MA5206 Notes

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Contents

1 Basic Fourier Analysis

Definition 1.1 (Schwartz Function)**.** *Suppose a function $f : \mathbb{R}^n \to \mathbb{R}^n$, $f \in C^\infty$, for all $\alpha, \beta \in \mathbb{N}_0^n$,*

$$\sup_{x \in \mathbb{R}} |x^\alpha D^{(\beta)} f(x)| < \infty$$

then f is a ***Schwartz function****. Here $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.*

In this section we assume that $f(x)$ is a Schwartz function.

Definition 1.2 (Fourier Transform)**.** *Given $f(x)$, we define its* ***Fourier transformation*** *by*

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$$

where $\xi \in \mathbb{R}$.

We define the ***inverse Fourier transform*** *of f to be*

$$\check{f}(x) := \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} d\xi.$$

In particular, we have

$$f(x) = \frac{1}{2\pi} (\check{\hat{f}})(x).$$

Remark 1.3. *Since f is a Schwartz function, then*

$$|\hat{f}(\xi)| \leqslant \int_{\mathbb{R}} |f(x)||e^{-i\xi x}| dx = \int_{\mathbb{R}} f(x) dx < \infty.$$

So its Fourier transform is well-defined.

Lemma 1.4. $\widehat{(xf)}(\xi) = i \frac{d}{d\xi} \hat{f}(\xi)$.

Proof.

$$\begin{aligned}
\widehat{(xf)}(\xi) &= \int_{\mathbb{R}} x f(x) e^{-i\xi x} dx \\
&= \int_{\mathbb{R}} f(x) x e^{-i\xi x} dx \\
&= \int_{\mathbb{R}} f(x) \frac{d}{d\xi}(e^{-i\xi x}) \cdot \frac{1}{-i} dx \\
&= -\frac{1}{i} \frac{d}{d\xi} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx \\
&= i \frac{d}{d\xi} \hat{f}(\xi)
\end{aligned}$$

$\square$

Lemma 1.5. $\left(\widehat{\frac{df}{dx}}\right)(\xi) = i\xi\hat{f}(\xi)$.

Proof.

$$\begin{aligned}
\left(\widehat{\frac{df}{dx}}\right)(\xi) &= \int_{\mathbb{R}} \frac{df}{dx} e^{-i\xi x} dx \\
&= \int_{\mathbb{R}} e^{-i\xi x} df \\
&= f e^{-i\xi x}|_{-\infty}^{\infty} - \int_{\mathbb{R}} f(x) d\left(e^{-i\xi x}\right) \quad \text{(by integration by part)} \\
&= (0-0) - \int_{\mathbb{R}} f(x) \cdot -i\xi e^{-i\xi x} dx \quad \text{(f is Schwartz)} \\
&= i\xi \int_{\mathbb{R}} f(x) e^{-\xi x} dx \\
&= i\xi \hat{f}(\xi)
\end{aligned}$$

□

Theorem 1.6 (Plancherel's Theorem)**.** $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \frac{d\xi}{2\pi}$.

The Heisenberg Uncertainty Principle: In quantum mechanics, ξ is called the moment variable and x is the position variable. Let $f(x) \in \mathbb{R}$ be the wave function satisfying

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1.$$

The expected value of the square of the position is

$$(\bar{x})^2 = \int_{-\infty}^{\infty} |x f(x)|^2 dx.$$

The expected value of the square of the momentum is

$$(\bar{\xi})^2 = \int_{-\infty}^{\infty} |\xi \hat{f}(\xi)|^2 \frac{d\xi}{2\pi}.$$

The uncertainty principle asserts

$$|\bar{x} \cdot \bar{\xi}| \geqslant \frac{1}{2}.$$

Proof. We have

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} xf(x)f'(x)dx\right| &\leqslant \left[\int_{-\infty}^{\infty} |xf(x)|^2 dx\right]^{1/2} \left[\int_{-\infty}^{\infty} |f'(x)|^2 dx\right]^{1/2} && \text{(by Hölder's Inequality)} \\
&= |\bar{x}| \cdot \left[\int_{-\infty}^{\infty} |f'(x)|^2 dx\right]^{1/2} \\
&= |\bar{x}| \cdot \left[\int_{-\infty}^{\infty} |(\widehat{f'})(\xi)|^2 \frac{d\xi}{2\pi}\right]^{1/2} && \text{(by Plancherel's)} \\
&= |\bar{x}| \cdot \left[\int_{-\infty}^{\infty} |i\xi \hat{f}(\xi)|^2 \frac{d\xi}{2\pi}\right]^{1/2} && \text{(by lemma)} \\
&= |\bar{x}| \cdot \left[\int_{-\infty}^{\infty} |\xi \hat{f}(\xi)|^2 \frac{d\xi}{2\pi}\right]^{1/2} \\
&= |\bar{x}| \cdot \left[(\bar{\xi})^2\right]^{1/2} \\
&= |\bar{x}\bar{\xi}|
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
\int_{-\infty}^{\infty} xf(x)f'(x)dx &= \int_{-\infty}^{\infty} xf(x)df(x) \\
&= \int_{-\infty}^{\infty} \frac{x}{2} d[f(x)]^2 \\
&= \frac{x}{2}[f(x)]^2\Big|_{-\infty}^{\infty} - \frac{1}{2}\int_{-\infty}^{\infty} [f(x)]^2 dx && \text{(by integration by part)} \\
&= 0 - \frac{1}{2} && \text{(Schwartz function + given condition)} \\
&= -\frac{1}{2}.
\end{aligned}
$$

□

Definition 1.7 (Fourier Transform in $\mathbb{R}^n$)**.** *For Fourier Transformation in $\mathbb{R}^n$, with $x, \xi \in \mathbb{R}^n$. For $f(x) \in L^1(\mathbb{R}^n)$, we can define its* ***Fourier Transform*** *$\hat{f}(\xi)$ to be*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i(\xi \cdot x)} f(x) d\mathbf{x}.$$

Remark 1.8. *If $x = (x_1, x_2, \cdots, x_n)$, $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$,*

$$\hat{f}(\xi_1, \xi_2, \cdots, \xi_n) = \int_{\mathbb{R}^n} e^{-i(\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n)} f(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n.$$

Proposition 1.9.

1.

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) d\xi.$$

2.
$$\widehat{f \cdot g}(\xi) = \hat{f} * \hat{g}(\xi).$$

3.
$$\widehat{f * g}(\xi) = \hat{f} \cdot \hat{g}(\xi).$$

Definition 1.10 (Convolution)**.** *Suppose $f, g : \mathbb{R}^n \to \mathbb{R}^n$, then f* ***convolution*** *g, denoted by $f * g$ is defined to be*
$$(f * g)(t) = \int_{\mathbb{R}^n} f(x)g(t-x)dx = \int_{\mathbb{R}^n} f(t-x)g(x)dx.$$

Using Fourier transform to solve some linear PDE:
We try to solve the following linear heat equation using Fourier Transform
$$\begin{cases} u_t - u_{xx} = 0 & (1) \\ u(0,x) = g(x) & (2) \end{cases}.$$

Take Fourier transform of (1), with respect to x, we have
$$u_t - u_{xx} = 0 \Rightarrow \widehat{u_t} - \widehat{u_{xx}} = \hat{0} \Rightarrow \frac{d}{dt}\hat{u}(t,\xi) - (i \cdot \xi)^2 \hat{u}(t,\xi) = 0.$$

Hence the original PDE problem becomes an ODE problem:
$$\frac{d}{dt}\hat{u}(t,\xi) + \xi^2 \hat{u}(t,\xi) = 0.$$

And doing Fourier transform on (2), we have
$$\hat{u}(0,\xi) = \hat{g}(\xi).$$

Now solving the ODE we get
$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{g}(\xi).$$

Through
$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\hat{f}(\xi)d\xi,$$

the Inverse Fourier Transform, dented $\frac{1}{2\pi}[\hat{f}(\xi)]^\vee$, we get

$$\begin{aligned}
u(t,x) &= \frac{1}{2\pi}[\hat{u}(t,\xi)]^\vee(x) \\
&= \frac{1}{2\pi}\left[e^{-t|\xi|^2}\hat{g}(\xi)\right]^\vee(x) \\
&= \frac{1}{2\pi}[(\hat{F}\cdot\hat{g})(\xi)]^\vee(x) && \text{(if we can find such F and g)} \\
\implies \hat{u}(t,\xi) &= (\hat{F}\cdot\hat{g})(\xi) \\
u(t,x) &= F * g(x)
\end{aligned}$$

If we choose $g(x) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{(ax)^2}{2}}$, and let $a = \frac{1}{\sqrt{2t}}$, then $\hat{g}(\xi) = e^{-tk^2}$. Hence $F(x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$. Then

$$u(t,x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}g(y)dy.$$

Lemma 1.11. *Suppose* $f(x) = e^{-\frac{x^2}{2}}$, *then* $\hat{f}(\xi) = \sqrt{2\pi}e^{-\frac{\xi^2}{2}}$.

Proof. Note $f'(x) = -xe^{-\frac{x^2}{2}} = -xf(x)$. Then taking Fourier Transformation on both sides we get

$$\begin{aligned}
i\xi\hat{f}(\xi) &= -i\frac{d}{d\xi}\hat{f}(\xi) \\
\xi\hat{f}(\xi) + \frac{d}{d\xi}\hat{f}(\xi) &= 0
\end{aligned}$$

By solving the ODE, we get

$$\hat{f}(\xi) = Ce^{-\frac{\xi^2}{2}}.$$

Let $\xi = 0$, then

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\frac{x^2}{2}}dx = \sqrt{2\pi} = C.$$

□

Lemma 1.12. $\widehat{f(ax)} = \frac{1}{|a|^n}\hat{f}\left(\frac{\xi}{a}\right)$, *where* $a \neq 0$.

Corollary 1.12.1. *Let* $f(x) = e^{-\frac{(ax)^2}{2}}$, *then*

$$\hat{f}(x) = \frac{\sqrt{2\pi}}{|a|}e^{-\frac{\xi^2}{2a^2}},$$

where $a \neq 0$.

Solving Schrödinger's Equation:

It is the equation of the form

$$\begin{cases} -i\partial_t\psi(t,x) + \frac{1}{2}\triangle\psi(t,x) = 0, & (1) \\ \psi(0,x) = \psi_0(x) \in L^2(\mathbb{R}^n). & (2) \end{cases}$$

where $\psi(t,x) : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{C}$.

Apply Fourier transform to (1) and (2) gives

$$\begin{cases} -i\partial_t \hat{\psi}(t,\xi) - \frac{1}{2}|\xi|^2 \hat{\psi}(t,\xi) = 0 \\ \hat{\psi}(0,\xi) = \hat{\psi_0}(\xi). \end{cases}$$

Where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$. Solving the system, we have

$$\hat{\psi}(t,\xi) = e^{i\frac{|\xi|^2}{2}t}\hat{\psi_0}(\xi).$$

So

$$\psi(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x} e^{i\frac{|\xi|^2}{2}t}\hat{\psi_0}(t)d\xi.$$

Then

$$\psi(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)} e^{i\frac{|\xi|^2}{2}t}\psi_0(y)dyd\xi$$

We observe that the solution ψ conserves L^2 norm.

$$\begin{aligned} \|\psi(t,x)\|^2_{L^2(\mathbb{R}^n)} &= \frac{1}{(2\pi)^n}\|\hat{\psi}(t,\xi)\|^2_{L^2(\mathbb{R}^n)} && \text{(By Plancherel's Theorem)} \\ &= \frac{1}{(2\pi)^n}\|e^{i|\xi|^2 t}\hat{\psi_0}(\xi)\|^2 \\ &= \frac{1}{(2\pi)^n}\|\hat{\psi_0}(\xi)\|^2 && \text{(By Plancherel's Theorem)} \\ &= \|\psi_0(x)\|^2 = 1 \end{aligned}$$

Furthermore, we also have conservation of energy, that is, if we define

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\psi(t,x)|^2 dx = \frac{1}{2}\int_{\mathbb{R}^n} (|\partial_{x_1}\psi|^2 + \cdots + |\partial_{x_n}\psi|^2)dx_1\cdots dx_n,$$

then $E(t) = E(0)$ for all $t \geqslant 0$.

Proof. Take Fourier transform on $\|\partial_{x_i}\psi(t,x)\|^2$, we have

$$\begin{aligned}
\|\partial_{x_i}\psi(t,x)\|^2 &= \frac{1}{(2\pi)^n}\|i\xi_i\hat{\psi}(t,\xi)\|^2 \\
&= \frac{1}{(2\pi)^n}\|\xi_i e^{i\frac{|\xi|^2}{2}t}\hat{\psi}_0(\xi)\|^2 \\
&= \frac{1}{(2\pi)^n}\|\xi_i^2 e^{i\frac{|\xi|^2}{2}t}\hat{\psi}_0(\xi)\cdot e^{-i\frac{|\xi|^2}{2}t}\overline{\hat{\psi}_0(\xi)}\| \\
&= \frac{1}{(2\pi)^n}\|\xi_i\hat{\psi}_0(\xi)\|^2 \\
&= \|\partial_{x_i}\psi_0(t,x)\|^2
\end{aligned}$$

□

Next, we study the Schrödinger equation with potential $V(x):\mathbb{R}^n\to\mathbb{R}$,

$$\begin{cases} -i\partial_t\psi(t,x)+\frac{1}{2}\triangle\psi(t,x)-V(x)\psi(t,x)=0 \\ \psi(0,x)=\psi_0(x)\in L^2(\mathbb{R}^n) \end{cases}.$$

Remark 1.13. *Some notable choices to our potential function $V(x)$: $V(x)=\frac{1}{2}|x|^2$, quantum harmonic oscillators; $V(x)=\frac{-z}{|x|}$, potential energy given by atomic nuclear of the hydrogen atom.*

We claim that $\frac{d}{dt}\|\psi(t,x)\|^2_{L^2(\mathbb{R}^n)}=0$, i.e., the energy given by the L^2-norm of ψ is conserved.

Proof.

$$\begin{aligned}
\frac{d}{dt}\|\psi(t,x)\|^2 &= \frac{d}{dt}\int_{\mathbb{R}^n}\psi\bar{\psi}dx \\
&= \int_{\mathbb{R}^n}\partial_t\psi\bar{\psi}+\psi\partial_t\bar{\psi}dx \\
&= -\int_{\mathbb{R}^n}(\frac{i}{2}\triangle\psi-iV(x)\psi)\bar{\psi}+\psi(-\frac{i}{2}\Delta\bar{\psi}+iV(x)\bar{\psi})dx \\
&= -\int_{\mathbb{R}^n}\frac{i}{2}(\triangle\psi\bar{\psi}-\psi\triangle\bar{\psi})dx \\
&= \frac{i}{2}\int_{\mathbb{R}^n}\nabla\psi\nabla\bar{\psi}-\frac{i}{2}\int_{\partial\mathbb{R}^n}\frac{\partial\psi}{\partial\vec{n}}\cdot\bar{\psi}-\frac{i}{2}\int_{\mathbb{R}^n}\nabla\psi\nabla\bar{\psi}+\frac{i}{2}\int_{\partial\mathbb{R}^n}\frac{\partial\overline{\psi}}{\partial\vec{n}}\cdot\psi \quad \text{(Green's Formula)} \\
&= 0
\end{aligned}$$

The last equality holds as the two surfaces integral vanishes on $\partial\mathbb{R}^n$ by Elliptic Regularity Theorem. □

2 Sobolev Spaces

2.1 Convolution and smooth approximation

Definition 2.1 (Standard Mollifier). *We define the **standard mollifier** $\eta \in C^\infty(\mathbb{R}^n)$ to be the bump function*

$$\eta(x) := \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geqslant 1 \end{cases}.$$

One can choose $C > 0$, s.t., $\int_{R^n} \eta dx = 1$.

We define the η_ϵ as follows:

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right) \in C_c^\infty.$$

*We then call η the **mollifier** with parameter $\epsilon > 0$.*

Remark 2.2. *The support of $\eta_\epsilon \subset B(0,\epsilon)$. It is also easy to verify that*

$$\int_{\mathbb{R}^n} \eta_\epsilon dx = \int_{\mathbb{R}^n} \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right) dx = 1.$$

Remark 2.3. *In fact, it is not very hard to verify that $\{\eta_\epsilon\}_{\epsilon>0}$ is an approximation to identity.*

Definition 2.4 (Mollification). *If $f : U \to \mathbb{R}$ is in $L^p(U)$ (or more generally $L^p_{\text{loc}}(U)$), we define its **mollification** by*

$$f_\epsilon(x) = \eta_\epsilon * f(x).$$

That is

$$f_\epsilon(x) = \int \eta_\epsilon(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta_\epsilon(y)f(x-y)dy.$$

Theorem 2.5 (Property of Mollifiers). *Let $f : U \to \mathbb{R}$ be a $L^p_{\text{loc}}(U)$ function, then*

1. *$f_\epsilon \in C^\infty(U_\epsilon)$, not U since the support of f_ϵ is slightly larger (size ϵ) than the support of f. Where*

$$U_\epsilon = \{x \in U \,:\, d(x, \partial U) > \epsilon\}.$$

2. *$f_\epsilon \to f$ a.e. as $\epsilon \to 0$.*

3. *If $f \in C(U)$, then f_ϵ converges to f uniformly on any compact subset of U.*

4. *If $1 \leqslant p < \infty$ and $f \in L^p_{\text{loc}}(U)$, then $f_\epsilon \to f$ in $L^p_{\text{loc}}(U)$.*

Proof. Fix $x \in U_\epsilon$, $i \in \{1, \cdots, d\}$ and $x + he_i \in U_\epsilon$. We show that

$$\partial_{x_i}(\eta_\epsilon * f)(x) = (\partial_{x_i}\eta * f)(x)$$

then by induction, we can show that $f_\epsilon \in C^\infty$.

The difference quotient

$$\begin{aligned}
\frac{f_\epsilon(x+he_i)-f_\epsilon(x)}{h} &= \frac{1}{\epsilon^d}\int_U \frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right)-\eta\left(\frac{x-y}{\epsilon}\right)\right)f(y)dy \\
&= \frac{1}{\epsilon^d}\int_V \frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right)-\eta\left(\frac{x-y}{\epsilon}\right)\right)f(y)dy
\end{aligned}$$

for some open set $V \subset\subset U$ (if ϵ is small enough), since η is supported in $B(0,1)$. Then we have

$$\frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right)-\eta\left(\frac{x-y}{\epsilon}\right)\right) \to \frac{1}{\epsilon}\partial_{x_i}\eta\left(\frac{x-y}{\epsilon}\right)$$

in V uniformly, as $h \to 0$. In particular, by the boundedness of $\partial_{x_i} f_\epsilon(x)$, we may use dominated convergence. Hence the difference quotient converges to the limit

$$\int_U \partial_{x_i}\eta_\epsilon(x-y)f(y)dy.$$

Next, we prove pointwise convergence. By definition, we have

$$\begin{aligned}
|f_\epsilon(x)-f(x)| &= \left|\int_{B(x,\epsilon)} \eta_\epsilon(x-y)(f(y)-f(x))dy\right| \\
&\leqslant \frac{1}{\epsilon^d}\int_{B(x,\epsilon)} \eta\left(\frac{x-y}{\epsilon}\right)|f(y)-f(x)|dy \\
&\leqslant C \fint_{B(x,\epsilon)} |f(y)-f(x)|dy
\end{aligned}$$

The last term converges to 0 for a.e. $x \in U$ by the Lebesgue's Differentiation Theorem. Moreover, if f is continuous, then given any subset $V \subset\subset U$, we can choose an open set W such that $V \subset\subset W \subset\subset U$. Then f is uniformly continuous in $\overline{W}$, so when ϵ is small enough, the difference $|f_\epsilon(x)-f(x)|$ is uniformly bounded. Hence we have uniform convergence.

Lastly we prove the convergence in L^p_{loc}. Let $V \subset\subset U$, choose $V \subset\subset W \subset\subset U$. We first prove that

$f_\epsilon \in L^p_{\text{loc}}(U)$ for $1 \leqslant p < \infty$. Fix $x \in V$, we have

$$
\begin{aligned}
|f_\epsilon(x)| &= \left| \int_{B(x,\epsilon)} \eta_\epsilon(x-y) f(y) dy \right| \\
&\leqslant \int_{B(x,\epsilon)} \eta_\epsilon^{1-\frac{1}{p}} \eta_\epsilon^{\frac{1}{p}} |f(y)| dy \\
&\leqslant \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y) dy \right)^{1-\frac{1}{p}} \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y) |f(y)|^p dy \right)^{\frac{1}{p}} \\
&= 1 \cdot \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y) |f(y)|^p dy \right)^{\frac{1}{p}}
\end{aligned}
$$

Now for $f \in C(W)$, we integrate both sides on V and use Fubini's theorem to interchange the order of integration to get

$$
\begin{aligned}
\int_V |f_\epsilon(x)|^p dx &\leqslant \int_V \int_{B(x,\epsilon)} \eta_\epsilon(x-y) |f(y)|^p dy dx \\
&\leqslant \int_W |f(y)|^p \left(\int_{B(y,\epsilon)} \eta_\epsilon(x-y) dx \right) dy \\
&= \int_W |f|^p < \infty.
\end{aligned}
$$

In particular, we conclude

$$
\|f_\epsilon\|_{L^p(V)} \leqslant \|f\|_{L^p(W)}.
$$

when ϵ is small enough.

Finally, the convergence in L^p norm is obtained by approximating f by $g \in C(W)$. That is, fixing V, W as above, and $\delta > 0$, we can find $g \in C(W)$ such that $\|f - g\|_{L^p(W)} < \delta$. Then

$$
\begin{aligned}
\|f_\epsilon - f\|_{L^P(V)} &\leqslant \|f_\epsilon - g_\epsilon\|_{L^p(V)} + \|g_\epsilon - g\|_{L^p(V)} + \|g - f\|_{L^p(V)} \\
&\leqslant 2\|f - g\|_{L^p(W)} + \|g_\epsilon - g\|_{L^p(V)}
\end{aligned}
$$

Then

$$
\limsup_{\epsilon \to 0} \|f_\epsilon - f\|_{L^p(V)} \leqslant 2\delta.
$$

□

2.2 Sobolev Space With Enough Regular Functions

Given $f : \mathbb{R}^n \to \mathbb{R}$ and let $1 \leqslant p \leqslant \infty$.

Definition 2.6 ($L^p(\mathbb{R}^n)$)**.** *We say $f \in L^p(\mathbb{R}^n)$ if*

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f|^p dx_1 dx_2 \cdots dx_n\right)^{1/p} < +\infty$$

when p is a real number

$$\|f\|_{L^\infty(\mathbb{R}^n)} := \operatorname{esssup}_{x\in\mathbb{R}^n} |f(x)| < \infty$$

when $p = \infty$.

Definition 2.7 ($W^{1,p}(\mathbb{R}^n)$)**.** *We say that a function f belongs to the Sobolev space $W^{1,p}(\mathbb{R}^n)$ if*

$$\|f\|_{W^{1,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|Df\|_{L^p(\mathbb{R}^n)} < +\infty.$$

Remark 2.8. *If $f \in W^{1,p}(\mathbb{R}^n)$, then for $1 \leqslant i \leqslant n$, $\partial_{x_i} f \in L^p(\mathbb{R}^n)$.*

Definition 2.9 ($W^{k,p}(\mathbb{R}^n)$)**.** *We say that a function f belongs to the Sobolev space $W^{k,p}(\mathbb{R}^n)$ with $k \in \mathbb{N}^+$, if all its partial derivative up to order k belongs to $L^p(\mathbb{R}^n)$. In this case, we define its norm to be*

$$\|f\|_{W^{k,p}(\mathbb{R}^n)} := \sum_{s=0}^{k} \left(\int_{\mathbb{R}^n} |\nabla^s f|^p dx_1 dx_2 \cdots dx_n\right)^{1/p} < +\infty.$$

For the special case where $p = 2$, we will denote the space $W^{s,2}(\mathbb{R}^n)$ to be $H^s(\mathbb{R}^n)$ and the corresponding norm

$$\|f\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f|^2 dx_1 \cdots dx_n\right)^{1/2} + \left(\int_{\mathbb{R}^n} |\nabla f|^2 dx_1 \cdots dx_n\right)^{1/2} + \cdots + \left(\int_{\mathbb{R}^n} |\nabla^s f|^2 dx_1 \cdots dx_n\right)^{1/2}.$$

In this special case, $H^s(\mathbb{R}^n)$ in fact will be a Hilbert space.

Remark 2.10. *We can define $\nabla^s f$ in several ways, one of which is the sum up all the possible combination of partial derivatives of order s.*

Remark 2.11. *Notice $f \in H^s(\mathbb{R}^n)$ iff*

$$\int_{\mathbb{R}^n} |f|^2 dx_1 \cdots dx_n + \int_{\mathbb{R}^n} |\nabla f|^2 dx_1 \cdots dx_n + \cdots + \int_{\mathbb{R}^n} |\nabla^s f|^2 dx_1 \cdots dx_n < +\infty.$$

Then applying Fourier Transform, we have this is true iff

$$\int_{\mathbb{R}^n} [1 + |\xi|^2 + \cdots + |\xi|^{2s}][\hat{f}(\xi)]^2 d\xi_1 d\xi_2 \cdots d\xi_n < +\infty,$$

where $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$. By analysing the size of $|\xi|$, we realizes this happens if and only if

$$\int_{\mathbb{R}^n} [1 + |\xi|^{2s}][\hat{f}(\xi)]^2 d\xi_1 \cdots d\xi_n < +\infty \Leftrightarrow \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi_1 \cdots d\xi_n < +\infty.$$

In this way, we can also define fractional Sobolev space, that is when the number s is not necessarily an integer.

Definition 2.12 ($\dot{H}^s(\mathbb{R}^n)$)**.** *We say f belongs to the Homogeneous Sobolev Space $\dot{H}^s(\mathbb{R}^n)$ if*

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |\nabla^s f|^2 dx_1 \cdots dx_n\right)^{1/2} < +\infty.$$

In this case, we allows s to be non-integers, and we define the norm by

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}|^2 d\xi\right)^{1/2}.$$

Theorem 2.13 (Sobolev's Lemma)**.** *Let $s > \frac{n}{2}$, then*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leqslant c\|f\|_{H^s(\mathbb{R}^n)}.$$

This is to say that we can bounded the L^∞ norm of f by sacrificing derivative up to $\frac{n}{2}^{th}$ order.

Remark 2.14. *Recall that if $1 \leqslant p < q \leqslant +\infty$, $\|f\|_{L^p(\Omega)} \leqslant \|f\|_{L^q(\Omega)} \cdot C(\Omega)$, where C is a function based only on Ω. In particular, if Ω is bounded, then $L^q(\Omega) \subset L^p(\Omega)$.*

Proof. WLOG we may assume that $f \in C_c^\infty(\mathbb{R}^n)$, since C_c^∞ functions are dense in $H^s(\mathbb{R}^n)$. By Fourier Transform, we have:

$$\begin{aligned}
\|f(x)\|_{L^\infty(\mathbb{R}^n)} &= \left\|\int_{R^n} e^{ix\cdot\xi} \hat{f}(\xi) d\xi_1 d\xi_2 \cdots d\xi_n\right\|_{L^\infty(\mathbb{R}^n)} \\
&\lesssim \int_{\mathbb{R}^n} |\hat{f}(\xi)| d\xi \\
&= \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\frac{s}{2}} (1+|\xi|^2)^{\frac{s}{2}} |\hat{f}(\xi)| d\xi \\
&\leqslant \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi\right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&\equiv \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi\right)^{1/2} \cdot \|f\|_{H^s(\mathbb{R}^n)} \\
&\leqslant c\|f\|_{H^s(\mathbb{R}^n)}
\end{aligned}$$

The last line follows from the following:

$$\begin{aligned}
\int_{\mathbb{R}^n}(1+|\xi|^2)^{-s}d\xi &= \int_{B_1(0)}(1+|\xi|^2)^{-s}d\xi + \int_{\mathbb{R}^n-B_1(0)}(1+|\xi|^2)^{-s}d\xi \\
&\leqslant \int_{B_1(0)} 1d\xi + \int_{\mathbb{R}^n-B_1(0)} \frac{1}{|\xi|^{2s}}d\xi \\
&= |B_1(0)| + C\int_1^\infty \frac{1}{r^{2s}}r^{n-1}dr \\
&\text{(by changing into Spherical Coordinate)} \\
&= |B_1(0)| + C\int_1^\infty r^{n-1-2s}dr \\
&= |B_1(0)| + C' \qquad (2s > n)
\end{aligned}$$

□

Theorem 2.15. *If $s > k + \frac{n}{2}$, $k \in \mathbb{N}$, then*

$$\sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{L^\infty(\mathbb{R}^n)} \leqslant C_{s,n}\|f\|_{H^s(\mathbb{R}^n)} \quad (*)$$

where $C_{s,n}$ is independent of f (Only depends on s and n).

