Contents

1 Topologies and Metrics

1.1 Topological Spaces

Definition 1.1.1 ▶ Topology

A **topology** on a set X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ such that

- $\varnothing, X \in \mathcal{T}$;
- for any index set I, if $\{X_i : i \in I\} \subseteq \mathcal{P}(\mathcal{T})$, then $\bigcup_{i \in I} X_i \in \mathcal{T}$;
- for any $X_1, X_2, \cdots, X_n \in \mathcal{T}$, $\bigcap_{i=1}^{n} X_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is said to be a **topological space**. A subset $Y \subseteq X$ is **open** if $Y \in \mathcal{T}$.

We define some commonly used topologies:

1. The *trivial topology*: $\{\varnothing, X\}$ — only $\varnothing$ and X itself are open.
2. The *discrete topology*: $\mathcal{P}(X)$ — everything is open.
3. The *co-finite topology*: $\{X \setminus U : U \subseteq X \text{ is finite}\} \cup \{\varnothing\}$ — any open set either is empty or has a finite complement.
4. The *co-countable topology*: $\{X \setminus U : U \subseteq X \text{ is countable}\} \cup \{\varnothing\}$ — any open set either is empty or has a countable complement.

The set $\{(-\alpha, \alpha) : \alpha > 0\} \cup \{\mathbb{R}, \varnothing\}$ defines a topology on $\mathbb{R}$. This example also demonstrates why it is crucial to only consider closure under finite intersections when defining a topology, because

$$\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = [-1, 1],$$

which should be closed. We now seek a systematic method to generate a topology given any set. The idea here is to make use of a *cover*.

Definition 1.1.2 ▶ Basis

A **basis** for a topology on X is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that

- for any $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$;
- for any $x \in X$ and $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

It may be useful to see a basis as a cover of a set with the second additional property as stated in the above definition. Notice that the first property of the basis $\mathcal{B}$ is basically saying that

$$X \subseteq \bigcup \mathcal{B},$$

i.e., $\mathcal{B}$ is a cover of X.

Given any basis $\mathcal{B}$ for some topology on X, a set generated by $\mathcal{B}$ can be defined as

$$\mathcal{T} := \{U \subseteq X : \text{ for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\},$$

i.e., $\mathcal{T}$ is the collection of all sets which can be covered by the basic sets. We will show that $\mathcal{T}$ is a topology on X. First, it is clear that $\varnothing, X \in \mathcal{T}$.

Let I be an index set and $\{X_i : i \in I\} \subseteq \mathcal{P}(\mathcal{T})$ be any collection of subsets of X. Notice that for any $x \in \bigcup_{i \in I} X_i$, there exists some $j \in I$ such that $x \in X_j \subseteq \mathcal{T}$. According to our construction, this means that there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq X_j \subseteq \mathcal{T}$. Therefore, $\bigcup_{i \in I} X_i \subseteq \mathcal{T}$ as desired.

To prove that $\mathcal{T}$ is closed under finite intersection, we consider the following lemma:

Lemma 1.1.3 ▶ Finite Intersection of Elements in Basis Is Covered

Let $\mathcal{B}$ be a basis for a topology on X and $B_1, B_2, \cdots, B_n \in \mathcal{B}$, then for any $x \in \bigcap_{i=1}^n B_i$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcap_{i=1}^n B_i$.

Proof. The case where $n = 1$ is trivial by taking $B = B_1$. Suppose that there is some integer $k \geq 1$ such that for any $B_1, B_2, \cdots, B_k \in \mathcal{B}$ and any $x \in \bigcap_{i=1}^k B_i$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcap_{i=1}^k B_i$. Take any $B_{k+1} \in \mathcal{B}$. It is clear that for any $x \in \bigcap_{i=1}^{k+1} B_i$, there exists some $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^{k} B_i.$$

Notice that $x \in B_{k+1} \in \mathcal{B}$, so we know that $x \in B \cap B_{k+1}$. By Definition 1.1.2, this means that there exists some $B' \in \mathcal{B}$ such that

$$x \in B' \subseteq B \cap B_{k+1} \subseteq \bigcap_{i=1}^{k+1} B_i.$$

□

Now, suppose $X_1, X_2, \cdots, X_n \in \mathcal{T}$ are finitely many subsets of X. Take any $x \in \bigcap_{i=1}^n X_i$. It is clear that $x \in X_i$ for each $i = 1, 2, \cdots, n$. Therefore, for each $i = 1, 2, \cdots, n$, there exists some $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq X_i$. By Lemma 1.1.3, this means that there exists some set $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^{n} B_i \subseteq \bigcap_{i=1}^{n} X_i.$$

Therefore, $\bigcap_{i=1}^{n} X_i \in \mathcal{T}$. So this set $\mathcal{T}$ generated by $\mathcal{B}$ is indeed a topology on X.

The following proposition further shows that the topology generated by a basis $\mathcal{B}$ is the set of all possible unions of elements in $\mathcal{B}$:

Proposition 1.1.4 ▸ Equivalent Construction of Topologies Generated from Bases

Let X be any set. If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X, then

$$\mathcal{T} = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \in \mathcal{P}(\mathcal{B}) \right\}.$$

Proof. Denote

$$\mathcal{T}_{\mathcal{B}} := \{U \subseteq X : \text{ for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\}.$$

It suffices to prove that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. Take any $T \in \mathcal{T}$, then there exists some $V \in \mathcal{P}(\mathcal{B})$ such that $T = \bigcup_{A \in \mathcal{V}} A$. This means that for every $t \in T$, there exists some $B_t \in \mathcal{V}$ such that $t \in B_t \subseteq T$. Therefore, $T \in \mathcal{T}_{\mathcal{B}}$. Conversely, for any $S \in \mathcal{T}_{\mathcal{B}}$, there exists some $B_s \in \mathcal{B}$ for each $s \in S$ such that $s \in B_s$. Denote $U := \{B_s : s \in S\} \in \mathcal{P}(\mathcal{B})$, then it is clear that $S \subseteq \bigcup_{B \in U} B$. Since $B_s \subseteq S$ for each $s \in S$, we have $\bigcup_{B \in U} B \subseteq S$, which implies that $S = \bigcup_{B \in U} B$. This means that $S \in \mathcal{T}$. Therefore, $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$, which means that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. □

Next, we define a special topology in Euclidean spaces using open balls.

Definition 1.1.5 ▸ Standard Topology

For any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and any $r > 0$. Denote the Euclidean open ball centred at $\boldsymbol{x}$ with radius r by

$$B_r(\boldsymbol{x}) := \left\{ \boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n : \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} < r \right\}$$

The **standard topology** on $\mathbb{R}^n$ is the set generated by the basis

$$\mathcal{B} := \{B_r(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n, r \in \mathbb{R}^+\}.$$

It may be helpful to actually show that this set $\mathcal{B}$ is indeed a basis of a topology on $\mathbb{R}^n$. The fact that $\mathcal{B}$ covers $\mathbb{R}^n$ is trivial enough. Take any $\boldsymbol{x} \in \mathbb{R}^n$ and balls $B_\alpha(\boldsymbol{x}_1), B_\beta(\boldsymbol{x}_2) \in \mathcal{B}$ such that $\boldsymbol{x} \in B_\alpha(\boldsymbol{x}_1) \cap B_\beta(\boldsymbol{x}_2)$ (the existence of these 2 balls is again trivial enough). Take

$$r = \min\{\alpha - \|\boldsymbol{x} - \boldsymbol{x}_1\|, \beta - \|\boldsymbol{x} - \boldsymbol{x}_2\|\}.$$

Clearly, $r > 0$ and $\boldsymbol{x} \in B_r(\boldsymbol{x})$, so we are done.

Now, we discuss the analogue of the subset relation in topologies.

Definition 1.1.6 ▸ Fineness and Coarseness

Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on some set X. We say that $\mathcal{T}$ is **finer** than $\mathcal{T}'$, or equivalently, that $\mathcal{T}'$ is **coarser** than $\mathcal{T}$, if $\mathcal{T}' \subseteq \mathcal{T}$.

Following this definition, a topology $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if everything open in $\mathcal{T}_2$ remains open in $\mathcal{T}_1$.

Observe that any topology of X must be a subset of $\mathcal{P}(X)$, which is the discrete topology on X, so the discrete topology is the finest topology on a set.

Remark. For any basis $\mathcal{B}$ for a topology on X, the topology generated by $\mathcal{B}$ is the coarsest topology containing $\mathcal{B}$.

The above remark is easy to verify. Let $\mathcal{T}$ be any topology on X with $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}_{\mathcal{B}}$ be the topology generated by $\mathcal{B}$. For any $T \in \mathcal{T}_{\mathcal{B}}$, by Proposition 1.1.4, there exists some $\mathcal{V} \subseteq \mathcal{B}$ such that $T = \bigcup_{A \in \mathcal{V}} A$. Note that $A \in \mathcal{T}$ for all $A \in \mathcal{V}$, so by Definition 1.1.1, $T \in \mathcal{T}$ and so $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$ as desired.

This motivates us to consider fineness in terms of bases.

Proposition 1.1.7 ▸ Fineness in Terms of Bases

Let $\mathcal{B}$ and $\mathcal{B}'$ generate topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for every $B \in \mathcal{B}$ and any $x \in B$, there exists some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B$.

Proof. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$, then $\mathcal{T} \subseteq \mathcal{T}'$. Take any $B \in \mathcal{B}$, then by Proposition 1.1.4, $B \in \mathcal{T}$, which means that $B \in \mathcal{T}'$. Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$, by Definition 1.1.2 for any $x \in B$, there exists some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B$.

Suppose conversely that for every $B \in \mathcal{B}$ and any $x \in B$, there is some $B_x \in \mathcal{B}'$ such that $x \in B' \subseteq B$. Take any $T \in \mathcal{T}$, for each $x \in T$, by Definition 1.1.2 there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq T$, and so we can find some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B \subseteq T$, so $T \in \mathcal{T}'$. Therefore, $\mathcal{T} \subseteq \mathcal{T}'$ and so $\mathcal{T}'$ is finer than $\mathcal{T}$. □

Notice that in a metric space, because there are uncountably many basic open balls $B_r(x)$ centred at each point, we can reduce the condition to that there exists some $B'_{r'}(x) \in \mathcal{B}'$ such that $B'_{r'}(x) \subseteq B_r(x)(x) \in \mathcal{B}$ for each $B_r(x) \in \mathcal{B}$.

Recall that every basis of a topology on X is an open cover of X consisting only of subsets of X. Therefore, the union of the elements in the basis is essentially X itself. This motivates us to propose another way to generate a topology on a set.

Definition 1.1.8 ▸ Sub-basis

A **sub-basis** of X is a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ such that $\bigcup_{A \in \mathcal{S}} A = X$.

Remark. Every basis is a sub-basis.

For an arbitrary set X, let $\mathcal{S}$ be a sub-basis and denote the collection of all finite subsets of $\mathcal{P}(\mathcal{S})$ as $\mathcal{F}_{\mathcal{S}}$. Define

$$\mathcal{U}_{\mathcal{S}} \coloneqq \left\{ \bigcap_{A \in F} A : F \in \mathcal{F}_{\mathcal{S}} \right\}$$

to be the collection of all finite intersections of sets in $\mathcal{S}$. The topology generated by a sub-basis of X is given by

$$\mathcal{T} \coloneqq \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{U}_{\mathcal{S}} \right\}.$$

We shall show that $\mathcal{T}$ is indeed a topology on X by considering the following proposition:

Proposition 1.1.9 ▸ Finite Intersections of Sets in a Sub-basis Form a Basis

Let $\mathcal{S}$ be a sub-basis for a set X and let $\mathcal{U}_{\mathcal{S}}$ be the set of all finite intersections of sets in $\mathcal{S}$, then $\mathcal{U}_{\mathcal{S}}$ is a basis of a topology on X.

Proof. Take any $x \in X$. By Definition 1.1.8, we have $x \in \bigcup_{A \in \mathcal{S}} A$. Therefore, there exists some $A \in \mathcal{S} \subseteq \mathcal{P}(X)$ such that $x \in A$. For any $x \in X$ and $B_1, B_2 \in \mathcal{U}_{\mathcal{S}}$ such that $x \in B_1 \cap B_2$, notice that $B_1 \cap B_2$ is a finite intersection of sets in $\mathcal{S}$, so $B_1 \cap B_2 \in \mathcal{U}_{\mathcal{S}}$. Therefore, by Definition 1.1.2, $\mathcal{U}_{\mathcal{S}}$ is a basis. □

With Propositions 1.1.9 and 1.1.4, it is clear that $\mathcal{T}$ as constructed above is a topology on X.

1.2 Metric Spaces

A metric space is a topological space augmented with the concept of distance.

Definition 1.2.1 ▶ Metric

A **metric** on a set S is a function $d: S \times S \to \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, y) = 0$ if and only if $x = y$ (definiteness);
- $d(x, y) = d(x, y)$ for all $x, y \in S$ (symmetry);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

Remark. A metric is sometimes also called a *distance function*.

A metric generalises the notion of distance in Euclidean spaces. We can weaken the above axioms to arrive at the following definition:

Definition 1.2.2 ▶ Pseudo-metric

A **pseudo-metric** on a set S is a function $d: S \times S \to \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, x) = 0$ for all $x \in S$;
- $d(x, y) = d(x, y)$ for all $x, y \in S$ (symmetry);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

The key difference between a pseudo-metric and a metric is that a pseudo-metric only requires that every element is at 0 distance away from itself, whereas a metric requires that every element is **the only element** that is at 0 distance away from itself.

By dropping the requirement on symmetry, we obtain the following definition:

Definition 1.2.3 ▶ Quasi-metric

A **quasi-metric** on a set S is a function $d: S \times S \to \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, y) = 0$ if and only if $x = y$ (definiteness);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

We equip a set with a metric to generalise the Euclidean spaces.

Definition 1.2.4 ▶ Metirc Space

A **metric space** (S, d) is a set S together with a metric d on S.

The most basic example of a metric is the *discrete metric* defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

over any set X, which essentially is just a characteristic function.

Recall that in an inner product space (V, g) over some field $\mathbb{F}$, we can define the length of any $\boldsymbol{v} \in V$ as

$$\|\boldsymbol{v}\| = \sqrt{g(\boldsymbol{v}, \boldsymbol{v})}.$$

Definition 1.2.5 ▸ Norm

Let V be a vector space, the **norm** over V is a mapping $\|\cdot\| : V \to \mathbb{R}$ such that

- $\|\boldsymbol{v}\| \geq 0$ for all $\boldsymbol{v} \in V$.
- $\|\lambda \boldsymbol{v}\| = |\lambda| \|\boldsymbol{v}\|$ for all $\boldsymbol{v} \in V$ and all $\lambda \in \mathbb{R}$.
- $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$.

This length function induces a metric over V given by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|.$$

In the Euclidean space $\mathbb{R}^n$, a usual definition for distance is

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{n} (y_i - x_i)^2\right]^{\frac{1}{2}}.$$

Note that $(\mathbb{R}^n, d_2)$ is a metric space, where d_2 is known as the *Euclidean distance*. In general, we can prove that for any $p \in \mathbb{N}^+$,

$$d_p(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{n} \|y_i - x_i\|^p\right]^{\frac{1}{p}}$$

is a metric over $\mathbb{F}^n$ for any inner product space $(\mathbb{F}^n, g)$ where $\mathbb{F}$ is a field.

Proposition 1.2.6 ▸ L^p-norm Induces a Metric

For any $p \in \mathbb{N}^+$, the L^p-norm $\|\cdot\|_p$ over $\mathbb{F}^n$ induces a metric $d_p : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{R}$ by

$$d_p(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^n \|y_i - x_i\|^p\right]^{\frac{1}{p}}.$$

Proof. It is obvious that $d_p(\boldsymbol{x}, \boldsymbol{y}) \geq 0$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^n$ and it is definite and symmetric. The definiteness is because the the summation is over non-negative terms. It suffices to prove that for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{F}^n$, we have

$$\left(\sum_{i=1}^n \|x_i - z_i\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n \|x_i - y_i\|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n \|y_i - z_i\|^p\right)^{\frac{1}{p}}.$$

Notice that when $p = 1$, since $\|x_i - z_i\| \leq \|x_i - y_i\| + \|y_i - z_i\|$ for all $i = 1, 2, \cdots, n$, the inequality is clearly true. Suppose $p > 1$. Let $\boldsymbol{a} = \boldsymbol{x} - \boldsymbol{y}$ and $\boldsymbol{b} = \boldsymbol{y} - \boldsymbol{z}$. Consider

$$\begin{aligned}\sum_{i=1}^n \|x_i - z_i\|^p &= \sum_{i=1}^n \|a_i + b_i\|^p \\ &\leq \sum_{i=1}^n (\|a_i\| + \|b_i\|)^p \\ &= \sum_{i=1}^n \|a_i\| (\|a_i\| + \|b_i\|)^{p-1} + \sum_{i=1}^n \|b_i\| (\|a_i\| + \|b_i\|)^{p-1}.\end{aligned}$$

Let $q = \frac{p}{p-1} > 1$. By Hölder's inequality, we have

$$\begin{aligned}\sum_{i=1}^n \|a_i\| (\|a_i\| + \|b_i\|)^{p-1} &\leq \left(\sum_{i=1}^n \|a_i\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (\|a_i\| + \|b_i\|)^{(p-1)q}\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^n \|a_i\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (\|a_i\| + \|b_i\|)^p\right)^{\frac{1}{q}}.\end{aligned}$$

Similarly,

$$\sum_{i=1}^n \|b_i\| (\|a_i\| + \|b_i\|)^{p-1} \leq \left(\sum_{i=1}^n \|b_i\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (\|a_i\| + \|b_i\|)^p\right)^{\frac{1}{q}}.$$

Therefore,

$$\sum_{i=1}^{n}(\|a_i\|+\|b_i\|)^p \le \left(\sum_{i=1}^{n}(\|a_i\|+\|b_i\|)^p\right)^{\frac{1}{q}}\left[\left(\sum_{i=1}^{n}\|a_i\|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}\|b_i\|^p\right)^{\frac{1}{p}}\right],$$

which implies that

$$\begin{aligned}\left(\sum_{i=1}^{n}\|x_i-z_i\|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{n}\|a_i+b_i\|^p\right)^{\frac{1}{p}}\\ &\le \left(\sum_{i=1}^{n}(\|a_i\|+\|b_i\|)^p\right)^{\frac{1}{p}}\\ &= \left(\sum_{i=1}^{n}(\|a_i\|+\|b_i\|)^p\right)^{1-\frac{1}{q}}\\ &\le \left(\sum_{i=1}^{n}\|a_i\|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}\|b_i\|^p\right)^{\frac{1}{p}}\\ &= \left(\sum_{i=1}^{n}\|x_i-y_i\|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}\|y_i-z_i\|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore, d_p is a metric for all $p \ge 1$. □

Furthermore, notice that

$$\max_{i\in\mathbb{N}^+, i\le n}\|y_i-x_i\|^p \le \sum_{i=1}^{n}\|y_i-x_i\|^p \le n\max_{i\in\mathbb{N}^+, i\le n}\|y_i-x_i\|^p.$$

Taking the p-th root on all three parts, we have

$$\max_{i\in\mathbb{N}^+, i\le n}\|y_i-x_i\| \le \left[\sum_{i=1}^{n}\|y_i-x_i\|^p\right]^{\frac{1}{p}} \le n^{\frac{1}{p}}\max_{i\in\mathbb{N}^+, i\le n}\|y_i-x_i\|.$$

By Squeeze Theorem, this allows us to define

$$d_\infty(\boldsymbol{x},\boldsymbol{y}) = \lim_{p\to\infty} d_p(\boldsymbol{x},\boldsymbol{y}) = \max_{i\in\mathbb{N}^+, i\le n}\|y_i-x_i\|.$$

$d_\infty(\boldsymbol{x},\boldsymbol{y})$ can be alternatively written as $\|\boldsymbol{x}-\boldsymbol{y}\|_\infty$, which is known as the *infinite norm*.

The p-adic numbers can be defined from the following lemma:

Lemma 1.2.7 ▶ p-adic Numbers

Let p be any prime number. For all $x \in \mathbb{Q} \setminus \{0\}$, there exists a unique $k \in \mathbb{Z}$ such that

$$x = \frac{p^k r}{s}, \qquad r, s \in \mathbb{Z}$$

with $p \nmid r, s$ and $s \neq 0$.

The *p-adic norm* is defined as

$$|x|_p = \begin{cases} p^{-k} & \text{if } x = \frac{p^k r}{s} \\ 0 & \text{if } x = 0 \end{cases},$$

which induces a metric over $\mathbb{Q}$ defined by

$$d(x, y) = |x - y|_p.$$

Proposition 1.2.8 ▶ Verification of the p-adic Norm and Metric

Let p be a prime number and define

$$|x|_p = \begin{cases} p^{-k} & \textit{if } x = \frac{p^k r}{s} \\ 0 & \textit{if } x = 0 \end{cases}$$

where $s, r \in \mathbb{Z}$ with $s \neq 0$ such that $p \nmid r, s$, then $|\cdot|_p$ is a norm and $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ defined by $d(x, y) = |x - y|_p$ is a metric over $\mathbb{Q}$.

Proof. It is obvious that $|x|_p \geq 0$ for all $x \in \mathbb{Q}$. Note that $|x|_p = 0$ if and only if $x = 0$ because $p^{-k} \neq 0$ for all primes p and any $k \in \mathbb{Z}$. Therefore, $|\cdot|_p$ is positive definite. Fix any $x \in \mathbb{Q}$ and $\lambda \in \mathbb{Q}$. Without loss of generality, suppose $x, \lambda \neq 0$ because otherwise clearly $|\lambda x|_p = |\lambda|_p |x|_p = 0$. Note that there exist unique $k_1, k_2 \in \mathbb{Z}$ such that

$$x = p^{k_1} \frac{r_1}{s_1}, \qquad \lambda = p^{k_2} \frac{r_2}{s_2}$$

for some integers r_1, s_1, r_2, s_2 not divisible by p where $s_1, s_2 \neq 0$. Therefore,

$$\lambda x = p^{k_1 + k_2} \frac{r_1 r_2}{s_1 s_2}.$$

It is clear that $p \nmid r_1 r_2$ and $p \nmid s_1 s_2$, so by definition we have

$$|\lambda x|_p = p^{-k_1-k_2} = |\lambda|_p |x|_p.$$

Take $a, b \in \mathbb{Q}$ to be arbitrary rational numbers. If one of them is 0, then it is obvious that $|a+b|_p \leq |a|_p + |b|_p$. Suppose $a, b \neq 0$, then there exist unique $k_1, k_2 \in \mathbb{Z}$ such that

$$a = p^{k_1} \frac{r_1}{s_1}, \qquad b = p^{k_2} \frac{r_2}{s_2}$$

for some $r_1, r_2, s_1, s_2 \in \mathbb{Z}$ which are not divisible by p. Define $k := \min\{k_1, k_2\}$, then

$$\begin{aligned} a + b &= p^k \left(p^{k_1-k} \frac{r_1}{s_1} + p^{k_2-k} \frac{r_2}{s_2} \right) \\ &= p^k \frac{p^{k_1-k} r_1 s_2 + p^{k_2-k} r_2 s_1}{s_1 s_2}. \end{aligned}$$

Clearly, $p \nmid s_1 s_2$. If $p \nmid (p^{k_1-k} r_1 s_2 + p^{k_2-k} r_2 s_1)$, then we are done. Otherwise, write

$$p^{k_1-k} r_1 s_2 + p^{k_2-k} r_2 s_1 = p^{k_0} q$$

for some $k_0, q \in \mathbb{N}^+$ such that $p \nmid q$, then

$$a + b = p^{k+k_0} \frac{q}{s_1 s_2}.$$

Since $k + k_0 \geq k = \min\{k_1, k_2\}$, this means that

$$|a+b|_p \leq p^{-k} \leq p^{-k_1} + p^{-k_2} = |a|_p + |b|_p.$$

Therefore, $|\cdot|_p$ is a norm. Note that this means that d is clearly positive definite, i.e.,

$$d(x, y) = |x - y|_p \geq 0$$

for all $x, y \in \mathbb{Q}$ with equality attained if and only if $x = y$. Since $|-1|_p = p^0 = 1$, we have

$$d(x, y) = |x - y|_p = |y - x|_p = d(y, x),$$

and so d is symmetric. Take any $x, y, z \in \mathbb{Q}$ and consider

$$d(x, y) + d(y, z) = |x - y|_p + |y - z|_p \geq |x - y + y - z|_p = |x - z|_p = d(x, z),$$

so d satisfies triangle inequality. Therefore, d is a metric. □

We can show that the p-adic metric satisfies

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}$$

for all $x, y, z \in \mathbb{Q}$. Such a metric is known as an *ultra-metric*.

Given any metric space, the metric will induce a distance between subsets of the space.

Definition 1.2.9 ▸ Distance between Subsets

Let (X,d) be a metric space and $A, B \subseteq X$ be non-empty. The **distance** between A and B is defined as

$$d(A,B) := \inf\{d(x,y) : (x,y) \in A \times B\}.$$

Note that by Definition 1.2.9, we essentially measure the "closeness" between two subsets A and B by the distance between the closest pair $(a,b) \in A \times B$. However, this may not be a very robust measure because two drastically different subsets could share a single common point, leading to a zero distance.

Instead, we consider the following construction: given any sets A and B, if every point $a \in A$ is close to some point $b \in B$ and vice versa, then A and B should be close. Therefore, from either set, we take the distance from each point to the closest point in the other set and find the maximum of these distances.

Definition 1.2.10 ▸ Hausdorff Metric

Let (X,d) be a metric space. The **Hausdorff metric** is defined as

$$d_H(A,B) := \max\left\{\sup_{a\in A} d(\{a\},B), \sup_{b\in B} d(A,\{b\})\right\}.$$

Equivalently, we have the following definition:

Theorem 1.2.11 ▸ Equivalent Definition of Hausdorff Metric

Let (X,d) be a metric space and define

$$B_\epsilon(Y) := \bigcup_{y\in Y} B_\epsilon(y).$$

The mapping $d_H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$ defined by

$$d_H(A,B) = \inf\{\epsilon \geq 0 : A \subseteq B_\epsilon(B) \text{ and } B \subseteq B_\epsilon(A)\}$$

is the Hausdorff metric.

Additionally, we may wish to define a measure for the size of a subset in a metric space.

Definition 1.2.12 ▸ Diameter

Let (X, d) be a metric space. The **diameter** of a set $A \subseteq X$ is defined as

$$\operatorname{diam}(A) := \sup\{d(x, y) : (x, y) \in A \times A\}.$$

The set A is **bounded** if $\operatorname{diam}(A)$ is finite.

The name "diameter" is not a coincidence with the diameter of a graph. Specifically, if we consider a graph $G = (V, E)$, the pair (V, d) forms a metric space with $d(u, v)$ being the usual distance between two vertices in G defined as the size of the shortest u-v path in G. It is clear that d is indeed a metric.

Now, let us consider the subgraph $H \subseteq G$ induced by any $U \subseteq V$ and check the eccentricity for H, i.e.,

$$\epsilon(u) = \max\{d_H(u, u') : u' \in U\} \qquad \text{for all } u \in U.$$

Now, the diameters for H can be computed as

$$\begin{aligned} \operatorname{diam}(H) &= \max\{\epsilon(u) : u \in U\} \\ &= \sup\{d_H(u, u') : (u, u') \in U \times U\}, \end{aligned}$$

and this obviously agrees with Definition 1.2.12!

Recall that in Definition 1.1.5, we use Euclidean open balls to construct a basis for a topology on $\mathbb{R}^n$. We can generalise this idea in any metric space.

Proposition 1.2.13 ▸ Metric Induces a Basis

Let (X, d) be a metric space. Define

$$B_r(x) := \{y \in X : d(x, y) < r\},$$

then collection

$$\mathcal{B}_d := \{B_r(x) : x \in X, r \in \mathbb{R}^+\}$$

is a basis for a topology on X.

Proof. Notice that for any $x \in X$, we have $x \in B_1(x) \in \mathcal{B}_d$. Let $B_p(x_1), B_q(x_2) \in \mathcal{B}_d$ be such that $x \in B_p(x) \cap B_q(x)$. Take $k = \min\{p - d(x, x_1), q - d(x, x_2)\}$, then clearly $k > 0$ and we can find $B_k(x) \subseteq B_p(x) \cap B_q(x)$ such that $x \in B_k(x) \in \mathcal{B}_d$. Therefore, $\mathcal{B}_d$ is a basis for a topology on X. $\square$

Since we can obtain a basis from a metric, it follows naturally that we can generate a topology using this induced basis.

Definition 1.2.14 ▸ Metrisable Topology

Let (X, d) be a metric space. A topology $\mathcal{T}$ on X is **metrisable**, or **induced** by d, if it is generated by $\mathcal{B}_d$

We can verify that the discrete topology $\mathcal{P}(X)$ is induced by the discrete metric. Let the discrete metric on X be χ, then it is easy to see that

$$B_r(x) = \begin{cases} \{x\} & \text{if } 0 < r \leq 1 \\ X & \text{if } r > 1 \end{cases}.$$

Therefore,

$$\mathcal{B}_\chi = \{X\} \cup \big\{\{x\} : x \in X\big\}.$$

Let $\mathcal{T}_\chi$ be the topology on X generated by $\mathcal{B}_\chi$, then it suffices to prove that $\mathcal{P}(X) \subseteq \mathcal{T}_\chi$. Take any $U \in \mathcal{P}(X)$, then for any $u \in U$, we have $u \in \{u\} \subseteq U$. Clearly $\{u\} \in \mathcal{B}_\chi$, so $\mathcal{T}_\chi = \mathcal{P}(X)$ is the discrete topology indeed.

It is worth noting that different metrics may not necessarily induce different topologies. An intuition is that if two metrics generate the same set of open balls, then they should induce the same topology.

Proposition 1.2.15 ▸ Equivalence of Metrics

Let d and d' be metrics on X inducing topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively. If there exists constants $C_1, C_2 \neq 0$ such that

$$C_1 d(x, y) \leq d'(x, y) \leq C_2 d(x, y)$$

for all $x, y \in X$, then $\mathcal{T} = \mathcal{T}'$.

Proof. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases induced by d and d' respectively. Take any $B_r(x) \in \mathcal{B}$, then for any $y \in B_r(x)$, we have $d(x, y) < r$. Take

$$\ell := \frac{1}{2}|r - d(x, y)| \leq \min\{d(x, y), r\},$$

then $B_\ell(y) \subseteq B_r(x)$. Take $k := \frac{1}{2C_2}\ell$ and consider $B'_k(y) \subseteq \mathcal{B}'$. For each $z \in B'_k(y)$, we have

$$d'(y, z) \leq C_2 d(y, z) < \ell$$

and so $y \in B'_k(y) \subseteq B_r(x)$. By Proposition 1.1.7, $\mathcal{T} \subseteq \mathcal{T}'$. By a similar argument, we can show that $\mathcal{T}' \subseteq \mathcal{T}$. Therefore, $\mathcal{T} = \mathcal{T}'$. □

In particular, we can use this result to prove that all L^p metrics are equivalent, and thus for Euclidean spaces, the following result extends Definition 1.1.5:

Corollary 1.2.16 ▸ Every L^p-metric Induces the Standard Topology

Let $\mathcal{T}$ be the standard topology on $\mathbb{R}^n$, then $\mathcal{T}$ is induced by any L^p-metric d_p.

Proof. For any $p \in \mathbb{N}^+$, notice that

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p \leq \sum_{i=1}^n \|y_i - x_i\|^p \leq n \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p.$$

Taking the p-th root yields

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\| \leq \left[\sum_{i=1}^n \|y_i - x_i\|^p\right]^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

This means that

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_p(\boldsymbol{x}, \boldsymbol{y}) \leq n^{\frac{1}{p}} d_\infty(\boldsymbol{x}, \boldsymbol{y}).$$

By Proposition 1.2.15, since $\mathcal{T}$ is induced by d_2, all L^p-metrics induce $\mathcal{T}$. □

Here, the fact that

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_p(\boldsymbol{x}, \boldsymbol{y}) \leq n^{\frac{1}{p}} d_\infty(\boldsymbol{x}, \boldsymbol{y})$$

means that all L^p-metrics are equivalent over the same space.

1.3 Subspace Topologies

Topologies over a space can be "inherited" by its subspaces.

Definition 1.3.1 ▸ Subspace Topology

Let $(Y, \mathcal{T}_Y)$ be a topological space and $X \subseteq Y$ be some subset. The collection

$$\mathcal{T}_X := \{U \cap X : U \in \mathcal{T}_Y\}$$

is the **subspace topology** on X.

We may check that $\mathcal{T}_X$ defined as such is indeed a topology on X. First, by taking $U = \varnothing$ and $U = Y$ respectively, we know that $\varnothing, X \in \mathcal{T}_X$. For any $U \in \mathcal{T}_Y$, we have $Y \setminus U \in \mathcal{T}_Y$ and so

$$X \setminus (U \cap X) = (Y \setminus U) \cap X \in \mathcal{T}_X.$$

For any $\mathcal{V} \subseteq \mathcal{T}_X$, we define a subset $\mathcal{U}_{\mathcal{V}} \subseteq \mathcal{T}_Y$ such that for each $V \in \mathcal{V}$ there is a unique $U_V \in \mathcal{U}_{\mathcal{V}}$ such that $V = U_V \cap X$. Then,

$$\begin{aligned} \bigcup_{A \in \mathcal{V}} A &= \bigcup_{B \in \mathcal{U}_{\mathcal{V}}} (B \cap X) \\ &= \left(\bigcup_{B \in \mathcal{U}_{\mathcal{V}}} B \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

Let $X_1, X_2, \cdots, X_n \in \mathcal{T}_X$ and define $X_i = U_i \cap X$ where $U_i \in \mathcal{T}_Y$ for $i = 1, 2, \cdots, n$, then

$$\begin{aligned} \bigcap_{i=1}^{n} X_i &= \bigcap_{i=1}^{n} (U_i \cap X) \\ &= \left(\bigcap_{i=1}^{n} U_i \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

So $\mathcal{T}_X$ is really a topology on X. Intuitively, the following holds:

Proposition 1.3.2 ▸ Basis for a Subspace

Let $(Y, \mathcal{T}_Y)$ be a topological space and $\mathcal{T}_X$ be the subspace topology on some $X \subseteq Y$. If $\mathcal{B}_Y$ is a basis of $\mathcal{T}_Y$, then

$$\mathcal{B}_X := \{B \cap X : B \in \mathcal{B}_Y\}$$

is a basis of $\mathcal{T}_X$.

Proof. We first prove that $\mathcal{B}_X$ is a basis. Take any $x \in X \subseteq Y$. Note that there exists some $B \in \mathcal{B}_Y$ such that $x \in B$. Take $B \cap X \in \mathcal{B}_X$, then $x \in B \cap X$. For any $B_1, B_2 \in \mathcal{B}_X$ with $x \in B_1 \cap B_2$, we write $B_1 := B_1' \cap X$ and $B_2 := B_2' \cap X$ where $B_1', B_2' \in \mathcal{B}_Y$, then we have $x \in B_1' \cap B_2'$. This means that there is some $B \in \mathcal{B}_Y$ such that $x \in B \subseteq B_1' \cap B_2'$. Write $B' := B \cap X \in \mathcal{B}_X$, then for each $b \in B'$, we know that $b \in B_1' \cap B_2'$ and $b \in X$, which implies that $b \in B_1 \cap B_2$. Therefore, $x \in B' \subseteq B_1 \cap B_2$. This means that $\mathcal{B}_X$ is a basis of a topology on X.

