MA3211S Notes

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Contents

1 Topological Spaces

1.1 Basics of Topology

Lemma 1.1 *The closure and boundary of a set A is closed and the interior of a set A is open. A closed set containing A contains $\overline{A}$, an open set that is contained in A is contained in $\mathring{A}$.*

Definition: Let (X, τ) be a topological space and let $x \in X$. A neighbourhood of x is a set N that contains an open set containing x.

Lemma 1.2 *Suppose (X, d) is a metric space, and $Y \subseteq X$. Then the subspace topology of Y is equivalent to the topology induced by the metric $d|_Y$.*

Definition: Let (X, τ) be a topological space and $\sim$ be an equivalence relation on X. Denote the set $X/\sim$ by Y and let $q : X \to Y$ be the equivalence map. The quotient topology on Y is

$$\{U \in Y \,:\, q^{-1}(U) \in \tau\}.$$

Remark: $q^{-1}(U)$ is the union of the equivalence classes in U. q is continuous and the quotient topology is the finest topology making it so.

Lemma 1.3 *Suppose X, Y, Z are topological spaces, then*

$$X \times (Y \times Z) \cong X \times Y \times Z.$$

And $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}^2$. In particular, $\mathbb{R}^{n+m} \cong \mathbb{R}^n \times \mathbb{R}^m$.

Lemma 1.4 *Suppose X is a topological space and $Y \subset X$. Then Y is compact in the subspace topology iff Y is compact in X (any open cover of Y in X admits a finite subcover).*

Proposition 1.5 *Every compact subset of a Hausdorff topological space is closed.*

Theorem 1.6 (Tychonoff) *The product of any collection of compact sets is compact with respect to the product topology.*

Theorem 1.7 *A continuous function maps compact sets to compact sets and maps connected sets to connected sets.*

Proposition 1.8 *Let X be a compact metric space, let Y be a metric space and let $f : X \to Y$ be continuous. Then f is uniformly continuous.*

Definition: let X be a metric space and $K, Y \subset X$, then the distance between K and Y is defined to be $d(K, Y) = \inf\{d(x, y) \,:\, x \in K,\, y \in Y\}$.

Lemma 1.9 *Let $K \subset X$ be compact and $Y \subset X$ be closed. Then $\exists x \in K$, s.t.,*

$$d(x, Y) = \inf\{d(x, y) \,:\, x \in K,\, y \in Y\}.$$

Lemma 1.10 *If X is a topological spaces. Then the following are equivalent:*

1. *X is connected;*
2. *Every continuous $f : X \to \mathbb{Z}$ is constant.*
3. *Every continuous $f : X \to \mathbb{Q}$ is constant.*
4. *The only subsets of X both open and closed are $\emptyset$ and X.*

Theorem 1.11 (Baire Category Theorem) *Let X be a complete metric space. Then any countable intersection of dense open subsets of X is dense in X.*

1.2 Curves, Paths, Domains

Definition: let X be a topological space and let $x, y \in X$. A (continuous) path from x to y is a continuous function $\phi : [a, b] \to X$ such that $\phi(a) = x$ and $\phi(b) = y$.
Definition: X is path connected if $\forall x, y \in X$, $\exists$ a path from x to y.

Proposition 1.12 *Path-connectedness implies connectedness.*

Definition: let X be a topological space and $\phi : [a, b] \to X$ and $\psi : [c, d] \to X$ be continuous with $\phi(b) = \psi(c)$. Then the join of ϕ and ψ, written $\phi \vee \psi$ is defined as $\phi \vee \psi : [a, b + d - c] \to X$,

$$\phi \vee \psi : t \mapsto \begin{cases} \phi(t), & a \leq t < b \\ \psi(t + c - b), & b \leq t \leq b + d - c \end{cases}.$$

Definition: the reverse of ϕ, wrtten $-\phi$, is the path $-\phi : [-b, -a] \to X$ with $t \mapsto \phi(-t)$.
Notation: we write $x \to y$ if there exists a continuous path in X from x to y. Note this is an equivalence relation.
Definition: the equivalence class of $\to$ are called path-components.
Definition: let $X \subseteq \mathbb{R}^n$. Then a polygonal path is a path $\phi : [a, b] \Rightarrow X$ which is piecewise linear, that is $\exists a = x_0 < x_1 < \cdots < x_n = b$ such that $\phi((1 - t)x_{i-1} + tx_i) = (1 - t)\phi(x_{i-1}) + t\phi(x_i)$ for $t \in [0, 1]$ and $1 \leq i \leq n$. X is said to be polygonally connected if any two points of X can be joined by polygonal paths.
Notation: we write $x \twoheadrightarrow y$ if there exists a polygonal paths in X from x to y. Note $\twoheadrightarrow$ is also an equivalence relation.

Theorem 1.13 *Let $X \subset \mathbb{R}^n$ be open. Then the following are equivalent:*

1. *X is connected;*
2. *X is path-connected;*
3. *X is polygonally connected.*

Remark: if $X \subset \mathbb{R}^n$ is open, then all path components of X are open.

Lemma 1.14 *Let $n \geq 2$ and $X \subset \mathbb{R}^n$ be a compact topological space. Then X^c has only open path-components and precisely one of these is unbounded.*

Definition: a path $\phi : [a, b] \to \mathbb{C}$ is C^1 if the function ϕ is continuously differentiable with the appropriate one-sides limits at the endpoints a and b. In particular ϕ and ϕ' are bounded.
Definition: two paths $\phi : [a, b] \to \mathbb{C}$ and $\psi : [c, d] \Rightarrow \mathbb{C}$ are equivalent if $\exists$ a C^1 function $\gamma : [a, b] \to [c, d]$ with $\gamma'(t) > 0$ $\forall t \in [a, b]$, $\gamma(a) = c$, $\gamma(b) = d$ and $\phi(t) = \psi(\gamma(t))$ $\forall t \in [a, b]$. Note equivalence of path is an equivalence relation.
Definition: if ϕ is a path, $\phi : [a, b] \to \mathbb{C}$, the track ϕ^* of ϕ is defined by $\phi^* = \{\phi(t) \in \mathbb{C} : a \leq t \leq b\}$.
Definition: a path $\phi : [a, b] \to \mathbb{C}$ is piecewise C^1 if $\exists a = x_0 < x_1 < \cdots < x_n = b$ such that the restriction of ϕ to $[x_{i-1}, x_i]$, $\phi|_{[x_{i-1}, x_i]}$ is C^1. Equivalently, ϕ is piecewise C^1 if it can be written as the joint of finitely many C^1 paths.
Definition: two piecewise paths are equivalent if we can find $a = x_0 < x_1 < \cdots < x_n = b$ and $c = y_0 < y_1 < \cdots < y_n = d$ such that for all $1 \leq i \leq n$, $\phi|_{[x_{i-1}, x_i]}$ and $\psi|_{[y_{i-1}, y_i]}$ are equivalent.
Definition: a piecewise C^1 path $\phi : [a, b] \to \mathbb{C}$ is closed if $\phi(a) = \phi(b)$. It is simple if $\phi(x) = \phi(y) \Rightarrow \{x, y\} \subset \{a, b\}$ or $x = y$.

Definition: a domain is a connected open subset of $\mathbb{C}$.
Definition: given $x, y \in \mathbb{C}$, let $[x \to y]$ be the path $\phi : [0, 1] \Rightarrow \mathbb{C}$, $\phi(t) = (1 - t)x + ty$.
Definition: a domain D is convex if $x, y \in D \Rightarrow [x \to y]^* \subset D$.
Definition a domain D is star-shaped if $\exists z_0 \in D$ such that $[z_0 \to z]^* \subset D$ $\forall z \in D$.
Note every non-empty convex domain is a star domain.

1.3 Homotopy and Elementary Deformation

Definition: let $\phi : [0, 1] \to D$ and $\psi : [0, 1] \to D$ be piecewise C^1 closed paths in a domain D. A homotopy from ϕ to ψ is a function $\gamma : [0, 1]^2 \to D$ such that

- γ is continuous;
- $\gamma(0, t) = \phi(t)$ $\forall t \in [0, 1]$;
- $\gamma(1, t) = \psi(t)$ $\forall t \in [0, 1]$;
- $\forall s \in [0, 1]$, then path $\gamma_s(t)$ defined by $\gamma_s(t) = \gamma(s, t)$ is closed and piecewise C^1.

Definition: ψ is said to be an elementary deformation of ϕ if $\exists 0 = x_0 < x_1 < \cdots < x_n = 1$ and convex open subsets $C_1, \cdots, C_n \subset D$ such that $x_{i-1} \leq t \leq x_i \Rightarrow \phi(t) \in C_i, \psi(t) \in C_i$.

Proposition 1.15 *Let $\phi : [0, 1] \to D$ and $\psi : [0, 1] \to D$ be homotopic in a domain D. Then $\exists \phi = \phi_0, \phi_1, \cdots, \phi_n = \psi$ such that ϕ_i is an elementary deformation of ϕ_{i-1}.*

Definition: let D be a domain. A closed path ϕ is contractible if it is homotopic to a constant path.
Definition: a domain D is simply connected if every closed path is contractible.

1.4 Space of Complex Functions

Proposition 1.16 *$\mathscr{C}(X)$ which is the space of all continuous complex valued function defined on X, equipped with the topology induced by the* sup *norm is complete.*

Proposition 1.17 *Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of holomorphic functions on D which converges uniformly, then it will converge to a holomorphic function f.*

Proposition 1.18 *Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of holomorphic function which converges in L_1 or L_2, then they converge locally uniformly.*

Lemma 1.19 *Let D be a domain, $\phi : [a,b] \to D$ be a path and $f_n : D \to \mathbb{C}$ be continuous. Suppose $f_n \to f$ uniformly on ϕ^*. Then*

$$\int_{\phi} f_n(z)dz \to \int_{\phi} f(z)dz.$$

2 Complex Numbers

- A complex number z has the form $a + bi$, $(a, b \in \mathbb{R})$, and i is the number such that $i^2 = -1$.
- The conjugate of the number z of the form $a + bi$ is $a - bi$ and is denoted by $\overline{z}$.
- The modulus of z is given by $(z\overline{z})^{1/2} = \sqrt{a^2 + b^2}$.
- $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$.
- One can write a complex number in polar form: $z = r(\cos\theta + i\sin\theta)$, where r is the modulus of z and θ is the argument of z which can be calculated by $\tan\theta = \frac{b}{a}$.
- One can further rewrite a complex number using the Euler's Formula: $z = re^{i\theta}$, since $e^{i\theta} = \cos\theta + i\sin\theta$.
- $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] = (a_1b_1 - a_2b_2) + i(a_1b_2 + a_2b_1)$.
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$.
- $z^n = r^n(\cos n\theta + i\sin n\theta)$.
- $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$ for $k = 0, 1, \cdots, n-1$.

Suppose z, w are complex numbers, then the following are true:

- $|z| \geq 0$
- $|z| = 0 \Leftrightarrow z = 0$
- $|\mathrm{Re}\, z| \leq |z|$

- $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
- $|zw| = |z||w|$
- $|z + w| \leq |z| + |w|$
- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{z} \cdot \overline{w} = \overline{z \cdot w}$
- $\overline{z}/\overline{w} = \overline{\frac{z}{w}}$
- $z\overline{z}$ is real and non-negative
- $z\overline{z} = |z|^2 = |\overline{z}|^2 = |z^2|$
- $\overline{z^n} = \overline{z}^n$

Note: Suppose a, b, x are complex numbers, then **it is NOT necessarily the case** that $(x^a)^b = x^{ab} = (x^b)^a$. Counterexample : $e^{-2\pi} = e^{2\pi i^2} = (e^{2\pi i})^i = 1^i = 1$, a contradiction.

Problem: The solution to $az + b\bar{z} + c = 0$.

Problem: The center of the circumcircle of a triangle (Answer by Scott).

Riemann Sphere:

It is desirable to introduce a geometric model in which all points of the extended plane have a concrete representative. To this end we consider the unit sphere S whose equation in three-dimensional space is $x_1^2 + x_2^2 + x_3^2 = 1$. With every point on S, except (0,0,1), we can associate a complex number

$$z = \frac{x_1 + ix_2}{1 - x_3}, \tag{24}$$

and this correspondence is one to one. Indeed, from (24) we obtain

$$|z|^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3},$$

and hence

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}. \tag{25}$$

Further computation yields

$$\begin{aligned} x_1 &= \frac{z + \bar{z}}{1 + |z|^2} \\ x_2 &= \frac{z - \bar{z}}{i(1 + |z|^2)}. \end{aligned} \tag{26}$$

Problem: Radius of the spherical image of a circle.

Suppose three points z_1, z_2, z_3 determines an oriented angle, where z_2 is the vertex, and the orientation is from the side z_1z_2 to z_3z_2. Then the size of the angle is given by $\arg \frac{z_2 - z_3}{z_2 - z_1} = \arg \frac{z_3 - z_2}{z_1 - z_2}$.

Two z_1, z_2 are orthogonal in the complex plane iff $z_1\bar{z}_2 + z_2\bar{z}_1 = 0$.
Four points z_1, z_2, z_3, z_4 ($z_1z_2z_3z_4$ is a quadrilateral) are concyclic or colinear iff

$$\frac{z_1 - z_4}{z_1 - z_2} \Big/ \frac{z_3 - z_4}{z_3 - z_2}$$

is a real number.

Suppose $f(z)$ is analytic in a simply connected region D, then the following are equivalent:

1. $f(z)$ is a constant function on D;
2. $f'(z) = 0$ in D;
3. $\overline{f(z)}$ is analytic in D;

4. $|f(z)|$ is constant in D;

5. $\operatorname{Re} f(z)$ or $\operatorname{Im} f(z)$ is constant in D.

We define the exponential function e^z by

$$e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!}.$$

The following are true about e^z:

1. Then
$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

2. Note e^z is holomorphic on the entire complex plane, with $(e^z)' = e^z$, and there doesn't exists z such that $e^z = 0$.