Proof. If $s > k + \frac{n}{2}$, then for the $\alpha' s$ derivative, $|\alpha| \leqslant k$, we have that $s - \alpha > \frac{n}{2}$. Hence we can apply Sobolev's lemma to get the result. □

Remark 2.16. *Given $s > k + \frac{n}{2}$, if $f \in H^s(\mathbb{R}^n)$, using $(*)$, we have*

$$\sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{L^\infty(\mathbb{R}^n)} \leqslant C_s\|f\|_{H^s(\mathbb{R}^n)} < +\infty.$$

Recall: $\|f\|_{C^k(\mathbb{R}^n)} := \sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{L^\infty(\mathbb{R}^n)}$ if f is sufficiently regular. Now if $\|f\|_X \leqslant C\|f\|_Y$, it implies $f \in Y$ then $f \in X$. Then by Theorem 2.15 given $s > k + \frac{n}{2}$, $f \in H^s(\mathbb{R}^n) \Rightarrow f \in C^k(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^n)$. This is sometimes called **Sobolev Embedding**.

2.3 Gagliardo-Nirenberg-Sobolev Inequality

In this section, we aim to prove the Gagliardo-Nirenberg-Sobolev Inequality, which is stated below: Assume $1 \leqslant p < n$, let $q = \frac{np}{n-p} > p$. Then there exists a constant $C_{p,n}$ independent of u such that

$$\|u\|_{L^q(\mathbb{R}^n)} \leqslant C_{p,n}\|Du\|_{L^p(\mathbb{R}^n)} \quad (*)$$

for all $u \in W^{1,p}(\mathbb{R}^n)$.

Remark 2.17. *The choice for q cannot be arbitrary, q must be of the form $\frac{np}{n-p} \Leftrightarrow 1 - \frac{n}{p} + \frac{n}{q} = 0$. As we will see from the following analysis.*

Choose a function $u \in C_c^\infty(\mathbb{R}^n)$, $u \not\equiv 0$. Define $u_\lambda(x) := u(\lambda x)$, $x \in \mathbb{R}^n$, $\lambda > 0$. Apply $(*)$ to $u_\lambda(x)$, we expect

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leqslant C_{p,n}\|D(u_\lambda)\|_{L^P(\mathbb{R}^n)} \quad (**)$$

Then

$$\begin{aligned}(LHS)^q &= \int_{\mathbb{R}^n} |u_\lambda|^q dx_1 dx_2 \cdots dx_n \\ &= \int_{\mathbb{R}^n} |u(\lambda x)|^q dx_1 \cdots dx_n \\ &= \int_{\mathbb{R}^n} \lambda^{-n}|u(\lambda x)|^q d(\lambda x_1) \cdots d(\lambda x_n) \\ &= \lambda^{-n} \int_{\mathbb{R}^n} |u(y)|^q dy_1 \cdots dy_n\end{aligned}$$

For the RHS without the coefficient, we have

$$\begin{aligned}(RHS)^p &= \int_{\mathbb{R}^n} |D(u_\lambda)|^p dx \\ &= \int_{\mathbb{R}^n} |D(u(\lambda x))|^p dx \\ &= \int_{\mathbb{R}^n} |\lambda \cdot (Du)(\lambda x)|^p dx \\ &= \int_{\mathbb{R}^n} \lambda^p \cdot \lambda^{-n}|(Du)(\lambda x)|dx|^p d(\lambda x_1) \cdots d(\lambda x_n) \\ &= \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |Du(y)|^p dy_1 \cdots dy_n\end{aligned}$$

Now we can rewrite above identities and we have

$$\lambda^{\frac{-n}{q}}\|u\|_{L^q(\mathbb{R}^n)} \leqslant C_{p,n} \cdot \frac{\lambda}{\lambda^{\frac{n}{p}}}\|Du\|_{L^p(\mathbb{R}^n)} \quad (***)$$

Suppose the inequality hold for $(*)$, then $(***)$ must also hold and vice versa. $(***)$ holds iff

$$\|u\|_{L^q(\mathbb{R}^n)} \leqslant C_{p,n}\lambda^{1-\frac{n}{p}+\frac{n}{q}}\|Du\|_{L_p(\mathbb{R}^n)}. \tag{1}$$

Since λ can be arbitrary positive number, then $\lambda^{1-\frac{n}{p}+\frac{n}{q}}$ must be 1, otherwise the write hand side of (1) can be arbitrarily small. Hence, for both inequality to holds at the same time, we must have $1 - \frac{n}{p} + \frac{n}{q} = 0$.

Definition 2.18 (Sobolev Conjugate)**.** *If $1 \leqslant p < n$, we define the* ***Sobolev Conjugate of*** *p to be*

$$p^* := \frac{np}{n-p} \;\; (> p).$$

Remark 2.19. *By definition we have*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Recall that by generalized Hölder's inequality, we have

$$\int_{\mathbb{R}^n} f_1 f_2 \cdots f_{n-1} dx \leqslant \|f_1\|_{L^{n-1}} \|f_2\|_{L^{n-1}} \cdots \|f_{n-1}\|_{L^{n-1}}.$$

Proof of Gagliardo-Nirenberg-Sobolev Inequality:

Step 1: Assume that $p = 1$, $n = 2$, then $p^* = 2$. We show in this special case that

$$\|u\|_{L^2(\mathbb{R}^2)} \lesssim \|Du\|_{L^1(\mathbb{R}^2)}.$$

Observe that

$$u(x_1, x_2) \leqslant \int_{-\infty}^{x} |(\partial_{x_1} u)(y_1, x_2) dy_1| \leqslant \int_{-\infty}^{\infty} |D_u|(y_1, x_2) dy_1.$$

Similarly, we have $u(x_1, x_2) = \int_{-\infty}^{\infty} |Du|(x_1, y_2) dy_2$.

Then

$$\begin{aligned}
\int_{-\infty}^{\infty} |u|^2 dx_1 &= \int_{-\infty}^{\infty} |u| \cdot |u| dx_1 \\
&\leqslant \int_{\infty}^{\infty} \left(\int_{-\infty}^{\infty} |Du|(y_1, x_2) dy_1 \right) \left(\int_{-\infty}^{\infty} |Du|(x_1, y_2) dy_2 \right) dx_1 \\
&= \int_{-\infty}^{\infty} |Du|(y_1, x_2) dy_1 \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|(x_1, y_2) dy_2 dx_1
\end{aligned}$$

Where the last line follows from Fubini's Theorem. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^2 dx_1 dx_2 &\leqslant \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|(y_1, x_2) dy_1 dx_2 \right) \cdot \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|(x_1, y_2) dy_2 dx_1 \right) \\
&= \|Du\|_{L^1(\mathbb{R}^2)}^2
\end{aligned}$$

Hence $\|u\|_{L^2(\mathbb{R}^n)} \leqslant \|Du\|_{L^1(\mathbb{R}^2)}$.

Step 2: Next, let $n \in \mathbb{Z}^+$ be arbitrary. We want to prove that

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n} \leqslant C \|Du\|_{L^1(\mathbb{R}^n)}.$$

Note we can approximate u with compact support functions, hence WLOG we can assume that u has compact support. Since u have compact support, for each $i = 1, \cdots, n$ and $x \in \mathbb{R}^n$, we have

$$u(x_1, x_2, \cdots, x_n) \leqslant \int_{-\infty}^{\infty} |D_u(x_1, \cdots, y_i, \cdots, x_n)| dy_i.$$

Then

$$
\begin{aligned}
\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx &\leqslant \int_{\mathbb{R}^n} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du|(x_1, \cdots, y_i, \cdots, x_n) dy_i \right)^{\frac{1}{n-1}} dx. \\
&= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du| dy_i \right)^{\frac{1}{n-1}} dx_1 dx_2 \cdots dx_n \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} |Du|(y_1, x_2, \cdots, x_n) dy_1 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du| dy_i \right)^{\frac{1}{n-1}} dx_1 \right) dx_2 \cdots dx_n \\
&\leqslant \int_{\mathbb{R}^{n-1}} \left(\int |Du| dy_1 \right)^{\frac{1}{n-1}} \cdot \prod_{i=2}^{n} \left(\int (\int |Du| dy_i) dx_1 \right)^{\frac{1}{n-1}} dx_2 \cdots dx_n \quad \text{(By Hölder's)} \\
&\text{let } I_1 = \int_{-\infty}^{\infty} |Du| dy_1 \;\; I_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i \\
&= \int_{\mathbb{R}^{n-1}} \left[I_1^{\frac{1}{n-1}} I_3^{\frac{1}{n-1}} \cdots I_n^{\frac{1}{n-1}} \left(\int \int |Du| dx_1 dy_2 \right)^{\frac{1}{n-1}} \right] dx_2 \cdots dx_n \\
&= \int_{\mathbb{R}^{n-2}} \left[(\int \int |Du| dx_1 dy_2)^{\frac{1}{n-1}} \cdot \int_{-\infty}^{\infty} \prod_{\substack{i=1 \\ i \neq 2}}^{n} I_i^{\frac{1}{n-1}} dx_2 \right] dx_3 \cdots dx_n \\
&\leqslant \int_{\mathbb{R}^{n-2}} \left[\int \int |Du| dx_1 dy_2)^{\frac{1}{n-1}} (\int \int |Du| dy_1 dx_2)^{\frac{1}{n-1}} \cdot \prod_{i=3}^{n} (\int \int \int |Du| dx_1 dx_2 dy_i)^{\frac{1}{n-1}} \right] dx_3 \cdots dx_n
\end{aligned}
$$

Continuing this manner, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx_1 \cdots dx_n \\
&\leqslant \prod_{i=1}^{n} \left(\int \cdots \int |Du| dx_1 \cdots dy_i \cdots dx_n \right)^{\frac{1}{n-1}} \\
&= \left(\int_{\mathbb{R}^n} |Du| dx \right)^{\frac{n}{n-1}}
\end{aligned}
$$

Hence we arrive at

$$
\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} = \left(\int \int \cdots \int |u|^{\frac{n}{n-1}} dx_1 \cdots dx_n \right)^{\frac{n-1}{n}} \lesssim \|Du\|_{L^1(\mathbb{R}^n)} \quad (\bigstar).
$$

Step 3: now for $1 < p < n$, we apply ($\bigstar$) to $|u|^\gamma$ with γ to be selected. We have,

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} dx\right)^{\frac{n-1}{n}} &\leqslant C_n \int_{\mathbb{R}^n} |D|u|^\gamma| dx \\
&= C_n \cdot \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |Du| dx && \text{(chain rule)} \\
&\leqslant C_n \cdot \gamma \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\cdot\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\
&(\|fg\|_{L^1} \leqslant \|g\|_{L^p}\|f\|_{L^{p'}},\ \frac{1}{p} + \frac{1}{p'} = 1.)
\end{aligned}$$

Consider choosing γ, such that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1}$, so $\gamma = \frac{p(n-1)}{n-p} > 1$. Then $\frac{\gamma n}{n-1} = \frac{p(n-1)}{n-p} \cdot \frac{n}{n-1} = \frac{np}{n-p} = p^*$. Then we get

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{n-1}{n} - \frac{p-1}{p}} \leqslant C_n \cdot \gamma \cdot \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{\frac{1}{p}}.$$

Observe

$$\frac{n-1}{n} - \frac{p-1}{p} = \frac{p(n-1) - n(p-1)}{np} = \frac{np - p - np + n}{np} = \frac{n-p}{np} = \frac{1}{p^*}.$$

Hence we conclude that

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{1}{p^*}} \leqslant C_n \cdot \gamma \cdot \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{\frac{1}{p}}$$

Which gives the desired inequality.

Remark 2.20. *We comment that for $n > 1$, consider $N = B(0,1)$, and $u = \log\log(1 + \frac{1}{|x|})$, one can check that $u \in W^{1,n}(N)$, but u is not in $L^\infty(N)$. And it is not true that*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leqslant C_n \|Du\|_{L^n(\mathbb{R}^n)}.$$

So the Gagliardo-Nirenberg-Sobolev Inequality does not hold when $p = n$.

Theorem 2.21. *Suppose $U \subset \mathbb{R}^n$ is bounded and open, with $\partial U \in C^1$. Then if $u \in W^{1,p}(U)$ with $1 \leqslant p < n$, $u \in L^{p^*}(U)$ with the estimate*

$$\|u\|_{L^{p^*}(U)} \leqslant C \cdot \|u\|_{W^{1,p}(U)}$$

where C depends only on p, n and u.

Proof. Since ∂U is C^1, by the Sobolev extension theorem, there exists an extension $E_u = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ such that

1. $\bar{u} = u$ in U;
2. $\bar{u}$ has compact support in a larger region;
3. $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C \cdot \|u\|_{W^{1,p}(U)}$.

Because $\bar{u}$ has compact support and $\bar{u} \in W^{1,p}(\mathbb{R}^n)$, there exists a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ $(m = 1, 2, \cdots)$ such that $u_m \to \bar{u}$ in $W^{1,p}(\mathbb{R}^n)$. Now we use the G-N-S for $u_m - u_l$ and we obtain

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C \cdot \|Du_m - Du_l\|_{L^p(\mathbb{R}^n)} \text{ for all } l, m \geqslant 1.$$

Thus u_m is also a Cauchy sequence in $L^{p^*}(\mathbb{R}^n)$ and $u_m \to \bar{u}$ in $L^{p^*}(\mathbb{R}^n)$. Now we use G-N-S again for u_m and we have

$$\frac{1}{2}\|u\|_{L^{p^*}(U)} \leqslant \frac{1}{2}\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leqslant \lim \|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leqslant \lim C \cdot \|Du_m\|_{L^p(\mathbb{R}^n)} \leqslant 2C\|D\bar{u}\|_{L^p(\mathbb{R}^n)} \leqslant \tilde{C}\|u\|_{W^{1,p}(U)}.$$

□

2.4 Distributional and Weak Derivative

Definition 2.22 ($L^p_{loc}(U)$)**.** *We say $f \in L^p_{loc}(U)$, if for any compact subset $K \subset U \subset \mathbb{R}^n$, $f \in L^p(K)$.*

Example:

1. $\frac{1}{x}$ is not in $L^1([1, +\infty))$, but $\frac{1}{x}$ is in $L^1_{loc}([1, +\infty))$
2. e^x and $\ln|x|$ $in L^1_{loc}(\mathbb{R})$.
3. $x^{-1} \notin L^1_{loc}(\mathbb{R})$.
4. $x^\gamma \in L^1_{loc}((0, +\infty))$ for all $\gamma \in \mathbb{R}$
5. $|x|^{-\gamma} \in L^1_{loc}(\mathbb{R}^n)$ iff $\gamma < n$.

.

Notation: we let $C_C^\infty(\Omega)$ denote the set of all smooth function on $\Omega \subset \mathbb{R}^n$ with compact support. We use $\text{supp}(\phi)$ to denote the closure of $\{x : \phi(x) \neq 0\}$.

Definition 2.23 (Distribution)**.** *A **distribution** on $\Omega \subseteq \mathbb{R}^n$ is a linear functional Λ defined on $C_c^\infty(\Omega)$, such that $\forall$ compact $K \subseteq \Omega$, $\exists N \geqslant 0$, $\exists c > 0$ such that $|\Lambda(\phi)| \leqslant C\|\phi\|_{C^N}$, where* $\text{supp}(\phi) \subset K$.

Definition 2.24 (Λ_f)**.** *Given $f \in L^1_{loc}(\mathbb{R})$, we define a linear functional, $\Lambda_f : C_C^\infty(\mathbb{R}) \to \mathbb{R}$ by*

$$\Lambda_f(\phi) := \int_{\mathbb{R}} f(x)\phi(x)dx, \quad \forall \phi \in C_C^\infty(\mathbb{R})$$

Remark 2.25. *If* $\text{supp}(\phi) \subseteq [a, b]$, *we have*

$$|\Lambda_f(\phi)| = |\int_{\mathbb{R}} f(x)\phi(x)dx| \leqslant \int_a^b |f(x)|dx \cdot \|\phi\|_{C^0}.$$

Λ_f *is well-defined. And clearly, it is linear:*

$$\Lambda_f(\lambda_1\phi_1 + \lambda_2\phi_2) = \lambda_1[\Lambda f(\phi_1)] + \lambda_2[\Lambda_f(\phi_2)].$$

Consider the case where f is continuously differentiable. Then f' also defines a linear functional on $C_C^\infty(\mathbb{R})$, namely,

$$\Lambda_{f'}(\phi) := \int_{\mathbb{R}} f'(x)\phi(x)dx = \phi(x)f(x)|_{-\infty}^{\infty} - \int_{\mathbb{R}} f(x)\phi'(x)dx = -\int_{\mathbb{R}} f(x)\phi'(x)dx.$$

Since supp$(\phi) \subseteq [a,b]$, we have $|\Lambda_{f'}(\phi)| \leqslant (\int_a^b |f(x)|dx) \cdot \|\phi\|_{C^1}$. This gives rise to the definition of distributional and weak derivative.

Definition 2.26 (Distributional and Weak Derivative on $\mathbb{R}^1$)**.** *Given $k \in \mathbb{N}^+$, the **distributional derivative of order** k **of** $f \in L^1_{loc}(\mathbb{R})$ is defined to be the linear functional*

$$\Lambda_{D^k f}(\phi) := (-1)^k \int_{\mathbb{R}} f(x) D^k\phi(x)dx.$$

If there exists a $g \in L^1_{loc}(\mathbb{R})$ such that $\Lambda_g = \Lambda_{D^k f}$, namely,

$$\int_R g(x)\phi(x)dx = (-1)^k \int_{\mathbb{R}} f(x)D^k\phi(x)dx$$

*for all $\phi \in C_C^\infty(\mathbb{R})$, then we call g to be the **weak derivative of order** k **of** f.*

Notation: multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is an $n-$tuple, its length is $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Then

$$D^\alpha f := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f.$$

Definition 2.27 (Distributional and Weak Derivative on $\mathbb{R}^n$)**.** *let $f \in L^1_{loc}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ and Λ_f be the corresponding distribution. Given a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$. We define the $\alpha-$**th distributional derivative** to be the linear functional*

$$D^\alpha \Lambda_f(\phi) := (-1)^{|\alpha|}\Lambda_f(D^\alpha \phi).$$

*We say that $g \in L^1_{loc}(\Omega)$ is the $\alpha-$**th weak derivative of** f if and only if $D^\alpha \Lambda_f = \Lambda_g$, i.e.,*

$$\int f D^\alpha \phi dx = (-1)^{|\alpha|} \int g\phi dx \text{ for all } \phi \in C_C^\infty(\Omega).$$

Remark 2.28. *For every $f \in L^1_{loc}(\Omega)$ and every multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, the distribution Λf always admits distributional derivative (by the next lemma we in fact can show that the distributional derivative exists for arbitrary distribution). However, the weak derivative may not exist.*

Lemma 2.29. *Let Λ be an arbitrary distribution, then $D^\alpha \Lambda$ is a distribution.*

Proof. Λ is linear, and D^α is also linear, then $(-1)^{|\alpha|}\Lambda \circ D^\alpha$ is also linear. So we just need to show $D^\alpha \Lambda$ is bounded.
Let K be any compact set in Ω and let ϕ be a test function with support contained in K,

$$|(D^\alpha \Lambda)(\phi)| = |(-1)^{|\alpha|}\Lambda(D^\alpha \phi)| = |\Lambda(D^\alpha \phi)| \leqslant C \cdot \|D^\alpha \phi\|_{C^N} \leqslant C \cdot \|\phi\|_{C^{N+|\alpha|}}.$$

□

Lemma 2.30 (Uniqueness of Weak Derivative)**.** *Assume $f \in L^1_{loc}(\Omega)$ and let $g, \tilde{g} \in L^1_{loc}(\Omega)$ be the $\alpha-$th weak derivative of f, so that*

$$\int f D^\alpha \phi dx = (-1)^{|\alpha|} \int g\phi dx = (-1)^{|\alpha|} \int \tilde{g}\phi dx$$

for all test functions $\phi \in C_c^\infty(n)$. Then $g(x) = \tilde{g}(x)$ for almost everywhere $x \in \Omega$.

Proof. By assumption, $g - \tilde{g} \in L^1_{loc}(\Omega)$ is such that $\int (g - \tilde{g})\phi dx = 0$ for all $\phi \in C_c^\infty(\Omega)$. Then by mollification, we would have $g - \tilde{g} = 0$ a.e. □

Lemma 2.31. *Assume that $f \in L^1_{loc}(\Omega)$ has weak derivative $D^\alpha f$ for every $|\alpha| \leqslant k$. Then for every par of multi-indices, α, β, with $|\alpha| + |\beta| \leqslant k$, one has*

$$D^\alpha(D^\beta f) = D^\beta(D^\alpha f) = D^{\alpha+\beta} f.$$

Proof. For any $\phi \in C_C^\infty(\Omega)$, $D^\beta(\phi) \in C_C^\infty(\Omega)$, then

$$\begin{aligned}
\int_\Omega D^\alpha f D^\beta \phi dx &\overset{\text{Def } D^\alpha f}{=} (-1)^{|\alpha|} \int_\Omega f(D^{\alpha+\beta}\phi) dx \\
&\overset{\text{Def } D^{\alpha+\beta} f}{=} (-1)^{|\alpha|}(-1)^{|\alpha|+|\beta|} \int_\Omega (D^{\alpha+\beta} f)\phi dx. \\
\Rightarrow \ D^\beta(D^\alpha f) &= D^{\alpha+\beta} f
\end{aligned}$$

By symmetry we have $D^\alpha(D^\beta f) = D^{\beta+\alpha} f$. Since $\alpha + \beta = \beta + \alpha$, then $D^{\alpha+\beta} f = D^{\beta+\alpha} f$ concludes the proof of the lemma. □

Lemma 2.32 (Convergence of weak derivatives)**.** *Consider a sequence of functions $f_n \in L^1_{loc}(\Omega)$, for a fixed multi-index α, assume that f_n admits the weak derivative $g_n = D^\alpha f_n$. If $f_n \to f$ and $g_n \to g$ in $L^1_{loc}(\Omega)$. Then $g = D^\alpha f$.*

Proof. For every $\phi \in C_c^\infty(\Omega)$, since $g_n \to g$ in $L^1_{loc}(\Omega)$, we have

$$\begin{aligned}
\left| \int_\Omega g\phi dx - \lim_{n\to\infty} \int_\Omega g_n \phi dx \right| &= \left| \lim_{n\to\infty} \int_\Omega (g - g_n)\phi dx \right| \\
&\leqslant \lim_{n\to\infty} \|\phi\|_{C^0} \int_K |g - g_n| dx \\
&\to 0 \ \text{ as } n \to \infty.
\end{aligned}$$

So we have

$$\begin{aligned}\int_\Omega g\phi dx &= \lim_{n\to\infty}\int_\Omega g_n\phi dx\\ &= \lim_{n\to\infty}(-1)^{|\alpha|}\int_\Omega f_n D^\alpha\phi dx\\ &= (-1)^{|\alpha|}\lim_{n\to\infty}\int_\Omega f D^\alpha\phi dx,\end{aligned}$$

Where the last line follows from a similar argument. Hence

$$g = D^\alpha f$$

is the weak derivative. □

2.5 Definition of Sobolev Spaces

Definition 2.33 (Sobolev Space $W^{k,p}$)**.** *Consider an open set $\Omega \subseteq \mathbb{R}^n$, fix $p \in [1+\infty]$, and let k be a non-negative integer. The* ***Sobolev Space*** *$W^{k,p}(\Omega)$ is the space of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for every multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \leqslant k$, the weak derivative $D^\alpha u$ exists and belongs to $L^P(\Omega)$.*

Moreover, we define

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|\leqslant k}\int_\Omega |D^\alpha u|^p dx\right)^{1/p}$$

if $1 \leqslant p < \infty$. If $p = \infty$, then

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha|\leqslant k} \operatorname{esssup}_{x\in\Omega} |D^\alpha u| = \sum_{|\alpha|\leqslant k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

where ess sup *is the essential supremum.*

Remark 2.34. *Given $f \in L^p(\Omega)$. Let $U_f^{ess} := \{a \in \mathbb{R} \cup \{+\infty\} : \mu(f^{-1}(a,\infty)) = 0\}$, i.e., the set of essential upper bound. Then if $a \in U_f^{ess}$, it means that, with μ, as the measure, $f^{-1}(a,\infty)$ is a set with measure zero. The* ***essential supremum*** *of f is defined to be*

$$\operatorname{esssup} f := \inf U_f^{ess}.$$

Remark 2.35. *The following norm is equivalent to the Sobolev norm in the definition when $1 \leqslant p < \infty$:*

$$\|u\|'_{W^{k,p}(\Omega)} = \sum_{|\alpha|\leqslant k} \|D^\alpha u\|_{L^p(\Omega)}.$$

When $p = \infty$, the following norm is equivalent to the norm in the definition:

$$\|u\|'_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \leqslant k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

Remark 2.36. *When $p = 2$, $H^k(\Omega) := W^{k,2}(\Omega)$ is a Hilbert Space and when endowed with the following inner product. Given $u, v \in H^k(\Omega)$, define their inner product by*

$$\langle u, v \rangle_{H^k} := \sum_{|\alpha| \leqslant k} \int_\Omega D^\alpha u D^\alpha v dx.$$

When $k = 0$, $H^0(\Omega) = L^2(\Omega)$ is a Hilbert Space, and the inner product of $f, g \in L^2(\Omega)$ is just $\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f \cdot g dx$.

Definition 2.37 ($W_0^{k,p}(\Omega)$)**.** *Let $\Omega \subseteq \mathbb{R}^n$. The space $W_0^{k,p}(\Omega) \subseteq W^{k,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. I.e., $u \in W_0^{k,p}(\Omega)$ if and only if there exists a sequence of functions $u \in C_c^\infty(\Omega)$, s.t., $\|u - u_n\|_{W^{k,p}(\Omega)} \to 0$.*

We let $W_{loc}^{k,p}(\Omega)$ denotes the space of functions which are locally in $W^{k,p}$, that is, $f \in W_{loc}^{k,p}$ if and only if for every $K \subseteq \Omega$, K compact, $f \in W^{k,p}(\bar{K})$.

Example: Let $\Omega = B(0,1)$, then $f(x) \equiv 1 \in W^{k,p}(\Omega)$, however by the Zero-Trace Theorem $f(x) \notin W_0^{k,p}(\Omega)$.

Theorem 2.38.

1. *Each Sobolev space $W^{k,p}(\Omega)$ with $k \in \mathbb{N}$ and $1 \leqslant p \leqslant +\infty$ is a Banach spaces.*
2. *The space $W_0^{k,p}(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$. Hence it is a Banach space, with the same norm.*
3. *The space $H^k(\Omega)$ and $H_0^k(\Omega) = W_0^{k,2}(\Omega)$ are Hilbert Space.*

Proof.

1. A Cauchy sequence in $W^{k,p}(\Omega)$ will be Cauchy in $L^p(\Omega)$ which admits a limit. Then make use of Proposition (2.32).
2. $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$, hence is closed.
3. $H^k(\Omega)$ is a Banach space, and its norm is precisely induced by the inner product we defined. $H_0^k(\Omega)$ is the closed subspace of a Hilbert space.

$\square$

2.6 Sobolev Inequality

First type of Embedding (when $p < n$): by Gagliardo-Nirenberg-Sobolev inequality, with $\Omega \subseteq \mathbb{R}^n$ and $1 \leqslant p < n$, we have

$$\|u\|_{L^{p^*}(\Omega)} \leqslant \|u\|_{W^{1,p}(\Omega)}$$

where $p^* = \frac{np}{n-p}$. Hence, we have the embedding $W^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$.

Definition 2.39 (Hölder Space)**.** *Let $\Omega \subset \mathbb{R}^n$, we define the* ***Hölder Space*** *$C^{0,\gamma}(\Omega)$ to be the spaces of all bounded continuous functions u defined on Ω such that*

$$\sup_{\substack{x \neq y \\ x,y \in \Omega}} \left[\frac{|u(x) - u(y)|}{|x - y|^\gamma} \right] < \infty.$$

In particular, any $u \in C^{0,\gamma}(\Omega)$ is uniformly continuous, as

$$|u(x) - u(y)| \leqslant |x - y|^\gamma \cdot k.$$

For functions in the $C^{0,\gamma}(\Omega)$, we define the ***Hölder norm*** *to be*

$$\|u\|_{C^{0,\gamma}(\Omega)} := \sup_{\substack{x \neq y \\ x,y \in \Omega}} \left[\frac{|u(x) - u(y)|}{|x - y|^\gamma} \right] + \|u\|_{C^0(\Omega)}.$$

Remark 2.40. *In the special case where $\gamma = 1$, then the Hölder space $C^{0,1}(\Omega)$ is precisely the space of Lipschitz functions.*

Second type of Embedding (when $p > n$): for $n < p \leqslant \infty$, with $\gamma = 1 - \frac{n}{p} > 0$, we have $W^{1,p}(\Omega) \cap C^1(\mathbb{R}^n) \subseteq C^{0,\gamma}(\mathbb{R}^n)$. This is guaranteed by Morrey's Inequality.

Theorem 2.41 (Morrey's Inequality)**.** *Assume $n < p \leqslant \infty$, then there exists a constant $C_{p,n}$, s.t.,*

$$\|u\|_{C^{0,\gamma}(\Omega)} \leqslant C_{p,n} \cdot \|u\|_{W^{1,p}(\Omega)}$$

for all $u \in L^p(\Omega) \cap C^1(\mathbb{R})$, where $\gamma := 1 - \frac{n}{p}$.