We then prove that $\mathcal{T}_X$ is generated by $\mathcal{B}_X$. Let $\mathcal{T}$ be the topology generated by $\mathcal{B}_X$. By Proposition 1.1.4, we have

$$\mathcal{T} = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{B}_X \right\}.$$

Similarly, we can write

$$\mathcal{T}_Y = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{B}_Y \right\}.$$

Take any $T \in \mathcal{T}_X$, then there exists some $\mathcal{V} \subseteq \mathcal{B}_Y$ such that

$$\begin{aligned} T &= \left(\bigcup_{A \in \mathcal{V}} A \right) \cap X \\ &= \bigcup_{A \in \mathcal{V}} A \cap X \\ &\in \mathcal{T}. \end{aligned}$$

Therefore, $\mathcal{T}_X \subseteq \mathcal{T}$. Conversely, take any $T' \in \mathcal{T}$, there exists some $\mathcal{U} \subseteq \mathcal{B}_Y$ such that

$$\begin{aligned} T' &= \bigcup_{B \in \mathcal{U}} (B \cap X) \\ &= \left(\bigcup_{B \in \mathcal{U}} B \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

Therefore, $\mathcal{T} \subseteq \mathcal{T}_X$ and so $\mathcal{T}_X = \mathcal{T}$. □

Naturally, every **proper** open subset in a subspace is still open in the containing space. The only edge case remains to be the subspace itself, which is addressed in the following result:

Proposition 1.3.3 ▶ Superspace Preserve Open Sets

Let $(Y, \mathcal{T}_Y)$ be a topological space. If $X \subseteq Y$ is open in Y and $U \subseteq X$ is open in X, then U is open in Y.

Proof. Let $\mathcal{T}_X$ be the subspace topology on X. Since U is open in X, we have $U \in \mathcal{T}_X$. By Definition 1.3.1, there exists some $V \in \mathcal{T}_Y$ such that $U = V \cap X$. However, $U \subseteq X$, so $U = V \in \mathcal{T}_Y$, which means that U is open in Y. □

We can do a similar manipulation with metric spaces and induce a metric on a subspace.

Definition 1.3.4 ▶ Subspace Metric

Let (X, d) be a metric space. The **subspace metric** of some $A \subseteq X$ is the restriction of d to A, denoted as

$$d_A(x, y) = d(x, y), \qquad \text{for all } x, y \in A.$$

Naturally, the following result is true:

Proposition 1.3.5 ▶ Subspace Metric Induces Subspace Topology

Let (X, d) be a metric space. The topology induced by the subspace metric d_A on some subspace $A \subseteq X$ is the subspace topology on A with respect to the metrisable topology on X.

Proof. Let $\mathcal{T}_d$ and $\mathcal{T}_{d_A}$ be topologies induced by d on X with basis $\mathcal{B}_d$ and by d_A on A with basis $\mathcal{B}_{d_A}$ respectively. Let $\mathcal{T}_A$ be the subspace topology on A with basis $\mathcal{B}_A$. Take any $B_A \in \mathcal{B}_A$, then there exists $B_r(x) \in \mathcal{B}_d$ such that $B_A = B_r(x) \cap A$. For any $y \in B_A$, consider the ball

$$B_{r'}(y) := \{z \in A : d_A(z, y) < r'\} \in \mathcal{B}_{d_A}.$$

Note that $y \in B_{r'}(y)$, so by Proposition 1.1.7, we have $\mathcal{T}_A \subseteq \mathcal{T}_{d_A}$. Conversely, for any $B_r(x) \in \mathcal{B}_{d_A}$, there exists some $B_{r'}(x) \in \mathcal{B}_d$ such that $B_r(x) \subseteq B_{r'}(x)$. Notice that $B_r(x) \subseteq A$, so for any $y \in B_r(x)$, we have $y \in B_{r'}(x) \cap A \in \mathcal{B}_A$. Therefore, by Proposition 1.1.7, $\mathcal{T}_{d_A} \subseteq \mathcal{T}_A$ and so $\mathcal{T}_{d_A} = \mathcal{T}_A$. □

1.4 Closed Sets

The opposite of an open set is a closed set.

Definition 1.4.1 ▶ Closed Set

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is **closed** if $X \setminus A \in \mathcal{T}$.

A set might be open and closed simultaneously. For example, every set X is both open and closed in itself.

It is also possible that a set is neither open nor closed. For example, $\mathbb{Q}$ is not open in $\mathbb{R}$ because it is not the union of open balls in $\mathbb{R}$. It is also not closed because its complement is also not the union of open balls in $\mathbb{R}$. This result stems from the fact that both $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{R}$.

Since closed sets are the complements of open sets, they should exhibit opposite properties regarding closure under intersection and union.

Proposition 1.4.2 ▶ Arbitrary Intersection and Finite Union of Closed Sets Are Closed

Let $(X, \mathcal{T})$ be a topological space, then

1. *if $\mathcal{G} := \{G_\alpha : \alpha \in I\}$ is a family of closed set in X with respect to some index set I, then $\bigcap_{\alpha \in I} G_\alpha$ is closed in X;*
2. *if $G_1, G_2, \cdots, G_n$ are closed in X, then $\bigcup_{i=1}^n G_i$ is closed in X.*

Proof. Notice that

$$X \setminus \bigcap_{\alpha \in I} G_\alpha = \bigcup_{\alpha \in G} X \setminus G_\alpha.$$

Since $X \setminus G_\alpha$ is open in X for all $\alpha \in I$, this means that $X \setminus \bigcap_{\alpha \in I} G_\alpha$ is open in X, and so $\bigcap_{\alpha \in I} G_\alpha$ is closed in X. Notice also that

$$X \setminus \bigcup_{i=1}^{n} G_i = \bigcap_{i=1}^{n} X \setminus G_i.$$

By a similar argument $\bigcup_{i=1}^n G_i$ is closed in X. □

Recall that an open set in a subspace is formed by intersecting an open set in the containing space with the subspace. For closed sets, it is intuitive that the same construction should still work.

Proposition 1.4.3 ▶ Closed Sets in Subspace Topology

Let $Y \subseteq X$, then $A \subseteq Y$ is closed in Y if and only if there exists some closed set $G \subseteq X$ such that $A = G \cap Y$.

Proof. Suppose that A is closed in Y, then $Y \setminus A$ is open in Y. Therefore, there exists some open set $B \subseteq X$ such that $Y \setminus A = B \cap Y$. Take $G \coloneqq X \setminus B$, then

$$G \cap Y = A.$$

Suppose conversely that there exists some closed set $G \subseteq X$ such that $A = G \cap Y$. Consider

$$Y \setminus (G \cap Y) = (X \setminus G) \cap Y.$$

Notice that $X \setminus G$ is open in X, so $Y \cap A$ is open in Y, i.e., A is closed in Y. □

The following result is analogous to Proposition 1.3.3:

Proposition 1.4.4 ▸ Superspace Preserves Closed Sets

If $Y \subseteq X$ is closed in X and $A \subseteq Y$ is closed in Y, then A is closed in X.

Proof. Consider $X \setminus A = X \setminus Y \cup Y \setminus A$. Since Y is closed in X, this means that $X \setminus Y$ is open in X. Note also that $Y \setminus A$ is open in Y. By Proposition 1.3.3, $Y \setminus A$ is open in X. Therefore, $X \setminus A$ is open in X and so A is closed in X. □

In an open set, there is an open ball — however small — centred at each point of the set which is contained in the set. This means that an open set has no clear notion of a "boundary". However, for closed sets, it is intuitive that these sets have a clear boundary.

Definition 1.4.5 ▸ Interior, Closure and Boundary

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The **interior** of A is

$$\mathring{A} \coloneqq \bigcup_{\substack{U \in \mathcal{T} \\ U \subseteq A}} A \cap U.$$

The **closure** of A is

$$\overline{A} \coloneqq \bigcap_{\substack{X \setminus G \in \mathcal{T} \\ A \subseteq G}} G.$$

The **boundary** of A is

$$\partial A = \overline{A} \setminus \mathring{A}.$$

We interpret the above definition as follows:

- $\mathring{A}$ is the union of all open sets contained by A;
- $\overline{A}$ is the smallest closed set in X which contains A.

To see the second point, let $C \subseteq X$ be any closed set in X containing A. Take any $a \in \overline{A}$, then since a is contained by all closed sets containing A, it is clear that $a \in C$, which implies that $\overline{A} \subseteq C$.

Remark.

1. $\mathring{A} \subseteq A \subseteq \overline{A}$.
2. $\mathring{A} = A$ if and only if A is open in X.
3. $\overline{A} = A$ if and only if A is closed in X.

The above results come in handy to prove the existence of sets which are neither open nor closed. Consider

$$A \coloneqq \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}.$$

Observe that with respect to the Euclidean topology on $\mathbb{R}$, the only open subset of A is $\varnothing$, so $\mathring{A} = \varnothing \neq A$, which implies that A is not open.

To find $\overline{A}$ requires a bit of work. Intuitively, the boundary ∂A separates A from the rest of the space X. Therefore, it is natural that every open neighbourhood of a point $a \in \partial A$ has to intersect with A.

Now, this means that we can shrink such an open neighbourhood indefinitely while maintaining a non-empty intersection. If this intersection contains more than the point a itself, then this analogously means that we can find points in A which are **arbitrarily close** to a.

Definition 1.4.6 ▶ Limit Point

Let $(X, \mathcal{T})$ be a topological space. For any $A \subseteq X$, a point $x \in X$ is a **limit point** of A if for every open set $U \subseteq X$ containing x,

$$(A \setminus \{x\}) \cap U \neq \varnothing.$$

Note that this definition is essentially saying that **every open neighbourhood of a limit point of a set A contains at least one other element in A.**

Intuitively, $\overline{A}$ should contain all limit points of A because every point outside of $\overline{A}$ is at some "distance" away from ∂A. Formally, for every $a \in X \setminus \overline{A}$, since $X \setminus \overline{A}$ is open, we can always find some open ball $B \subseteq X$ such that $a \in B$. Clearly, $B \cap (A \setminus \{a\}) = \varnothing$ because $A \subseteq \overline{A}$.

We know that $A \subseteq \overline{A}$, so the above simple result motivates us to conclude that $\overline{A}$ contains nothing else but A and all limit points of A.

Proposition 1.4.7 ▶ A Set with All Its Limit Points Is Its Closure

Let $(X,\mathcal{T})$ be a topological space. For any $A \subseteq X$,

1. *$x \in \overline{A}$ if and only if for any open set $U \subseteq X$ containing x, $U \cap A \neq \varnothing$;*
2. *if A' is the set of limit points of A, then $\overline{A} = A \cup A'$.*

Proof. We will prove the first statement by considering the contrapositive, i.e., we prove that $x \in X \setminus \overline{A}$ if and only if there exists some open set $U \subseteq X$ containing x such that $U \cap A = \varnothing$. The sufficiency is trivial because $X \setminus \overline{A}$ is open in X such that $\left(X \setminus \overline{A}\right) \cap A = \varnothing$. Take $U \subseteq X$ to be an open set in X with $x \in U$ and $U \cap A = \varnothing$. This means that $A \subseteq X \setminus U$ which is open, and so $\overline{A} \subseteq X \setminus U$. Therefore, $x \in X \setminus \overline{A}$.

Take any $a \in A'$, then for every open set $U \subseteq X$ with $a \in U$, we have

$$U \cap A \supseteq U \cap (A \setminus \{a\}) \neq \varnothing.$$

Therefore, $A \cup A' \subseteq \overline{A}$. Take any $x \in \overline{A}$, we shall prove that if $x \notin A'$, then $x \in A$. Since x is not a limit point of A, there exists some open set $V \subseteq X$ containing x such that $(A \setminus \{x\}) \cap V = \varnothing$. However, $x \in \overline{A}$ implies that $V \cap A \neq \varnothing$, so $x \in A$. This means that $\overline{A} \subseteq A \cup A'$ and so $A \cup A' = \overline{A}$. □

Now let us rewind and consider our set

$$A \coloneqq \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}$$

again. It is easy to verify that 0 is the only limit point of A, so by Proposition 1.4.7,

$$\overline{A} = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\} \neq A.$$

Therefore, A is neither open nor closed in $\mathbb{R}$. In fact, we have $A \subseteq \overline{A} = \partial A$, so this also shows that $\partial A \neq A'$ in general.

It is a good exercise for us to verify that $\overline{A}$ is indeed closed here. We claim that

$$\mathbb{R} \setminus \overline{A} = (-\infty, 0) \cup (1, +\infty) \cup \bigcup_{n \in \mathbb{N}^+} \left(\frac{1}{n+1}, \frac{1}{n}\right).$$

Take any $a \in (0,1)$ such that $a \neq \frac{1}{n}$ for all $n \in \mathbb{N}^+$, then there exists some $n_a \in \mathbb{N}^+$ such that

$$a \in \left(\frac{1}{n_a + 1}, \frac{1}{n_a}\right),$$

and so the equality holds as desired.

Another way to interpret the closure of a set A is that it contains all points in A and all points which are "arbitrarily close" to some point in A. We now formalise this notion as *density*.

Definition 1.4.8 ▸ Dense Set

Let X be a topological space and $A \subseteq X$ be any subset, then A is **dense** in X if for any $x \in X$ and any open neighbourhood $U \subseteq X$ of x, we have $A \cap U \neq \varnothing$.

Naturally, everything in $\overline{A}$ is either directly included by A or "almost" included by A, so A is dense in $\overline{A}$. This motivates the following result:

Proposition 1.4.9 ▸ Equivalent Definitions for Dense Sets

For any topological space X and any subset $S \subseteq X$, then the following are equivalent:

- *S is dense in X;*
- $X \mathring{\setminus} S = \varnothing$;
- $\overline{S} = X$.

Proof. Note that it is equivalent to proving the equivalence of the following statements:

- $\overline{S} \neq X$.
- $X \mathring{\setminus} S \neq \varnothing$.
- S is not dense in X.

Suppose that $\overline{S} \neq X$, then there exists some $x \in X \setminus \overline{S} \subseteq X \setminus S$. Note that $\overline{S}$ is closed, so $X \setminus \overline{S}$ is open in X, which means that $X \setminus \overline{S} \subseteq X \mathring{\setminus} S$. Therefore, $x \in X \mathring{\setminus} S$ and so $X \mathring{\setminus} S \neq \varnothing$.

Suppose that $X \mathring{\setminus} S \neq \varnothing$, then there exists some non-empty open set $U \subseteq X \setminus S$. Fix some $x \in U \subseteq X$, then U is a neighbourhood of x but $U \cap S = \varnothing$. Therefore, S is not dense.

Lastly, it suffices to prove that if $\overline{S} = X$, then S is dense in X. Take any $x \in X$ and let $U \subseteq X$ be an arbitrary neighbourhood of x. If $x \in S$ then we are done. Otherwise, x must be a limit point of S. Therefore, $U \cap (S \setminus \{x\}) \neq \varnothing$, which implies that $U \cap S \neq \varnothing$, and so S is dense in X. □

1.5 Convergence and Continuity

Recall that in Definition 1.4.6, a limit point of A is defined as a point a such that any open set containing a shares at least one other common point with A. Analogously, we may view such phenomenon as a having a "neighbour" in A.

Definition 1.5.1 ▸ Neighbourhood

Let $(X, \mathcal{T})$ be a topological space. An open set $U \subseteq X$ is called a **neighbourhood** of some $x \in X$ if $x \in U$.

Now let us generalise the idea of *convergence*. Intuitively, we think of the statement $x_i \to x$ as the fact that x_i will become **arbitrarily close** to x in the long-run. Equivalently, this means that no matter how small a neighbourhood we choose for x, the "tail" of the x_i's will always fall in this neighbourhood.

Definition 1.5.2 ▸ Convergence

A sequence $\{x_i\}_{i=1}^{\infty}$ of points in a topological space $(X, \mathcal{T})$ **converges** to $x \in X$ if for any neighbourhood $U \subseteq X$ containing x, there exists some $N \in \mathbb{N}^+$ such that $x_k \in U$ for all $k > N$, denoted as $x_i \to x$. x is said to be the **limit** of $\{x_i\}_{i=1}^{\infty}$.

It is important to distinguish between limit and limit points. For example, consider the constant sequence $\{1\}_{i=1}^{\infty}$. Clearly, $x_i \to 1$ but one may check that 1 is not a limit point for this sequence.

On the other direction, it is also not true in general that for every limit point of a set, we can find a sequence converging to it. For example, consider

$$A \coloneqq \mathbb{N}^+ \cup \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\},$$

which obviously does not converge but has a limit point at 0.

In a metric space, we can make use of the metric to describe convergence in a more quantitative way.

Theorem 1.5.3 ▸ Convergence in Metric Spaces

Let (X, d) be a metric space. A sequence $\{x_i\}_{i=1}^{\infty}$ in X converges to x if and only if for every $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that $d(x_i, x) < \epsilon$ for all $i > N$.

Proof. Suppose that $x_i \to x$ as $i \to \infty$. For all $\epsilon > 0$, take the open ball $B_\epsilon(x) \subseteq X$. Clearly, $B_\epsilon(x)$ is a neighbourhood of x. By Definition 1.5.2, there exists some $N \in \mathbb{N}^+$ such that $x_i \in B_\epsilon(x)$ for all $i > N$, i.e., $d(x_i, x) < \epsilon$ for all $i > N$. Conversely, suppose that for every $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that $d(x_i, x) < \epsilon$ for all $i > N$. Let $U \subseteq X$ be any neighbourhood containing x. Note that U is open in X, so by Theorem 1.1.4, there exists some open ball $B_r(x) \subseteq U$ such that $x \in B_r(x)$.

Therefore, there exists some $M \in \mathbb{N}^+$ such that $d(x_i, x) < r$, i.e., $x_i \in B_r(x) \subseteq U$, for all $i > M$. Therefore, $x_i \to x$. $\square$

We can also generalise the notion of *continuity*. Recall that in real analysis, a function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. Therefore, if we fix any open neighbourhood U for a, we would expect the pre-image $f^{-1}(U)$ to be open.

Definition 1.5.4 ▶ Continuous Map

Let X and Y be topological spaces. A map $f : X \to Y$ is **continuous** if for any open set $U \subseteq Y$, the pre-image $f^{-1}(U)$ is open in X.

A map is continuous if and only if pre-images of open sets are open, but an open set may not necessarily have an open image under a continuous map. Here are 2 examples:

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \mathrm{e}^{-|x|}$ is clearly continuous because

$$f^{-1}((a, b)) = \begin{cases} (-\ln b, -\ln a) & \text{if } -\ln b \geq 0 \\ (\ln a, -\ln a) & \text{if } -\ln b < 0 < -\ln a \\ \varnothing & \text{if } 0 \geq -\ln a \end{cases}.$$

   However, $f(\mathbb{R}) = (0, 1]$ which is neither open nor closed.

2. $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ \frac{1}{x+1} & \text{otherwise} \end{cases}$$

   is continuous but $f\big((-\infty, 0)\big) = \{1\}$ is closed and $f\big([0, +\infty)\big) = (0, 1]$ is not closed.

Suppose $\mathcal{T}_X$ and $\mathcal{T}_Y$ are topologies on X and Y respectively. Definition 1.5.4 basically says that for all $U \in \mathcal{T}_Y$, we have $f^{-1}(U) \in \mathcal{T}_X$. The following proposition gives an equivalent definition for continuity in terms of sub-bases.

Proposition 1.5.5 ▶ Equivalent Definition of Continuity

If $\mathcal{S}$ is a sub-basis for a topology on some set Y, then for any topological space X, a map $f : X \to Y$ is continuous if and only if $f^{-1}(S)$ is open in X for any $S \in \mathcal{S}$.

Proof. Suppose that f is continuous. Note that any $S \in \mathcal{S}$ is open in Y, so by Definition 1.5.4, $f^{-1}(S)$ is open in X. Suppose conversely that $f^{-1}(S)$ is open in X for any $S \in \mathcal{S}$. Take any open set $U \subseteq Y$. By Propositions 1.3.2 and 1.1.4, there exists finite subsets $\mathcal{U}_i \subseteq \mathcal{P}(\mathcal{S})$ where $i \in I$ for some index set I such that

$$U = \bigcup_{i \in I} \left(\bigcap_{S \in \mathcal{U}_i} S \right).$$

Therefore,

$$f^{-1}(U) = \bigcup_{i \in I} \left(\bigcap_{S \in \mathcal{U}_i} f^{-1}(S) \right),$$

which is clearly open in X. Therefore, f is continuous. □

A trivial example for continuous maps is the *constant map* $f : X \to Y$ such that $f(x) = y_0$ for some fixed $y_0 \in Y$. This is simply because

$$f^{-1}(U) = \begin{cases} X & \text{if } y_0 \in U \\ \varnothing & \text{otherwise} \end{cases}.$$

The following result should be very intuitive:

Proposition 1.5.6 ▸ Composition Preserves Continuity

Let X, Y, Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then $g \circ f$ is continuous.

It is also clear that for any topological space X, the *inclusion map* $f : A \to X$ for any $A \subseteq X$ such that $f(a) = a$ is continuous. Analogously, if $f : X \to Y$ is continuous, then the restriction $f|_A : A \to Y$ for any subspace $A \subseteq X$ is also continuous.

Proposition 1.5.7 ▸ Properties of Continuous Maps

Let X and Y be topological spaces. For any map $f : X \to Y$, the followings are equivalent:

1. *f is continuous;*
2. *for all $A \subseteq X$, $f\left(\overline{A}\right) \subseteq \overline{f(A)}$;*
3. *for any closed set $B \subseteq Y$, $f^{-1}(B)$ is closed in X;*
4. *for any $x \in X$ and any open set $V \subseteq Y$ with $f(x) \in V$, there exists an open set $U \subseteq X$ such that $x \in U$ and $f(U) \subseteq V$.*

Proof. Suppose that f is continuous. We claim that $\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$. Observe that $X = f^{-1}(Y)$, so

$$X \setminus f^{-1}\left(\overline{f(A)}\right) = f^{-1}(Y) \setminus f^{-1}\left(\overline{f(A)}\right) = f^{-1}\left(Y \setminus \overline{f(A)}\right)$$

Since $\overline{f(A)}$ is closed in Y, so by Definition 1.5.4, we know that $f^{-1}\left(\overline{f(A)}\right)$ is closed in X. For any $a \in A$, we have

$$f(a) \in f(A) \subseteq \overline{f(A)},$$

and so $A \subseteq f^{-1}\left(\overline{f(A)}\right)$. Recall that $\overline{A}$ is the smallest closed set containing A, so $\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$ as desired. Therefore, $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.

Suppose that $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for any $A \subseteq X$. For any closed set $B \subseteq Y$, we have $B = \overline{B}$. Notice that

$$f\left(\overline{f^{-1}(B)}\right) \subseteq \overline{f(f^{-1}(B))} = B = f\left(f^{-1}(B)\right),$$

so $\overline{f^{-1}(B)} \subseteq f^{-1}(B)$. This implies that $\overline{f^{-1}(B)} = f^{-1}(B)$, and so $f^{-1}(B)$ is closed in X.

Suppose that $f^{-1}(B)$ is closed in X for any closed set $B \subseteq Y$. Take any $x \in X$ and any open set $V \subseteq Y$ with $f(x) \in V$. Since $Y \setminus V$ is closed in Y, $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is closed in X. Therefore, $f^{-1}(V)$ is open in X. It is clear that $x \in f^{-1}(V)$ and $f\left(f^{-1}(V)\right) \subseteq V$.

Suppose that for any $x \in X$ and any open set $V \subseteq Y$ with $f(x) \in V$, there exists an open set $U \subseteq X$ such that $x \in U$ and $f(U) \subseteq V$. For any open set $V \subseteq Y$, consider

$$f^{-1}(V) = \{x \in X : f(x) \in V\}.$$

For each $x \in f^{-1}(V)$, fix some open set $U_x \subseteq X$ such that $x \in U_x$ and $f(U_x) \subseteq V$. Notice that this means that $U_x \subseteq f^{-1}(V)$, and so

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

is open in X. Therefore, f is continuous. $\square$

Next, we introduce a lemma which specifies a methodology to construct a continuous map from two different continuous maps.

Lemma 1.5.8 ▸ Pasting Lemma

Let X and Y be topological spaces such that $X = A \cup B$ for some closed sets A and B. If $f : A \to Y$ and $g : B \to Y$ are continuous and $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}.$$

Proof. Let $U \subseteq Y$ be any open set. Then it is clear that $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$. Since f and g are continuous, both $f^{-1}(U)$ and $g^{-1}(U)$ are open in X, so it follows that $h^{-1}(U)$ is open in X. Therefore, h is continuous. □

Observe that the continuity of a function $f : X \to Y$ actually depends on our choice of topologies on X and Y. On the other hand, this also means that every function f could induce a topology on X such that it is continuous.

Definition 1.5.9 ▸ Pull-Back Topology

Let $\mathcal{T}_Y$ be a topology on Y and let $f : X \to Y$. The **pull-back topology** on X is defined as

$$\mathcal{T}_X \coloneqq \left\{ f^{-1}(U) : U \in \mathcal{T}_Y \right\}.$$

Note that the pull-back topology is the coarsest topology on X such that f is a continuous map. To verify this, let $\mathcal{T}$ be any topology on X such that f is continuous. Take any $T \in \mathcal{T}_X$, then there exists some $U \in \mathcal{T}_Y$ such that $T = f^{-1}(U)$. However, this means that $T \in \mathcal{T}$ since f is continuous with respect to $\mathcal{T}$. This shows that $\mathcal{T}_X \subseteq \mathcal{T}$.

We can define continuity in metric spaces quantitatively as follows:

Definition 1.5.10 ▸ Continuity in Metric Spaces

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is **continuous** at a point $x_0 \in X$ if for all $\epsilon > 0$, there exists some $\delta > 0$ such that $d_Y\big(f(x), f(y)\big) < \epsilon$ whenever $d_X(x_0, y) < \delta$.

One can show that Definition 1.5.4 aligns with this definition in metric spaces.

Proposition 1.5.11 ▸ Equivalence of Continuity in General Spaces

Let X be a topological space and (Y, d) be a metric space, then a map $f : X \to Y$ is continuous if and only if for all $\epsilon > 0$, there exists some open neighbourhood $U_x \subseteq X$ of *all $x \in X$ such that $d\big(f(x), f(z)\big) < \epsilon$ for all $z \in U_x$.*

Proof. Suppose that f is continuous. For any $\epsilon > 0$ and any $x \in X$, $d\big(f(x)\big)$, take

$$U_x := f^{-1}\Big(B_\epsilon\big(f(x)\big)\Big)$$

which is an open neighbourhood of x in X, then $f(z) \in B_\epsilon\big(f(x)\big)$ for all $z \in U_x$, and so $d\big(f(x), f(z)\big) < \epsilon$. Conversely, suppose that for all $\epsilon > 0$, there exists some open neighbourhood $U_x \subseteq X$ of all $x \in X$ such that $d\big(f(x), f(z)\big) < \epsilon$ for all $z \in U_x$. Note that it suffices to prove that $f^{-1}\big(B_\epsilon(y)\big)$ is open for all $y \in Y$ and all $\epsilon > 0$. Take any $x \in f^{-1}\big(B_\epsilon(y)\big)$, then there exists some open neighbourhood U_x of x such that $f(U_x) \subseteq B_\epsilon(y)$, which implies that $U_x \subseteq f^{-1}\big(B_\epsilon(y)\big)$. Therefore,

$$f^{-1}\big(B_\epsilon(y)\big) = \bigcup_{f(x) \in B_\epsilon(y)} U_x$$

which is open, and so f is continuous. □

Note that in the above definition, the radius of the open neighbourhood of x_0 is dependent on the choice of x_0. By removing such dependence, we achieve a stronger version of continuity.

Definition 1.5.12 ▸ Uniform Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is **uniformly continuous** on X if for all $\epsilon > 0$, there exists some $\delta > 0$ such that $d_Y\big(f(x), f(y)\big) < \epsilon$ whenever $d_X(x, y) < \delta$.

Essentially, uniform continuity describes a phenomenon where the choice of δ is irrelevant to the point in the function's domain.

We wish to use the following proposition to characterise all uniformly continuous functions:

Proposition 1.5.13 ▸ Uniform Continuity Characterisation

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is uniformly continuous if and only if for any sequences $\{x_i\}_i^\infty$ and $\{y_i\}_i^\infty$ in X such that $\lim_{i\to\infty} d_X(x_i, y_i) = 0$, we have $\lim_{i\to\infty} d_Y\big(f(x_i), f(y_i)\big) = 0$.

Proof. It suffices to prove the "if" direction only because the other direction is trivial from Definition 1.5.12. We shall consider the contrapositive statement. Suppose that there exist sequences $\{x_i\}_i^\infty$ and $\{y_i\}_i^\infty$ with $\lim_{i\to\infty} d_X(x_i, y_i) = 0$ such that $\lim_{i\to\infty} d_Y(f(x_i), f(y_i)) \neq 0$, then for all $\delta > 0$, there exists some $N \in \mathbb{N}^+$ such that $d_X(x_i, y_i) > 0$ for all $i > N$. However, notice that there exists some $\epsilon > 0$ such that for all $M \in \mathbb{N}^+$, there exists some $m > M$ with $d_Y(f(x_m), f(y_m)) \geq \epsilon$. This means that for any $\epsilon > 0$, we can find some k such that for all $\delta > 0$, we have $d_X(x_k, y_k) < \delta$ but $d_Y(f(x_k), f(y_k)) \geq \epsilon$. By Definition 1.5.12, this implies that f is not uniformly continuous. □

Recall that previously, we have defined convergence for sequences. In particular, we can consider a sequence of functions. Note that, if we say that $\{f_n\}_{n=1}^\infty$ converges to some function f, it might have two different interpretations. First, it might be the case where for any fixed x, we have $f_n(x)$ converging to $f(x)$; second it is also possible that the values of f_n and f becomes infinitesimally close when n is large over all possible x. We will formalise this distinction as follows:

Definition 1.5.14 ▸ Point-wise and Uniform Convergence

Let $\{f_n\}_{n=1}^\infty$ be a sequence of maps from X to a metric space (Y, d). We say that $\{f_n\}_{n=1}^\infty$ **converges point-wisely** to $f : X \to Y$ if for any $x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$, and that $\{f_n\}_{n=1}^\infty$ **converges uniformly** to $f : X \to Y$ if for any $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that for all $n \geq N$ and any $x \in X$, $d(f_n(x), f(x)) < \epsilon$.

Clearly, uniform convergence implies point-wise convergence, but the converse is not true in general. In particular, consider $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. It is clear that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \leq x < 1 \end{cases}.$$

Therefore, $\{f_n\}_{n=1}^\infty$ converges point-wisely to $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \leq x < 1 \end{cases}$$

but such convergence is not uniform. This example also shows that the limit of a point-wisely convergent sequence of continuous functions need not be continuous. In fact, we can show that the limit of a uniformly convergent sequence of continuous functions has to be continuous.

Proposition 1.5.15 ▸ Uniform Convergence Implies Continuity

Let $\{f_n\}_{n=1}^\infty$ be a sequence of maps from a metric space (X, d_X) to a metric space (Y, d_Y). If $\{f_n\}_{n=1}^\infty$ converges uniformly to $f : X \to Y$, then f is continuous.

Proof. Since $\{f_n\}_{n=1}^\infty$ converges uniformly to f, for any $x \in X$ and any $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that $d_Y\big(f_n(x), f(x)\big) < \frac{\epsilon}{3}$ for all $n > N$. Since f_n is continuous, for each $x_0 \in X$ there exists some $\delta > 0$ such that $d_Y\big(f_n(x_0), f_n(x)\big) < \frac{\epsilon}{3}$ whenever $d_X(x_0, x) < \delta$. Notice that for all $x \in X$ such that $d_X(x_0, x) < \delta$, we have

$$\begin{aligned} d_Y\big(f(x_0), f(x)\big) &\leq d_Y\big(f(x_0), f_n(x_0)\big) + d_Y\big(f_n(x_0), f_n(x)\big) + d_Y\big(f_n(x), f(x)\big) \\ &< \epsilon. \end{aligned}$$

Therefore, f is continuous on X. $\square$

In other words, a function is continuous if there exists some sequence of continuous functions which converges to it uniformly.

1.6 Product Space

Consider an indexed family of sets $\{X_\alpha\}_{\alpha\in\Lambda}$, then an element in their Cartesian product is a tuple where the k-th component is from X_k. This means that from each element in the Cartesian product, we can "project" it back to its k-th component.

Definition 1.6.1 ▸ Projection

Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a sequence of non-empty sets, consider the product space $\prod_{\alpha\in\Lambda} X_\alpha$. The **projection** on the β-th factor is

$$\pi_{X_\beta} : \prod_{\alpha\in\Lambda} X_\alpha \to X_\beta$$

such that $\pi_{X_\beta}(\boldsymbol{x}) = x_\beta$.

Note that for any $\beta \in \Lambda$ and any $U \subseteq X_\beta$, we have

$$\pi_{X_\beta}^{-1}(U) = \left\{\boldsymbol{x} \in \prod_{\alpha\in\Lambda} X_\alpha : x_\beta \in U\right\}.$$

In other words, the pre-image of projection for some set U is the set of all tuples in the product space whose β-th component is in U. The projection map helps define a topology over the product space.

Definition 1.6.2 ▶ Product Topology

Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\Lambda}$ be a sequence of non-empty topological spaces. The **product topology** is the topology generated by the sub-basis

$$\mathcal{S} := \left\{\pi_{X_\alpha}^{-1}(U_\alpha) : \alpha \in \Lambda, U_\alpha \in \mathcal{T}_\alpha\right\}.$$

Intuitively, the product topology is formed by all tuples whose α-th component is contained in some open set in the slice in the α-th dimension. Now, fix any $k \in \mathbb{N}^+$ and take any k slices indexed $\alpha_1, \alpha_2, \cdots, \alpha_k$. The **finite intersection** of the pre-images of these slices are exactly all tuples whose α_i-th component is in an open set in the α_i-th slice. Note that if Λ is finite, this is equivalent to taking all the slices, but setting the open sets in the irrelevant slices to be the entire space.

There is another perspective to look at the product space. Consider $\mathbb{R}^n$. If we take one open interval I_i from each dimension, then their product $\prod_{i=1}^n I_i$ is an n-dimensional cuboid or "box". So in general, if we take the product of open sets in each dimension, the resultant set in the product space is a union of many such boxes.

Definition 1.6.3 ▶ Box Topology

Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\Lambda}$ be a sequence of non-empty topological spaces. The **box topology** is the topology generated by the basis

$$\mathcal{B} := \left\{\prod_{\alpha\in\Lambda} U_\alpha : U_\alpha \in \mathcal{T}_\alpha\right\}.$$

We may check that the set $\mathcal{B}$ in Definition 1.6.3 is indeed a basis. Take any $\boldsymbol{x} \in \prod_{\alpha\in\Lambda} X_\alpha$, then for each $\alpha \in \Lambda$, there exists some $U_\alpha \in \mathcal{T}_\alpha$ such that $x_\alpha \in U_\alpha$, which implies that

$$\boldsymbol{x} \in \prod_{\alpha\in\Lambda} U_\alpha.$$

If $V := \prod_{\alpha\in\Lambda} U_\alpha$ and $U := \prod_{\alpha\in\Lambda} U_\alpha$ are such that $\boldsymbol{x} \in U \cap V$, then $x_\alpha \in U_\alpha \cap V_\alpha$ for each $\alpha \in \Lambda$. Clearly, $U_\alpha \cap V_\alpha \in \mathcal{T}_\alpha$, so we can take $W := \prod_{\alpha\in\Lambda} U_\alpha \cap V_\alpha$ such that

$$\boldsymbol{x} \in W \subseteq V \cap U.$$

Remark. For finite product spaces, the product topology and the box topology are the same. However, they are different in general if the product is infinite.

As a counter example to illustrate the above remark, consider $\{X_n\}_{n\in\mathbb{N}^+}$ be a sequence of countably infinitely many pair-wise disjoint topological spaces such that every space contains at least one non-empty proper open set U_n. Take

$$U := \prod_{n=1}^{\infty} U_n,$$

then U is clearly open with respect to the box topology but not the product topology. In particular, let $\pi(\mathbb{N}^+)$ be a permutation of $\mathbb{N}^+$, then each open set with respect to the product topology can be written as

$$U := \prod_{i=1}^{n} U_{\pi(\mathbb{N}^+)_i} \times \prod_{i=n+1}^{\infty} X_{\pi(\mathbb{N}^+)_i}$$

for some $n \in \mathbb{N}^+$, where $U_{\pi(\mathbb{N}^+)_i}$ is open in $X_{\pi(\mathbb{N}^+)_i}$. This also shows that the box topology is **finer** than the product topology. One product space constructed in a similar manner is the *Hilbert cube* $[0,1]^{\mathbb{N}}$.

Observe that by using the box topology, we can easily decompose $\mathbb{R}^n$ into the product of n copies $\mathbb{R}$ while maintaining the topological structure, i.e., the product topology induced on the product space is the standard topology on the product space. However, we can actually generalise this further.