3. $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

4. e^z is a periodic function with period $2\pi i$. $e^{z_1} = e^{z_2}$ iff $z_1 = z_2 + 2k\pi i$, $k = 0, \pm 1, \pm 2, \cdots$.

We define the following trigonometric functions:

1. $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$;

2. $\cos z = \frac{e^{iz}+e^{iz}}{2i}$;

3. $\tan z = \frac{\sin z}{\cos z} = \frac{1}{i} \cdot \frac{e^{2iz}-1}{e^{2iz}+1}$.

We can also define cot, sec and csc in the usual way. They will satisfy the following properties:

1. The derivative of the trigonometric functions are equal to the derivative of these function restricted to the real plane.

2. Any trigonometric identity that holds for real numbers will also holds for complex numbers.

3. The zeros of $\sin z$ are $z = n\pi$, $n \in \mathbb{Z}$, and the zeros for $\cos z$ are $z = \left(n + \frac{1}{2}\right)\pi$, $n \in \mathbb{Z}$.

4. If $\tan(z + \omega) = \tan z$, then $\omega = k\pi$, $k \in \mathbb{Z}$.

We define the following hyperbolic trigonometric functions:

1. $\sinh z = \frac{e^z-e^{-z}}{2}$;

2. $\cosh z = \frac{e^z+e^{-z}}{2}$;

3. $\tanh z = \frac{\sinh z}{\cosh z}$;

4. $\coth z = \frac{\cosh z}{\sinh z}$;

5. $\operatorname{sech} z = \frac{1}{\cosh z}$;

6. $\operatorname{csch} z = \frac{1}{\sinh z}$.

Suppose $z = re^{i\theta}$, then
$$w = \sqrt[n]{z} = \sqrt[n]{|z|}e^{i\frac{\theta+2k\pi}{n}}.$$
Suppose $z = re^{i\theta}$, then $\operatorname{Ln}(z) = \ln|z| + i(\arg z + 2k\pi) = \ln r + i(\theta + 2k\pi), \quad k \in \mathbb{Z}$. Using Ln, we can define the inverse trignometric functions and inverse hyperbolic trigonometric functions:

1. $\operatorname{Arcsin} z = \frac{1}{i}\operatorname{Ln}(iz + \sqrt{1-z^2})$;
2. $\operatorname{Arccos} z = \frac{1}{i}\operatorname{Ln}(z + i\sqrt{1-z^2})$;
3. $\operatorname{Arctan} z = \frac{1}{2i}\operatorname{Ln}\frac{1+iz}{1-iz}$;
4. $\operatorname{Arcsinh} z = \operatorname{Ln}(z + \sqrt{z^2+1})$;
5. $\operatorname{Arccosh} z = \operatorname{Ln}(z + \sqrt{z^2-1})$;
6. $\operatorname{Artanh} z = \frac{1}{2}\operatorname{Ln}\frac{1+z}{1-z}$.

Let $z \in \mathbb{C}$, then the following are true:

1. $\overline{e^z} = e^{\bar{z}}$.
2. $\overline{\sin(z)} = \sin(\bar{z})$.
3. $\overline{\cos(z)} = \cos(\bar{z})$.
4. $\lim\limits_{z\to 0}\frac{\sin z}{z} = 1$.
5. $\lim\limits_{z\to 0}\frac{e^z-1}{z} = 1$.

3 Complex Functions

3.1 Holomorphic and Harmonic Functions

Definition: let $f : D \to \mathbb{C}$, $z_0 \in D$, then f is differentiable at z_0 if
$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$
exists, and we call the limit the derivative of f at z_0, denoted $f'(z_0)$.
Definition: we say a function f is holomorphic at a point z_0 if it is differentiable in a neighbourhood of z_0. If D is a domain, and $f(z)$ is differentiable for every point in D, then we say f is holomorphic or $\mathbb{C}-$analytic in D. We say f is holomorphic on $\overline{D}$, if it is holomorphic in a domain containing $\overline{D}$.

Definition: suppose $f : D \to \mathbb{C}$ is a complex-valued function, then we define the partial derivative of f with respect to z and $\bar{z}$ as follows:

$$\frac{\partial f}{\partial z} = \lim_{h\to 0} \frac{f(z+h,\bar{z}) - f(z,\bar{z})}{h};$$
$$\frac{\partial f}{\partial \bar{z}} = \lim_{h\to 0} \frac{f(z,\bar{z}+h) - f(z,\bar{z})}{h}.$$

Lemma 3.1

1. $\frac{\partial}{\partial x} = \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)$.
2. $\frac{\partial}{\partial y} = \frac{1}{2i}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$.
3. $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$.
4. $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.
5. $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$.

Definition: a function $f : D \to \mathbb{C}$ is C^1 if $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ exist and are continuous. f is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$, then in this case $\frac{\partial f}{\partial z} = f'$.
Definition: a function $f : D \to \mathbb{C}$ is harmonic if $\Delta f = 0$, which happens if and only if $\frac{\partial^2 h}{\partial z \partial \bar{z}} = 0 \Leftrightarrow \frac{\partial h}{\partial z}$ is holomorphic.

Cauchy-Riemann Equation:
Let $f = u + iv$, u, v real. Then

$$\frac{\partial}{\partial \bar{z}} = 0 \;\Leftrightarrow u_x = v_y, u_y = -v_x.$$

Suppose $u_x = v_y, u_y = -v_x$ and the partial derivatives are continuous or $f \in C^1$, then f is holomorphic, and

$$f' = u_x + iv_x = v_y + iv_x = u_x - iu_y = v_y - iu_y.$$

Cauchy-Riemann Equation in Polar Form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial u}{\partial \theta}, \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

In this case,

$$f'(z) = (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right).$$

Definition: f is antiholomorphic, if $\bar{f}$ is holomorphic, which happens iff $\frac{\partial f}{\partial z} = 0$. In particular,

$$\frac{\partial f(z)}{\partial z} = \frac{\partial \overline{f(z)}}{\partial \bar{z}}.$$

If f is antiholomorphic, then f is also harmonic.

Lemma 3.2 *Suppose $h = u + iv$ is harmonic, where u, v are real functions, then u is harmonic and v is harmonic.*

Definition: we say v is a harmonic conjuagate of u if $u + iv$ is holomorphic.

Lemma 3.3 *If $u : D \to \mathbb{R}$ is harmonic, D is simply connected, then $\exists$ a harmonic conjugate of u up to an additive constant.*

To find the harmonic conjuagate of a real function u, we express it in terms of z and $\bar{z}$, or $\frac{1}{2}(f(z) + \overline{f(z)})$. Then u must be the real part of $f(z)$. I.e., if $\exists f$ holomorphic, s.t., $u = \mathrm{Re}\,(f)$, then $u = \frac{f+\bar{f}}{2}$. And $v = \mathrm{Im}(f)$.

Definition: if $u : D \to \mathbb{R}$ is C^2, if $\Delta u \geq 0$, we say that u is subharmonic.
If $f : D \to \mathbb{C}$ is holomorphic, then $|f|^2$ is subharmonic, since $\frac{\partial^2}{\partial z \partial \bar{z}}|f|^2 = |f'|^2 \geq 0$.
Alternatively, $f : U \to \mathbb{R}$ is subharmonic on U, if $\forall$ domain D with $\bar{D} \subset U$, and $\forall v : D \to \mathbb{R}$ harmonic, $\bar{D} \to \mathbb{R}$ continuous, if $u \leq v$ on bD, then $u \leq v$ on D.
Example: $|f|$, $|f| + |g|$, $\log|f|$, $\max(|f|, |g|)$.

Proposition 3.4 *Suppose U and V are open sets in the complex plane. If $f : U \to V$ and $g : V \to \mathbb{C}$ are two C^1 functions. Let $h = g \circ f$, then*

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}$$
$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}.$$

4 Complex Integrals

4.1 Basics of Complex Integral

Lemma 4.1 *if $f : [a, b] \to \mathbb{C}$ is Riemann integral, i.e, if both its real and imaginary parts are Riemann integral. Then*

$$\int_a^b f(t)dt = \int_a^b \mathrm{Re}(f(t))dt + i\int_a^b \mathrm{Im}(f(t))dt.$$

Lemma 4.2 *If $f : [a, b] \to \mathbb{C}$ is continuous, then*

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt.$$

Definition: let D be a domain, $f : D \to \mathbb{C}$ be continuous and $\phi : [a, b] \to D$ be a C^1 path. Then the integral of f along ϕ is defined as

$$\int_\phi f(z)dz = \int_a^b f(\phi(t))\phi'(t)dt.$$

If ϕ is piecewise C^1, then we take the integral to be the sum of individuals integrals on each C^1 components.

Lemma 4.3 *Equivalent paths give the same integral.*

Lemma 4.4 *Let D be $\mathbb{C}^*$, and $f(z) = z^n$ for some $n \in \mathbb{Z}$, $\phi : [0, 2\pi] \Rightarrow D$ be $t \mapsto e^{it}$. Then*

$$\int_\phi f(z)dz = \begin{cases} 2\pi i, & \text{if } n = -1, \\ 0 & \text{otherwise} \end{cases}.$$

Definition: let D be a domain and $\phi : [a,b] \to C$ be a C^1 path. Then the length of ϕ, $L(\phi)$ is defined as

$$L(\phi) = \int_a^b |\phi'(t)|dt = \int_\phi |dz|.$$

Lemma 4.5 *Let D be a domain, $f : D \to \mathbb{C}$ be continuous and $\phi : [a,b] \Rightarrow D$ be a C^1 path. Then*

$$\left| \int_\phi f(z)dz \right| \leq \sup_{z \in \phi^*} |f(z)| L(\phi).$$

Proposition 4.6 (Fundamental Theorem of Calculus) *Let D be a domain and let $f : D \to \mathbb{C}$ be continuous. Suppose f has an antiderivative F (i.e., a function $F(z)$ such that $F'(z) = f(z)$ $\forall z \in D$). Let $\phi : [a,b] \to D$ be a path. Then*

$$\int_\phi f(z)dz = F(\phi(b)) - F(\phi(a)).$$

Corollary 4.6.1 *If D is a domain, $f : D \to \mathbb{C}$ is continuous with antiderivative F and ϕ is a closed path, then $\int_\phi f(z)dz = 0$.*

Proposition 4.7 *Let D be a star-domain and $f : D \to \mathbb{C}$ be continuous. Then the following are equivalent:*

1. *f has an antiderivative F on D.*
2. *$\int_\phi f(z)dz = 0$ for all closed paths ϕ in D.*
3. *$\int_{\partial T} f(z)dz = 0$ for the boundary ∂T of any triangle T such that $\overline{T} \subset D$.*

Theorem 4.8 (Cauchy's Theorem for Triangles) *Let D be a domain and T be a triangle lying entirely in D. If $f : D \to \mathbb{C}$ is analytic, then*

$$\int_{\partial T} f(z)dz = 0.$$

Corollary 4.8.1 (Cauchy's Theorem for a star-domain) *Let D be a star-domain and $f : D \to \mathbb{C}$ be $\mathbb{C}$-analytic. Then $\int_\phi f(z)dz = 0$ for all closed paths ϕ in D.*

Lemma 4.9 *Let D be a domain, $f : D \to \mathbb{C}$ be $\mathbb{C}-$analytic, $\phi : [0,1] \to D$ be a closed path and ψ be an elementary deformation of ϕ. Then*

$$\int_\phi f(z)dz = \int_\psi f(z)dz.$$

Corollary 4.9.1 *Let D be a domain, $f : D \to \mathbb{C}$ be $\mathbb{C}-$analytic and ϕ, ψ be homotopic closed paths in D. Then*

$$\int_\phi f(z)dz = \int_\psi f(z)dz.$$

4.2 Cauchy's integral

Lemma 4.10 *Let D be a domain and $f : D \to \mathbb{C}$ be $\mathbb{C}-$analytic. If the closed path ϕ is contractible, then*

$$\int_\phi f(z)dz = 0.$$

If D is simply connected then $\int_\phi f(z)dz = 0$ for all closed paths ϕ.

Theorem 4.11 (Cauchy's Integral Formula) *Let D be a domain and let $f : D \to \mathbb{C}$ be $\mathbb{C}-$analytic. Let z_0, r be such that $\overline{B_r(z_0)} \subset D$. Then $\forall z \in B_r(z_0)$,*

$$f(z) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(w)}{w-z}dw.$$

Corollary 4.11.1 *If the function $f(z)$ is $\mathbb{C}-$analytic in $B_r(z_0)$, and continuous on $\overline{B_r(z_0)}$, then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})d\theta.$$

Theorem 4.12 *Let D be a domain and $f : D \to \mathbb{C}$ be $\mathbb{C}-$analytic. Then f is infinitely differentiable inside $B_r(z_0)$, $\overline{B_r(z_0)} \subset D$ and $\forall z \in B_r(z_0)$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{|z-z_0|=r} \frac{f(w)}{(w-z)^{n+1}}dw.$$

Corollary 4.12.1 *The derivative of an $\mathbb{C}-$analytic function is also $\mathbb{C}-$analytic.*

Theorem 4.13 (Cauchy's Inequality) *Let D be a domain, and $f(z)$ is $\mathbb{C}-$analytic in D, $a \in D$, $\overline{B_R(a)} \subset D$. Then*

$$|f^{(n)}(a)| \leq \frac{n!M(R)}{R^n},$$

where $M(R) = \max\limits_{|z-a|=R} |f(z)|$, $n = 1, 2, \cdots$, in particular, $|f'(a)| \leq \frac{M}{R}$.

