned}$$

   Hence writing out the explicit formula for $\Phi([f])$ and $\Phi([g])$, we get that the two are equal.

Lastly, we show that Φ is a homomorphism. Then $\Phi \circ j_1 = \phi_1$, $\Phi \circ j_2 = \phi_2$ as desired. Since Φ is well-defined, independent of the choice of the representing loop and subdivision. For any $[f], [g] \in \pi_1(x, x_0)$, we can subdivide the interval $[0, 1]$, and partition the loops f, g, then we can show that $\Phi([f] * [g]) = \Phi([f]) \cdot \Phi([g])$ in H. $\square$

Example:

1. The Fundamental group of the figure 8 is the free group of rank 2, i.e., $F(x, y) = \mathbb{Z} * \mathbb{Z}$.
2. The fundamental group of the theta space is the free group of rank 2, i.e., $F(x, y)$. Since the theta space and the figure 8 are both the deformation retract of the two punctured plane.

3.2 Graphs

Definition 3.5 (Arc and Graph) *An **arc** is a topological space homeomorphic to the unit interval* $[0, 1]$. *If A is an arc, its end points p, q are the unique points of $\partial A = A \setminus \operatorname{int}(A)$ corresponding to* 0, 1 *under the homeomorphism. Its interior is* $\operatorname{int}(A) = A \setminus \partial A = A \setminus \{p, q\}$.

*A **graph** is a topological space X satisfying the following: it is either the singleton, or there exists a collection $\{A_\alpha\}_{\alpha \in J}$ of subspaces of X called the **edges** of X, such that*

- *For each α, the subspace A_α is an arc.*
- *For each $\alpha \neq \beta$, either $A_\alpha \cap A_\beta = \emptyset$ or $A_\alpha \cap A_\beta = \partial A_\alpha \cap \partial A_\beta$ is a single point.*
- *$X = \bigcup\limits_{a} A_\alpha$ and is given with the colimit topology (union topology), i.e., a subset of X is open (resp. closed) in X iff its intersection with each A_α is open (resp. closed) in A_α. In this case, we also say the topology of X is coherent with the subsapce A_α.*

*The set $X^0 := \bigcup\limits_{\alpha} \partial A_\alpha$ is called the **vertices** of X. We say that X is a finite graph iff X^0 is finite.*

Lemma 3.6 *Let X be a graph, if $C \subseteq X$ is a union of edges and vertices, then C is closed in X.*

Proof: For each α, the intersection $C \cap A_\alpha$ is either empty, or is A_α or a subset of ∂A_α, so it is closed in A_α. $\square$

Corollary 3.6.1 *Any subset of vertices is closed in X; any union of edges is closed in X. In particular, X^0 with subspace topology is discrete.*

Lemma 3.7 *Every graph is normal, hence Hausdorff, as a topological space.*

Proof: Let $B, C \subseteq X$ be disjoint closed subsets. Since $X^0 \setminus C$ is closed in X, we replace B by $B \cup (X^0 \cup C)$, so one can assume that $X^0 \subseteq B \cup C$. For each α, there exists $U_\alpha, V_\alpha \subseteq A_\alpha$ which are disjoint and open sets such that $B \cap A_\alpha \subseteq U_\alpha$, $C \cap A_\alpha \subseteq V_\alpha$ (Since $[0, 1]$ is normal, and the intersections are disjoint closed).

Let $U = \bigcup\limits_{\alpha} U_\alpha$, $V = \bigcup\limits_{\alpha} V_\alpha$. Then $B \subseteq U$, $C \subseteq V$, and $U, V \subseteq X$ are disjoint and open sets such that. Hence X is normal. $\square$

Definition 3.8 (Subgraph) *Let X be a graph. A* ***subgraph of*** *X is a subset $Y \subseteq X$ which is a union of edges given with the subspace topology from X.*

Lemma 3.9 *If Y is a subgraph of X then $Y \subseteq X$ is closed in X and the subspace topology of Y from X is the colimit topology from its edges. Hence Y is itself a graph.*

Proof: Since Y is a union of edges, it is closed in X.

Let $D \subseteq Y$ be a subset. If D is closed for the subspace topology of Y from X, then D is closed in X, so for each edge A_β of Y, the intersection $D \cap A_\beta$ is closed in A_β, so D is closed for the colimit topology from the edges of Y.

If D is closed for the colimit topology from the edges of Y, then for each edge A_α of X, the intersection $D \cap A_\alpha$ is close in A_α if $A_\alpha \subseteq Y$ or is either empty of $\subseteq \partial A_\alpha$ if $A_\alpha \not\subseteq Y$, still closed in A_α, so D is closed in X. Then D is closed for the subspace topology of Y from X. $\square$

Lemma 3.10 *Let X be a graph. For any compact (resp. compact connected) subset $C \subseteq X$, there exists a finite (resp. finite connected) subgraph $Y \subseteq X$ such that $C \subseteq Y$.*

Proof: Let $C \subseteq X$ be a compact subset. Then $C \cap X^0$ is compact and discrete, hence finite.

For each edge A_α of X, choose a point $x_\alpha \in C \cap \mathrm{int}(A_\alpha)$ if the intersection is nonempty. Then any subset of $B = \{x_\alpha\}$ is closed in X, so $B \subseteq C$ is compact and discrete, hence finite. Let Y be the union of the following edges of X:

- for each $x \in C \cap X^0$, choose an edge A_α with x as a vertex.
- for each α such that $C \cap \mathrm{int}(A_\alpha) \neq \emptyset$, the edge A_α.

Then Y is a finite subgraph of X such that $C \subseteq Y$.

If C is connected, then Y is a union of connected edges each of which intersects C, so Y is connected. $\square$

Lemma 3.11 *Every graph is locally path connected and locally simply connected (hence semilocally simply connected), i.e. for any path connected graph, there exists a universal covering space.*

Theorem 3.12 *Let X be a graph. Let $p : E \to X$ be a covering space of X. Then E is a graph. The edges of E are the path components of $p^{-1}(A_\alpha)$ where A_α ranges over all edges of X.*

3.3 The Fundamental Group of a Wedge of Circles

Definition 3.13 (Wedge Of Circles) *A* ***wedge of circles*** *is a topological space X satisfying the following: there exists a collection $\{S_\alpha\}_{\alpha \in J}$ of subspaces of X, called the* ***circles*** *of X, such that*

- *for each α, the subspace S_α is homeomorphic to the unit circle.*

- *there exists $p \in X$ such that for each $\alpha \neq \beta$, one has $S_\alpha \cap S_\beta = \{p\}$.*

- *$X = \bigcup_\alpha S_\alpha$ and is given with the colimit topology (union topology), i.e., a subset of X is open (resp. closed) in X iff its intersection with each S_α is open (resp. closed) in S_α. In this case, we also say the topology of X is coherent with the subsapce S_α.*

Example:

- Let X be a wedge of finitely many circles $S_1, \cdots, S_n$. Then X is homeomorphic to $C_1 \cup \cdots \cup C_n \subseteq \mathbb{R}^2$ given with the subspace topology of $\mathbb{R}^2$, where $C_i \subseteq \mathbb{R}^2$ is the circle of radius i with center at $(i, 0)$.