Proposition 1.6.4 ▸ Standard Topology Decomposition on Euclidean Spaces

Let $n \in \mathbb{Z}^+$ and $n = \sum_{i=1}^{k} m_i$ with $m_i \in \mathbb{Z}^+$ for $i = 1, 2, \cdots, k$. Let $\mathcal{T}_i$ be the standard topology on $\mathbb{R}^{m_i}$ and $\mathcal{T}$ be the product topology induced by the $\mathcal{T}_i$'s, then $\mathcal{T}$ is the standard topology on $\mathbb{R}^n$.

Proof. Let $\mathcal{B}_n$ be the standard basis generating the standard topology $\mathcal{T}_n$ over $\mathbb{R}^n$ for each $n \in \mathbb{N}^+$. Take any $B_r^{(n)}(\boldsymbol{x}) \in \mathcal{B}_n$. For any $\boldsymbol{y} \in B_r^{(n)}(\boldsymbol{x})$, define $t = r - \|\boldsymbol{x} - \boldsymbol{y}\|$ and consider, for each $i = 1, 2, \cdots, k$,

$$B_t^{(m_i)}\left(\pi_{\mathbb{R}^{m_i}}(\boldsymbol{y})\right) \in \mathcal{B}_{m_i} \subseteq \mathcal{T}_i$$

to be an open ball centred at the projection of $\boldsymbol{y}$ to the m_i-th factor. Denote

$$B_t := \prod_{i=1}^{k} B_t^{(m_i)}\left(\pi_{\mathbb{R}^{m_i}}(\boldsymbol{y})\right).$$

Clearly, B_t is contained by the basis for the product topology induced by the $\mathcal{T}_i$'s. For any $\boldsymbol{p} \in B_t$, we have

$$\|\boldsymbol{p} - \boldsymbol{x}\| \leq \|\boldsymbol{p} - \boldsymbol{y}\| + \|\boldsymbol{x} - \boldsymbol{y}\| < r.$$

Therefore, $\boldsymbol{y} \in B_t \subseteq B_r^{(n)}(\boldsymbol{x})$. By Proposition 1.1.7, $\mathcal{T} \subseteq \mathcal{T}_n$. Conversely, take an arbitrary $U_i \in \mathcal{T}_i$ for each $i = 1, 2, \cdots, k$. Notice that for any $\boldsymbol{x} \in \prod_{i=1}^k U_i$, there is some open ball $B_{r_i}^{(m_i)}(\pi_{\mathbb{R}^{m_i}}(\boldsymbol{x})) \in \mathcal{B}_{m_i}$, centred at the projection of $\boldsymbol{x}$ to the m_i-th factor, such that $B_{r_i}^{(m_i)}(\pi_{\mathbb{R}^{m_i}}(\boldsymbol{x})) \subseteq U_i$. Take

$$r := \min_{1 \leq i \leq k} r_i$$

and consider the open ball $B_r^{(n)}(\boldsymbol{x}) \in \mathcal{B}_n$. For any $\boldsymbol{y} \in B_r^{(n)}(\boldsymbol{x})$, consider

$$\|\boldsymbol{y} - \boldsymbol{x}\| = \sum_{i=1}^k \left\|\pi_{\mathbb{R}^{m_i}}(\boldsymbol{y}) - \pi_{\mathbb{R}^{m_i}}(\boldsymbol{x})\right\| < r \leq \min_{1 \leq i \leq k} r_i.$$

Therefore, for every positive integer $i \leq k$, we have

$$\left\|\pi_{\mathbb{R}^{m_i}}(\boldsymbol{y}) - \pi_{\mathbb{R}^{m_i}}(\boldsymbol{x})\right\| < r_i,$$

which implies that $\pi_{\mathbb{R}^{m_i}}(\boldsymbol{y}) \in B_{r_i}^{(m_i)}(\pi_{\mathbb{R}^{m_i}}(\boldsymbol{x})) \subseteq U_i$. Therefore, $\boldsymbol{y} \in \prod_{i=1}^k U_i$ and so $\mathcal{T}_n \subseteq \mathcal{T}$ by Proposition 1.1.7. Therefore, $\mathcal{T} = \mathcal{T}_n$. ☐

The importance of the product topology is that it is the coarsest topology to ensure the existence of continuous projection maps.

Proposition 1.6.5 ▸ Product Topology Guarantees Continuous Projection

Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \Lambda}$ be a sequence of non-empty topological spaces. The product topology on $\prod_{\alpha \in \Lambda} X_\alpha$ is the coarsest topology such that π_{X_α} is continuous for all $\alpha \in \Lambda$.

Proof. Let $\mathcal{S}$ be the sub-basis generating the product topology and let $U \subseteq X_\alpha$ be open, then $\pi_{X_\alpha}^{-1}(U) \in \mathcal{S}$ and therefore is open. Therefore, π_{X_α} is continuous with respect to the product topology.

Let $\mathcal{T}$ be any topology on $\prod_{\alpha \in \Lambda} X_\alpha$ such that π_{X_α} is continuous, then for any open set $U \in \mathcal{T}$, the pre-image $\pi_{X_\alpha}^{-1}(U) \in \mathcal{S}$ must be open, so $\mathcal{S} \subseteq \mathcal{T}$. Notice that this is equivalent to $\mathcal{T}$ containing

$$\mathcal{B} := \left\{\bigcap_{\alpha \in \Lambda} \pi_{X_\alpha}^{-1}(U_\alpha) : \Lambda' \subseteq \Lambda \text{ is finite}, U_\alpha \in \mathcal{T}_\alpha\right\}.$$

However, $\mathcal{B}$ generates the product topology, which is the coarsest topology containing $\mathcal{B}$, so $\mathcal{T}$ must be finer than the product topology. ☐

Analogously, we can construct continuous functions over the product space using continuous functions defined on each of the individual spaces.

Proposition 1.6.6 ▸ Continuous of Functions over Product Spaces

Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\Lambda}$ be a sequence of non-empty topological spaces and let Y be a topological space. For any $\alpha \in \Lambda$, define $f_\alpha : Y \to X_\alpha$, then $f : Y \to \prod_{\alpha\in\Lambda} X_\alpha$ defined by

$$f(y) = (f_\alpha(y))_{\alpha\in\Lambda}$$

is continuous if and only if f_α is continuous for all $\alpha \in \Lambda$.

Proof. Suppose that f is continuous. Notice that for all $\alpha \in \Lambda$,

$$f_\alpha(y) = \pi_{X_\alpha}(f(y))$$

for all $y \in Y$. Therefore, $f_\alpha = \pi_{X_\alpha} \circ f$. By Proposition 1.6.5, π_{X_α} is continuous, and so f_α is continuous for all $\alpha \in \Lambda$.

Conversely, suppose that f_α is continuous for all $\alpha \in \Lambda$. Notice that the basis for the product topology is given by

$$\mathcal{B} := \left\{ \bigcap_{\alpha\in\Lambda} \pi_{X_\alpha}^{-1}(U_\alpha) : \Lambda' \subseteq \Lambda \text{ is finite}, U_\alpha \in \mathcal{T}_\alpha \right\}.$$

For any finite $\Lambda' \subseteq \Lambda$, we have

$$\begin{aligned} f^{-1}\left(\bigcap_{\alpha\in\Lambda} \pi_{X_\alpha}^{-1}(U_\alpha) \right) &= \bigcap_{\alpha\in\Lambda} f^{-1}\left(\pi_{X_\alpha}^{-1}(U_\alpha)\right) \\ &= \bigcap_{\alpha\in\Lambda} \left(\pi_{X_\alpha} \circ f\right)^{-1}(U_\alpha) \\ &= \bigcap_{\alpha\in\Lambda} f_\alpha^{-1}(U_\alpha), \end{aligned}$$

which is open. Therefore, f is continuous. □

Notice that it is important to equip the product space with the product topology but not the box topology. Define $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ such that $f(x) = (x, x, \cdots)$. Note that every component of $f(x)$ is obtained by the identity map which is continuous. We claim that f is not continuous with respect to the box topology. Consider

$$U := \prod_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right),$$

then clearly $f(0) = \mathbf{0} \in U$. Suppose on contrary that f is continuous with respect to the box topology, then $f^{-1}(U)$ should be open, which means that there exists some $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq f^{-1}(U)$. This means that $f\left(\frac{\epsilon}{2}\right) \in U$, which is not possible because there exists $N \in \mathbb{N}^+$ such that $\frac{1}{n} < \frac{\epsilon}{2}$ for all $n \geq N$.

Recall that a binary arithmetic operation on a set Y is simply a map $h\colon Y \times Y \to Y$. If there are two continuous maps $f, g\colon X \to Y$, then they are essentially the component maps of the map $F\colon X \to Y \times Y$ defined by

$$F(x) = \big(f(x), g(x)\big).$$

If h is continuous, then we can construct a continuous map $H = h \circ F$, which is fundamentally just applying the binary operation on two continuous maps.

Corollary 1.6.7 ▶ Arithmetic of Continuous Functions

Let X be a topological space and let $f, g\colon X \to \mathbb{R}$ be continuous functions, then the functions $f + g$, $f - g$ and fg are continuous. If $0 \notin g(X)$, then $\frac{f}{g}$ is continuous.

Note that our definition of the product topology relies on a sub-basis, which can be too convoluted when we try to represent an open set in the product space. To simplify this, we propose a way to identify the basis for the product topology.

Proposition 1.6.8 ▶ Basis for Finite Product Topology

Let $\{(X_i, \mathcal{T}_i)\}_{i=1}^n$ be a finite family of topological spaces and let $\mathcal{B}_i$ be the basis for $\mathcal{T}_i$ for all $i = 1, 2, \cdots, n$, then $\prod_{i=1}^n \mathcal{B}_i$ is a basis for the product topology over $\prod_{i=1}^n X_i$.

Proof. Let $\mathcal{T}_{\mathcal{B}}$ be the topology generated by $\mathcal{B} \coloneqq \prod_{i=1}^n \mathcal{B}_i$ and $\mathcal{T}$ be the product topology generated by the sub-basis

$$\mathcal{S} \coloneqq \left\{\pi_{X_i}^{-1}(U_i) : i = 1, 2, \cdots, n, U_i \in \mathcal{T}_i\right\}.$$

Note that the basis induced by $\mathcal{S}$ is

$$\mathcal{B}_{\mathcal{S}} \coloneqq \left\{\bigcap_{i \in I} \pi_{X_i}^{-1}(U_i) : I \subseteq \{1, 2, \cdots, n\}, U_i \in \mathcal{T}_i\right\}.$$

Take any $B \coloneqq \prod_{i=1}^n B_i \in \mathcal{B}$ and any $\boldsymbol{x} \in B$, then $\pi_{X_i}(\boldsymbol{x}) \in B_i$ for each $i = 1, 2, \cdots, n$.

Take

$$B_{\mathcal{S}} := \bigcap_{i=1}^{n} \pi_{X_i}^{-1}(B_i) \in \mathcal{B}_{\mathcal{S}},$$

then for each $\boldsymbol{y} \in B_{\mathcal{S}}$, we have $\pi_{X_i}(\boldsymbol{y}) \in B_i$ for all $i = 1, 2, \cdots, n$ and so $\boldsymbol{y} \in B$. Therefore, $\boldsymbol{x} \in B_{\mathcal{S}} \subseteq B$. Similarly, we can prove that $B \subseteq \mathcal{B}_{\mathcal{S}}$, so by Proposition 1.1.7, we have $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$. □

We can generalise this basis to infinite product spaces:

Proposition 1.6.9 ▸ Natural Basis for Product Topology

If $\{X_i\}_{i\in I}$ is an indexed family of topological spaces and $\mathcal{F}$ is the set of all finite subsets of I, then

$$\mathcal{B} := \bigcup_{F\in\mathcal{F}} \left\{ \prod_{i\in I\setminus F} X_i \times \prod_{i\in F} U_i : U_i \text{ is open in } X_i \right\}$$

is a basis for $X := \prod_{i\in I} X_i$.

Proof. By definition, X is generated by the sub-basis

$$\mathcal{S} := \bigcup_{i\in I} \left\{ \pi_{X_i}^{-1}(U_i) : U_i \text{ is open in } X_i \right\},$$

so it suffices to prove that $\mathcal{S}$ generates $\mathcal{B}$. Take any $i \in I$ and any open set $U_i \in X_i$, it is clear that

$$\pi_{X_i}^{-1}(U_i) = U_i \times \prod_{j\in I\setminus\{i\}} X_j \in \mathcal{B},$$

so $\mathcal{S} \subseteq \mathcal{B}$. However, this means that for any $S_1, S_2 \in \mathcal{S}$, we have $S_1 \cap S_2 \in \mathcal{B}$, which means that $\mathcal{B}$ contains all finite intersections of the sets in $\mathcal{S}$. Take any $B \in \mathcal{B}$, then there exists some finite subset $F \subseteq I$ such that

$$B = \prod_{i\in I\setminus F} X_i \times \prod_{i\in F} U_i = \bigcap_{i\in F} \pi_{X_i}^{-1}(U_i),$$

where each U_i is open in X_i. Therefore, every $B \in \mathcal{B}$ is a finite intersection of the sets in $\mathcal{S}$. Therefore, $\mathcal{S}$ generates $\mathcal{B}$ and so $\mathcal{B}$ is a basis for X. □

However, notice that every open set in a topological space can be expressed by the basis of its topology, so we can further transform this construction to the following:

Proposition 1.6.10 ▸ Product Space Basis Induced from Factor Space Bases

If $\{X_i\}_{i\in I}$ is an indexed family of topological spaces where $\mathcal{B}_i$ is the basis for X_i. Let $\mathcal{F}$ be the set of all finite subsets of I. Define

$$\mathcal{B}_F := \left\{\prod_{i\in I\setminus F} X_i \times \prod_{i\in F} B_i : B_i \in \mathcal{B}_i\right\}$$

for each $F \in \mathcal{F}$, then

$$\mathcal{B} := \bigcup_{F\in\mathcal{F}} \mathcal{B}_F$$

is a basis for $X := \prod_{i\in I} X_i$.

Proof. Let

$$\mathcal{B}' := \bigcup_{F\in\mathcal{F}} \left\{\prod_{i\in I\setminus F} X_i \times \prod_{i\in F} U_i : U_i \text{ is open in } X_i\right\},$$

then $\mathcal{B}'$ is the basis for X by Proposition 1.6.9. Observe that $\mathcal{B} \subseteq \mathcal{B}'$, so it suffices to prove that for each $U \in \mathcal{B}'$ and any $\boldsymbol{u} \in U$, there exists some $B \in \mathcal{B}$ with $\boldsymbol{u} \in B \subseteq U$. Take any

$$U := \prod_{i\in I\setminus F} X_i \times \prod_{i\in F} U_i \in \mathcal{B}'$$

where each U_i is open in X_i and consider an arbitrary $\boldsymbol{x} \in U$. For all $i \in F$, there exists some $B_i \in \mathcal{B}_i$ such that

$$\pi_{X_i}(\boldsymbol{x}) \in B_i \subseteq U_i.$$

Consider

$$B := \prod_{i\in I\setminus F} X_i \times \prod_{i\in F} B_i \subseteq U_i.$$

Notice that $\boldsymbol{x} \in B \in \mathcal{B}$, so $\mathcal{B}$ and $\mathcal{B}'$ both generate X. $\square$

Let us combine the product space with subspace topologies. Intuitively, the subspace topology on product space should be the product of subspace topologies.

Proposition 1.6.11 ▸ Subspace Topology on Product Spaces

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $A \subseteq X$, $B \subseteq Y$ and $\mathcal{T}_{A\times B}$ be the subspace topology on $A \times B$ induced by the product topology. If $\mathcal{T}'_{A\times B}$ be the product topology induced by the subspace topologies on A and B, then $\mathcal{T}_{A\times B} = \mathcal{T}'_{A\times B}$.

Proof. Let $\mathcal{T}_A$ and $\mathcal{T}_B$ be the subspace topologies on A and B respectively and $\mathcal{T}_{A\times B}$ be the product topology on $A \times B$, then $\mathcal{T}_{A\times B}$ is generated by the basis

$$\mathcal{B}_{A\times B} := \{U_A \times U_B : U_A \in \mathcal{T}_A, U_B \in \mathcal{T}_B\}.$$

Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the subspace topologies on X and Y respectively and $\mathcal{T}_{X\times Y}$ be the product topology on $X \times Y$, then $\mathcal{T}_{X\times Y}$ is generated by the basis

$$\mathcal{B}_{X\times Y} := \{U_X \times U_Y : U_X \in \mathcal{T}_X, U_Y \in \mathcal{T}_Y\}.$$

let $\mathcal{T}$ be the subspace topology on $A \times B$, then $\mathcal{T}$ is generated by the basis

$$\mathcal{B} := \{(U_X \times U_Y) \cap (A \times B) : U_X \in \mathcal{T}_X, U_Y \in \mathcal{T}_Y\}.$$

Take any $U_A \times U_B \in \mathcal{B}_{A\times B}$, then there exists some $U_X \in \mathcal{T}_X$ and $U_Y \in \mathcal{T}_Y$ such that $U_A = U_X \cap A$ and $U_B = U_Y \cap B$. Therefore,

$$U_A \times U_B = (U_X \cap A) \times (U_Y \cap B) = (U_X \times U_Y) \cap (A \times B) \in \mathcal{B},$$

and so $\mathcal{B}_{A\times B} \subseteq \mathcal{B}$. Take any $(U_X \times U_Y) \cap (A \times B) \in \mathcal{B}$. Notice that $U_X \cap A \in \mathcal{T}_A$ and $U_Y \cap B \in \mathcal{T}_B$, so

$$(U_X \times U_Y) \cap (A \times B) = (U_X \cap A) \times (U_Y \cap B) \in \mathcal{B}_{A\times B}.$$

Therefore, $\mathcal{B} \subseteq \mathcal{B}_{A\times B}$ and so $\mathcal{B}_{A\times B} = \mathcal{B}$. Therefore, $\mathcal{T} = \mathcal{T}_{A\times B}$. □

Lastly, let us discuss some additional properties of the product space for metric spaces. Recall that we have discussed the L^p-metric in the previous sections. This metric can be extended to the product space.

Definition 1.6.12 ▶ L^p-metric on a Product Space

Let $(X_1, d_{X_1}), (X_2, d_{X_2}), \cdots, (X_n, d_{X_n})$ be metric spaces, we define

$$d_1\big((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)\big) := \sum_{i=1}^{n} d_{X_i}(x_i, y_i)$$
$$d_\infty\big((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)\big) := \max_{1\le i\le n} d_{X_i}(x_i, y_i).$$

It is easy to see that the two metrics are equivalent in the product space by Proposition 1.2.15. In fact, we shall analogously prove that they both induce the product topology.

Proposition 1.6.13 ▸ Metric for Product Topology

Let $\left\{\left(X_i, d_{X_i}\right)\right\}_{i=1}^{n}$ be a sequence of topological spaces, then d_1 and d_∞ both induce the product topology over $\prod_{i=1}^{n} X_i$.

Proof. For any $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i=1}^{n} X_i$, it it clear that

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_1(\boldsymbol{x}, \boldsymbol{y}) \leq n d_\infty(\boldsymbol{x}, \boldsymbol{y}).$$

By Proposition 1.2.15, it suffices to prove that d_∞ induces the product topology. Let

$$\begin{aligned} B_r^\infty(\boldsymbol{x}) &:= \left\{\boldsymbol{y} \in \prod_{i=1}^{n} X_i : d_\infty(\boldsymbol{x}, \boldsymbol{y}) < r\right\} \\ &= \left\{\prod_{i=1}^{n} B_{r_i}\left(\pi_{X_i}^{-1}(\boldsymbol{x})\right) : i = 1, 2, \cdots, n, r_i \leq r\right\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_\infty &:= \left\{B_r^\infty(\boldsymbol{x}) : \boldsymbol{x} \in \prod_{i=1}^{n} X_i, r \in \mathbb{R}^+\right\} \\ &= \prod_{i=1}^{n} \{B_r(x) : x \in X_i, r \in \mathbb{R}^+\}. \end{aligned}$$

By Proposition 1.6.10, $\mathcal{B}_\infty$ is clearly a basis for the product topology. □

Note that our above construction only applies to finite product spaces. In infinite product spaces, we need to address the possible un-boundedness of the metric we choose. To do this, we can normalise the metric.

Proposition 1.6.14 ▸ Normlised Metric Induces the Metrisable Topology

Let (X, d) be a metric space and define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x, y) := \frac{d(x, y)}{1 + d(x, y)},$$

then ρ is a metric on X and $\operatorname{diam}_\rho(X) < 1$. Let $\mathcal{T}_\rho$ and $\mathcal{T}_d$ be the topologies on X induced by ρ and d respectively, then $\mathcal{T}_\rho = \mathcal{T}_d$.

Proof. We first show that ρ is a metric. The non-negativity and symmetry are obvious, so it suffices to prove the triangle inequality. Consider

$$f(t) = \frac{t}{1+t}.$$

One may check that f is an increasing concave function with $f(0) = 0$. Therefore,

$$\frac{b}{a+b} f(a+b) = \frac{a}{a+b} f(0) + \frac{b}{a+b} f(a+b) \le f(b).$$

By symmetry, $\frac{a}{a+b} f(a+b) \le f(a)$, and so $f(a) + f(b) \ge f(a+b)$. Therefore, for any $x, y, z \in X$,

$$\begin{aligned}
\rho(x,y) + \rho(y,z) &= f\big(d(x,y)\big) + f\big(d(y,z)\big) \\
&\ge f\big(d(x,y) + d(y,z)\big) \\
&\ge f\big(d(x,z)\big) \\
&= \rho(x,z).
\end{aligned}$$

It is clear that $\operatorname{diam}_\rho(X) = \max_{(x,y)\in X^2} \rho(x,y) < 1$. Notice that

$$\begin{aligned}
2\rho(x,y) &= \frac{2d(x,y)}{1+d(x,y)} \\
&> d(x,y) \\
&> \rho(x,y)
\end{aligned}$$

for all $x, y \in X$. Therefore, $\mathcal{T}_d = \mathcal{T}_\rho$ by Proposition 1.2.15. □

With the above result, we can now obtain a well-behaved metric over infinite product spaces and therefore extend our discussion to these spaces.

Proposition 1.6.15 ▸ Metrisable Topology on Infinite Product

Let $\left\{\left(X, d_{X_i}\right)\right\}_{i=1}^{\infty}$ be a sequence of metric spaces and define $\rho_i : X_i \times X_i \to \mathbb{R}$ by

$$\rho_i(x,y) \coloneqq \frac{d_{X_i}(x,y)}{1 + d_{X_i}(x,y)},$$

then $d : \prod_{i=1}^{\infty} X_i \times \prod_{i=1}^{\infty} X_i \to \mathbb{R}$ defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) \coloneqq \sup\left\{\frac{\rho_i(x_i, y_i)}{i} : i \in \mathbb{Z}^+\right\}$$

is a metric inducing the product topology on $\prod_{i=1}^{\infty} X_i$.

Proof. Let $\mathcal{T}_d$ and $\mathcal{T}_\rho$ be the topology induced by d and the product topology with respect to the metrisable topologies induced by ρ_i's respectively. Define $\rho_i' = \frac{1}{i}\rho_i$. Let $\mathcal{B}_d$ be the basis induced by the metric d and $\mathcal{B}_i$ be the basis induced by the metric ρ_i'. Take any $B_r^d(\boldsymbol{x}) \in \mathcal{B}_d$. For any $\boldsymbol{y} \in B_r^d(\boldsymbol{x})$, we have

$$\frac{\rho_i(x_i, y_i)}{i} \leq d(\boldsymbol{x}, \boldsymbol{y}) < r$$

for all $i \in \mathbb{Z}^+$. Therefore, $y_i \in B_r^{\rho_i'}(x_i)$ and so

$$B_r^d(\boldsymbol{x}) \subseteq \prod_{i=1}^{\infty} B_r^{\rho_i'}(x_i)$$

for any $\boldsymbol{x} \in \prod_{i=1}^{\infty} X_i$. Notice that for any $r > 0$, there exists some $N \in \mathbb{N}^+$ such that $\frac{1}{i} < \frac{r}{2}$ for all $i \geq N$. By Proposition 1.6.14, $\operatorname{diam}(\rho_i) < 1$ for all $i \in \mathbb{N}^+$, so

$$\operatorname{diam}\left(\frac{1}{i}\rho_i\right) = \frac{1}{i}\operatorname{diam}(\rho_i) < \frac{r}{2}$$

for all $i \geq N$. Therefore, $B_r^{\rho_i'}(\boldsymbol{x}_i) = X_i$ for any $\boldsymbol{x}_i \in X_i$ and

$$\sup\left\{\frac{\rho_i(x_i, y_i)}{i} : i \geq N\right\} \leq \frac{r}{2} < r.$$

Take any $\boldsymbol{y} \in \prod_{i=1}^{\infty} B_r^{\rho_i'}(x_i)$, we have

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{y}) &= \sup\left\{\frac{\rho_i(x_i, y_i)}{i} : i \in \mathbb{Z}^+\right\} \\ &= \max\left\{\max\left\{\frac{\rho_i(x_i, y_i)}{i} : 0 < i < N\right\}, \sup\left\{\frac{\rho_i(x_i, y_i)}{i} : i \geq N\right\}\right\} \\ &< r, \end{aligned}$$

and so $\boldsymbol{y} \in B_r^d(\boldsymbol{x})$. This means that for any $\boldsymbol{x} \in \prod_{i=1}^{\infty} X_i$,

$$B_r^d(\boldsymbol{x}) = \prod_{i=1}^{N-1} B_r^{\rho_i'}(x_i) \times \prod_{i=N}^{\infty} X_i.$$

By Proposition 1.6.10, the basis of $\mathcal{T}_\rho$ is

$$\mathcal{B}_\rho := \bigcup_{F \in \mathcal{F}} \left\{\prod_{i \in F} B_r^{\rho_i'}(x_i) \times \prod_{i \in \mathbb{N}^+ \setminus F} X_i : x_i \in X_i, r \in \mathbb{R}^+\right\}$$

where $\mathcal{F}$ is the set of all finite subsets of $\mathbb{N}^+$. This means that every open ball in the metrisable topology induced by d is open in the product topology on $\prod_{i=1}^{\infty} X_i$, and so $\mathcal{T}_d \subseteq \mathcal{T}_\rho$. Take any $N \in \mathbb{N}^+$ and any finite subset

$$K \coloneqq \{k_1, k_2, \cdots, k_{N-1}\} \subseteq \mathbb{N}^+.$$

For any $r_1, r_2, \cdots, r_{N-1} \in \mathbb{R}^+$, take

$$B \coloneqq \prod_{i=0} B_{r_i}^{N-1}\left(x_{k_i}\right) \times \prod_{i \in \mathbb{N}^+ \setminus K} X_i \in \mathcal{B}_\rho.$$

Let $r \coloneqq \min\{r_1, r_2, \cdots, r_{N-1}\} > 0$, then for any $\boldsymbol{y} \in B$, we have

$$\boldsymbol{y} \in B_r^d(\boldsymbol{x}) \subseteq B,$$

and so $\mathcal{T}_\rho \subseteq \mathcal{T}_d$. Therefore, $\mathcal{T}_d = \mathcal{T}_\rho$ is the product topology. □

1.7 Quotient Spaces

If there are product topologies, then by right there should also be quotient topologies. The term "quotient" arises from the quotient of a set by an equivalence relation.

Notice that every equivalence relation induces a partition and every partition also induces an equivalence relation. If $\sim$ is an equivalence relation on X, then $X/\sim$ is a partition of X by its equivalence classes. On the other hand, if X^* is a partition of a set X, then an equivalence relation $\sim$ on X can be defined as $x \sim y$ if and only if there exists some $S \in X^*$ such that $x, y \in S$.

If $X/\sim$ is a partition for a space X, our focus is then a map $p \colon X \to X/\sim$ such that $x \in p(x)$, i.e., p is an "indexing" map for the components in the partition.

Definition 1.7.1 ▸ Canonical Projection Map

Let X be a set and $X/\sim$ be a partition of X, then a surjective map $p \colon X \to X/\sim$ is a **canonical projection map** if $x \in p(x)$.

Note that under a canonical projection map p, for each $S \in X/\sim$, we have $S = p^{-1}(S)$. This means that an equivalence class in the quotient set $X/\sim$ is open if and only if its pre-image is open.

Definition 1.7.2 ▸ Quotient Map

Let X and Y be topological spaces. A surjective map $p: X \to Y$ is a **quotient map** if $V \subseteq Y$ is open if and only if $p^{-1}(V) \subseteq X$ is open.

Note that $p\big(p^{-1}(V)\big) = V$, so this definition is essentially saying that a quotient map p is such that **pre-images of open sets are open** and **pre-images of non-open sets are not open**. Note that this has nothing to do with whether an open set will be sent to an open image!

Definition 1.7.3 ▸ Open and Closed Maps

A continuous map $f: X \to Y$ is **open** if $f(U)$ is open in Y for any open set $U \subseteq X$, and **closed** if $f(V)$ is closed in Y for any closed set $V \subseteq X$.

There are many edge cases for open and/or closed maps.

1. A map that is closed but not open: consider $f: [0,1] \cup [2,3] \to [0,2]$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ x-1 & \text{if } x \in [2,3] \end{cases}.$$

   Take any closed set A, then both $A_1 := A \cap [0,1]$ and $A_2 := A \cap [2,3]$ are closed. Therefore, both $f(A_1)$ and $f(A_2)$ are closed in $[0,2]$. Therefore,

$$f(A) = f(A_1) \cup f(A_2)$$

   is closed, and so f is closed. However, notice that $(0,1]$ is open in $[0,1] \cup [2,3]$ but

$$f\big((0,1]\big) = (0,1] \subseteq [0,2]$$

   is not open in $[0,2]$. Therefore, f is not open.

2. A map that is open but not closed: consider $g: (0,1) \cup (2,3) \to (0,2)$ defined by

$$g(x) = \begin{cases} x & \text{if } x \in (0,1) \\ x-1 & \text{if } x \in (2,3) \end{cases}.$$

   Take any open set B, then both $B_1 := B \cap (0,1)$ and $B_2 := B \cap (2,3)$ are open. Therefore, both $g(B_1)$ and $g(B_2)$ are open in $(0,2)$. Therefore,

$$g(A) = g(A_1) \cup g(A_2)$$

is open, and so g is open. However, notice that $(0,1)$ is closed in $(0,1)\cup(2,3)$ but

$$g\big((0,1)\big)=(0,1)\subseteq(0,2)$$

is open in $(0,2)$. Therefore, g is not closed.

3. A map that is neither open nor closed: let $X\coloneqq\big([0,\infty)\times\mathbb{R}\big)\cup\big(\mathbb{R}\times\{0\}\big)$ and consider $h\colon X\to\mathbb{R}$ defined by $h(x,y)=x$. Note that $C_1\coloneqq\mathbb{R}_0^+\times(1,2)$ is open in X but $h(C_1)=\mathbb{R}_0^+$ is closed in $\mathbb{R}$, and that $C_2\coloneqq\left\{\left(x,\frac{1}{x}\right):x\in\mathbb{R}^+\right\}$ is closed in X but $h(C_2)=\mathbb{R}^+$ is open in $\mathbb{R}$. Therefore, h is neither open nor closed.
4. A map that is both open and closed: let X be equipped with the trivial topology, then the identity map is both open and closed.

Note that the maps in the first two examples are continuous and the map in the last example is not continuous, so continuity is not related to the openness of maps in general.

Remark.

- If a surjective continuous map is open or closed, then it is a quotient map.
- Quotient map, open map and closed map are all preserved under composition.

Let us re-visit the third example where

$$X\coloneqq\big([0,\infty)\times\mathbb{R}\big)\cup\big(\mathbb{R}\times\{0\}\big)$$

and we consider the projection $\pi_1\colon X\to\mathbb{R}$ defined by $\pi_1(x,y)=x$. It is interesting that this function is surjective, continuous, neither open nor closed, but it **is a quotient map**! For every open set $(a,b)\in\mathbb{R}$, we have

$$\pi_1^{-1}\big((a,b)\big)=\big((a,b)\times\mathbb{R}\big)\cap X$$

which is open. For every set $A\subseteq\mathbb{R}$ with $\pi_1^{-1}(A)$ being open, we have A to be open.

One feature of functions which lead to this seemingly strange behaviour is the fact that the pre-image of $f(X)$ is often a superset of X, so with forward mapping, the function fails to "capture all the information" in a set. To fix this issue, we restrict our consideration to only a specific type of sets.

Definition 1.7.4 ▸ Saturated Set

Let $f\colon X\to Y$ be a surjective continuous map. A set $A\subseteq X$ is **saturated** with respect to f if $A=f^{-1}(S)$ for some $S\subseteq Y$.

Equivalently, this means that $A=f^{-1}\big(f(A)\big)$, i.e., a saturated set contains everything that could be mapped to its own image.

Proposition 1.7.5 ▸ Equivalent Definition for Quotient Maps

Let $f : X \to Y$ be a surjective continuous map, then f is a quotient map if and only if $f(A)$ is open (closed) in Y whenever A is a saturated open (closed) set with respect to f in X.

Proof. Take any saturated open set $A \subseteq X$, then $A = f^{-1}\big(f(A)\big)$. By definition 1.7.2, since $f^{-1}\big(f(A)\big)$ is open, $f(A)$ must be open in Y. Conversely, suppose that $f(A)$ is open in Y for all saturated open set $A \subseteq X$. Let $U \subseteq Y$ such that $f^{-1}(U)$ is open in X. Notice that $f^{-1}(U)$ is saturated. Therefore, $f\big(f^{-1}(U)\big) = U$ is open in Y. Since f is surjective, it is a quotient map. □

Due to the nice property of saturated sets, by intuition a quotient map should be preserved under restriction onto a saturated sub-domain.

Proposition 1.7.6 ▸ Restriction of Quotient Map to Saturated Open Sets

If $f : X \to Y$ is a continuous quotient map and $A \subseteq X$ is a saturated open (closed) set with respect to f, then $f|_A : A \to f(A)$ is a quotient map.

Proof. Let $B \subseteq A$ be open and saturated with respect to $f|_A$, then $B = f|_A^{-1}\big(f|_A(B)\big)$. Note that

$$f|_A(B) = f(B \cap A) = f(B)$$

for any $B \subseteq A$. Since B is open,by Proposition 1.7.5, $f(B)$ is open. This implies that $f|_A$ is a quotient map. □

Now, whether a map is a quotient map depends on the topology we choose because we need to be aware of the open and closed sets in the space. Therefore, we may be interested in the types of topologies which can induce a quotient map.

Definition 1.7.7 ▸ Quotient Topology

Let X and Y be sets and $p : X \to Y$ be a surjective map. A topology on Y is a **quotient topology** with respect to p if p is a quotient map with respect to the topology.

Let us consider the canonical projection map $p : X \to X/\sim$ again. Assuming Axiom of Choice, we can fix a choice function $c : X/\sim \to X$ to represent each equivalence class $[x]_\sim$ by $c\left([x]_\sim\right)$. Define $X^* \coloneqq c(X/\sim)$, then $X^* \subseteq X$. This allows us to define a *quotient space*.

Definition 1.7.8 ▶ Quotient Space

Let X be a topological space with partition $X/\sim$. Let $c: X/\sim \to X$ be a choice function and $X^* := c(X/\sim)$. Let $p: X \to X^*$ be a surjective map such that $x \in c^{-1}(p(x))$. If $\mathcal{T}^*$ is a quotient topology on X^*, then $(X^*, \mathcal{T}^*)$ is the **quotient space** of X.

The advantage of assuming Axiom of Choice here is that without the choice function, we cannot reduce each equivalence class into a singleton, and so there could be multiple topologies such that the canonical projection $p: X \to X/\sim$ is a quotient map. To deal with that, we have to define the quotient topology as the finest topology to guarantee the continuity of p as a quotient map. However, with the choice function, we can actually guarantee the uniqueness of this quotient topology!

Proposition 1.7.9 ▶ Existence and Uniqueness of Quotient Topology

Let $p: X \to A$ be surjective for some $A \subseteq X$, then there exists a unique topology $\mathcal{T}_A$ such that p is a quotient map.