Corollary 4.13.1 *If f is entire, $f(z) = o(|z|^{\alpha+1})$, when $z \to \infty$. Then f is a polynomial of degree $\leq \alpha$.*

Theorem 4.14 (Liouville's Theorem) *Every bounded entire function is constant.*

Corollary 4.14.1 *Every holomorphic function on the extended complex plane is constant.*

Corollary 4.14.2 (Fundamental Theorem of Algebra) *Every non-constant polynomial has at least one root in $\mathbb{C}$.*

Theorem 4.15 *Suppose $f(\xi)$ is continuous on a piecewise smooth curve C, then*

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}d\xi \quad (z \notin \mathbb{C})$$

defines a function $F(z)$ which is analytic on $D \setminus C$, and

$$F^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi.$$

Theorem 4.16 (Schwarz Lemma Generalized) *Suppose $f(z)$ is analytic in $|z| < R$, $f(0) = 0$ with order λ, $|f(z)| \leq M < +\infty$, then $|f(z)| \leq \frac{M}{R^\lambda}|z|^\lambda$, $|z| < R$, and $|\frac{f^{(\lambda)}(0)}{\lambda!}| \leq \frac{M}{R^\lambda}$. In addition if any of the equality holds, then*

$$f(z) = \frac{M}{R^\lambda}e^{ia}z^\lambda, \quad a \in \mathbb{R}.$$

Theorem 4.17 (Schwarz-Pick Theorem) *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then for all $z_1, z_2 \in \mathbb{D}$ and for all $z \in \mathbb{D}$, we have*

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)}\right| \leq \left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right|, \quad \text{and} \quad |f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

Proof: Consider $M(z) = \frac{z_1 - z}{1 - \bar{z_1}z}$, and $\varphi(z) = \frac{f(z_1) - z}{1 - \overline{f(z_1)}z}$. Since $M(z_1) = 0$ and the Möbius transformation is invertible, the composition $\varphi(f(M^{-1}(z)))$ maps 0 to 0 and the unit disk is mapped into itself. Thus we can apply Schwar'z Lemma, which is to say

$$\left|\varphi(f(M^{-1}(z)))\right| = \left|\frac{f(z_1) - f(M^{-1}(z))}{1 - \overline{f(z_1)}f(M^{-1}(z))}\right| \leq |z|.$$

Letting $z_2 = M^{-1}(z)$ yields the desired conclusion. For the derivative part, just rearrange and let $z_2 \to z_1$. $\square$

Theorem 4.18 (Borel-Carathéodory Theorem) *Let f be analytic on a closed disk with radius R centered at the origin and suppose $r < R$, then*

$$\max_{|z| \leq r} |f(z)| \leq \frac{2r}{R - r} \sup_{|z| \leq R} \operatorname{Re} f(z) + \frac{R + r}{R - r}|f(0)|.$$

Proof: If $f(z)$ is a constant, then the claim trivially follows, hence assume f is non-constant. And let $f(0) = 0$ by offsetting, define $A = \sup_{|z| \leq \mathbb{R}} \operatorname{Re} f(z)$, then $A > 0$. In addition, $\operatorname{Re} f(0) = \frac{1}{2\pi} \int_{|z|=R} \operatorname{Re} f(z)dz$. Now f maps into the half-plane P to the left of the line $x = A$. Consider the map $w \mapsto \frac{w}{A} - 1$ which sends P to the standard left half-plane, $w \mapsto R\frac{w+1}{w-1}$ which send the plane to circle with radius R. The composite, which maps 0 to 0, is $w \mapsto \frac{Rw}{w-2A}$. From Schwarz' Lemma, we have $\frac{|R(f)|}{|f(z) - 2A|} \leq |z|$. Take $|z| \leq r$, the above becomes $R|f(z)| \leq r|f(z) - 2A| \leq r|f(z)| + 2Ar$ so $|f(z)| \leq \frac{2Ar}{R-r}$. In general case, just need to add $f(0)$ and we will have the claim. $\square$

Corollary 4.18.1 *If the real part or imaginary part of $f(z)$ is $o(|z|^{\alpha+1})$, $\alpha \in \mathbb{Z}^+$, and f is entire, then f is a polynomial of degree $\leq a$.*

4.3 Morera's Theorem

Theorem 4.19 (Morera's Theorem) *Let D be a star-shaped domain and $f : D \to \mathbb{C}$ be continuous. if*

$$\int_{\partial T} f(z)dz = 0$$

for all triangles $T \subset D$ then f is $\mathbb{C}-$analytic.

Remark: now let D be an arbitrary domain and let $z \in D$. Since $\exists \epsilon > 0$ such that $B_\epsilon(z) \subset D$ and $B_\epsilon(z)$ is star-shaped, one can easily extend Morera's theorem to any domain.

5 Taylor and Laurent's Series

5.1 Power Series

Lemma 5.1 *Consider a power series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$*. If this sum converges for some* z *with* $|z-z_0| = 0$*, then it converges for all* w *with* $|w-z_0| < 0$ *and for any* $r < p$*. Moreover, then convergence is uniform in* $\overline{B_r(z_0)}$*.*

Definition: the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is

$$R = \sup\{r : \exists z \text{ such that } |z-z_0| = r \text{ and } \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ converges}\}.$$

Lemma 5.2 *Let* $l = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$*, or* $l = \lim \sqrt[n]{|a_n|}$ *or* $l = \limsup \sqrt[n]{|a_n|}$*, then the radius of convergence of* $\sum a_n(z-z_0)^n$ *is*

$$R = \begin{cases} \frac{1}{l}, & l \neq 0, l \neq +\infty; \\ 0, & l = \infty; \\ \infty, & l = 0 \end{cases}.$$

Definition: the series $\sum f_n(x)$ is called normal convergent on S if

$$\sum_{n=0}^{\infty} \sup_{x\in S} |f_n(x)| < \infty.$$

Definition: the series $\sum f_n(x)$ is called locally normal convergent on X if for every $x \in X$, exists an open neighbourhood of x, denoted U, such that $\sum f_n|_U$ is normal convergent.
Definition: the series $\sum f_n(x)$ is called compact normal convergent on X if for every compact subset K of X, $\sum f_n|_K$ is normal convergent.

Lemma 5.3 *The power series* $\sum a_n(z-z_0)$ *with radius of convergence* R *is compact normal convergent and locally normal convergent in* $B_R(z_0)$*.*

Lemma 5.4 *Let* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$*. Suppose there exists a sequence* $(z_k)_{k=1}^{\infty} \to z_0$*,* $z_k \neq z_0$ *and* $f(z_k) = g(z_k)$*. Then* $a_n = b_n$ *for all* n*.*

Lemma 5.5 *Let* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *with radius of convergence* R*. Then the antiderivative of* f*,*

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$$

and the derivative of f*,*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

are both $\mathbb{C}-$*analytic in* $B_R(z_0)$*, in particular, they have the same radius of convergence.*

5.2 Taylor Series

Theorem 5.6 (Taylor's Theorem) *Let D be a domain and then $f : D \to \mathbb{C}$ be $\mathbb{C}$-analytic. Let $z_0 \in D$ and let R be such that $B_R(z_0) \subset D$. Then there exist unique coefficients $(a_n)_{n=0}^{\infty}$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \;\; a_n = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi, \;\; \forall z \in B_R(z_0).$$

Corollary 5.6.1 (Identity Theorem / Uniqueness Theorem) *Let D be a domain and $f, g : D \to \mathbb{C}$ be $\mathbb{C}-$analytic. Suppose $z_k \to z_0 (\in D)$, $z_k \neq z_0$ and $f(z_k) = g(z_k)$ for all k. Then $f(z) = g(z)$ $\forall z \in D$. In particular, the zeros of a non-constant $\mathbb{C}$-analytic function is isolated.*

Corollary 5.6.2 *Suppose $f(z)$ is holomorphic on a bounded domain D and f is not a constant function. Then $f(z) = z_0$ has only finitely many solutions in D, where $z_0 \in \mathbb{C}$.*

Theorem 5.7 *Suppose the power series $\sum\limits_{n=0}^{\infty} c_n(z-a)^n$ has a radius of convergence $R > 0$, and*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n, \;\; (z \in K : |z-a| < R),$$

Then $f(z)$ has at least one singularity on the circle $C : |z-a| = R$, i.e., there does not exists a function $F(z)$ such that it is equal to $f(z)$ $\forall |z-a| < R$ but analytic at every point of C.

Definition: suppose $f(z)$ is holomorphic in D and $\exists z_0$ in D, s.t., $f(z_0) = 0$, then we call z_0 a zero of $f(z)$. If f is not constantly 0 on D, then $\exists m \in \mathbb{Z}^+$, s.t., $f(z) = (z-z_0)^m g(z)$, where $g(z_0) \neq 0$ and $g(z)$ is analytic on a neighbourhood of z_0. Then we call m the order of the zero.

Theorem 5.8 (Maximum Modulus Principle) *Let D be a domain and $f : D \to \mathbb{C}$ be $\mathbb{C}$-analytic. Suppose $|f|$ has a local maximum. Then f is constant.*

Corollary 5.8.1 (Minimum Modulus Principle) *Suppose $f(z)$ is not a constant on D, and $f(z)$ is not 0. If f is holomorphic on D, then the minimum of $|f(z)|$ cannot be obtained on D.*

Corollary 5.8.2 *Suppose D is the interior of a closed curve C, and $f(z)$ is holomorphic in D, and continuous on $\overline{D}$. If $|f(z)|$ is constant on C and $f(z)$ is not a constant function, then $f(z)$ has at least one zero on D.*

5.3 Laurent Series

Theorem 5.9 (Laurent's Theorme) *Let D be the (non-empty) domain $\{z : a < |z-z_0| < b\}$ and let $f : D \to \mathbb{C}$ be $\mathbb{C}-$analytic. Then there exists unique coefficients $(a_n)_{n\in\mathbb{Z}}$ such that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \;\; a_n = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi, \;\; \forall z \in D.$$

Definition: the principle part of f at z_0 is defined to be the function

$$g(z) = \sum_{n=-\infty}^{-1} a_n(z - z_0)^n.$$

Theorem 5.10 (L' Hópital) *Suppose a is a removable singularity of a pole for f, g, and $g \not\equiv 0$, then*

$$\lim_{z \to a} \frac{f}{g} = \lim_{z \to a} \frac{f'}{g'}.$$

Proof: Since a is removable or a pole for f, g then f, g is holomorphic in a neighbourhood of a that does not include a. Then by Laurent's Series, we have $f = (z-a)^k \tilde{f}$, $g = (z-a)^l \tilde{g}$, where $\tilde{f}, \tilde{g}$ holomorphic, $\tilde{f}(a), \tilde{g}(a) \neq 0$, $k, l \leq 0$. Then

$$\lim_{z \to a} \frac{f}{g} = \lim_{z \to a} \frac{(z-a)^k \tilde{f}}{(z-a)^l \tilde{g}} = \begin{cases} 0 & \text{if } k > l \\ \frac{\tilde{f}(a)}{\tilde{g}(a)} & \text{if } k = l \\ \infty & \text{if } k < l \end{cases}.$$

And if we take the derivative, then it is clear that the same result holds. □

5.4 Singularities

Theorem 5.11 (Riemann's Removable Singularity Theorem) *Consider a domain D with $z_0 \in D$. Let $f : D \setminus \{z_0\} \to \mathbb{C}$ be analytic. If f is bounded near z_0, then $f \to a$ as $z \to z_0$ for some $a \in \mathbb{C}$, and the function*

$$g(z) = \begin{cases} f(z), & z \neq z_0 \\ a, & z = z_0 \end{cases}$$

is $\mathbb{C}-$analytic.

Proposition 5.12 *Let D be a domain, $z_0 \in D$ and $f : D \setminus \{z_0\} \to \mathbb{C}$ be $\mathbb{C}-$analytic. Suppose $|f(z)| \to \infty$ as $z \to z_0$. Then there are unique integer $k \geq 1$ and unique $\mathbb{C}-$ analytic function $g : D \to \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^{-k} g(z)$ when $z \neq z_0$.*

Corollary 5.12.1 *Suppose $f : D \to \mathbb{C}$, and $z_0 \in D$ is not an essential singularity or non-isolated singularity. Then $\exists k \in \mathbb{Z}$, s.t., $f(z) = (z - z_0)^k g(z)$, where $g(z_0) \neq 0$ and g is holomorphic in a neighbourhood of z_0.*

Definition: a point $z_0 \in D$ is a singular point of f if in every neighbourhood of z_0 exists some points that f is holomorphic at those points.

Definition: let D be a domain, $z_0 \in D$ and $f : D \setminus \{z_0\} \to \mathbb{C}$ be $\mathbb{C}$-analytic. Then z_0 is an isolated singularity of f:

- If f is bounded in a neighbourhood of z_0, the singularity is called removable.
- If $|f(z)| \to \infty$ as $z \to z_0$ and if k is the integer from the previous proposition, the singularity is a pole of degree $|k|$.

- Other singularity are called essential.

Definition: for a single valued function, a singularity that is not an isolated singularity is known as a non-isolated singular point.