Proof. We first consider the case where $n = 1$, for $p > 1$ and $u(x) \in W^{1,p}(\mathbb{R})$. WLOG, we may assume

$u \in C^{\infty}$, then we have

$$\begin{aligned}
u(y) - u(x) &= \int_x^y u'(z)dz \\
|u(y) - u(x)| &\leqslant \int_x^y |u'(z)|dz \\
&\leqslant \left(\int_a^b |u'(z)|^p dz\right)^{1/p} \left(\int_x^y 1dz\right)^{1-1/p} \\
&\leqslant \|u\|_{W^{1,p}(\mathbb{R})} \cdot |y-x|^{1-1/p} \\
&= \|u\|_{W^{1,p}(\mathbb{R})} \cdot |y-x|^{1-1/p}
\end{aligned}$$

Hence

$$\frac{|u(x) - u(y)|}{|x-y|^{1-\frac{1}{p}}} \leqslant \|u\|_{W^{1,p}(\mathbb{R})}.$$

Next, we prove the inequality for the case $n \geqslant 2$. We first make the following claim:

Claim: Choose any ball $B(x,r) \subset \mathbb{R}^n$, there exists a constant C depending only on n, such that

$$⨍_{B(x,r)} |u(y) - u(x)|dy \leqslant C_n \cdot \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}} dy.$$

proof of claim: For every $y \in B(x,r)$, $y = x + sw$ for some $w \in S^{n-1}$ and $0 \leqslant s < r$.

$$\begin{aligned}
|u(x+sw) - u(x)| &= |\int_0^s \frac{d}{dt} u(x+tw)dt| \\
&= |\int_0^s Du(x+tw) \cdot wdt| \\
&\leqslant \int_0^s |Du(x+tw)|dt \\
\text{Hence } \int_{\partial B(0,1)} |u(x+sw) - u(x)|dS^{n-1} &\leqslant \int_0^s \int_{\partial B(0,1)} |Du(x+tw)|dS^{n-1}dt \\
&= \int_0^s \int_{\partial B(0,1)} |Du(x+tw)| \cdot \frac{t^{n-1}}{t^{n-1}} dS^{n-1}dt
\end{aligned}$$

Let $y = x + tw$, so $t = |y - x|$, then

$$\begin{aligned}
\int_0^s \int_{\partial B(0,1)} |Du(x+tw)| \cdot \frac{t^{n-1}}{t^{n-1}} dS^{n-1}dt &= \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}} dy \\
&\leqslant \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy \qquad (*)
\end{aligned}$$

Multiply s^{n-1} on both sides of $(*)$ and integrate from 0 to r with respect to s, then

$$\int_0^r s^{n-1}\int_{\partial B(0,1)}|u(x+sw)-u(x)|dS^{n-1}ds \leqslant \int_0^r s^{n-1}\int_{B(x,r)}\frac{|Du(y)|}{|x-y|^{n-1}}dyds$$
$$\int_{B(x,r)}|u(y)-u(x)|dy \leqslant \frac{r^n}{n}\int_{B(x,r)}\frac{|Du(y)|}{|x-y|^{n-1}}dy$$

This finishes the proof of the claim.

Now fix arbitrary $x \in \mathbb{R}^n$ and fix $r \in \mathbb{R}$. We have

$$\begin{aligned}
|u(x)| &= \fint_{B(x,r)} u(x)dy \\
&= \frac{\int_{B(x,r)} u(x)-u(y)+u(y)dy}{|B(x,r)|} \\
&\leqslant \frac{\int_{B(x,r)}|u(x)-u(y)|dy}{|B(x,r)|}+\frac{\int_{B(x,r)} u(y)dy}{|B(x,r)|} \\
&\lesssim \int_{B(x,r)}\frac{|Du(y)|}{|x-y|^{n-1}}dy + C\|u\|_{L^p(\Omega)} && \text{(By Above Claim)} \\
&\leqslant \left(\int_{B(x,r)}|Du(y)|^p dy\right)^{\frac{1}{p}}\cdot\left(\int_{B(x,r)}\frac{dy}{|x-y|^{(n-1)\cdot\frac{p}{p-1}}}\right)^{\frac{p-1}{p}} + C\|u\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

We need to check that $(n-1)\cdot\frac{p}{p-1} < n$ so the integral converges. Since $p > n$, then $pn-p < np-n$, then $\frac{p}{p-1} < \frac{n}{n-1}$, so $(n-1)\cdot\frac{p}{p-1} < n$. So $|u(x)| \leqslant C_{n,p}(\|Du\|_{L^p(\Omega)} + \|u\|_{L^p(\mathbb{R}^n)})$. This proves that $\|u\|_{C^0(\Omega)} < \infty$ can be controlled by the $W^{1,p}$-norm, in particular, u is bounded.

Next, we fix any two points $x, y \in \mathbb{R}^n$, and write $r := |x-y|$, and let $W = B(x,r)\cap B(y,r)$. Then we have

$$u(x)-u(y) = \frac{\int_W [(u(x)-u(y)]dz}{|W|}$$

This implies

$$|u(x)-u(y)| \leqslant \frac{\int_W |u(x)-u(z)|dz}{|W|} + \frac{\int_W |u(y)-u(z)|dz}{|W|}$$

But then

$$\begin{aligned}
\frac{\int_W |u(x)-u(z)|dz}{|W|} &\leqslant c\cdot\frac{\int_{B(x,r)}|u(x)-u(z)|dz}{|B(x,r)|} \\
&\leqslant c'\int_{B(x,r)}\frac{|Du(z)|}{|x-z|^{n-1}}dz && \text{(By Above Claim)} \\
&\leqslant c'\left(\int_{B(x,r)}|Du|^p dz\right)^{\frac{1}{p}}\cdot\left(\int_{B(x,r)}\frac{dz}{|x-z|^{(n-1)\frac{p}{p-1}}}\right)^{\frac{p-1}{p}} && \text{(Spherical Coordinate)} \\
&\leqslant c'\|Du\|_{L^p(\mathbb{R}^n)}\cdot r^{1-\frac{n}{p}} \\
&\leqslant c'\|u\|_{W^{1,p}(\mathbb{R}^n)}|x-y|^{\gamma}
\end{aligned}$$

Similarly, we can do the estimate for the second term. Combining the two parts, we would get

$$\frac{|u(x)-u(y)|}{|x-y|^{\gamma}}\leqslant C'\cdot\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

We note that the above proof also holds for the case where $p=\infty$, its just when we apply the Hölder's inequality, we use the Hölder conjugates ∞ and 1. Hence together with the bounded on $\|u\|_{C^0(\mathbb{R}^n)}$, we conclude that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)}\leqslant C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

□

Definition 2.42 (Compact Embedding)**.** *Let X and Y be Banach spaces. For $X\subset Y$, we say that X* ***is compactly embedded in*** *Y, written $X\subset\subset Y$, provided*

1. *$\|x\|_Y\leqslant C\cdot\|x\|_X$, $(x\in X)$ for some constant C.*
2. *If $\{u_m\}_{m=1}^{\infty}$ is a bounded sequence in X, then there exists a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$, which converges in Y. If this happens, we say each bounded sequence in X is precompact in Y.*

Theorem 2.43 (Rellich-Kondrachev Compactness Theorem)**.** *Assume U is a bounded open subset of $\mathbb{R}^n$ and ∂U is C^1. Suppose $1\leqslant p<n$. Then $W^{1,p}(U)\subset\subset L^q(U)$ for each $1\leqslant q<p^*$.*

Proof. Fix $1\leqslant q<p^*$ and we have $W^{1,p}(U)\subset L^{p^*}(U)\subset L^q(U)$, $\|u\|_{L^q(U)}\leqslant C\|u\|_{W^{1,p}}(U)$, by G-N-S inequality and Hölder's Inequality.

Using Sobolev Extension Theorem, WLOG, we can assume $U=\mathbb{R}^n$ and $\{u_m\}_{m=1}^{\infty}$ all have compact support in some bounded open set $V\subset\mathbb{R}^n$. We also have $\sup\limits_m\|u_m\|_{W^{1,p}(V)}<\infty$.

We define the smooth functions $u_m^{\epsilon}:=\eta_{\epsilon}*u_m$ $(\epsilon>0,\ m=1,2,\cdots)$. Where η_ϵ is the standard mollifier. We may suppose the functions $\{u_m^{\epsilon}\}_{m=1}^{\infty}$ all have support in V as well. By Hölder's Inequality, we still have $\sup\limits_m\|u_m^{\epsilon}\|_{W^{1,p}(V)}<+\infty$. We find convergent subsequences for $\{u_m^{\epsilon}(x)\}$ first. Then use them to construct a convergent subsequence for $\{u_m(x)\}$.

We show $u_m^\epsilon \to u_m$ in $L^q(V)$ as $\epsilon \to 0$ uniformly in m (We know for each m, $u_m^\epsilon \to u_m$ in $L^q(V)$). To prove this, we first note if $u_m \in W^{1,p}(V)$, then for a.e. $x \in V$, we have

$$\begin{aligned} u_m^\epsilon(x) - u_m(x) &= \int_{B(0,1)} \eta(y)(u_m(x-\epsilon y) - u_m(x))dy \\ &= \int_{B(0,1)} \eta(y)\left[\int_0^1 \frac{d}{dt}(u_m(x-\epsilon ty))dt\right]dy \\ &= -\epsilon\int_{B(0,1)} \eta(y)\left[\int_0^1 Du_m(x-\epsilon ty)\cdot ydt\right]dy \end{aligned}$$

Thus by Fubini's Theorem, we have

$$\int_V |u_m^\epsilon(x) - u_m(x)|dx \leqslant \epsilon\int_{B(0,1)} \eta(y)\int_0^1 \left(\int_V |Du_m(x-\epsilon ty)|dx\right)dtdy.$$

Since u_m is compact support, then the inner integral is no more than $\int_V |Du_m(z)|dz$, so we have

$$\int_V |u_m^\epsilon(x) - u_m(x)|dx \lesssim \epsilon\int_V |Du_m(z)|dz$$

Hence,

$$\|u_m^\epsilon - u_m\|_{L^1(V)} \lesssim \epsilon\|Du_m\|_{L^1(v)} \lesssim \epsilon\|Du_m\|_{L^p(v)}.$$

This shows that $u_m^\epsilon \to u_m$ in $L^1(V)$ uniformly in m. Since $1 \leqslant q < p^*$, by the interpolation inequality for L^p-norms, we have $\|f\|_{L^q(V)} \leqslant \|f\|_{L^1(V)}^\theta \|f\|_{L^{p*}(V)}^{1-\theta}$, $1 \leqslant q < p^*$, where $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$, $0 < \theta < 1$. So

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leqslant \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta \|u_m^\epsilon - u_m\|_{L^{p*}(V)}^{1-\theta}.$$

The G-N-S inequality gives that

$$\|u_m^\epsilon - u_m\|_{L^{p*}(V)} \leqslant D\|u_m^\epsilon - u_m\|_{W^{1,p}(V)}.$$

Since

$$\|u_m^\epsilon - u_m\|_{W^{1,p}(V)} \leqslant \|u_m^\epsilon\|_{W^{1,p}(V)} + \|u_m\|_{W^{1,p}(V)}$$

The last term is uniformly bounded by assumption. The term $\|u_m^\epsilon\|_{W^{1,p}(V)}$ is bounded by a constant multiple of $\|u_m\|_{W^{1,p}(V)}$ using Fubini's Theorem. Hence we conclude that

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leqslant C\epsilon^\theta.$$

Since $p < p^*$, then $\theta > 0$, so we havethat $u_m^\epsilon \to u_m$ in $L^p(V)$ uniformly in m.

We claim for each fixed $\epsilon > 0$, the sequence $\{u_m^\epsilon\}_{m=1}^\infty$ is uniformly bounded and equicontinuous. So the assumption for Arzela Ascoli Theorem is met.

For $x \in \mathbb{R}^n$, then

$$\begin{aligned} |u_m^\epsilon(x)| &\leqslant \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u_m(y)|dy \\ &\leqslant \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \\ &\leqslant \frac{C}{\epsilon^n} \cdot \|u_m\|_{L^p(V)} \\ &\leqslant \frac{C'}{\epsilon^n} < \infty \end{aligned}$$

On the other hand,

$$\begin{aligned} |Du_m^\epsilon(x)| &\leqslant \int_{B(x,\epsilon)} |D\eta_\epsilon(x-y)||u_m(y)|dy \\ &\leqslant \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \cdot \|u_m\|_{L^1(V)} \\ &\leqslant \frac{C}{\epsilon^{n+1}} \cdot \|u_n\|_{L^p(V)} \\ &\leqslant \frac{C'}{\epsilon^{n+1}} < \infty \end{aligned}$$

which shows that $\{u_m^\epsilon\}$ is equicontinuous.

Then by Arzela-Ascoli Theorem, we are able to obtain a convergent subsequence $\{u_{m_j}^\epsilon\}_{j=1}^\infty$ converges uniformly on V and

$$\limsup_{j,k\to\infty} \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q(V)} = 0.$$

Recall that or $1 \leqslant q < p^*$, we have

$$\|u_{m_j}^\epsilon - u_{m_j}\|_{L^q(V)} \leqslant C\epsilon^\theta \quad (\theta > 0).$$

With this we select ϵ such that $\|u_m^\epsilon - u_m\|_{L^q(V)} \leqslant \frac{\delta}{2}$ for all $m = 1, 2, \cdots$. Since for any fixed ϵ, we have $\limsup_{j,k\to\infty} \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q(V)} = 0$,

$$\|u_{m_j}^\epsilon - u_{m_j}\|_{L^q(V)} \leqslant \frac{\delta}{2} \quad \text{and} \quad |u_{m_k}^\epsilon - u_{m_k}\|_{L^q(V)} \leqslant \frac{\delta}{2}.$$

Hence, by triangle inequality, for any $\delta > 0$, we have

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} < \delta. \tag{2}$$

Let $\delta = 1, \frac{1}{2}, \frac{1}{3}, \cdots$, construct the subsequence $\{u_{\delta,m_j}\}$ recursively such that Equation (2) holds, and use a diagonal argument to extract a subsequence $\{u_{m_c}\}_{c=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ satisfying

$$\limsup_{l,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

Since $L^q(V)$ is complete, then this subsequence converges to a limit. $\square$

Notation: we denote $(u)_U = \frac{1}{|U|}\int_U u dy = ⨍_U u dy$.

Theorem 2.44 (Poincaré's Inequality). *Let U be a bounded connected, open subset of $\mathbb{R}^n$, with $\partial U \in C^1$. Assume $1 \leqslant q \leqslant \infty$. Then there exists a constant C, depending only on n, p and U, such that*

$$\|u - (u)_U\|_{L^p(U)} \leqslant C \cdot \|Du\|_{L^p(U)}$$

for each function $u \in W^{1,p}(U)$.

Proof. We may use the Rellich-Kondrachev Compactness Theorem to prove for the special case where $1 \leqslant p < n$:

We argue by contradiction. Assume the stated estimate is false, then there would exists for each integer $k \in \mathbb{N}$, there exists a function $u_k \in W^{1,p}(U)$ such that $\|u_k - (u_k)_U\|_{L^p(U)} > k\|Du_k\|_{L^p(U)}$. We normalize by defining

$$v_k := \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}}, \quad (k = 1, 2, \cdots)$$

Then $(v_k)_U = 0$, $\|v_k\|_{L^P(U)} = 1$. Also $\|Dv_k\|_{L^p(U)} < \frac{1}{k}$. Hence the sequence$\{v_k\}_{k=1}^{\infty}$ is bounded in $W^{1,p}(U)$. Since $1 \leqslant p < p^*$, we have $W^{1,p}(U) \subset\subset L^p(U)$. Thus there exists a subsequence $\{v_{k_j}\}_{j=1}^{\infty} \subset \{v_k\}_{k=1}^{\infty}$ and a function $v \in L^p(U)$, s.t., $v_{k_j} \to v$ in $L^p(U)$ which implies $v_{k_j} \to v$ in $L^1(U)$.

Recall $(v_k)_U = 0$, $\|v_k\|_{L^p(U)} = 1$, then $\|v\|_{L^p(U)} = 1$. We show $(v)_U = 0$. By Hölder's Inequality

$$\int_U v dx = \int_U (v - v_{k_j}) dx + \int_U v_{k_j} dx \lesssim \|v - v_{k_j}\|_{L^p(U)} \to 0.$$

Also, recall $\|Dv_k\|_{L^p(U)} < \frac{1}{k}$, $k = 1, 2, \cdots$. Since

$$\lim_{k_j\to\infty} \int_U (v - v_{k_j})\phi_{x_i} dx \leqslant \lim_{k_j\to\infty} \|v - v_{k_j}\|_{L^p(U)} \|\phi_{x_i}\|_{L^{p'}(U)} \to 0.$$

Then

$$\begin{aligned}\int_U v\phi_{x_i} dx &= \lim_{k_j\to\infty}\int_U v_{k_j}\phi_{x_i} dx\\ &= -\lim_{k_j\to\infty}\int_U v_{k_j,x_i}\phi dx\\ &\leqslant \lim_{k_j\to\infty}\|v_{k_j,x_i}\|_{L^p(U)}\|\phi\|_{L^{p'}(U)}\\ &= 0\end{aligned}$$

This shows $Dv = 0$ (in the sense of weak derivative). This shows that v is a constant function a.e. (since U is connected, so there is only one constant). This contradicts $(v)_U = 0$ and $\|v\|_{L^p(U)} = 1$. □

Notation: we denote $(u)_{x,r} = ⨍_{B(x,r)} u dy$.

Corollary 2.44.1 (Poincaré's Inequality for a Ball)**.** *Assume $1 \leqslant p \leqslant \infty$, then there exists a constant C depending on n and p such that*

$$\|u - (u)_{x,r}|_{L^p(B(x,r))} \leqslant C_{n,p}\cdot r\cdot \|D_u\|_{L^p(B(x,r))}.$$

Proof. For $z \in B(x,r)$, we can write $z = x + ry$ with $y \in B(0,1)$. Then

$$u(z) = w(x + ry).$$

Apply the Poincaré's inequality to $u(x + ry)$ with $y \in B(0,1)$, we have

$$\int_{B(0,1)} |u(x+ry) - (u)_{x,r}|^p dy \leqslant C'(n,p)\int_{B(0,1)} |D_y u(x+ry)|^p dy$$

Since

$$⨍_{B(0,1)} u(x+ry)dy = ⨍_{B(x,r)} u(z)dz.$$

We continue to compute

$$\begin{aligned}\int_{B(0,1)} |D_y u(x+ry)|^p dy &= \int_{B(x,r)} r^{-n}|rD_z u(z)|^p r^n dy_1\cdots y_n)\\ &= r^{p-n}\int_{B(z,r)} |Du(z)|^p dz.\end{aligned}$$

Similarly,

$$\int_{B(0,1)} |u(x+ry) - (u)_{x,r}|^p dy = \frac{1}{r^n}\int_{B(x,r)} |u(z) - (u)_{x,r}|^p dz.$$

Then we can see that the statement in the Corollary follows accordingly. □

2.7 Sobolev Embedding

Theorem 2.45 (Sobolev Embedding Theorem)**.** *Let Ω be a bounded domain with C^m boundary in $\mathbb{R}^N$. $1 \leqslant q < \infty$, $m \in \mathbb{Z}_{\geqslant 0}$, then the embedding*

$$W^{m,q}(\Omega) \hookrightarrow L^r(\Omega), \quad \frac{1}{r} \geqslant \frac{1}{q} - \frac{m}{N}\ (mq < N), \tag{3}$$

and $\forall j \in \mathbb{N}$,

$$W^{m+j,q}(\Omega) \hookrightarrow C^{j,\lambda}(\bar{\Omega}), \quad 0 < \lambda \leqslant m - \frac{N}{q}\ (mq > N) \tag{4}$$

are continuous.

In particular, when $m = 1$, then the embedding

$$W^{1,q}(\Omega) \hookrightarrow L^r(\Omega), \quad r \leqslant q^*(N > q), \tag{5}$$

and

$$W^{1,q}(\Omega) \hookrightarrow C(\bar{\Omega}), \quad (q > N) \tag{6}$$

are continuous.

3 Littlewood-Paley Theory

Recall for $f(x) \in L^1(\mathbb{R}^n)$. Its Fourier transform $\hat{f}(\xi)$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

Usually, we refer x as the physical space variable and ξ to be the frequency space variable.

The Inverse Fourier Transform is given by

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)d\xi = [\hat{f}(f)]^\vee(x) \quad a.e.$$

We also have the following useful properties:

1. $\widehat{fg}(\xi) = \hat{f} * \hat{g}(\xi)$.
2. $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.
3. (Plancherel Identity) $\langle f, g\rangle_{L^2_x(\mathbb{R}^n)} = \langle \hat{f}, \hat{g}\rangle_{L^2_\xi(\mathbb{R}^n)}$. In particular, $\|f\|_{L^2_x(\mathbb{R}^n)} = \|\hat{f}\|_{L^2_\xi(\mathbb{R}^n)}$.

3.1 Littlewood-Paley Decomposition(LPD)

Let us always assume that all the functions we consider here is of L^1, so its Fourier transform exists. The idea of Littlewood-Paley Decomposition is based on the frequency space localization. The technique of localization is familiar for physical space. For example, if we choose any $C_c^\infty(\mathbb{R}^n)$ function, $\chi(x)$, which is supported on $B_r(x_0)$ and also impose the condition $\chi(x)$ to be 1 on $B_{\frac{r}{2}}(x_0)$. That is $\chi(x)$ is a $C_c^\infty(\mathbb{R}^n)$ function satisfying

$$\chi(x) := \begin{cases} 1, & x \in B_{\frac{r}{2}}(x_0) \\ 0, & x \notin B_{\frac{r}{2}} \end{cases}.$$

Then for any function defined on $\mathbb{R}^n$, the function $\chi(x)f(x)$ gives a physical space localization of f around the point x_0.

We can apply the same idea for frequency space. Given a domain D in frequency space, one can choose a smooth function $\chi(\xi)$ supported on D and define $\pi_D f(x)$ in such a way that

$$\widehat{\pi_D f(x)} := \chi(\xi)\hat{f}(\xi).$$

The function $\pi_D f(x)$ is called a frequency space localization of f over D. In particular, we can compute $\pi_D f(x)$ explicitly using inverse Fourier Transform:

$$\pi_D f(x) = [\chi(\xi)\hat{f}(\xi)]^\vee(x) = \check{\chi} * f(x).$$

We will make use of this idea in the Littlewood-Paley Decomposition.

Littlewood-Paley Decomposition:
We say that $\phi(\xi)$ is a real **radial bump function** if $\phi(\xi) \in C_c^\infty(\mathbb{R}^n)$ is radial and

$$\phi(\xi) = \begin{cases} 1, & |\xi| \leqslant 1 \\ 0, & |\xi| \geqslant 2 \end{cases}.$$

Define

$$\psi(\xi) := \phi(\xi) - \phi(2\xi).$$

Then the functions $\psi\left(\frac{\xi}{2^k}\right)$ with $k \in \mathbb{Z}^+ \cup \mathbb{Z}^- \cup \{0\}$ is supported on the set $\{2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$. We can consider $\psi\left(\frac{\xi}{2^k}\right)$ an example of dyadic decomposition.

Proposition 3.1. *Whenever $\xi \neq 0$,*

$$\sum_{k \in \mathbb{Z}} \psi\left(\frac{\xi}{2^k}\right) = 1$$

Proof. We compute directly, first notice that

$$\begin{aligned}
&\psi\left(\frac{\xi}{2}\right) + \psi(\xi) + \psi(2\xi) \\
&= \left[\phi\left(\frac{\xi}{2}\right) - \phi(\xi)\right] + [\phi(\xi) - \phi(2\xi)] + [\phi(2\xi) - \phi(4\xi)] \\
&= \phi\left(\frac{\xi}{2}\right) - \phi(4\xi)
\end{aligned}$$

Similarly, we can compute that for any $k_1, k_2 \in \mathbb{Z}^+$

$$\psi\left(\frac{\xi}{2^{k_1}}\right) + \cdots + \psi(2^{k_2}\xi) = \phi\left(\frac{\xi}{2^{k_1}}\right) - \phi(4^{k_2}\xi),$$

Now for any $\xi \in \mathbb{R}^n$, for large enough k_1,k_2, $\phi(\xi/2^{k_1}) = 1$, and $\phi(2^{k_2}\xi) = 0$. Hence we conclude that for large enough k_1, k_2,

$$\psi\left(\frac{\xi}{2^{k_1}}\right) + \cdots + \psi(2^{k_2}\xi) = 1.$$

Letting $k_1, k_2 \to \infty$, we get the desired result. $\square$

Corollary 3.1.1. *For any $k \in \mathbb{Z}$, $\phi\left(\frac{\xi}{2^k}\right) = \sum_{j \leqslant k} \psi\left(\frac{\xi}{2^j}\right)$, if $\xi \neq 0$.*

Definition 3.2 (Littlewood-Paley Projection)**.** *We define the* ***Littlewood-Paley projection operators*** *P_k, $P_{\leqslant k}$ by $f \mapsto P_k f$, such that*

$$\widehat{P_k f}(\xi) := \psi\left(\frac{\xi}{2^k}\right) \hat{f}(\xi),$$

i.e., a projection to the annulus, $\{|f| \sim 2^k\}$, and $f \mapsto P_{\leqslant k} f$, such that

$$\widehat{P_{\leqslant k} f}(\xi) := \phi\left(\frac{\xi}{2^k}\right) \hat{f}(\xi),$$

i.,e. a projection to the ball $\{|f| \lesssim 2^k\}$.

The explicit definition of the Littlewood-Paley projection operators in physical space are given as below:

$$P_k f = \left[\psi\left(\frac{\xi}{2^k}\right) \hat{f}(\xi)\right]^{\vee} = m_k * f(x),$$

where

$$m_k(x) = \left[\psi\left(\frac{\xi}{2^k}\right)\right]^{\vee}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi\left(\frac{\xi}{2^k}\right) d\xi$$

is the inverse Fourier transform of $\psi(\xi/2^k)$.

Let $m(x) = \check{\psi}(x)$, then we can represent m_k in terms of m. We consider the change of variable $\xi' := \frac{\xi}{2^k}$, then

$$\begin{aligned}
m_k(x) &= \frac{1}{(2\pi)^n} \cdot 2^{kn} \int_{\mathbb{R}^n} e^{i2^k x\cdot\xi'} \psi(\xi') d\xi' \\
&= 2^{kn} \cdot \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(2^k x)\cdot\xi'} \psi(\xi') d\xi' \\
&= 2^{kn} \check{\psi}(2^k x) \\
&= 2^{kn} m(2^k x),
\end{aligned}$$

Lemma 3.3. *Using notation as above, then the quantity $\|m_k(x)\|_{L^1(\mathbb{R}^n)}$ is finite and independent of k.*

Proof. Using the fact $m_k(x) = 2^{kn} m(2^k x)$, we have,

$$\int_{\mathbb{R}^n} |m_k(x)| dx_1 dx_2 \cdots dx_n = \int_{\mathbb{R}^n} |2^{kn} m(2^k x)| dx_1 dx_2 \cdots dx_n.$$

Let $y = 2^k x$, then by a change of variable, we see no matter what k is, they are equal to

$$\int_{\mathbb{R}^n} |m(y)| dy_1 dy_2 \cdots dy_n < +\infty.$$

Now since $m(x) = \check{\psi}(x)$, and ψ is a Schwartz function. Since we know the Fourier Transform Operator $\mathcal{F}$ is an automorphism on the Schwartz Space, then $\check{\psi}(x)$ is Schwartz, hence $m(x) \in L^1$. □

Proposition 3.4. *Suppose $f \in L^2(\mathbb{R}^n)$. As $M, N \to +\infty$, then*

$$\left\| f - \sum_{-M \leqslant k \leqslant n} P_k f \right\|_{L^2_x(\mathbb{R}^n)} \to 0$$

Proof. We proceed to prove that for any $f \in L^2(\mathbb{R}^n)$, we have $f = \sum\limits_{k\in\mathbb{Z}} P_k f$, where the summation is in L^2 sense.
By Plancherel's formula, we have

$$\|f - \sum_{-M\leqslant k\leqslant n} P_k f\|_{L^2_x(\mathbb{R}^n)} = \|\hat{f} - \sum_{-M\leqslant k\leqslant N} \widehat{P_k f}\|_{L^2_\xi(\mathbb{R}^n)}$$

Note except at $\xi = 0$, we have

$$\begin{aligned}\sum_{-M\leqslant k\leqslant N} \widehat{P_k f}(\xi) &= \sum_{k\leqslant N} \widehat{P_k f}(\xi) - \sum_{k<-M} \widehat{P_k f}(\xi) \\ &= \widehat{P_{\leqslant N} f}(\xi) - P_{\leqslant \widehat{-(M+1)}} f(\xi) \\ &= \left[\phi\left(\frac{\xi}{2^N}\right) - \phi\left(\frac{\xi}{2^{-(M+1)}}\right)\right]\hat{f}(\xi)\end{aligned}$$

Hence,

$$\begin{aligned}\|f - \sum_{-M\leqslant k\leqslant N} P_k f\|_{L^2_x(\mathbb{R}^n)} &= \|\hat{f} - \phi(2^{-N})\hat{f} + \phi(2^{M+1})\hat{f}\|_{L^2_\xi(\mathbb{R}^n)} \\ &\leqslant \|[1-\phi(2^{-N})]\hat{f}\|_{L^2_\xi(\mathbb{R}^n)} + \|\phi(2^{M+1})\hat{f}\|_{L^2_\xi(\mathbb{R}^n)} \\ &\leqslant \left(\int_{|\xi|\geqslant 2^N} |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} + \left(\int_{|\xi|\leqslant 2^{-M}} |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\ &\longrightarrow 0 \quad \text{as} \quad M, N \to +\infty\end{aligned}$$

□

Remark 3.5. *By Plancherel's formula, this Proposition shows that* $\hat{f}(\xi) = \sum\limits_{k\in\mathbb{Z}} \widehat{P_k f}(\xi)$ *in* $L^2_\xi(\mathbb{R}^n)$ *sense, for any* $\xi \neq 0$.