Proof. Define

$$\mathcal{T}_A := \left\{U \subseteq A : p^{-1}(U) \subseteq X \text{ is open}\right\}.$$

Observe that $\varnothing, A \in \mathcal{T}_A$. Note that for any index set Λ such that $U_\alpha \in \mathcal{T}_A$ for all $\alpha \in \Lambda$,

$$p^{-1}\left(\bigcup_{\alpha\in\Lambda} U_\alpha\right) = \bigcup_{\alpha\in\Lambda} p^{-1}(U_\alpha)$$

is open is X, so $\bigcup_{\alpha\in\Lambda} U_\alpha \in \mathcal{T}_A$. Similarly, one may check that $\bigcup_{i=1}^{n} U_i \in \mathcal{T}_A$ for any $n \in \mathbb{N}^+$ such that $U_i \in \mathcal{T}_A$ for all $i = 1, 2, \cdots, n$. Let $\mathcal{T}'$ be any topology on A such that p is a quotient map. For any $U \subseteq A$, notice that $U \in \mathcal{T}'$ if and only if $p^{-1}(U) \subseteq X$ is open, i.e., $U \in \mathcal{T}'$ if and only if $U \in \mathcal{T}_A$. Therefore, $\mathcal{T}' = \mathcal{T}_A$, which means that $\mathcal{T}_A$ is unique. $\square$

Interestingly, this quotient topology we construct is nothing but a pull-back topology.

1.8 Topological Embeddings

The relationship between a "container" space and a "contained" space in topological spaces can be quite peculiar. On one hand, we can define such relationship using the classic subset-superset relationship. On the other hand, we see that what defines a topological space is not so much about the actual elements contained in the space, but the structure exhibited by the space, which can be preserved under certain transformations.

In the context of topological spaces, this means that open spaces should be preserved.

Definition 1.8.1 ▸ Homeomorphism

A function $h\colon X \to Y$ between topological spaces is a **homeomorphism** if it is a bijective open map.

In other words, a bijection h between topological spaces is a homeomorphism if both h and h^{-1} are continuous. Sometimes, h is known as *bi-continuous*. Roughly speaking, a homeomorphism is a reversible deformation such that open sets are still open before and after the deformation.

Remark. Since both bijective maps and open maps are preserved under composition, homeomorphism is preserved under composition.

One merit of having a homeomorphism is that it helps construct a structure-preserving map that "projects" a space into another possibly bigger space.

Definition 1.8.2 ▸ Topological Embedding

Let X and Y be topological spaces. A continuous injective function $f\colon X \hookrightarrow Y$ is a **topological embedding** if it is homeomorphic onto $f(X)$.

The utility of topological embeddings is that they allow us to transform an unfamiliar or highly abstract topological space into something we are more comfortable with, without altering the topological structures of the space. Therefore, anything we do with the image will be reflected in the original space in the same way. A good candidate for this "space that we are comfortable with" is the Euclidean spaces.

In a metric space, we can further generalise the notion of a structure-preserving map into maps which preserve the distance between points.

Definition 1.8.3 ▸ Isometric Embedding

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is an **isometric embedding** if for all $a, b \in X$,

$$d_Y\big(f(a), f(b)\big) = d_X(a, b).$$

If f is surjective, then f is said to be an **isometry**.

2

Countability Axioms

2.1 Second Countable Spaces

Note that a topological space can be completely captured by its basis, so an important property of a topological space is the countability of its basis. Roughly speaking, the countability of the basis represents how "easy" it is to generate all open sets in the space using some fixed open sets.

Definition 2.1.1 ▶ Second Countable Space

A topological space X is **second countable** if it has a countable basis.

Remark. Equivalently, a topological space X is second countable if and only if for every open set $S \subseteq X$, there exists a countable collection $\mathcal{U}_S$ of open sets in X such that

$$S = \bigcup_{U \in \mathcal{U}_S} U.$$

It is important to note that the countability of a space's basis does not depend on the countability of the space itself. For example, $\mathbb{R}^n$ is an uncountable space, but its standard topology has a countable basis

$$\{B_r(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{Q}^n, r \in \mathbb{Q}\}.$$

Even for the uncountable product space $\mathbb{R}^\omega$, there exists a countable basis for the product topology, given by

$$\left\{\prod_{i \in I}(a_i, b_i) \times \prod_{i \in \mathbb{Z} \setminus I} \mathbb{R} : (a_i, b_i) \in \mathbb{Q}^2, I \subseteq \mathbb{Z} \text{ is finite}\right\}.$$

Since a second countable space can be "covered" by countably many open sets, it is not surprising that the density of the space is at most countable.

Proposition 2.1.2 ▶ Dense Countable Subspaces in Second Countable Spaces

If X is a second countable space, then there exists a countable subset $A \subseteq X$ such that A is dense in X. The converse is true if X is metrisable.

Proof. Suppose X is second countable and let $\mathcal{B} := \{B_i : i \in \mathbb{N}^+\}$ be a countable basis for X. For each $i \in \mathbb{N}^+$, fix some $x_i \in B_i$ and define $A := \{x_i : i \in \mathbb{N}^+\}$. It is clear that A is countable. For every open set $U \subseteq X$, there exists some $n \in \mathbb{N}^+$ such that $B_n \subseteq U$. Since $\{x_n\} \subseteq A \cap B_n$, we have $A \cap U \neq \varnothing$ and so A is dense in X.
Conversely, let $A \subseteq X$ be a countable dense subset. Define

$$\mathcal{B} := \left\{B_{\frac{1}{n}}(a) : a \in A, n \in \mathbb{N}^+\right\}$$

which is countable. We claim that $\mathcal{B}$ is a basis for the metrisable topology on X. Note that for any $x \in X$, then there exists some $\epsilon > 0$ such that $x \in B_\epsilon(x)$. Notice that there exists $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \epsilon$. Since A is dense in X, there exists some $a \in A$ such that $a \in B_{\frac{1}{N}}(x)$, which implies that $x \in B_{\frac{1}{N}}(a) \in \mathcal{B}$, and so $\mathcal{B}$ is a basis for the metrisable topology on X. Therefore, X is second countable. □

Intuitively, second countability is preserved in subspace topologies because we need fewer basic open sets to cover a subspace.

Proposition 2.1.3 ▸ Subspace Topologies Preserve Second Countability

If X is a second countable space, then every $Y \subseteq X$ is second countable with respect to the subspace topology.

Proof. Let $\mathcal{B}_X$ be a countable basis for X, then clearly

$$\mathcal{B}_Y := \{B \cap Y : B \in \mathcal{B}_X\}$$

is a countable basis for Y. Therefore, Y is second countable. □

It is natural that second countability is preserved under finite product. However, countably infinite product spaces still preserve second countability.

Proposition 2.1.4 ▸ Countable Product Topologies Preserve Second Countability

If $\{X_n\}_{n\in\mathbb{N}^+}$ is a family of second countable spaces and

$$X := \prod_{n\in\mathbb{N}^+} X_n$$

is the countable product space equipped with the product topology, then X is second countable.

Proof. Let $\mathcal{B}_i$ be a countable basis for X_i. Let $\mathcal{F}$ be the set of all finite subsets of $\mathbb{N}^+$, then by Proposition 1.6.10,

$$\mathcal{B} := \bigcup_{F \in \mathcal{F}} \left\{ \prod_{i \in \mathbb{N}^+ \setminus F} X_i \times \prod_{i \in F} B_i : B_i \in \mathcal{B}_i \right\}$$

is a basis for X. Define $f : \mathcal{B}_F \to \prod_{i \in F} \mathcal{B}_i$ by $f(U) = (B_i)_{i \in F}$ if and only if

$$U = \prod_{i \in \mathbb{N}^+ \setminus F} X_i \times \prod_{i \in F} B_i.$$

It is clear that f is a bijection and so for all $F \in \mathcal{F}$,

$$\left\{ \prod_{i \in \mathbb{N}^+ \setminus F} X_i \times \prod_{i \in F} B_i : B_i \in \mathcal{B}_i \right\} \approx \prod_{i \in F} \mathcal{B}_i,$$

which is countable because it is a finite Cartesian product over countable sets. Therefore, $\mathcal{B}$ is countable because it is a countable union of countable sets. Therefore, X is second countable. □

2.2 First Countable Spaces

We can also discuss countability in a "local" context. Consider a random point x in some topological space. Suppose that U is an open set containing x. Intuitively, we can reduce U by taking an open subset of U containing x. We do this repeatedly until we find some minimal open subset of U containing x. Now, we can perform this operation to every open set containing x in the space and collect all these minimal open subsets. The countability of this collection offers a different criterion of space categorisation.

Definition 2.2.1 ▶ Countable Basis

Let X be a topological space. For all $x \in X$, a **countable basis of x** is a countable collection $\mathcal{B}$ of open sets in X containing x such that for any open set $Y \subseteq X$ containing x, there exists some $B \in \mathcal{B}$ with $B \subseteq Y$.

The next definition is a "local" version of second countable spaces.

Definition 2.2.2 ▶ First Countable Space

A topological space X is **first countable** if every $x \in X$ has a countable basis.

Intuitively, every second countable space is first countable. Suppose $\mathcal{U}$ is a countable basis for the space, then for each point x in the space, we can simply take $\{U \in \mathcal{U} : x \in U\}$ as a countable basis of x.

We claim that all metric spaces are first countable, because for any metric space X and any $x \in X$, we can take its countable basis as

$$\mathcal{B} := \left\{B_{\frac{1}{i}}(x) : i \in \mathbb{Z}^+\right\}.$$

This above example also helps show the existence of first countable spaces which are not second countable. Let $\mathbb{R}$ be equipped with the discrete metric, then the metrisable space induced has no countable basis because it has the discrete topology, but is obviously first countable.

Consider some uncountable set Y equipped with the co-finite topology. For any $y \in Y$, suppose it has a countable basis $\mathcal{B}$. Note that in the co-finite topology, a set is open if and only if its complement is finite. Therefore, there exists some finite set $F_i \subseteq Y$ such that there exists some $B_i \in \mathcal{B}$ such that $Y \setminus F_i = B_i$. Notice that

$$Y \setminus \left(\{y\} \cup \bigcup_{i=1}^{\infty} F_i\right) \neq \varnothing.$$

Take some $z \in Y \setminus \left(\{y\} \cup \bigcup_{i=1}^{\infty} F_i\right)$, then $U := Y \setminus \{z\}$ is open such that $y \in U$. Note that for all $B_i \in \mathcal{B}$, we have $z \in B_i$. This means that $B_i \nsubseteq U$ for all $i \in \mathbb{Z}^+$ because otherwise $y \in U$, which is not possible. Therefore, $\mathcal{B}$ is not a countable basis.

Suppose that x is a random point in a topological space with a countable basis $\mathcal{B}$. Fix any $B \in \mathcal{B}$, then $x \in B$. Notice that for any open set U containing x, it is not possible that U and B are disjoint. Therefore, there exists some open subset of B which contains x. This motivates the following natural question: can we always re-construct the countable basis such that for any pair of sets in the basis, one must be a subset of the other?

Proposition 2.2.3 ▸ Construction of Countable Basis

Let $x \in X$ be a point with a countable basis $\mathcal{B}$, then there exists some countable basis $\mathcal{B}'$ of x such that for any $B_1, B_2 \in \mathcal{B}'$, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

Proof. Take any $B_1, B_2 \in \mathcal{B}$ such that $B_1 \nsubseteq B_2$ and $B_2 \nsubseteq B_1$, then $B_1 \cap B_2$ is open and $x \in B_1 \cap B_2$. Therefore, for any open set U containing B_2, clearly $B_1 \cap B_2 \subseteq U$. Therefore, we can repeat this process until we obtain a countable basis $\mathcal{B}'$ such that for any $B_1', B_2' \in \mathcal{B}'$, we have either $B_1' \subseteq B_2'$ or $B_2' \subseteq B_1'$. □

Based on this small proposition, we introduce the following result to help us identify spaces which are not first countable:

Proposition 2.2.4 ▸ Closure Characterisation of First Countable Spaces

Let X be a topological space and $A \subseteq X$. If there exists a sequence $\{x_i\}_{i\in\mathbb{N}^+} \subseteq A$ such that $x_i \to x$, then $x \in \overline{A}$. The converse is true if X is first countable.

Proof. Suppose that there exists a sequence $\{x_i\}_{i\in\mathbb{N}^+} \subseteq A$ such that $x_i \to x$. Notice that $\overline{A} = A \cup A'$, then it suffices to prove that $x \notin A$ implies $x \in A'$. Suppose that $x \notin A$. Note that for any open set $U \subseteq X$ such that $x \in U$, there exists some $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. In particular, this means that $x_N \in U \cap (A \setminus \{x\})$. Therefore, x is a limit point of A.

Conversely, suppose that $x \in \overline{A}$. Since X is first countable, x has some countable basis $\mathcal{B} := \{B_i : i \in \mathbb{N}^+\}$. By Proposition 2.2.3, we can assume that $B_j \subseteq B_i$ whenever $j \geq i$. For each $i \in \mathbb{N}^+$, take

$$x_i \in \bigcap_{j=1}^{i} (B_j \cap A),$$

then $\{x_i\}_{i=\mathbb{N}^+}$ is a sequence in A. Let U be any open set containing x, then there exists some $B_N \in \mathcal{B}$ such that $x \in B_N \subseteq U$. However, notice that for all $i \geq N$, we have $x_i \in B_N \subseteq U$, and so $x_i \to x$. $\square$

In the above result, first countability is crucial for us to construct the desired sequence.

We can re-write this result using continuous maps.

Proposition 2.2.5 ▸ Continuous Map Characterisation of First Countable Spaces

Let X be a topological space. If $f : X \to Y$ is continuous, then for any sequence $\{x_i\}_{i\in\mathbb{N}^+}$ with $x_i \to x$, then $f(x_i) \to f(x)$. The converse is true if X is first countable.

Proof. Suppose that $f : X \to Y$ is continuous. Let $U \subseteq Y$ be an open set such that $f(x) \in U$, then $x \in f^{-1}(U)$, which is open. Notice that there exists some $N \in \mathbb{N}^+$ such that $x_i \in f^{-1}(U)$ for all $i \geq N$, i.e., $f(x_i) \in U$ for all $i \geq N$. Therefore, $f(x_i) \to f(x)$.

Conversely, suppose that for any sequence $\{x_i\}_{i\in\mathbb{N}^+}$ with $x_i \to x$, then $f(x_i) \to f(x)$. Take any $A \subseteq X$ with $x \in \overline{A}$. Since X is first countable, by Proposition 2.2.4, there exists a sequence $\{y_i\}_{i\in\mathbb{N}^+}$ in A such that $y_i \to x$. This means that $f(y_i) \to f(x)$. Since $f(y_i) \in f(A)$, this means that $f(x) \in \overline{f(A)}$, and so $f\left(\overline{A}\right) \subseteq \overline{f(A)}$. By Proposition 1.5.7, f is continuous. $\square$

Similar to second countable spaces, first countability should also be "inherited" in subspaces.

Proposition 2.2.6 ▸ Subspaces Preserve First Countability

If X is first countable, then every subspace $Y \subseteq X$ with the subspace topology is first countable.

Proof. Take any $y \in Y$ and let $V \subseteq Y$ be any open set in Y containing y, then there exists some open set $U \subseteq X$ such that $V = U \cap Y$. Since X is first countable, there exists a countable collection $\mathcal{B}$ of open sets in X containing y such that there exists some $B_y \in \mathcal{B}$ with $y \in B_y \subseteq U$. Define

$$\mathcal{B}' \coloneqq \{B \cap Y : B \in \mathcal{B}\},$$

then it is clear that $\mathcal{B}'$ is a countable collection of open sets in Y containing y. Notice that $B_y \cap Y \in \mathcal{B}'$ is such that $y \in B_y \cap Y \subseteq V$, so Y is first countable. □

Without surprise, countable product will preserve first countability as expected.

Proposition 2.2.7 ▸ Countable Product Preserves First Countability

If $\{X_n\}_{n\in\mathbb{N}^+}$ is a family of first countable spaces and

$$X \coloneqq \prod_{n\in\mathbb{N}^+} X_n,$$

then X is first countable.

Proof. Let $\mathcal{B}_i$ be a basis for X_i and define $X \coloneqq \prod_{i\in\mathbb{N}^+} X_i$. Let $\mathcal{F}$ be the set of all finite subsets of $\mathbb{N}^+$, then by Proposition 1.6.10,

$$\mathcal{B} \coloneqq \bigcup_{F\in\mathcal{F}} \left\{ \prod_{i\in\mathbb{N}^+\setminus F} X_i \times \prod_{i\in F} B_i : B_i \in \mathcal{B}_i \right\}$$

is a basis for X. Take any $\boldsymbol{x} \in X$. By first countability, for each $i \in \mathbb{N}^+$, there exists a countable basis $\mathcal{B}_{X_i} \subseteq \mathcal{B}_i$ at $\pi_{X_i}(\boldsymbol{x})$. Define

$$\mathcal{B}_{\boldsymbol{x}} \coloneqq \bigcup_{F\in\mathcal{F}} \left\{ \prod_{i\in\mathbb{N}^+\setminus F} X_i \times \prod_{i\in F} B_i : B_i \in \mathcal{B}_{X_i} \right\} \subseteq \mathcal{B},$$

which is countable. Let $U \subseteq X$ be any open set containing $\boldsymbol{x}$, then there exists

some $F \in \mathcal{F}$ such that
$$\boldsymbol{x} \in \prod_{i \in \mathbb{N}^+ \setminus F} X_i \times \prod_{i \in F} B_i \in \mathcal{B}.$$
By first countability, for each $i \in F$, there exists some $B_{X_i} \in \mathcal{B}_{X_i}$ such that
$$\pi_{X_i}^{-1}(\boldsymbol{x}) \in B_{X_i} \subseteq B_i,$$
and so
$$\boldsymbol{x} \in \prod_{i \in \mathbb{N}^+ \setminus F} X_i \times \prod_{i \in F} B_{X_i} \in \mathcal{B}_{\boldsymbol{x}}.$$
Notice that
$$\prod_{i \in \mathbb{N}^+ \setminus F} X_i \times \prod_{i \in F} B_{X_i} \subseteq \prod_{i \in \mathbb{N}^+ \setminus F} X_i \times \prod_{i \in F} B_i \subseteq U,$$
so $\mathcal{B}_{\boldsymbol{x}}$ is a countable basis at $\boldsymbol{x}$. Therefore, X is first countable. □

2.3 Lindelof Spaces

Now, based on some intuition, we see that certain spaces are "tight" in a sense that they have a clearly defined boundary, whereas some other spaces like $\mathbb{R}$ can extend indefinitely. One idea to measure the "tightness" of a topological space is to consider the number of open sets needed to cover the entire space.

First, notice that given any topological space, we can always partition the space into a collection of open sets in the space.

Definition 2.3.1 ▸ Open Cover

Let X be a topological space. An **open cover** is a collection of open sets $\{U_\alpha\}_{\alpha \in \Lambda}$ such that
$$\bigcup_{\alpha \in \Lambda} U_\alpha = X.$$

The first step here is to check if the space can be covered by countably many open sets.

Definition 2.3.2 ▸ Lindelof Space

A topological space X is **Lindelof** if every open cover of X contains a countable subcover.

Intuitively, a countable basis will induce a countable cover of the space.

Proposition 2.3.3 ▶ Second Countable Spaces Are Lindelof

Every second countable space X is Lindelof. The converse is true if X is metrisable.

Proof. Suppose that X is second countable, then there exists a countable basis

$$\mathcal{B} := \{B_i : i \in \mathbb{N}^+\}$$

for X and define

$$I := \{i \in \mathbb{N}^+ : \text{there exists some } U \in \mathcal{U} \text{ such that } B_i \subseteq U\}.$$

For each $i \in I$, pick some $U_i \in \mathcal{U}$ with $B_i \subseteq U_i$. Note that for any $x \in X$, there exists some $U \in \mathcal{U}$ with $x \in U$. Since $\mathcal{B}$ is a basis, there exists some $i \in \mathbb{N}^+$ such that $B_i \subseteq U$ with $x \in B_i$, which means that $i \in I$ and so $x \in B_i \subseteq U_i$. Therefore, $\{U_i\}_{i\in I}$ is a countable sub-cover for X and so X is Lindelof. Conversely, suppose that X is a metrisable Lindelof space and let

$$\mathcal{V}_n := \left\{B_{\frac{1}{n}}(x) : x \in X\right\}$$

for each $n \in \mathbb{N}^+$, then clearly each $\mathcal{V}_n$ is an open cover for X. For each $n \in \mathbb{N}^+$, fix a countable sub-cover $\mathcal{V}_n' \subseteq \mathcal{V}_n$ and define

$$\mathcal{B} := \bigcup_{n\in\mathbb{N}^+} \mathcal{V}_n',$$

which is countable. For any open set $U \subseteq X$ and any $x \in U$, there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. Take $N \in \mathbb{N}^+$ with $\frac{1}{N} < \frac{\epsilon}{2}$, then since $\mathcal{V}_N'$ covers X, there exists some $y \in X$ such that $x \in B_{\frac{1}{N}}(y) \in \mathcal{V}_N' \subseteq \mathcal{B}$. Observe that $y \in B_{\frac{\epsilon}{2}}(x) \subseteq U$, so this means that $\mathcal{B}$ is a basis for the metrisable topology on X and so X is second countable. □

3 Separation Axioms

The *separation axioms* are a series of conditions imposed on topological spaces to better specify the notion of *disjoint* sets and *distinguishable* points. The motivation comes from the fact that knowing two points are distinct is not enough to differentiate one from the other in a topological space setting. For example, if $x \neq y$ are *distinct* points in a topological space X, it is completely possible that every open set containing x also contains y and vice versa. Therefore, though unequal, there is no practical way to seclude x and y into disjoint regions topologically.

3.1 T_1 Spaces

T_1 spaces define properly the notion of separation in a topological sense. Before we introduce the full definition, it might be useful to consider a few preliminary notions.

Definition 3.1.1 ▸ Distinguishable Points

Two points x and y in a topological space are **topologically distinguishable** if one of them has an open neighbourhood which does not contain the other.

Note that being distinguishable alone is not sufficient to "separate" two points. Topologically speaking, we think of "separation" as the ability to seclude two objects into disjoint open regions.

Definition 3.1.2 ▸ Separated Points

Two points x and y in a topological space are **separated** if there exists open neighbourhoods U, V with $x \in U$ and $y \in V$ such that $x \notin V$ and $y \notin U$.

In other words, two points are separated if each has an open neighbourhood which does not contain the other.

Definition 3.1.3 ▸ T_1 Spaces

A topological space X is T_1 if for any $x, y \in X$ with $x \neq y$, there exists an open neighbourhood $U \subseteq X$ such that $x \in U$ but $y \notin U$.

It is clear then that a T_1 space is a space in which every two distinct points are separated.

Intuitively, if we fix any point x in a T_1 space, then we can find some open sets to cover every point in the space other than x, which means that the isolated singleton $\{x\}$ is closed.

Proposition 3.1.4 ▸ Characterisation of T_1 Spaces

A topological space X is T_1 if and only if for all $x \in X$, the singleton set $\{x\}$ is closed.

Proof. Suppose that X is T_1 and take any $x \in X$. Let $y \in X \setminus \{x\}$ be any point. Notice that there exists some open set $V_y \subseteq X$ such that $y \in V_y$ but $x \notin V_y$. Clearly,

$$X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} V_y$$

is open, and so $\{x\}$ is closed. Conversely, suppose that for all $x \in X$, the set $\{x\}$ is closed. For any $x \in X$, take any $y \in X$ with $x \neq y$. Notice that $y \in X \setminus \{x\}$, which is open. Therefore, X is T_1. □

Since T_1 spaces guarantee pair-wise distinguishability, it is "easy" to preserve the structure of a T_1 space under transformations. In fact, recall that in Definition 1.8.2, we defined a structure-preserving map to embed a space into a different space. In a T_1 space, we will prove that the existence of such a map is guaranteed.

Theorem 3.1.5 ▸ Embedding Theorem

Let X be a T_1 space. If $\{f_\alpha\}_{\alpha \in \Lambda}$ is a family of continuous functions from X to $\mathbb{R}$ such that for any $x \in X$ and any open set $U \subseteq X$ with $x \in U$, there exists some $\alpha \in \Lambda$ with $f_\alpha(x) > 0$ and $f_\alpha(X \setminus U) = \{0\}$, then the map $F : X \to \mathbb{R}^\Lambda$ defined by

$$\pi_\alpha\big(F(x)\big) = f_\alpha(x)$$

is a topological embedding.

Proof. By Proposition 1.6.5, F is continuous because all of its component maps, the f_α's, are continuous. Take any $x, y \in X$ with $x \neq y$. Since X is T_1, by Proposition 3.1.4, $X \setminus \{y\}$ is open and contains x. Therefore, there exists some $\alpha \in \Lambda$ with $f_\alpha(x) > 0$ and $f_\alpha(y) = 0$ because $y \in X \setminus (X \setminus \{y\})$. Therefore, $F(x) \neq F(y)$ and so F is injective. In particular, this shows that F is a continuous bijection onto $F(X)$, so it suffices to further prove that F is an open map. Take any open set $U \subseteq X$ and any $z \in F(U)$. Let $x \in U$ be such that $F(x) = z$. Note that there exists some $\alpha \in \Lambda$ such that $f_\alpha(x) > 0$ and $f(X \setminus U) = 0$. Define

$$V_z \coloneqq \pi_\alpha^{-1}\big((0, \infty)\big) \subseteq \mathbb{R}^\Lambda, \qquad W_z = V_z \cap F(X).$$

Note that V_z is open, so $W_z \subseteq F(X)$ is open in $F(X)$. Consider

$$\pi_\alpha(\mathbf{z}) = f_\alpha(x) > 0,$$

so $\mathbf{z} \in V_z \subseteq W_z$. For any $y \in X \setminus U$, we have

$$\pi_\alpha\big(F(y)\big) = f_\alpha(y) = 0,$$

so $F(y) \in F(X) \setminus V_z$, which implies that $F(X \setminus U) \subseteq F(X) \setminus V_z$. Since F is injective,

$$\begin{aligned} F(X) \setminus F(U) &= F(X \setminus U) \\ &\subseteq F(X) \setminus V_z \\ &= F(X) \setminus W_z. \end{aligned}$$

Therefore, $W_z \subseteq F(U)$, which implies that

$$F(U) = \bigcup_{z \in F(U)} W_z$$

which is open in $F(X)$. Therefore, F is a topological embedding. □

3.2 Hausdorff Spaces

T_1 separation can be strengthened by requiring that distinct points have disjoint open neighbourhoods.

Definition 3.2.1 ▸ Hausdorff Space

A topological space X is T_2 or **Hausdorff** if for any $x, y \in X$ with $x \neq y$, there exist open neighbourhoods $U, V \subseteq X$ with $x \in U$ and $y \in V$ such that $U \cap V = \varnothing$.

It is easy to see that every Hausdorff space is T_1. We can check that every metric space is Hausdorff, as for any $x \neq y$, we can always take

$$r \coloneqq \frac{1}{3} d(x, y),$$

and so the r-neighbourhoods of x and y are always disjoint. One can also easily check that the discrete topology of any set is Hausdorff. Hence, the following result is easy to show:

Proposition 3.2.2 ▸ Finite Sets in Metric Spaces Are Closed

Let X be a metric space, then all finite sets in X are closed.

As another example, we prove that a finite co-finite topological space is always Hausdorff.

Proposition 3.2.3 ▸ Separation in Co-finite Topology

Let X be a topological space with the co-finite topology $\mathcal{T}$, then X is T_1. If X is finite, then X is Hausdorff.

Proof. Take any $x, y \in X$ with $x \neq y$. Clearly, $x \in X \setminus \{y\} \in \mathcal{T}$. Therefore, X is T_1. If X is finite, then $X \setminus \{y\}$ is finite and so $\{y\} \in \mathcal{T}$. Therefore, X is Hausdorff.

We shall also prove that if X is infinite, then X is not Hausdorff. Take any $x \in X$ and any $U \in \mathcal{T}$ with $x \in U$, then $X \setminus U$ is finite. For all $y \in X$ with $y \neq x$, we have $y \in X \setminus U$. However, for all $V \subseteq X \setminus U$, we have $V \notin \mathcal{T}$ because $X \setminus V$ is infinite. Therefore, X is not Hausdorff. □

Since finite product preserves open sets, it is expected that finite product of Hausdorff spaces is still Hausdorff.

Proposition 3.2.4 ▸ Finite Product Preserves Hausdorff Spaces

The product of two Hausdorff spaces is Hausdorff.

Proof. Let X and Y be Hausdorff spaces and take any $(x_1, y_1), (x_2, y_2) \in X \times Y$ such that $(x_1, y_1) \neq (x_2, y_2)$. Without loss of generality, assume $x_1 \neq x_2$, then there exists open sets $U_1, U_2 \subseteq X$ such that $U_1 \cap U_2 = \varnothing$, $x_1 \in U_1$ and $x_2 \in U_2$. Note that there exists some open sets $V_1, V_2 \subseteq Y$ such that $y_1 \in V_1$ and $y_2 \in V_2$, and so

$$(x_1, y_1) \in U_1 \times V_1 \quad \text{and} \quad (x_2, y_2) \in U_2 \times V_2.$$

Observe that $U_1 \times V_1$ and $U_2 \times V_2$ both are open in $X \times Y$ and that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = \varnothing,$$

so $X \times Y$ is Hausdorff. □

An interesting characterisation of Hausdorff spaces makes use of a notion called “diagonal”, which is an abstraction of the diagonal of polygons in geometry.

Definition 3.2.5 ▶ Diagonal

Let X be a topological space, then the **diagonal** of X is defined as

$$\Delta := \{(x, x) : x \in X\}.$$

The characterisation states the following:

Proposition 3.2.6 ▶ Characterisation of Hausdorff Spaces with Closed Diagonals

A topological space X is Hausdorff if and only if its diagonal is closed in $X \times X$ with the product topology.

Proof. Define

$$\Delta' := X \times X \setminus \Delta = \{(x, y) \in X \times X : x \neq y\}.$$

Suppose that X is Hausdorff, then for any $(x, y) \in \Delta'$, there exist open sets $U_x, U_y \subseteq X$ with $U_x \cap U_y = \varnothing$ such that $x \in U_x$ and $y \in U_y$. Therefore, $(x, y) \in U_x \times U_y$, which is open in $X \times X$. Notice that

$$\Delta' = \bigcup_{(x,y)\in\Delta'} U_x \times U_y,$$

so Δ' is open in $X \times X$. Therefore, Δ is closed. Conversely, suppose that Δ is closed, then Δ' is open. Take any $x, y \in X$ with $x \neq y$, then $(x, y) \in \Delta'$ and so there exist some open sets $V_x, V_y \subseteq X$ such that $(x, y) \in V_x \times V_y \subseteq \Delta'$. We claim that $V_x \cap V_y = \varnothing$. Suppose on contrary that there exists some $z \in V_x \cap V_y$, then $(z, z) \in V_x \times V_y \subseteq \Delta'$ and so $z \neq z$, which is not possible. Therefore, V_x and V_y are disjoint open sets in X such that $x \in V_x$ and $y \in V_y$, and so X is Hausdorff. □

3.3 Regular Spaces

Suppose we fix a point x in a T_1 space X and use an open set $U \subseteq X$ to seclude x from the rest of the space, then $X \setminus U$ is closed. In a more ideal situation, we may wish to aim for a secluding open set which is as small as possible.

Definition 3.3.1 ▶ Regular Space

A T_1 topological space X is **regular** or T_3 if for any $x \in X$ and any closed set $B \in X$ with $x \notin B$, there exists disjoint open sets $U, V \subseteq X$ with $x \in U$ and $B \subseteq V$.

Notice that every regular space is T_1. Roughly speaking, if we seclude a point x in a T_1 space X using an open set U, then the space is regular if we can "shrink" U into a smaller open set V and "expand" $X \setminus U$ in to a bigger set which is open.

Proposition 3.3.2 ▸ Characterisation of Regular Spaces

A topological space X is regular if and only if for any $x \in X$ and any open set $U \subseteq X$ with $x \in U$, there exists an open set $V \subseteq X$ with $x \in V$ such that $\overline{V} \subseteq U$.

Proof. Suppose that X is a regular space. Take any $x \in X$ with any open set $U \subseteq X$ containing x, then $B := X \setminus U$ is a closed set not containing x. By Definition 3.3.1, there exists disjoint open sets $V, W \subseteq X$ such that $x \in V$ and $B \subseteq W$. This means that

$$V \subseteq X \setminus W \subseteq X \setminus B = U.$$

Note that $X \setminus W$ is closed, so $\overline{V} \subseteq X \setminus W \subseteq U$. □

3.4 Normal Spaces

Note that in a regular space, we are able to separate every point from an arbitrary closed set. A strengthened version of this property is that we can also separate every pair of disjoint closed sets in the space.

Definition 3.4.1 ▸ Normal Space

A T_1 topological space X is **normal** or T_4 if for any disjoint closed sets $A, B \subseteq X$, there exists disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

Notice that every normal space is T_1. In fact, a normal space is regular, a regular space is Hausdorff, and a Hausdorff space is T_1.

Proposition 3.4.2 ▸ Relationship between Separation Axioms

Every T_4 space is T_3, every T_3 space is T_2 and every T_2 space is T_1.

Proof. Let X be a topological space. Suppose that X is T_4. Note that X is T_1, so $\{x\}$ is closed for all $x \in X$ by Proposition 3.1.4. Take any $x \in X$ and any closed set $B \subseteq X$ with $x \notin B$, then $\{x\}$ and B are disjoint closed sets. Therefore, there exist disjoint open sets $U, V \subseteq X$ such that $\{x\} \subseteq U$ and $B \subseteq V$ since X is T_4. This means that $x \in U$, which implies that X is T_3.

Suppose that X is T_3, then X is T_1. Take any $x, y \in X$ with $x \neq y$, then by Proposition 3.1.4, $\{y\}$ is closed in X and clearly $x \notin \{y\}$. Therefore, there exist disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $\{y\} \in V$. This means that $y \in V$ and so X

is T_2.

Suppose that X is T_2, then for any $x, y \in X$ with $x \neq y$, there exist disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$. Notice that $y \notin U$ and so X is T_1. □

We can characterise a normal space using the following result:

Proposition 3.4.3 ▶ Characterisation of Normal Spaces

A topological space X is normal if and only if for any closed set $A \subseteq X$ and any open set $U \subseteq X$ with $A \subseteq U$, there exists some open set $V \subseteq X$ such that $A \subseteq V$ and $\overline{V} \subseteq U$.

Proof. Suppose that X is normal. Take any closed subset $A \subseteq X$ and any open set U with $A \subseteq U$. Note that $B := X \setminus U$ is closed and $A \cap B = \varnothing$, so there exists disjoint open sets $W_A, W_B \subseteq X$ such that $A \subseteq W_A$ and $B \subseteq W_B$. Since $W_A \cap W_B = \varnothing$, we have $W_A \subseteq X \backslash W_B$. However, $X \backslash W_B$ is closed, and so $\overline{W_A} \subseteq X \backslash W_B$. Take $V := W_A$, then $A \subseteq V$ and

$$\overline{V} = \overline{W_A} \subseteq X \setminus W_B \subseteq X \setminus B = U.$$

Conversely, suppose that for every closed $A \subseteq X$ and every open $U \subseteq X$ with $A \subseteq U$, there exists an open set $V \subseteq X$ with $A \subseteq V$ such that $\overline{V} \subseteq U$. Let $A, B \subseteq X$ be any disjoint closed sets with $A \cap B = \varnothing$ and take $U := X \setminus B$, then there exists some open set $V \subseteq X$ with $A \subseteq V$ and $\overline{V} \subseteq U$. Take $W := X \setminus \overline{V}$ which is open, then $B \subseteq W$ because $B \cap \overline{V} \subseteq B \cap U = \varnothing$. Therefore, X is normal. □

Recall that we have argued that every metric space is Hausdorff using a simple argument. In fact, we can directly prove a much stronger result that every metric space is normal.

Proposition 3.4.4 ▶ Every Metrisable Space Is Normal

Every metrisable space is normal.