Proposition 5.13 *If a is an isolated singularity of $f(z)$, then any of the following is equivalent to a being a removable singularity:*

- *The principle part of $f(z)$ at a is zero;*
- $\lim_{z \to a} f(z) = b$*;*
- *$f(z)$ is bounded at neighbourhood of a excluding a.*

Proposition 5.14 *If a is an isolated singularity of $f(z)$, then any of the following is equivalent to a being a pole of order k:*

- *The principle part of $f(z)$ at a is*
$$\frac{c_{-k}}{(z-a)^k} + \cdots + \frac{c_{-1}}{z-a}, \quad (c_{-m} \neq 0).$$
- *for a punctured neighbourhood of a,*
$$f(z) = \frac{\lambda(z)}{(z-a)^m},$$
where $\lambda(z)$ is analytic in a neighbourhood of a and $\lambda(a) \neq 0$;
- *$g(z) = \frac{1}{f(z)}$ has a zero of order m at a (by letting $g(a) = 0$). Note the converse of this also holds.*

Lemma 5.15 *Suppose f has an essential singularity at z_0, then $\lim_{z \to z_0} f(z)$ does not exist. z_0 is an essential singularity if and only if $a_n \neq 0$ for infinitely many negative n.*

Proposition 5.16 *Suppose $f : D \to \mathbb{C}$, and f is holomorphic on $D \backslash \{z_0\}$. If $\forall r > 0$, $f(D \backslash \{z_0\} \cap B(z_0, r) \supset \mathbb{C} \backslash \{p\}$, where $p \in \mathbb{C}$, then $\{z_0\}$ is an essential singularity of f.*

Proof: Suppose not, then one can argue by contradiction by construction a sequence $\{z_n\}$ that approaches to z_0, but $\{f(z_n)\}$ does not converge. □

Lemma 5.17 *Suppose z_0 is a non-removable singularity for f, then $e^{f(z)}$ has an essential singularity at z_0.*

Proposition 5.18 *Suppose $f(z)$ is not constantly 0 and $z = a$ is a holomorphic point or a pole for f. $\varphi(z)$ has an essential singularity at a, then $z = a$ is an essential singularity for $\varphi(z) \pm f(z)$, $\varphi(z) \cdot (f(z))^n$, $n \in \mathbb{Z}$.*

Theorem 5.19 (Casorati-Weierstrass Theorem) *Let D be a domain, $z_0 \in D$ and $f : D \setminus \{z_0\} \to \mathbb{C}$ be $\mathbb{C}-$analytic with an essential singularity at z_0. Then for every $w \in \mathbb{C}$, $\exists z_n \to z_0$ such that $f(z_n) \to w$. In particular, take $D = C^*$ and $z_0 = \infty$, we get that the image of any non-constant entire function is dense in $\mathbb{C}$*

Theorem 5.20 (Little Picard's Theorem) *$\forall f : \mathbb{C} \to \mathbb{C} \setminus \{a, b\}$, if f is holomorphic, then f is a constant. I.e., if f is entire and non-constant, then $C \setminus f(C)$ contains at most one point.*

Theorem 5.21 (Great Picard's Theorem) *If an analytic function has an essential singularity at a point then on any punctured neighborhood of f, it takes on all possible complex values, with at most one single exception, infinitely often. Any entire, non-polynomial function attains all possible complex values infinitely often, with at most one exception.*

Theorem 5.22 *$f(z)$ is a rational function if and only if $f(z)$ only has poles (no other type of singularity) on the extended complex plane.*

6 Winding Number and Residues

6.1 Continuous branches

Definition: $z \in \mathbb{C}^*$, w is a value of $\log z$ if $e^w = z$. If $z = re^{i\theta}$, then $w = \log r + i(\theta + 2k\pi)$, where $k \in \mathbb{Z}$. w is unique modulo $2i\pi$.
Definition: if $\theta \in \mathbb{R}$, s.t., $z = |z|e^{i\theta}$, we say that θ is a value of $\arg(z)$ (Unique mod 2π).
Definition: θ is called the principle value of $\arg(z)$ if $-\pi < \theta \leq \pi$. $w = a + ib$ is principle value of $\log z$ if $-\pi < b \leq \pi$.
Definition: $D \subset \mathbb{C}^*$ is a domain, a continuous branch of $\log z$ is a continuous function $L : D \to \mathbb{C}$ such that $L(z)$ is a value of $\log(z)$, $\forall z \in D$.

Lemma 6.1 *Suppose D is simply connected $f : D \to \mathbb{C}$ is holomorphic. Then there exists an antiderivative of f on D.*

Proof: Let $F(z) = \int_\phi f(w)dw$, where $\phi \subset D$ is an arbitrary piecewise C^1 path from z_0 to z (Note $F(z)$ is well-defined by Cauchy's Theorem). Then $F'(z) = \lim\limits_{h \to 0} \frac{F(z+h)-F(z)}{h} = f(h)$. □

Lemma 6.2 *Suppose $D \subset \mathbb{C}^*$ is simply connected, $z_0 \in D$, w_0 is a value of $\log z_0$. Then there exists a unique continuous branch $L : D \to \mathbb{C}$ of $\log z$ such that $L(z_0) = w_0$.*

Proof: Define $L(z) = \int_\phi \frac{dw}{w} + w_0$ (where ϕ is any path from z_0 to z), then $L'(z) = \frac{1}{z}$. We want to show L is a branch of $\log z$, i.e., $e^{L(z)} = z$, $\forall z \in D$.
Consider $g(z) = ze^{-L(z)}$, then $g'(z) = e^{-L(z)} - zL'(z)e^{-L(z)} = 0$, so $g \equiv c$. $g(z_0) = z_0 e^{-L(z_0)} = z_0 \cdot \frac{1}{z_0} = 1$, so $e^{L(z)} = z$, $\forall z$. □

Lemma 6.3 *Let $z_0 \in \mathbb{C}$ and $\Phi : [a, b] \to \mathbb{C} \setminus \{z_0\}$ be a path. Then $\exists L : [a, b] \to \mathbb{C}$, s.t., $L \in \mathscr{C}^0$, and $L(t)$ is a value of $\log(\phi(t) - z_0)$. Moreover L is unique modulo $2i\pi\mathbb{Z}$.*

Proof: If $\tilde{L}$ is another such function. Then $\tilde{L} - L : [a, b] \to \mathbb{C}$ is continuous and $\tilde{L}(t) - L(t) \in 2i\pi\mathbb{Z}$. Then $\tilde{L} - L \equiv$ constant in $2i\pi\mathbb{Z}$.
Now to prove the existence.
Case one: $\Phi^* \subset D \subset \mathbb{C} \setminus \{z_0\}$, where D is simply connected. Then $\exists l : D \to \mathbb{C}$ which is a branch of $\mathrm{Log}(z - z_0)$. And we can choose to define $L(t) = l(\phi(t))$.
Case two: Write $\Phi = \Phi_1 + \Phi_2 + \cdots + \Phi_n$, where each Φ_n is inside a convex set. Then choose $L_i : [t_{i+1}, t_i] \to \mathbb{C}$, where $L_i(t) =$ a value of $\log(\Phi(t) - z_0)$ (Can be done by case 1). Further, we can choose $L_i(t_i) = L_{i+1}(t_i)$. And we define $L = L_i$ on $[t_{i-1}, t_i]$. $\square$

Proposition 6.4 *Suppose $n \in \mathbb{Z}$, then $\exists f : C \to \mathbb{R}_- \to \mathbb{C}$ holomorphic, such that $f(z)^n = z$.*

Proof: Define $f(z) = e^{\frac{1}{n}L(Z)}$, where $L(z)$ is a holomorphic branch of log. Clear f is holomorphic in Ω, then $[f(z)]^n = e^{L(z)} = z$. $\square$

6.2 Residues

Definition: let D be a simply connected domain, take $z_0 \in \mathbb{C} \setminus D$ and consider a path $\phi : [a, b] \to D$. The change (or variation) in $\log(z - z_0)$ is defined as $L(\phi(b)) - L(\phi(a))$ where L is any continuous branch of $\log(z - z_0)$ on D. Note that this is well-defined, and from the way that L was produced, it is also equal to

$$\int_\phi \frac{dz}{z - z_0}.$$

Now suppose D is any domain, then we can split ϕ into pieces $\phi_1, \phi_2, \cdots, \phi_n$, such that for each $i \in \{1, \cdots, n\}$, we have $\phi_i^* \subset C_i$ where C_i is some convex set.
Definition: the change in $\log(z - z_0)$ along ϕ is defined to be the sum of the changes in log for each of the ϕ_i. Thus the change in log is also equal to

$$\int_\phi \frac{dz}{z - z_0}.$$

Remark: When we chose continuous branches L_i of $\log(z - z_0)$ in each C_i, we could, by adding suitable constants, ensure that $L_i(\phi(x_i)) = L_{i+1}(\phi(x_i))$. If we do that, then the change along ϕ is

$$\sum_{i=1}^{n} L_i(\phi(x_i)) - L_i(\phi(x_{i-1})) = L_n(\phi(b)) - L_1(\phi(a)).$$

In particular, if ϕ is closed, this must be $2\pi k i$ for some $k \in \mathbb{Z}$.

Definition: the winding number of a closed path ϕ about z_0 is defined as this k (in the remark above). It is denoted as $w(\phi, z)$ or $\operatorname{Ind}(\phi, z)$ and is equal to

$$\frac{1}{2\pi i}\int_\phi \frac{dw}{w-z}.$$

Remark: Ind: $\mathbb{C}\setminus\phi^* \to \mathbb{Z}$, $\operatorname{Ind}(\phi, z) = \frac{1}{2i\pi}\int_\Phi \frac{dw}{w-z}dz$ is $\mathscr{C}^0$, hence Ind is constant on each connected component of $\mathbb{C}\setminus\phi^*$.

Remark:

$$\lim_{z\to\infty} \operatorname{Ind}(\Phi, z) = \lim_{z\to\infty}\frac{1}{2i\pi}\int_\Phi \frac{dw}{w-z}dz = 0.$$

So $\operatorname{Ind}(\Phi, \cdot) \equiv 0$ on the unbounded connected component.

Remark: $\operatorname{Ind}(\phi, z_0)$ is in fact a $\mathbb{C}-$analytic function of z for $z \notin \phi^*$, the derivative is

$$\frac{1}{2\pi i}\int_\phi \frac{dw}{(w-z)^2} = 0.$$

Definition: the residue of f at z is defined as a_{-1} and is written $\operatorname{Res}(f, z)$, where a_{-1} is the coefficient of $(w-z)^{-1}$ in the Laurent expansion. Note

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=\delta} f(z)dz.$$

Theorem 6.5 *Suppose $D \subset \mathbb{C}$ is simply connected, $Z \subset D$ is a finite set ,and $f : D\setminus Z \to \mathbb{C}$ is holomorphic. Φ is a closed path in $D\setminus Z$, then*

$$\int_\Phi f(z)dz = 2i\pi\sum_{z\in Z}\operatorname{Ind}(\Phi, z)\cdot\operatorname{Res}(f, z).$$

Proof: Define $h(z) = f(z) - \sum_{a\in Z} g_a$, that is we remove all singularities of f, where g_a is the principle part of the f at a (By Laurent's Series). So h is holomorphic in D, then

$$\int_\Phi h(z)dz = 0.$$

Hence

$$\int_\Phi f = \sum_{a\in Z}\int_\Phi g_a(z)dz.$$

It suffices to show that $\int_\Phi g_a(z)dz = 2i\pi\operatorname{Ind}(\Phi, a)\cdot\operatorname{Res}(f, a)$.

Note $g_a : \mathbb{C}\setminus\{a\} \to \mathbb{C}$ is holomorphic, and $g_a = \sum_{n<0} a_n(z-a)^n$. So

$$\begin{aligned}\int_\Phi g_a(z)dz &= \int_\Phi \sum_{n<0} a_n(z-a)^n dz\\ &= \sum_{n<0} a_n\int_\Phi (z-a)^n dz\\ &= 2i\pi\operatorname{Ind}(\Phi, a)\cdot a_{-1}\\ &= 2i\pi\operatorname{Ind}(\Phi, a)\cdot\operatorname{Res}(f, a).\end{aligned}$$

□

Remark: $\mathrm{Ind}(\Phi, a) \cdot \mathrm{Res}(f, a)$, if f is holomorphic at b, then $\mathrm{Res}(f, b) = 0$. So we can replace Z with a larger set, then the formula

$$\int_\Phi f = \sum_{a \in Z} \mathrm{Ind}(\Phi, a) \cdot \mathrm{Res}(f, a)$$

will still hold.
Remark: in the theorem, we assume that Z is finite, but it enough to assume that Z is locally finite in D.

Proposition 6.6 *Suppose a is a pole of $f(z)$ with degree n, recall*

$$f(z) = \frac{\varphi(z)}{(z-a)^n},$$

then

$$\mathrm{Res}(f, a) = \frac{\varphi^{(n-1)}(a)}{(n-1)!}.$$

Or equivalently, if a is an order n pole for $f(z)$.

$$\mathrm{Res}(f, a) = \frac{1}{(n-1)!}((z-a)^n f(z))^{(n-1)}.$$

Proposition 6.7 *Let a be a degree 1 pole for $f(z) = \frac{\varphi(z)}{\psi(z)}$, and $\varphi(z)$, $\psi(z)$ is holomorphic at a, with $\varphi(a) \neq 0$, $\psi(a) = 0$, $\psi'(a) \neq (0)$, then*

$$\mathrm{Res}(f, a) = \frac{\varphi(a)}{\psi'(a)}.$$

6.3 Calculating Real Integrals Using Residue

- Calculating $\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$ type integral: we set $\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2i}$, $d\theta = \frac{dz}{iz}$. And the integral becomes

$$\int_{|z|=1} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}$$

- Calculating $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx$ type integral:

Lemma 6.8 *Suppose $f(z)$ is continuous along $S_R : z = Re^{i\theta}$ $(\theta_1 \leq \theta \leq \theta_2)$ for R large enough, and*

$$\lim_{R\to+\infty} zf(z) = \lambda,$$

then

$$\lim_{R\to\infty} \int_{S_R} f(z)dz = i(\theta_2 - \theta_1)\lambda.$$

Theorem 6.9 *If $f(z) = \frac{P(z)}{Q(z)}$ and $P(z), Q(z)$ are relatively prime and the degree of Q is at least greater than or equal to 2+ the degree of P and $Q(z) \neq 0$ on the real line, then*

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{\mathrm{Im}\, z>0} \mathrm{Res}(f, z).$$

- Calculating $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{imx} dx$ type integral:

Lemma 6.10 (Jordan) *Suppose $g(z)$ is continuous along the semicircle $\Gamma_R : z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and R is large enough, and*

$$\lim_{R \to +\infty} g(z) = 0$$

then

$$\lim_{R \to \infty} \int_{\Gamma_R} g(z) e^{imz} dz = 0, \quad (m > 0).$$

Theorem 6.11 *Let $g(z) = \frac{P(z)}{Q(z)}$, and $P(z)$, $Q(z)$ are relatively prime with the following condition:*

- *the order of $Q(z)$ is higher than the order of $P(z)$.;*
- *$Q(z) \neq 0$ on the real line.*
- *$m > 0$.*

Then

$$\int_{-\infty}^{\infty} g(x) e^{imx} dx = 2\pi i \sum_{\operatorname{Im} z_0 > 0} \operatorname{Res}[g(z) e^{imz}, z_0].$$

- Integrals with singularity on the path:

Lemma 6.12 *Suppose $f(z)$ is continuous along the arc $S_r : z - a = re^{i\theta}$, $\theta_1 \leq \theta \leq \theta_2$ and r is sufficiently small, and*

$$\lim_{r \to 0} (z - a) f(z) = \lambda,$$

then

$$\lim_{r \to 0} \int_{S_r} f(z) dz = i(\theta_2 - \theta_1)\lambda.$$

6.4 Principle of the Argument

Definition: recall f is holomorphic at a, or a is a pole of f. Then $\exists! g : D \to \mathbb{C}$ holomorphic, $g(a) \neq 0$ and $k \in \mathbb{Z}$, s.t., $f(z) = (z-a)^k g(z)$. Define the order of f at a, $\operatorname{ord}(f, a) = k$. Note $\operatorname{ord}(f, a) = 0$ iff a is not a zero or pole for f.