- The infinite earing given with the subspace topology of $\mathbb{R}^2$ is not a wedge of circles, because a wedge of circles is homeomorphic (by cutting each circle into three pieces) to a graph hence is locally simply connected, but the infinite earing is not semilocally simply connected.

Lemma 3.14 *For any set J, there exists a wedge of circle indexed by J.*

Proof: For each $\alpha \in J$, let S_α be a copy of the unit circle and choose $b_\alpha \in S_\alpha$. Let $E = \bigsqcup_{\alpha \in J} S_\alpha$ given with the colimit topology, and let $P = \bigsqcup_{\alpha \in J} \{b_\alpha\}$, closed in E.

Let $X = E/\sim$, the quotient set of E modulo the equivalence relation $\sim$ collapsing P: $e_1 \sim e_2$ iff $e_1 = e_2$ or $e_1, e_2 \in P$, given with the quotient topology via the quotient map $\pi : E \to X$. So S_α is homeomorphic to $\pi(S_\alpha) \subseteq X$ (for each α). Then X is a wedge of circles $\{\pi(S_\alpha); \alpha \in J\}$ indexed by J. $\square$

Lemma 3.15 *Let X be a wedge of circles $\{S_\alpha : \alpha \in J\}$, then*

1. *X is normal, hence Hausdorff, as a topological space.*

2. *For any compact subset $C \subseteq X$, there exists a finite subset $J_0 \subseteq J$ such that $C \subseteq \bigcup_{\alpha \in J_0} S_\alpha$, i.e., C is contained in the union of finitely many of the circles.*

Proof: Since X is homeomorphic to a graph, this follows from Lemma (3.7) and Lemma (3.10). $\square$

Theorem 3.16 *Let $n \in \mathbb{N}_{\geq 0}$ be a natural number. Let X be a wedge of n many circles $S_1, \cdots, S_n$ with common point p. Then $\pi_1(X, p)$ is the free product of the groups $\pi_1(S_i, p)$ hence a free group of rank n.*

Proof: By induction on n. The case of $n = 0$ or 1 is immediate. So suppose $n > 1$. For each $i \in \{1, \cdots, n\}$, choose a point $q_i \in S_i \setminus \{p\}$ and let $W_i := S_i \setminus \{q_i\}$. Let $U = S_1 \cup W_2 \cup \cdots \cup W_n$, $V = W_1 \cup S_2 \cup \cdots \cup S_n$, so $X = U \cap V$ and $U \cap V = W_1 \cup \cdots \cup W_n$ and U, V are open in X. Each W_i has $\{p\}$ as deformation retract, Hence U has S_1 as a deformation retract, V has $S_2 \cup \cdots \cup S_n$ as deformation retract and $U \cap V$ has $\{p\}$ as deformation retract.

Hence by the Selfert-Van Kampen Theorem, $\pi_1(X, p) = \pi_1(U, p) * \pi_1(V, p)$ is the free product of groups $\pi_1(S_i, p)$ for all $i \in \{1, \cdots, n\}$. $\square$

Theorem 3.17 *For any set J. Let X be the wedge of the circles $\{S_\alpha : \alpha \in J\}$ with common point p. Then $\pi_1(X,p)$ is a free group of rank $|J|$. More precisely: for each $\alpha \in J$, let f_α be a loop in S_α representing a generator of $\pi_1(S_a,p)$. and let $i_\alpha : \pi_1(S_\alpha,p) \to \pi_1(X,p)$ be the homomorphism induced by inclusion. Then each i_α is injective, and*

$$\{i_\alpha([f_\alpha]) : \alpha \in J\}$$

is a set of free generators of $\pi_1(X,p)$.

Proof: First note that by Lemma 3.15, if f is a loop in X based at p, then f lies in the union of finitely many of the circles S_α. If f and g are two such loops which is path homotopic in X, then they are path homotopic in the union of finitely many S_α's ($I \times I$ is compact, so the image of the homotopy is compact). So if f is a loop in S_β which is path homotopic in X to the constant path at p, then there exists a finite subset $J_0 \subseteq J$ such that f is path homotopic in $X_0 := \bigcup_{\alpha \in J_0} S_\alpha$ to the constant path at p. Since i_β factorizes as $\pi_1(S_\beta,p) \to \pi_1(X_0,p) \to \pi_1(X,p)$ and $\pi_1(S_\beta,p) \to \pi_1(X_0,p)$ is injective by Theorem 3.16, then f is path homotopic in S_β to the constant path at p. Hence for each $\beta \in J$, the homomorphism i_β is injective.

Next, we show that the image groups $i_\alpha(\pi_1(S_\alpha,p)) \subseteq \pi_1(X,p)$ for $\alpha \in J$, generates $\pi_1(X,p)$.
If f is a loop in X based at p, then there exists a finite subset $J_0 \subseteq J$ such that f lies in $X_0 := \bigcup_{\alpha \in J_0} S_\alpha$. So $[f]$ lies in the image of $\pi_1(X_0,p) \to \pi_1(X,p)$. Since $\pi_1(X_0,p)$ is generated by $i_\alpha(\pi_1(S,\alpha,p))$ for $\alpha \in J_0$ by Theorem 3.16, it follows that $\pi_1(X,p)$ is generated by $i_\alpha(\pi_1(S_\alpha,p))$, for $\alpha \in J$, i.e., the canonical homomorphism (induced by the i_α's) $h : *_{\alpha \in J}\pi_1(S_\alpha,p) \to \pi_1(X,p)$ is surjective.

Lastly, suppose w is a reduced word in the kernel of h. Then there exists a finite subset $J_0 \subseteq J$ such that w lies in $*_{\alpha \in J_0}\pi_1(S_\alpha,p)$ regarded as subgroup of $*_{\alpha \in J}\pi_1(S_\alpha,p)$. So w is represented by a loop f in $X_0 := \bigcup_{\alpha \in J_0} S_\alpha$ based at p which is path homotopic in X to the constant path at p. Enlarging the finite subset J_0 if necessary, we may assume that f is path homotopic in X_0 to the constant path at p. Then by Theorem 3.16, $w = [f]$ is trivial in $\pi_1(X_0,p) = *_{\alpha \in J_0}\pi_1(S_\alpha,p)$ and so in $*_{\alpha \in J}\pi_1(S_\alpha,p)$. □

3.4 Computation

Let $B^2 \subseteq \mathbb{R}^2$ be the unit ball, $S^1 = \partial B^2$, and $p \in S^1$. Let 0 be the origin.

Theorem 3.18 (Adjoining a Two-cell) *Let X be a Hausdorff path connected space. Let $A \subseteq X$ be a closed path-connected subspace. Let $h : B^2 \to X$ be a continuous map such that*

- *h maps $\operatorname{Int} B^2$ bijectively onto $X - A$.*
- *h maps S^1 into A (not necessarily injective or surjective).*

So $X = A \cup h(\operatorname{Int} B^2)$. Let $a = h(p) \in A$. Then the homomorphism induced by inclusion $i_* : \pi_1(A,a) \to \pi_1(X,a)$ is surjective. The kernel is the normal subgroup of $\pi_1(A,a)$ generated by the image of $h_* : \pi_1(S^1,p) \to \pi_1(A,a)$. I.e., if $[f]$ generates $\pi_1(S^1,p)$, then $\pi_1(X,a)$ is obtained from $\pi_1(A,a)$ by "killing off" $h_*[f]$.