Proof. Let (X, d) be a metric space and $A, B \subseteq X$ be disjoint closed sets in the metrisable topology. For each $a \in A$, there exists some $\epsilon_a > 0$ such that $B_{\epsilon_a}(a) \subseteq A$. Similarly, for each $b \in B$, there exists some $\epsilon_b > 0$ such that $B_{\epsilon_b}(b) \subseteq B$. Define

$$U := \bigcup_{a \in A} B_{\epsilon_a}(a), \qquad V := \bigcup_{b \in B} B_{\epsilon_b}(b),$$

then $A \subseteq U$ and $B \subseteq V$, where both U and V are open. We claim that $U \cap V = \varnothing$. Suppose on contrary that there exists some $x \in U \cap V$, then there exists some $\alpha \in A$ and $\beta \in B$ such that $x \in B_{\epsilon_\alpha}(\alpha) \cap B_{\epsilon_\beta}(\beta) \subseteq A \cap B = \varnothing$, which is a contradiction. Therefore, X is normal. □

Notice that we get regularity from a normal space for free. Now in the converse direction, we wish to investigate what is the condition which will lead to normality from a regular space.

Proposition 3.4.5 ▸ Second Countable Regular Spaces Are Normal

Every second countable regular space is normal.

Proof. Let X be a second countable regular space and $A, B \subseteq X$ be disjoint closed sets. By Definition 3.3.1, for every $a \in A$, there exists some open set $U_a \subseteq X$ with $a \in U_a$ but $B \cap U_a = \varnothing$. By Proposition 3.3.2, there exists some open set $V_a \subseteq X$ such that

$$a \in V_a \subseteq \overline{V_a} \subseteq U_a$$

for all $a \in A$. Fix a countable basis $\mathcal{B}$ for X, then there exists some $B_a \in \mathcal{B}$ such that $a \in B_a \subseteq V_a$ for all $a \in A$. Define

$$\mathcal{U}_A \coloneqq \{B_a \in \mathcal{B} : a \in A\} \subseteq \mathcal{B},$$

then $\mathcal{U}_A$ is a countable open cover of A, where $B \cap \overline{B_a} = \varnothing$ for all $B_a \in \mathcal{U}_A$. Similarly, we can construct a countable open cover $\mathcal{U}_B \subseteq \mathcal{B}$ for B such that $A \cap \overline{B_b} = \varnothing$ for all $B_b \in \mathcal{U}_B$. Re-write

$$\mathcal{U}_A \coloneqq \{A_i : i \in I\}, \qquad \mathcal{U}_B \coloneqq \{B_j : j \in J\}$$

where $I, J \subseteq \mathbb{N}^+$. Without loss of generality, assume $I \subseteq J$. For each $i \in I$, define

$$A_i' \coloneqq A_i \setminus \bigcup_{j \leq i} \overline{B_j}$$

and for each $j \in J$, define

$$B_j' \coloneqq B_j \setminus \bigcup_{i \leq j} \overline{A_j}.$$

Consider

$$U_A \coloneqq \bigcup_{i \in I} A_i', \qquad U_B \coloneqq \bigcup_{j \in J} B_j',$$

which are both open. Notice that $\overline{B_i} \cap A \neq \varnothing$ for all $i \in I$, so $A_i \cap A = A_i' \cap A$. Therefore,

$$A = \bigcup_{i \in I} (A_i \cap A) = \bigcup_{i \in I} (A_i' \cap A) \subseteq \bigcup_{i \in I} A_i' = U_A.$$

Similarly, $B \subseteq U_B$. We claim that $U_A \cap U_B = \varnothing$. Suppose on contrary that there exists some $x \in U_A \cap U_B$, then there exists some $i \in I$ and $j \in J$ such that $x \in A'_i \cap B'_j$. Without loss of generality, assume that $i \leq j$. This means that $B'_j \cap A_i = \varnothing$ but $A'_i \subseteq A_i$, which is not possible. Therefore, X is normal. $\square$

3.5 Separation by Continuous Functions

Through a different perspective, if two sets are distinguishable, we can "move" from one set to another via a series of continuous transformations. This motivates us to define separation using continuous functions.

Definition 3.5.1 ▸ Separation by a Continuous Function

Let X be a topological space. For any $A, B \subseteq X$, they are **separated by a continuous function** if there exists a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

We see that separation by continuous functions offers a quantitative description for disjointness. With this, we can define the following characterisations:

Definition 3.5.2 ▸ Completely Regular Space

A T_1 topological space X is **completely regular** if for every $x \in X$ and every closed set $A \subseteq X$ with $x \notin A$, $\{x\}$ and A are separated by a continuous function.

Similarly, we can define *completely normal spaces*:

Definition 3.5.3 ▸ Completely Normal Space

A T_1 topological space X is **completely normal** if any disjoint closed sets $A, B \subseteq X$ are separated by a continuous function.

As the names suggest, we expect the following to be true:

Proposition 3.5.4 ▸ Complete Regularity and Complete Normality

Every completely regular space is regular and every completely normal space is normal.

Proof. Let X be a completely regular space. Take any $x \in X$ and any closed set $A \subseteq X$ with $x \notin A$, then there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$

and $f(A) = \{1\}$. Clearly,

$$A \subseteq f^{-1}\left(\left(\frac{1}{2}, 1\right]\right), \qquad x \in f^{-1}\left(\left[0, \frac{1}{2}\right)\right),$$

where both pre-images are open in X. It is obvious that they are disjoint because otherwise there exists some $x \in X$ such that $f(x) > \frac{1}{2}$ and $f(x) < \frac{1}{2}$, which is ridiculous. The case where X is completely normal is similar and is left to the reader as an exercise. $\square$

A truly marvellous aspect of normal spaces is that their normality is "always complete".

Lemma 3.5.5 ▸ Urysohn's Lemma

Every normal space is completely normal.

Proof. For each $n \in \mathbb{N}^+$, let Q_n be an increasing sequence in $\mathbb{Q} \cap [0, 1]$ such that for every $x \in Q_n$, we have $x = \frac{y}{n}$ where y and n are co-prime. Define a sequence

$$\{r_n\}_{n\in\mathbb{N}^+} \coloneqq \{1, 0\} \cup \bigcup_{n=2}^{\infty} Q_n.$$

For every $n \in \mathbb{N}^+$, let $P_n \coloneqq \{r_1, r_2, \cdots, r_n\}$. Take any disjoint closed sets $A, B \subseteq X$ and define $U_1 \coloneqq X \setminus B$ which is an open set containing A. Since X is normal, by Proposition 3.4.3, there exists some open set $U_0 \subseteq X$ such that $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. Suppose that there exists some integer $n \geq 2$ such that for all $p, q \in P_n$ with $p < q$, we have $\overline{U_p} \subseteq U_q$. Consider $\{r_{n+1}\} = P_{n+1} \setminus P_n$ and define

$$r' \coloneqq \max\{r \in P_n : r < r_{n+1}\}, \qquad r'' \coloneqq \min\{r \in P_n : r > r_{n+1}\}.$$

Notice that $r_{n+1} \neq 0, 1$ and so both r' and r'' are well-defined. Since $r' < r''$, we have $\overline{U_{r'}} \subseteq U_{r''}$. Since X is normal, by Proposition 3.4.3, there exists some open set $U_r \subseteq X$ such that $\overline{U_{r'}} \subseteq U_r \subseteq \overline{U_r} \subseteq U_{r''}$. Therefore, for all $p, q \in P_{n+1}$, we have $\overline{U_p} \subseteq U_q$ whenever $p < q$. Define $U_p = \varnothing$ for all $p < 0$ and $U_p = X$ for all $p > 1$, then by mathematical induction, there exists a family of open sets $\{U_p\}_{p\in\mathbb{Q}}$ in X such that $\overline{U_p} \subseteq U_q$ whenever $p < q$, $B \cap U_1 = \varnothing$ and $A \subseteq U_0$. For every $x \in X$, define

$$A_x \coloneqq \{p \in Q : x \in U_p\},$$

then $(1, \infty) \subseteq A_x \subseteq [0, \infty)$. This means that $\inf A_x \in [0, 1]$ for all $x \in X$. Define a map $f : X \to [0, 1]$ by $f(x) = \inf A_x$. We claim that $f(x) \leq p$ for all $x \in \overline{U_p}$ and that $f(x) \geq p$ for all $x \notin U_p$. Suppose that $x \in \overline{U_p}$, then for all $q \in \mathbb{Q} \cap (p, \infty)$, we have $x \in U_q$ and so $q \in A_x$. Therefore,

$$f(x) = \inf A_x \leq \inf\{q \in \mathbb{Q} : q > p\} = p.$$

On the other hand, if $x \notin U_p$, then for every $q < p$, we have $x \notin U_q$ and so $q \notin A_x$. Therefore,

$$f(x) = \inf A_x \geq \inf\{q \in \mathbb{Q} : q > p\} = p.$$

Now, for any $x \in A$, we have $x \in U_0$ and so $f(x) \leq 0$. Since $f(x) \in [0,1]$, this means that $f(A) = \{0\}$. Similarly, we have $f(B) = \{1\}$. Let $(a,b) \in \mathbb{R}$ be any open interval. If for all $x \in X$, we have $f(x) \notin (a,b)$, then $f^{-1}\big((a,b)\big) = \varnothing$ which is open. Otherwise, for every $x \in X$ with $f(x) \in (a,b)$, there exists $a', b' \in \mathbb{Q}$ such that

$$a < a' < f(x) < b' < b.$$

Let $U_x := U_{b'} \setminus \overline{U_{a'}} \neq \varnothing$, which is open in X. For every $y \in U_x$, we have $y \in U_{b'} \subseteq \overline{U_{b'}}$ and $y \notin U_{a'}$. Therefore, $a' \leq f(y) \leq b'$ and so

$$f(U_x) \subseteq [a', b'] \subseteq (a,b).$$

Therefore,

$$f^{-1}\big((a,b)\big) = \bigcup_{x \in f^{-1}\left((a,b)\right)} U_x,$$

which is open in X. Therefore, f is continuous and so X is completely normal. □

Recall that in Proposition 3.4.4, we proved that every metrisable space is normal. This in turn demonstrates how metrisability is a very good property in topological spaces.

Theorem 3.5.6 ▶ Urysohn's Metrisation Theorem

Every second countable regular topological space is metrisable.

Proof. By Proposition 3.4.5, X is normal and so by Lemma 3.5.5, X is completely normal. Fix $\mathcal{B} := \{B_i : i \in \mathbb{N}^+\}$ as a countable basis for X. Take

$$\Lambda := \left\{(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : \overline{B_i} \subseteq B_j\right\}$$

which is countable. For each $(i,j) \in \Lambda$, notice that $\overline{B_i}$ and $X \setminus B_j$ are disjoint closed sets, and so by Lemma 3.5.5, there exists a map $f_{ij} : X \to [0,1]$ such that $f\left(\overline{B_i}\right) = \{1\}$ and $f\left(X \setminus B_j\right) = \{0\}$. Take any $x \in X$ and any open set $U \subseteq X$ with $x \in U$. Note that there exists some $B_j \in \mathcal{B}$ is such that $x \in B_j \subseteq U$. Since X is regular, by Proposition 3.3.2, there exists some open set $V \subseteq X$ such that $x \in V \subseteq \overline{V} \subseteq B_j$. Note that there exists some $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq V \subseteq \overline{B_j}$ and so $(i, j) \in \Lambda$. Since $x \in B_i$, we have $f_{ij}(x) = 1$ and $f_{ij}(X \setminus U) = \{0\}$ because $X \setminus U \subseteq X \setminus B_j$. By Theorem 3.1.5, the map $F : X \to \mathbb{R}^\omega$ defined by

$$\pi_{ij}(F(x)) = f_{ij}(x)$$

is a topological embedding and so it is homeomorphic onto $F(X) \subseteq \mathbb{R}^\omega$. Observe that $F(X)$ is metrisable. Let d_F be the metric on $F(X)$, then $d : X \times X \to \mathbb{R}$ defined by

$$d(x, y) := d_F(F(x), F(y))$$

is a metric on X. Therefore, X is metrisable. □

4

Compactness

4.1 Compact Spaces

We next introduce the notion of compact spaces, which are an attempt to generalise the notion of **closed and bounded** subsets in Euclidean spaces. The idea here is to define compactness as "having no missing end point".

The intuition here is that if a space is not compact, i.e., it can extend indefinitely without an end, then there is a way to partition the space into infinitely many open subsets. On the other hand, if the space has a clearly defined boundary, then every possible partition must be finite.

Definition 4.1.1 ▸ Compact Space

Let X be a topological space. X is **compact** if every open cover of X contains a finite sub-cover for X.

Remark. $Y \subseteq X$ is a compact subspace if and only if for every collection $\mathcal{U}$ of open sets in Y such that $Y \subseteq \bigcup_{U \in \mathcal{U}} U$, there exists a finite sub-collection $\mathcal{U}' \subseteq \mathcal{U}$ such that $Y \subseteq \bigcup_{U \in \mathcal{U}'} U$.

Although compactness is motivated by the closed and bounded subsets in Euclidean spaces, we will see that in general, a closed and bounded subset in a topological space might not be compact.

Consider $\mathbb{Q}$ with the standard topology and $Q := \left[-\sqrt{2}, \sqrt{2}\right] \cap \mathbb{Q}$, which is closed and bounded. We claim that the subset is not compact.

Consider $\left\{\left(-\sqrt{2} + \frac{1}{n}, \sqrt{2} - \frac{1}{n}\right) : n \in \mathbb{N}^+\right\}$. We claim that it is an open cover for Q. Take any $q \in Q$ and let

$$\delta := \min\left\{\sqrt{2} - q, q + \sqrt{2}\right\} > 0.$$

Clearly, there exists some $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \delta$, and so $q \in \left(-\sqrt{2} + \frac{1}{N}, \sqrt{2} - \frac{1}{N}\right)$. Therefore,

$$Q \subseteq \bigcup_{n \in \mathbb{N}^+} \left(-\sqrt{2} + \frac{1}{n}, \sqrt{2} - \frac{1}{n}\right).$$

Let $S \subseteq \mathbb{N}^+$ be any finite subset and let $M := \max S$, then

$$\bigcup_{n \in S} \left(-\sqrt{2} + \frac{1}{n}, \sqrt{2} - \frac{1}{n}\right) = \left(-\sqrt{2} + \frac{1}{M}, \sqrt{2} - \frac{1}{M}\right).$$

However, there exists some rational number $q \in \left[\sqrt{2} - \frac{1}{M}, \sqrt{2}\right]$, which means that

$$\left\{\left(-\sqrt{2} + \frac{1}{n}, \sqrt{2} - \frac{1}{n}\right) : n \in S\right\}$$

is not a sub-cover for any finite $S \subseteq \mathbb{N}^+$. Therefore, Q is not compact.

Note that previously we have shown that a closed set contains all of its limit points. Therefore, it is intuitive that if a closed set is contained by a compact space, this closed set should be compact as well.

Proposition 4.1.2 ▸ Closed Subspaces of Compact Spaces Are Compact

Every closed subspace of a compact space is compact.

Proof. Let X be a compact topological space and $Y \subseteq X$ be a closed subspace. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of Y with $U_\alpha \subseteq X$ for all $\alpha \in \Lambda$, then $\{U_\alpha\}_{\alpha \in \Lambda} \cup \{X \setminus Y\}$ is an open cover for X because Y is closed. Since X is compact, we can find a finite sub-cover $\mathcal{U} \subseteq \{U_\alpha\}_{\alpha \in \Lambda} \cup \{X \setminus Y\}$. Clearly, $\mathcal{U} \setminus \{X \setminus Y\} \subseteq \{U_\alpha\}_{\alpha \in \Lambda}$ is a finite sub-cover for Y. Therefore, Y is compact. □

One may be tempted to think that compact subspaces must be closed. However, this is not true in general. Consider X to be an infinite topological space equipped with the co-finite topology. For any $Y \subseteq X$, let $\mathcal{U}$ be an open cover for Y. Take any $U \in \mathcal{U}$, then clearly $Y \setminus U$ is finite. For each $y \in Y \setminus U$, there exists some $U_y \in \mathcal{U}$ such that $y \in U_y$. Therefore,

$$\{U\} \cup \{U_y : y \in Y \setminus U\}$$

is a finite sub-cover for Y. Therefore, every subset of X is compact. However, $Y \subseteq X$ is closed if and only if Y is finite.

Proposition 4.1.3 ▸ Sets in Co-finite Topology Are Compact

Let X be a topological space equipped with the co-finite topology, then every $Y \subsetneq X$ is compact and it is closed if and only if it is finite.

Proof. Take any $Y \subsetneq X$ and let $\mathcal{U}$ be an open cover for Y. Fix some $U \in \mathcal{U}$, since U is open in X with respect to the co-finite topology, $X \setminus U$ is finite. Note that $Y \setminus U \subseteq X \setminus U$, so $Y \setminus U$ is finite. For each $y \in Y \setminus U$, there exists some $U_y \in \mathcal{U}$ such that $y \in U_y$, otherwise $\mathcal{U}$ does not cover Y. Therefore,

$$\{U\} \cup \{U_y : y \in Y \setminus U\}$$

is a finite sub-cover for Y. Therefore, every subset of X is compact. If Y is finite, then $X \setminus Y$ is open by definition of the co-finite topology, so Y is closed. Conversely, suppose $Y \subsetneq X$ is closed, then $X \setminus Y$ is an non-empty open set, and so Y is finite. $\square$

In fact, even the closure of a compact subspace may not be compact. A classic counter example makes use of the *particular point topology*.

Definition 4.1.4 ▸ Particular Point Topology

For any set X, the **particular point topology** on X with respect to some $p \in X$ is defined as

$$\mathcal{T}_p := \{U \subseteq X : p \in U\} \cup \{\varnothing\}.$$

Fix some $n \in \mathbb{N}$ and define $\mathcal{T}_n$ to be the particular topology on $\mathbb{N}$ with respect to n. Note that for any open cover $\mathcal{U}$ for $\{n\}$, we can take any $U_0 \in \mathcal{U}$ to form a finite sub-cover $\{U_0\}$ because $n \in U_0$. Therefore, $\{n\}$ is compact.

We claim that $\overline{\{n\}} = \mathbb{N}$ because if on contrary $\overline{\{n\}} = \mathbb{N} \setminus X$ for some non-empty $X \in \mathcal{T}_n$, then $n \notin X$ which is not possible. Now consider

$$\{\{m, n\} : m \in \mathbb{N}, m \neq n\},$$

which is an open cover for $\overline{\{n\}}$ but clearly it does not contain a finite sub-cover. Therefore, $\overline{\{n\}}$ is not compact.

To conclude closed-ness from compactness, we require some additional assumptions.

Proposition 4.1.5 ▸ Compact Subspaces of Hausdorff Spaces Are Closed

Every compact subspace of a Hausdorff space is closed.

Proof. Let X be a Hausdorff space and $Y \subseteq X$ be a compact subspace. Let $Z := X \setminus Y$. Fix any $z \in Z$, then for each $y \in Y$, there exists disjoint open sets $U_y, V_y \subseteq X$ such that $z \in U_y$ and $y \in V_y$. Note that $\{V_y : y \in Y\}$ is an open cover for Y, so there exists some finite subset $F \subseteq Y$ such that $\{V_y : y \in F\}$ is a finite sub-cover for Y. Take

$$U_z := \bigcap_{y \in F} U_y,$$

then $U_z \cap V_y = \varnothing$ for all $y \in F$, and so $U_z \cap Y = \varnothing$. Note that U_z is an open neighbourhood of z, so

$$Z = \bigcup_{z \in Z} U_z$$

is open in X. Therefore, $Y = X \setminus Z$ is closed in X. □

A compact space possesses many nice properties, one of which is with regard to the separability of the space. Recall that a normal space is regular and a regular space is Hausdorff, but to go in the converse direction is not so simple. However, for compact spaces, the converse direction will hold.

Proposition 4.1.6 ▸ Separability of Compact Spaces

If X is a compact space, then X is T_4 if and only if it is T_3, and it is T_3 if and only if it is Hausdorff.

Proof. By Proposition 3.4.2, every T_4 space is T_3 and every T_3 space is Hausdorff, so it suffices to prove that X being Hausdorff implies that X is T_3 and X being T_3 implies that X is T_4. Suppose X is Hausdorff. Take any $x \in X$ and any closed set $B \subseteq X$ with $x \notin B$. Note that for every $b \in B$, there exists disjoint open sets $U_b, V_b \subseteq X$ such that $b \in U_b$ and $x \in V_b$. Note that $\{U_b : b \in B\}$ is an open cover of B. Since B is closed and X is compact, by Proposition 4.1.2, B is compact and so there exists a finite subset $A \subseteq B$ such that $\{U_a : a \in A\}$ is a finite sub-cover of B. Define

$$V := \bigcap_{a \in A} V_a, \qquad U := \bigcup_{a \in A} U_a,$$

then both U and V are open in X and $U \cap V = \varnothing$. Note that $x \in V$ and $B \subseteq U$, so X is T_3. Next, suppose that X is T_3 and take any disjoint closed sets $A, B \subseteq X$. Since X is compact, by Proposition 4.1.2, both A and B are compact. For each $a \in A$, since X is T_3, there exists disjoint open sets $U_a, V_a \subseteq X$ such that $a \in U_a$ and $B \subseteq V_a$. Note that $\{U_a : a \in A\}$ is an open cover of A, and so there exists some finite subset $C \subseteq A$ such that $\{U_c : c \in C\}$ is a finite sub-cover of A. Define

$$U := \bigcup_{c \in C} U_c, \qquad V := \bigcap_{c \in C} V_c,$$

then both U and V are open in X and $U \cap V = \varnothing$. Note that $A \subseteq U$ and $B \subseteq V$, so X is T_4. □

Intuitively, a continuous map should send a compact set to a compact image because open sets, and hence open covers, are preserved by the map.

Proposition 4.1.7 ▸ Continuous Map Induces Compact Image

Let $f : X \to Y$ be continuous, if X is compact, then $f(X)$ is compact.

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover for $f(X)$. Notice that for every $x \in X$, we have $f(x) \in U_\lambda$ for some $\lambda \in \Lambda$ and so $x \in f^{-1}(U_\lambda)$ which is open in X because f is continuous. Therefore, $\{f^{-1}(U_\alpha)\}_{\alpha\in\Lambda}$ is an open cover for X. Since X is compact, there exists some finite $I \subseteq \Lambda$ such that $\{f^{-1}(U_i)\}_{i\in I}$ is a finite sub-cover for X. For any $y \in f(X)$, there exists some $x_y \in X$ such that $f(x_y) = y$. Notice that there exists some $j \in I$ such that $x_y \in f^{-1}(U_j)$, so $y \in U_j$. Therefore,

$$f(X) \subseteq \bigcup_{i\in I} U_i$$

and so $f(X)$ is compact. $\square$

Next, we consider the following observation: if $X := (x_1, x_2)$ and $Y := [y_1, y_2]$ are intervals in $\mathbb{R}$, let $N := X \times Y$ be a rectangular region in $\mathbb{R}^2$. It is easy to see that if N contains a line segment $((x_0, y_1), (x_0, y_2))$ for some $x_0 \in X$, then N contains a "tube"

$$T := (x_0 - \delta, x_0 + \delta) \times [y_1, y_2]$$

for some $\delta > 0$. This is generalised to the following lemma:

Lemma 4.1.8 ▸ Tube Lemma

Let X be a topological space and Y be a compact topological space. If $N \subseteq X \times Y$ is an open set that contains $\{(x_0, y) : y \in Y\}$, then N contains $W \times Y$ for some open $W \subseteq X$ that contains x_0.

Proof. Let $\mathcal{B}$ be the basis generating the product topology. Define

$$\mathcal{A} := \{U \times V \in \mathcal{B} : U \times V \subseteq N, x_0 \in U\}.$$

Since N is open, for any $y \in Y$, there exists some $U_y \times V_y \in \mathcal{B}$ such that

$$(x_0, y) \in U_y \times V_y \subseteq N.$$

Therefore, $\mathcal{A}$ is an open cover for $\{x_0\} \times Y$. Note that $\{x_0\} \times Y$ is compact because of the compactness of Y. Therefore, there exists a finite sub-cover

$$\{U_1 \times V_1, \cdots, U_n \times V_n\} \subseteq \mathcal{A}$$

of $\{x_0\} \times Y$ where $\bigcup_{i=1}^n V_i = Y$. Take

$$W := \bigcap_{i=1}^n U_i \subseteq X,$$

then W is open and $x_0 \in W$. Since $W \subseteq U_i$ for all $i = 1, 2, \cdots, n$, we have

$$W \times Y = \bigcup_{i=1}^n (W \times V_i) \subseteq \bigcup_{i=1}^n (U_i \times V_i) \subseteq N.$$

□

The compactness of Y is crucial for the Tube lemma to hold. Consider

$$S := \left\{(x, y) \in \mathbb{R}^2 : |x| < \frac{1}{y^2 + 1}\right\} \subseteq \mathbb{R}^2,$$

which is open and contains $\{0\} \times \mathbb{R}$, but does not contain a tube.

Applying the Tube Lemma to Cartesian products yields the following result:

Corollary 4.1.9 ▸ Compactness of Cartesian Product

If X and Y are compact topological spaces, then $X \times Y$ is compact.

Proof. Let $\mathcal{A}$ be an open cover of $X \times Y$, then $\mathcal{A}$ is also an open cover of $\{x\} \times Y$ for any $x \in X$. Since Y is compact, there exists some finite subset $\mathcal{A}_x \subseteq \mathcal{A}$ such that

$$\{x\} \times Y \subseteq \bigcup_{A \in \mathcal{A}_x} A := N_x.$$

Note that N_x is open in $X \times Y$. By Lemma 4.1.8, there exists some open subset $W_x \subseteq X$ such that $x \in W_x$ and $W_x \times Y \subseteq N_x$. Clearly, $\{W_x : x \in X\}$ is an open cover for X. Since X is compact, this means that there exists $x_1, x_2, \cdots, x_n \in X$ such that

$$X = \bigcup_{i=1}^n W_{x_i}.$$

Take $\mathcal{A}' \coloneqq \bigcup_{i=1}^{n} \mathcal{A}_{x_i} \subseteq \mathcal{A}$ which is finite, then

$$\begin{aligned} X \times Y &\subseteq \bigcup_{i=1}^{n} (W_{x_i} \times Y) \\ &\subseteq \bigcup_{i=1}^{n} N_{x_i} \\ &= \bigcup_{i=1}^{n} \bigcup_{A \in \mathcal{A}_{x_i}} A \\ &= \bigcup_{A \in \mathcal{A}'} A. \end{aligned}$$

Therefore, $\mathcal{A}'$ is a finite sub-cover for $X \times Y$ and so $X \times Y$ is compact. □

Compactness is also related to the notion of a *graph* of a function. We will first define this concept in topological terms.

Definition 4.1.10 ▸ Graph of A Function

For any function $f : X \to Y$, the set

$$\Gamma_f = \{(x, f(x)) : x \in X\} \subseteq X \times Y$$

is called the **graph** of f.

Recall that in elementary real analysis, we have defined continuity at "end points" by the existence of one-side limits. This translates to the closure of the function's graph, and in fact is closely related to the compactness of the range.

Proposition 4.1.11 ▸ Closure of Graphs of Functions

Let $f : X \to Y$ be a function. If X is any space and Y is a compact Hausdorff space, then Γ_f is closed if and only if f is continuous.

Proof. Suppose that Γ_f is closed. Let $V \subseteq Y$ be a closed set, then $X \times V$ is closed in $X \times Y$. Define

$$G \coloneqq \Gamma_f \cap (X \times V) = f^{-1}(V) \times f\left(f^{-1}(V)\right).$$

Note that G is closed in $X \times Y$. Take any $x \in X \setminus \pi_X(G)$, then $(x, y) \in (X \times Y) \setminus G$ for all $y \in Y$. Note that $(X \times Y) \setminus G$ is open and Y is compact, so by Lemma 4.1.8, there exists some open set $W_x \subseteq X$ with $x \in W_x$ such that $W_x \times Y \subseteq (X \times Y) \setminus G$.

Therefore, $W_x \subseteq X \setminus \pi_X(G)$ and so

$$X \setminus \pi_X(G) = \bigcup_{x \in X \setminus \pi_X(G)} W_x,$$

which is open. Therefore, $\pi_X(G) = f^{-1}(V)$ is closed in X. Therefore, by Proposition 1.5.7, f is continuous. Conversely, suppose that f is continuous, then it suffices to prove that $\Gamma_f = \overline{\Gamma_f}$. Suppose on contrary that $\Gamma_f \neq \overline{\Gamma_f}$. Take

$$(x, y) \in \overline{\Gamma_f} \setminus \Gamma_f \subseteq X \times Y$$

such that $y \neq f(x)$. Since Y is Hausdorff, there exists disjoint open sets $U, V \subseteq Y$ such that $y \in U$ and $f(x) \in V$. Let $W := f^{-1}(V) \neq \varnothing$, which is open in X, and so $W \times U$ is an open neighbourhood of (x, y). Note that (x, y) is a limit point of Γ_f, so

$$\Gamma_f \cap (W \times U \setminus \{(x, y)\}) \neq \varnothing.$$

Take

$$(x', y') \in \Gamma_f \cap \left(W \times U \setminus \{(x, y)\}\right),$$

then $y' = f(x') \in U$. However, $f(x') \in f(W) \subseteq V$, so $U \cap V \neq \varnothing$, which is a contradiction. Therefore, Γ_f must be closed. □

At another perspective, we can re-formulate the definition of compactness using closed sets. This require us to use something known as the *finite intersection property*.

Definition 4.1.12 ▸ Finite Intersection Property

Let X be some set. A collection $\mathcal{G} \subseteq \mathcal{P}(X)$ has the **finite intersection property** if for any finite sub-collection $\mathcal{G}' \subseteq \mathcal{G}$, we have

$$\bigcap_{G \in \mathcal{G}'} G \neq \varnothing.$$

The following result characterises compactness using the finite intersection property:

Proposition 4.1.13 ▸ Equivalent Definition of Compactness

Let X be a topological space, then X is compact if and only if for any collection of closed sets $\mathcal{G} \subseteq \mathcal{P}(X)$ with the finite intersection property, we have

$$\bigcap_{G \in \mathcal{G}} G \neq \varnothing.$$

Proof. Suppose that for any collection $\mathcal{G} \subseteq \mathcal{P}(X)$ of closed sets in X with the finite intersection property, we have $\bigcap_{G\in\mathcal{G}} G \neq \varnothing$. Let $\mathcal{U}$ be an open cover of X and consider

$$\mathcal{G} := \{X \setminus U : U \in \mathcal{U}\}.$$

Note that $\mathcal{G}$ is a collection of closed sets and

$$\bigcap_{G\in\mathcal{G}} G = \bigcap_{U\in\mathcal{U}} X \setminus U = X \setminus \bigcup_{U\in\mathcal{U}} U = \varnothing.$$

This implies that $\mathcal{G}$ does not have the finite intersection property, i.e., there exists open sets $U_1, U_2, \cdots, U_n \in \mathcal{U}$ such that

$$\bigcap_{i=1}^{n} X \setminus U_i = X \setminus \bigcup_{i=1}^{n} U_i = \varnothing.$$

This means that $\{U_1, U_2, \cdots, U_n\} \subseteq \mathcal{U}$ is a finite sub-cover for X and so X is compact. Conversely suppose X is compact and let $\mathcal{G}$ be a collection of closed sets in X such that

$$\bigcap_{G\in\mathcal{G}} G = \varnothing.$$

Define

$$\mathcal{U} := \{X \setminus G : G \in \mathcal{G}\},$$

then clearly

$$\bigcup_{U\in\mathcal{U}} U = X \setminus \bigcap_{G\in\mathcal{G}} G = X,$$

so $\mathcal{U}$ is an open cover of X. Since X is compact, there exists $G_1, G_2, \cdots, G_n \in \mathcal{G}$ such that

$$\bigcup_{i=1}^{n} X \setminus G_i = X \setminus \bigcap_{i=1}^{n} G_i = X.$$

Therefore, $\mathcal{G}$ does not have the finite intersection property. □

An intuitive example of a collection with finite intersection property is **nested closed intervals** in Euclidean spaces. This can be generalised as the following corollary:

Corollary 4.1.14 ▸ Intersection of Nested Closed Sets in Compact Spaces

Let X be a compact topological space and $\{G_i\}_{i=1}^{\infty}$ be a sequence of closed sets in X such *that $G_{i+1} \subseteq G_i$ for all $i \in \mathbb{N}^+$, then*

$$\bigcap_{i=1}^{\infty} G_i \neq \varnothing.$$

Lastly, we discuss the cardinality of compact spaces. Intuitively, a space is unlikely to be countable if its points are **dense**. To visualise the density of points, it might be helpful to think of **sparse points** as points around which there is some **empty space**.

Definition 4.1.15 ▸ Isolated Point

A point x in a topological space X is **isolated** if $\{x\}$ is open in X.

We can establish an analogous connection between the density of points distributed in a space and the number of isolated points in the space. Naturally, a space contains "fewer" points if there are lots of isolated points.

Proposition 4.1.16 ▸ Characterisation of Uncountable Topological Spaces

Let X be a non-empty compact Hausdorff space. If X has no isolated point, then X is uncountable.

Proof. Suppose on contrary that X is countable, then we can write

$$X \coloneqq \{x_i : i \in I\}$$

for some $I \subseteq \mathbb{N}^+$. We consider the following lemma:

Lemma 4.1.17 ▸ Non-isolated Points in Hausdorff Spaces

Let X be a Hausdorff space. If $U \subseteq X$ is open and non-empty and $x \in X$ is not isolated, then there exists some open and non-empty subset $V \subseteq U$ with $x \notin \overline{V}$.

Proof. Note that $U \setminus \{x\}$ is open and non-empty because otherwise $\{x\}$ is open. Fix some $y \in U \setminus \{x\}$, then since X is Hausdorff, we can find some disjoint open sets $W_x, W_y \subseteq X$ such that $x \in W_x$ and $y \in W_y$. Take $V \coloneqq W_y \cap U \subseteq U$, then V is non-empty and open. Note that $X \setminus W_x$ is a closed set containing V, so $\overline{V} \subseteq X \setminus W_{x_i}$. This means that $x \notin \overline{V}$. □

By Lemma 4.1.17, there exists some open set $V_1 \subseteq X$ such that $x_1 \notin \overline{V_1}$. For each integer $i \geq 2$, we can find some open set $V_i \subseteq V_{i-1}$ such that $x_i \notin \overline{V_i}$. By Corollary

4.1.14, we know that

$$\bigcap_{i \in I} \overline{V_i} \neq \varnothing.$$

However, for every $x_i \in X$, there exists some V_i such that $x_i \notin \overline{V_i}$, which is a contradiction. $\square$

A natural thought is that the product space of compact spaces should be compact. However, it turns out the proof of this innocent claim is not so straightforward. We first define some preliminary concepts.

Definition 4.1.18 ▶ Partial Order and Total Order

A **partial order** on a set X is a binary relation $\leq$ on X such that

- $x \leq x$;
- for any $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$;
- if $x \leq y$ and $y \leq z$, then $x \leq z$.

A **total order** on a set X is a partial order on X such that for every $(x, y) \in X^2$, we have $x \leq y$ or $y \leq x$.

Note that a partial order is not necessarily applicable to any pair of elements in the set. However, we can always purposefully choose a subset such that the restricted order becomes a total order.

Definition 4.1.19 ▶ Chain

Let X be a set with a partial order $\leq$. A **chain** in X is a set $\mathcal{C} \subseteq \mathcal{P}(X)$ such that $\leq$ restricted to $\mathcal{C}$ is a total order.

An important result in set theory is *Zorn's lemma*:

Lemma 4.1.20 ▶ Zorn's Lemma

Let X be a set and let $\leq$ be a partial order on X. If every chain $\mathcal{C}$ in X has an upper bound in X, then X contains a $\leq$-maximal element.

Zorn's lemma helps prove the following result:

Proposition 4.1.21 ▶ Maximal Collection with the Finite Intersection Property

Let X be a set and $\mathcal{A} \subseteq \mathcal{P}(X)$ have the finite intersection property, then there exists some maximal $\mathcal{D} \subseteq \mathcal{P}(X)$ with the finite intersection property such that $\mathcal{A} \subseteq \mathcal{D}$.