Remark 3.6. *for any subset* $J \subset \mathbb{Z}$*, it is convenient to define*

$$P_J := \sum_{k\in J} P_k.$$

Since

$$P_{\leqslant k} = \sum_{j\leqslant k} P_J,$$

then

$$P_{\leqslant k} = P_J, \quad \textit{where } J = \{j \in \mathbb{Z} \,|\, j \leqslant k\}.$$

Theorem 3.7 (Properties of Little-Wood Paley Projection Operator)**.** *We have the following properties for the Little-Wood Paley Projection Operator:*

1. *(Almost Orthogonality)*

- *The operators P_k's are self-adjoint operators on L^2.*
- $P_{k_1}P_{k_2} = 0$ *whenever* $|k_1 - k_2| \geqslant 2$.
- $\|f\|_{L^2}^2 = \|\sum_{k\in\mathbb{Z}} P_k f\|_{L^2}^2 \approx \sum_{k\in\mathbb{Z}} \|P_k f\|_2^2$.

2. *(L^P-boundedness)* $\forall 1 \leqslant p \leqslant \infty$, $\forall J = [M, N] \subseteq \mathbb{Z}$, *we have*

$$\|P_J f\|_{L^p} \lesssim \|f\|_{L^p}.$$

3. *(Bound on Derivatives)* $\forall 1 \leqslant p \leqslant \infty$, *we have*
   - $\|\partial_j(P_k f)\|_p \lesssim 2^k\|f\|_p$;
   - $2^k\|P_k f\|_p \lesssim \|\partial f\|_p = \sum_{j=1}^{n} \|\partial_j f\|_p$.

4. *(Bernstein Inequality)* $\forall 1 \leqslant p \leqslant q \leqslant \infty$, *we have*
   - $\|P_k f\|_q \leqslant 2^{nk\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p$;
   - $\|P_{\leqslant 0} f\|_q \leqslant \|f\|_p$.

5. *(Commutator Estimates) For* $f, g \in C_c^\infty(\mathbb{R}^n)$, *define*

$$[P_k, f]g := P_k(fg) - fP_kg,$$

then for $1 \leqslant p \leqslant \infty$,

$$\|[P_k, f]g\|_p \lesssim 2^{-k}\|\nabla f\|_\infty\|g\|_p.$$

6. *(Littlewood-Paley Inequality) Define the Littlewoord-Paley Square function* $\mathcal{S}$ *by*

$$\mathcal{S}f(x) := \left(\sum_{k\in\mathbb{Z}} |P_k f|^2\right)^{1/2}.$$

Then for any $1 < p < \infty$ *and any* $f \in C_c^\infty(\mathbb{R}^n)$, *there exists constants* $c_1, c_2 > 0$ *independent of* f, *such that*

$$c_1\|\mathcal{S}f\|_p \leqslant \|f\|_p \leqslant c_2\|\mathcal{S}f\|_p.$$

Proof. **Proof for Almost Orthogonality:** We first show that for every $k \in \mathbb{Z}$, the operator P_k is self-adjoint, i.e.,

$$\langle P_k f, g\rangle = \langle f, P_k g\rangle.$$

$\forall f, g \in L^2(\mathbb{R}^n)$. By Plancherel's Identity, we have

$$\begin{aligned}\langle P_k f, g\rangle &= \langle \widehat{P_k f}, \hat{g}\rangle \\ &= \langle \psi\left(\frac{\cdot}{2^k}\right)\hat{f}, \hat{g}\rangle \\ &= \langle \hat{f}, \widehat{P_k g}\rangle \\ &= \langle f, P_k g\rangle.\end{aligned}$$

Next, we proceed to show that $P_{k_1}P_{k_2} = 0$, whenever $|k_1 - k_2| \geqslant 2$.

$$\widehat{P_{k_1}P_{k_2}f} = \widehat{P_{k_1}(P_{k_2}f)} = \psi\left(\frac{\xi}{2^{k_1}}\right)\widehat{P_{k_2}f} = \psi\left(\frac{\xi}{2^{k_1}}\right)\psi\left(\frac{\xi}{2^{k_2}}\right)\hat{f}(\xi).$$

Recall $\operatorname{supp}\psi\left(\frac{\xi}{2^k}\right) = \{2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$. So if $|k_1 - k_2| \geqslant 2$, $\psi(\frac{\xi}{2^{k_1}})\psi(\frac{\xi}{2^{k_2}}) \equiv 0$, as the supports of the two function are disjoint. Therefore, when $|k_1 - k_2| \geqslant 2$,

$$\widehat{P_{k_1}P_{k_2}f}(\xi) \equiv 0 \Longrightarrow P_{k_1}P_{k_2}f = 0.$$

Lastly, we prove that $\|f\|_2^2 \approx \sum\limits_{k\in\mathbb{Z}} \|P_k f\|_2^2$, i.e., $\|f\|_2^2$ and $\sum\limits_{k\in\mathbb{Z}} \|P_k f\|_2^2$ bounds each other by a uniform constant. To show there exists $C > 0$, such that $\|f\|_2^2 \leqslant C\sum\limits_k \|P_k f\|_2^2$, we have the following estimates:

$$\begin{aligned}\|f\|_2^2 &= \|\sum_{k\in\mathbb{Z}} P_k f\|_2^2 \\ &= \langle\sum_k P_k f, \sum_{k'} P_{k'} f\rangle \\ &= \sum_{k,k'}\langle P_k f, P_{k'} f\rangle\end{aligned}$$

Since P_k is self-adjoint and $P_k P_{k'}$ is the zero operator if $|k - k'| \geqslant 2$, then we have

$$\begin{aligned}\|f\|_2^2 &= \sum_{|k-k'|\leqslant 1}\langle P_k f, P_{k'} f\rangle \\ &\leqslant \sum_{|k-k'|\leqslant 1}\|P_k f\|_2\|P_{k'} f\|_2 \\ &\leqslant \sum_{|k-k'|\leqslant 1}\frac{1}{2}(\|P_k f\|_2^2 + \|P_{k'} f\|_2^2) \\ &\leqslant \sum_k \frac{1}{2}(3\|P_k f\|_2^2 + \|P_{k-1} f\|_2^2 + \|P_k f\|_2^2 + \|P_{k+1} f\|_2^2) \\ &\leqslant \frac{6}{2}\sum_k \|P_k f\|_2^2 = 3\sum_k \|P_k f\|_2^2\end{aligned}$$

To show there exists C such that $\sum\limits_k \|P_k f\|_2^2 \leqslant C\|f\|_2^2$, we have the following calculations:

$$\begin{aligned}
\sum_k \|P_k f\|_2^2 &= \sum_k \|\widehat{P_k} f\|_2^2 && \text{(Plancherel)} \\
&= \sum_k \int_{\mathbb{R}^n} \left|\psi\left(\frac{\xi}{2^k}\right)\hat{f}(\xi)\right|^2 d\xi \\
&\leqslant \sum_k \int_{2^{k-1}\leqslant|\xi|\leqslant 2^{k+1}} |\hat{f}(\xi)|^2 d\xi \\
&\leqslant 2\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi \\
&= 2\|f\|_2^2
\end{aligned}$$

So

$$\frac{1}{2}\sum_k \|P_k f\|_2^2 \leqslant \|f\|_2^2 \leqslant 3\sum_k \|P_k f\|_2^2$$

as desired.

Proof of L^p-Boundedness: Let $J = [M, V] \subset \mathbb{Z}$, we want to show $\forall 1 \leqslant p \leqslant \infty$, $\|P_J f\|_p \lesssim \|f\|_p$.

Since

$$\|P_J f\|_p = \|P_{\leqslant N} f - P_{\leqslant M-1} f\|_p \leqslant \|P_{\leqslant N} f\|_p + \|P_{\leqslant M-1} f\|_p.$$

It suffices to prove for the case $J = (-\infty, K) \subset \mathbb{Z}$.

Recall that $\widehat{P_{\leqslant k} f}(\xi) = \phi(\frac{\xi}{2^k})\hat{f}(\xi)$. Then by the Inverse Fourier Transformation, we have

$$P_{\leqslant k} f = \bar{m}_k * f = \int_{\mathbb{R}^n} \bar{m}_k(x-y) f(y) dy,$$

where $\bar{m}(k)(x) = \left[\check{\phi}\left(\frac{\cdot}{2^k}\right)\right](x)$. Let

$$\bar{m}(x) = [\check{\phi}(\xi)](x).$$

Then by change of variable, we can show $\bar{m}_k(x) = 2^{nk}\bar{m}(2^k x)$.

We also recall Young's Inequality: $\|\tilde{g} * \tilde{f}\|_{L^r} \leqslant \|\tilde{g}\|_{L^q}\|\tilde{f}\|_{L^p}$ if $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{p}$. Then by Young's inequality, we have $\|P_{\leqslant k}f\|_p = \|\bar{m}_k * f\|_p \leqslant \|\bar{m}_k\|_1\|f\|_p$. We compute $\|\bar{m}_k\|_1$:

$$\begin{aligned}\|\bar{m}_k\|_1 &= \int_{\mathbb{R}^n} 2^{nk}\bar{m}(2^k x)dx_1\cdots dx_n \\ &= \int_{\mathbb{R}^n} \bar{m}(2^k x)(2^k dx_1)\cdots(2^k dx_n) \\ &= \int_{\mathbb{R}^n} \bar{m}(y)dy\end{aligned}$$

Since $\bar{m}$ is the inverse fourier transform of a Schwarz Function, then $\bar{m}$ is a Schwarz function, so $\|\bar{m}\|_1$ is finite. Hence, we have $\|P_{\leqslant k}f\|_p \lesssim \|f\|_p$.

Proof of Bounds on Derivatives: To prove $\|\partial(P_k f)\|_p \lesssim 2^k\|f\|^p$, recall we have shown $P_k f = m_k * f$ and $m_k(\cdot) = 2^{nk}m(2^k\cdot)$ where $m(x) = [\check{\psi}(\xi)](x)$. Then taking partial derivative with respect to x_j, we have:

$$\begin{aligned}\partial_j(P_k f) &= \partial_j[m_k * f] \\ &= \partial_j[2^{nk}m(2^k\cdot) * f] \\ &= 2^k\cdot 2^{nk}(\partial_j m)(2^k\cdot) * f \\ \implies \|\partial_j(P_k f)\|_p &= \|2^k\cdot 2^{nk}(\partial_j m)(2^k\cdot) * f\|_p \\ &\leqslant 2^k\|2^{nk}(\partial_j m)(2^k\cdot)\|_{L^1}\|f\|_p && \text{(Young's Inequality)} \\ &\lesssim 2^k\|f\|_p\end{aligned}$$

Since m is a Schwarz function, then $\partial_j m$ is also a Schwarz function. Notice by a change of variable, $\|2^{n_k}(\partial_j m)(2^k\cdot)\|_{L^1} = \|\partial_j m\|_{L^1}$ is uniformly bounded in k.

To prove $2^k\|P_k f\|_p \lesssim \|\partial f\|_p$. We observe that

$$\hat{f}(\xi) := \sum_{j=1}^n \frac{\xi_j}{i|\xi|^2}\widehat{\partial_{x_j} f}(\xi),$$

for all $\xi \neq 0$. This is because

$$\widehat{\partial_{x_j} f}(\xi) = i\xi_j\hat{f}(\xi).$$

Consequently,

$$\begin{aligned}\widehat{P_k f}(\xi) &= \psi\left(\frac{\xi}{2^k}\right)\hat{f}(\xi) \\ &= \sum_{j=1}^n \frac{\xi_j}{i|\xi|^2}\psi\left(\frac{\xi}{2^k}\right)\widehat{\partial_{x_j} f}(\xi)\end{aligned}$$

To make the factor 2^k appear on both sides, we multiply 2^k on both sides, so

$$\begin{aligned}
2^k \widehat{P_k f} &= \sum_{j=1}^{n} 2^k \cdot \frac{\xi_j}{i|\xi|^2} \psi\left(\frac{\xi}{2^k}\right) \widehat{\partial_{x_j} f} \\
&= \sum_{j=1}^{n} \frac{\frac{\xi_j}{2^k}}{i|\frac{\xi}{2^k}|^2} \psi\left(\frac{\xi}{2^k}\right) \widehat{\partial_j f} \\
&= \sum_{j=1}^{n} \chi_j\left(\frac{\xi}{2^k}\right) \widehat{\partial_j f}
\end{aligned}$$

where we set

$$\chi_j(\xi) := \frac{\xi_j}{i|\xi|^2} \psi(\xi).$$

Notice that χ_j is also a Schwarz function. Now taking inverse Fourier Transform, we have

$$2^k P_k f = \sum_{j=1}^{n} \left[\chi_j(2^{-k}\cdot)\right]^\vee * \partial_j f$$

We claim $\| \left[X_j(2^{-k}\cdot)\right]^\vee \|_1 \lesssim C$, then by Young's inequality, we would have

$$2^k \|P_k f\|_p \leqslant \sum_{j=1}^{n} \| \left[\chi_j(2^{-k}\cdot)\right]^\vee \|_1 \|\partial_j f\|_p \lesssim \|\partial f\|_p.$$

So it remains to prove the claim.

Let

$$h_j(x) := [\check{\chi}_j(\xi)](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \chi_j(\xi) d\xi.$$

Since χ_j is Schwarz, then h_j is Schwarz, so $h_j(x) \in L^1(\mathbb{R}^n)$. Now by definition of inverse Fourier Transform,

$$\begin{aligned}
[\chi_j(2^{-k}\cdot)]^\vee(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \chi_j\left(\frac{\xi}{2^k}\right) d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} 2^{nk} e^{i2^k x\cdot\frac{\xi}{2^k}} \chi_j\left(\frac{\xi}{2^k}\right) (\frac{1}{2^k} d\xi_1) \cdots (\frac{1}{2^k} d\xi_n) \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} 2^{nk} \cdot e^{i2^k x\cdot\eta} \chi_j(\eta) d\eta \\
&= 2^{nk} h_j(2^k x)
\end{aligned}$$

Therefore,by a change of variable, we have

$$\begin{aligned}
\|[\chi_j(2^{-k}\cdot)]^\vee\|_{L^1} &= \int_{\mathbb{R}^n} 2^{nk} |h_j(2^k x)| dx \\
&= \int_{\mathbb{R}^n} |h_j(y)| dy
\end{aligned}$$

which is uniformly bounded by a constant.

Proof of Bernstein Inequality: WTS for any $1 \leqslant p \leqslant q \leqslant \infty$,

$$\|P_k f\|_p \lesssim 2^{nk(\frac{1}{p}-\frac{1}{q})}\|f\|_p.$$

Recall that$P_k f = m_k * f$, then by Young's Inequality, we have

$$\|P_k f\|_q = \|m_k * f\|_q \leqslant \|m_k\|_r \|f\|_p \quad \text{where} \quad 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}.$$

Also recall that $m_k(x) = 2^{nk} m(2^k x)$ and $\|m(\cdot)\|_r$ is finite. Then

$$\begin{aligned}
\|m_k\|_r &= \left(\int_{\mathbb{R}^n} \left(2^{nk}|m(2^k x)|\right)^r dx\right)^{\frac{1}{r}} \\
&= 2^{nk}\left(\int_{\mathbb{R}^n} |m(2^k x)|^r dx\right)^{\frac{1}{r}} \\
&= 2^{nk}\left(\int_{\mathbb{R}^n} |m(y)|^r 2^{-nk} dy_1 \cdots dy_n\right)^{\frac{1}{r}} \\
&= 2^{nk} \cdot 2^{nk \cdot \frac{-1}{r}} \|m\|_r \\
&= 2^{nk(1-\frac{1}{r})}\|m\|_r \\
&= 2^{nk(\frac{1}{p}-\frac{1}{q})}\|m\|_r
\end{aligned}$$

This shows

$$\|P_k f\|_q \leqslant 2^{nk\left(\frac{1}{p}-\frac{1}{q}\right)}\|m\|_r \|f\|_p$$

which concludes the proof.

Proof of Commutator Estimates: We want to show that

$$\|P_k(fg) - f P_k g\|_p = \|[P_k, f]g\|_p \lesssim 2^{-k}\|\nabla f\|_\infty \|g\|_p$$

for $f, g \in \mathbb{C}_c^\infty(\mathbb{R}^n)$ and $1 \leqslant p \leqslant \infty$.

Since $P_k f = m_k * f$, we have

$$\begin{aligned}
P_k(fg)(x) - f(x)P_k g(x) &= m_k * fg - f \cdot (m_k * g) \\
&= \int_{\mathbb{R}^n} m_k(x-y) f(y) g(y) dy - f(x) \int_{\mathbb{R}^n} m_k(x-y) g(y) dy \\
&= \int_{\mathbb{R}^n} m_k(x-y)[f(y) - f(x)] g(y) dy
\end{aligned}$$

Since f is C_c^∞, then by the intermediate value theorem, we have

$$|f(y) - f(x)| \leqslant \|\nabla f\|_\infty (x - y)$$

So

$$\begin{aligned} P_k(fg)(x) - f(x)P_kg(x) &\leqslant \|\nabla f\|_\infty \int_{\mathbb{R}^n} |m_k(x-y)| \cdot |x-y| \cdot |g(y)| dy \\ &= \|\nabla f\|_\infty \int_{\mathbb{R}^n} 2^{nk} |m(2^k y)| \cdot |y| \cdot |g(x-y)| dy. \end{aligned}$$

Where the last line follows from a change of variables.

Next, set

$$\int_{\mathbb{R}^n} 2^{nk} |m(2^k y)| \cdot |y| \cdot |g(x-y)| dy = F * g(x),$$

consequently, by Young's Inequality, we obtain:

$$\begin{aligned} \|P_k(fg) - f \cdot P_kg\|_{L^p(\mathbb{R}^n, x)} &\leqslant \|\nabla f\|_\infty \|g(x-y)\|_{L^p(\mathbb{R}^n, x)} \cdot \int_{\mathbb{R}^n} 2^{nk} |m(2^k y)| |y| dy_1 \cdots dy_n \\ &= \|\nabla f\|_\infty \|g\|_p \int_{\mathbb{R}^n} 2^{(n-1)k} |m(z)| |z| \cdot 2^{-nk} dz \\ &= \|\nabla f\|_\infty \|g\|_p \cdot 2^{-k} \|m(z) \cdot z\|_1 \\ &\lesssim 2^{-k} \cdot \|\nabla f\|_\infty \cdot \|g\|_p \end{aligned}$$

Since $m(z) \cdot z$ is also a Schwarz function.

Proof of Littlewood-Paley Inequality: We first prove $\|Sf\|_{L^p} \lesssim \|f\|_{L^p}$ by introducing the vector-valued linear operator

$$\tilde{S}f(x) = \{P_k f(x)\}_{k \in \mathbb{Z}}.$$

Then we note that $\tilde{S}$ has a vector valued kernel

$$\tilde{K}(x) := \{m_k(x)\}_{k \in \mathbb{Z}} = \left\{2^{nk} m(2^k(x))\right\}_{k \in \mathbb{Z}},$$

where m is the inverse Fourier Transform of ψ. That is

$$\tilde{S}f(x) = \{m_k * f(x)\}_{k \in \mathbb{Z}}.$$

We show that

$$|\tilde{K}(x)| \lesssim |x|^{-n}, \quad |\partial \tilde{K}(x)| \lesssim |x|^{-n-1}, \quad \forall x \neq y.$$

This follows from the fact that each m is Schwarz, since m is Schwarz. $\tilde{S} : L^2 \to L^2$ is bounded. Then

Proposition (4.35) implies that

$$\|Sf\|_{L^p} = \|\|\tilde{S}f\|_{\ell^2}\|_{L^p} \lesssim \|f\|_{L^p}.$$

Next we prove $\|f\|_{L^p} \lesssim \|Sf\|_{L^p}$ by duality of L^p norms. For any Schwartz function g, by using $P_k P_{k'} = 0$ for $|k - k'| \geqslant 2$, the Cauchy - Schwarz inequality, and the Hölder inequality, we have

$$\begin{aligned}
\int f(x)g(x)dx &= \int \sum_{k,k' \in \mathbb{Z}} P_k f(x) P_{k'} g(x) dx \\
&= \int \sum_{|k-k'| \leqslant 1} P_k f(x) P_{k'} g(x) dx \\
&\lesssim \int \left(\sum_k |P_k f(x)|^2 \right)^{\frac{1}{2}} \left(\sum_{k'} |P_{k'} g(x)|^2 \right)^{\frac{1}{2}} dx \\
&\lesssim \|Sf\|_{L^p} \|Sg\|_{L^{p'}} \lesssim \|Sf\|_{L^p} \|g\|_{L^{p'}},
\end{aligned}$$

where $1/p + 1/p' = 1$. This implies $\|f\|_{L^p} \lesssim \|Sf\|_{L^p}$. □

Proposition 3.8. *Recall that*

$$\|f\|_{\dot{H}^s(\mathbb{R}^2)} := \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}|^2 d\xi \right)^{1/2}.$$

Then

$$\|f\|_{\dot{H}^s}^2 \approx \sum_{k \in \mathbb{Z}} 2^{2ks} \|P_k f\|_2^2 \approx \sum_{k \in \mathbb{Z}} \|P_k f\|_{\dot{H}^s}^2.$$

Remark 3.9. *Recall that* $\|f\|_{L^2}^2 \approx \sum_{k \in \mathbb{Z}} \|P_k f\|_2^2$.

Remark 3.10. *Let* $\{a_k\}$ *be the sequence given by*

$$a_k = 2^{ks} \|P_k f\|_2.$$

Then the Proposition says

$$\|f\|_{\dot{H}^s} = \|a_k\|_{\ell^2}.$$

Remark 3.11. *The following is also true:*

$$\|f\|_{H^s(\mathbb{R}^n)}^2 \approx \sum_{k \in \mathbb{Z}} (1 + 2^k)^{2s} \|P_k f\|_2^2.$$

Proof. Since $\operatorname{supp}\widehat{P_k f} = \{2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$, we have

$$\begin{aligned}
\|P_k f\|_{\dot{H}^s}^2 &= \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{P_k f}(\xi)|^2 d\xi \\
&= \int_{2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}} |\xi|^{2s} |\widehat{P_k f}(\xi)|^2 d\xi. \\
&\approx 2^{2ks} \int_{2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}} |\widehat{P_k f}(\xi)|^2 d\xi \\
&= 2^{2ks} \int_{\mathbb{R}^n} |\widehat{P_k f}(\xi)|^2 d\xi \\
&= 2^{2ks} \|P_k f\|_2^2,
\end{aligned}$$

where the last line follows from Plancherel's Identity. Then

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} 2^{2ks} \|P_k f\|_2^2 &\approx \sum_{k \in \mathbb{Z}} \|P_k f\|_{\hat{H}^s}^2 \\
&= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{P_k f}(\xi)|^2 d\xi \\
&\leqslant 2 \sum_{k \in \mathbb{Z}} \int_{2^k \leqslant |\xi| \leqslant 2^{k+1}} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \\
&= 2\|f\|_{\dot{H}^s(\mathbb{R}^n)}^2.
\end{aligned}$$

Where the third line follows from the definition of $\widehat{P_k f}$.

Next, we proceed to show $\|f\|_{\dot{H}^s}^2 \lesssim \sum\limits_{k\in\mathbb{Z}} \|P_k f\|_{\dot{H}^s}^2$. Since $\dot{H}^s$ is an inner product space, we have

$$\begin{aligned}
\|f\|_{\dot{H}^s}^2 &= \langle f, f\rangle_{\dot{H}^s} \\
&= \langle \sum_k P_k f, \sum_{k'} P_{k'} f\rangle_{\dot{H}^s} \\
&= \sum_k \sum_{k'} \langle P_k f, P_{k'} f\rangle_{\dot{H}^s} \\
&= \sum_k \sum_{k'} \langle |\xi|^s \psi(\frac{\xi}{2^k})\hat{f}, |\xi|^s \psi(\frac{\xi}{2^{k'}})\hat{f}\rangle_{L^2} \\
&\quad (\text{let } k' = k + k'', \text{ with } k'' = -1, 0, 1) \\
&\approx \sum_{k''=-1}^{1} \sum_{k\in\mathbb{Z}} \langle 2^{sk} P_k f, 2^{s(k+k'')} P_{k+k''} f\rangle_{L^2} \\
&\overset{\text{Hölder}}{\leqslant} \sum_{k''=-1}^{1} \sum_{k\in\mathbb{Z}} 2^{sk} \|P_k f\|_{L^2} \cdot 2^{s(k+k'')} \|P_{k+k''} f\|_{L^2} \\
&\approx \sum_{k''=-1}^{1} \left(\sum_{k\in\mathbb{Z}} \|P_k f\|_{\dot{H}^s} \cdot \|P_{k+k''} f\|_{\dot{H}^s} \right) \\
&\leqslant \sum_{k''=-1}^{1} \frac{1}{2} \left(\sum_{k\in\mathbb{Z}} \|P_k f\|_{\dot{H}^s}^2 + \sum_{k\in\mathbb{Z}} \|P_{k+k''} f\|_{\dot{H}^s}^2 \right) \\
&\lesssim 3 \sum_{k\in\mathbb{Z}} \|P_k(f)\|_{\dot{H}^s}^2
\end{aligned}$$

Since $\sum_k \|P_k f\|_{\dot{H}^s}^2 \approx \sum_{k\in\mathbb{Z}} 2^{2ks} \|P_k f\|_2^2$, we have the desired result. □

Definition 3.12 (Besov Space)**.** *We say that the f belongs to the Besov space $B_{p,q}^s$ if the following corresponding norm are finite:*

$$\|f\|_{B_{2,1}^s} := \sum_{k\in\mathbb{Z}} (1+2^k)^s \|P_k f\|_2 = \|(1+2^k)^s \|P_k f\|_2\|_{\ell^1}$$

$$\|f\|_{H^s} = \|f\|_{B_{2,2}^s} = \left(\sum_{k\in Z} (1+2^k)^{2s} \|P_k f\|_2^2 \right)^{\frac{1}{2}} = \|(1+2^k)^s \|P_k f\|_2\|_{\ell^2}.$$

$$\|f\|_{B_{p,q}^s} = \|(1+2^k)^s \|P_k f\|_p\|_{\ell^q} = \left[\sum_{k\in\mathbb{Z}} \left((1+2^k)^s \|P_k f\|_{L_x^p(\mathbb{R}^n)} \right)^q \right]^{\frac{1}{q}}.$$

3.2 Applications of Littlewood-Paley Operator

3.2.1 An Interpolation Inequality

In this section, we will use the Littlewood-paley operator to prove that the following interpolation inequality is correct:

For $f \in C_0^\infty(\mathbb{R}^n)$ and any integer $1 \leqslant i \leqslant m-1$, we have

$$\|\partial^i f\|_{L^p} \lesssim \|f\|_{L^p}^{1-\frac{i}{m}} \|\partial^m f\|_{L^p}^{\frac{i}{m}}.$$

Recall Littlewood-Paley (3): $\forall 1 \leqslant p \leqslant \infty$, we have

- $\|\partial_j (P_k f)\|_p \lesssim 2^k \|f\|_p$;
- $2^k \|P_k f\|_p \lesssim \|\partial f\|_p = \sum_{j=1}^n \|\partial_j f\|_p$.

Proof. Due to the third property of Littlewood-Paley, we have for $i \in \mathbb{N}$,

$$\|\partial P_k f\|_p \lesssim 2^{ki} \|f\|_p \tag{*1}$$
$$2^{ki} \|P_k f\|_p \lesssim \|\partial^i f\|_p \tag{*2}$$

We decompose $f = P_{\leqslant k} f + P_{>k} f =: f_{\leqslant k} + f_{>k}$. Since ∂^i is a linear operator, we have

$$\partial^i f = \partial^i f_{\leqslant k} + \partial^i f_{>k}$$
$$\|\partial^i f\|_p \leqslant \underbrace{\|\partial^i f_{\leqslant k}\|_p}_{(1)} + \underbrace{\|\partial^i f_{>k}\|_p}_{(2)}$$

We estimate (1)and (2) separately. Intuitively, the lower frequency part should be bounded by lower order derivative and for the higher frequency part should be bounded by higher order derivative.