Theorem 6.13 (Principle of the Argument) *Let D be a simply connected domain and let $f : D \to \mathbb{C}$ be $\mathbb{C}$-analytic except at finitely many poles. Suppose also that f has finitely many zeros in D, and let the zeros and poles be $z_1, \cdots, z_k$. Let ϕ be a closed path in D such that $\phi^* \cap \{z_1, \cdots, z_k\} = \emptyset$. Then*

$$\frac{1}{2\pi i} \int_{\Phi} \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{k} \operatorname{ord}(f, z_j) \operatorname{Ind}(\Phi, z_j).$$

Corollary 6.13.1 *Let D be a simply connected domain and let $f : D \to \mathbb{C}$ be $\mathbb{C}$-analytic except at finitely many poles. Suppose also that f has finitely many zeros in D, and let the zeros and poles be $z_1, \cdots, z_k$. Let ϕ be a closed path in D such that $\phi^* \cap \{z_1, \cdots, z_k\} = \emptyset$. g is analytic on D, then*

$$\frac{1}{2\pi i} \int_{\Phi} g(z) \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{k} g(z_j) \operatorname{ord}(f, z_j) \operatorname{Ind}(\Phi, z_j).$$

Remark: let $z_1, z_2, \cdots, z_n$ be the zeros and poles of f where we repeat a zero $|m|$ times if it is of m, then the above formula can be written

$$\frac{1}{2\pi i}\int_{\Phi} g(z)\frac{f'(z)}{f(z)}dz = \sum_{i=1}^{n} g(z_i)\,\mathrm{ord}(f, z_j)\,\mathrm{Ind}(\Phi, z_j).$$

Remark: We can extend the summation set. The theorem also holds if $f \not\equiv 0$ (the zero set is locally finite), and the set of poles is locally finite.

Definition: the Logarithmic differentiation of a meromorphic function f is defined to be $\frac{f'}{f}$. Idea $(\log f)' = \frac{f'}{f}$. It is clear that logarithmic derivative is additive

$$\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}.$$

The geometric interpretation of principle of the argument is thus the change in $\arg f(z)$.

Notation: we write $ZP(f,\phi)$ for $\frac{1}{2\pi i}\int_{\phi}\frac{f'(z)}{f(z)}dz$.
Notice

$$\frac{1}{2\pi i}\int_{\phi}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{f\circ\phi}\frac{dz}{z} = \mathrm{Ind}(f\circ\phi, 0).$$

So $ZP(f,\phi)$ is always an integer.

Theorem 6.14 (Rouché's Theorem) *Let D be a simply connected domain, ϕ be a closed path in D and f and g be functions with the following properties:*

1. *f and g are $\mathbb{C}-$analytic on D except for finitely many poles, none of which lie on ϕ^*.*
2. *f and $f+g$ have finitely many zeros, none of which lie on ϕ^*.*
3. *$|g(z)| < |f(z)|\ \forall z \in \phi^*$.*

Then $ZP(f+g,\phi) = ZP(f,\phi)$. In particular, if f, g has no poles, then the number of zeros counting multiplicity for both functions are equal in the interior of ϕ.

Corollary 6.14.1 *Every degree n polynomial, $n \geq 1$ has exactly n roots counting multiplicity.*

Corollary 6.14.2 *Suppose $p(z)$ is a degree n polynomial, i.e.,*

$$p(z) = a_0 z^n + \cdots + a_t z^{n-t} + \cdots + a_n, \ \ a_0 \neq 0.$$

And $|a_t| > |a_0| + \cdots + |a_{t-1}| + |a_{t+1}| + \cdots + |a_n|$, then $p(z)$ has exactly $n-t$ zeros in the unit circle.

Corollary 6.14.3 *Suppose $f(z)$ is analytic in $B_r(0)$, and continuous on $\overline{B_r(0)}$, and on $\partial B_r(0)$, we have $|f(z)| < r$. Then there exists a unique point z_0, s.t., $f(z_0) = z_0$, for any $z_0 \in B_r(0)$.*

7 Conformal Mapping

7.1 Conformal

Theorem 7.1 (Local Mapping Theorem) *Let D be a domain, $z_0 \in D$ and $f : D \to \mathbb{C}$ be $\mathbb{C}-$analytic and non-constant. Then for $\epsilon > 0$ sufficiently small, there exists $\delta > 0$ such that whenever $0 < |w - w_0| < \delta$, there are exactly k values of z such that $0 < |z - z_0| < \epsilon$ and $f(z) = w$, where $k = \operatorname{ord}(f - w_0, z_0)$.*

Corollary 7.1.1 *Suppose $w = f(z)$ is holomorphic at z_0 and $f'(z_0) \neq 0$, then $f(z)$ is univalent and holomorphic in a neighbourhood of z_0.*

Theorem 7.2 (Open Mapping Theorem) *Let D be a domain and $f : D \to \mathbb{C}$ be $\mathbb{C}$-analytic and non-constant. Then if $U \subset D$ is open, $f(U)$ is open.*

Corollary 7.2.1 *If $f(z)$ is a univalent holomorphic function in a domain D, then $G = f(D)$ is a domain.*

Definition: a function $f : D \to C$ is angle preserving at z_0, if it preserves angles between directed curves through z_0 and orientations. We say f is a angle preserving map if f is conformal at every point $z_0 \in D$.

Theorem 7.3 *Suppose $f : D \to \mathbb{C}$ is holomorphic in D, then it is conformal at points where the derivative doesn't vanish.*

Corollary 7.3.1 *Suppose $f(z)$ is univalent and holomorphic in D, then $f(z)$ is angle preserving in D.*

Definition: a function $f : D \to \mathbb{C}$ is called a conformal if it is angle-preserving and univalent in D. We say $f : D \to G$ is a conformal map, or f maps D to G conformally if f is bijective and conformal.

Proposition 7.4 *Suppose $f : D \to \mathbb{C}$ is univalent and holomorphicm then*

1. *$f(z)$ maps D conformally to $G = f(D)$.*
2. *The inverse function of $w = f(z)$, which is $z = f^{-1}(w)$ is univalent and holomorphic in G, and*

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)} \quad (z_0 \in D,\, w_0 = f(z_0) \in G).$$

Conversely, if $w = f(z)$ is a conformal map from D to G, then $w = f(z)$ is univalent and holomorphic in D.

Proposition 7.5 *Suppose $f(z)$ is a univalent holomorphic function that maps a domain D conformally to a domain G, then the area of G is equal to*

$$A = \iint_D |f'(z)|^2 dxdy \ (z = x + iy).$$

Proof: We consider $f(z)$ as a transformation from D to G, and notice $|f'(z)|^2$ is the Jacobian of the map $(x, y) \mapsto (u(x, y), v(x, y))$. $\square$

7.2 Möbius Transformation

Definition: a linear transformation or Möbius transformation is a map of the following form:

$$L(z) = w = \frac{az+b}{cz+d}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0.$$

Moreover, we often extend the definition of $L(z)$ to the extended complex plane:

1. If $c \neq 0$, then we define $L(\frac{-d}{c}) = -\infty$, and $L(\infty) = \frac{a}{c}$;
2. If $c = 0$, we define $L(\infty) = \infty$.

Lemma 7.6 *A Möbius transformation $L(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ is a conformal map from the extended complex plane to the extended complex plane, with the inverse transformation:*

$$z = \frac{-dw+b}{cw-a}.$$

A Möbius transformation can be seen as the composition of two simpler linear transformation:

1. $w = kz + h = \rho e^{i\theta} z + h$, $k \neq 0$, $\rho \in \mathbb{R} \neq 0$. I.e., it is the map that rotates the plane anticlockwise by θ, then scale with respect to the origin by a factor of ρ, then translate by h.
2. $w = \frac{1}{z}$. This maps can be decomposed into two simpler transformations $\omega = \frac{1}{\bar{z}}$, then $w = \bar{\omega}$. ω is symmetric with z about the unit disk. w is symmetric with ω about the real line. We say 0 is symmetric with ∞ about the unit disk.

Definition: let z_1, z_2, z_3, z_4 be four distinct points on the extended complex plane, then we define their cross ratio, (z_1, z_2, z_3, z_4), to be

$$(z_1, z_2, z_3, z_4) = \frac{z_4 - z_1}{z_4 - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

If one of the point is ∞, then we take the term in the ratio containing that point to be 1.

Proposition 7.7 *Suppose z_1, z_2, z_3, z_4 are four distinct points on the extended complex plane, $L(z)$ is a Möbius transformation. Then the cross ratio of z_1, z_2, z_3, z_4 is preserved under L, that is*

$$(z_1, z_2, z_3, z_4) = (L(z_1), L(z_2), L(z_3), L(z_4)) = (w_1, w_2, w_3, w_4).$$

Corollary 7.7.1 *A Möbius transformation is uniquely determined by the image of three distinct points z_1, z_2, z_3. Moreover, if $w_i = L(z_i)$, $i = 1, 2, 3$, then L is given by*

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

7.2.1 Preservation of Generalized Circles and Symmetry

Definition: we define a circle on the complex plane or a line on the complex plane to be generalized circles.

Theorem 7.8 *Möbius transformation map generalized circles to generalized circles.*

Proof: Note every Möbius transformation is a composition of the two simpler linear transformations mentioned earlier. For the type one linear transformation, it is clear that it maps generalized circles to generalized circles. Now for the type two linear transformation, a generalized circle can be written in the following form:

$$Az\bar{z} + \bar{\beta}z + \beta\bar{z} + C = 0$$

where A, C are real and $|\beta|^2 > AC$ (if $A = 0$, the equation represents a straight line; otherwise, it represents a circle). Then under $w = \frac{1}{z}$, the image is

$$Cw\bar{w} + \bar{\beta}\bar{w} + \beta w + A = 0,$$

which is a circle or line. □

Remark: a straight line can be seen as a circle passing through ∞.
Remark: suppose the Möbius transformation $w = L(z)$ maps a bounded circle C to a straight line, then it must map a point z_0 on C to the point ∞.
Remark: since Möbius transformation is a conformal map, it preserves orientations. So let C be a circle and z_1, z_2, z_3 be three distinct points on C. Suppose C is mapped to D, then z_1, z_2, z_3 are mapped to three distinct points w_1, w_2, w_3 on D. Now if z_4 is any point different from z_1, z_2, z_3, then it is either to the left or to the right of z_1, z_2, z_3, then its image w_4 under the transformation will preserve this orientation, that is w_4 will either be to the left or to the right of w_1, w_2, w_3 depending on the position of z_4 relative to z_1, z_2, z_3.

Definition: two points z_1, z_2 are symmetric about the circle $\gamma : |z - a| = R$ if z_1, z_2 lies on the same ray passing through a, and

$$|z_1 - a| \cdot |z_2 - a| = R^2.$$

In addition, we define a to be symmetric to ∞ with respect to γ.
Note: $|z_1 - a| \cdot |z_2 - a| = R^2$ can be written as

$$z_2 - a = \frac{R^2}{\overline{z_1 - a}}.$$

Proposition 7.9 *Two points $z_1, z_2 \in \overline{C}$ are symmetric about the generalized circle γ, if and only if any generalized circle passing through z_1 and z_2 is orthogonal with γ.*

Proof: $\Rightarrow$: If z_1, z_2 are symmetric about a straight line, then it is clear that any circle passing through z_1, z_2 is orthogonal to the line of symmetry. So we assume z_1, z_2 are symmetric about the circle $\gamma = |z - a| = R$, then any straight line passing through z_1 and z_2 is orthogonal to γ, as a radius of γ lies on the line.
Now assume δ is a circle passing through z_1, z_2. Let $a\xi$ be a tangent of δ, where ξ is the tangential point on δ. Then

$$|\xi - a|^2 = |z_1 - a||z_2 - a| \quad \text{(power of a point)}.$$

But z_1, z_2 is symmetric about γ, so $|z_1 - a||z_2 - a| = R^2$, i.e., $|\xi - a| = R$, that is $a\xi$ is a radius of γ. So δ is orthogonal to δ.

$\Leftarrow$: First consider the case where γ is a straight line. Then construct a circle such that $z_1 z_2$ forms a diame-

ter. Since γ is orthogonal to this circle, then we must have that z_1, z_2 is symmetric about γ.
Next suppose any generalized circle passing through z_1, z_2 is orthogonal to the circle $\gamma : |z-a|^2 = R$. Then let δ be arbitrary circle passing through z_1, z_2, we have δ is orthogonal to γ. Let they intersect at ξ, then the radius $a\xi$ of γ is tangent to δ. Now since the straight line z_1z_2 must pass through a because it is orthogonal to γ, then using a power of a point again, we have

$$R^2 = |\xi - a|^2 = |z_1 - a||z_2 - a|.$$

Hence z_1, z_2 is symmetric about the circle γ. □

Proposition 7.10 *Suppose $z_1, z_2 \in \overline{C}$ are symmetric about the generalized circle γ, and $w = L(z)$ is a Möbius transformation. Then $w_1 = L(z_1), w_2 = L(z_2)$ are two points symmetric about the generalized circle $\Gamma = L(\gamma)$.*

Proof: Suppose Δ is a generalized circle on passing through w_1, w_2. Then there must exists generalized circle δ passing through z_1, z_2 such that $\Delta = L(\delta)$ (the inverse transformation of L is still a Möbius transformatio). As z_1, z_2 is symmetric about γ, then δ is orthogonal to γ. But $w = L(z)$ is conformal hence angle preserving, then $\Delta = L(\delta)$ and $\Gamma = L(\gamma)$ is still orthogonal. Hence w_1, w_2 is symmetric about Γ. □

7.2.2 Linear Transformations Between Different Domains

Proposition 7.11 *Linear transformations that map the upper half plane to the upper plane has the form*

$$w = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. For such a transformation, it maps the real line to the real line, the point $z = \frac{-b}{a}$ to 0 and maps the lower half plane to the lower half plane.