Proof. Define

$$\mathscr{P}_{\mathcal{A}} := \{\mathcal{U} \subseteq \mathcal{P}(X) : \mathcal{A} \subseteq \mathcal{U} \text{ and } \mathcal{U} \text{ has the finite intersection property}\},$$

then $\mathscr{P}_{\mathcal{A}}$ is a partially ordered set by $\subseteq$. Let $\mathscr{C}_{\mathcal{A}} \subseteq \mathscr{P}_{\mathcal{A}}$ be any chain and define

$$\mathcal{U}_{\mathcal{A}} := \bigcup_{\mathcal{U} \in \mathscr{C}_{\mathcal{A}}} \mathcal{U}.$$

Let $U_1, U_2, \cdots, U_n \in \mathcal{U}_{\mathcal{A}}$ for some $n \in \mathbb{N}^+$, then for each $i = 1, 2, \cdots, n$, there exists some $\mathcal{U}_i \in \mathscr{C}_{\mathcal{A}}$ such that $U_i \in \mathcal{U}_i$. Note that $\mathscr{C}_{\mathcal{A}}$ is totally ordered, so there exists some $\mathcal{U}_0 \in \mathscr{C}_{\mathcal{A}}$ such that $\mathcal{U}_i \subseteq \mathcal{U}_0$ for all $i = 1, 2, \cdots, n$. Therefore, for all $i = 1, 2, \cdots, n$, we have $U_i \in \mathcal{U}_0 \in \mathscr{P}_{\mathcal{A}}$, and so

$$\bigcap_{i=1}^{n} U_i \neq \varnothing.$$

Note that $\mathcal{A} \subseteq \mathcal{U}_{\mathcal{A}}$, so $\mathcal{U}_{\mathcal{A}} \in \mathscr{P}_{\mathcal{A}}$. Clearly, $\mathcal{U}_{\mathcal{A}}$ is an upper bound for $\mathscr{C}_{\mathcal{A}}$. By Lemma 4.1.20, there exists a $\subseteq$-maximal element $\mathcal{D} \in \mathscr{P}_{\mathcal{A}}$. □

Following this, we can show that such maximal containing collections exhibit certain special properties.

Proposition 4.1.22 ▶ Characterisation of the Maximal Containing Collection

Let X be a set and $\mathcal{A} \subseteq \mathcal{P}(X)$ be a collection with the finite intersection property. For each maximal collection $\mathcal{D} \subseteq \mathcal{P}(X)$ with $\mathcal{A} \subseteq \mathcal{D}$ with the finite intersection property, $\mathcal{D}$ is closed under finite intersection and for all $A \subseteq X$ with $A \cap D \neq \varnothing$ for all $D \in \mathcal{D}$, we have $A \in \mathcal{D}$.

Proof. Let $D_1, D_2, \cdots, D_n \in \mathcal{D}$ for some $n \in \mathbb{N}^+$ and define $D := \bigcap_{i=1}^{n} D_i$. Take any $D_1', D_2', \cdots, D_m' \in \mathcal{D}$ for some $m \in \mathbb{N}^+$ and consider

$$D \cap \bigcap_{i=1}^{m} D_i' = \left(\bigcap_{i=1}^{n} D_i\right) \cap \left(\bigcap_{i=1}^{m} D_i'\right) \neq \varnothing.$$

Therefore, $\mathcal{D} \cup \{D\}$ has a finite intersection property and $\mathcal{A} \subseteq \mathcal{D} \cup \{D\}$. By maximality of $\mathcal{D}$, we must have $D \in \mathcal{D}$, and so $\mathcal{D}$ is closed under finite intersection. Let $A \subseteq X$ be such that $A \cap D \neq \varnothing$ for all $D \in \mathcal{D}$, then for any $D_1, D_2, \cdots, D_k \in \mathcal{D}$ where $k \in \mathbb{N}^+$, we have

$$A \cap \bigcap_{i=1}^{k} D_i \neq \varnothing,$$

which means that $\mathcal{D} \cup \{A\}$ contains $\mathcal{A}$ and has the finite intersection property. By maximality of $\mathcal{D}$, this means that $A \in \mathcal{D}$. □

All of the above results combined together lead to the following important major theorem:

Theorem 4.1.23 ▸ Tychonoff's Theorem

If $\{X_\alpha\}_{\alpha \in \Lambda}$ is a family of compact topological spaces, then $X := \prod_{\alpha \in \Lambda} X_\alpha$ is compact.

Proof. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ have the finite intersection property. By Proposition 4.1.21, there exists a maximal collection $\mathcal{D} \subseteq \mathcal{P}(X)$ containing $\mathcal{A}$, which has the finite intersection property. For every $\alpha \in \Lambda$, define

$$\left\{\overline{\pi_\alpha(D)} : D \in \mathcal{D}\right\} \subseteq \mathcal{P}(X_\alpha),$$

which has the finite intersection property. Since X_α is compact, by Proposition **??**,

$$\Pi_\alpha := \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)} \neq \varnothing.$$

For each $\alpha \in \Lambda$, pick some $x_\alpha \in \Pi_\alpha$ and define $\boldsymbol{x} := (x_\alpha)_{\alpha \in \Lambda}$. Let $U \subseteq X$ be an open neighbourhood for $\boldsymbol{x}$, then there exists an open neighbourhood $U_\alpha \subseteq X_\alpha$ for x_α for each $\alpha \in \Lambda$ such that $U = \prod_{\alpha \in \Lambda} U_\alpha$. Note that for all $D \in \mathcal{D}$,

$$\overline{\pi_\alpha(D)} \cap U_\alpha \neq \varnothing.$$

Therefore,

$$\pi_\alpha(D) \cap U_\alpha \neq \varnothing,$$

and so

$$D \cap \pi_\alpha^{-1}(U_\alpha) \neq \varnothing.$$

By Proposition 4.1.22, $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{D}$ and for all finite $I \subseteq \Lambda$,

$$U \subseteq \bigcap_{i \in I} \pi_i^{-1}(U_i) \in \mathcal{D}.$$

Therefore, for all $D \in \mathcal{D}$, we have $U \cap D \neq \varnothing$ and so $\boldsymbol{x}$ is a limit point of every $D \in \mathcal{D}$. This means that

$$\boldsymbol{x} \in \bigcap_{D \in \mathcal{D}} \overline{D} \subseteq \bigcap_{A \in \mathcal{A}} \overline{A} = \bigcap_{A \in \mathcal{A}} A \neq \varnothing.$$

By Proposition **??**, X is compact. □

4.2 Limit Point and Sequentially Compact Spaces

Let us re-visit the motivation of compactness: a space with no missing end points (i.e., end points which are infinitely far away). Now, another way to interpret this statement is that if we fix an infinite subset in the space, then we can always find a limit point of the subset in the space.

Definition 4.2.1 ▶ Limit Point Compactness

A topological space X is **limit point compact** if every infinite subset of X has a limit point in X.

Note that it is equivalent to say that every subset of a limit point compact space which has no limit point in the space must be finite.

It is possible for a non-compact space to be limit point compact. However, a compact space must be limit point compact.

Proposition 4.2.2 ▶ Compactness Implies Limit Point Compactness

Any compact topological space is limit point compact.

Proof. Let X be a compact topological space and $A \subseteq X$ be a subset without any limit point in X, then it suffices to prove that A is finite. Notice that $A = \overline{A}$, so A is closed in X. By Proposition 4.1.2, A is compact. For every $a \in A$, since a is not a limit point, by Definition 1.4.6 there exists some open set $U_a \subseteq X$ with $a \in U_a$ but

$$(A \setminus \{a\}) \cap U_a = \varnothing.$$

Clearly, this means that $U_a \cap A = \{a\}$. Therefore, $\{a\}$ is open in A. This means that

$$\mathcal{A} \coloneqq \big\{\{a\} : a \in A\big\}$$

is an open cover of A. Since A is compact, there exists $a_1, a_2, \cdots, a_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} \{a_i\},$$

and so A is finite. □

We can further strengthen the condition by changing "existence of limit points" into convergence.

Definition 4.2.3 ▸ Sequential Compactness

A topological space X is **sequentially compact** if every sequence in X has a convergent subsequence.

Intuitively, sequential compactness should be a stronger result than limit point compactness because not every limit point is the limit of some sequence.

Proposition 4.2.4 ▸ Sequential Compactness Implies Limit Point Compactness

Any sequentially compact topological space is limit point compact, but the converse is not true.

Proof. Let X be sequentially compact and let $A \subseteq X$ be an infinite subset. Note that there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A, and so we can find a convergent subsequence $\{a_{n_i}\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} a_{n_i} = a \in X$. It is clear that a is a limit point of A and so A is limit point compact. □

However, the converse of this statement is false.

Proposition 4.2.5 ▸ Limit Point Compactness Is Weaker

There exists a topological space X which is limit point compact but not sequentially compact.

Proof. Take $X = \mathbb{R}$ and define

$$\mathcal{R} \coloneqq \{(a, \infty) : a \in \mathbb{R}\}.$$

We first check that $\mathcal{R}$ is a basis for a topology on $\mathbb{R}$. It is clear that for any $x \in \mathbb{R}$, we have $x \in (x-1, \infty) \in \mathcal{R}$. Take any $x \in \mathbb{R}$ and any $a, b \in \mathbb{R}$ with $x \in (a, \infty) \cap (b, \infty)$. Without loss of generality, assume that $a < b$ and so $x \in (b, \infty) \subseteq (a, \infty) \cap (b, \infty)$. Therefore, $\mathcal{R}$ is a basis. Let $A \subseteq \mathbb{R}$ be any infinite subset. Fix any $a \in A$, then there exists some $(a', \infty) \subseteq A$ such that $a \in (a', \infty)$. Take $\ell \coloneqq \frac{a+a'}{2}$, then for any open neighbourhood U of ℓ, there exists some $u < \ell$ such that $\ell \in (u, \infty) \subseteq U$. Therefore,

$$U \cap (A \setminus \{a\}) \subseteq (u, \infty) \cap (a', \infty) \neq \varnothing,$$

and so ℓ is a limit point of A. Therefore, X is limit point compact. Consider the sequence $\{-n\}_{n\in\mathbb{N}}$. We claim that the sequence has no convergent subsequence. Suppose on contrary that $\{-n_k\}_{k\in\mathbb{N}}$ is a convergent subsequence with

$$\lim_{k\to\infty} -n_k = N,$$

then $(\lfloor N-1\rfloor, \infty)$ is an open neighbourhood of N but for all $k > -\lfloor N-1\rfloor$, we have

$$-n_k \leq -k < \lfloor N-1\rfloor,$$

which is a contradiction. Therefore, $\{-n\}_{n\in\mathbb{N}}$ has no convergent subsequence and so X is not sequentially compact. □

4.3 Totally Bounded Spaces

Suppose we construct an open cover for a space X, then intuitively, we could use some open sets which are "big" enough to cover the space, such that all the "small" subsets of the space can be contained by some single open set in the cover.

Definition 4.3.1 ▸ Lebesgue Number

Let $\mathcal{U}$ be an open cover for a metric space X, then $\delta > 0$ is called a **Lebesgue number** for $\mathcal{U}$ if for all $S \subseteq X$ with $\operatorname{diam}(S) < \delta$, there exists some $U \in \mathcal{U}$ such that $S \subseteq U$.

The existence of Lebesgue number is not guaranteed in general topological spaces. For example, for each $n \in \mathbb{Z}$, define $A_n :\ (n, n+1)$ and

$$B_n := \begin{cases} (-1,1) & \text{if } n = 0 \\ \left(n - \frac{1}{n}, n + \frac{1}{n}\right) & \text{otherwise} \end{cases}.$$

Clearly, $\{A_n : n \in \mathbb{Z}\} \cup \{B_n : n \in \mathbb{Z}\}$ is an open cover for $\mathbb{R}$. However, for any $\delta > 0$, there exists some $N \in \mathbb{Z}^+$ such that $\frac{2}{N} < \delta$. Take

$$S := \left(N + 1 - \frac{1}{N}, N + 1 + \frac{1}{N}\right) \subseteq \mathbb{R}.$$

Clearly, $\operatorname{diam}(S) = \frac{2}{N} < \delta$. Note that $N \leq N+1-\frac{1}{N} < N+1$ and $N+1 < N+1+\frac{1}{N} \leq N+2$, so there does not exist any $n \in \mathbb{Z}$ such that $S \subseteq A_n$. Since $N > 0$, we have $S \not\subseteq B_0$. Notice that if $n - \frac{1}{n} \leq N + 1 - \frac{1}{N}$, then $n + \frac{1}{n} < N + 1 + \frac{1}{N}$, so $S \not\subseteq B_n$ for all $n \in \mathbb{Z}$. Therefore, δ is not a Lebesgue number and so $\{A_n : n \in \mathbb{Z}\} \cup \{B_n : n \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$ that does not have a Lebesgue number.

However, it can be shown that Lebesgue number always exists for sequentially compact spaces.

Proposition 4.3.2 ▶ Sequentially Compact Spaces Guarantee Lebesgue Number

Every open cover of a sequentially compact metric space has a Lebesgue number.

Proof. Let X be sequentially compact. Suppose on contrary that there exists some open cover $\mathcal{U}$ of X without a Lebesgue number, then for every $n \in \mathbb{N}^+$, there exists some $S_n \subseteq X$ with $\operatorname{diam}(S_n) < \frac{1}{n}$ but $S_n \nsubseteq U$ for all $U \in \mathcal{U}$. For each $n \in \mathbb{N}^+$, pick some $x_n \in S_n$, then the sequence $\{x_n\}_{n\in\mathbb{N}^+}$ has a convergent subsequence $\{x_{n_i}\}_{i\in\mathbb{N}^+}$. Let

$$\lim_{i\to\infty} x_{n_i} = x \in U$$

for some $U \in \mathcal{U}$, then there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. Note that there exists some $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \frac{\epsilon}{2}$ and $x_{n_i} \in B_{\frac{\epsilon}{2}}(x)$ for all $i > N$. Note that for all $i > N$, we have $n_i \geq i > N$, so

$$\operatorname{diam}(S_{n_i}) < \frac{1}{n_i} < \frac{1}{N} < \frac{\epsilon}{2}.$$

Therefore,

$$S_{n_i} \subseteq B_{\frac{\epsilon}{2}}(x_{n_i}) \subseteq B_\epsilon(x) \subseteq U,$$

which is a contradiction. □

Suppose that we have constructed an open cover with a Lebesgue number δ. Clearly, we can partition any space into subsets with diameter strictly less than δ. This means that for each of the subset, we can first fix some U containing it, and find an open ball containing U in the space. In a more ideal case, such open balls will have their radius upper-bounded by an arbitrarily small number. Intuitively, this means that we will be able to cover the space with balls which are “small” enough.

Definition 4.3.3 ▶ Totally Bounded Space

A metric space X is **totally bounded** if for all $\epsilon > 0$, there exists a finite cover of X by open balls of radius ϵ.

Previously, we have seen that sequential compactness is closely linked to the existence of Lebesgue numbers. Therefore, it is not so surprising that sequential compactness leads to total boundedness.

Proposition 4.3.4 ▸ Sequential Compactness Implies Total Boundedness

Every sequentially compact metrisable topological space is totally bounded.

Proof. We shall prove the contrapositive. Suppose that X is a metric space that is not totally bounded, then there exists some $\epsilon > 0$ such that X cannot be covered with finitely many open ϵ-balls. This means that for any $n \in \mathbb{N}$ and any $x_1, x_2, \cdots, x_n \in X$,

$$X \setminus \bigcup_{i=1}^{n} B_\epsilon(x_i) \neq \varnothing.$$

Now, fix some $x_1 \in X$ and for each $i = 2, 3, \cdots$, pick some

$$x_i \in X \setminus \bigcup_{j=1}^{i} B_\epsilon(x_j),$$

then $\{x_i\}_{i\in\mathbb{N}^+}$ is a sequence in X. Note that for any $i \neq j$, we have $x_j \notin B_\epsilon(x_i)$ and so for all $x \in X$, the open ball $B_{\frac{\epsilon}{2}}(x)$ contains at most one of $x_1, x_2, \cdots$. Therefore, the sequence contains no convergent subsequence and so X is not sequentially compact. □

A nice thing about Propositions 4.3.2 and 4.3.4 is that they apply to any compact metric space in general, which is a result of the following equivalence:

Proposition 4.3.5 ▸ Compactness of Metrisable Spaces

If X is a metrisable topological space, then the followings are equivalent:

1. *X is compact;*
2. *X is limit point compact;*
3. *X is sequentially compact.*

Proof. (1) $\implies$ (2) is by Proposition 4.2.2. Now, suppose X is limit point compact and let $\{x_i\}_{i\in\mathbb{N}}$ be any sequence in X. Define

$$A \coloneqq \{x_i : i \in \mathbb{N}\}.$$

If A is finite, then there exists some x_k which appears countably infinite times in the sequence, and so $\{x_k\}_{i\in\mathbb{N}}$ is a convergent subsequence. If A is infinite, then it has a limit point x. Since X is a metric space, it is first countable and T_1. Fix a countable basis $\mathcal{B} \coloneqq \{B_i : i \in \mathbb{N}^+\}$ for x such that $B_{i+1} \subseteq B_i$ for all $i \in \mathbb{N}^+$, then

$$B_i \cap (A \setminus \{x\}) \neq \varnothing.$$

We claim that $B_i \cap (A \setminus \{x\})$ is infinite. Suppose on contrary that it is finite, then we can write

$$B_i \cap (A \setminus \{x\}) = \{a_1, a_2, \cdots, a_n\}$$

for some $n \in \mathbb{N}^+$. Since X is T_1, for each $k = 1, 2, \cdots, n$, there exists some $B_{j_k} \in \mathcal{B}$ such that $x \in B_{j_k}$ but $a_k \notin B_{j_k}$. Therefore,

$$B := B_i \cap \bigcap_{k=1}^{n} B_{j_k} \subseteq X$$

is open but $B \cap (A \setminus \{x\}) = \varnothing$, which is a contradiction. For each $i \in \mathbb{N}^+$, pick some $x_{n_i} \in B_i \cap (A \setminus \{x\})$, and so $\{x_{n_i}\}_{i \in \mathbb{N}^+}$ is a convergent subsequence. Therefore, X is sequentially compact.

Suppose that X is sequentially compact and let $\mathcal{U}$ be an open cover for X. By Proposition 4.3.2, $\mathcal{U}$ has a Lebesgue number $\delta > 0$. By Proposition 4.3.4, there exists a finite cover

$$\left\{B_{\frac{\delta}{3}}(x_i) : i = 1, 2, \cdots, n\right\}$$

for X. For every $i = 1, 2, \cdots, n$, we have

$$\operatorname{diam}\left(B_{\frac{\delta}{3}}(x_i)\right) = \frac{2\delta}{3} < \delta,$$

and so there exists some $U_i \in \mathcal{U}$ such that $B_{\frac{\delta}{3}}(x_i) \subseteq U_i$. Therefore,

$$\{U_i : i = 1, 2, \cdots, n\}$$

is a finite sub-cover for X and so X is compact. □

By exploiting this equivalence in metric spaces, we can obtain the following result:

Proposition 4.3.6 ▸ Compact Spaces Induce Uniform Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.

Proof. For any $\epsilon > 0$, define

$$\mathcal{B}_\epsilon := \left\{B_{\frac{\epsilon}{2}}(y) : y \in Y\right\}, \qquad \mathcal{F}_\epsilon := \left\{f^{-1}(B) : B \in \mathcal{B}_\epsilon\right\}.$$

Note that $\mathcal{B}_\epsilon$ is an open cover for Y. Since f is continuous, $\mathcal{F}_\epsilon$ is an open cover for X.

Since X is compact, by Proposition 4.3.5, X is sequentially compact and by Proposition 4.3.2, $\mathcal{F}_\epsilon$ has a Lebesgue number $\delta > 0$. For any $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, there exists some open set $S \subseteq X$ such that $\operatorname{diam}(S) < \delta$ and $x_1, x_2 \in S$. Therefore, there exists some $B_{\frac{\epsilon}{2}}(y) \in \mathcal{B}_\epsilon$ such that $x_1, x_2 \in f^{-1}\left(B_{\frac{\epsilon}{2}}(y)\right)$. This means that $d_Y\big(f(x_1), f(x_2)\big) < \epsilon$ and so f is uniformly continuous. □

An intuition regarding totally bounded spaces is that: since we can cover the entire space with finitely many balls, then the "size" of the space should not be too large.

Proposition 4.3.7 ▸ Totally Bounded Spaces Have Finite Diameter

If a metric space X is totally bounded, then $\operatorname{diam}(X)$ *is finite.*

Proof. Fix some $\epsilon > 0$, then we can find a finite cover with open ϵ-balls $\{B_1, B_2, \cdots, B_k\}$ for X. For each $i = 1, 2, \cdots, k$, pick some $x_i \in B_i$ and define

$$M := \max\left\{d\left(x_i, x_j\right) : i, j = 1, 2, \cdots, k\right\}.$$

For all $x, y \in X$, there exist B_i, B_j with $x \in B_i$ and $y \in B_j$. Therefore,

$$d(x, y) \leq d(x, x_i) + d\left(x_i, x_j\right) + d\left(x_j, y\right) \leq M + 2\epsilon.$$

Therefore, $\operatorname{diam}(X) \leq M + 2\epsilon$ is finite. □

Intuitively, bi-Lipschitz spaces should preserve totally bounded properties because they induce the same topology.

Proposition 4.3.8 ▸ Bi-Lipschitz Spaces Preserve Totally Bounded Spaces

If (X, d) and (X', d') are bi-Lipschitz, then (X, d) is totally bounded if and only if (X', d') is totally bounded.

Proof. By Proposition 1.2.15, d and d' induce the same topology, and so (X, d) is totally bounded if and only if (X, d') is totally bounded. □

Recall that every metric can be normalised to $[0, 1]$, so it is natural that a totally bounded metric space is still totally bounded after such normalisation.

Corollary 4.3.9 ▸ Normalisation Preserves Totally Bounded Spaces

If (X, d) is a metric space and $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$, then (X, ρ) is totally bounded if and only if (X, d) is totally bounded.

Proof. By Proposition 1.6.14, $\frac{1}{2}\rho(x,y) \leq \rho(x,y) < d(x,y) < 2\rho(x,y)$ for all $x, y \in X$. Therefore, (X, ρ) and (X, d) are bi-Lipschitz. By Proposition 4.3.8, (X, ρ) is totally bounded if and only if (X, d) is totally bounded. □

Lastly, in Euclidean spaces, it is intuitive that every bounded space is totally bounded. We can push this result further by generalising to all L^p-metrics.

Proposition 4.3.10 ▸ Bounded Subspaces of L^p Spaces Are Totally Bounded

Every subspace X of $(\mathbb{R}^n, \ell_p)$ is totally bounded if and only if X is bounded.

Proof. By Proposition 4.3.7, it is clear that every totally bounded subspace $X \subseteq \mathbb{R}^n$ is bounded. Therefore, by Corollary 1.2.16, it suffices to prove that every bounded subspace X of $(\mathbb{R}^n, \ell_2)$ is totally bounded. Since X is bounded, there exists some closed ball $B_M(\boldsymbol{x}_0)$ for some $\boldsymbol{x}_0 := (x_0, x_0, \cdots, x_0) \in \mathbb{R}^n$ such that

$$X \subseteq B_M(\boldsymbol{x}_0) \subseteq [x_0 - M, x_0 + M]^n.$$

For any $\epsilon > 0$, consider

$$N := \left\{\boldsymbol{x}_0 + \frac{\epsilon}{2}(k_1, k_2, \cdots, k_n) : k_2, \cdots, k_n \in \mathbb{Z}, \frac{\epsilon}{2}k_i \leq M \text{ for all } i = 1, 2, \cdots, n\right\}.$$

Note that N is finite. Define

$$\mathcal{B}_\epsilon := \{B_\epsilon(\boldsymbol{x}_N) : \boldsymbol{x}_N \in N\},$$

which is a finite cover for X. Therefore, X is totally bounded. □

4.4 Complete Spaces

In elementary real analysis, we examine the notion of convergence with Cauchy sequences. Recall that in $\mathbb{R}^n$, every Cauchy sequence is convergent. However, in general metric spaces, this might not be the case.

For example, consider the sequence $\{a_n\}_{n\in\mathbb{N}^+}$ defined by

$$a_n := \left(1 + \frac{1}{n}\right)^n,$$

which is a Cauchy sequence. However, $\lim_{n\to\infty} a_n = \mathrm{e}$, so the sequence converges in $\mathbb{R}$ but not in $\mathbb{Q}$ with respect to the standard metric.

Informally, such behaviour is caused by the fact that there are "break points" in $\mathbb{Q}$ but not in $\mathbb{R}$, so in a sense $\mathbb{R}$ is a "more complete" space.

Definition 4.4.1 ▸ Complete Space

A metric space X is **complete** if every Cauchy sequence in X is convergent.

To characterise a complete metric space, it is useful to examine the conditions under which a Cauchy sequence is convergent.

Proposition 4.4.2 ▸ Convergence of Cauchy Sequences

A Cauchy sequence converges if and only if it has a convergent subsequence.

Proof. Since every sequence is its own subsequence, it suffices to prove that a Cauchy sequence converges if it has a convergent subsequence. Let $\{a_n\}_{n\in\mathbb{N}^+}$ be a Cauchy sequence in a metric space (X,d), then for any $\epsilon > 0$, there exists some $N_1 \in \mathbb{N}^+$ such that for all $p, q > N_1$,

$$d\left(a_p, a_q\right) < \frac{\epsilon}{2}.$$

Suppose that $\{a_{n_k}\}_{k\in\mathbb{N}^+}$ is a convergent subsequence. Let

$$L \coloneqq \lim_{k\to\infty} a_{n_k},$$

then for any $\epsilon > 0$, there exists some $N_2 \in \mathbb{N}^+$ such that $d\left(a_{n_k}, L\right) < \frac{\epsilon}{2}$ for all $k > N_2$. Take $N \coloneqq \max\{N_1, N_2\}$. Notice that for all $k > N$, we have $n_k \geq k > N$ and so

$$\begin{aligned} d\left(a_k, L\right) &\leq d\left(a_k, a_{n_k}\right) + d\left(a_{n_k}, L\right) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n = L$ and so $\{a_n\}_{n\in\mathbb{N}^+}$ converges. □

Note that whether a sequence is Cauchy depends on the metric we choose to equip the space with. The following result shows an important connection between Cauchy sequences and bi-Lipschitz spaces.

Proposition 4.4.3 ▸ Bi-Lipschitz Spaces Preserve Cauchy Sequences

If (X, d) and (X', d') are bi-Lipschitz, then a sequence is Cauchy in (X, d) if and only if it is Cauchy in (X', d').

Proof. Since (X, d) and (X, d') are bi-Lipschitz, there exists some $A > 1$ such that

$$\frac{1}{A} d(x, y) < d'(x, y) < Ad(x, y)$$

for all $x, y \in X$. Let $\{a_n\}_{n\in\mathbb{N}^+}$ be a Cauchy sequence in (X, d), then for any $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that for all $m, n > N$,

$$d(a_m, a_n) < \frac{\epsilon}{A}.$$

Therefore, for all $m, n > N$, we have

$$d'(a_m, a_n) < Ad(a_m, a_n) < \epsilon,$$

which means that $\{a_n\}_{n\in\mathbb{N}^+}$ is Cauchy in (X, d'). Conversely, let $\{b_n\}_{n\in\mathbb{N}^+}$ be a Cauchy sequence in (X, d'), then for any $\epsilon' > 0$, there exists some $N' \in \mathbb{N}^+$ such that for all $m, n > N'$,

$$d'(b_m, b_n) < \frac{\epsilon'}{A}.$$

Therefore, for all $m, n > N'$, we have

$$d(b_m, b_n) < Ad'(b_m, b_n) < \epsilon',$$

and so $\{b_n\}_{n\in\mathbb{N}^+}$ is Cauchy in (X, d). □

Using these concepts, we shall re-visit the Euclidean space $\mathbb{R}^n$. First, let us argue that $\mathbb{R}^n$ is complete with respect to the L^p-metric for $p \geq 1$.

Take any Cauchy sequence $\{x_n\}_{n\in\mathbb{N}^+}$ in $\mathbb{R}^n$. It is useful to prove that every Cauchy sequence must be bounded.

Proposition 4.4.4 ▸ Every Cauchy Sequence Is Bounded

If $\{x_n\}_{n\in\mathbb{N}^+}$ is a Cauchy sequence in (X, d), then there exists some $M > 0$ such that for all $n \in \mathbb{N}^+$, we have $d(x_n, 0) \leq M$.

Proof. Fix some $\epsilon > 0$, then there exists some $N \in \mathbb{N}^+$ such that for all $m, n \geq N$, we have $d(x_m, x_n) < \epsilon$. Take

$$M := \max\{d(x_n, 0) : n = 1, 2, \cdots, N\} + \epsilon,$$

then for any integer $n > N$, we have

$$d(x_n, 0) \leq d(0, x_N) + (x_N, x_n) \leq M,$$

and so $\{x_n\}_{n\in\mathbb{N}^+}$ is bounded. $\square$

With this result, it is clear that for any Cauchy sequence $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}^+}$ in $\mathbb{R}^n$ and for all $i \in \mathbb{N}^+$, we have $\boldsymbol{x}_i \in \overline{B_M(\mathbf{0})}$ for some $M > 0$. Note that $\overline{B_M(\mathbf{0})}$ is a compact ball in $\mathbb{R}^n$ and so it must be sequentially compact by Proposition 4.3.5. Therefore, $\{\boldsymbol{x}_i\}_{i\in\mathbb{N}^+}$ has a convergent subsequence and so it is convergent by Proposition 4.4.2. Therefore, $\mathbb{R}^n$ is indeed complete with respect to the L^p-metric for $p \geq 1$.

Lastly, recall that in Proposition 1.6.15, we defined a metric over the infinite product space $\mathbb{R}^\omega$. We shall prove that this infinite product space is still complete.

Proposition 4.4.5 ▸ Completeness of Infinite Product Euclidean Space

Let d be the standard metric on $\mathbb{R}$ and define ρ to be a metric on $\mathbb{R}$ by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Define $D : \mathbb{R}^\omega \to \mathbb{R}$ to be a metric by

$$D(\boldsymbol{x}, \boldsymbol{y}) = \sup\left\{\frac{\rho(\pi_k(\boldsymbol{x}), \pi_k(\boldsymbol{y}))}{k} : k \in \mathbb{Z}^+\right\},$$

then $(\mathbb{R}^\omega, D)$ is complete.

Proof. Let $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}^+}$ be a Cauchy sequence in $\mathbb{R}^\omega$. Define

$$p_{k_n} := \pi_k(\boldsymbol{a}_n).$$

We claim that $\{p_{k_n}\}_{n\in\mathbb{N}^+}$ is convergent in $(\mathbb{R}, \rho)$ for all $k \in \mathbb{N}^+$. Take any $\epsilon > 0$ and any $k \in \mathbb{N}^+$, then there exists some $N_k \in \mathbb{N}^+$ such that for all $m, n > N_k$, we have $D(\boldsymbol{a}_m, \boldsymbol{a}_n) < \frac{\epsilon}{k}$. Therefore,

$$\rho\left(p_{k_m}, p_{k_n}\right) = \rho(\pi_k(\boldsymbol{x}), \pi_k(\boldsymbol{y})) \leq kD(\boldsymbol{a}_m, \boldsymbol{a}_n) < \epsilon.$$

This means that $\{p_{k_n}\}_{n\in\mathbb{N}^+}$ is Cauchy in $(\mathbb{R}, \rho)$ for all $k \in \mathbb{N}^+$. Define

$$B := \{y \in \mathbb{R} : \rho(x, y) \leq 1\}.$$

Note that $\rho(x, y) < 1$ for all $x, y \in \mathbb{R}$, so $p_{k_n} \in B$ for all $k, n \in \mathbb{N}^+$. Note that B is compact and so sequentially compact in $\mathbb{R}$, which means that $\{p_{k_n}\}_{n\in\mathbb{N}^+}$ is convergent in $(\mathbb{R}, \rho)$. Define

$$L_k := \lim_{n\to\infty} p_{k_n}.$$

We claim that $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}^+}$ converges to $\boldsymbol{a}$ where

$$\pi_k(\boldsymbol{a}) = L_k.$$

Observe that for any $\epsilon > 0$, there exists some $K \in \mathbb{N}^+$ such that $0 < \frac{1}{K} < \epsilon$, so for all $k > K$, we have

$$\frac{\rho(p_{k_n}, L_k)}{k} \leq \frac{1}{k} < \frac{1}{K} < \epsilon$$

for all $n \in \mathbb{N}^+$. Notice that for all $k \in \mathbb{N}^+$, for all $\epsilon > 0$, there exists some $M_k \in \mathbb{N}^+$ such that for all $n > M_k$, we have

$$\rho(p_{k_n}, L_k) < k\epsilon.$$

Define

$$M := \max\{M_1, M_2, \cdots, M_K\},$$

then for all $n > M$, we have

$$\begin{aligned} D(\boldsymbol{a}_n, \boldsymbol{a}) &= \sup\left\{\frac{\rho(p_{k_n}, L_k)}{k} : k \in \mathbb{N}^+\right\} \\ &< \epsilon. \end{aligned}$$

Therefore, $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}^+}$ converges to $\boldsymbol{a}$ in $\mathbb{R}^\omega$. Therefore, $\mathbb{R}^\omega$ is complete. □

Another thing we wish to investigate is whether completeness can be inherited through subspaces. It turns out that we only require some small additional condition to make this work.

Proposition 4.4.6 ▸ Closed Subspaces Preserve Completeness

A subspace of a complete metric space is complete if and only if it is closed.

Proof. Let (X, d) be a complete metric space and $A \subseteq X$ be a subspace. Suppose that A is closed, then $A = \overline{A}$ contains all of its limit points. Let $\{a_n\}_{n\in\mathbb{N}^+}$ be any Cauchy sequence in $A \subseteq X$, then since X is complete, the sequence is convergent. Let $a := \lim_{n\to\infty} a_n$, then a must be a limit point of A and so $a \in \overline{A} = A$, which means that $\{a_n\}_{n\in\mathbb{N}^+}$ is convergent in A. Therefore, A is complete.

Conversely, suppose that A is complete, we shall prove that A is closed by considering the contrapositive statement. Suppose that A is not closed, then there exists some $a \in \overline{A} \setminus A$ which is a limit point of A. Notice that $B_{\frac{1}{n}}(a) \cap A \neq \varnothing$ for all $n \in \mathbb{N}^+$. For each $n \in \mathbb{N}^+$, pick some $a_n \in B_{\frac{1}{n}}(a) \cap A$. Note that for any $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that for all $n \geq N$, we have $\frac{1}{n} < \frac{\epsilon}{2}$, and so for all $m, n \geq N$, we have

$$d(a_m, a_n) \leq d(a_m, a) + d(a, a_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $\{a_n\}_{n \in \mathbb{N}^+}$ is a Cauchy sequence in A but it does not converge in A, which means that A is not complete. □

In an elementary level, something we commonly say is that an interval is compact if and only if it is closed and bounded. By now, it should become clear that completeness is a generalisation for closed-ness, and that totally boundedness is a generalisation for boundedness.

Proposition 4.4.7 ▸ Characterisation of Compact Metric Spaces

A metric space (X, d) is compact if and only it is complete and totally bounded.

Proof. Suppose that (X, d) is compact, then by Proposition 4.3.5, X is sequentially compact and so every Cauchy sequence in X has a convergent subsequence in X. By Proposition 4.4.2, this means that every Cauchy sequence in X is convergent and so X is complete. Notice that for any $\epsilon > 0$, there exists an open cover for X with open balls of radius ϵ. Since X is compact, there exists a finite sub-cover with open balls of radius ϵ and so X is totally bounded.

Conversely, suppose that (X, d) is complete and totally bounded. Let $\{x_n\}_{n \in \mathbb{N}^+}$ be any sequence in X. Since X is totally bounded, there exists some $y_0 \in X$ such that

$$B_1(y_0) \cap \{x_n\}_{n \in \mathbb{N}^+} \neq \varnothing.$$

Similarly, for each $n \in \mathbb{N}^+$, there exists some $y_n \in X$ such that

$$B_{2^{-n}}(y_n) \cap \left(B_{2^{-(n-1)}}(y_{n-1}) \cap \{x_n\}_{n \in \mathbb{N}^+}\right) \neq \varnothing.$$

Pick some $x_{k_n} \in B_{2^{-n}}(y_n)$ where $k_n \in \mathbb{N}^+$. For any $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that for all $n \geq N$, we have $2^{-n} < \epsilon$. Therefore, for any $m, n > N$, we have

$$d(x_m, x_n) \leq 2^{-N} < \epsilon,$$

and so $\{x_{k_n}\}_{n\in\mathbb{N}^+}$ is a Cauchy subsequence. Since X is complete, the subsequence is convergent and so X is sequentially compact. By Proposition 4.3.5, X is compact. $\square$

With this very sophisticated result, we can now turn back and prove something which appears to be very naïve (but surprisingly hard to prove):

Theorem 4.4.8 ▸ Heine-Borel Theorem

A subspace $G \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. Note that $\mathbb{R}^n$ is complete. By Proposition 4.4.6, G is complete if and only if it is closed. By Proposition 4.3.10, G is totally bounded if and only if it is bounded. By Proposition 4.4.7, G is compact if and only if it is complete and totally bounded, and so G is compact if and only if it is closed and bounded. $\square$

4.5 Compactification

In the previous section in the study of compactness, we have been treating a space as a single entity. However, another perspective is to view the space as a collection of many singleton "points". This motivates us to define compactness point-wise.