Proof: Let $V := X - A = h(\operatorname{Int} B^2)$ which is open, $x_0 := h(0) \in V$, $u := X - x_0$ which is also open and path connected (X is Hausdorff), so $X = U \cup V$ and $U \cap V = V - x_0$ is path connected (homeomorphic to $\operatorname{Int} B^2 - 0$).

The map $h \times id : B^2 \times I \to h(B^2) \times I$ is closed. This is because $B^2 \times I$ is compact, so any closed subset of $B^2 \times I$ is compact, hence the image under the continuous map is compact. Since $h(B^2) \times I \subseteq X \times I$ is Hausdorff, then the image is closed. So the topology of $h(B^2) \times I$ is the quotient topology from $B^2 \times I$. The topology of $(h(B^2) - x_0) \times I$ is also the quotient topology from $(B^2 - 0) \times I$.

The deformation retraction of $(B^2 - 0)$ onto S^1 then induces a deformation retraction of $(h(B^2) - x_0)$ onto $h(S^1) \subseteq A$ extended to a deformation retraction of $X - x_0$ onto A by keeping each point of A fixed.

Thus A is a deformation retraction of $U := X - x_0$. So the homomorphism $i_* : \pi_1(A, a) \to \pi_1(U, a)$ is an isomorphism (however, $a \notin U \cap V$). Choose any $b \in U \cap V$. Let $q = h^{-1}(b) \in \operatorname{Int} B^2 - 0$. Let γ be any path in $B^2 - 0$ from q to p inside S^1. Then $\delta = h \circ \gamma$ is a path in $U \cap V$ from b to a in A, and the diagram

$$\begin{tikzcd}
 & \pi_1(U,b) \arrow[r] \arrow[d, "\hat{\delta}"] & \pi_1(X,b) \arrow[d, "\hat{\delta}"] \\
\pi_1(A,a) \arrow[r, "\simeq"] & \pi_1(U,a) \arrow[r] & \pi_1(X,a)
\end{tikzcd}$$

commutes.

Let f be a loop of S^1 such that $[f]$ generates $\pi_1(S^1, p)$. Then $g = h \circ f$ is a loop in U based at a. $\delta * g * \bar{\delta} = h \circ (\gamma * f * \bar{\gamma})$ is a loop in U based at b. So $\hat{\delta}^{-1}[g] = [\delta * g * \bar{\delta}] \in \pi_1(U, b)$ is the image of a generator of $\pi_1(U \cap V, b)$.

By the Seifert-van Kampen Theorem (3.3.2), the homomorphism induced by inclusion $\pi_1(U, b) \to \pi_1(X, b)$ is surjective, and the kernel is the normal subgroup of $\pi_1(U, b)$ generated by the image of (a generator of) $\pi_1(U \cap V, b)$. Hence the homomorphism $\pi_1(U, a) \to \pi_1(X, a)$ is surjective, and the kernel is the normal subgroup of $\pi_1(U, a)$ generated by $h_*[f] = [g]$. $\square$

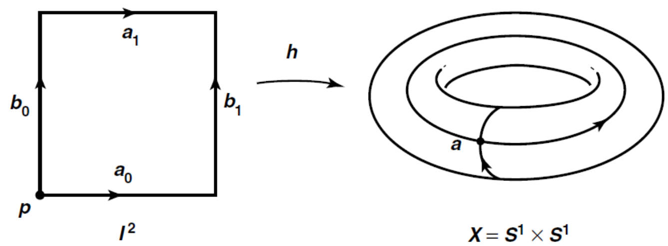
Corollary 3.18.1 *The fundamental group of the Torus have the presentation*

$$\langle \alpha, \beta \,|\, \alpha\beta\alpha^{-1}\beta^{-1} \rangle$$

a free abelian group of rank 2.

Proof: Let $X = S^1 \times S^1$ be the torus, and let $h : I^2 \to X$ be obtained by restricting the standard covering map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$. Let p be the point $(0, 0)$ of Bd I^2, let $a = h(p)$, and let $A = h(\text{Bd } I^2)$.

The space A is the wedge of two circles, so the fundamental group of A is free. Indeed, if we let a_0 be the path $a_0(t) = (t, 0)$ and b_0 be the path $b_0(t) = (0, t)$ in Bd I^2, then the paths $\alpha = h \circ a_0$ and $\beta = h \circ b_0$ are loops in A such that $[\alpha]$ and $[\beta]$ form a system of free generators for $\pi_1(A, a)$.



Now let a_1 and b_1 be the paths $a_1(t) = (t, 1)$ and $b_1(t) = (1, t)$ in Bd I^2. Consider the loop f in Bd I^2 defined by the equation

$$f = a_0 * (b_1 * (\overline{a_1} * \overline{b_0})).$$

Then f represents a generator of $\pi_1(\text{Bd } I^2, p)$; and the loop $g = h \circ f$ equals the product $\alpha * (\beta * (\overline{\alpha} * \overline{\beta}))$. Theorem 72.1 tells us that $\pi_1(X, a)$ is the quotient of the free group on the free generators $[\alpha]$ and $[\beta]$ by the least normal subgroup containing the element $[\alpha][\beta][\alpha]^{-1}[\beta]^{-1}$. $\square$

Theorem 3.19 *Let G be a group; let x be an element of G and N be the least normal subgroup of G containing x. Then if there is a normal, path-connected space whose fundamental group is isomorphic to G, then there is a normal path-connected space whose fundamental group is isomorphic to G/N.*

Proof: Let A denote the normal path connected space having fundamental group isomorphic to G, let x be represented by some loop f in A. Then f induces a continuous map $\tilde{f} : S^1 \to A$.

We consider the **adjunction space** formed from the disjoint union of the norm path-connected space A and the unit ball B^2 by means of a continuous map $\tilde{f}$. I.e., $X = A \sqcup B^2 / \sim$, such that $r \in A$ and $y \in B^2$ are equivalent iff $f(y) = r$ and equip X with the quotient topology. Let $\pi : A \sqcup B^2 \to X$ denote the projection map.

Since A is normal, then we can show X is Hausdorff. It is clear that $\pi(A)$ is closed and path-connected subspace of X, and $\pi|_A : A \to \pi(A)$ is in fact a homeomorphism. We can also see that there is a continuous map $h : B^2 \to X$ that maps Int B^2 bijectively onto $X - \pi(A)$ and maps $S^1 = \partial B^2$ into $\pi(A)$. By Theorem 3.18, there is an isomorphism between

$$\pi_1(\pi(A), a))/N \to \pi_1(X, a)$$

where N is the least normal subgroup of $\pi_1(\pi(A), a)$ containing the image of $\tilde{f}_* : \pi_1(S^1, p) \to \pi_1(\pi(A), a)$, which is just N. Hence $\pi_1(X, a) \cong G/N$. $\square$

Definition 3.20 (n-fold dunce cap) *Let n be a positive integer with $n > 1$. Let $\mu_n \subseteq S^1$ be the (finite) subgroup of the nth roots of unity. The **n-fold dunce cap** is the quotient space*

$$X_n = B^2 / \mu_n$$

of the unit ball B^2 modulo the action of μ_n by rotation given with the quotient topology. I.e., $\forall x, y \in B^2$, $x \sim y$ iff $x = y$ or $x, y \in S^1$ and $\exists p \in \mu_n$, s.t., $x = \mu_y$.