Proof: Suppose $w = \frac{az+b}{cz+d}$ satisfies the conditions in the proposition, then then

$$\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2} \neq 0.$$

Hence $z \mapsto w$ is a conformal map.

$$\begin{aligned}\operatorname{Im} w &= \frac{1}{2i}(w - \bar{w}) \\ &= \frac{1}{2i}\left(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right) \\ &= \frac{1}{2i}\frac{ad-bc}{|cz+d|^2}(z-\bar{z}) \\ &= \frac{ad-bc}{|cz+d|^2}\operatorname{Im} z\end{aligned}$$

So it maps the upper half plane to the upper half plane, the lower half plane to the lower half plane, and the real line to the real line. It is also clear that $z = \frac{-b}{a}$ is mapped to 0.

Conversely, from the above formulation, we see that if $ad - bc$ is not real then it will not map the upper half plane to the upper half plane. So if a linear transformation maps the upper half plane to the lower half plane, then we must have $ad - bc > 0$. Now since the real line is the boundary of the upper half plane, then by a continuity argument, there must exists some point on $\mathbb{R}$ that is mapped to 0, so $\frac{-b}{a} \in \mathbb{R}$. Then we can easily show that a, b, c, d has argument that differ by $-k\pi i$, hence by dividing $\arg a$ on both the numerator and denominator, we conclude that a, b, c, d can be written as real numbers. $\square$

Corollary 7.11.1 *A linear transformation that maps the upper half plane to the lower half plane has the form*

$$w = \frac{az + b}{cz + d},$$

$a, b, c, d \in \mathbb{R}$ *and* $ad - bc < 0$.

Proposition 7.12 *A linear transformation that maps the upper half plane* $\operatorname{Im} z > 0$ *to the unit disk and maps the point* $z = a$ *(*$\operatorname{Im} a > 0$*) to* $w = 0$*, must have the form*

$$w = e^{i\theta} \frac{z - a}{z - \bar{a}} \quad (\operatorname{Im} a > 0).$$

Proof: Since linear transformation preserves symmetry, $\bar{a}$ which is symmetric to a about the real line should be mapped to ∞, as ∞ is symmetric to 0 about the unit circle. Hence the map should have the form

$$w = k \frac{z - a}{z - \bar{a}}.$$

We can easily deduce $|k| = 1$ by considering a point z on the real line, it should be mapped to the unit disk by continuity argument (note any linear transformation is an automorphism from $\bar{\mathbb{C}}$ to $\bar{\mathbb{C}}$, hence bijective and continuous). And lastly, so

$$w = e^{i\theta} \frac{z - a}{z - \bar{a}} \quad (\operatorname{Im} a > 0).$$

$\square$

Corollary 7.12.1 *A linear transformation that maps the lower half plane to* $\mathbb{D}$ *and maps the point* $z = a$ *(*$\operatorname{Im} a < 0$*) to* $w = 0$*, must have the form*

$$w = e^{i\theta} \frac{z - a}{z - \bar{a}} \quad (\operatorname{Im} a < 0).$$

Corollary 7.12.2 *A linear transformation that maps the right half plane to* $\mathbb{D}$ *and maps the point* $z = a$ *(*$\operatorname{Re} a > 0$*) to* $w = 0$*, must have the form*

$$w = e^{i\theta} \frac{z - a}{z + a}.$$

Corollary 7.12.3 *A linear transformation that maps* $\mathbb{D}$ *to the right half plane, must have the form*

$$w = \frac{\bar{a} e^{i\theta} z + a}{1 - e^{i\theta} z},$$

where $\operatorname{Re} a > 0$.

Proposition 7.13 *A linear transformation that maps* $\mathbb{D}$ *to* $\mathbb{D}$ *and* $z = a$ *(*$|a| < 1$*) to* $w = 0$ *must have the form*

$$w = e^{i\theta} \frac{z - a}{1 - \bar{a} z}.$$

Proof: Since a and $\frac{1}{\bar{a}}$ is symmetric about the unit circle, then $\frac{1}{\bar{a}}$ must be mapped to ∞, so

$$w = k\frac{z-a}{z-\frac{1}{\bar{a}}} = k_1\frac{z-a}{1-\bar{a}z}.$$

Then take any point one the unit disk, it must be mapped to the unit disk by a continuity argument, so we can conclude $|k_1| = e^{i\theta}$. It is easy to check that such map indeed map the unit disk to the unit disk. $\square$

Corollary 7.13.1 *A linear transformation of the form*

$$w = \frac{1-\bar{a}z}{z-a}$$

maps $\mathbb{D}$ *to* $|w| > 1$.

Proposition 7.14 *Consider* $\Phi_a : \mathbb{D} \to \mathbb{D}$, $a \in \mathbb{D}$, *then* $\Phi_a = \frac{a-z}{1-\bar{a}z}$ *is an involution, that is* $\Phi_a \circ \Phi_a = id$.

Proof: Direct Verification. $\square$

7.3 Elementary Conformal Mappings

Proposition 7.15 *Suppose* $n \in \mathbb{N}^+$, *then the map* $w = z^n$ *maps the domain* $D : 0 < \arg z < \alpha$ *to* $G : 0 < \arg w < n\alpha$, $0 < \alpha \leq \frac{2\pi}{n}$. *The inverse of the* $w = z^n$, $z = \sqrt[n]{w}$, *maps the domain* $G : 0 < \arg w < n\alpha$ *to* $D : 0 < \arg z < \alpha$. *However, by taking the different branches of the* log, *we can make* $z = \sqrt[n]{w}$ *map to different domains.*

Proposition 7.16 *The exponential function* $w = e^z$ *maps the domain* $D : 0 < \operatorname{Im} z < h$, $0 < h \leq 2\pi$ *conformally to the domain* $G : 0 < \arg w < h$.

Proposition 7.17 *Suppose two generalized circles intersect at points* a, b *at an angle of* $\frac{\pi}{n}$, *then there exists* k *such that*

$$w = \left(k\frac{z-a}{z-b}\right)^n$$

maps the region bounded by the two generalized circles to the upper half plane.

Proof: We first consider a linear transformation that maps the two generalized circles to two rays starting from the origin, that is we apply

$$\xi = k\frac{z-a}{z-b}$$

where k is a constant chosen such that the image is the region described by $0 < \arg\xi < \frac{\pi}{n}$, since we know linear transformation preserves angles, we can always find such k. Then consider $w = \xi^n$, which maps the region conformally to the upper half plane. Hence we have

$$w = \left(k\frac{z-a}{z-b}\right)^n$$

is the desired conformal map. $\square$

Proposition 7.18 *Suppose circles (does not include lines) are tangent at a point a, then there is a conformal map of the form*

$$w = e^{\frac{cz+d}{z-a}}$$

which maps the moon-shaped region bounded by the two circles to the upper half plane.

Proof: Firstly, consider the map

$$\xi = \frac{cz+d}{z-a}$$

which maps the two circles to two parallel lines. Suitably choose c and d, we can make sure the region between the two parallel lines is $0 < \operatorname{Im}\xi < \pi$. Then apply $w = e^{\xi}$, we have the desired map. □

7.4 Schwarz-Christoffel Transformation

Theorem 7.19 *Suppose*

1. *P_n is a bounded $n-$gon with vertices $A_1, A_2, \cdots, A_n$, and interior angles $\alpha_1\pi, \alpha_2\pi, \cdots, \alpha_n\pi$, $0 < \alpha_j < 2$, $j = 1, 2, \cdots, n$.*
2. *$w = f(z)$ maps the upper half plane $\operatorname{Im} z > 0$ conformally to P_n.*
3. *The points $a_j \in \mathbb{R}$ are mapped to A_j satisfies*

$$-\infty < a_1 < a_2 < \cdots < a_j < \cdots < a_n < +\infty.$$

Then

$$f(z) = C\int_{z_0}^{z} (z-a_1)^{\alpha_1 - a}(z-a_2)^{\alpha_2-1}\cdots(z-a_n)^{\alpha_n-1}dz + C_1$$

where z_0, C, C_1 are constants in $\mathbb{C}$.

Remark: It is clear that $\sum\limits_{j=1}^{n} a_j = n-2$.
Remark: $z = f^{-1}(w)$ maps P_n conformally to the upper half plane. Or more explicitly,

$$z = C'\int_{w_0'}^{w} \frac{dw}{(w-A_1)^{1-\alpha_1}(w-A_2)^{1-\alpha_2}\cdots(w-A_n)^{1-\alpha_n}} + C_1'.$$

Lemma 7.20 *Let L_1, L_2 be two straight lines on $\mathbb{C}$, then the angle of their intersection at ∞ is the negative of the angle of their second intersection. In particular, if two lines are parallel, then the angle of their intersection at ∞ is* 0.

Proof: Suppose L_1, L_2 are two rays starting from ∞ and intersect at the finite point ξ_0 (We can take $\xi_0 \neq 0$ WLOG). Then apply transformation $w = \frac{1}{z}$, the image of L_1 and L_2 are two curves Γ_1 and Γ_2 that pass through the point 0 and $\frac{1}{\xi_0}$. Note the angle of intersection at these intersections are negative of each other. Since $w = \frac{1}{z}$ is angle-preserviging, then we conclude that the statement we want to prove holds. □

Degenerate Case for Schwarz-Christoffel Transformation:

1. Suppose the $n-$gon P_n has a vertex being mapped to infinity, say A_n is mapped to $a_n = \infty$. Then we can apply a linear transformation
$$\xi = -\frac{1}{z} + a_n'$$
that maps the upper half plane to the upper half plane and points $a_1, \cdots, a_{n-1}$ and $a_n = \infty$ to finite points $a_1', a_2', \cdots, a_n'$. (Note a_n' is arbitrary constant. If one of a_j is zero, then we take $\xi = a_n' - \frac{1}{z-a}$, where a is distinct from a_j.)
Then apply a change of variable to the formula in Schwarz-Christoffel Transformation, we have
$$\begin{aligned}
w &= C' \int_{\xi_0}^{\xi} (\xi - a_1')^{\alpha_1 - 1} (\xi - a_2')^{\alpha_2 - 1} \cdots (\xi - a_n')^{\alpha_n - 1} d\xi + C_1' \\
&= C' \int_{z_0}^{z} \left(a_n' - a_1' - \frac{1}{z}\right)^{\alpha_1 - 1} \left(a_n' - a_2' - \frac{1}{z}\right)^{\alpha_2 - 1} \cdots \left(-\frac{1}{z}\right)^{\alpha_n - 1} \frac{dz}{z^2} + C_1' \\
&= C' \int_{z_0}^{z} [(a_n' - a_1')z - 1]^{\alpha_1 - 1} [(a_n' - a_2')z - 1]^{\alpha_2 - 1} \cdots [(a_n' - a_{n-1}')z - 1]^{\alpha_{n-1} - 1} (-1)^{\alpha_n - 1} \cdot \frac{dz}{z^{\alpha_1 + \alpha_2 + \cdots + \alpha_n - n + 2}} + C_1' \\
&= C \int_{z_0}
\end{aligned}$$
where $a_j^0 = \frac{1}{a_n' - a_j'}$, $j = 1, \cdots, n-1$ are real constants. If we denote a_j^0 as a_j, then we have
$$w = C \int_{z_0}^{z} (z - a_1)^{\alpha_1 - a} (z - a_2)^{\alpha_2 - 1} \cdots (z - a_{n-1})^{\alpha_{n-1} - 1} dz + C_1.$$
I.e., if one of the vertex of P_n is infinity, then we can discard the factor containing that term in the integral.

2. Definition: an $n-$gon P_n that has one or more vertex lying at ∞ is called a generalized $n-$gon. For generalized $n-$gons we can get a similar formula for Schwarz-Christoffel Transformation:
$$f(z) = C \int_{z_0}^{z} (z - a_1)^{\alpha_1 - a} (z - a_2)^{\alpha_2 - 1} \cdots (z - a_n)^{\alpha_n - 1} dz + C_1$$
However, if two edges intersect at infinity, then the angle $\alpha_j \pi$ is equal to the negative of the angle of their second intersection. In this case, we still have $\sum_{i=1}^{n} \alpha_i = n - 2$.

7.5 Riemann Mapping Theorem

Theorem 7.21 (Riemann Mapping Theorem) *Let D be a simply connected domain on the extended complex plane such that its boundary contains more than one point, then there exists a conformal map $f(z)$ between D and the unit disk $\mathbb{D}$. In addition, if it was further given that*
$$f(a) = 0, \;\; f'(a) > 0 \;\; (a \in D),$$
then the map is uniquely determined.

Proof: Uniqueness: Suppose conformal map $f_1(z)$ also satisfies the condition $f_1(a) = 0$ and $f_1'(a) > 0$. Then let

$$\Phi(w) = f_1[f^{-1}(w)]$$

which is univalent and holomorphic in the unit disk and satisfies the condition

$$\begin{aligned} \Phi(0) &= f_1[f^{-1}(0)] = f_1(a) = 0 \\ |\Phi(w)| &\leq 1 \\ \Phi'(0) &= \frac{f_1'(a)}{f'(a)} > 0 \end{aligned}$$

Then by Schwarz Lemma, we have $|\Phi(w)| \leq |w|$. Similarly, we have that $|\Phi^{-1}(w)| \leq |w|$. But then this implies $|w| = |\phi(w)|$, so $\Phi(w) = e^{i\theta}w$ by Schwarz Lemma again. And since $\Phi'(0) > 0$, then $e^{i\theta} = 1$, so $f_1(z) = f(z)$. $\square$

Remark: a confromal map between two simply connected domains D and G is uniquely determined by

$$f(a) = b, \ \arg f'(a) = \theta,$$

$a \in D$, $b \in G$ and $\theta \in \mathbb{R}$.
Remark: the uniqueness condition can also be expressed by

$$f(\xi_i) = \eta_i \quad (i = 1, 2, 3),$$

where ξ_i and η_i are three distinct points on D and G respectively that have the same orientation (both clockwise or anticlockwise).