Definition 4.5.1 ▸ Local Compactness

A topological space X is **locally compact** at $x \in X$ if there exists a compact set $K \subseteq X$ and an open set $U \subseteq X$ such that $x \in U \subseteq K$. We say that X is locally compact if it is locally compact at every $x \in X$.

An interpretation of local compactness at a point x is that we can find a strictly wider compact neighbourhood that contains x.

Clearly, $\mathbb{R}^n$ is locally compact because every $\boldsymbol{x} \in \mathbb{R}^n$ is contained in $B_\epsilon(\boldsymbol{x}) \subseteq \overline{B_\epsilon(\boldsymbol{x})}$ which is compact for any $\epsilon > 0$.

However, $\mathbb{Q}$ with respect to the standard topology is not locally compact. Let $U \subseteq \mathbb{Q}$ be any open set, then there exists $a, b \in \mathbb{Q}$ with $a < b$ such that $(a, b) \cap \mathbb{Q} \subseteq U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists some $x \in \mathbb{Q}^c$ such that there is a sequence $\{x_n\}_{n\in\mathbb{N}^+}$ in $\mathbb{Q}$ converging to x. This means that every convergent subsequence of $\{x_n\}_{n\in\mathbb{N}^+}$ converges to x. Therefore, for any subset $C \subseteq \mathbb{Q}$ with $U \subseteq C$, it contains a sequence which does not have any convergent subsequence in itself. Therefore, C is not compact, and so $\mathbb{Q}$ is not locally compact.

We can also show that $\mathbb{R}^\omega$ with the product topology is not locally compact. Let $U \subseteq \mathbb{R}^\omega$ be open. We claim that every set $C \subseteq \mathbb{R}^\omega$ containing U is not compact. Suppose on contrary that there exists some compact $C \subseteq \mathbb{R}^\omega$ with $U \subseteq C$. By Propositions 3.4.4 and 3.4.2, since $\mathbb{R}^\omega$ is metrisable, it is T_4 and therefore Hausdorff. By Proposition 4.1.5, this means that C is closed in $\mathbb{R}^\omega$. By Proposition 1.6.9, there exists some finite subset $I \subseteq \mathbb{N}^+$ and an open interval $(a_i, b_i) \subseteq \mathbb{R}$ for each $i \in I$ such that

$$B \coloneqq \prod_{i \in I} (a_i, b_i) \times \prod_{i \in \mathbb{N}^+ \setminus I} \mathbb{R} \subseteq U \subseteq C.$$

This implies that

$$\overline{B} = \prod_{i \in I} [a_i, b_i] \times \prod_{i \in \mathbb{N}^+ \setminus I} \mathbb{R} \subseteq C.$$

By Proposition 4.1.2, $\overline{B}$ is compact. However, the sequence $\{\boldsymbol{x}_n\}_{n \in \mathbb{N}^+}$ in $\overline{B}$ defined by

$$\pi_i(\boldsymbol{x}_n) = \begin{cases} a_i & \text{if } i \in I \\ n & \text{if } i \in \mathbb{N}^+ \setminus I \end{cases}$$

does not have any convergent subsequence in $\overline{B}$. Therefore, $\overline{B}$ is not sequentially compact, which is a contradiction.

Is there any connection between locally compact spaces and compact spaces? In fact, by going one step further, we may wish to find a way to transform a general space into a compact Hausdorff space. Such a transformation should preserve the structure of the pre-image space.

Definition 4.5.2 ▸ Compactification

Let X be a topological space and Y be a compact Hausdorff space, then Y is a **compactification** of X if there exists a map $h : X \to Y$ which is homeomorphic onto $h(X)$ such that $\overline{h(X)} = Y$. If $Y \setminus h(X)$ is a singleton set, then Y is a **one-point compactification** of X.

In Hausdorff spaces, it is then relatively easy to construct a compactification.

Proposition 4.5.3 ▸ Characterisation of Locally Compact Hausdorff Spaces

A topological space X is locally compact and Hausdorff if and only if there exists a compact Hausdorff space Y and a map $h_Y : X \to Y$ such that h_Y is homeomorphic onto $h_Y(X)$ and $Y \setminus h_Y(X)$ is a singleton set.

Proof. Suppose that X is locally compact and Hausdorff. Fix a topology $\mathcal{T}_X$ on X and let $Y \coloneqq X \cup \{p\}$ for some $p \notin X$ and define

$$\mathcal{A} \coloneqq \{Y \setminus C : \ C \text{ is compact in } X\}.$$

We claim that $\mathcal{T}_Y \coloneqq \mathcal{T}_X \cup \mathcal{A}$ is a topology on Y. Let $K_1, K_2, \cdots, K_n$ be compact sets in X, then $\bigcup_{i=1}^{n} K_i$ is compact in X. Therefore,

$$\bigcap_{i=1}^{n} (Y \setminus K_i) = Y \setminus \bigcup_{i=1}^{n} K_i \in \mathcal{A}.$$

Let $\{K_\alpha\}_{\alpha \in \Lambda}$ be a collection of compact sets in X, then $\bigcap_{\alpha \in \Lambda} K_\alpha$ is closed in K_α for any $\alpha \in \Lambda$ and so compact in X. Therefore,

$$\bigcup_{\alpha \in \Lambda} (Y \setminus K_\alpha) = Y \setminus \bigcap_{\alpha \in \Lambda} K_\alpha \in \mathcal{A}.$$

Take any $T \in \mathcal{T}_X$ and any compact set $K \subseteq X$, then $K \setminus T \subseteq K$ is closed and so compact in X. Therefore,

$$\begin{aligned} T \cup (Y \setminus K) &= Y \setminus (K \setminus T) \in \mathcal{A}, \\ T \cap (Y \setminus K) &= T \setminus K \in \mathcal{T}_X. \end{aligned}$$

It is obvious that $Y, \varnothing \in \mathcal{T}_Y$. Therefore, $\mathcal{T}_Y$ is a topology on Y. Define $h_Y : X \to Y$ by $h_Y(x) = x$, then clearly h_Y is homeomorphic onto $h_Y(X)$ and $Y \setminus h_Y(X) = \{p\}$ is a singleton set. Let $\mathcal{U} \subseteq \mathcal{T}_Y$ be an open cover for Y, then there exists some $U_0 \in \mathcal{U}$ such that $p \in U_0$. Notice that $U_0 \in \mathcal{A}$, so there exists some compact set $K_0 \subseteq X$ such that $U_0 = Y \setminus K_0$. Note that $\mathcal{U}$ is an open cover for K_0, so there exists a finite sub-cover $U_1, U_2, \cdots, U_n$ for K_0. Therefore, $U_0, U_1, \cdots, U_n$ is a finite sub-cover for Y, and so Y is compact. Take any $x \in X$, then since X is locally compact, there exists some open set $U \in \mathcal{T}_X$ and compact set $K \subseteq X$ such that $x \in U \subseteq K$. Clearly, $p \in Y \setminus K$, which is open in Y and disjoint from U. Therefore, Y is Hausdorff.

Conversely, suppose that Y is a compact Hausdorff space Y such that there exists a map $h_Y : X \to Y$ which is homeomorphic onto $h_Y(X)$ such that $Y \setminus h_Y(X) = \{p\}$ is a singleton set. Note that $h_Y(X) \subseteq Y$ must be Hausdorff. Since h_Y is homeomorphic onto $h_Y(X)$, we know that X is Hausdorff. Take any $x \in X$, then there exist disjoint open sets $U, V \subseteq Y$ such that $x \in U$ and $p \in V$. Note that $U \subseteq Y \setminus V$, which is closed and so compact since Y is compact. Therefore, X is locally compact. $\square$

Essentially, the above proposition is basically saying that if we add a new point to a locally compact Hausdorff space, the space will become a compact Hausdorff space. Intuitively, this new point can be anything, so all such spaces should be homeomorphic to each other by just re-mapping the new points.

Proposition 4.5.4 ▸ Uniqueness of One-Point Compactification

If X is a locally compact Hausdorff space and Y and Y' are compact Hausdorff spaces such that there exist maps $h_Y : X \to Y$ and $h_{Y'} : X \to Y'$ homeomorphic onto $h_Y(X)$ and $h_{Y'}(X)$ respectively such that $Y \setminus h_Y(X)$ and $Y' \setminus h_{Y'}(X)$ are singleton sets, then there exists a homeomorphism $f : Y \to Y'$ such that

$$f|_{h_Y(X)} = h_{Y'} \circ h_Y^{-1}|_{h_Y(X)}.$$

Proof. Note that both $h_Y(X)$ and $h_{Y'}(X)$ are open in Y. Let

$$\{p\} = Y \setminus h_Y(X), \qquad \{p'\} = Y' \setminus h_{Y'}(X)$$

and define $f : Y \to Y'$ by

$$f(y) = \begin{cases} p' & \text{if } y = p \\ h_{Y'}\left(h_Y^{-1}(y)\right) & \text{if } y \in h_Y(X) \end{cases}.$$

Clearly, f is a bijection with

$$f^{-1}(y) = \begin{cases} p & \text{if } y = p' \\ h_Y\left(h_{Y'}^{-1}(y)\right) & \text{if } y \in h_{Y'}(X) \end{cases}.$$

Let $U \subseteq Y'$ be any open set. If $p' \notin U$, then U is open in $h_{Y'}(X)$. Therefore, $h_{Y'}^{-1}(U)$ is open in X, and so $f^{-1}(U) = h_Y\left(h_{Y'}^{-1}(U)\right)$ is open in $h_Y(X)$. Since $h_Y(X)$ is open in Y, this implies that $f^{-1}(U)$ is open in Y. If $p' \in U$, define $K \coloneqq Y' \setminus U$ which is closed and therefore compact in Y' by Proposition 4.1.2. It is clear that K is compact in $h_{Y'}(X)$. By Proposition 4.1.7, $f^{-1}(K) = h_Y\left(h_{Y'}^{-1}(K)\right)$ is compact in $h_Y(X)$ and so compact in Y. Since Y is Hausdorff, by Proposition 4.1.5, $f^{-1}(K)$ is closed in Y, and so

$$f^{-1}(U) = f^{-1}(Y' \setminus K) = Y \setminus f^{-1}(K),$$

which is open in Y. Therefore, f is continuous. Similarly, f^{-1} is continuous, and so f is a homeomorphism. ☐

It is no surprise that such a space Y is a one-point compactification for X, provided that X is not already compact.

Proposition 4.5.5 ▸ Compactification of Hausdorff Spaces

Let X be a non-compact locally compact Hausdorff space and Y be a compact Hausdorff space. If a map $h_Y : X \to Y$ is homeomorphic onto $h_Y(X)$ and $Y \setminus h_Y(X)$ is a singleton set, then Y is a one-point compactification of X.

Proof. By Proposition 4.5.4, it suffices to consider $h_Y(x) = x$ and

$$\mathcal{T}_Y := \mathcal{T}_X \cup \{Y \setminus C : C \text{ is compact in } X\}$$

as the topology on Y, where $\mathcal{T}_X$ is the topology on X. Let $Y = X \cup \{p\}$ for some $p \notin X$. We claim that p is a limit point of X. Suppose on contrary that p is not a limit point of X, then there exists some open set $U \subseteq Y$ with $p \in U$ but $U \cap X = \varnothing$. This means that $U = \{p\}$, and so $X = Y \setminus U$ is compact, which is a contradiction. □

One-point compactification has important geometric implications. Define

$$\mathbb{S}^2 := \left\{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| = 1\right\}$$

to be the 2-dimensional unit sphere and

$$\mathbb{D} := \left\{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\| < 1\right\}$$

to be the open unit disc, we can prove that both $\overline{\mathbb{D}}$ and $\mathbb{S}^2$ are the one-point compactification for $\mathbb{D}$. In fact, since $\mathbb{D} \approx \mathbb{R}^2$ due to the existence of the bijection $f : \mathbb{R}^2 \to \mathbb{D}$ by

$$f(\boldsymbol{v}) = \frac{\boldsymbol{v}}{1 + \|\boldsymbol{v}\|},$$

both $\overline{\mathbb{D}}$ and $\mathbb{S}^2$ are the one-point compactification for $\mathbb{R}^n$ for all $n \in \mathbb{N}^+$.

Since compactification is closely related to Hausdorff spaces and locally compact spaces, it is not surprising that we can make use of it to check if a Hausdorff space is locally compact.

Proposition 4.5.6 ▸ Characterisation of Locally Compact Spaces

A Hausdorff space X is locally compact if and only if for any $x \in X$ and any open set $U \subseteq X$ with $x \in U$, there exists an open compact set $V \subseteq X$ such that $x \in V \subseteq U$.

Proof. By Proposition 4.5.3, define $Y := X \cup \{p\}$ to be compact and

$$\mathcal{T}_Y := \mathcal{T}_X \cup \{Y \setminus K : K \text{ is compact in } X\}$$

as the topology on Y, where $\mathcal{T}_X$ is the topology on X. Take any $x \in X$ and any open set $U \subseteq X$ with $x \in U$, then $K := Y \setminus U$ is closed and so compact because Y is compact. Note that $x \notin K$. By Proposition 4.1.6, Y is T_3. Therefore, there exist disjoint open sets $V, W \subseteq Y$ such that $x \in V$ and $K \subseteq W$. Since $p \in K \subseteq W$, we have

$$X \setminus W = Y \setminus W,$$

which is compact and so closed in X. Note that $V \subseteq Y \setminus W = X \setminus W$, so

$$x \in V \subseteq \overline{V} \subseteq X \setminus W \subseteq X \setminus K = U,$$

which implies that X is locally compact. The converse is trivial and is left to the reader as an exercise. □

This result can be utilised to check if a subspace of a locally compact space is locally compact.

Corollary 4.5.7 ▸ Local Compactness of Subspaces

Let X be a locally compact space and $A \subseteq X$ be a subspace, then A is locally compact if

1. *A is closed, or*
2. *A is open and X is Hausdorff.*

Proof. Suppose that A is closed. For any $x \in A \subseteq X$, there exists an open set $U \subseteq X$ and a compact set $K \subseteq X$ such that $x \in U \subseteq K$. Note that $A \cap K$ is closed in K, so by Proposition 4.1.2, $A \cap K$ is compact in K. Therefore, $A \cap K$ is compact in X, and so it is compact in A. Note that $A \cap U$ is open and $x \in A \cap U \subseteq A \cap K$, so A is locally compact.

Otherwise, suppose that A is open and X is Hausdorff, then for any $x \in A \subseteq X$, by Proposition 4.5.6, there exists an open compact set $V \subseteq X$ for every open neighbourhood $U \subseteq X$ of x such that $x \in V \subseteq U$. Note that $A \cap V$ is open and compact in A and $A \cap U$ is open in A, so $x \in A \cap V \subseteq A \cap U$. By Proposition 4.5.6, A is locally compact. □

By using the above results, we propose the last conclusion:

Corollary 4.5.8 ▸ Locally Compact Hausdorff Spaces and Homeomorphisms

A topological space X is locally compact and Hausdorff if and only if there exists a compact Hausdorff space Y such that X is homeomorphic to some open subset $A \subseteq Y$.

Proof. Suppose that there exists a compact Hausdorff space Y such that X is homeomorphic to some open subset $A \subseteq Y$. Note that by Corollary 4.5.7, A is locally compact and so is X. Conversely, suppose that X is locally compact, then by Proposition 4.5.4, there exists a compact Hausdorff space $Y := X \cup \{p\}$ for some $p \notin X$ such that X is homeomorphic to $X \subseteq Y$ which is open in Y. □

4.6 Metric Completion

Consider $f : X \to Y$ to be a map. It is clear that we can write f as a set of ordered pairs from $X \times Y$. By treating X as an index set, every such map f can be represented as a point from the product space

$$Y^X := \prod_{x \in X} Y.$$

Using the projection map, we can write

$$f(x) = \pi_x(f).$$

Clearly, Y^X is a topological space when equipped with the product topology. Therefore, we can define a suitable metric on it to "measure" the distance between maps.

Definition 4.6.1 ▸ Uniform Metric and Uniform Topology

Let (Y, d) be a metric space and ρ be a metric on Y defined by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

The **uniform metric** on Y^Λ is defined by

$$\overline{\rho}(\boldsymbol{x}, \boldsymbol{y}) = \sup\left\{\rho\big(\pi_\alpha(\boldsymbol{x}), \pi_\alpha(\boldsymbol{y})\big) : \alpha \in \Lambda\right\}.$$

The **uniform topology** on Y^Λ is the topology generated by $\overline{\rho}$.

Remark. The uniform topology on $\mathbb{R}^\Lambda$ is finer than the product topology and coarser than the box topology.

Note that the uniform topology, product topology and box topology are pair-wise unequal on $\mathbb{R}^\omega$.

Note that $\overline{\rho}$ is a metric on Y^X because for any two maps $f, g \in Y^X$, we have

$$\overline{\rho}(f, g) = \sup\{\rho(f(x), g(x)) : x \in X\},$$

i.e., we measure the distance between maps by taking the maximum distance between corresponding points.

The following result examines the completeness of the uniform metric spaces:

Proposition 4.6.2 ▸ Completeness of the Uniform Metric Spaces

If (Y, d) is a complete metric space, then $(Y^\Lambda, \overline{\rho})$ is complete for any index set Λ.

Proof. Note that (Y, d) and (Y, ρ) are bi-Lipschitz, so it suffices to prove that (Y, ρ) being complete implies that $(Y^\Lambda, \overline{\rho})$ is complete. Suppose that (Y, ρ) is complete and let $\{f_n\}_{n\in\mathbb{N}^+}$ be a Cauchy sequence. Note that for any $m, n \in \mathbb{N}^+$ and every $\alpha \in \Lambda$,

$$\overline{\rho}(f_m, f_n) = \sup\{\rho(f_m(\lambda), f_n(\lambda)) : \lambda \in \Lambda\} \geq \rho(f_m(\alpha), f_n(\alpha)),$$

so $\{f_n(\alpha)\}_{n\in\mathbb{N}^+}$ is a Cauchy sequence in (Y, ρ). Since (Y, ρ) is complete, there exists some $y_\alpha \in Y$ for each $\alpha \in \Lambda$ such that

$$\lim_{n\to\infty} f_n(\alpha) = y_\alpha.$$

Take $f \in Y^\Lambda$ with $f(\alpha) = y_\alpha$. Note that for every $\epsilon > 0$, there exists some $N_1 \in \mathbb{N}^+$ such that for all $m, n > N_1$, we have

$$\rho(f_m(\alpha), f_n(\alpha)) \leq \overline{\rho}(f_m, f_n) < \frac{\epsilon}{2}$$

for all $\alpha \in \Lambda$. Notice that there exists $N_2 \in \mathbb{N}^+$ such that for all $m > N_2$, we have $\rho(f_m(\alpha), y_\alpha) < \frac{\epsilon}{2}$. Therefore, whenever $m, n > \max\{N_1, N_2\}$,

$$\begin{aligned}\overline{\rho}(f_n, f) &\leq \rho(f_n(\alpha), f(\alpha)) \\ &\leq \rho(f_n(\alpha), f_m(\alpha)) + \rho(f_m(\alpha), y_\alpha) \\ &< \epsilon,\end{aligned}$$

and so $\lim_{n\to\infty} f_n = f$. Therefore, $(Y^\Lambda, \overline{\rho})$ is complete. □

Usually, given a topological space X and a metric space (Y, d), we denote $\mathcal{C}(X, Y)$ to be the set of all continuous maps from X to Y and $\mathcal{B}(X, Y)$ to be the set of all bounded maps (i.e., maps $f : X \to Y$ such that $\operatorname{diam}(f(X))$ is finite) from X to Y.

Proposition 4.6.3 ▶ Closedness and Completness of $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$

Both $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$ are closed with respect to the uniform topology on Y^Λ. If (Y,d) is complete, then both $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$ are complete.

Proof. By Proposition 4.6.2, $\left(Y^X, \overline{\rho}\right)$ is complete. By Proposition 4.4.6, it suffices to prove that both $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$ are closed in $\left(Y^X, \overline{\rho}\right)$. Note that both $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$ are metrisable and so they are first countable. Let $f \in \overline{\mathcal{C}(X,Y)} \setminus \mathcal{C}(X,Y)$ be a limit point. By Proposition 2.2.4, there exists a sequence $\{f_n\}_{n \in \mathbb{N}^+}$ in $\mathcal{C}(X,Y)$ such that $\lim_{n\to\infty} f_n = f$. Note that this means that $\{f_n\}_{n \in \mathbb{N}^+}$ converges uniformly. By Proposition 1.5.15, f is continuous and so $f \in \mathcal{C}(X,Y)$. Therefore, $\mathcal{C}(X,Y)$ is closed. Similarly, if $g \in \overline{\mathcal{B}(X,Y)}$, then there exists a sequence $\{g_n\}_{n \in \mathbb{N}^+}$ in $\mathcal{B}(X,Y)$ which converges uniformly to g, which means that there exists some $N \in \mathbb{N}^+$ such that for all $x \in X$,

$$\rho\big(g_N(x), g(x)\big) \leq \overline{\rho}(g_N, g) < \frac{1}{2}.$$

Since $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$, we have

$$d\big(g_N(x), g(x)\big) < \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

for all $x \in X$. Notice that there exists some $M > 0$ such that $d\big(g_N(a), g_N(b)\big) < M$ for all $a, b \in X$. Therefore, for all $a, b \in X$,

$$\begin{aligned} d\big(g(a), g(b)\big) &\leq d\big(g(a), g_N(a)\big) + d\big(g_N(a), g_N(b)\big) + d\big(g_N(b), g(b)\big) \\ &< M + 2, \end{aligned}$$

which implies that $g \in \mathcal{B}(X,Y)$, and so $\mathcal{B}(X,Y)$ is closed. $\square$

For any two bounded maps $f, g \in \mathcal{B}(X,Y)$, it is clear that $d\big(f(x), g(x)\big)$ is bounded. Therefore, it makes sense to talk about its supremum.

Definition 4.6.4 ▶ Supremum Metric

The **supremum metric** on $\mathcal{B}(X,Y)$ is defined by

$$d_{\sup}(f,g) = \sup\big\{d\big(f(x), g(x)\big) : x \in X\big\}.$$

With some computation, one can show that

$$\overline{\rho}(f,g) = \frac{d_{\sup}(f,g)}{1 + d_{\sup}(f,g)},$$

and so if (Y, d) is complete, this implies that $\left(\mathcal{B}(X, Y), d_{\sup}\right)$ is complete.

Notice that if X is compact, then every continuous function has a compact range and so

$$C(X, Y) \subseteq \mathcal{B}(X, Y).$$

Since $\mathcal{C}(X, Y)$ is closed in Y^X, this means that $\mathcal{C}(X, Y)$ is closed in $\mathcal{B}(X, Y)$. In this case, we can restrict $d_{\sup}$ to make $\left(\mathcal{C}(X, Y), d_{\sup}\right)$ complete as well when (Y, d) is complete.

Recall that in Definition 1.8.3, we defined an analogous notion of homeomorphic maps in metric spaces which preserves distances. While an isometry preserves distances, it does not necessarily retain the "size" of the spaces. Therefore, as a further step, we may wish to construct a "good" isometry such that the image of the domain is not too far away from the co-domain.

Definition 4.6.5 ▸ Metric Completion

Let (X, d_X) be a metric space. The **metric completion** of X is a complete metric space (Y, d_Y) with an isometric embedding $\phi : X \to Y$ such that $\overline{\phi(X)} = Y$.

We will show that a metric completion always exists, and it is in fact unique up to isometry.

Theorem 4.6.6 ▸ Existence and Uniqueness of Metric Completion

For every metric space (X, d_X), there exists a complete metric space (Y, d_Y) and an isometric embedding $\phi : X \to Y$ such that $\phi(X)$ is dense in Y. Furthermore, for any complete metric space $(Y', d_{Y'})$ and any isometric embedding $\phi' : X \to Y'$ with $\overline{\phi'(X)} = Y'$, there exists an isometry $f : Y \to Y'$ such that

$$f|_{\phi(X)} = \phi' \circ \phi^{-1}.$$

Proof. Fix some $x_0 \in X$. For every $a \in X$, define $\phi_a : X \to \mathbb{R}$ by

$$\phi_a(x) = d_X(x, a) - d_X(x, x_0).$$

Note that

$$d(x, a) + d(a, x_0) \geq d(x, x_0),$$
$$d(x, x_0) + d(a, x_0) \geq d(x, a),$$

so we have

$$-d(a, x_0) \leq d(x, a) - d(x, x_0) \leq d(a, x_0).$$

Therefore, for every $x \in X$, we have $|\phi_a(x)| \leq d(a, x_0)$, and so $\phi_a \in \mathcal{B}(X, Y)$. Define a map $\phi\colon (X, d_X) \to (\mathcal{B}(X, Y), d_{\sup})$ by $\phi(a) = \phi_a$. Take any $a, b \in X$ and consider

$$\begin{aligned} d_{\sup}(\phi_a, \phi_b) &= \sup\{|\phi_a(x), \phi_b(x)|\colon x \in X\} \\ &= \sup\{|d_X(x, a) - d_X(x, b)|\colon x \in X\} \\ &\leq d_X(a, b) \\ &= |d_X(b, a) - d_X(b, b)| \\ &\leq \sup\{|d_X(x, a) - d_X(x, b)|\colon x \in X\} \\ &= d_{\sup}(\phi_a, \phi_b). \end{aligned}$$

Therefore, $d_{\sup}(\phi_a, \phi_b) = d_X(a, b)$ and so $d_{\sup}$ is an isometric embedding. Take

$$Y := \overline{\phi(X)} \quad \text{and} \quad d_Y := d_{\sup}|_{\overline{\phi(X)}},$$

then by Propositions 4.4.6 and 4.6.3, Y is complete and ϕ is an isometry onto Y. Take any complete metric space $(Y', d_{Y'})$ and any isometric embedding $\phi'\colon X \to Y'$ such that $\overline{\phi'(X)} = Y'$. Let $\mathcal{S}$ be the collection of all Cauchy sequences on $\phi(X)$ and define $\sim$ to be an equivalence relation on $\mathcal{S}$ such that $\{a_n\}_{n \in \mathbb{N}^+} \sim \{b_n\}_{n \in \mathbb{N}^+}$ if and only if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Consider $\mathcal{A} := \mathcal{S}/\sim$ and define $m\colon \mathcal{A}^2 \to \mathbb{R}$ by

$$m\left([\{a_n\}_{n\in\mathbb{N}^+}]_\sim, [\{b_n\}_{n\in\mathbb{N}^+}]_\sim\right) = d_Y\left(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right).$$

One may check that m is a metric. Define $g\colon \mathcal{A} \to Y$ by

$$g\left([\{a_n\}_{n\in\mathbb{N}^+}]_\sim\right) = \lim_{n\to\infty} a_n.$$

One may check that g is an isometry. Define $\mathcal{A}'$, m' and g' on $\phi'(X)$ similarly and consider $F\colon \mathcal{A} \to \mathcal{A}'$ defined by

$$F\left([\{x_n\}_{n\in\mathbb{N}^+}]_\sim\right) = \left[\{\phi'(\phi(x_n))\}_{n\in\mathbb{N}^+}\right]_\sim := \Phi_x.$$

One can check that F is well-defined because $\phi' \circ \phi\colon \phi(X) \to \phi'(X)$ is an isometry.

Consider

$$\begin{aligned} m'(\Phi_a, \Phi_b) &= d_Y\left(\lim_{n\to\infty} \phi'(\phi(a_n)), \lim_{n\to\infty} \phi'(\phi(b_n))\right) \\ &= \lim_{n\to\infty} d_Y\big(\phi'(\phi(a_n)), \phi'(\phi(b_n))\big) \\ &= \lim_{n\to\infty} d_Y(a_n, b_n) \\ &= d_Y\left(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right) \\ &= m\left([\{a_n\}_{n\in\mathbb{N}^+}]_\sim, [\{b_n\}_{n\in\mathbb{N}^+}]_\sim\right). \end{aligned}$$

Define $f := g' \circ F \circ g^{-1}$, then f is an isometry. For each $y \in \phi(X)$, we have

$$\begin{aligned} f(x) &= g'\big(F([\{x\}_{n\in\mathbb{N}^+}]_\sim)\big) \\ &= g'\left([\{\phi'(\phi(x))\}_{n\in\mathbb{N}^+}]_\sim\right) \\ &= \phi'(\phi(x)). \end{aligned}$$

Therefore, $f|_{\phi(X)} = \phi' \circ \phi^{-1}$. □

4.7 Equicontinuity

Recall that in Definition 1.5.14, we defined the notion of uniform convergence. The idea of uniform convergence is that as $n \to \infty$, the sequence of functions will stay within a **small and confined** neighbourhood from the limit. In general topological spaces, this "confinedness" can be described using compact sets.

Definition 4.7.1 ▸ Compact Convergence

Let X be a topological space and (Y, d) be a metric space. A sequence of functions $\{f_n\}_{n\in\mathbb{N}^+}$ from X to Y is said to **converge compactly** to $f : X \to Y$ if for every compact subset $K \subseteq X$,

$$\lim_{n\to\infty} \sup_{x\in K} d\big(f_n(x), f(x)\big) = 0.$$

Remark. Equivalently, $\{f_n\}_{n\in\mathbb{N}^+}$ converges compactly to f if and only if $\{f_n|_K\}_{n\in\mathbb{N}^+}$ converges uniformly to $f|_K$.

Note that compact convergence alone does not guarantee that $\{f_n\}_{n\in\mathbb{N}^+}$ will converge in Y^X. However, we can define a special type of topologies to establish the correspondence.

Definition 4.7.2 ▸ Topology of Compact Convergence

Let X be a topological space and (Y, d) be a metric space. For each $f \in Y^X$ and each compact set $K \subseteq X$, define

$$B(K, f, \epsilon) := \{g \in Y^X : d_{\sup}(f|_K, g|_K) < \epsilon\}$$

where $\epsilon > 0$. The topology generated by

$$\mathcal{B} := \{B(K, f, \epsilon) : f \in Y^X, K \subseteq X \text{ is compact}, \epsilon > 0\}$$

is called the **topology of compact convergence** on Y^X.

Intuitively, each basic open ball $B(K, f, \epsilon)$ can be thought of the collection of all the functions $g \in Y^X$ which, when restricted to K, are within a distance of ϵ from f. This topology is so named because $\{f_n\}_{n\in\mathbb{N}}$ in Y^X converges to f with respect to the topology if and only if $\{f_n|_K\}_{n\in\mathbb{N}}$ converges to $f|_K$ uniformly for any compact set $K \subseteq X$.

Now, to describe the convergent behaviour of functions, we need to confine their images into some "regions" in the co-domain. The following definition provides one tool to do this.

Definition 4.7.3 ▸ Compact-Open Topology

Let X and Y be topological spaces. For every compact set $K \subseteq X$ and open set $U \subseteq Y$, define

$$S(K, U) := \{g \in \mathcal{C}(X, Y) : g(K) \subseteq U\}.$$

The topology generated by the sub-basis

$$\mathcal{S} := \{S(K, U) : K \subseteq X \text{ is compact}, U \subseteq Y \text{ is open}\}$$

is the **compact-open topology** on $\mathcal{C}(X, Y)$.

Intuitively, the compact-open topology consists of continuous functions in Y^X where the images of compact sets are contained in an open neighbourhood. With some well-defined metric, every topology of compact convergence restricted to continuous functions is actually a compact-open topology, which makes sense because we would expect a compactly convergent function to have images confined within open sets for every compact pre-image.

Proposition 4.7.4 ▸ Topology of Compact Convergence over Continuous Maps

Let X is a topological space and (Y, d) is a metric space. Let $\mathcal{T}_K$ be the topology of compact convergence and $\mathcal{T}_O$ be the compact-open topology on $\mathcal{C}(X, Y)$, then $\mathcal{T}_K = \mathcal{T}_O$.

Proof. Take any $B(K, f, \epsilon) \subseteq \mathcal{T}_K$. For every $x \in X$, let $V_x \subseteq X$ be an open set defined by

$$V_x := f^{-1}\left(B^d_{\frac{\epsilon}{3}}(f(x))\right).$$

Since f is continuous, by Proposition 1.5.7,

$$f\left(\overline{V_x}\right) \subseteq \overline{f(V_x)} \subseteq B^d_{\frac{\epsilon}{2}}(f(x)).$$

Notice that $\{V_x : x \in K\}$ is an open cover for K. Since K is compact, there exists a finite sub-cover $\left\{V_{x_1}, V_{x_2}, \cdots, V_{x_n}\right\}$ for K. Define $K_i := K \cap \overline{V_{x_i}}$ for each $i = 1, 2, \cdots, n$, which is closed and thus compact in K. Consider

$$U := \bigcap_{i=1}^{n} S\left(K_i, B^d_{\frac{\epsilon}{2}}(f(x_i))\right),$$

which is a basic open set in $\mathcal{T}_O$. Note that for every $i = 1, 2, \cdots, n$,

$$f(K_i) \subseteq f\left(\overline{V_{x_i}}\right) \subseteq B^d_{\frac{\epsilon}{2}}(f(x_i)),$$

which means that $f \in U$. For every $g \in U$ and every $x \in K$, we have $x \in K_i$ for all $i = 1, 2, \cdots, n$, and so

$$\begin{aligned} d(g(x), f(x)) &\leq d(g(x), f(x_i)) + d(f(x_i), f(x)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $d_{\sup}\left(g|_K, f|_K\right) < \epsilon$ and so $g \in B(K, f, \epsilon)$. Therefore, $f \in U \subseteq B(K, f, \epsilon)$, and so we have $\mathcal{T}_K \subseteq \mathcal{T}_O$. Take any compact set $K \subseteq X$ and any open set $U \subseteq Y$. For every $f \in S(K, U)$, since f is continuous, $f(K)$ is compact by Proposition 4.1.7. Define

$$\epsilon_f := \inf\{d(x, y) : x \in f(K), y \in Y \setminus U\}.$$

We claim that $\epsilon_f > 0$. Suppose on contrary that $\epsilon_f = 0$, then for every $n \in \mathbb{N}^+$, there exists some $x_n \in f(K)$ and some $y_n \in Y \setminus U$ such that $d(x_n, y_n) < \frac{1}{n}$. Since $f(K)$ is a metrisable compact space, it is sequentially compact by Proposition 4.3.5, and so there exists a convergent subsequence $\{x_{k_n}\}_{n\in\mathbb{N}^+}$ of $\{x_n\}_{n\in\mathbb{N}^+}$. Let $x := \lim_{n\to\infty} x_{k_n}$. Consider

$$0 \leq d\left(x, y_{k_n}\right) \leq d\left(x, x_{k_n}\right) + d\left(x_{k_n}, y_{k_n}\right).$$

By Squeeze Theorem, this means that x is a limit point of $Y \setminus U$. However, $Y \setminus U$ is closed, which means that $x \in (Y \setminus U) \cap f(K)$. This is not possible as $f(K) \subseteq U$. Notice that for every $g \in B(K, f, \epsilon) \subseteq \mathcal{T}_K$, we have $d\big(f(x), g(x)\big) < \epsilon_f$ for all $x \in K$ and so $g(K) \subseteq U$ by the definition of ϵ_f. Therefore,

$$f \in B\left(K, f, \epsilon_f\right) \subseteq S(K, U),$$

and so $\mathcal{T}_O \subseteq \mathcal{T}_K$. In conclusion, $\mathcal{T}_K = \mathcal{T}_O$. □

Both compact-open topology and topology of compact convergence are deeply related with compact sets in a space. More strictly, we define a type of spaces in which the open sets are coherent in the space itself and in the compact subspaces.

Definition 4.7.5 ▸ Compactly Generated Space

A topological space X is **compactly generated** if A is open in X if and only if $A \cap K$ is open in K for every compact set K in X.