Remark 3.20.1 *The quotient map $\pi : B^2 \to X_n$ is closed. X_n is compact Hausdorff path- connected. $X_2 = B^2/\mu_2$ is homeomorphic to the projective plane $P^2 = S^2/\pm 1$.*

Theorem 3.21 *The fundamental group of the n-fold dunce cap is cyclic of order n, i.e. has the presentation $\langle a \,|\, a^n \rangle$.*

Proof: Let $h : B^2 \to X$ be the quotient map,where X is the n-fold dunce cap. Set $A = h(S^1)$. Let $p = (1, 0) \in S^1$ and let $a = h(p)$. Then h maps the arc C of S^1 running from p to $r(p)$ onto A (r is the generator of μ_n); it identifies the end points of C but is otherwise injective. Therefore, A is homeomorphic to a circle, so its fundamental group is infinite cyclic. Indeed, if γ is the path

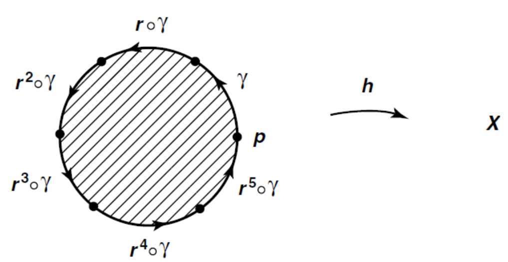
$$\gamma(t) = (\cos(2\pi t/n), \sin(2\pi t/n))$$

in S^1 from p to $r(p)$, then $\alpha = h \circ \gamma$ represents a generator of $\pi_1(A, a)$.

Now the class of the loop

$$f = \gamma * ((r \circ \gamma) * (r^2 \circ \gamma) * \cdots * (r^{n-1} \circ \gamma)))$$

generates $\pi_1(S^1, p)$. Since $h(r^m(x)) = h(x)$ for all x and m, the loop $h \circ f$ equals the n-fold product $\alpha * (\alpha * (\cdots * \alpha))$.



□

Theorem 3.22 *If G is a finitely presented group, then there is a compact Hausdorff space X whose fundamental group is isomorphic to G.*

Proof: We sketch the proof. Suppose G is generated by $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then let A denote the wedge of n circles whose fundamental group is the free group of order n. Now let $B_1, \cdots, B_n$ be the relations of B. Suppose $B_1 = x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k}$, where $x_i \in \{\alpha_1, \cdots, \alpha_n\}$. By abuse of notation, we use α_j to denote the jth circle in the wedge. Then consider the continuous map $f : S^1 \to A$ induced by the path looping the x_1 circle l_1 times, then x_2 circle l_2 times and so on. Then consider the adjunction space X of X and B^2 by means of the continuous map f. By Theorem 3.18, we get the fundamental group of X is

$$\langle \alpha_1, \cdots, \alpha_n \,|\, B_1 \rangle.$$

Lastly, by induction, we could keep adding relations to the presentation of the fundamental group. □

3.5 Fundamental Group of a Graph

Definition 3.23 *Let X be a graph.*

- *An edge e of X is* ***oriented*** *iff it is given with an ordering of ∂e from the initial vertex to the final vertex.*
- *An* ***edge path*** *from x_0 to x_n in X is a finite sequence $x_0, \cdots, x_n$ of vertices in X such that for each $i = 1, \cdots, n$, there exists an edge e_i of X with $\partial e = \{x_{i-1}, x_i\}$. Equivalently (when $n > 0$): it is a finite sequence $e_1, \cdots, e_n$ of oriented edges of X such that for each $i = 1, \cdots, n-1$, the final vertex of e_i is the initial vertex of e_{i+1}.*

An edge path is called ***closed*** *iff $x_0 = x_n$.*

An edge path is ***not reduced*** *iff there exists $i = 1, \cdots, n-1$ such that the oriented edges e_i, e_{i+1} are the same edge (i.e., $\partial e_i = \partial e_{i+1}$). In this case, reducing the edge path by deleting e_i and e_{i+1} and we still have an edge path.*

Given an oriented edge e, let $f_e : I \overset{\text{positive linear}}{\longrightarrow} e \hookrightarrow X$ be the composite path.

Given an edge path $e_1, \cdots, e_n$ from x_0 to x_n, the corresponding path from x_0 to x_n is $f = f_1 * (f_2 * (\cdots * f_n)$ where $f_i = f_{e_i}$. It is a loop iff the edge path is closed.

Lemma 3.24 *A graph X is connected as a topological space if and only if each pair of vertices x, y in X, there exists an (reduced) edge path in X from x to y.*

Definition 3.25 (Trees) *A* ***tree*** *is a graph which is connected and contains no closed reduced edge path. A tree in a graph X is a subgroup of X which is a tree.*

Lemma 3.26 *Let X be a graph and let T be a tree in X.*

- *For any edge A of X such that $T \cap A$ is a single vertex, the union $T \cup A$ is a tree in X.*
- *If T is finite and consists of at least one edge, then there exists a tree T_0 in X and an edge A of X such that $T_0 \cap A$ is a single vertex and $T = T_0 \cup A$.*

Theorem 3.27 *A tree is simply connected as a topological space, i.e., the fundamental group of a tree is trivial.*

Proof: We first consider the case where T is a finite tree. If T consists of a single vertex, then T is simply connected. If T has n edges with $n > 0$, there is an edge A of T such that $T = T_0 \cup A$, where T_0 is a tree with $N - 1$ edges and $T_0 \cap A$ is a single vertex. Then T_0 is a deformation retraction of T, since T_0 is simply connected by the induction hypothesis, so is T.

To prove the general case, let f be a loop in T. The image set of f is compact and connected, so it is contained in a finite connected subgraph Y of T. Now Y contains no closed reduced edge paths, because T contains none, thus Y is a tree. Since Y is finite, it is simply connected. Hence f is path homotopic to the constant paths in Y hence in T. $\square$

Definition 3.28 *Let X be a graph, a* ***maximal tree*** *in X is a tree in X such that there does not exists a tree in X that properly contains T.*

Theorem 3.29 *Let X be a connected graph. A tree T in X is maximal if and only if $X^0 \subseteq T$, i.e., T contains all the vertices of X.*

Proof: Suppose T is a tree in X that contains all the vertices of X. If Y is a subgraph of X that properly contains T, we show that Y contains a closed reduced edge path, this will show that T is maximal. Let A be an edge of Y that is not in T; by hypothesis, the end points a and b of A belong to T. Since T is connected, we can choose a reduced edge path $e_1, \cdots, e_n$ in T from a to b. If we follow this sequence by the edge A oriented from b to a, we obtain a closed reduced edge path in Y.