For (1), we have

$$\|\partial^i P_{\leqslant k} f\|_p \leqslant \sum_{k' \leqslant k} \|\partial^i P_{k'} f\|_p \overset{(*1)}{\lesssim} \sum_{k' \leqslant k} 2^{k' i} \|f\|_p.$$

Then we have the following

$$\sum_{k' \leqslant k} 2^{(k'-k)i} 2^{ki} \|f\|_p = 2^{ki} \|f\|_p \left(\sum_{k' \leqslant k} 2^{(k'-k)i} \right) \lesssim 2^{ki} \|f\|_p$$

For (2), since $P_{k'}f = P_{k'}(P_{k'-1} + P_{k'} + P_{k'+1})f \approx P_{k'}P_{k'}f$. Then we have the following estimate:

$$\begin{aligned}
(2) = \|\partial^i f_{>k}\|_p &\leqslant \sum_{k' \geqslant k} \|\partial^i P_{k'} f\|_p \\
&\approx \sum_{k' \geqslant k} \|\partial^i P_{k'} P_{k'} f\|_p \\
&\overset{(*1)}{\lesssim} \sum_{k' \geqslant k} 2^{ki} \|P_{k'} f\|_p \leqslant \sum_{k' \geqslant k} 2^{k'i} \|P_{k'} f\|_p \\
&\leqslant \sum_{k' \geqslant k} 2^{k'(i-m)} 2^{k'm} \|P_{k'} f\|_p \overset{(*2)}{\lesssim} \sum_{k' \geqslant k} 2^{k'(i-m)} \|\partial^m f\|_p \\
&\lesssim 2^{k(i-m)} \|\partial^m f\|_p
\end{aligned}$$

Thus, $\|\partial^i f\|_p \lesssim 2^{ki}\|f\|_p + 2^{k(i-m)}\|\partial^m f\|_p$ for all $k \in \mathbb{Z}$. We want to find the best k so we get the desired inequality. Since we know $\|\partial^i f\|_p \lesssim \lambda^i \|f\|_p + \lambda^{i-m}\|\partial^m f\|_p$ for any $\lambda \in 2^{\mathbb{Z}}$, by AM-GM, we try $\lambda_0^i \|f\|_p = \lambda_0^{i-m}\|\partial^m f\|_p$. Then

$$\lambda_0 := \left(\frac{\|\partial^m f\|_p}{\|f\|_p} \right)^{\frac{1}{m}}.$$

Under this choice of λ_0, we have

$$\lambda_0^i \|f\|_p = \left(\frac{\|\partial^m f\|_p}{\|f\|_p} \right)^{\frac{i}{m}} \|f\|_p = \|f\|_p^{1-\frac{i}{m}} \|\partial^m f\|_p^{\frac{i}{m}}$$

Now for this fixed λ_0, if $\lambda_0 \in 2^{\mathbb{Z}}$, then we done, otherwise, there exists $\lambda \in 2^{\mathbb{Z}}$, s.t., $\lambda < \lambda_0 < 2\lambda$. Then $\lambda^i < \lambda_0^i$, $\lambda^{i-m} = (2\lambda)^{i-m} \cdot 2^{m-i} < \lambda_0^{i-m} \cdot 2^{m-i}$. So

$$\lambda^i \|f\|_p + \lambda^{i-m}\|\partial^m f\|_p \leqslant \lambda_0^i \|f\|_p + \lambda_0^{i-m} \cdot 2^{m-i} \|\partial^m f\|_p \leqslant 2^{m-i} \cdot 2 \|f\|_p^{1-\frac{i}{m}} \|\partial^m f\|_p^{\frac{i}{m}}.$$

This concludes the proof of the interpolation inequality.

□

3.2.2 Non-Sharp Sobolev Inequality

In this section, we will prove the following result:

Let $f \in C_0^\infty(\mathbb{R}^n)$ and let $1 \leqslant p < q < \infty$ with $\frac{1}{p} - \frac{m}{n} < \frac{1}{q} \Leftrightarrow \frac{1}{p} - \frac{1}{q} = \frac{m}{n} - \epsilon\,(\epsilon > 0)$. Then we have

$$\|f\|_{L^q} \lesssim \|f\|_{L^p} + \|\partial^m f\|_{L^p}.$$

Remark 3.13. *Recall the that by Sobolev Embedding, we know the statement is true even if* $\frac{1}{p} - \frac{m}{n} = \frac{1}{q}$.

Proof. We write $f = P_{\leqslant 0} f + \sum\limits_{k\in\mathbb{N}} P_k f =: f_{\leqslant 0} + \sum\limits_{k\geqslant 1} f_k$. Then

$$\|f\|_q \leqslant \underbrace{\|f_{\leqslant 0}\|_q}_{(1)} + \underbrace{\sum_{k>0} \|f_k\|_q}_{(2)}$$

Recall by Bernstein's Inequality, we have $\|P_k f\|_q \lesssim 2^{nk(\frac{1}{p}-\frac{1}{q})}\|f\|_p$, so $\|P_{\leqslant 0} f\|_q \lesssim \|f\|_p$. Recall we also have the following bound on derivatives: $\|\partial^i P_k f\|_p \lesssim 2^{ki}\|f\|_p$, so $2^{ki}\|P_k f\|_p \lesssim \|\partial^i f\|_p$. Then

$$\begin{aligned}
(1) &= \|f_{\leqslant 0}\|_q \leqslant \|f\|_p \\
(2) &\approx \sum_{k>0} \|P_k P_k f\|_{L^q} \\
&\leqslant \sum_{k>0} 2^{nk(\frac{1}{p}-\frac{1}{q})}\|P_k f\|_p \\
&= \sum_{k>0} 2^{nk(\frac{m}{n}-\epsilon)}\|P_k f\|_p \\
&= \sum_{k>0} 2^{-kn\epsilon} \cdot 2^{km}\|P_k f\|_p \\
&\leqslant \sum_{k>0} 2^{-kn\epsilon}\|\partial^m f\|_p \\
&\leqslant \sum_{k>0} (2^{-n\epsilon})^k \cdot \|\partial^m f\|_p \\
&\lesssim \|\partial^m f\|_p
\end{aligned}$$

Hence we have

$$\|f\|_q \leqslant (1) + (2) \lesssim \|f\|_p + \|\partial^m f\|_p.$$

□

3.2.3 Product Estimate

In this section, we will prove that the following estimate holds true: for all $s > 0$ and $f, g \in H^s(\mathbb{R}^n)$,

$$\|fg\|_{H^s} \leqslant \|f\|_{L^\infty}\|g\|_{H^s} + \|g\|_{L^\infty}\|f\|_{H^s} \quad (*)$$

Remark 3.14. *If $s = 0$, then $\|fg\|_{L^2} \leqslant \|f\|_{L^\infty}\|g\|_{L^2} + \|f\|_{L^2}\|g\|_{L^\infty}$ clearly holds.*

Remark 3.15. *For $s > \frac{n}{2}$, by Sobolev inequality, we have $\|f\|_\infty \leqslant \|f\|_{H^s}$. Then $(*)$ implies that $\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}$. Hence for $s > \frac{n}{2}$, H^s is an algebra.*

Remark 3.16 (Product Estimate)**.** *Let f, g be two functions on $\mathbb{R}^n$. Consider*

$$P_k(fg) = \sum_{k',k''\in\mathbb{Z}} P_k(P_{k'} f \cdot P_{k''} g),$$

as we can write $f = \sum_{k' \in \mathbb{Z}} P_{k'} f$ *and* $g = \sum_{k'' \in \mathbb{Z}} P_{k''} g$*. Since* $P_{k'} f$ *has Fourier support in* $D' = \{2^{k'-1} \leqslant |\xi| \leqslant 2^{k'+1}\}$*,* $P_{k''}(f)$ *has Fourier support in* $D'' = \{2^{k''-1} \leqslant |\xi| \leqslant 2^{k''+1}\}$*. Then* $P_{k'} f \cdot P_{k''} g$ *has Fourier support in* $D' + D''$ *(by the property of convolution). So for* $P_k(P_{k'} f P_{k''} g)$ *not being zero, we need* $D' + D''$ *intersect* $\{2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$*.*

Notation: Write $f_k := P_k f$, $f_{<k} := P_{<k} f$, $f_J := P_J f$ with $J \subset \mathbb{Z}$ being an interval.

Lemma 3.17. *Given functions* f, g*, we have the following decomposition:*

$$P_k(f, g) = HH_k(f, g) + LL_k(f, g) + LH_k(f, g) + HL_k(f, g).$$

Here

$$\begin{aligned}
HH_k(f, g) &= P_k(P_{k+4} f \cdot P_{k+6} g) + P_k(P_{k+6} f \cdot P_{k+4} g) + \sum_{\substack{k', k'' > k+5 \\ |k' - k''| \leqslant 3}} P_k(P_{k'} f \cdot P_{k''} g) \\
LL_k(f, g) &= P_k(f_{[k-5,k+5]} \cdot g_{[k-5,k+5]}) \\
LH_k(f, g) &= P_k(f_{\leqslant k-5} \cdot g_{[k-3,k+3]}) \\
HL_k(f, g) &= P_k(f_{[k-3,k+3]} \cdot g_{\leqslant k-5}).
\end{aligned}$$

Remark 3.18. *We can write this schematically, i.e.,*

$$P_k(fg) = HH_k(f, g) + LL_k(f, g) + LH_k(f, g) + HL_k(f, g)$$

where

$$\begin{aligned}
HH_k(f, g) &= P_k\Big(\sum_{m > k} f_m \cdot g_m\Big) \\
LL_k(f, g) &= P_k(f_{<k} \cdot g_{<k}) \\
LH_k(f, g) &= P_k(f_{<k} \cdot g_k) \\
HL_k(f, g) &= P_k(f_k \cdot g_{<k})
\end{aligned}$$

Proof of Product Estimate:
By Proposition (3.8)

$$\begin{aligned}
\|fg\|_{\dot{H}^s}^2 &\approx \sum_k 2^{2ks} \|P_k(fg)\|_{L^2}^2 \\
&\leqslant \underbrace{\sum_k 2^{2ks} \|P_k(f_{<k} g)\|_{L^2}^2}_{I} + \underbrace{\sum_k 2^{2ks} \|P_k(f_{\geqslant k} g)\|_{L^2}^2}_{II}
\end{aligned}$$

We bound I and II separately:

$$
\begin{aligned}
II &\leqslant \sum_k 2^{2ks} \|f_{\geqslant k} g\|_{L^2}^2 \\
&\leqslant \sum_k 2^{2ks} \|f_{\geqslant k}\|_{L^2}^2 \|g\|_{L^\infty}^2 \\
&\leqslant \|g\|_{L^\infty}^2 \sum_k \sum_{k' \geqslant k} 2^{2ks} \|f_{k'}\|_{L^2}^2 \\
&= \|g\|_{L^\infty}^2 \sum_k \sum_{k' \geqslant k} 2^{2(k-k')s} 2^{2k's} \|f_{k'}\|_{L^2}^2 \\
&= \|g\|_{L^\infty}^2 \sum_{k'} \left(\sum_{k \leqslant k} 2^{2(k-k')s} \right) \|2^{k's} f_{k'}\|_{L^2}^2 \\
&\leqslant C_s \|g\|_{L^\infty}^2 \sum_{k'} \|2^{k's} f_{k'}\|_{L^2}^2 \\
&\approx C_s \|g\|_{L^\infty}^2 \|f\|_{\dot{H}^s}^2.
\end{aligned}
$$

To estimate I, we need to control the following terms first:

$$
\begin{aligned}
\sum_k 2^{2ks} \|P_k(f_{[k-3,k]} g)\|_{L^2}^2 &\lesssim \|g\|_{L^\infty}^2 \sum_k 2^{2ks} \|f_{[k-3,k]}\|_{L^2}^2 \\
&\leqslant \|g\|_{L^\infty}^2 \sum_k 2^{2ks} \sum_{k'=k-3}^{k} \|f_{k'}\|_{L^2}^2 \\
&\lesssim \|g\|_{L^\infty}^2 \sum_{k'} 2^{2k's} \|f_{k'}\|_{L^2}^2 \\
&\approx \|g\|_{L^\infty}^2 \|f\|_{\dot{H}^s}^2
\end{aligned}
$$

For the cases in I not cover above, we have the following estimate:

$$
\begin{aligned}
P_k(f_{\leqslant k-3} g) &= \sum_{k'} P_k(f_{\leqslant k-3} g'_k) \\
&= \sum_{k' < k-2} P_k(f_{\leqslant k-3} g_{k'}) + \sum_{k-2 \leqslant k' \leqslant k+2} P_k(f_{\leqslant k-3} g_{k'}) + \sum_{k' > k+2} P_k(f_{\leqslant k+3} g_{k'}) \\
&= 0 + \sum_{k-2 \leqslant k' \leqslant k+2} P_k(f_{\leqslant k-3} g_{k'}) + 0 \\
&= \sum_{k-2 \leqslant k' \leqslant k+2} P_k(f_{\leqslant k-3} g_{k'}).
\end{aligned}
$$

So

$$
\begin{aligned}
&\sum_k 2^{2ks} \|P_k(f_{\leqslant k-3} g)\|_{L^2}^2 \\
&= \sum_k \sum_{k-2 \leqslant k' \leqslant k+2} 2^{2ks} \|P_k(f_{\leqslant k-3} g_{k'})\|_{L^2}^2 \\
&\leqslant \sum_k \sum_{k-2 \leqslant k' \leqslant k+2} 2^{2ks} \|f_{\leqslant k-3} g_{k'}\|_{L^2}^2 \\
&\lesssim \|f_{\leqslant k-3}\|_{L^\infty}^2 \sum_k \sum_{k-2 \leqslant k' \leqslant k+2} 2^{2ks} \|P_{k'} g\|_{L^2}^2 \\
&\approx \|f\|_{L^\infty}^2 \sum_{k'} 2^{k's} \|P_{k'} g\|_{L^2}^2 \\
&\leqslant \|f\|_{L^\infty} \|g\|_{\dot{H}^s}
\end{aligned}
$$

Combining the three estimates, we get the desired result.

3.2.4 Wente's Inequality

In this section, we will see another application of Littlewood-Paley Theory by proving the Wente's Inequality. We first need to prove the following Lemma:

Lemma 3.19. *We have the following control of norms:*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^{n/2}_{2,1}} = \sum_{k \in \mathbb{Z}} 2^{\frac{n}{2}k} \|P_k f\|_{L^2(\mathbb{R}^n)}.$$

Remark 3.20. *Recall for $1 \leqslant p < q < \infty$, $\frac{1}{p} - \frac{m}{n} < \frac{1}{q}$, we have*

$$\|f\|_{L^q(\mathbb{R}^n)} \leqslant \|f\|_{L^p} + \|\partial^m f\|_{L^p}.$$

Now if we let $q = \infty$, $m = \frac{n}{2}$, and $p = 2$, then

$$\frac{1}{p} - \frac{m}{n} = 0 \nless 0 = \frac{1}{q}.$$

We know

$$\|f\|_{L^\infty(\mathbb{R}^N)} \nleqslant \|f\|_{H^{n/2}(\mathbb{R}^n)} = \|f\|_{B^{n/2}_{2,2}(\mathbb{R}^n)}$$

and $B^{n/2}_{2,2}(\mathbb{R}^n) \supset B^{n/2}_{2,1}(\mathbb{R}^n)$. So the lemma would not be true for the $\dot{B}^{n/2}_{2,2}$ norm and $B^{n/2}_{2,1}(\mathbb{R}^n)$ norm is in some sense the "best" we can do.

Proof. By Bernstein Inequality, by selecting $q = \infty$, $p = 2$, we have

$$\|P_k f\|_{L^\infty} \leqslant 2^{kn \cdot \frac{1}{2}} \|f\|_{L^2}.$$

Since we know $P_k f \approx P_k P_k f$, then

$$\|P_k f\|_{L^\infty} \approx \|P_k P_k f\|_{L^\infty} \leqslant 2^{kn\cdot\frac{1}{2}} \|P_k f\|_{L^2}.$$

This gives

$$\|f\|_{L^\infty} = \|\sum_k P_k f\|_{L^\infty} \leqslant \sum_k \|P_k f\|_{L^\infty} \leqslant \sum_k 2^{kn\cdot\frac{1}{2}} \|P_k f\|_{L^2} = \|f\|_{\dot{B}^{n/2}_{2,1}}.$$

□

Lemma 3.21 (Discrete Version of the Young's Inequality)**.** *Let $f(k) \in L^1(\mathbb{Z})$ and $g(k), h(k) \in L^2(\mathbb{Z})$, then*

$$\sum_{k,\ell} f(k-l) g(l) h(k) \leqslant \|f\|_{\ell^1} \|g\|_{\ell^2} \|h\|_{\ell^2}.$$

Proof. See Appendix Young's Inequality for Convolution Alternative Form. □

Theorem 3.22 (Wente's Inequality)**.** *In $\mathbb{R}^2$, suppose $f, g \in \dot{H}(\mathbb{R}^2)$. Suppose u solves the following PDE:*

$$\Delta u = \partial_x f \partial_y g - \partial_y f \partial_x g, \tag{7}$$

then $u \in L^\infty(\mathbb{R}^n)$.

Remark 3.23. *$\partial_x f \partial_y g - \partial_y f \partial_x g \in L^1(\mathbb{R}^n)$ since $f, g \in \dot{H}(\mathbb{R}^2)$. Then for u satisfying Equation (9), we have (not rigorously) $u \in W^{2,1}(\mathbb{R}^2)$. Let $q = \infty$, $m = 2$, $q = 1$, by the standard Sobolev Embedding, we cannot get the result stated in the Theorem. So we need to make use of the special structure of the stated PDE.*

Proof. We make use of the following facts:

- If $\Delta f = h$, and if $h \in B^{s-2}_{2,1}(\mathbb{R}^n)$, we have $f \in B^s_{2,1}(\mathbb{R}^n)$.
- If $h \in L^p(\mathbb{R}^n)$ with $1 < p < +\infty$, then $f \in W^{2,p}(\mathbb{R}^n)$.
- if $h \in C^{0,\gamma}(\mathbb{R}^n)$, with $0 < \gamma < 1$, then $f \in C^{0,\gamma}(\mathbb{R}^n)$.

Then we claim that:

$$\|f\|_{B^s_{2,1}(\mathbb{R}^n)} \cong \|\Delta f\|_{B^{s-2}_{2,1}(\mathbb{R}^n)}.$$

proof of Claim:

$$\begin{aligned}
\|\Delta f\|_{\dot{B}^{s-2}_{2,1}(\mathbb{R}^n)} &= \sum_k 2^{k(s-2)} \|P_k \Delta f\|_2 \\
&= \sum_k 2^{k(s-2)} \|\Delta(P_k f)\|_2 && [P_k(\Delta f) = \Delta(P_k f)] \\
&\approx \sum_k 2^{k(s-2)} \cdot 2^{2k} \|P_k f\|_2 \\
&= \sum_k 2^{ks} \|P_k f\|_2 \\
&= \|f\|_{\dot{B}^s_{2,1}}
\end{aligned}$$

Using this claim, suffices to show that $\partial_x f \partial_y g - \partial_y f \partial_x g \in \dot{B}^{-1}_{2,1}$. If this is the case, then $u \in \dot{B}^{1}_{2,1}(\mathbb{R}^2)$, and by Lemma 3.19, we would have $u \in L^\infty(\mathbb{R}^2)$.

Now we proceed to show $df \wedge dg = \partial_x f \partial_y g - \partial_y f \partial_x g \in \dot{B}^{-1}_{2,1}$. By decomposing f, g into different frequency regions, and notice that the differential operator commutes with the Littlewood-Paley Projection operator, we obtain

$$df \wedge dg = \sum_k (LH_k + HL_k + HH_k + LL_K),$$

where

$$\begin{aligned} LH_k &= P_k(dP_{<k} f \wedge dP_k g) \\ HL_k &= P_k(dP_k f \wedge dP_{<k} g) \\ HH_K &= P_k\left(\sum_{m \geqslant k} dP_m f \wedge P_m g\right) \\ LL_k &= P_k\left(\sum_{m<k} dP_m f \wedge dP_m g\right) \end{aligned}$$

Since LH_k and HL_k are anti-symmetric, then we can bound them in the same way.

Bounding LH_k and HL_k: By definition of LH_k,

$$\begin{aligned} 2^{-k}\|LH_k\|_{L^2(\mathbb{R}^2)} &= 2^{-k}\|dP_{<k} f \wedge dP_k g\|_{L^2} \\ &\leqslant 2^{-k}\sum_{\ell<k}\|dP_\ell f \wedge dP_k g\|_{L^2} \\ &\lesssim 2^{-k}\sum_{\ell<k}\|dP_\ell f\|_{L^\infty}\|dP_k g\|_{L^2} \\ &\lesssim \sum_{\ell<k} 2^{\ell-k}\|DP_l f\|_{L^2}\|DP_k f\|_{L^2} \end{aligned}$$

The last inequality follows from Bernstein's Inequality as follows:

$$\begin{aligned} \|dP_\ell f\|_{L^\infty} &= \|P_\ell df\|_{L^\infty} \\ &\approx \|P_\ell P_\ell df\|_{L^\infty} \\ &\lesssim 2^\ell\|P_\ell df\|_{L^2} \\ &= 2^\ell\|Dp_\ell f\|_{L^2}. \end{aligned}$$

Next, by Lemma (3.21), we conclude

$$\begin{aligned}
&\sum_k 2^{-k}\|LH_k\|_{L^2} \\
&\leqslant \sum_k \sum_{\ell<k} \underbrace{2^{\ell-k}}_{f(k-\ell)} \underbrace{\|DP_\ell f\|_{L^2}}_{g(\ell)} \underbrace{\|DP_k g\|_{L^2}}_{h(k)} \\
&\lesssim \sum_{\substack{h<0 \\ h\in\mathbb{Z}}} 2^h \left(\sum_\ell \|DP_\ell f\|_{L^2}^2\right)^{1/2} \left(\sum_k \|DP_k f\|_{L^2}^2\right)^{1/2} && \text{(by Lemma 3.21)} \\
&\approx \|Df\|_{L^2}\|Dg\|_{L^2} < \infty
\end{aligned}$$

where the last line follows from Littlewood-Paley Inequality.

Bounding HH_k: we use the following identity:

$$df \wedge dg = d(f \wedge dg) \tag{1}$$

Then

$$\begin{aligned}
HH_k &= \sum_{m\geqslant k} P_k(dP_m f \wedge dP_m g) \\
&\overset{(1)}{=} \sum_{m\geqslant k} P_k d(P_m f \wedge dP_m g) \\
&= \sum_{m\geqslant k} dP_k(P_m f \wedge dP_m g).
\end{aligned}$$

Then by taking the norm on both sides, we have

$$\begin{aligned}
\|HH_k\|_{L^2} &\leqslant \sum_{m\geqslant k} \|dP_k(P_m f \wedge dP_m g)\|_{L^2} \\
&\leqslant \sum_{m\geqslant k} 2^k \|P_k(P_m f \wedge dP_m g)\|_{L^2} && \text{(Bound on Derivative)} \\
&\leqslant \sum_{m\geqslant k} 2^k 2^k \|P_m f \wedge dP_m g\|_{L^1} && \text{(Bernstein)} \\
&\leqslant \sum_{m\geqslant k} 2^{2k} \|P_m f\|_{L^2} \|DP_m g\|_{L^2} \\
&\leqslant \sum_{m\geqslant k} 2^{2k-m} 2^m \|P_m f\|_{L^2} \cdot \|DP_m g\|_{L^2} \\
&\leqslant \sum_{m\geqslant k} 2^{2k-m} \|DP_m f\|_{L^2} \|DP_m g\|_{L^2} && \text{(Bernstein)}
\end{aligned}$$

Therefore

$$2^{-k}\|HH_k\|_{L^2} \leqslant \sum_{m\geqslant k} 2^{k-m} \|DP_m f\|_{L^2} \|DP_m g\|_{L^2}.$$

Hence

$$
\begin{aligned}
\sum_k 2^{-k}\|HH_k\|_{L^2} &\leqslant \sum_k \sum_{m\geqslant k} 2^{k-m}\|DP_m f\|_{L^2}\|DP_m g\|_{L^2} \\
&= \sum_m \underbrace{\left(\sum_{k\leqslant m} 2^{k-m}\right)}_{<\infty} \|DP_m f\|_{L^2}\|DP_m g\|_{L^2} \\
&\lesssim \sum_m \|DP_m f\|_{L^2}\|DP_m g\|_{L^2} \\
&\leqslant \left(\sum_m \|DP_m f\|_{L^2}^2\right)^{1/2}\left(\sum_m \|DP_m g\|_{L^2}^2\right)^{1/2} \\
&\lesssim \|Df\|_{L^2}\|Dg\|_{L^2} < \infty.
\end{aligned}
$$

Bounding LL_k**:** We proceed directly,

$$
\begin{aligned}
\sum_k 2^{-k}\|LL_k\|_{L^2} &= \sum_k 2^{-k}\left\| P_k\left(\sum_{\ell\leqslant k} dP_\ell f \wedge dP_\ell g\right)\right\|_{L^2} \\
&\leqslant \sum_k \sum_{\ell\leqslant k} 2^{-k}\|dP_\ell f \wedge dP_\ell g\|_{L^2} && (L^p - \text{boundedness}) \\
&= \sum_k \sum_{\ell\leqslant k} 2^{-k}\|dP_\ell f\|_{L^\infty}\|dP_\ell g\|_{L^2} \\
&= \sum_k \sum_{\ell\leqslant k} 2^{-k+\ell}\|dP_\ell f\|_{L^2}\|dP_\ell g\|_{L^2} && (\text{Bernstein}) \\
&= \sum_\ell \left(\sum_{k\geqslant \ell} 2^{-k+\ell}\right)\|dP_\ell f\|_{L^2}\|dP_\ell g\|_{L^2} \\
&\lesssim \left(\sum_\ell \|dP_\ell f\|_{L^2}^2\right)^{1/2}\left(\sum_\ell \|dP_\ell g\|_{L^2}^2\right)^{1/2} \\
&\lesssim \|Df\|_{L^2}\|Dg\|_{L^2} < \infty.
\end{aligned}
$$

By combining the bonds on HL_k, LH_k, HH_k, LL_k, we conclude that $df \wedge dg \in B^{-1}_{2,1}(\mathbb{R}^2)$. This completes the proof of the theorem. $\square$

4 Calderon-Zygmund Theory

4.1 Weak-Type Inequality

Definition 4.1 (Weak Operator). *Let (X, μ) and (Y, ν) be measure spaces. Let T be an operator from $L^p(X, \mu)$ into the space of measurable functions from Y to $\mathbb{C}$. We say that the operator T is* ***weak*** (p, q) *with $q < \infty$, if for all $\lambda > 0$ and $f \in L^p(X, \mu)$, we have*

$$\nu(\{y \in Y : |Tf(y)| > \lambda\}) \lesssim \left(\frac{\|f\|_p}{\lambda}\right)^q.$$

We say T is ***weak*** (p, ∞) *if*

$$\|Tf\|_{L^\infty} \lesssim \|f\|_{L^p}.$$

Definition 4.2 (Strong Operator). *We say that T is* ***strong*** (p, q) *if it is a bounded operator from $L^p(X, \mu)$ to $L^q(Y, \mu)$, i.e.,*

$$\|Tf\|_{L^q(Y)} \leqslant C\|f\|_{L^P(x)}.$$

Remark 4.3. *If $q = \infty$, weak (p, ∞) is just strong (p, ∞), namely $\|Tf\|_\infty \leqslant C\|f\|_p$.*

Proposition 4.4. *If T is strong (p, q), then it is weak (p, q).*

Proof. For any $\lambda > 0$, we define $E_\lambda = \{y \in Y : |Tf(y)| > \lambda\}$. Then

$$\begin{aligned}
\mu(E_\lambda) &= \int_{E_\lambda} 1 d\mu \\
&\leqslant \int_{E_\lambda} \left|\frac{Tf(x)}{\lambda}\right|^q d\mu \\
&= \frac{1}{\lambda^q}\int_{E_\lambda} |Tf(x)|^q d\mu \\
&\leqslant \frac{1}{\lambda^q}\|Tf\|_q^q \\
&\leqslant \frac{1}{\lambda^q}(C\|f\|_p)^q \\
&= (\frac{C\|f\|_p}{\lambda})^q
\end{aligned}$$

□

Definition 4.5 (Distribution function). *Let (X, μ) be measurable space and $f : X \to \mathbb{C}$ be a measurable function. We define $a_f : (0, \infty) \mapsto [0, \infty]$, given by $a_f(\lambda) := \mu(\{x \in X : |f(x)| > \lambda\})$ as the* ***distribution function of*** *f (associated with μ).*

Proposition 4.6. *Let $\phi : [0, \infty) \mapsto [0, \infty)$ be differentiable, increasing and $\phi(0) = 0$. Then*

$$\int_X \phi(|f(x)|)d\mu = \int_0^\infty \phi'(\lambda)a_f(\lambda)d\lambda.$$

In particular, if $\phi(\lambda) = \lambda^p$, $p > 0$, we have

$$\|f(x)\|_{L_x^p}^p = \int_X |f(x)|^p d\mu = \int_0^\infty p\lambda^{p-1} a_f(\lambda) d\lambda.$$

Proof.

$$\begin{aligned}
\int_0^\infty \phi'(\lambda) \cdot a_f(\lambda) d\lambda &= \int_0^\infty \phi'(\lambda) \left(\int_X 1_{\{|f(x)| > \lambda\}} d\mu \right) d\lambda \\
&= \int_0^\infty \int_X 1_{\{|f(x)| > \lambda\}} d\mu \cdot \phi'(\lambda) d\lambda \\
&= \int_X \int_0^{|f(x)|} \phi'(\lambda) d\lambda d\mu \\
&= \int_X \phi(\lambda)|_{\lambda=0}^{\lambda=|f(x)|} d\mu \\
&= \int_X \phi(|f(x)|) d\mu.
\end{aligned}$$

□

Definition 4.7 (Sublinear Functions). *An operator T from a vector space of measurable function to a vector space of measurable functions is* ***sublinear*** *if the following is true*

- $|T(f_0 + f_1)(x)| \leqslant |Tf_0(x)| + |Tf_1(x)|$
- $|T(\lambda f)(x)| = |\lambda||Tf(x)|$, $\lambda \in \mathbb{C}$.