Corollary 7.21.1 *A conformal map that maps $\mathbb{D}$ to $\mathbb{D}$ must be a linear transformation. Moreover, if $a \in \mathbb{D}$ is mapped to 0, then this map must have the form*

$$w = e^{i\theta}\frac{z-a}{1-\bar{a}z}.$$

Proof: It is clear that $e^{i\theta}\dfrac{z-a}{1-\bar{a}z}$ is a conformal map between $\mathbb{D}$ to $\mathbb{D}$ and maps a to 0. Then by the uniqueness condition of the Riemann Mapping Theorem, we can easily get that such conformal map must be of this form. $\square$

Theorem 7.22 (Riemann Uniformization) *Let $\Omega \subset \mathbb{C}$ domain that is not equal to $\mathbb{C}$. If Ω is simply connected (or simply connected Riemann surface that isn't bilohormorphic to $\mathbb{P}^1, \mathbb{C}$), then $\exists h : \Omega \xrightarrow{\sim} \mathbb{D}$ that is biholomorphic.*

Remark: $\frac{az+b}{cz+d} : \mathbb{P}^1 \to \mathbb{P}^1$ is biholomorphic, $\forall a, b, c, d$, s.t., $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. In addition, any biholomorphic map from $\mathbb{P}^1 \to \mathbb{P}^1$ is of the form.

Theorem 7.23 (Boundary Correspondence Theorem) *Suppose*

1. *D and G are two domains enclosed by closed curves C and Γ.*
2. *$w = f(z)$ maps D conformally to G.*

Then there exists $F(z)$ such that $F(z) = f(z)$ in D and continuous on $\bar{D} = D + C$. Moreover, $F(z)$ maps C continuously and bijectively to Γ.

Theorem 7.24 (Converse of Boundary Correspondence Theorem) *Suppose a simply connected domain D and G are the interior of two closed curve C and Γ. Let $w = f(z)$ satisfy the following:*

1. *$w = f(z)$ is holomorphic in D and continuous on $D + C$.*
2. *$w = f(z)$ maps C bijectively to Γ.*

Then $w = f(z)$ is univalent in D and $G = f(D)$, that is $f(z)$ is a conformal map between D to G.

Proof: Suppose w_0 is an arbitrary point in D, we prove $w_0 \in f(D)$ and the equation $f(z) - w_0 = 0$ has exactly one root in C. By Principle of The Argument, we have

$$ZP(f(z) - w_0, C) = \frac{1}{2\pi}\Delta_\Gamma \arg(w - w_0).$$

But since f is holomorphic, then $f(z) - w_0$ has no poles in D, so $ZP(f(z) - w_0, C)$ is the order of zeros. Now since $w = f(z)$ maps C bijectively to Γ, then $\frac{1}{2\pi}\Delta_\Gamma \arg(w - w_0) = \pm 1$ (w_0 is in the interior of Γ, so Γ winds around w_0 exactly once), and in this case it can only be 1 as there is no poles. So we conclude that $f(z) - w_0$ has exactly one root in D. Similarly we can show that if w_0 is outside Γ, then the equation has no root. Lastly, if $w_0 \in \Gamma$, then by open mapping theorem, we conclude that some point outside γ must have a preimage in D, which is not possible. $\square$

Corollary 7.24.1 *A multi-connected domain cannot be conformally mapped to a simply connected domain.*

8 More on Harmonic Functions

8.1 Mean Value and Inequalities

Proposition 8.1 *Let f be a holomorphic function on the unit disk $\mathbb{D}$ such that $|f(z)| \leq 1$ when $|z| < 1$, then*

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}$$

for all $|z| < 1$.

Proposition 8.2 *Let f be holomorphic in a neighbourhood of $\bar{D}$, then*

$$f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \operatorname{Re} f(e^{it}) dt.$$

Proof: Decompose $\operatorname{Re} f(e^{it})$ as $\frac{1}{2}(f(e^{it}) + \overline{f(e^{it})})$, then proceed with Taylor Expansion or other means. □

Proposition 8.3 *Let u be a harmonic function on the disk $|\xi - z_0| < R$ and continuous on $|\xi - z_0| \leq R$, then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{i\varphi}) d\varphi.$$

Proposition 8.4 (Maximum Principle for Harmonic Functions) *Suppose $u(z)$ is a harmonic function in D and obtains maximum or minimum in some interior point of D, then $u(z)$ is constant.*

Corollary 8.4.1 *Suppose $u(z)$ is harmonic in a bounded domain D and continuous on $\overline{D}$. $u(z) \leq M$ for all $z \in \partial D$, then for any $z \in D$, we must have $u(z) \leq M$, equality holds iff u is a constant function on $\bar{D}$.*

Corollary 8.4.2 *Let v be a harmonic function in $\mathbb{C}$. Assume that v is bounded, then v is a constant.*

8.2 Property of Harmonic Functions From Holomorphic Functions

Proposition 8.5 *Suppose $u(z)$ is a harmonic function in D and the disk $|z - a| < R$ is contained in D, then when $z = a + re^{i\theta}$, $r < R$, we have*

$$u(z) = \sum_{n=0}^{\infty} r^n (a_n \cos n\theta + b \sin n\theta)$$

for some complex constant a_n, b_n. Moreover, the expansion is unique.

Proof: Let $f(z) = u(z) + iv(z)$ be holomorphic on $|z - a| < R$, then $u(z) = \frac{f(z) + \overline{f(z)}}{2}$. Next, consider the Taylor expansion, we get the desired equality. □

Proposition 8.6 *Suppose f is entire and not a constant, let $u(z) = \operatorname{Re} f(z)$, then for any $a \in \mathbb{R}$, $\exists z_0 \in \mathbb{C}$, s.t., $u(z_0) = a$. Moreover, if we assume Picard's Theorem, this value can be taken infinitely many times by $u(z)$.*

Proof: Consider $|e^f| = e^u$. Since u is harmonic in $\mathbb{C}$, then it cannot be bounded from above or below (consider $\pm u$), thus for any $m, M \in \mathbb{R} > 0$, there exists $z_0, z_1 \in \mathbb{C}$, s.t., $e^{u(z_0)} < m$ and $e^{u(z_1)} > M$. Thus the image of $|e^f|$ should be $(0, \infty)$ as $|e^f|$ is continuous. Hence we conclude that u takes every single real values. Moreover, if we assume Picard's Theorem, then it is easy that $u(z)$ takes any value infinitely many times. □

Proposition 8.7 *Suppose u is a harmonic function in D, and the ball $B(z_0, r) \subset D$, $r > 0$, $z_0 \in D$. If u is a constant on $B(z_0, r)$, then u is a constant in D.*

Proof: Follows from the uniqueness theorem of holomorphic functions. Notice if a point is path connected to any point in $B(z_0, r)$, then it must take the same value (The track of the path is compact). □

Proposition 8.8 *Suppose $u(z)$ is a harmonic function in D, $z_0 \in D$, and $u(z_0) = a$. Let U be any neighbourhood of z_0, there exists infinitely many $z \in U$, s.t., $u(z) = a$.*

Proof: Let $f = u + iv$ be holomorphic in a ball $B(z_0, r)$, s.t., $B(z_0, r) \subset U$. Then by the open mapping theorem, we easily get the result. □

8.3 Poisson Kernel and Dirichlet Problem

Theorem 8.9 (Poisson Kernel) *Let u be a harmonic function in the disk $K : |z| < R$ and continuous on $\overline{K} : |z| \leq R$, then for any $z \in K$, let $z = re^{i\varphi}$, then*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} d\theta.$$

Proof: Since K is simply connected, then let $f(z) = u(z) + iv(z)$, s.t., f is holomorphic. Then by Cauchy integration formula,we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|\xi|=R'} \frac{f(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(R'e^{i\theta}) \frac{R'e^{i\theta}}{R'e^{i\theta} - re^{i\varphi}} d\theta \end{aligned}$$

Where $r < R' < R$. Next consider the point z^* which is symmetric to z about the circle $|\xi| = R'$, then

$$z^* = \frac{(R')^2}{\bar{z}} = \frac{(R')^2 e^{i\varphi}}{r}.$$

Then since z^* is outside the circle $|\xi| = R'$, we have

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_{|\xi|=R'} \frac{f(\xi)}{\xi - z^*} d\xi \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(R'e^{i\theta}) \frac{re^{i\theta}}{re^{i\theta} - R'e^{i\varphi}} d\theta. \end{aligned}$$

Then we have

$$f(z) = \frac{1}{2\pi} f(R'e^{i\theta}) \left[\frac{R'e^{i\theta}}{R'e^{i\theta} - re^{i\varphi}} - \frac{re^{i\theta}}{re^{i\theta} - R'e^{i\varphi}} \right] d\theta.$$

Then by some calculation we have

$$u(z)+iv(z)=f(z)=\frac{1}{2\pi}\int_0^{2\pi}f(R'e^{i\theta})\frac{(R')^2-r^2}{(R')^2-2R'r\cos(\theta-\varphi)+r^2}d\theta.$$

Then by taking the real parts, we obtain

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}u(R'e^{i\theta})\frac{(R')^2-r^2}{(R')^2-2R'r\cos(\theta-\varphi)+r^2}d\theta.$$

Now letting $R'\to R$, the value of the integral does not change (as it is equal to $u(z)$). Since $u(z)$ is continuous on $\bar{K}$, then we have that

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}u(Re^{i\theta})\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}d\theta.$$

□

Corollary 8.9.1 *Let u be a harmonic function in a neighbourhood of $\bar{D}$, then*

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|z-e^{it}|^2}u(e^{it})dt.$$

Corollary 8.9.2 (Harnack's Inequality) *Let u be a harmonic function in neighbrouhood of $\bar{D}$, and assume that $u\geq 0$ on $\{|z|=1\}$, then*

$$\frac{1-|z|}{1+|z|}u(0)\leq u(z)\leq\frac{1+|z|}{1-|z|}u(0)\quad \textit{for } |z|<1.$$

Proof: Note in Corollary $\frac{1-|z|^2}{|z-e^{it}|^2}$, we have

$$\frac{1-|z|}{1+|z|}\leq\frac{1-|z|^2}{|z-e^{it}|^2}\leq\frac{1+|z|}{1-|z|}.$$

□

Dirichlet Problem: we want to find a harmonic function $u(z)$ on D and $u(z)$ continuous on $\bar{D}$, s.t., for any $\xi\in\partial D$, we have

$$u(\xi)=\tilde{u}(\xi),$$

where $\tilde{u}(\xi)$ are known values.

Proposition 8.10 *For a given domain D and boundary values $\tilde{u}(\xi)$, there cannot be more than one solution to the Dirichlet Problem.*

Proof: Suppose u_1 and u_2 are two solutions to the Dirichlet problem, then u_1-u_2 is harmonic in D and continuous on $\bar{D}$. On ∂D, $u_1(z)-u_2(z)\equiv 0$, so by the maximum principle of harmonic functions, $u_1(z)-u_2(z)$ must be equal to 0 on $\bar{D}$. □

Proposition 8.11 (Dirichlet Problem on Unit Disk) *The solution of the Dirichlet Problem on the unit disk is given by the Poisson Kernel for the unit disk, that is*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|z-e^{it}|^2} u(e^{it})dt.$$

Proposition 8.12 (Dirichlet Problem on Upper Half Plane) *Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half-plane. Let $u : \mathbb{H} \cup \mathbb{R} \to \mathbb{R}$ be a continuous function such that u is harmonic inside $\mathbb{H}$ and*

$$\lim_{z\in\overline{\mathbb{H}},\, z\to\infty} u(z) \text{ exists and is finite.}$$

Then for any $z_0 \in \mathbb{H}$, we have

$$u(z_0) = \operatorname{Re}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(t)}{t-z_0}dt = \frac{1}{\pi}\int_{-\infty}^{\infty} u(t)\cdot\frac{ydt}{(t-x)^2+y^2}.$$

Hence if the value of $u(t)$ is known for all $t \in \mathbb{R}$, then the value of function u is uniquely determined in $\mathbb{H}$.

Proof: Suppose we want to find the value of $u(z)$, we consider the conformal mapping

$$\omega = \frac{\xi - z}{\xi - \bar{z}}$$

which maps $\mathbb{H}$ to $\mathbb{D}$ and z to 0. If we let $U(\omega) = u[\xi(\omega)]$ and $U(1) = \lim_{z\in\overline{\mathbb{H}},\, z\to\infty} u(z)$. Then U is harmonic in $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$, then

$$U(0) = \frac{1}{2\pi}\int_0^{2\pi} U(\omega)d\tau.$$

Where τ is the angle of ω, so $e^{i\tau} = \omega = \frac{t-z}{t-\bar{z}}$ where $t \in \mathbb{R}$ (as $\mathbb{R}$ is mapped to $|\omega| = 1$ by under the conformal map). Taking differentials on both sides, we get

$$e^{i\tau}d\tau = \frac{2y}{(t-\bar{z})^2}dt \quad (z = x+iy).$$

Hence

$$d\tau = \frac{2ydt}{(t-z)(t-\bar{z})} = \frac{2ydt}{(t-x)^2+y^2}.$$

Then substitute $t = \xi^{-1}(\omega)$, we get

$$u(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} u(t)\cdot\frac{ydt}{(t-x)^2+y^2} = \operatorname{Re}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(t)}{t-z_0}dt.$$

$\square$

9 Analytic Continuation

9.1 Power Series Extension

Definition: suppose the function $f(z)$ is holomorphic in D and $D \subsetneq G$. If $F(z)$ is holomorphic in G and $F(z) = f(z)$, $\forall z \in D$, then we say $f(z)$ can be extend to be analytic on G, and call $F(z)$ to be the analytic continuation of $f(z)$ in G.
Note by the Uniqueness Theorem, the analytic continuation of a function to a domain must be unique.