Now let T be a tree in X that does not contain all the vertices of X. We show T is not maximal. Let x_0 be a vertex of T. Since X is connected, we may choose an edge path in X from x_0 to a vertex of T, specified by the sequence of vertices $x_0, \cdots, x_n$. Let i be the smallest index such that $x_i \notin T$. Let A be the edge of X with vertices x_{i-1} and x_i. Then $T \cup A$ is a tree in X by the preceding lemma, and $T \cup A$ properly contains T. $\square$

Theorem 3.30 *Any tree T_0 in X is contained in a maximal tree in X. In particular, any graph X contains a maximal tree.*

Proof: Apply Zorn's Lemma. $\square$

Theorem 3.31 (Fundamental Group of A Graph) *Let X be a connected graph. Let $T \subseteq X$ be a maximal tree in X. Let $J \subseteq X$ be the collection of edges of X not in T. Then the fundamental group of X is a free group on the set J: for any vertex $x_0 \in T$, we have $\pi_1(X, x_0) = F(J)$.*

Theorem 3.32 (Nielsen-Schreier Theorem) *Let F be a free group and let $H \subseteq F$ be any subgroup. Then H is a free group.*

Proof: Let $\{\alpha \mid \alpha \in J\}$ be a system of free generators for F. Let X be a wedge of the circles $\{S_\alpha : \alpha \in J\}$ with common point x_0. We may identify F with $\pi_1(X, x_0)$. Since X is homeomorphic to a graph, then it is path connected, locally path connected, and locally simply connected (hence semilocally simply connected). Hence corresponding to the subgroup $H \subseteq F$, there exists a covering space $p : E \to X$ and $e_0 \in p^{-1}(x_0)$ such that $H = p_*(\pi_1(E, e_0))$ as subgroups of $F = \pi_1(X, x_0)$ by Theorem 2.14. Note that p_* induces an isomorphism $\pi_1(E, e_0) \to H$. Recall the covering space E is homeomorphic to a connected graph. Hence $\pi_1(E, e_0)$ is a free group. $\square$

Definition 3.33 (Euler Number) *Let X be a finite graph, the **Euler number** of X is*

$$\chi(X) := (\textit{number of vertices}) - (\textit{number of edges}).$$

Remark 3.33.1 *Note this is a topological invariant of (space homeomorphic to) graphs.*

Lemma 3.34 *If T is a finite tree, then $\chi(T) = 1$.*

Proof: We proceed by induction on $n :=$ number of edges in T. When $n = 0$ or $n = 1$, then it is clear. When $n > 1$, write $T = T_0 \cup A$ with $T_0 \subseteq T$ a tree, $A \subseteq T$ is an edge, such that $T_0 \cap A$ is a single vertex. Then T_0 has $n - 1$ edges, so $\chi(T_0) = 1$ by induction hypothesis. Since T has one more edge and one more vertex than T_0, it follows that $\chi(T) = \chi(T_0) = 1$. $\square$

Lemma 3.35 *Let X be a finite connected graph. The rank of the fundamental group of X as a free group is $1 - \chi(X)$.*

Proof: Given X, let $T \subseteq X$ be a maximal tree in X. Let $J \subseteq X$ be the collection of edges of X not in T, then the fundamental group of X is a free group on the set J, hence of rank $n := |J|$. But X and T has exactly the same vertex set, X has n more edges than T. Hence $\chi(X) = \chi(T) - n = 1 - n$, so (rank n) $= 1 - \chi(X)$. $\square$

Theorem 3.36 (Schreier Index Formula) *Let F be a free group of rank $n+1$, in particular $n \geq -1$. Let $H \subseteq F$ be a subgroup of finite index k in F. Then H is a free group of rank $kn + 1$.*

4 Surfaces

Definition 4.1 (Topological n-manifold) *A **toplogical n-manifold** is a topological space X such that*

- *X is locally homeomorphic to $\mathbb{R}^n$, i.e., for any $x \in X$, there exists an open neighbourhood $U \subseteq X$ of x such that U is homeomorphic to $\mathbb{R}^n$ (or $B_{\mathbb{R}^n}(1)$).*
- *X is Hausdorff.*
- *X is second-countable.*

*Then n is called the **dimension** of the manifold.*

*A **surface** is a connected topological 2-manifold.*

4.1 Fundamental Groups of Surfaces

Definition 4.2 *For any $n \geq 3$, the **regular n-gon** P_n is the convex hull in $\mathbb{R}^2$ of the points $p_k = \left(\cos \frac{2\pi k}{n}, \sin \frac{2\pi k}{n}\right)$, $k = 0, 1, \cdots, n$, $(p_n = p_0)$.*

*A n-**sided polygonal region** in the plane is any subset P of $\mathbb{R}^2$ which is given with a homeomorphism from a regular n-gon P_n. The **vertices** and **edges** of P are the images of those of P_n.*

*An **orientation** of an edge e of P is an ordering of ∂e from the initial point to the final point. We define ∂P as images of the boundaries of P_n, define $\mathring{P}$ to be $P - \partial P$ as the image of the interior of P_n.*

Definition 4.3 *Let $P = P_n$ be a n-sided polygonal region in the plane. Let S be a set (elements of S are labels). A **labelling** of the edges of P is a map from the set of edges of P to S.*

*Given an orientation of each edge of P and a labelling of the edges of P, the corresponding **labelling scheme of length** n is the sequence of labels with exponents $+1$ or -1:*

$$w = (a_{i_1})^{e_1} (a_{i_2})^{e_2} \cdots (a_{i_n})^{e_n}$$

where for each k

- *a_{i_k} is the label assigned to the edge $p_{k-1}p_k$.*
- *$e_k = \pm 1$ according to the orientation assigned to this edge goes from p_{k-1} to p_k or the reverse.*

We define the equivalence relation $\sim_w$ on P as follows: $x \sim_w y$ iff $x = y$ or there exists edges e_x, e_y of P with the same label such that $x \in e_x$, $y \in e_y$ and $y = h(x)$, where h is the unique orientation preserving (i.e. positive) linear map from e_x to e_y.

The space obtained by pasting the edges of P together according to the given orientations and labelling (given the labelling scheme w) is the quotient set $X = P/\sim_w$ of P modulo the equivalence relation $\sim_w$ given with the quotient *topology.*

Given a finite number $P_1, \cdots, P_k$ of disjoint polygonal regions, along with orientations and a labelling of their edges, one can form a quotient space X by pasting the edges of these regions together in exactly the same way.

Lemma 4.4 *Let $\pi : E \to X$ be a continuous surjective closed map. Suppose E is normal,and X is given with the quotient topology via π. Then X is normal.*

Proof: Let $A, B \subseteq$ be disjoint closed subsets in X. Then $\pi^{-1}(A)$, $\pi^{-1}(B) \subseteq E$ are disjoint closed subsets in E, since π is continuous. Choose disjoint open $U, V \subseteq E$ such that $\pi^{-1}(A) \subseteq U$, $\pi^{-1}(B) \subseteq V$, this is possible since E is normal. Let $C = E - U$, $D = E - V$, which are both closed in E. Then $\pi(C)$ and $\pi(D)$ are closed in X, with $\pi(C)$ disjoint from A and $\pi(D)$ disjoint from B. So $U_0 = X - \pi(C)$, $V_0 = X - \pi(D)$ are open in X with $A \subseteq U_0$ and $B \subseteq V_0$ and U_0, V_0 disjoint. □

Let $P = P_n$ be a n-sides polygonal region in the plane. Let $X = P/\sim_w$ be the space obtained by pasting the edges of P together according to a labelling scheme w.

Theorem 4.5 *The space X is compact Hausdorff, path connected and second countable.*

Proof: Since X is the continuous surjective image of P (compact, path connected, second countable), it is compact, path connected and second countable.