Intuitively, we can flip the condition in the above definition to base on closed sets.

Proposition 4.7.6 ▸ Alternative Definition for Compactly Generated Spaces

A topological space X is compactly generated if A is closed in X if and only if $A \cap K$ is closed in K for every compact set K in X.

Proof. Let $U \subseteq X$ be any open set, then $X \setminus U$ is closed. Therefore, $(X \setminus U) \cap K$ is closed in K, and so

$$U \cap K = K \setminus \big((X \setminus U) \cap K\big)$$

is open in K. Conversely, suppose that $U \subseteq X$ is such that $U \cap K$ is open in K, then

$$K \setminus U \cap K = K \cap (X \setminus U)$$

is closed in K. This means that $X \setminus U$ is closed in X, and so U is open in X. Therefore, X is compactly generated. □

Let us now try to characterise compactly generated spaces.

Proposition 4.7.7 ▸ Sufficient Conditions for Compactly Generated Spaces

A topological space X is compactly generated if it is locally compact or first countable.

Proof. Suppose that X is locally compact. Take any $A \subseteq X$ such that $A \cap K$ is open in K for every compact set $K \subseteq X$. For every $a \in A$, let $U_a \subseteq X$ be any open neighbourhood of a, then since X is locally compact, there exists some compact set $K_a \subseteq X$ be a compact set such that $a \in U_a \subseteq K_a$, then

$$A \cap U_a = (A \cap K_a) \cap U_a$$

is open in U_a, and so $A \cap U_a$ is open in X. Notice that $a \in A \cap U_a \subseteq A$, so

$$A = \bigcup_{a \in A} (A \cap U_a)$$

is open in X. Therefore, X is compactly generated.

Suppose that X is first countable and take $B \subseteq X$ such that $B \cap K$ is closed in K for every compact set $K \subseteq X$. Take any $b \in \overline{B}$. Since X is first countable, by Proposition 2.2.4, there exists a sequence $\{b_n\}_{n \in \mathbb{N}^+}$ in B such that $\lim_{n \to \infty} b_n = b$. take

$$K := \{b_n : n \in \mathbb{N}^+\} \cup \{b\},$$

then K is compact and so $B \cap K$ is closed. Therefore, since b is a limit point for $B \cap K$, we have $b \in B \cap K$ and so $b \in B$. This means that $\overline{B} = B$, and so B is closed. By Proposition 4.7.6, X is compactly generated. □

The name "compactly generated" suggests that such a space is completely characterised by its compact subsets. This has important implications on how we could characterise continuity over the space.

Proposition 4.7.8 ▸ Continuity in Compactly Generated Spaces

Let X and Y be topological spaces. If X is compactly generated, then a map $f : X \to Y$ is continuous if and only if for each compact set $K \subseteq X$, $f|_K$ is continuous.

Proof. The converse direction is trivial, so it suffices to prove for the sufficiency. Suppose that $f|_K$ is continuous for any compact set $K \subseteq X$. Take any open set $V \subseteq Y$, then for any compact set $K \subseteq X$,

$$f^{-1}(V) \cap K = f|_K^{-1}(V),$$

which is open in K. Since X is compactly generated, this means that $f^{-1}(V)$ is open in X, and so f is continuous. □

This leads to the following corollary:

Corollary 4.7.9 ▶ The Set of Continuous Maps Is Closed

Let X be a compactly generated space and (Y, d) be a metric space, then $\mathcal{C}(X, Y) \subseteq Y^X$ is closed under the topology of compact convergence.

Proof. Let $f \in Y^X$ be any limit point of $\mathcal{C}(X, Y)$. It suffices to show that $f \in \mathcal{C}(X, Y)$, i.e., f is continuous. Note that $B\left(K, f, \frac{1}{n}\right)$ is a basic open neighbourhood of f in Y^X for every compact set $K \subseteq X$ and every $n \in \mathbb{N}^+$. Therefore,

$$B\left(K, f, \frac{1}{n}\right) \cap \mathcal{C}(X, Y) \neq \varnothing.$$

This means that for each $n \in \mathbb{N}^+$, there exists some $f_n \in B\left(K, f, \frac{1}{n}\right) \cap \mathcal{C}(X, Y)$. Clearly, $\{f_n|_K\}_{n \in \mathbb{N}^+}$ converges to $f|_K$ uniformly, and so by Proposition 1.5.15, $f|_K$ is continuous. By Proposition 4.7.8, f is continuous. □

To aim for a finer control quantitatively for a family of functions, we investigate a way to describe the "variation" of the functions in a given neighbourhood. In the ideal scenario, all the functions in the family have the same variation, which is formally captured by the following definition:

Definition 4.7.10 ▶ Equicontinuity

Let X be a topological space, (Y, d) be a metric space and $\mathcal{Y} \subseteq \mathcal{C}(X, Y)$. We say that $\mathcal{Y}$ is **equicontinuous at $x_0 \in X$** if for all $\epsilon > 0$, there exists an open set $U \subseteq Y$ such that for all $f \in \mathcal{Y}$,

$$d\big(f(x_0), f(x)\big) < \epsilon$$

for all $x \in U$. We say that $\mathcal{Y}$ is **equicontinuous** if it is equicontinuous at every $x \in X$.

Intuitively, a family of functions is equicontinuous around a point x if they can be arbitrarily close to the point when evaluated in a carefully chosen neighbourhood. Naturally, this implies that the distance between the functions must be upper bounded, which motivates the following sufficient condition:

Proposition 4.7.11 ▶ Sufficient Condition for Equicontinuity

Let X be a topological space, (Y, d) be a metric space and $\overline{\rho}$ be the uniform metric on $\mathcal{C}(X, Y)$. If $\mathcal{Y} \subseteq \mathcal{C}(X, Y)$ is totally bounded with respect to $\overline{\rho}$, then it is equicontinuous.

Proof. Take any $x_0 \in X$ and any $\epsilon > 0$. Define $\delta := \frac{\epsilon}{1+\epsilon} > 0$. Since $\mathcal{Y}$ is totally bounded, there exists $f_1, f_2, \cdots, f_n \in \mathcal{Y}$ such that

$$\mathcal{Y} \subseteq \bigcup_{i=1}^{n} B_{\frac{\delta}{3}}(f_i)$$

for some $n \in \mathbb{N}^+$. For each $i = 1, 2, \cdots, n$, since f_i is continuous,

$$U_i := f_i^{-1}\left(B^{\rho}_{\frac{\delta}{3}}(x_0)\right)$$

is an open neighbourhood of x_0. Note that for all $x \in U_i$, we have

$$\rho\big(f_i(x), f_i(x_0)\big) < \frac{\delta}{3}.$$

Take

$$U := \bigcap_{i=1}^{n} U_i \subseteq X$$

which is an open neighbourhood of x_0. Take any $f \in \mathcal{Y}$. Notice that $f \in B_{\frac{\delta}{3}}(f_k)$ for some $k \in \{1, 2, \cdots, n\}$, so for all $x \in U$, we have

$$\begin{aligned}
\frac{d\big(f(x), f(x_0)\big)}{d\big(f(x), f(x_0)\big) + 1} &= \rho\big(f(x), f(x_0)\big) \\
&\leq \rho\big(f(x), f_i(x)\big) + \rho\big(f_i(x), f_i(x_0)\big) + \rho\big(f_i(x_0), f(x_0)\big) \\
&< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} \\
&= \frac{\epsilon}{\epsilon + 1}.
\end{aligned}$$

Since $x \mapsto \frac{x}{x+1}$ is strictly increasing, we have $d\big(f(x), f(x_0)\big) < \epsilon$, and so $\mathcal{Y}$ is equicontinuous. □

Referring back to the proof of Proposition 4.4.5, we see that if $\{f_n\}_{n \in \mathbb{N}^+}$ is a sequence in Y^X and that every component $\{\pi_x(f_n)\}_{n \in \mathbb{N}^+}$ converges, then $\{f_n\}_{n \in \mathbb{N}^+}$ converges in the product space, where the components of the limit are the limits of the component sequences. However, notice that $\pi_x(f_n) = f_n(x)$, so convergence in the product space with respect to the product topology is essentially point-wise convergence. Therefore, the product topology is sometimes referred to as the *topology of point-wise convergence*. In particular, each basic open ball centred at f in the product topology can be written as

$$B_f \coloneqq \prod_{x\in F} B_{\epsilon_i}(f(x)) \times \prod_{x\in X\setminus F} X$$

for some finite $F \subseteq X$. Note that F is compact and this is the same as $B(F, f, \epsilon)$. On the other hand, for every open ball $B(F, f, \epsilon)$, we can easily find a basic open set in the product topology to contain it.

On the other hand, it is easy to see that convergence with respect to the uniform topology is essentially uniform convergence.

In an equicontinuous family, however, since the functions have equal variation, we expect both types of convergence to be equivalent.

Proposition 4.7.12 ▸ Convergence of Equicontinuous Families

Let $\mathcal{G} \subseteq \mathcal{C}(X, Y)$ be an equicontinuous family. If $\mathcal{T}_K$ and $\mathcal{T}_P$ are the topology of compact convergence and the topology of point-wise convergence respectively on $\mathcal{G}$, then $\mathcal{T}_K = \mathcal{T}_P$.

Proof. It is clear that $\mathcal{T}_K \subseteq \mathcal{T}_P$. Let $g \in \mathcal{G}$ and consider the basic open set $B(K, g, \epsilon)$ for any compact set $K \subseteq X$ and any $\epsilon > 0$, then it suffices to prove that there exists a basic open set in $\mathcal{T}_P$ which is contained in $B(K, g, \epsilon) \cap \mathcal{G}$. Since $\mathcal{G}$ is equicontinuous, for every $x \in X$, there exists an open neighbourhood U_x of x such that for all $y \in U_x$,

$$d\big(f(x), f(y)\big) < \frac{\epsilon}{3}$$

for every $f \in \mathcal{G}$. Since K is compact, there exist $x_1, x_2, \cdots, x_n \in K$ such that

$$K = \bigcup_{i=1}^{n} U_{x_i}.$$

Notice that

$$g \in B\left(\{x_1, x_2, \cdots, x_n\}, g, \frac{\epsilon}{3}\right) \cap \mathcal{G} \subseteq \mathcal{T}_P.$$

For every $x \in K$, there exists some $i \in \{1, 2, \cdots, n\}$ such that $x \in U_{x_i}$ and so if

$$h \in B\left(\{x_1, x_2, \cdots, x_n\}, g, \frac{\epsilon}{3}\right) \cap \mathcal{G},$$

we have

$$\begin{aligned} d\big(h(x), g(x)\big) &\leq d\big(h(x), h(x_i)\big) + d\big(h(x_i), g(x_i)\big) + d\big(g(x_i), g(x)\big) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Therefore,

$$g \in B\left(\{x_1, x_2, \cdots, x_n\}, g, \frac{\epsilon}{3}\right) \cap \mathcal{G} \subseteq B(K, g, \epsilon).$$

□

Let us consider a special map known as the *evaluation map*, which takes in a function as a first class citizen together with a point, and output the function value at the point. Intuitively, this map is likely to be continuous if the functions in its domain are continuous.

Proposition 4.7.13 ▸ Continuity of Evaluation

Let X be a locally compact Hausdorff space and (Y, d) be a metric space. If $\mathcal{C}(X, Y)$ is equipped with the compact-open topology, then the evaluation map $e : X \times \mathcal{C}(X, Y) \to Y$ defined by

$$e(x, f) = f(x)$$

is continuous.

Proof. Let $V \subseteq Y$ be open. Take any $(x, f) \in e^{-1}(V)$, then $f(x) \in V$. Since f is continuous, $x \in f^{-1}(V)$ is open in X. Pick some open neighbourhood $U_{x,f}$ of x in $f^{-1}(V)$. Since X is locally compact, there exists a compact set K in $f^{-1}(V)$ such that

$$x \in U_{x,f} \subseteq \overline{U_{x,f}} \subseteq K \subseteq f^{-1}(V).$$

Since X is Hausdorff, by Proposition 4.1.2, $\overline{U_{x,f}}$ is compact in $f^{-1}(V)$, so $S\left(\overline{U_{x,f}}, V\right)$ is open in the compact-open topology and

$$U_{x,f} \times S\left(\overline{U_{x,f}}, V\right) \subseteq X \times \mathcal{C}(X, Y)$$

is open with respect to the product topology. Take any $(u, g) \in U_{x,f} \times S\left(\overline{U_{x,f}}, V\right)$, then $e(u, g) = g(u) \in V$ and so

$$(x, f) \in U_{x,f} \times S\left(\overline{U_{x,f}}, V\right) \subseteq e^{-1}(V).$$

Therefore,

$$e^{-1}(V) = \bigcup_{(x,f)\in e^{-1}(V)} U_{x,f} \times S\left(\overline{U_{x,f}}, V\right)$$

is open in the compact-open topology, so e is continuous. □

The above results combined allow us to prove the following major theorem:

Theorem 4.7.14 ▸ Arzela-Ascoli Theorem

Let X be a topological space and (Y, d) be a metric space. For each $\mathcal{Y} \subseteq \mathcal{C}(X, Y)$, define

$$\mathcal{Y}_a := \{f(a) : f \in \mathcal{Y}\}$$

for each $a \in X$. If $\mathcal{C}(X, Y)$ is equipped with the compact-open topology, then $\overline{\mathcal{Y}}$ is compact if it is equicontinuous and $\overline{\mathcal{Y}_a}$ is compact for all $a \in X$. The converse is true if X is locally compact and Hausdorff.

Proof. Let $\mathcal{G}$ be the closure of $\mathcal{Y}$ in Y^X. Note that both Y and Y^X are metric spaces and so they are Hausdorff. Since $\overline{\mathcal{Y}_a}$ is compact for all $a \in X$, by Theorem 4.1.23,

$$\mathcal{K} := \prod_{a\in X} \overline{\mathcal{Y}_a}$$

is compact in Y^X, and so it is closed by Proposition 4.1.5. Observe that $\mathcal{Y} \subseteq \mathcal{K}$, so we have $\mathcal{G} \subseteq \mathcal{K}$ and so by Proposition 4.1.2, $\mathcal{G}$ is compact. By Proposition 4.7.11, for every $x_0 \in X$ and every $\epsilon > 0$, there exists an open neighbourhood U of x_0 such that

$$d(f(x), f(x_0)) < \frac{\epsilon}{3}$$

for all $x \in U$ and all $f \in \mathcal{Y}$. Take any $g \in \mathcal{G}$. For each $x \in X$, define

$$\begin{aligned} V_x &:= B\left(\{x\}, g, \frac{\epsilon}{3}\right) \cap B\left(\{x_0\}, g, \frac{\epsilon}{3}\right) \\ &= \pi_x^{-1}\left(B_{\frac{\epsilon}{3}}(g(x))\right) \cap \pi_{x_0}^{-1}\left(B_{\frac{\epsilon}{3}}(g(x_0))\right) \\ &\subseteq Y^X, \end{aligned}$$

which is an open neighbourhood of g. By Proposition 1.4.7, we have $V_x \cap \mathcal{Y} \neq \varnothing$. Take any $h \in V_x \cap \mathcal{Y}$, then

$$\begin{aligned} d\big(g(x), g(x_0)\big) &\leq d\big(g(x), h(x)\big) + d\big(h(x), h(x_0)\big) + d\big(h(x_0), g(x_0)\big) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

One can check that $\mathcal{C}(X, Y)$ is Hausdorff with respect to the compact-open topology. By Proposition 4.7.12, $\mathcal{G}$ is with the topology of compact convergence, and so is a subspace of $\mathcal{C}(X, Y)$ by Proposition 4.7.4. Notice that $\mathcal{G}$ is compact in a Hausdorff space, so it is closed. Therefore, $\overline{\mathcal{Y}} \subseteq \mathcal{G}$, and so $\overline{\mathcal{Y}}$ is compact. □

5

Connectedness

5.1 Connected Spaces

Colloquially, when we are talking about a space or "region", we can distinguish between a "contiguous" region and a "disconnected" region based on whether one has to "jump" across "gaps" in order to traverse in the region. In this section, we will formalise this notion of connectedness.

Definition 5.1.1 ▶ Separation

Let X be a topological space. A **separation** of X is a pair of disjoint non-empty open subsets $U, V \subseteq X$ such that $U \cup V = X$. A space is **connected** if it does not contain any separation.

In other words, a separation is a bipartition of a space with non-empty open sets. This is intuitive because if 2 disjoint non-empty open sets can cover the space, then there must exists some gap in between the boundaries of the 2 sets.

Now, consider a disconnected space X formed by two connected non-empty spaces A and B. Intuitively, both A and B are also closed because their closure are equal to themselves. This inspires the following characterisation of a connected space:

Proposition 5.1.2 ▶ Characterisation of Connected Spaces

A topological space X is connected if and only if the only sets which are both open and closed in X are X and $\varnothing$.

Proof. We shall prove the contrapositive statement, i.e., a topological space X is disconnected if and only if there exists some non-empty proper subset of X which is both closed and open. Suppose that there exists some non-empty $U \subseteq X$ which is both open and closed such that $X \setminus U \neq \varnothing$. Note that $X \setminus U$ is open and disjoint from U and that $U \cup (X \setminus U) = X$, so X is disconnected. Conversely, suppose that X is disconnected, then there exists disjoint non-empty open subsets $U, V \subseteq X$ such that $U \cup V = X$. Note that $U = X \setminus V \neq \varnothing$ and $V = X \setminus U \neq \varnothing$, so both U and V are also closed proper subsets of X. □

Intuitively, if we can find a separation in a space X, then any connected subspace of X

should not be “cut off” by the separation.

Proposition 5.1.3 ▸ Property of Conencted Subspaces

If U, V are a separation of a topological space X and $Y \subseteq X$ is a connected subspace, then either $Y \subseteq U$ or $Y \subseteq V$.

Proof. We shall prove the contrapositive statement. Let U, V be a separation of X, then $U = X \setminus V$. Suppose that $Y \nsubseteq U$ and $Y \nsubseteq V$, then it suffices to prove that Y is not connected. Since $Y \nsubseteq U$, then

$$Y \cap (X \setminus U) = Y \cap V \neq \varnothing.$$

Similarly, $Y \cap U \neq \varnothing$. Since $U \cup V = X$, we have

$$(Y \cap U) \cup (Y \cap V) = Y.$$

Note that both $Y \cap U$ and $Y \cap V$ are open in Y, so they form a separation of Y. Therefore, Y is not connected, and so if Y is connected, then either $Y \subseteq U$ or $Y \subseteq V$. □

Naturally, connectedness should be transitive, i.e., if two connected subsets are not disjoint, then their union should be connected.

Proposition 5.1.4 ▸ Connectedness of the Union of Connected Subspaces

If $\{A_\alpha\}_{\alpha \in \Lambda}$ is a collection of connected subsets in a topological space X such that

$$\bigcap_{\alpha \in \Lambda} A_\alpha \neq \varnothing,$$

then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is connected.

Proof. Suppose on contrary that $\bigcup_{\alpha \in \Lambda}$ is disconnected, then there exists non-empty disjoint open sets $U, V \subseteq X$ such that $\bigcup_{\alpha \in \Lambda} A_\alpha \subseteq U \cup V$. Without loss of generality, suppose that $U \cap \bigcap_{\alpha \in \Lambda} A_\alpha \neq \varnothing$ and take $p \in U \cap \bigcap_{\alpha \in \Lambda} A_\alpha$. Note that $p \in A_\alpha$ for all $\alpha \in \Lambda$, so $p \in A_\alpha \cap U \neq \varnothing$ for all $\alpha \in \Lambda$. We consider two cases. If there exists some $a \in A_\alpha \setminus U$ for some $\alpha \in \Lambda$, then $a \in V$. However, since A_α is connected, this means that there exists some $b \in A_\alpha \setminus (U \cup V)$, which implies that

$$\bigcup_{\alpha \in \Lambda} A_\alpha \nsubseteq U \cup V$$

and is a contradiction. If $A_\alpha \subseteq U$ for all $\alpha \in \Lambda$, then

$$\bigcup_{\alpha\in\Lambda} A_\alpha \subseteq U$$

and so

$$\bigcup_{\alpha\in\Lambda} A_\alpha \cap V = \varnothing.$$

Since $A_\alpha \neq \varnothing$ for all $\alpha \in \Lambda$, this means that $V = \varnothing$, which is a contradiction. ☐

By intuition, if we slightly expand the boundary of a connected subset, then the resultant set should remain connected.

Proposition 5.1.5 ▸ Slightly Larger Supersets of Connected Sets Are Connected

If A is connected in a topological space X and $A \subseteq B \subseteq \overline{A}$ for some $B \subseteq X$, then B is connected.

Proof. Suppose on contrary that B is disconnected, then there exist disjoint non-empty open sets $U, V \subseteq X$ such that $B \subseteq U \cup V$. By Proposition 5.1.3, without loss of generality, assume that $A \subseteq U$, then $B \subseteq \overline{A} \subseteq \overline{U}$. Note that $V \subseteq X \setminus \overline{U}$, so $V \cap B = \varnothing$, which is a contradiction. ☐

Recall that continuous maps preserve open sets, so it is also natural that the image of a connected set under a continuous map is also connected.

Proposition 5.1.6 ▸ Continuous Maps Preserve Connectedness

If $f : X \to Y$ is continuous and $A \subseteq X$ is connected, then $f(A)$ is connected.

Proof. We shall prove the contrapositive statement. Suppose that $f(A)$ is disconnected, then there exists non-empty disjoint open sets $U, V \subseteq Y$ such that

$$f(A) = \big(U \cap f(A)\big) \cup \big(V \cap f(A)\big).$$

Note that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint non-empty open sets in X and

$$A \subseteq f^{-1}(U) \cup f^{-1}(V),$$

so A is disconnected. ☐

Lastly, if we take the Cartesian product of two connected spaces, then this should not result in any new "gap".

Proposition 5.1.7 ▸ Cartesian Products of Connected Spaces Are Connected

If X and Y are connected topological spaces, then $X \times Y$ is connected.

Proof. We shall prove the contrapositive statement. Suppose that $X \times Y$ is disconnected, then there exists non-empty disjoint open sets $U, V \subseteq X \times Y$ such that

$$X \times Y = U \cup V.$$

Note that there exists non-empty open sets $U_X, V_X \subseteq X$ and non-empty open sets $U_Y, V_Y \subseteq Y$ such that $U = U_X \times U_Y$ and $V = V_X \times V_Y$. We claim that U_X and V_X are disjoint or U_Y and V_Y are disjoint because otherwise U and V are not disjoint. Without loss of generality, assume that U_X and V_X are disjoint. Notice that for every $(x, y) \in U \cup Y$, either $x \in U_X$ or $x \in V_X$, so $X = U_X \cup V_X$ and is disconnected. $\square$

For those with background in graph theory, the discussion by now should have been clear that connectedness in topological spaces is a generalisation of connectedness in graphs. For example:

1. 5.1.1 is analogous to the fact that a graph is disconnected if and only if we can break it into two subgraphs sharing no common vertex such that there is no edge between their vertices in the original graph.
2. 5.1.2 is analogous to the fact that if a graph can be broken up into two subgraphs sharing no common vertex such that there is no edge between their vertices in the original graph, then any connected subgraph must be contained by one of the two subgraphs.
3. 5.1.3 is analogous to the fact that if we augment two connected graphs by identifying some vertices, then the resulting graph is still connected.

Therefore, it is not surprising that we can generalise the notion of a *path* to topological spaces.

Definition 5.1.8 ▸ Path

Let X be a topological space. For any $x, y \in X$, a **path** from x to y is a continuous map $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. A space is **path connected** if there is a path between any two points in the space.

Clearly, path connectedness is a stronger version of connectedness.

Proposition 5.1.9 ▶ Path Connectedness Implies Connectedness

A space is connected if it is path connected.

Proof. We shall prove the contrapositive statement. Let X be a disconnected space, then there exists non-empty disjoint open sets $U, V \subseteq X$ such that $X = U \cup V$. Take some $u \in U$ and some $v \in V$. Suppose on contrary that there exists a continuous map $f: [a,b] \to X$ such that $f(a) = u$ and $f(b) = v$. Notice that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint non-empty open sets, so there exists some $x \in [a,b]$ such that there is no $y \in X$ with $f(x) = y$, which is not possible. Therefore, X is not path connected. □

One might wonder if the converse of the above result is true. Intuitively, it should not be true. Specifically, we consider the following counter example known as the *topologist's sine curve*: define

$$S := \left\{(x, y) \in \mathbb{R}^2 : y = \sin\frac{2\pi}{x}, x \in (0,1]\right\}$$

and let $f: (0,1] \to S$ be defined as

$$f(t) = \left(t, \sin\frac{2\pi}{t}\right).$$

One may check that $S = f\big((0,1]\big)$ is path connected and thus connected. Now, consider

$$\overline{S} = S \cup (\{0\} \times [-1,1]),$$

which is still connected but not path connected.

Proposition 5.1.10 ▶ Topologist's Sine Curve Is Not Path Connected

Let S be defined as

$$S = \left\{(x, y) \in \mathbb{R}^2 : y = \sin\left(\frac{2\pi}{x}\right), 0 < x \leq 1\right\}$$

and define

$$\overline{S} = S \cup (\{0\} \times [-1,1]),$$

then $\overline{S}$ is not path connected.

Proof. It suffices to show that there exists two points in $\overline{S}$ such that there is no path between them. We shall prove that there is no path from $(0,0)$ to any point $(x, y) \in S$. Suppose on contrary that there exists some $f: [a,b] \to \overline{S}$ which is continuous such that $f(a) = (0,0)$ and $f(b) = (x,y)$, then $f^{-1}(\{0\} \times [-1,1])$ is closed in $[a,b]$. Take

$$M \coloneqq \max f^{-1}(\{0\} \times [-1,1])$$

and consider $g\colon [M,b] \to \overline{S}$. Notice that there exists some $y_0 \in [-1,1]$ such that $g(M) = (0, y_0)$ and $g\big((M,b]\big) \subseteq S$. Without loss of generality, assume $y_0 \geq 0$. Write $g(t) = \big(x(t), y(t)\big)$, then $x(M) = 0$ and for all $t \in (M,b]$, we have $x(t) > 0$ and $y(t) = \sin \frac{2\pi}{x(t)}$. Note that

$$\lim_{t \to 0^+} y(t) = y_0 \geq 0$$

because y is a continuous projection. For each $n \in \mathbb{N}^+$, there exists some z_n with

$$0 < z_n < x\left(\frac{1}{n}\right)$$

such that $\sin \frac{2\pi}{z_n} = (-1)^n$. Note that x is continuous because it is a projection, so there exists some $t_n \in \left(0, \frac{1}{n}\right)$ such that $x(t_n) = z_n$. This means that $\{t_n\}_{n \in \mathbb{N}^+}$ is a sequence such that $y(t_n) = (-1)^n$ for all $n \in \mathbb{N}^+$. Now, notice that for any $\delta > 0$, there exists some $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \delta$ and there exists some $k > N$ such that

$$|y(t_k) - y_0| > \frac{1 - y_0}{2} > 0,$$

which means that $y(t)$ does not converge to y_0 as $t \to 0^+$ and is a contradiction. Therefore, $\overline{S}$ must not be path connected. □

5.2 Connected Components

Recall that in a graph, connectedness between vertices is an equivalence relation. Analogously, we can define a relation between two points in a topological space based on whether there exists a connected set containing both points.

One may check that such a relation is indeed an equivalence relation. In particular, we can partition a space using the equivalence classes.

Definition 5.2.1 ▸ Connected Component

Let $\sim$ be a binary relation on a topological space X such that $x \sim y$ if and only if there exists a connected subset $C \subseteq X$ with $x, y \in C$. The **connected components** of X are the equivalence classes in the quotient $X/\sim$.

Intuitively, $X/\sim$ contains all maximal connected subsets of X, which leads to the following obvious result:

Proposition 5.2.2 ▶ Connected Components Are Connected

Every connected component of a topological space is connected.

Proof. Let $C \subseteq X$ be a connected component and fix some $x_0 \in C$. For each $x \in C$, there exists some connected subset $A_x \subseteq C$ with $x_0, x \in A_x$. Therefore,

$$x_0 \in \bigcap_{x \in C} A_x \neq \varnothing.$$

Notice that $C = \bigcup_{x \in C} A_x$, so by Proposition 5.1.4, C is connected. □

Similarly, we can define another equivalence relation using path connectedness.

Definition 5.2.3 ▶ Path Component

Let $\overset{p}{\sim}$ be a binary relation on a topological space X such that $x \overset{p}{\sim} y$ if and only if there exists a path from x to y in X. The **path components** of X are the equivalence classes in the quotient $X/\overset{p}{\sim}$.

Again, one may check that $X/\overset{p}{\sim}$ contains all maximal path-connected subsets of X.

Proposition 5.2.4 ▶ Path Components Are Path Connected

Every path component of a topological space is path connected.

Proof. Let X be a topological space and $P \subseteq X$ be a path component in X. By definition, $P = [x_0]_{\overset{p}{\sim}}$ for some $x_0 \in X$, where $[x_0]_{\overset{p}{\sim}}$ is the equivalence class of x_0 under the equivalence relation $\overset{p}{\sim}$. Take any $x, y \in P$, then $x \overset{p}{\sim} x_0$ and $y \overset{p}{\sim} x_0$. By transitivity, we have $x \overset{p}{\sim} y$ and so there exists a path from x to y. Therefore, P is path connected. □

Recall that path connectedness is a stronger version of connectedness, so it is natural that every path component is contained in some connected component.

Proposition 5.2.5 ▶ Connected Components Contain Path Components

Every path component of a topological space is contained in some connected component.

Proof. Let P be a path component of a topological space X. By Proposition 5.2.4, P is path connected and by Proposition 5.1.9, P is connected. Note that for every $x \in P$, we have $x \sim x_0$ and so $P \subseteq [x_0]_\sim$, where $[x_0]_\sim$ is the equivalence class of x_0 under the equivalence relation $\sim$, which is a connected component of X. $\square$

Similar to our discussion of compactness, we can define connectedness in a local perspective.

Definition 5.2.6 ▶ Local Connectedness

A topological space X is **locally (path) connected at $x \in X$** if for all open sets $U \subseteq X$ containing x, there exists a (path) connected open set $V \subseteq X$ such that $x \in V \subseteq U$. The space X is **locally (path) connected** if it is locally connected at every $x \in X$.

Even though local connectedness is "local", connectedness does not imply local connectedness.

Proposition 5.2.7 ▶ Topologist's Sine Curve Is Not Locally Connected

Let S be defined as

$$S = \left\{(x, y) \in \mathbb{R}^2 : y = \sin\left(\frac{2\pi}{x}\right), 0 < x \leq 1\right\}$$

and define

$$\overline{S} = S \cup (\{0\} \times [-1, 1]),$$

then $\overline{S}$ is not locally connected.

Proof. It suffices to show that there exists an open subset $U \subseteq \overline{S}$ which contains some $\boldsymbol{p} \in \overline{S}$ such that all open subsets of U are disconnected. Take $\boldsymbol{p} \coloneqq (0, 0)$ and consider

$$U \coloneqq B_{\frac{1}{2}}(\boldsymbol{p}) \cap \overline{S}$$

which is open in $\overline{S}$ containing $\boldsymbol{p}$. Let $W \subseteq U$ be any open set containing $\boldsymbol{p}$, then there exists some open set

$$V \subseteq B_{\frac{1}{2}}(\boldsymbol{p})$$

such that $W = V \cap \overline{S}$. Note that V contains some $B_\epsilon(\boldsymbol{p})$ where $0 < \epsilon < \frac{1}{2}$. Take

$$\delta \coloneqq \frac{\sqrt{2}}{2}\epsilon,$$

then V contains the open square $(-\delta, \delta)^2$. Define

$$D_\delta \coloneqq \left\{x \in (0, \delta) : \left|\sin\frac{2\pi}{x}\right| < \delta\right\},$$

then
$$W = \{(0, y) \in \mathbb{R}^2 : -\delta < y < \delta\} \cup \left\{\left(x, \sin\frac{2\pi}{x}\right) \in \mathbb{R}^2 : x \in D_\delta\right\}.$$
We claim that W is disconnected. Notice that there exists some $x_0 \in (0, \delta)$ such that $\sin\frac{2\pi}{x} = 1$. Consider
$$X_1 \coloneqq W \cap \left((-\infty, x_0) \times \mathbb{R}^2\right),$$
$$X_2 \coloneqq W \cap \left((x_0, +\infty) \times \mathbb{R}^2\right).$$
Since $\{(0, y) \in \mathbb{R}^2 : -\delta < y < \delta\} \subseteq X_1$, we know that $X_1 \neq \varnothing$. Since there exists some $x' \in (x_0, \delta)$ such that $\left|\sin\frac{2\pi}{x}\right| < \delta$, we know that $X_2 \neq \varnothing$. Furthermore,
$$(\{x_0\} \times \mathbb{R}) \cap W = \varnothing,$$
so $X_1 \cup X_2 = W$. Since both X_1 and X_2 are open, W is disconnected. Therefore, $\overline{S}$ is not locally connected. □

In the ideal case, since every point in a topological space is contained by some (path) connected component, the space is naturally locally (path) connected if each of these components is open. This motivates the following characterisation:

Proposition 5.2.8 ▸ Characterisation of Locally (Path) Connected Spaces

A topological space X is locally (path) connected if and only if for any open subset $U \subseteq X$, every (path) connected component of U is open.

Proof. Let $U \subseteq X$ be any open set and $C \subseteq U$ be a (path) connected component. Since X is locally (path) connected, then for each $c \in C \subseteq U$, there exists a (path) connected open set $V_c \subseteq X$ such that $c \in V_c \subseteq U$. Note that $V_c \subseteq C$, so $C = \bigcup_{c \in C} V_c$ is open. Conversely, suppose that for any open subset $U \subseteq X$, every (path) connected component of U is open. Take any $x \in X$, then there exists some open set U containing x. Since the (path) connected components of U form a partition, there exists some open (path) connected component $C_x \subseteq U$ with $x \in C_x$. By Propositions 5.2.2 and 5.2.4, C_x is (path) connected and so X is locally (path) connected. □

It turns out that there is no need to differentiate between local connectedness and local path connectedness because the components induced by the two relations are the same in a local context.

Proposition 5.2.9 ▶ Connected Components in Locally Path Connected Spaces

If X is a locally path connected topological space, then $X/\sim = X/\overset{p}{\sim}$.

Proof. Suppose that P is a path component of X but not a connected component. Note that by Proposition 5.2.5, there exists some connected component $C \subseteq X$ such that $Q \coloneqq C \setminus P \neq \varnothing$. By Proposition 5.2.8, since X is locally path connected, both P and C are open, and so Q is also open. Therefore, $C = P \cup Q$ where P and Q are disjoint non-empty open sets, which is a contradiction because C is connected by Proposition 5.2.2. □

Note that connectedness is an equivalence relation which induces a partition of the space by its connected components, so it is natural to link connectedness to quotient topologies.

Proposition 5.2.10 ▶ Quotient Topology over Connected Components Is Discrete

If X is a locally path connected space and $\widetilde{X} \coloneqq X/\sim$ are the connected components of X, then the quotient topology on $\widetilde{X}$ is discrete.

Proof. Since the quotient topology is unique, it suffices to prove that the discrete topology $\mathcal{P}\left(\widetilde{X}\right)$ is a quotient topology with respect to $p\colon X \to \widetilde{X}$ such that $x \in p(x)$, which is equivalent to proving that p is a quotient map with respect to $\mathcal{P}\left(\widetilde{X}\right)$. Take any open set $\mathcal{S} \in \mathcal{P}\left(\widetilde{X}\right)$, then $p^{-1}(\mathcal{S}) = \bigcup_{C \in \mathcal{S}} C$. Note that each $C \in \mathcal{S}$ is a connected component of X. Since X is locally path connected, then each $C \in \mathcal{S}$ is a path component of X. Note that X is open in X, so every $C \in \mathcal{S}$ is open in X. This implies that $p^{-1}(\mathcal{S})$ is open in X. Conversely, take any open set $U \subseteq X$, then

$$p(U) = \left\{C \in \widetilde{X} : C \cap U \neq \varnothing\right\} \in \mathcal{P}\left(\widetilde{X}\right),$$

which is clearly open in $\widetilde{X}$. Therefore, p is a quotient map and so $\mathcal{P}\left(\widetilde{X}\right)$ is the quotient topology. □