Theorem 4.8 (Marcinkiewicz Interpolation Theorem). *Let (X, μ) and (Y, μ) be measure spaces, $1 \leqslant p_0 < p_1 \leqslant \infty$ and T be a sublinear operator from $L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$ to the measurable functions on Y. If T is weak (p_0, p_0) and weak (p_1, p_1), then T is strong (p, p) for $p_0 < p < p_1$, i.e.,*

$$\|Tf\|_p^p \lesssim \|f\|_p^p.$$

Proof. Given $f \in L^p$, the idea is to decompose f as $f_0 + f_1 \in L^{p_0} + L^{p_1}$, this is possible since $p_0 < p < p_1$. For any $\lambda > 0$, with a constant $c > 0$ which we will fix later, set

$$\begin{aligned}
f_0 &:= f \cdot \chi_{\{x : |f(x)| > c\lambda\}} \\
f_1 &:= f \cdot \chi_{\{x : |f(x)| \leqslant c\lambda\}}
\end{aligned}$$

Then we can verify $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$:

$$\begin{aligned}
\|f_0\|_{p_0}^{p_0} &= \int |f_0|^{p_0} \\
&= \int |f|^{p_0} \cdot \chi_{\{x:|f(x)|>c\lambda\}} \\
&= \int |f|^{p_0-p}|f|^p \cdot \chi_{\{x:|f(x)|>c\lambda\}} \\
&\leqslant (c\lambda)^{p_0-p} \int |f|^p \\
&= (cf)^{p_0-p}\|f\|_p^p.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|f_1\|_p^p &= \int |f_1|^{p_1} \\
&= \int |f|^{p_1} \cdot \chi_{\{x:|f(x)|\leqslant c\lambda\}} \\
&\leqslant \int |f|^p \cdot |f|^{p_1-p} \cdot \chi_{\{x:|f(x)|\leqslant c\lambda\}} \\
&\leqslant (c\lambda)^{p_1-p}\|f\|_p^p.
\end{aligned}$$

As $a_{T_f}(\lambda) := \mu(\{y \in Y \ : \ |Tf(y)| > \lambda\})$, since T is sublinear, we have

$$|Tf| = |T(f_0 + f_1)| \leqslant |Tf_1| + |Tf_0|.$$

Hence, we observe that

$$a_{Tf}(\lambda) \leqslant a_{Tf_1}\left(\frac{\lambda}{2}\right) + a_{T_{f_0}}\left(\frac{\lambda}{2}\right) \leqslant \mu(\{y \in Y \ : \ |Tf_0(y)| > \lambda/2\}) + \mu(\{y \in Y \ : \ |Tf_1(y)| > \lambda/2\}).$$

Next, we consider controlling $a_{Tf_0}(\frac{\lambda}{2})$ and $a_{Tf_1}(\frac{\lambda}{2})$ by consider two case, $p_1 < \infty$ and $p_1 = \infty$.

Case 1: $p_1 < \infty$. Then since T is weak (p_0, p_0) and weak (p_1, p_1), then there exists constants A_0 and A_1, such that

$$\begin{aligned}
a_{Tf_0}(\frac{\lambda}{2}) &\leqslant \left(\frac{A_0}{\lambda/2}\|f_0\|_{p_0}\right)^{p_0} \\
a_{Tf_1}(\frac{\lambda}{2}) &\leqslant \left(\frac{A_1}{\lambda/2}\|f_1\|_{p_1}\right)^{p_1}
\end{aligned}$$

Hence by Proposition (4.6),

$$\begin{aligned}
\|Tf\|_p^p &= p\int_0^\infty \lambda^{p-1} a_{Tf}(\lambda)d\lambda \\
&\leqslant p\int_0^\infty \lambda^{p-1} a_{Tf_0}(\frac{\lambda}{2})d\lambda + p\int_0^\infty \lambda^{p-1} a_{Tf_1}(\frac{\lambda}{2})d\lambda \\
&\leqslant p\int_0^\infty \lambda^{p-1}(\frac{A_0}{\lambda/2}\|f_0\|_{p_0})^{p_0} d\lambda + p\int_0^\infty \lambda^{p-1}(\frac{A_1}{\lambda/2}\|f_1\|_{p_1})^{p} d\lambda \\
&= p\int_0^\infty \lambda^{p-1-p_0}(2A_0)^{p_0}\int_{\{x:|f(x)|>c\lambda\}} |f(x)|^{p_0} d\mu d\lambda + p\int_0^\infty \lambda^{p-1-p_1}(2A_1)^{p_1}\int_{\{x:|f(x)|\leqslant c\lambda\}} |f(x)|^{p_1} d\mu d\lambda \\
&= p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\int_0^{|f(x)|/c} \lambda^{p-1-p_0} d\lambda d\mu + p(2A_1)^{p_1}\int_X |f(x)|^{p_1}\int_{|f(x)|/c}^\infty \lambda^{p-1-p_1} d\lambda d\mu \\
&= p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\cdot \frac{1}{p-p_0}\left(\frac{|f(x)|}{c}\right)^{p-p_0} d\mu + p(2A_1)^{p_1}\int_X |f(x)|^{p_1}\cdot \frac{1}{p_1-p}\left(\frac{|f(x)|}{c}\right)^{p-p_1} d\mu \\
&= p(2A_0)^{p_0}\frac{1}{p-p_0}\left(\frac{1}{c}\right)^{p-p_0}\int_X |f(x)|^p d\mu + p(2A_1)^{p_1}\cdot\frac{1}{p_1-p}\left(\frac{1}{c}\right)^{p-p_1}\int_X |f(x)|^p d\mu \\
&\lesssim \|f\|_p^p.
\end{aligned}$$

Case 2: $p_1 = \infty$. Then by assumption, we can find constant A_1, such that $\|Tg\|_\infty \leqslant A_1\|g\|_\infty$. Next, we decompose f as $f_0 + f_1$ by

$$f_0 := f\cdot\chi_{\{x:|f(x)|>\frac{1}{2A_1}\lambda\}}$$

$$f_1 := f\cdot\chi_{\{x:|f(x)|\leqslant\frac{1}{2A_1}\lambda\}}$$

That is we let $c = \frac{1}{2A_1}$. Then we know $f_0 \in L^{p_0}$ and $f_1 \in L^\infty$ per previous analysis.

From T is weak (p_0, p_0), we have

$$a_{Tf_0}\left(\frac{\lambda}{2}\right) \leqslant \left(\frac{A_0}{\lambda/2}\|f_0\|_{p_0}\right)^{p_0}.$$

We also note that

$$\|Tf_1\|_\infty \leqslant A_1\|f_1\|_\infty \leqslant A_1\cdot\frac{1}{2A_1}\lambda = \frac{\lambda}{2}.$$

This implies $a_{Tf_1}\left(\frac{\lambda}{2}\right) = 0$.

Therefore, we have the following estimates:

$$\begin{aligned}
\|Tf\|_p^p &= p\int_0^\infty \lambda^{p-1}a_{Tf}(\lambda)d\lambda \\
&\leqslant p\int_0^\infty \lambda^{p-1}a_{Tf_0}(\frac{\lambda}{2})d\lambda + p\int_0^\infty \lambda^{p-1}a_{Tf_1}(\frac{\lambda}{2})d\lambda \\
&\leqslant p\int_0^\infty \lambda^{p-1}(\frac{A_0}{\lambda/2}\|f_0\|p_0)^{p_0}d\lambda \\
&= p\int_0^\infty \lambda^{p-1-p_0}(2A_0)^{p_0}\int_{\{x:|f(x)|>c\lambda\}}|f(x)|^{p_0}d\mu d\lambda \\
&= p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\left(\int_0^{|f(x)|/c}\lambda^{p-1-p_0}d\lambda\right)d\mu \\
&= p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\cdot\frac{1}{p-p_0}\left(\frac{|f(x)|}{c}\right)^{p-p_0}d\mu \\
&= p(2A_0)^{p_0}\frac{1}{p-p_0}\left(\frac{1}{c}\right)^{p-p_0}\int_X |f(x)|^p d\mu \\
&\lesssim \|f\|_p^p
\end{aligned}$$

This concludes the proof of the Theorem. □

4.2 The Hardy-Littlewood Maximal Function

Definition 4.9 (Hardy-Littlewood Maximal Function)**.** *Let $B_r = B(0,r)$ be the Euclidean ball of radius r centered at the origin. The* ***Hardy-Littlewood maximal function (with balls)*** *of a locally integral function f in $\mathbb{R}^n$ is defined by*

$$Mf(x) := \sup_{r>0}\frac{1}{|B_r|}\int_{B_r}|f(x-y)|dy.$$

Similarly, we can define the ***Hardy-Littlewood maximal function (with cubes)****. That is, we let Q_r be the cube $[-r,r]^n$ and define*

$$M'f(x) := \sup_{r>0}\frac{1}{(2r)^n}\int_{Q_r}|f(x-y)|dy.$$

More generally, we can take the supremum over cubes that is not centered at x, hence we have the following ***Hardy-Littlewood maximal function (with non-centered cubes****:*

$$M''f(x) := \sup_{x\in Q}\left[\frac{1}{|Q|}\int_Q |f(y)|dy\right]$$

where Q is a cube. Then it clear that $M'f(x) \leqslant M''f(x)$.

Remark 4.10. *If $f \in L^1(\mathbb{R}^n)$, then*

$$\lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)| dy = 0.$$

Remark 4.11. *We will prove this operator is weak $(1,1)$ and weak (∞, ∞). So by the Marcinkiewicz Interpolation Theorem, the operator M is strong (p,p) for any $p \in (1, \infty)$.*

Proposition 4.12. *For $n \geqslant 1$, there exists constants c_n and C_N, depending only on n, such that*

$$c_n M' f(x) \leqslant M f(x) \leqslant C_n M' f(x)$$

for all $f \in L^1_{loc}(\mathbb{R}^n)$.

Proof. We show one direction, that is $c_n M' f(x) \leqslant M f(x)$. Given any ball B_r centered at 0 with radius r, then its circumscribing cube Q_r has side length $2r$. Then

$$\frac{1}{|B_r|} \int_{B_r} |f(x-y)| dy \leqslant \frac{1}{|B_r|} \int_{Q_r} |f(x-y)| dy = \frac{|Q_r|}{|B_r|} \frac{1}{|Q_r|} \int_{B_r} |f(x-y)| dy.$$

This show $c_n M' f(x) \leqslant M f(x)$, where $c_n = \frac{|B_r|}{|Q_r|}$ only depending on the dimension n. Similarly, by putting the cube in a ball, we establish the other direction of the inequality. □

Definition 4.13 (Dyadic Cubes). *We define the* ***Dyadic cubes*** *in $\mathbb{R}^n$ in the following way: we first define Q_k as the family of cubes, open on the right, whose vertices are adjacent points of the lattice $(2^{-k}\mathbb{Z})^n$. Then we call any cube in the collection $\bigcup_{k \in \mathbb{Z}} Q_k$ to be a dyadic cube.*

The following properties is clear from the definition of Dyadic cubes:

Proposition 4.14. *From the construction of Dyadic cubes, they satisfy the following three properties:*

1. *Given $x \in \mathbb{R}^n$, there is a unique cube in each family Q_k, $k \in \mathbb{Z}$, which contains x.*
2. *Any two dyadic cube are either disjoint or one is wholly contained in the other.*
3. *A dyadic cube in Q_k is contained in a unique cube of each family Q_j, $j < k$ and Q_k contains 2^n dyadic cubes of Q_{k+1}.*

Definition 4.15 (Dyadic Maximal function). *Given any $f \in L^1_{loc}(\mathbb{R}^n)$, we define*

$$E_k f(x) := \sum_{Q \in Q_k} \left[\left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x) \right]$$

Given $f \in L^1_{loc}(\mathbb{R}^n)$, its ***dyadic maximal function*** *$M_d f(x)$ is defined as*

$$M_d f(x) := \sup_k |E_k f(x)|$$

Remark 4.16. *If Ω is the union of several cubes in $\mathbb{Q}_k$, then we have*

$$\int_\Omega E_k f = \int_\Omega f. \tag{8}$$

Suppose we can write $\Omega = \bigcup_{i \in I} P_i$ for some index set I, $P_i \in Q_k$. It is to be noted that this decomposition of Ω is unique. Since for all $P_i \in Q_k$ and $Q \in Q_k$, we either have $P_i = Q$ or $P_i \cap Q = \varnothing$, by the definition of E_k, we have

$$\begin{aligned}
\int_\Omega E_k f(x)\,dx &= \sum_{i \in I} \int_{P_i} \left[\sum_{Q \in \mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x) \right] dx \\
&= \sum_{i \in I} \sum_{Q \in \mathcal{Q}_k, P_i = Q} \int_{P_i} \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x)\,dx \\
&= \sum_{i \in I} \int_{P_i} \left(\frac{1}{|P_i|} \int_{P_i} f \right) \chi_{P_i}(x)\,dx \\
&= \sum_{i \in I} \left(\frac{1}{|P_i|} \int_{P_i} f \right) \int_{P_i} 1\,dx \\
&= \sum_{i \in I} \int_{P_i} f = \int_\Omega f.
\end{aligned}$$

Theorem 4.17. *The dyadic maximal function $M_d f(x)$ is weak $(1,1)$, namely*

$$m(\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}) \leqslant \frac{1}{\lambda} \|f\|_1$$

for all $\lambda \in \mathbb{R}^+$.

Remark 4.18. *Any function f can be decomposed in the following way:*

$$f = f_+ - f_-.$$

Then we note that

$$\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} \subset \{x \in \mathbb{R}^n : M_d f^+(x) > \lambda\} \cup \{x \in \mathbb{R}^n : M_d f^-(x) > \lambda\}.$$

So without the loss of generality, we may assume in the proof that f is nonnegative.

Proof. For any $\lambda \in \mathbb{R}^+$, we have $\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \bigcup\limits_k \Omega_k$, where

$$\Omega_k = \{x \in \mathbb{R}^n : E_k f(x) > \lambda \text{ and } E_j f(x) \leqslant \lambda \text{ if } j < k\}.$$

Since $f \in L^1$, we note $E_k f(x) \to 0$ as $k \to -\infty$. It is also clear from the definition of Ω_k that the Ω_k's are disjont. We claim that each Ω_k can be written as the union of cubes in Q_k. Since if $x \in \Omega_k$, then $x \in q_k \in Q_k$ for some dyadic cube q_k. Then for all $\tilde{x} \in q_k$, $\tilde{x} \in q_k$, $E_j(\tilde{x}) = \mathbb{E}_j(x)$ for any $j \leqslant k$. This shows $\tilde{x} \in \Omega_k$ hence

$q_k \subset \Omega_k$.

Hence

$$\begin{aligned} m(\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}) &= \left|\bigcup_k \Omega_k\right| \\ &= \sum_k |\Omega_k| \\ &\leqslant \sum_k \int_{\Omega_k} \frac{E_k f}{\lambda} \\ &= \sum_k \frac{1}{\lambda} \int_{\Omega_k} f && \text{(by 8)} \\ &= \sum_k \frac{1}{\lambda} \|f\|_{L^1(\mathbb{R}^n)} \end{aligned}$$

The last line follows since $\Omega_k's$ are disjoint. □

Proposition 4.19. *Suppose $f \in L^1(\mathbb{R}^n)$ and $f \geqslant 0$. Then for any $\lambda > 0$, we have*

$$m(\{x \in \mathbb{R}^n : M' f(x) > 4^n \lambda\}) \leqslant 2^n m(\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}).$$

Proof. We know

$$\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \bigcup_k \Omega_k = \bigcup_i C_i$$

where $\Omega_k = \{x \in \mathbb{R}^n : E_k f(x) > \lambda,\ E_j f(x) \leqslant \lambda, \forall j < k\}$ and C_i are sequence of disjoint cubes that form the Ω_k.

Let $2C_i$ be the cube with the same centers as C_i and let the sides be twice as long. We claim

$$\{x \in \mathbb{R}^n : M' f(x) > 4^n \lambda\} \subset \bigcup_k \bigcup_i (2C_i^k).$$

We first show that if $x \notin \bigcup_i (2C_i)$, then $M' f(x) \leqslant 4^n \lambda$. For such an x, let Q be any cube that centered at x. Let ℓ be the side length of the cube Q. Hence there exists $k \in \mathbb{Z}$ such that $2^{k-1} \leqslant \ell < 2^k$. Then Q intersects m dyadic cube in $\mathbb{Q}_{-k}$ with $m \leqslant 2^n$. Denote these dyadic cubes as $R_1, R_2, \cdots, R_m$. None of these are contained in any of C_i^k's, for otherwise we would have $x \in \bigcup_{k,i} (2C_i^k)$.

Hence for any of these R_i, we have $\frac{1}{|R_i|}\int_{R_i} f \leqslant \lambda$. So

$$\begin{aligned}
\frac{1}{|Q|}\int_Q f &= \frac{1}{|Q|}\sum_{i=1}^{m}\int_{Q\cap R_i} f \\
&\leqslant \frac{1}{|Q|}\sum_{i=1}^{m}\int_{R_i} f \\
&= \frac{1}{|Q|}\sum_{i=1}^{,}\frac{2^{kn}}{|R_i|}\int_{R_i} f \\
&= \frac{2^{kn}}{|Q|}\sum_{i=1}^{,}\frac{1}{|R_i|}\int_{R_i} f \\
&\leqslant \frac{2^{kn}}{2^{nk}\cdot 2^{-n}}2^n\cdot\lambda = 4^n\cdot\lambda.
\end{aligned}$$

This shows that if $M'f(x) > 4^n\lambda$, then $x \in \bigcup_{k,i}(2C_i^k)$. Since $m(2C_i^k) = 2^n(C_i^k)$, we conclude that

$$m(\{x \in \mathbb{R}^n : M'f(x) > 4^n\lambda\}) \leqslant 2^n m(\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}).$$

□

Theorem 4.20. *The operator M is weak $(1,1)$, weak (∞,∞) and strong (p,p) for any $p \in (1,\infty)$.*

Proof. Since

$$\frac{1}{B_r}\int_{B_r}|f(x-y)|dy \leqslant \frac{1}{B_r}\int_{B_r}\|f\|_{L^\infty}dy \leqslant \|f\|_{L^\infty}$$

So we see

$$\|Mf\|_{L^\infty} \leqslant \|f\|_{L^\infty},$$

that is M is strong (∞,∞), hence weak (∞,∞). Then by Marcinkiewicz Interpolation Theorem suffices to prove M is weak $(1,1)$. Given any $\lambda > 0$, let

$$E = \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$$

and

$$E' = \{x \in \mathbb{R}^n : M'f(x) > \lambda\}$$

Then we know $m(E) \approx m(E')$ by Proposition (4.12). So suffices to show that $m(E') \lesssim \frac{1}{\lambda}\|f\|_{L^1}$. We know by Theorem (4.17) that the operator M_d is weak $(1,1)$. Then

$$m(\{x \in \mathbb{R}^n : M_d f(x) > \frac{1}{4^n}\lambda)\} \lesssim \frac{1}{4^n\lambda}\|f\|_{L^1(\mathbb{R}^d)}.$$

Also by Proposition (4.19), we have

$$m(E) \leqslant 2^n m(\{x \in \mathbb{R}^n : M_d f(x) > \frac{1}{4^n}\lambda)\}$$

Hence we have

$$m(E') \lesssim \frac{1}{2^n \lambda}\|f\|_{L^1(\mathbb{R}^d)}$$

This shows M' is weak $(1,1)$, hence M is weak $(1,1)$. $\square$

We have another proof of Theorem (4.20) using the Vitali Covering Lemma (4.21).

Proof. Let m be the standard Lebesgue measure on $\mathbb{R}^n$. We show

$$m(\{x \in \mathbb{R}^n : |Mf(x)| > \lambda\}) \leqslant \frac{3^n}{\lambda}\|f\|_{L^1(\mathbb{R}^d)}$$

that is we show the operator M is weak $(1,1)$.

Recall $a_{Mf}(\lambda) = \{x : Mf(x) > \alpha\}$. Then by the definition of the Hardy-Littlewood Maximal function, for each $x \in a_{Mf}(\lambda)$ there exists a ball B_x centered at x such that

$$\frac{1}{m(B_x)}\int_{B_x} |f(y)|dy > \lambda.$$

Therefore, for such a ball B_x we have

$$m(B_x) < \frac{1}{\lambda}\int_{B_x} |f(y)|dy. \tag{9}$$

Fix an arbitrary compact subset K of $a_{Mf}(\lambda)$, we show

$$m(K) \leqslant \frac{3^n}{\lambda}\|f\|_{L^1(\mathbb{R}^d)}$$

this would imply

$$m(\{x \in \mathbb{R}^n : |Mf(x)| > \lambda\}) \leqslant \frac{3^n}{\lambda}\|f\|_{L^1(\mathbb{R}^d)}.$$

Since K is compact and is covered by $\bigcup_{x \in a_{Mf}(\lambda)} B_x$, we may select a finite subcover of K, say $K \subset \bigcup_{l=1}^N B_l$. The Vitali Covering lemma (4.21) guarantees the existence of a sub-collection $B_{i_1}, \cdots, B_{i_k}$ of disjoint balls with

$$m\left(\bigcup_{l=1}^N B_l\right) \leqslant 3^n \sum_{j=1}^k m(B_{i_j}). \tag{10}$$

Since the balls $B_{i_1}, \cdots, B_{i_k}$ are disjoint and satisfy (9) and (10), then

$$\begin{aligned} m(K) &\leqslant m\left(\bigcup_{l=1}^{N} B_l\right) \\ &= 3^n \sum_{j=1}^{k} m(B_{i_j}) \\ &= \frac{3^n}{\lambda} \sum_{j=1}^{k} \int_{B_{i_j}} |f(y)| dy \\ &= \frac{3^n}{\lambda} \int_{\bigcup B_{i_j}} |f(y)| dy \\ &\leqslant \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f(y)| dy \\ &= \frac{3^n}{\lambda} \|f\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

This finishes the proof of the theorem. □

Lemma 4.21 (Vitali Covering Lemma)**.** *Suppose $\mathcal{B} = \{B_1, B_2, \cdots, B_N\}$ is a finite collection of open balls in $\mathbb{R}^n$. Then there exists a disjoint sub-collection $B_{i_1}, B_{i_2}, \cdots, B_{i_k}$ of $\mathcal{B}$ that satisfies*

$$m\left(\bigcup_{l=1}^{N} B_l\right) \leqslant 3^n \sum_{j=1}^{k} m(B_{i_j}).$$

Proof. Firstly, pick a ball B_{i_1} in $\mathcal{B}$ with maximal radius, and then delete from $\mathcal{B}$ the ball B_{i_1}, as well as any balls that intersection B_{i_1}. Thus all the balls that are deleted are contained in the ball $\tilde{B}_{i_1}$ concentric with B_{i_1}, but with 3 times its radius.

The remaining balls yield a new collection B', for which we repeat the procedure. We pick B_{i_2} with largest radius in B' and delete from B' the ball B_{i_2} and any ball that intersects with it. Continuing this way we find, after at most N steps, a collection of disjoint balls $B_{i_1}, \cdots, B_{i_k}$. It is clear that the collection of balls we selected satisfies the desired inequality. □

4.3 Calderon-Zygmund Decomposition

Lemma 4.22 (Chebyshev's Inequality)**.** *If $f \in L^1(U)$ is nonnegative, where $U \subset \mathbb{R}^d$, then for any $\alpha > 0$, let $E_a := \{x : f(x) \geqslant a\}$, then*

$$m(E_a) \leqslant \frac{1}{a} \int_U f$$

where $m(E_a)$ denotes the standard Lebesgue measure on $\mathbb{R}^n$.

Proof. We can rewrite $m(E_a) = \int_U \chi_{E_a}$ We note that $f(x) \geqslant a\chi_{E_a}(x)$ for all $x \in E_a$. Hence

$$m(E_a) \leqslant \frac{1}{a}\int_U f.$$

□

Proposition 4.23. *Suppose $f \in L^1(\mathbb{R}^n)$ is nonnegative, then*

$$\lim_{k\to\infty} E_k f(x) = f(x) \quad a.e.$$

Proof. Let

$$C_\alpha := \left\{ x \,:\, \limsup_{k\to\infty} \left| \sum_{Q\in Q_k} \left[\left(\frac{1}{|Q|}\int_Q f(y)dy \right) \chi_Q(x) \right] - f(x) \right| > 2\alpha \right\}.$$

If we can show that for each $\alpha > 0$, the set has measure zero, then the set of all x such that $\lim\limits_{k\to\infty} E_k f(x) \neq f(x)$ is just the union

$$\bigcup_{j\in\mathbb{N}_+} C_{\frac{1}{j}}$$

which is the countable union of measure zero set which is again measure zero.

We fix α. For each $\epsilon > 0$, let g be a continuous function in $\mathbb{R}^n$ with compact support such that $\|f - g\|_{L^1}(\mathbb{R}^d) < \epsilon$. Then the continuity of g implies that

$$\lim_{k\to\infty} \sum_{Q\in Q_k} \left[\left(\frac{1}{|Q|}\int_Q g(y)dy \right) \chi_Q(x) \right] = g(x) \quad \forall x \in \mathbb{R}^n. \tag{11}$$

We can rewrite

$$\begin{aligned} &\sum_{Q\in Q_k} \left[\left(\frac{1}{|Q|}\int_Q f(y)dy \right) \chi_Q(x) \right] - f(x) \\ =& \sum_{Q\in Q_k} \left[\left(\frac{1}{|Q|}\int_Q f(y) - g(y) \right) \chi_Q(x) \right] + \sum_{Q\in Q_k} \left[\left(\frac{1}{|Q|}\int_Q g \right) \chi_Q(x) \right] - g(x) + g(x) - f(x) \end{aligned}$$

By 11, we have

$$\limsup_{k\to\infty} \sum_{Q\in Q_k} \left[\left(\frac{1}{|Q|}\int_Q g \right) \chi_Q(x) \right] - g(x) = 0.$$

Hence we have

$$\limsup_{k\to\infty} \left| \sum_{Q\in Q_k} \left[\left(\frac{1}{|Q|}\int_Q f(y)dy \right) \chi_Q(x) \right] - f(x) \right| \leqslant M_d(f-g)(x) - |g(x) - f(x)|.$$

Let

$$F_\alpha^1 := \{x \in \mathbb{R}^n : M_d(f-g)(x) > \alpha\}$$
$$F_\alpha^2 := \{x \in \mathbb{R}^n : |g(x) - f(x)| > \alpha\}$$

Then we see that $C_\alpha \subset F_\alpha^1 \cup F_\alpha^2$.

Next the fact $\|f-g\|_{L^1}(\mathbb{R}^d) < \epsilon$ and the operator M_d is weak $(1,1)$, we have

$$m(F_\alpha^1) \leqslant \frac{A}{\alpha}\|f-g\|_{L^1(\mathbb{R}^n)} = \frac{A}{\alpha}\epsilon.$$

By Chebyshev's inequality, since $|g-f|$ is nonnegative, we have

$$m(F_\alpha^2) \leqslant \frac{1}{\alpha}\|g-f\|_{L^1(\mathbb{R}^n)} = \frac{1}{\alpha}\epsilon.$$

This shows

$$m(C_\alpha) \leqslant (\frac{A}{\alpha}+1)\epsilon$$

Since ϵ is arbitrary, then letting $\epsilon \to 0$, we conclude $m(C_\alpha) = 0$. This finishes the proof of this proposition. □

Theorem 4.24 (Calderon-Zygmund Decomposition). *Let f be an integrable function and be nonnegative. For any $\lambda > 0$, there exists a sequence of disjoint dyadic cubes such that*

1. $f(x) \leqslant \lambda$ *for almost every* $x \notin \bigcup_i C_j$.
2. $\left|\bigcup_j C_j\right| \leqslant \frac{1}{\lambda}\|f\|_{L^1}$.
3. $\lambda < \frac{1}{|C_j|}\int_{C_j} f \leqslant 2^n\lambda$.