To show X is Hausdorff, we show that the quotient map $\pi : P \to X$ is a closed map, and apply Lemma 4.4. Let $C \subseteq P$ be a closed subset of P. Then $\pi^{-1}\pi(C) = C \cup \bigcup_{e \in \{\textbf{edges of } P\}} \pi^{-1}\pi(C) \cap e$ in P, and for each edge e of P, $\pi^{-1}\pi(C) \cap e$ is equal to the union of $h_i(C \cap e_i)$ where e_i ranges over all edges of P with the same label as e and $h_i : e_i \to e$ is the unique orientation preserving linear map from e_i to e. Hence $\pi^{-1}\pi(C)$ is closed in P, hence $\pi(C)$ is closed in X. □

Let $w = (a_{i_1})^{e_1}(a_{i_2})^{e_2} \cdots (a_{i_n})^{e_n}$ be a labelling scheme, and $\pi : P \to X = P/\sim_w$ be the quotient map.

Theorem 4.6 *Suppose π maps all the vertices of P to a single point $x_0 \in X$. Let $\alpha_1, \cdots, \alpha_k$ be distinct labels (k many) in the labelling scheme w. Then $\pi_1(X, x_0)$ has the presentation*

$$\langle \alpha_1, \cdots, \alpha_k \,|\, (a_{i_1})^{e_1}(a_{i_2})^{e_2} \cdots (a_{i_n})^{e_n} \rangle$$

Proof: Let $A = \pi(\partial P) \subseteq X$, then A is a closed path-connected subspace of X. Since π maps all the vertices of P to $x_0 \in X$, and there are k many distinct labels of the edges of P, it follows that A is a wedge of k many circles S_i with common point x_0. For each i, choose an edge e_i labelled a_i, oriented counterclockwise in $\mathbb{R}^2$, and let f_i be the positive linear map $I \to e_i$, so $g_i = \pi \circ f_i$ is a loop in A based at x_0 such that $[g_i]$ generates $\pi_1(S_i, x_0)$.

By Theorem 3.16, $\pi_1(A, x_0)$ is a free group of rank k,and $[g_1], \cdots, [g_k]$ are free generatorsof $\pi_1(A, x_0)$. The continuous map $\pi : P \to X$ maps $\mathring{P}$ bijectively onto $X - A$, and maps ∂P into A. Thus by Theorem 3.18, the homomorphism induced by the inclusion $\pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective whose kernel is the normal subgroup of $\pi_1(A, x_0)$ generated by the image of $\pi_* : \pi_1(\partial P, \text{vertex}) \to \pi_1(A, x_0)$). Let f be the loop around ∂P once counterclockwise, so $[f]$ generates $\pi_1(\partial P, \text{vertex})$. Then $\pi \circ f$ is the loop $(g_{i_1})^{e_1} * \cdots (g_{i_n})^{e_n}$ in A, and $\pi_*[f] = [(g_{i_1})^{e_1} * \cdots (g_{i_n})^{e_n}]$ in $\pi_1(A, x_0)$. $\square$

Definition 4.7 (n-fold Torus) *For $n \geq 1$, the n-**fold torus** is the space obtained by pasting the edges of a $4n$-side polygonal region P according to the labelling scheme*

$$(a_1 b_1 a_1^{-1} b_1^{-1})(a_2 b_2 a_2^{-1} b_2^{-1}) \cdots (a_n b_n a_n^{-1} b_n^{-1})$$

*Also called the **connected sum of** n **tori**, denoted $\underbrace{T\#\cdots\#T}_{n}$.*

Corollary 4.7.1 *Let X denote the n-fold torus, $T\#\cdots\#T$. Then $\pi_1(X, x_0)$ has the presentation*

$$\langle \alpha_1, \beta_1, \cdots, \alpha_n, \beta_n \,|\, [\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_n, \beta_n] \rangle.$$

Definition 4.8 (m-fold projective plane) *For any $m > 1$, the m-**fold projective plane** is the space obtained by pasting the edges of a $2m$-sided polygonal region P according to the labelling scheme*

$$(a_1 a_1)(a_2 a_2) \cdots (a_m a_m).$$

*This space is also called the **connected sum of** m **projective planes**, denoted $P^2\#\cdots\#P^2$.*

Corollary 4.8.1 *Let X denote the m-fold projective plane, $P^2\#\cdots\#P^2$, then $\pi_1(X, x_0)$ has the presentation*

$$\langle \alpha_1, \cdots, \alpha_m \,|\, (\alpha_1)^2 (\alpha_2)^2 \cdots (\alpha_m)^2 \rangle.$$

Definition 4.9 (Klein Bottle) *The **Klein Bottle** is the space obtained by pasting the square according to the labelling scheme*

$$aba^{-1}b.$$

Remark 4.9.1 *In fact the Klein Bottle is homeomorphic to the space $P^2\#P^2$.*

4.2 Homology Of Surfaces

Definition 4.10 (Maximal Abelian Quotient) *The abelianization of a group G is the **maximal abelian quotient***

$$G^{ab} = G/[G, G].$$

Let X be a path connected topological space. The abelianized fundamental group of X is

$$\pi_1^{ab}(X) := \pi_1(X, x_0)^{ab} = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$$

for any choice of the base point $x_0 \in X$.

Lemma 4.11 *For any $x_1 \in X$, choose a path α in X from x_0 to x_1. The isomorphism $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$, $[f] \mapsto \hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$ induces the same isomorphism*

$$\pi_1(X, x_0)^{ab} \xrightarrow{\cong} \pi_1(X, x_1)^{ab}$$

independent of the choice of the path α.

Proof: If β is another path in X from x_0 to x_1, then $g = \alpha * \bar{\beta}$ is a loop in X at x_0. So $\hat{g} : \pi_1(X, x_0) \to \pi_1(X, x_0)$, $[f] \mapsto \hat{g}[f] = [g]^{-1} * [f] * [g]$ induces the identity isomorphism on $\pi_1(X, x_0)^{ab}$, hence $\hat{a} = \hat{\beta}$. □

Theorem 4.12 *Let F be a group. $N \subseteq F$ be a normal subgroup. Let $p : F \to F^{ab} = F/[F, F]$ be the quotient homomorphism. Then $F^{ab}/p(N) \cong (F/N)^{ab}$ canonically.*

Proof: Commutator commutes with surjective homomorphism. □

Corollary 4.12.1 *Let G be a group with the presentation $\langle a_1, \cdots, a_n \,|\, x \rangle$. Let $F = \langle a_1, \cdots, a_n \rangle$ be the free group generating G. $p : F \to F^{ab} = Z^n$ be the quotient homomorphism. Then $G^{ab} \cong Z^n/Z \cdot p(x)$ canonically.*

Proof: Write G as F/N where $N = \langle x \rangle \trianglelefteq F$ is the normal subgroup of F generated by x. And all conjugates of x maps to the same elements. □

Proposition 4.13 *Let X denote the n-fold torus $T\#\cdots\#T$, then $\pi_1^{ab}(X) \cong \mathbb{Z}^{2n}$ is free abelian of rank $2n$.*

Proof: Recall the $\pi_1(X, x_0)$ has the presentation

$$\langle \alpha_1, \beta_1, \cdots, \alpha_n, \beta_n \,|\, [\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_n, \beta_n] \rangle.$$

The element $[\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_n, \beta_n]$ maps to 0 in Z^{2n}. □