Remark 4.25. *The three properties says the following:*

1. *The "good parts" ($f(x) \leqslant \lambda$) is for most cubes.*
2. *"Bad parts" is confined in several cubes, and the size of the bad points is controlled.*
3. *The average of f in each bad cube is also controlled.*

Proof. We use $M_d f(x)$ and using the notation in the Proof of Theorem (4.17), we have

$$\{x \in \mathbb{R}^n : M_d(x) > \lambda\} = \bigcup_k \Omega_k.$$

And we also know that Ω_k can be decomposed into disjoint dyadic cubes $\{Q_k^j\}$ that lies in Q_k and these dyadic cubes form the family $\{C_i\}_{i\in\mathbb{N}}$. That is

$$\bigcup_{k\in\mathbb{Z}} \Omega_k = \bigcup_{k\in\mathbb{Z}} \left(\bigcup_j Q_k^j\right) = \bigcup_{i\in\mathbb{N}} C_i.$$

Since the $\Omega_k's$ are disjoint, then the $C_i's$ are still disjoint. Now if $x \notin \bigcup_i C_i$, then there does not exists any $k \in \mathbb{Z}$ such that $E_k f(x) > \lambda$. So

$$\limsup_{k \to \infty} E_k f(x) \leqslant \lambda$$

and by the fact that if $f \in L^1_{loc}(\mathbb{R}^n)$ is non-negative, then

$$\lim_{k \to \infty} E_k f(x) = f(x) \quad \text{a.e.}$$

We have for almost every $x \notin \bigcup_j Q_j$, $f(x) \leqslant \lambda$. So the sequence $\{C_i\}$ satisfies property 1. Next by Theorem 4.17, we have

$$\left| \bigcup_i C_i \right| = \left| \bigcup_k \Omega_k \right| = \mu |\{x \in \mathbb{R}^n \ : \ M_d f(x) > \lambda\}| \leqslant \frac{\|f\|_{L^1}}{\lambda}.$$

This verifies property 2. Lastly, if $x \in \bigcup_i C_i$, let $x \in C_i \subset \Omega_k$. Then $E_k f(x) > \lambda$ implies

$$\lambda < \frac{1}{|C_i|} \int_{C_i} f.$$

Next, if we assign $\tilde{C}_i$ to be the unique dyadic cube with side length 2^{-k+1} such that $C_i \subset \tilde{C}_i$. Then since $x \in \Omega_k$, we must have $E^{k-1} \leqslant \lambda$. So

$$\frac{1}{|\tilde{C}_i|} \int_{\tilde{C}_i} f \leqslant \lambda.$$

Since f is nonnegative, and $\tilde{C}_i$ has double the side length of C_i, then

$$\begin{aligned} \frac{1}{2^n |C_i|} \int_{C_i} f &\leqslant \frac{1}{2^n |C_i|} \int_{\tilde{C}_i} f \\ &= \frac{1}{|\tilde{C}_i|} \int_{\tilde{C}_i} f \leqslant \lambda. \end{aligned}$$

This verifies Property (3). $\square$

4.4 Hilbert Transform in $\mathbb{R}$

Definition 4.26 (Principle Value of $\frac{1}{x}$)**.** *Let $x \in \mathbb{R}$. The principle value of $\frac{1}{x}$ abbreviated p.v. $\frac{1}{x}$ is defined as*

$$p.v. \frac{1}{x}(\phi) := \lim_{\epsilon \to 0^+} \int_{|x| > \epsilon} \frac{\phi(x)}{x} dx$$

with ϕ being Schwarz function.

Remark 4.27. *Note that*

$$
\begin{aligned}
p.v.\frac{1}{x}(\phi) &= \lim_{\epsilon\to 0^+}\int_{|x|>\epsilon}\frac{\phi(x)}{x}dx \\
&= \lim_{\epsilon\to 0^+}\int_{\epsilon<|x|<1}\frac{\phi(x)}{x}dx + \int_{|x|\geqslant 1}\frac{\phi(x)}{x}dx
\end{aligned}
$$

We note

$$
\lim_{\epsilon\to 0^+}\int_{\epsilon\leqslant|x|<1}\frac{\phi(x)}{x}dx = \lim_{\epsilon\to 0^+}\int_{\epsilon<|x|<1}\frac{\phi(x)-\phi(0)}{x}dx.
$$

Since

$$
\lim_{\epsilon\to 0^+}\int_{\epsilon<|x|<1}\frac{\phi(0)}{x}dx = \lim_{\epsilon\to 0^+} 0 = 0.
$$

Then by the mean value theorem, we have

$$
\begin{aligned}
\lim_{\epsilon\to 0^+}\int_{\epsilon\leqslant|x|<1}\frac{\phi(x)}{x}dx &= \lim_{\epsilon\to 0^+}\int_{\epsilon<|x|<1}\phi'(\xi)dx \\
&\leqslant \|\phi'\|_{L^\infty}
\end{aligned}
$$

Now for the other term, we have

$$
\begin{aligned}
\int_{|x|\geqslant 1}\frac{\phi(x)}{x}dx &= \int_{|x|\geqslant 1}\frac{\phi(x)x}{x^2}dx \\
&\leqslant \|x\phi(x)\|_{L^\infty}\int_{|x|\geqslant 1}\frac{1}{x^2}dx < \infty
\end{aligned}
$$

Thus we conclude that

$$
\left|p.v.\frac{1}{x}(\phi)\right| \lesssim \|\phi'\|_{L^\infty} + \|x\phi\|_{L^\infty}
$$

Hence the principle value of $\frac{1}{x}$ is well-defined, in fact, this is a tempered distribution.

Definition 4.28 (Hilbert Transform)**.** *Let $f \in L^p(\mathbb{R})$, then we define its **Hilbert transform**, Hf by*

$$
\begin{aligned}
Hf(x) &:= \frac{1}{\pi}p.v.\frac{1}{x} * f(x) \\
&:= \frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_{|y|>\epsilon}\frac{f(x-y)}{y}dy
\end{aligned}
$$

Remark 4.29. *By definition,*

$$
\begin{aligned}
Hf(x) &= \frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_{|y|>\epsilon}\frac{f(x-y)}{y}dy \\
&= \frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_{|y|>\epsilon}\frac{f(y)}{x-y}dy
\end{aligned}
$$

Then if $x \notin \operatorname{supp} f$ (assume f is compactly supported), we have

$$Hf(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x-y} f(y) dy.$$

This is the case, since whenever $\epsilon < \operatorname{dist}(x, \operatorname{supp} f)$, we note that

$$\frac{1}{\pi} \int_{|x-y|<\epsilon} \frac{f(y)}{x-y} dy = 0$$

Next we state the following useful fact:
Fact:

$$\widehat{Hf}(\xi) = -i \cdot \operatorname{sgn}(\xi) \hat{f}(\xi).$$

As an consequence, we can easily establish the following results:

Lemma 4.30.

1. $\|Hf\|_{L^2} = \|f\|_{L^2}$.
2. $H(Hf) = -f$.
3. $\int Hf \cdot g = -\int f \cdot Hg$.

Proof. (1) is clear from the fact. For (2), we have

$$\widehat{H(Hf)} = -i \cdot \operatorname{sgn}(\xi) \widehat{Hf}(\xi) = -i \cdot \operatorname{sgn}(\xi) \cdot -i \operatorname{sgn}(\xi) \hat{f}(\xi) = -\hat{f}(\xi).$$

Taking inverse Fourier transform on both sides, we get the desired result.

For (3). We recall that

$$\begin{aligned}\langle f, g \rangle &:= \int f \cdot \bar{g} \\ &= \langle \hat{f}, \hat{g} \rangle.\end{aligned}$$

Then

$$\begin{aligned}\langle Hf, g \rangle &= \langle \widehat{Hf}, \hat{g} \rangle \\ &= \langle -i \cdot \operatorname{sgn}(\xi) \hat{f}, \hat{g} \rangle \\ &= -i \cdot \frac{1}{\overline{-i}} \langle \hat{f}, \widehat{Hg} \rangle \\ &= -\langle \hat{f}, \widehat{Hg} \rangle \\ &= -\langle f, Hg \rangle.\end{aligned}$$

$\square$

Theorem 4.31. *For f being a Schwarz function, the following assertions are true:*

1. *(Kolmogorov) H is weak $(1,1)$, that is*

$$m(\{x \in \mathbb{R} : |Hf(x)| > \lambda\}) \lesssim \frac{1}{\lambda}\|f\|_{L^1}.$$

2. *(M. Riese) H is strong (p,p) for $1 < p < \infty$, that is*

$$\|Hf\|_p \lesssim \|f\|_p.$$

Proof.

1. Fix $\lambda > 0$ and f non-negative. From the Calderon-Zygmund Decomposition (4.24) of f at height λ, we have a sequence of disjoint interval $\{I_j\}$, such that

   (a) $f(x) \leqslant \lambda$ for a.e. $x \notin \bigcup_j I_j$.

   (b) $\left|\bigcup_j I_j\right| \leqslant \frac{1}{\lambda}\|f\|_{L^1}$.

   (c) $\lambda < \frac{1}{|I_j|}\int_{I_j} f \leqslant 2\lambda$.

   We now decompose f as $f = g + b$ with

$$g(x) = \begin{cases} f(x) & x \notin \bigcup_j I_j \\ \frac{1}{|I_j|}\int_{I_j} f & x \in I_j \end{cases}$$

   In particular, we have $g(x) \leqslant 2\lambda$ a.e. Then we have $b(x) = \sum_j b_j(x)$, where

$$b_j(x) = \left(f(x) - \frac{1}{|I_j|}\int_{I_j} f\right)\chi_{I_j}(x).$$

   Then we note that $\int_{\mathbb{R}} b_j dx = 0$, so $\int_{\mathbb{R}} b dx = 0$ and $\int_{\mathbb{R}} g(x)dx = \|f\|_{L^1}$.

   Next, since $f = g + b$, $Hf \leqslant Hg + Hb$. Then

$$m(\{x \in \mathbb{R} : |Hf(x)| > \lambda\}) \leqslant \underbrace{m(\{x \in \mathbb{R} : |Hg(x)| > \lambda/2\})}_{(1)} + \underbrace{m(\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\})}_{(2)}$$

For (1), let $E_{\lambda/2} = \{x \in \mathbb{R} : |Hg(x)| > \lambda/2\}$. Then

$$\begin{aligned} m(E_{\lambda/2}) &= \int \chi^2_{E_{\lambda/2}} \\ &\leqslant \frac{4}{\lambda^2} \int |Hg(x)|^2 \\ &= \frac{4}{\lambda^2} \int |g(x)|^2 \\ &\leqslant \frac{4}{\lambda^2} \|g\|_{L^\infty} \int g(x) \\ &= \frac{8}{\lambda} \|g\|_{L^1} = \frac{8}{\lambda} \|f\|_{L^1}. \end{aligned}$$

To control (2). Let $2I_j$ be the interval with the same center as I_j and twice the length, and let $\Omega^* = \bigcup_j (2I_j)$. Then

$$m(\Omega^*) \leqslant 2 \left| \bigcup_j I_j \right| \leqslant \frac{2}{\lambda} \|f\|_{L^1(\mathbb{R})}.$$

Then

$$(2) = m(\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}) \leqslant m(\Omega^*) + m(\{x \notin \Omega^* : |Hb(x)| > \lambda/2\}).$$

Since $m(\Omega^*)$ is bounded by $\|f\|_{L^1(\mathbb{R})}$, then suffices to bound the second term:

$$\begin{aligned} & m(\{x \notin \Omega^* : |Hb(x)| > \lambda/2\}) \\ \leqslant{} & \frac{2}{\lambda} \int_{\mathbb{R} \setminus \Omega^*} |Hb(x)| dx \\ \leqslant{} & \frac{2}{\lambda} \sum_j \int_{\mathbb{R} \setminus \Omega^*} |Hb_j(x)| dx \\ \leqslant{} & \frac{2}{\lambda} \sum_j \int_{\mathbb{R} \setminus 2I_j} |Hb_j(x)| dx. \end{aligned}$$

When $x \notin 2I_j$, we have

$$\begin{aligned} Hb_j(x) &= \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y| > \epsilon} \frac{b_j(x-y)}{y} dy \\ &= \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|x-t| > \epsilon} b_j(t) \cdot \frac{1}{x-t} dt \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{b_j(t)}{x-t} dt \\ &= \int_{I_j} \frac{b_j(y)}{x-y} dy \end{aligned}$$

Hence

$$\int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|dx = \int_{\mathbb{R}\setminus 2I_j} \left| \int_{I_j} \frac{b_j(y)}{x-y} dy \right| dx$$
$$= \int_{\mathbb{R}\setminus 2I_j} \left| \int_{I_j} b_j(y) \left[\frac{1}{x-y} - \frac{1}{x-c_j} \right] dy \right| dx$$

where c_j is the center of I_j; this is true since $\int_{I_j} b_j(y)dy = 0$. To continue, we have

$$\begin{aligned}
\int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|dx &= \int_{\mathbb{R}\setminus 2I_j} \left| \int_{I_j} b_j(y) \frac{y-c_j}{(x-y)(x-c_j)} dy \right| dx \\
&\leqslant \int_{\mathbb{R}\setminus 2I_j} \int_{I_j} \left| b_j(y) \frac{y-c_j}{(x-y)(x-c_j)} \right| dydx \\
&= \int_{I_j} |b_j(y)| \left(\int_{\mathbb{R}\setminus 2I_j} \left| \frac{y-c_j}{(x-y)(x-c_j)} \right| dx \right) dy \\
&\lesssim \int_{I_j} |b_j(y)| \left(\int_{\mathbb{R}\setminus 2I_j} \frac{|I_j|}{(x-c_j)^2} dx \right) dy \\
&= \int_{I_j} 2|b_j(y)|dy \\
&= 2\int_{I_j} \left| \left[f(y) - \frac{1}{|I_j|} \int_{I_j} f \right] \chi_{I_j}(y) \right| dy \\
&\leqslant 2\int_{I_j} |f(y)|dy + 2 \cdot |I_j| \cdot \frac{1}{|I_j|} \int_{I_j} |f|dy \\
&= 4\int_{I_j} |f|dy
\end{aligned}$$

Consequently,

$$\sum_j \int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|dx \leqslant \sum_j 4\int_{I_j} |f|dy \leqslant 4\|f\|_{L^1(\mathbb{R})}$$

and this completes the proof of the Kolmogorov's Theorem.

2. Since $\|Hf\|_{L^2}^2 = \|f\|_{L^2}^2$, then we know that H is strong $(2,2)$. By (1), we know H is weak $(1,1)$. Then by Marcinkiewicz Interpolation Theorem (4.8), we know H is strong (p,p) for $1 < p < 2$.
Now if $p > 2$, we know by Lemma (4.30) $\int Hf \cdot g = -\int f \cdot Hg$ and the fact that $p' < 2$. From the strong (p',p') of H, we have
$$\|Hg\|_{p'} \leqslant C_{p'}\|g\|_{p'}.$$

For any $f \in L^p$, by the dual representation of the norm, we have

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}} \leqslant 1} \{|\int_{\mathbb{R}} f(x)g(x)dx| \, : \, \|g\|_{L^{p'}} \leqslant 1\}.$$

Then

$$\begin{aligned}
\|Hf\|_p &= \sup\left\{|\int_{\mathbb{R}} Hf \cdot gdx| \, : \, \|g\|_{L^{p'}} \leqslant 1\right\} \\
&= \sup\left\{|\int_{\mathbb{R}} f \cdot Hgdx| \, : \, \|g\|_{L^{p'}} \leqslant 1\right\} \\
&\leqslant \|f\|_{L^p} \sup\{\|Hg\|_{L^{p'}} \, : \, \|g\|_{L^{p'}} \leqslant 1\} \\
&\leqslant \|f\|_{L^p} C_{p'}
\end{aligned}$$

Where the third line follows from Hölder's Inequality.

□

4.5 Calderon-Zygmund Operator

To give some motivation, we start by stating the following fact: for any $u \in C_c^\infty(\mathbb{R}^n)$, we have

$$\sum_{i,j=1}^{n} \|\partial_i \partial_j u\|^2_{L^2(\mathbb{R}^n)} \approx \|\Delta u\|^2_{L^2(\mathbb{R}^n)}.$$

Proof. Using integration by parts, we have

$$\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \partial_i \partial_j u \cdot \partial_i \partial_j u = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \partial_i \partial_i u \cdot \partial_j \partial_j u = \int_{\mathbb{R}^n} \Delta u \cdot \Delta u$$

□

The fact is equivalent to saying that

$$\|\partial^2 u\|_{L^2(\mathbb{R}^n)} \lesssim \|\Delta u\|_{L^2(\mathbb{R}^n)}.$$

The natural question is to ask whether the similar kind of approximation holds if we replace the exponent by other numbers $p \in [1, \infty]$. We can show using the Calderon-Zygmund theory that

$$\|\partial^2 u\|_{L^p(\mathbb{R}^n)} \lesssim \|\Delta u\|_{L^p(\mathbb{R}^n)}$$

for $1 < p < \infty$.

Definition 4.32 (Calderon-Zygmund Operator)**.** *A **Calderon-Zygmund operator** T is a linear operator on $L^2(\mathbb{R}^n)$ if*

1. *T is bounded from L^2 to L^2.*
2. *There exists a measurable kernel K, such that for every $f \in L^2$ with compact support and $x \notin \operatorname{supp} f$, we have*
$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy.$$
3. *There exists constant $C > 1$ and $A > 0$ such that*
$$\int_{|x| \geqslant C|y|} |K(x-y) - K(x)|dx \leqslant A \tag{12}$$
uniformly in y. (For simplicity, one can take $C = 2$.)

Proposition 4.33. *Assume that the kernel $K(x)$ satisfies for all $x \neq 0$,*

$$|K(x)| \lesssim |x|^{-n}, \quad |\partial K(x)| \lesssim |x|^{-n-1},$$

then K satisfies the conditions (3) *given in Definition (4.32).*

Proof. Suppose the conditions in the proposition holds, then by the mean value theorem, we have

$$\int_{|x| \geqslant C|y|} |K(x-y) - K(x)|dx = \int_{|x| \geqslant C|y|} |\partial K(x - \theta y)| \cdot |y|dx$$

where $\theta \in [0, 1]$. We take $C = 2$, then $|x - \theta y| \approx |x|$. So

$$\begin{aligned}\int_{|x| \geqslant 2|y|} |K(x-y) - K(x)|dx &\lesssim \int_{|x| \geqslant 2|y|} \frac{1}{|x|^{n+1}} \cdot |y|dx_1 \cdots dx_n \\ &= \int_{|x| \geqslant 2|y|} \frac{|y|}{|x|^2} d|x| \\ &= |y| \int_{|x| \geqslant 2|y|} \frac{1}{|x|^2} d|x| \\ &\approx |y| \cdot \frac{1}{|y|} = 1\end{aligned}$$

where we have used polar coordinates in the second line. □

Example 1: In $\mathbb{R}$, the Hilbert transform is an example of Calderon-Zygmund Operator. Since

$$Hf(x) = \int_{\mathbb{R}} e^{i \cdot x \cdot \xi} \cdot -i \operatorname{sgn}(\xi) \hat{f}(\xi) d\xi$$

Then we have

$$\|Hf(x)\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$$

so H is a bounded operator from L^2 to L^2. For f with compact support and $x \notin \operatorname{supp} f$, we have

$$Hf(x) = c\int_{-\infty}^{\infty} \frac{1}{x-y} f(y)dy.$$

Clearly, we have

$$\left|\frac{1}{x}\right| \leqslant \frac{1}{|x|} \quad \text{and} \quad \left|\partial\frac{1}{x}\right| \lesssim \frac{1}{|x|^2}.$$

Example 2: Consider $\Delta u = f$ in $\mathbb{R}^n$ with $n \geqslant 3$ and f being smooth and compact supported. Then $u = K_n * f$ where

$$K_n(x) = C_n|x|^{2-n}.$$

Then if $x \notin \operatorname{supp} f$, it make sense to have the following expression:

$$\partial_i\partial_j u = \partial_i\partial_j(K_n * f) = \int_{\mathbb{R}^n} \partial_i\partial_j K_n(x-y)f(y)dy.$$

We define

$$R_{ij}f(x) := \int_{\mathbb{R}^n} \partial_i\partial_j K_n(x-y).$$

We claim that R_{ij} is a Calderon-Zygmund Operator. We first show that R_{ij} is a bounded operator from L^2 to L^2. By integration by parts, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^2 dx &= \int_{\mathbb{R}^n} \Delta u(x)\Delta u(x)dx \\
&= \int_{\mathbb{R}^n} \sum_{i=1}^n \partial_i\partial_i u \sum_{j=1}^n \partial_j\partial_j u dx \\
&= \sum_{i,j=1}^n \int_{\mathbb{R}^n} |\partial_i\partial_j u(x)|^2 dx \\
&= \sum_{i,j=1}^n \int_{\mathbb{R}^n} |R_{ij}(x)|dx.
\end{aligned}$$

It is also easy to verify that

$$|K(x)| = \left|\partial_i\partial_j(|x|^{2-n})\right| \lesssim \frac{1}{|x^n|} \quad \text{and} \quad |\partial K(x)| \lesssim \frac{1}{|x|^{n+1}}.$$

Theorem 4.34. *All Calderon-Zygmund operators are weak* $(1,1)$.

Proof. Let $f \in L^1(\mathbb{R}^n)$ be nonnegative and $\alpha > 0$. Then we have the following Calderon-Zygmund decomposition:

$$f = g + \sum_{Q\in\Omega} b_Q,$$

where

$$g(x) := \begin{cases} f(x), & \text{if } x \notin \Omega \\ f_Q := \frac{1}{|Q|}\int_Q f dx, & \text{if } x \in \mathbb{Q} \end{cases}$$

and

$$b_Q(x) = (f(x) - f_Q) \cdot \chi_Q(x).$$

such that

1. $\|g\|_{L^\infty} \lesssim \alpha$

2. $\operatorname{supp} b_Q \subset Q$ (a dyadic cube)

3. $\int b_Q(x)dx = 0$

4.
$$\|b_Q\|_{L^1} \lesssim \alpha|Q| \tag{13}$$

5.
$$\sum_{Q\in\Omega} |Q| \lesssim \frac{1}{\alpha}\|f\|_{L^1}. \tag{14}$$

We note that Property 4 is true since

$$\begin{aligned} \|b_Q\|_{L^1} &= \int_Q |f(x) - f_Q| \cdot \chi_Q(x)dx \\ &\approx \int_Q |f_Q| \\ &\leqslant \int_Q 2^n\alpha \\ &\lesssim \alpha|Q|. \end{aligned}$$

We start from

$$\{|Tf(x)| > \alpha\} \subset \underbrace{\left\{|Tg(x)| > \frac{\alpha}{2}\right\}}_{(1)} \cup \underbrace{\left\{|Tb(x)| > \frac{\alpha}{2}\right\}}_{(2)}$$

Hence it is enough the show that

$$m\left(\left\{|Tg(x)| > \frac{\alpha}{2}\right\}\right) \lesssim \frac{1}{\alpha}\|f\|_{L^1}$$
$$m\left(\left\{|Tb(x)| > \frac{\alpha}{2}\right\}\right) \lesssim \frac{1}{\alpha}\|f\|_{L^1}$$

We have

$$
\begin{aligned}
m\left(\left\{|Tg(x)| > \frac{\alpha}{2}\right\}\right) &\leqslant \frac{1}{\alpha^2}\|Tg\|_{L^2}^2 \\
&\lesssim \frac{1}{\alpha^2}\|g\|_{L^2}^2 \quad \text{(Chebyshev)} \\
&\lesssim \frac{1}{\alpha}\|g\|_{L^1} \quad \left(|Tg(x)| > \frac{\alpha}{2}\right) \\
&= \frac{1}{\alpha}\|f\|_{L^1}
\end{aligned}
$$

For the estimate of the measure of (2). Since the family of Q belongs to Ω is countable, we denote them by Q_j, $j \in \mathbb{N}$. For each Q_j, let y_j be be its center and take $\hat{Q}_j$ to be the cube with the same center but with sides expanded by a factor $2n^{\frac{1}{2}}$. Then we note that for any $x \notin \hat{Q}_j$,

$$
|x - y_j| \geqslant 2 \max_{y \in Q_j} |y - y_j|.
$$

Since

$$
|x - y_j| \geqslant 2n^{\frac{1}{2}} d \geqslant 2 \max_{y \in Q_j} |y - y_j|.
$$

Let $\hat{\Omega} := \bigcup_j \hat{Q}_j$ and F its complement. For simplicity, we also denote $b_j := b_{Q_j}$.

Since $\int b_j dy = 0$, for $x \in F = \mathbb{R}^n \backslash \hat{\Omega}$, we have

$$
T(b_j)(x) = \int_{Q_j} [K(x - y) - K(x - y_j)] b_j(y) dy
$$

Then

$$\begin{aligned}
\int_F |Tb(x)|dx &\leqslant \sum_j \int_{\mathbb{R}^n \backslash (\bigcup \hat{Q}_j)} |Tb_j(x)|dx \\
&\leqslant \sum_j \int_{\mathbb{R}^n \backslash \hat{Q}_j} |T(b_j)|dx \\
&\lesssim \sum_j \int_{\mathbb{R}^n \backslash \hat{Q}_j} \left(\int_{y \in Q_j} |K(x-y) - K(x-y_j)||b(y)|dy \right) dx \\
&= \sum_j \int_{y \in \hat{Q}_j} |b(y)| \left(\int_{\mathbb{R}^n \backslash \hat{Q}_j} |K(x-y) - K(x-y_j)|dx \right) dy \\
&= \sum_j \int_{y \in \hat{Q}_j} \int_{\mathbb{R}^n \backslash (\hat{Q}_j - y_j)} |K(x-(y-y_j)) - K(x)|dxdy \\
&\leqslant \sum_j \int_{y \in \hat{Q}_j} |b(y)| \left(\int_{|x| \geqslant 2|y-y_j|} |K(x-(y-y_j)) - K(x)|dx \right) dy \\
&\lesssim A \sum_j \int_{Q_j} |b(y)|dy \quad \text{(by 12)} \\
&\leqslant A \cdot \alpha \sum_j |Q_j| \quad \text{(by 13)} \\
&\leqslant A \cdot \alpha \cdot \frac{1}{\alpha} \|f\|_{L^1} \quad \text{(by 14)} \\
&\leqslant A \|f\|_{L^1}
\end{aligned}$$

Therefore by Chebyshev's inequality,

$$m(\{x \in \mathbb{R}^n \backslash \bigcup \hat{Q}_j \;:\; |Tb(x)| > \frac{\alpha}{2}\}) \leqslant \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\alpha}.$$

Moreover, by (14) again, we have

$$|\bigcup \hat{Q}_j| \lesssim |\bigcup Q_j| = \sum_{Q \in \Omega} |Q| \lesssim \frac{1}{\alpha} \|f\|_{L^1}$$

Consequently

$$m\left(\left\{|Tb(x)| > \frac{\alpha}{2}\right\}\right) \lesssim \frac{1}{\alpha} \|f\|_{L^1}.$$

This concludes the proof of the theorem. □

Proposition 4.35. *All Calderon-Zygmund operators are bounded from L^p into L^p, where $1 < p < \infty$. However, Calderon-Zygmund operators need not be strong $(1,1)$ or strong (∞, ∞).*

Proof. Duality of norms. □

5 Appendix

5.1 Notations

1. $\mathbb{R}_+^d = \{x \in \mathbb{R}^d : x_d > 0\}$ which is the open upper half-space.

2. $e_i = (0, \cdots, 0, 1, \cdots, 0)$ is the i^{th} standard coordinate vector.

3. $\check{B}(x,r) = B(x,r)\backslash\{x\}$ which is the punctured ball.

4. $S^{d-1} = \partial B_d(0,1)$ denotes the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$.

5. $\alpha(n)$ denotes the volume of unit ball in $\mathbb{R}^d$, which is given by

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ \frac{2\pi}{n} \times \alpha(n-2), & n \geqslant 2 \end{cases}.$$

   The surface area of unit sphere $S^{n-1} = \partial B_n(0,1)$ in $\mathbb{R}^n$ is $n\alpha(n)$. To derive the formula, either use spherical coordinates or induction by $n+2$.

6. Let U and V be open subsets of $\mathbb{R}^d$, then we write $V \Subset U$ if $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact. In this case, we say that V is compactly contained in U.

7. Given $T > 0$ and an open set $U \subset \mathbb{R}^d$, we denote the parabolic cylinder by $U_T := U \times (0,T]$ and its parabolic boundary by $\Gamma_T := \bar{U}_T \backslash U_T$.

8. The average of f over set E is denoted by

$$⨍_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu$$

   provided $\mu(E) > 0$. Similarly, we can define the average of f over the set ∂E.

9. Let $f, g \in \mathbb{R}^d$, then the their convolution $f * g$ is given by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy = \int_{\mathbb{R}^d} f(y)g(x-y)dy.$$

10. We write $f = O(g)$ as $x \to x_0$ if there exists a constant C such that $|f(x)| \leqslant C|g(x)|$ for all x sufficiently close to x_0. We write $f = o(g)$ as $x \to x_0$ if

$$\lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| = 0.$$

11. Given $\alpha = (\alpha_1, \cdots, \alpha_d) \subset \mathbb{N}^d$, then

$$\partial^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} u$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_k$.

12. For nonnegative integers, we write $\partial^k u(x) := \{\partial^\alpha u(x) : |\alpha| = k\}$ and

$$|\partial^k u| = \left(\sum_{|\alpha|=k} |\partial^\alpha u|^2 \right)^{1/2}.$$

Then $\|\partial^k u\|_{L^p(U)} = \| |\partial^k u|_{\ell^2} \|_{L^p(U)}$.

13. Divergence of $\mathbf{u}$: $\operatorname{Div} \mathbf{u} = tr(D\mathbf{u}) = \nabla \cdot u = \sum_{i=1}^{n} u^i_{x_i}$, where $u : \mathbb{R}^n \to \mathbb{R}^n$.

14. $\nabla^2 u$ represents the Hessian matrix of u. Laplacian of u: $\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \operatorname{tr}(D^2 u)$, this is precisely the trace of $\nabla^2 u$.

15. $u_r := \frac{x}{|x|} \cdot \nabla u$ represents the radial derivative of u.

16. $C(\bar{U})$ is the set of functions that is uniformly continuous on bounded subsets of U.

17. $C^k(\bar{U})$ is the subset of $C^k(U)$ such that $\partial^\alpha u$ is uniformly continuous on bounded subsets of U, $\forall 0 \leqslant |\alpha| \leqslant k$.

18. $C^\infty(U) = \bigcap_{k=0}^\infty C^k(U)$ and $C^\infty(\bar{U}) = \bigcap_{k=0}^\infty C^k(\bar{U})$.

5.2 Calculus

In this section, unless otherwise stated, we assume U is a bounded, open subset of $\mathbb{R}^d$ and ∂U is C^1. We denote $N = (N_1, \cdots, N_d)$ to be the unit outer normal vector of $\partial\Omega$.