Proposition 4.14 *Let X denote the m-fold projective plane, $P^2\#\cdots\#P^2$. Then $\pi_1^{ab}(X) \cong Z^{m-1} \oplus Z/2Z$ has torsion subgroup of order 2 and free quotient of rank $m - 1$*

Proof: $\pi_1(X, x_0)$ has the presentation

$$\langle \alpha_1, \cdots, \alpha_m \,|\, (\alpha_1)^2(\alpha_2)^2 \cdots (\alpha_m)^2 \rangle.$$

The element $(\alpha_1)^2 \cdots (\alpha_m)^2$ maps to $p(x) = 2(\alpha_1 + \cdots + \alpha_m)$ in Z^m. Change basis in Z^m to $\alpha_1, \cdots, \alpha_{m-1}$, $\beta = \alpha_1 + \cdots + \alpha_m$. Then $p(x) = 2\beta$ and so $\pi_1^{ab}(X) \cong \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{m-1} \oplus \mathbb{Z}\beta/2\beta$. □

Theorem 4.15 *The surfaces S^2, $\underbrace{T\#\cdots\#T}_{n\geq 1}$, $\underbrace{P^2\#\cdots\#P^2}_{m>1}$ are topologically distinct (not homeomorphic to others).*

Proof: Because their fundamental group has different abelianization. □

Elementary Operations on labelling schemes of disjoint polygonal regions.

1. Cutting;
2. Relabel;
3. Cyclic Permutation;
4. Flip;
5. Cancel;
6. Uncancel.

Definition 4.16 *Two labelling schemes for disjoint polygonal regions are* ***equivalent*** *iff one can be obtained from the other by a finite sequence of elementary operations.*

Remark 4.16.1 *The spaces obtained by pasting the edges of disjoint polygonal regions, according to equivalent labelling schemes are homeomorphic to each other*

4.3 The Classification Theorem

Definition 4.17 *A labelling scheme $w_1, \cdots, w_k$ of disjoint polygonal regions $P_1, \cdots, P_k$ is* ***proper*** *if each label appears exactly twice (regardless of orientation) in $w_1, \cdots, w_k$ iff the edges of $P_1, \cdots, P_k$ are pasted in pairs.*

Remark 4.17.1 *Note proper-ness is stable under elementary operations. If one applies any elementary opeartion toa proper scheme, one obtains a proper scheme.*

Definition 4.18 *Let w be a proper labelling scheme of a single polgonal region. Then w is of* ***torus type*** *iff each label appears exactly one with exponent $+1$ and one with -1. Otherwise, w is of* ***projective type****.*

Remark 4.18.1 *Of torus type or of projective type is stable under elementary operations.*

Notation: we use w, w', w'' to denote proper labelling schemes of a single polygonal region. We use w_i, y_i, z_i denote labelling schemes that may be empty. We use a, b, c denote labels.

Lemma 4.19 *If $w = [y_0]a[y_1]a[y_2]$, then $w \sim aa[y_0 y_1^{-1} y_2]$ (of the same length).*

Corollary 4.19.1 *If w is of projective type, then w is equivalent to a scheme of the same length of the form*

$$(a_1 a_1)(a_2 a_2) \cdots (a_k a_k) w_1 \quad k \geq 1$$

with w_1 of torus type or empty.

Lemma 4.20 *If $w = w_0(cc)(aba^{-1}b^{-1})w_1$, then $w \sim w_0(aabbcc)w_1$ (of the same length).*

Lemma 4.21 *Suppose $w = w_0[y_1]a[y_2]b[y_3]a^{-1}[y_4]b^{-1}[y_5]$, then $w \sim w_0 aba^{-1}b^{-1}[y_1 y_2 y_3 y_4 y_5]$.*

Lemma 4.22 *If $w = w_0 w_1$, where w_1 if of non-empty torus type with no two adjacent terms having the same label. Then $w \sim w_0 w_2$, where $w_2 = aba^{-1}b^{-1}w_3$ has the same length as w_1 with w_3 of torus type or empty.*

Proof: Brute Force Computation. □

Theorem 4.23 (Classification of Proper Labelling Schemes) *Let w be a proper labelling scheme of a single polygonal region of length ≥ 4, then w is equivalent to*

1. $aa^{-1}bb^{-1}$.
2. $abab$.
3. $(a_1a_1)\cdots(a_ma_m)$, $m \geq 2$.
4. $(a_1b_1a_1^{-1}b_1^{-1})\cdots(a_nb_na_n^{-1}b_n^{-1})$, $n \geq 1$.

Remark 4.23.1 *Recall:*

- S^2 *is given by the labelling scheme* $aa^{-1}bb^{-1}$.
- P^2 *is given by* $abab$.
- P_m *is given by* $(a_1a_1)\cdots(a_ma_m)$, $m \geq 2$.
- T_n *is given by* $(a_1b_1a_1^{-1}b_1^{-1})\cdots(a_nb_na_n^{-1}b_n^{-1})$, $n \geq 1$.

Proof: Suppose w is of torus type. If w is of length 4, then by relabelling, $w \sim aa^{-1}bb^{-1}$ or $aba^{-1}b^{-1}$ (no choice). Now proceed by induction on the length of w. Suppose the length of $w > 4$, If there exists two adjacent terms having the same label, then by cancelling, w is equivalent to a shorter scheme of torus type done by induction hypothesis. Otherwise, w is of torus type with no two adjacent terms having the same label. Then by Lemma (4.22), $w \sim aba^{-1}b^{-1}w_3$ of the same length, where w_3 (non-empty here because w of length > 4) is of torus type with no two adjacent terms having the same label. Then by induction hypothesis, w is of type (4).

Next, suppose w is of projective type, then we show w is of type (2) or (4). If w is of length 4, then $w \sim aabb$ by Corollary 4.19.1 (type 3) or $aab^{-1}b$ (type 2, by Lemma 4.19). We proceed by induction on the length of w. Assume w is of length greater than 4, then by Corollary 4.19.1, $w \sim (a_1a_1)(a_2a_2)\cdots(a_ka_k)w_1$ of the same length, where $k \geq 1$ and w_1 is of torus type of empty. If w_1 is empty, then we are done. If there exists in w_1 two adjacent terms having the same label, then by cancelling, w_1 is equivalent to a shorter scheme of torus type, and hence w is equivalent to a shorter scheme of projective type done by induction hypothesis. Otherwise, w_1 is of torus type with no two adjacent terms having the same label. By Lemma 4.22, $w \sim (a_1a_1)\cdots(a_ka_k)aba^{-1}b^{-1}w_2$ where w_2 is of torus type or empty. Then by Lemma 4.20, because $k \geq 1$, then $w \sim (a_1a_1)\cdots(a_{k-1}a_{k-1})aabb(a_ka_k)w_2$ of the same length. Then by induction hypothesis, this is of type (3). □

Theorem 4.24 (Topological Classification of Surfaces Obtained by Polygonal Regions) *Let X be the space obtained by pasting the edges in pairs from a single polygonal region. Then X is homeomorphic to the 2-sphere S^2, or the n-fold torus T_n ($n \geq 1$), or the m-fold projective plane P_m ($m \geq 1$).*