1. If ∂U is C^1, then along ∂U is defined the outward pointing unit normal vector field:

$$N = (N_1, \cdots, N_d).$$

The unit normal at any point $x_0 \in \partial U$ is $N(x_0) = N = (N_1(x_0), \cdots, N_d(x_0))$.
Suppose $u \in C^1(\bar{U})$, then the directional derivative in the direction ν is given by

$$\frac{\partial u}{\partial \nu} := \nu \cdot Du.$$

2. (Gauss-Green Theorem) Suppose $u \in C^1(\bar{U}, \mathbb{R})$. Then

$$\int_U \partial_{x_i} u dx = \int_{\partial U} u N_i dS, \quad (i = 1, \cdots, d).$$

Proof. We provide a sketch of the proof. Firstly prove the theorem for bounded rectangular regions in $\mathbb{R}^n$ by writing $u(x_1, \cdots, x_d) = \int_a^{x_d} \partial_{x_i} u(t, x_2, \cdots, x_d) dt$. Using this to prove the formula holds for Ω when Ω is the almost disjoint union of rectangles in $\mathbb{R}^d$. Lastly, any open $\Omega \subset \mathbb{R}^d$, can be approximated by almost disjoint union of rectangles in $\mathbb{R}^d$. By parsing to limit, we prove the desired result. □

3. (Divergence Theorem) for each $\mathbf{u} \in C^1(\bar{U}; \mathbb{R}^d)$, we have

$$\int_U \operatorname{Div} \mathbf{u} dx = \int_{\partial U} \mathbf{u} \cdot N dS.$$

Proof. From Gauss-Green Theorem, we have

$$\begin{aligned} \int_\Omega \operatorname{Div} \mathbf{u} dx &= \int_\Omega \sum_{i=1}^d \partial_{x_i} u_i dx \\ &= \int_{\partial\Omega} \sum_{i=1}^d u_i N_i dS \\ &= \int_{\partial\Omega} \mathbf{u} \cdot N dS. \end{aligned}$$

□

4. (Integration by parts) Let $u, v \in C^1(\bar{U})$. Then

$$\int_U u_{x_i} v dx = \int_{\partial U} uv N_i dS - \int_U u v_{x_i} dx \quad (i = 1, \cdots, d).$$

Proof. By Gauss Green's Theorem, we have

$$\int_\Omega \partial_{x_i}(uv) dx = \int_{\partial\Omega} uv N_i dS.$$

Then the theorem follows by noting that

$$\partial_{x_i}(uv) = v \partial_{x_i} u + u \partial_{x_i} v.$$

□

5. (Green's Formula) Let $u, v \in C^2(\bar{U})$. Then

(a) $\int_U \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS = \int_{\partial U} \nu \cdot Du dS$;

(b) $\int_U Dv \cdot Du dx = -\int_U u\Delta v dx + \int_{\partial U} \frac{\partial v}{\partial N} u dS$;

(c) $\int_U u\Delta v - v\Delta u dx = \int_{\partial U} u\frac{\partial v}{\partial N} - v\frac{\partial u}{\partial N} dS$.

Proof. For (1), we note that $\Delta u = \mathrm{div}(\nabla u)$, then the result directly follows from the Divergence Theorem.

For (2), we note that

$$\nabla u \cdot \nabla v = \sum_{i=1}^{d} u_{x_i} v_{x_i}.$$

Then by integration by parts, we have

$$\int_\Omega \nabla u \cdot \nabla v = -\int_\Omega \sum_{i=1}^{d} u v_{x_i x_i} dx + \int_{\partial\Omega} \sum_{i=1}^{d} u v_{x_i} N_i dx.$$

The right hand side of above equation is precisely the right hand side of the equality stated in the theorem. Lastly, notice that (3) directly follows from (2). ☐

6. Assume $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is harmonic in Ω, then we have

   (a) $\int_{\partial\Omega} \frac{\partial u}{\partial N} dS = 0$.

   (b) $\int_\Omega |\nabla u|^2 dx = \int_{\partial\Omega} u\frac{\partial u}{\partial N} dS$.

   Proof. Since u is harmonic, then $\Delta u = 0$. Then this follows directly from Green's formula. ☐

7. Let $N \geqslant 0$, then

$$\int_0^{\frac{\pi}{2}} \cos^N \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^N \theta d\theta = \begin{cases} \frac{N-1}{N}\frac{N-3}{N-2}\cdots\frac{1}{2}\frac{\pi}{2}, & \text{if } N \text{ is even} \\ \frac{N-1}{N}\frac{N-3}{N-2}\cdots\frac{2}{3}, & \text{if } N \text{ is odd.} \end{cases}$$

   Proof. The proof is by induction, and considering the cases $n = 0$ and $n = 1$, then use integration by parts to get the result for $n + 2$. ☐

8. (Polar coordinates) Let $f : \mathbb{R}^d \to \mathbb{R}$ be continuous and summable. Then

$$\int_{\mathbb{R}^d} f dx = \int_0^\infty \left(\int_{\partial B_d(x_0,r)} f dS \right) dr$$

   for each point $x_0 \in \mathbb{R}^d$. For each $R > 0$, and $x_0 \in \mathbb{R}^d$, then

$$\int_{B(x_0,R)} f dx = \int_0^R \left(\int_{\partial B_d(x_0,\rho)} f(y) dS_y \right) d\rho.$$

In particular, if f is a function not dependent on r, then

$$\frac{d}{dr}\left(\int_{B_d(x_0,r)} f dx\right) = \int_{\partial B(x_0,r)} f dS$$

for each $r > 0$.

9. (Co-area formula). Let $u : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz continuous and assume that for a.e. $r \in \mathbb{R}$, the level set

$$\{x \in \mathbb{R}^d \,|\, u(x) = r\}$$

is a smooth, $(d-1)$-dimensional hypersurface in $\mathbb{R}^n$. Suppose also $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and Lebesgue integrable. Then

$$\int_{\mathbb{R}^d} f|Du|dx = \int_{-\infty}^{\infty}\left(\int_{u=r} f dS\right) dr.$$

10. (Differentiating Moving Regions) Let $f = f(x,t)$ be a smooth function. Then

$$\frac{d}{dt}\int_{\Omega(t)} f dx = \int_{\partial\Omega(t)} f \cdot (v \cdot N) dS_x + \int_{\Omega(t)} \partial_t f dx,$$

where $v(x,t)$ is the velocity vector field indicating the speed at which every point on $\partial\Omega$ is moving.

5.3 Inequalities

1. (Young's Inequality) let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q} \quad (a, b > 0).$$

The equality holds if and only if $a^p = b^q$.

Proof. It is clear that when $a = 0$ or $b = 0$, the inequality holds. So assume $a > 0$ and $b > 0$, let $t = \frac{1}{p}$ and $\frac{1}{q} = 1 - t$. Since the log function is concave, we have

$$\ln(ta^p + (1-t)b^q) \geqslant t\ln(a^p) + (1-t)\ln(b^q) = \ln(a) + \ln(b) = \ln(ab).$$

Since ln is a strictly increasing function, we have $ta^p + (1-t)b^q \geqslant ab$, i.e.,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

With equality holding if and only if $a^p = b^q$. $\square$

2. (Young's Inequality with ϵ) let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leqslant \epsilon a^p + C(\epsilon) b^q \quad (a, b > 0, \epsilon > 0)$$

where $C(\epsilon) = (\epsilon p)^{-q/p} q^{-1}$.

Proof. By Young's inequality, we have that

$$\frac{ab}{\epsilon p} \leqslant \frac{a^p}{p} + \frac{1}{q}\left(\frac{b}{\epsilon p}\right)^q.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, we have the desired result after multiplying ϵp on both sides. $\square$

3. (Hölder's Inequality) Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. Then for $1 \leqslant p \leqslant \infty$, if f and g are measurable (real or complex) on Ω, then

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q.$$

When $p, q \in (1, \infty)$, and $f \in L^p$, $g \in L^q$, the equality holds iff $|f|^p$ and $|g|^q$ are linearly dependent in L^1, i.e., $\exists \alpha, \beta \geqslant 0$, not both zero such that $\alpha |f|^p = \beta |g|^q$ a.e.

Proof. If $p = 1$, and $q = \infty$, then $|fg| \leqslant \|f\|_\infty |g|$ almost everywhere, so the Hölder's inequality in this case follows from the monotonicity of the Lebesgue integral. Similarly we have the case for $p = \infty$ and $q = 1$.

Hence we assume $p, q \in (1, \infty)$. The case $\|f\|_p = 0$, $\|f\|_p = \infty$ or $\|g\|_q = 0$, $\|g\|_q = \infty$ are trivial. So WLOG, assume $\|f\|_p, \|g\|_q \in (0, \infty)$. Then dividing f and g by $\|f\|_p$ and $\|g\|_q$ respectively, we may assume

$$\|f\|_p = \|g\|_q = 1.$$

By Young's inequality, we have

$$|f(s)g(s)| \leqslant \frac{|f(s)|^p}{p} + \frac{|g(s)|^q}{q}, \quad \forall s \in \Omega.$$

Integrating both sides gives

$$\|fg\|_1 \leqslant \frac{\|f\|_p^p}{p} + \frac{\|g\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1,$$

which proves the claim.

Under the assumption $p \in (1, \infty)$, and $\|f\|_p = \|g\|_q$ are finite (can scale), equality holds if and only if $|f|^p = |g|^q$ almost everywhere. More generally, if $\|f\|_p$ and $\|g\|_q$ are in $(0, \infty)$, then Hölder's inequality becomes an equality if and only if there exists a real number $\alpha, \beta > 0$, namely

$$\alpha = \|g\|_q^q, \quad \beta = \|f\|_p^p,$$

such that

$$\alpha|f|^p = \beta|g|^q$$

almost everywhere. $\square$

4. (Hölder's Inequality For Sequences) Suppose (x_i), (y_i) are two sequences and $1 \leqslant p, q < \infty$ is such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{\infty} |x_i y_i| \leqslant \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{1/q}.$$

Proof. Direct consequence of Hölder's inequality. $\square$

5. (General Hölder Inequality) Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. Assume that $r \in (0, \infty]$ and $p_1, \cdots, p_n \in (0, \infty]$ such that

$$\sum_{k=1}^{n} \frac{1}{p_k} = \frac{1}{r}$$

where $\frac{1}{\infty}$ is interpreted as 0. Then for all measurable real or complex valued functions $f_1, \cdots, f_n$ defined on Ω,

$$\left\| \prod_{k=1}^{n} f_k \right\|_r \leqslant \prod_{k=1}^{n} \|f_k\|_{p_k}.$$

In particular, if $f_k \in L^{p_k}(\mu)$ for all $k \in \{1, \cdots, n\}$ then $\prod_{k=1}^{n} f_k \in L^r(\mu)$.

Proof. We prove by induction, it is clear that the statement holds for $n = 1$. Now for general n, suppose the statement holds for $n-1$ and WLOG let $p_1 \leqslant \cdots \leqslant p_n$.

Case 1: if $p_n = \infty$, then

$$\sum_{k=1}^{n-1} \frac{1}{p_k} = \frac{1}{r}.$$

Pulling out the essential supremum of $|f_n|$ and using the induction hypothesis, we get

$$\begin{aligned}\|f_1 \cdots f_n\|_r &\leqslant \|f_1 \cdots f_{n-1}\|_r \|f_n\|_\infty \\ &\leqslant \|f_1\|_{p_1} \cdots \|f_{n-1}\|_{p_{n-1}} \|f_n\|_\infty.\end{aligned}$$

Case 2: if $p_n < \infty$, then necessarily $r < \infty$ as well. Then set

$$p := \frac{p_n}{p_n - r}, \quad q := \frac{p_n}{r},$$

note (p, q) are Hölder conjugates in $(1, \infty)$. Then the Hölder inequality gives

$$\||f_1 \cdots f_{n-1}|^r |f_n|^r\|_1 \leqslant \||f_1 \cdots f_{n-1}|^r\|_p \||f_n|^r\|_q.$$

Raising to the power $\frac{1}{r}$, we get

$$\|f_1 \cdots f_n\|_r \leqslant \|f_1 \cdots f_{n-1}\|_{pr} \|f_n\|_{qr}.$$

Since $qr = p_n$, and

$$\sum_{k=1}^{n-1} \frac{1}{p_k} = \frac{1}{r} - \frac{1}{p_n} = \frac{p_n - r}{rp_n} = \frac{1}{pr},$$

then the desired statement follows. $\square$

6. (Jensen's Inequality) Assume $f : \mathbb{R}^m \to \mathbb{R}$ is convex and $U \subset \mathbb{R}^n$ is a bounded domain. Let $\mathbf{u} : U \to \mathbb{R}^m$ be summable. Then

$$f\left(⨍_U \mathbf{u} dx\right) \leqslant ⨍_U f(\mathbf{u}) dx.$$

Proof. Since f is convex, for each $p \in \mathbb{R}^n$, there exists $r \in \mathbb{R}^m$ such that

$$f(q) \geqslant f(p) + r \cdot (q - p)$$

for all $q \in \mathbb{R}^m$. Let $p = ⨍_U \mathbf{u} dy$, $q = \mathbf{u}(x)$, then

$$f(\mathbf{u}(x)) \geqslant f\left(⨍_U \mathbf{u} dy\right) + r \cdot \left(\mathbf{u}(x) - ⨍_U \mathbf{u} dy\right).$$

Integrate with respect to x over U gives the desired result. $\square$

7. (Minikowski's Inequality): Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. Then for $1 \leqslant p \leqslant \infty$, $f, g \in L^p$, then

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p.$$

When $1 < p < \infty$, equality holds if and only if $f \sim \lambda g$ for some $\lambda \geqslant 0$ or $g \sim 0$.

Proof. One can verify directly that the statement holds for $p = 1$ or $p = \infty$, so we assume that $1 < p < \infty$.

If $\|f + g\|_p = 0$, then the statement holds trivially. So also assume $\|f + g\|_p \neq 0$. Note

$$\begin{aligned} |f + g|^p &= |f + g||f + g|^{p-1} \\ &\leqslant (|f| + |g|)|f + g|^{p-1} \\ &= |f||f + g|^{p-1} + |g||f + g|^{p-1} \end{aligned}$$

Apply Holder's inequality, we have

$$\int |f||f+g|^{p-1}d\mu \leqslant \|f\|_p \cdot \|(f+g)^{p-1}\|_q,$$

where q is the Hölder conjugate of p. Then $p = (p-1)q$. So

$$\|(f+g)^{p-1}\|_q = \|f+g\|_p^{p-1}.$$

Then

$$\|f+g\|_p^p = \int |f+g|^p d\mu \leqslant \|f\|_p \cdot \|f+g\|_p^{p-1} + \|g\|_p \cdot \|f+g\|_p^{p-1}$$

Then as $\|f+g\|_p \neq 0$, then

$$\|f+g\|_p \leqslant \|f\|_p + \|g\|_p.$$

□

8. If $1 \leqslant p \leqslant \infty$, and p' is the Hölder conjugate of p, then $\|fg\|_{L^1} \leqslant \|f\|_{L^p}\|g\|_{L^{p'}}$. In particular, if $\mu(X) < \infty$ and $p < q$, then $L^q \subset L^p$, and $\|f\|_{L^p} \leqslant \|f\|_{L^q}\mu(X)^{\frac{1}{p}-\frac{1}{q}}$.
9. If $1 \leqslant p \leqslant q \leqslant r \leqslant \infty$, $L^p \cap L^r \subset L^q \subset L^p + L^r$.

Proof. Let f be a measurable function, we consider $f = f\chi_E + f\chi_{E^c}$ where $E := \{x\,;\,|f(x)| > 1\}$. Then we have

$$|f\chi_E|^p = |f|^p\chi_E \leqslant |f|^q\chi_E$$

and

$$|f|^q\chi_{E^c} \geqslant |f\chi_{E^c}|^r.$$

This shows that if $f \in L^q$, then $f \in L^p + L^r$. Next by comparing $|f\chi_E|^q$ with $|f\chi_E|^r$, and $|f\chi_{E^c}|^q$ with $|f\chi_{E^c}|^p$, we conclude that if $f \in L^p \cap L^r$, then $f \in L^q$. □

10. (Interpolation Enequality for L^p-norms) Assume $1 \leqslant s \leqslant r \leqslant t \leqslant \infty$ and

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t},$$

Suppose also $u \in L^s(U) \cap L^t(U)$. Then $u \in L^r(U)$, and

$$\|u\|_{L^r(U)} \leqslant \|u\|_{L^s(U)}^{\theta}\|u\|_{L^t(U)}^{1-\theta}.$$

11. If $1 \leqslant p \leqslant q \leqslant \infty$, then $\inf\{\mu(F) : F \in \mathscr{A}, F \subset X, \mu(F) > 0\} \geqslant c_0 > 0$ for some positive constant c_0, then $L^p \subset L^q$. In particular, $\ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z})$.

Proof. For the sake of simplicity, we assume $\|f\|_{L^p} = 1$ and $c_0 = 1$. Then for any $\epsilon > 0$, we see that

$$\mu\{x \in X : |f(x)| > 1+\epsilon\} \leqslant (1+\epsilon)^{-p} \int_X |f|^p d\mu < 1 \Rightarrow \mu\{x \in X : f(x)| > 1+\epsilon\} = 0$$

and thus $\mu\{x \in X : |f(x)| > 1\} = 0$. Then it is easy to see that $\int_X |f|^q d\mu \leqslant \int_X |f|^p d\mu = 1$. □

12. Suppose $\mu(X) < \infty$ and $f \in L^\infty(X)$. Then $f \in L^p(x)$ for any $p < \infty$ and

$$\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}.$$

The condition $\mu(X) < \infty$ can be removed if we additional assume $f \in L^p \cap L^\infty$ for some $p \geqslant 1$.

Proof. Since $\mu(x) < \infty$, then we have

$$\|f\|_{L^p} \leqslant \|f\|_{L^\infty} \mu(X)^{\frac{1}{p}}.$$

Then

$$\limsup_{p\to\infty} \|f\|_{L^p} \leqslant \|f\|_{L^\infty}.$$

On the other hand, given $\epsilon > 0$, we know there exists some $\delta > 0$ such that

$$\mu(\{x : |f(x)| \geqslant \|f\|_{L^\infty} - \epsilon\} \geqslant \delta$$

and hence

$$\int_X |f|^p d\mu \geqslant \delta(\|f\|_{L^\infty} - \epsilon)^p$$

Therefore, we know

$$\liminf_{p\to\infty} \|f\|_{L^p} \geqslant \|f\|_{L^\infty} - \epsilon.$$

In particular, this holds for general measure spaces. Letting $\epsilon \to 0$, we get the desired result.

Now if $\mu < \infty$, then by the interpolation inequality, we also have

$$\limsup_{p\to\infty} \|f\|_{L^p} \leqslant \|f\|_{L^\infty}.$$

□

13. (Young's Inequality for Convolution) suppose $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, where $1 \leqslant p, q, r \leqslant \infty$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, then

$$\|f * g\|_r \leqslant \|f\|_p \|g\|_q.$$

Proof. We note

$$\begin{aligned}
(f * g)(x) &= \int f(x-y)g(y)\mathrm{d}y \\
|(f * g)(x)| &\leqslant \int |f(x-y)| \cdot |g(y)|\mathrm{d}y \\
&= \int |f(x-y)|^{1+p/r-p/r} \cdot |g(y)|^{1+q/r-q/r}\mathrm{d}y \\
&= \int |f(x-y)|^{p/r} \cdot |g(y)|^{q/r} \cdot |f(x-y)|^{1-p/r} \cdot |g(y)|^{1-q/r}\mathrm{d}y \\
&= \int (|f(x-y)|^{p} \cdot |g(y)|^{q})^{1/r} \cdot |f(x-y)|^{(r-p)/r} \cdot |g(y)|^{(r-q)/r}\mathrm{d}y \\
&\leqslant \left\|(|f(x-y)|^{p} \cdot |g(y)|^{q})^{1/r}\right\|_r \cdot \left\||f(x-y)|^{(r-p)/r}\right\|_{\frac{pr}{r-p}} \cdot \left\||g(y)|^{(r-q)/r}\right\|_{\frac{qr}{r-q}} \\
&= \left\|(|f(x-y)|^{p} \cdot |g(y)|^{q})^{1/r}\right\|_r \cdot \|f\|_p^{\frac{r-p}{r}} \cdot \|g\|_q^{\frac{r-q}{r}}.
\end{aligned}$$

Note the second last inequality follows from the generalized Hölder's inequality since we have

$$\frac{1}{r} + \frac{r-p}{pr} + \frac{qr}{r-q} = 1.$$

Then

$$\begin{aligned}
\|f * g\|_r^r &\leqslant \int \left(\left\|(|f(x-y)|^{p} \cdot |g(y)|^{q})^{1/r}\right\|_r \cdot \|f\|_p^{\frac{r-p}{r}} \cdot \|g\|_q^{\frac{r-q}{r}}\right)^r dx \\
&= \|f\|_p^{r-p}\|g\|_q^{r-q} \int\int |g(y)|^q |f(x-y)|^p dydx \\
&= \|f\|_p^{r-p}\|g\|_q^{r-q} \int |g(y)|^q \left(\int |f(x-y)|^p dx\right) dy \\
&= \|f\|_p^{r-p}\|g\|_q^{r-q}\|g\|_q^q\|f\|_p^p \\
&= \|f\|_p^r\|g\|_q^r.
\end{aligned}$$

Hence

$$\|f * g\|_r \leqslant \|f\|_p\|g\|_q.$$

□

14. (Young's Inequality For Convolution Alternative Form): If $1 \leqslant p, q, r \leqslant \infty$ are such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2,$$

then

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y)dxdy\right| \leqslant \|f\|_p\|g\|_q\|h\|_r.$$

More generally, both Young's inequality for convolution and this alternative form can be generalized to unimodular groups such as $\mathbb{Z}$.

Proof. WLOG we may assume f, g, h are nonnegative and integrable. Then since $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y)dxdy\right| \\
&\leqslant \int\int |f(x)^p g(x-y)^q|^{1-\frac{1}{r}} |f(x)^p h(y)^r|^{1-\frac{1}{q}} |g(x-y)^q h(y)^r|^{1-\frac{1}{p}} dxdy \\
&\overset{\text{Hölder}}{\leqslant} \left(\int\int f(x)^p g(x-y)^q\right)^{1-\frac{1}{r}} \left(\int\int f(x)^p h(y)^r\right)^{1-\frac{1}{q}} \left(\int\int g(x-y)^q h(y)^r\right)^{1-\frac{1}{p}} \\
&= \|f\|_p \|g\|_q \|h\|_r.
\end{aligned}$$

Since we note that
$$p(2-\frac{1}{q}-\frac{1}{r}) = q(2-\frac{1}{p}-\frac{1}{r}) = r(2-\frac{1}{p}-\frac{1}{q}) = 1.$$
□

15. (Equivalent Definition of L^p Norms Via Duality) Suppose p and p' are Hölder conjugates with $1 \leqslant p' < \infty$. If $g \in L^{p'}$ then
$$\|g\|_{L^{p'}} = \|\phi_g\| = \sup\left\{\left|\int_\Omega fg\right| : \|f\|_{L^p} = 1\right\}.$$

Proof. If $\|g\|_{L^{p'}} = 0$, then the $g \equiv 0$ a.e., so the statement is trivial. Otherwise, we have:
$\geqslant$: by Hölder's inequality, we have
$$\|g\|_{L^{p'}}\|f\| \geqslant \|fg\|_{L^1}.$$
$\leqslant$: take
$$f = \frac{|g|^{p'-1}\operatorname{sgn}(g)}{\|g\|_{p'}^{p-1}}.$$
□

16. Suppose $1 < p < \infty$, μ is σ-finite and g is measurable on X such that $fg \in L^1$ for all f simple functions supported in a finite-measure set. Define
$$M_{p'}(g) := \sup\{|\int fg| \,:\, f \,\text{simple}\,,\, \|f\|_{L^p} = 1\} < \infty.$$

Then $g \in L^{p'}$ and $M_{p'}(g) = \|g\|_{L^{p'}}$.

Proof. Since we can approximate $f \in L^p$ with simple functions supported on finite-measure set when $1 < p < \infty$. □

17. (Duality of L^p). When $1 < p < \infty$, for each $\phi \in (L^p)^*$, there exists a $g \in L^{p'}$ such that $\phi(f) = \int fg$ for all $f \in L^p$, and hence $L^{p'}$ is isometrically isomorphic to $(L^p)^*$. When μ is σ-finite, then the same conclusion holds for $p = 1$. In particular, $L^p(X)$ is reflexive when $1 < p < \infty$.

18. (Minkowski's inequality for integrals) Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces and let $f : X \times Y \to \mathbb{R}$ be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function.

    (a) If $f \geqslant 0$ and $1 \leqslant p < \infty$, then

    $$\left[\int_X \left(\int_Y f(\mathbf{x}, \mathbf{y}) \mathrm{d}\nu(\mathbf{y})\right)^p \mathrm{d}\mu(\mathbf{x})\right]^{1/p} \leqslant \int_Y \left[\int_X f(\mathbf{x}, \mathbf{y})^p \mathrm{d}\mu(\mathbf{x})\right]^{1/p} \mathrm{d}\nu(\mathbf{y}).$$

    (b) If $1 \leqslant p \leqslant \infty$, $f(\cdot, \mathbf{y}) \in L^p(\nu)$ for a.e. $\mathbf{y} \in Y$ and $\mathbf{y} \mapsto \|f(\cdot, \mathbf{y})\|_{L^p}$ is in $L^1(\nu)$, then $f(\mathbf{x}, \cdot) \in L^1(\nu)$ for a.e. $\mathbf{x} \in X$, the function $\mathbf{x} \to \int_Y f(\mathbf{x}, \mathbf{y}) \mathrm{d}\nu(\mathbf{y})$ is in $L^p(\mu)$ and

    $$\left\|\int_Y f(\cdot, \mathbf{y}) \mathrm{d}\nu(\mathbf{y})\right\|_{L^p} \leqslant \int_Y \|f(\cdot, \mathbf{y})\|_{L^p} \mathrm{d}\nu(\mathbf{y}).$$

    Proof. We only prove (a). (b) is a direct consequence of (a) (with f replaced by $|f|$) and Fubini's theorem.

    When $p = 1$, (a) becomes Tonelli's theorem. When $1 < p < \infty$, let p' be the conjugate exponent to p and let $g \in L^{p'}(\mu)$ with $\|g\|_{L^{p'}} \leqslant 1$. Then by Tonelli's theorem and Hölder's inequality, we have

    $$\begin{aligned} \int_X \left(\int_Y f(\mathbf{x}, \mathbf{y}) \mathrm{d}\nu(\mathbf{y})\right) |g(\mathbf{x})| \mathrm{d}\mu(\mathbf{x}) &= \iint_{X \times Y} f(\mathbf{x}, \mathbf{y}) |g(\mathbf{x})| \mathrm{d}\mu(\mathbf{x}) \mathrm{d}\nu(\mathbf{y}) \\ &\leqslant \|g\|_{L^{p'}} \int_Y \left[\int_X f(\mathbf{x}, \mathbf{y})^p \mathrm{d}\mu(\mathbf{x})\right]^{1/p} \mathrm{d}\nu(\mathbf{y}). \end{aligned}$$

    Taking supremum of the left side over all $g \in L^{p'}(\mu)$ with $\|g\|_{L^{p'}} \leqslant 1$ leads to our desired inequality by the equivalence of norm using dual. $\square$

5.4 Fourier Transform

We Recall the following basic facts fromFourier transform:

1. (Riemann Lebesgue Lemma) For any $f \in L^1(\mathbb{R}^d)$, the Fourier transform $\hat{f} \in \mathbb{C}(\mathbb{R}^d)$ and satisfies $|\hat{f}(\xi)| \to 0$ as $|\xi| \to \infty$.

2. (Fourier Inversion Formula) If $f, \hat{f} \in L^1(\mathbb{R}^d)$, then there exists a function $f_0 \in C_0(\mathbb{R}^d)$ (continuous functions that vanish at infinity) such that $f = f_0$ a.e., and $f_0 = (\hat{f})^\vee = (\check{f})^\wedge$.

3. If $f \in L^1$ and $\hat{f} = 0$, then $f = 0$ a.e.

4. C_c^∞ and $\mathcal{S}$ are both dense in L^p

5. If T is an invertible linear transform of $\mathbb{R}^d$ and $S = (T^*)^{-1}$ is the inverse transpose. Then $\widehat{f \circ T} = |\det T|^{-1} \hat{f} \circ S$. In particular, we have

   - (Translation) $(f(x-h))^\vee(\xi) = e^{-h \cdot \xi} \hat{f}(\xi)$ for any $h \in \mathbb{R}^d$.
   - (Scaling) $(f(\lambda x))^\vee(\xi) = |\lambda|^{-d} \hat{f}(\xi/\lambda)$ for any $\lambda \in \mathbb{R}$.
   - (Symmetry) If $f, \hat{f} \in L^1$, then $\check{f}(\xi) = \hat{f}(-\xi)$.

6. If $f, g \in L^1$, then

$$\int_{\mathbb{R}^d} \hat{f}(x) g(x) dx = \int_{\mathbb{R}^d} f(\xi) \hat{g}(\xi) d\xi.$$

   Proof. Fubini's theorem. □

7. The Fourier transform is an automorphism of $\mathcal{S}$ onto itself.