Proof: Follows from Theorem 4.23. □

Definition 4.25 (Triangulation) *Let T be the regular 3-gon in $\mathbb{R}^2$. Let X be a topological spacee. A* ***curved triangle*** *in X is a subspace $A \subseteq X$ given with a homeomorphism $h : T \to A$. The* ***vertices and edges*** *of A are the images of those of T. A* ***2-dimensional triangulation*** *of X is a collection $\{A_\alpha\}_\alpha$ of subspaces of X satisfying the following:*

1. *For each α, the subspace A_α is a curved triangle in X (so homeomorphic to T);*
2. *for each $\alpha \neq \beta$, either $A_\alpha \cap A_\beta = \emptyset$ or a vertex of both A_α, A_β, or an edge of both A_α, A_β. And if $e = A_\alpha \cap A_\beta$ is an edge of both and $h_\alpha : T \to A_\alpha$, $h_\beta : T \to A_\beta$ are the given homeomorphisms, then $h_\beta^{-1} h_\alpha : h_\alpha^{-1}(e) \to h_\beta^{-1}(e)$ is a linear homeomorphism between edges of T;*
3. *$X = \bigcup_\alpha A_\alpha$ and is given with the colimit topology.*

The space X is ***triangulable*** *iff there exists a triangulation of X.*

Theorem 4.26 *Let X be a compact surface (not necessarily connected). Then X is homeomorphic to the space obtained by pasting the edges in pairs from a finite collection of disjoint triangular regions.*

Proof: Step 0: X is triangulable. This is the assertion of a Theorem by Tibor Radó (1925): Any compact surface (not necessarily connected) is triangulable (the proof uses the Jordan Curve Theorem, Hausdorff and second countability) .

Since X is compact, $X = A_1 \cup \cdots \cup A_n$ is a finite union of curved triangles. For each i, let $h_i : T_i \to A_i$ be the corresponding homeomorphism where the T_i's are disjoint regular 3-gon in $\mathbb{R}^2$, and if $e = A_i \cap A_j$ is an edge of both, then $h_j^{-1} h_i : h_i^{-1}(e) \to h_j^{-1}(e)$ is a linear homeomorphism.

Let $E = T_1 \cup \cdots \cup T_n \subseteq \mathbb{R}^2$, be the finite collection of disjoint triangular regions. The h_i's combine to define a surjective map $h : E \to X$. Since E is compact and X is Hausdorff, the map h is a closed map, so the topology of X is the quotient topology from E via h.

Step 1: For each edge e of a curved triangle A_i, there exists a unique different curved triangle A_j with e as an edge. This follows from the fact that each point on the edge in question must have an open neighborhood homeomorphic to the open disc in $\mathbb{R}^2$. Hence $h : E \to X$ pasts the edges of E in pairs.

Let e be an edge of any curved triangle A_i. Let p be an interior point of e.
Existence: Suppose not, then A_i is the only curved triangle with e as an edge. Also: A_i contains a neighborhood of p in X. Hence for every sufficiently small neighborhood W of p in X, the space W_p is contractible to a point. However, since X is a surface, it is locally homeomorphic to $\mathbb{R}^2$, so there exists a neighbourhood U of p in X homeomorphic to $\mathbb{R}^2$. But $U - p$ is homeomorphic to $\mathbb{R}^2 - 0$ which is not simply connected, so it is not contractible to a point, so we have a contradiction.

Uniqueness: Suppose not. Then there exists $k \geq 3$ curved triangles with e as an edge. Let C be their union. Then C contains a neighborhood of p in X. Let A be the union of all the edges of C different from e. Then

A is a deformation retract of $C - p$. Also A is a graph, the union of $k \geq 3$ arcs, each pair of which intersect only at their end points. Let B be the union of three of the arcs of A. Then B is a θ-space, whose fundamental group of B is $\mathbb{Z} * \mathbb{Z}$ which is not abelian. Note B is a retract of A: the map $r : A \to B$ mapping the arcs in B, identically to themselves and the arcs not in B homeomorphically onto one of the arcs in B, keeping the end points fixed is a retraction. Then r_* is a surjective homomorphism from fundamental group of A to fundamental group of B. So the fundamental group of A is not abelian, and the fundamental group of $C - p$ is not abelian.

Let W be a neighborhood of p in X, sufficiently small, so that $W \subseteq C$. Then W contains a homeomorphic copy C' of C inside itself, which implies that the fundamental group of $W - p$ is not abelian. However, since X is a surface, it is locally homeomorphic to $\mathbb{R}^2$, so there exists a neighborhood U of p in X homeomorphic to $\mathbb{R}^2$, but $U - p$ is a homeomorphic to $\mathbb{R}^2 - 0$ has abelian fundamental group $\mathbb{Z}$, which is a contradiction.

Step 2: Let w be the labelling scheme on the edges of E induced by h. Then w is a proper labelling scheme. From the definition of triangulation of X, it is clear that for any $x, y \in E$, one has if $x \sim_w y$ then $h(x) = h(y)$ in X, and $h(x) = h(y)$ in X when x, y are in the interior of T_i's or in the interior of the edges. However, it is not clear that $h(x) = h(y)$ in X would imply $x \sim_w y$ when x, y are vertices of triangles. This is we will show next.

For any curved triangles A_i, A_j, such that $v = A_i \cap A_j$ is a vertex of both, there exists a sequence of curved triangles with v as vertex, beginning with A_i and ending with A_j such that each triangle of the sequence intersecting its successors is an edge of both. Then X is indeed the space $E/\sim_w$ obtained by pasting the edges of E together according tot he proper labelling scheme w induced by h.

Let v be a vertex of any curved triangle. Define the relation $\sim$ on the set of curved triangles with v as vertex by setting: $A_i \sim A_j$ iff there exists a sequence of curved triangles with v as vertex beginning with A_i and ending with A_j, such that each triangle of the sequence intersecting its successor is an edge of both. It is clear that $\sim$ is an equivalence relation. We want to show that there is only one equivalence class.

Suppose not. Let B be the union of the curved triangles in one equivalence class, and let C be the union of all the others. Then $B \cup C$ contains a neighbourhood of v in X. Also, no triangle in B has an edge in common with any triangle in C, so $B \cap C = \{v\}$, so $(B \cup C) - v$ is not connected. Hence for every sufficiently small neighborhood W of v in X, the space $W - v$ is not connected. $\square$

Theorem 4.27 *Let X be a compact connected surface. Then X is homeomorphic to the space obtained by pasting the edges in pairs from a single polygonal region.*

Proof: By Theorem 4.26, there exists a finite collection of disjoint polygonal (triangular) regions $T_1, \cdots, T_n$ and a proper labelling scheme of these region, such that X is homeomorphic to the space obtained by pasting the edges of these regions according to the labelling scheme. If there exists two edges in distinct regions with the same label, then by flipping one of the regions if necessary, we can past these regions along these two edges, then the number of polygonal regions is reduced by 1, labelling scheme remains proper.

By induction, we are reduced to the situation where either one has only a single polygonal region with proper labelling scheme (in this case, we are done); or one has more than one polygonal region such that no two distinct regions have edges with the same label. However, the later case implies the resulting quotient space X is not connected, contradicting the hypothesis. $\square$

Theorem 4.28 (Topological Classification of Surfaces) *Let X be a compact connected surface, then X is homeomorphic to the 2-sphere S^2, or the n-fold torus T_n $(n \geq 1)$, or the m-fold projective plane P_m $(m \geq 1)$.*