Definition: suppose D is a domain in $\mathbb{C}$, $f(z)$ is a univalent holomorphic function in D, then the ordered pair D and the function, denoted $\{D, f(z)\}$ is called a function element or analytic function element. Two function element $\{D, f(z)\}$ and $\{G, g(z)\}$ are equal if $D = G$ and $f(z) = g(z)$, $\forall z \in D$.

Theorem 9.1 *Suppose $\{D_1, f_1(z)\}$, $\{D_2, f_2(z)\}$ are two function element such that*

1. *$D_{12} := D_1 \cap D_2$ is a non-empty domain;*
2. *$f_1(z) = f_2(z)$, $\forall z \in D_{12}$.*

Then $\{D_1 \cup D_2, F(z)\}$ is also a function element, where

$$F(z) = \begin{cases} f_1(z), & z \in D_1 \setminus D_{12} \\ f_2(z), & z \in D_2 \setminus D_{12} \\ f_1(z) = f_2(z), & z \in D_{12} \end{cases}.$$

Definition: If $\{D_1, f_1(z)\}$ and $\{D_2, f_2(z)\}$ satisfies the hypothesis in the above theorem, then $\{D_1, f_1(z)\}$ and $\{D_2, f_2(z)\}$ are said to be analytic continuations of one another. A chain of analytic continuations is a finite sequence of function elements $\{D_1, f_1(z)\}, \{D_2, f_2(z)\}, \cdots, \{D_n, f_n(z)\}$ such that each consecutive pair are analytic continuations of one another.

Next we discuss a common way to extend a holomorphic function to a large domain.
Suppose $\{D, f(z)\}$ is a function element, and $z_1 \in D$. Then $f(z)$ has a Taylor expansion at z_1,

$$f_1(z) = \sum_{n=0}^{\infty} c_n^{(1)} (z - z_1)^n, \quad (*)$$

where $c_n^{(1)} = \frac{1}{n!} f^{(n)}(z_1)$. Suppose the radius of convergence of this power series is $+\infty$, then the power series converges everywhere on the complex plane. Thus it is the analytic continuation of $f(z)$ to the entire complex plane.
Suppose on the other hand, the radius of convergence of the power series $(*)$ is finite, denote it R_1 and $\Gamma_1 = \{|z - z_1| < R_1 : z \in \mathbb{C}\}$ does not lie entirely in D (if Γ_1 is in D, then we pick another point $z_1 \in D$ and repeat the same process), then we can take a point $z_2 \in \Gamma_1$, $z_2 \neq z_1$, and find the Taylor Expansion of $f_1(z)$ centered at z_2:

$$f_2(z) = \sum_{n=0}^{\infty} c_n^{(2)} (z - z_2)^n,$$

where $c_n^{(2)} = \frac{1}{n!} f_1^{(n)}(z_2)$. If the radius of convergence of $f_2(z)$ is R_2, then R_2 must satisfies the following inequality:

$$R_2 \geq R_1 - |z_2 - z_1|$$

However, R_2 cannot be larger than $R_1 + |z_2 - z_1|$, as it cannot contain the entire Γ_1 (We know $f_1(z)$ does not converge at some point on Γ_1).

Now we have two cases:

- Case 1: Suppose $R_2 = R_1 - |z_2 - z_1|$, then $\Gamma_2 = \{|z - z_2| < R_2 : z \in \mathbb{C}\}$ lies entirely in Γ_1, so we cannot extend $f_1(z)$ along the direction $z_1 z_2$. Hence we conclude that that common tangent ξ of $\partial\Gamma_1$ and $\partial\Gamma_2$ is a singularity of $f_1(z)$.
  This is because on $\partial\Gamma_2$, there must be at least one singularity of $f_2(z)$, however, all points on $\partial\Gamma_2$ lies in Γ_1 except ξ, hence all those points are holomorphic points of $f_1(z)$, thus non-singular point for $f_2(z)$. Then it must be the case that ξ is a singularity for $f_2(z)$ hence also for $f_1(z)$.
- Case 2: $R_2 > R_1 - |z_2 - z_1|$, then $\Gamma_2 = \{|z - z_2| < R_2 : z \in \mathbb{C}\}$ is not entirely in Γ_1. Thus we can extend $f_1(z)$ hence f to a larger domain, as $\{\Gamma_2, f_2(z)\}$ and $\{\Gamma_1, f_1(z)\}$ are analytic continuation of each other.
  Similarly, we can repeat the process by taking a point z_3 on $\Gamma_1 \cup \Gamma_2$, and extend f to $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. We keep repeating this process until we can no longer can keep any point on the extended domain which will make the domain of holomorphicity larger. At this point, we have extend the function f along all possible directions.

Remark: a function cannot be extended in any direct unless the radius of convergence of f at any point is ∞, as there is at least one singularity of the function on the circle of convergence.
Remark: It is possible that the function cannot be extended in any direction. For example: $f(z) = \sum\limits_{n=1}^{\infty} z^{2^n}$ cannot be extended to the exterior of the unit disk.

9.2 Analytic Continuation Along a Curve and Reflection Principle

Theorem 9.2 (Painlevé's theorem on analytic continuation) *Suppose $\{D_1, f_1(z)\}$ and $\{D_2, f_2(z)\}$ are two function elements satisfying the following:*

1. *D_1 and D_2 do not intersect but the boundary of D_1 and D_2 shares a common curve; we denote this curve without its boundary point to be Γ;*
2. *$f_1(z)$ is continuous on $D_1 \cup \Gamma$, $f_2(z)$ is continuous on $D_2 \cup \Gamma$;*
3. *Along Γ, $f_1(z) = f_2(z)$;*

Then $\{D_1 \cup \Gamma \cup D_2, F(z)\}$ is also a function element, where

$$F(z) = \begin{cases} f_1(z), & z \in D_1 \\ f_2(z), & z \in D_2 \\ f_1(z) = f_2(z), & z \in \Gamma \end{cases}.$$

Proof: Morera's Theorem. □

Theorem 9.3 (Schwarz Reflection Principle) *Suppose D^+ and D^- are two symmetric domains about the real axis, and their boundary contains a common line segment L on the real axis. $\{D^+, f(z)\}$ is a function element, continuous on $D^+ \cup L$ and takes real values on L. Define*

$$F(z) := \begin{cases} f(z), & z \in D^+ \cup L \\ \overline{f(\bar{z})}, & z \in D^- \end{cases}$$

Then $F(z)$ is holomorphic in $D^+ \cup L \cup D^-$.

Corollary 9.3.1 *Let $\Omega = \Omega^+ \cup L \cup \Omega^-$, where Ω^+ and Ω^- is symmetric about the real line L. $g : \Omega \to \mathbb{C}$ is holomorphic, and g is real on L, then $g(z) = \overline{g(\bar{z})}$, $\forall z \in \Omega$.*

Corollary 9.3.2 *Suppose D^+ and D^- are two symmetric domains about the real axis, and their boundary contains a common line segment L on the real axis. Also let G^+ and G^- be two symmetric domains about the real axis, and their boundary contains a common line S. Suppose $f(z)$ is a univalent holomorphic function in $D^{\cup}$ and continuous on $D^+ \cup L$, s.t., f maps D^+ conformally to G^+ and maps L bijectively to S. Let*

$$F(z) := \begin{cases} f(z), & z \in D^+ \cup L \\ \overline{f(\bar{z})}, & z \in D^- \end{cases}$$

Then F is univalent holomorphic in $D^+ \cup L \cup D^-$, and maps $D^+ \cup L \cup D^-$ conformally to $G^+ \cup L \cup G^-$.

Theorem 9.4 (Reflection Principle General Form) *Suppose the following holds:*

1. *d and d^* are domains that are symmetric domains about an arc or straight line s, such that their boundaries contain s.*

2. *g and g^* are domains that are symmetric domains about an arc or straight line t, such that their boundaries contain t.*

3. *$w = f_1(z)$ is univalent holomorphic in d, continuous on $d + s$, and maps d conformally to g, maps s bijectively to t.*

Then there exists a function $w = F(z)$ such that the following holds:

1. *$w = F(z)$ is univalent holomorphic in $d \cup s \cup d^*$, and map $d \cup s \cup d^*$ conformally to $g \cup t$ $cupg^*$;*

2. *$F(z) = f_1(z)$ in d;*

3. *$F(z) = f_2(z)$ in d^*; where $\{d^*, f_2(z)\}$ is the analytic continuation of $\{d, f_1(z)\}$ across the curve s.*

Proof: Consider the linear transformation

$$\xi = \frac{az+b}{cz+e}, \quad \omega = \frac{\alpha\omega + \beta}{\gamma\omega + \epsilon}$$

which maps s and t to a line segment on the real axis, and map d and g to D and G respectively, where D, G satisfies the condition in Corollary 9.3.2.
The inverse transformation of ξ and ω are

$$z = -\frac{e\xi + b}{c\xi - a}, \ \ w = \frac{-\epsilon\omega + \beta}{\gamma\omega - \alpha}$$

Substitute this into $w = f_1(z)$, we get a conformal map $\omega = f_1^*(\xi)$ from D to G.
Define the map

$$f_2^*(\xi) = \overline{f_1^*(\bar{\xi})}.$$

Then by Corollary 9.3.2, f_2^* maps D^* conformally to G^*. So the function

$$\omega = f^*(\xi) = \begin{cases} f_1^*(\xi), & \xi \in D, \\ f_2^*(\xi), & \xi \in D^*, \\ f_1^*(\xi) = f_2^*(\xi), & \xi \in S \end{cases}$$

is univalent and holomorphic in $D \cup S \cup D^*$, and map $D \cup S \cup D^*$ conformally to $G \cup T \cup G^*$.
Then substitute

$$\xi = \frac{az + b}{cz + e}, \ \ \omega = \frac{\alpha\omega + \beta}{\gamma\omega + \epsilon}$$

to $\omega = f_2^*(\xi)$, so we get a function $w = f_2(z)$ in d^*. It is univalent and holomorphic in d^*, and maps d^* conformally to g^*. Thus

$$w = F(z) = \begin{cases} f_1(z), & z \in d \\ f_2(z), & z \in d^*, \\ f_1(z) = f_2(z), & z \in s \end{cases}$$

is univalent holomorphic in $d \cup s \cup d^*$ and maps $d \cup s \cup d^*$ conformally to $g \cup t \cup g^*$. It is also clear that $\{d^*, f_2(z)\}$ is the analytic continuation of $\{d, f_1(z)\}$ across the curve s. $\square$

10 Infinite Sums and Infinite Products

10.1 Infinite Sums

Definition: let Ω be a domain in $\mathbb{C}$, and (f_n) a sequence on Ω, We say that $\sum_{n\geq 1} f_n$ converges uniformly on Ω, if $\left(\sum\limits_{n=1}^{N} f_n\right)_{N\geq 1}$ converges uniformly on Ω to some function f. If this is the case, we write $f = \sum\limits_{n=1}^{\infty} f_n$. On can see this happens if and only if

$$\sup_{x\in\Omega}\left|f - \sum_{i=1}^{N} f_i(x)\right| \to 0.$$

Definition: we say that $\sum\limits_{n\geq 1} f_n$ converge normally on Ω if $\sum\limits_{n\geq 1} \sup\limits_{\Omega} |f_n| < +\infty$. One can see that normal convergence on Ω implies uniform convergence on Ω.
Definition: we say $\sum f_n$ converges locally uniformly (resp. normally) on Ω if $\sum f_n$ converge uniformly (resp.

normally) on every compact sets in Ω. I.e., for every point $a \in \Omega$, there is a neighbourhood U_a of a on which the series converges uniformly.

Proposition 10.1 *An infinite sum converges pointwise iff its real part and imaginary part converges pointwise. It is absolutely convergent if and only the real part and imaginary part of the sum converges absolutely.*

Proposition 10.2 *Uniform convergence preserve continuity: if $f_n \in \mathscr{C}^0$, $\sum f_n$ converges uniformly, then $\sum f_n \in \mathscr{C}^0$.*

Theorem 10.3 *$f_n : \Omega \to \mathbb{C}$ are sequence of holomorphic functions. Assume $\sum f_n$ converges locally uniformly, then $\sum f_n$ is holomorphic and $\sum f_n'$ converges locally uniformly and $(\sum f_n)' = \sum f_n'$.*

Theorem 10.4 *$f_n : \Omega \to \mathbb{C}$ are sequence of holomorphic functions. Assume $\sum f_n \to f$ locally normally, then $\sum f_n' \to f'$ locally normally.*

Corollary 10.4.1 *$f_n : \Omega \to \mathbb{C}$ are sequence of holomorphic functions. Assume $\sum f_n \to f$ normally, then $\sum f_n' \to f'$ locally normally.*

Definition: let $t > 0$, $s \in \mathbb{C}$, then $t^s = e^{s \log t}$, where $\log t$ is real log.
Definition: let $a = (a_n)$, $a \in \mathbb{C}$, define $L(a, s) = \sum\limits_{n \geq 1} a_n n^{-s}$. Series of this form is called Dirichilet's series.
Definition: given $L(a, s)$, define

$$\tau = \limsup_{n \to \infty} \frac{\log |a_n|}{\log n}.$$

Proposition 10.5 *$\forall \delta > 0$, $L(a, s)$ converges normally on the half-plane $\{\mathrm{Re}(s) > \tau + 1 + \delta\}$ to a holomorphic function. So $L(a, s)$ is holomorphic in $\{\mathrm{Re}(s) > \tau + 1\}$.*

Definition: $\Omega \subset \mathbb{C}$ be a domain, and $Z \subset \Omega$ is locally finite. If $f : \Omega \setminus Z \to \mathbb{C}$ is holomorphic, $\forall a \in Z$, a is either removable or a pole, then we say f is meromorphic function on Ω.
Remark: if f is meromorphic, then $f : \Omega \to \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a holomorphic map.

Theorem 10.6 *f is meromorphic in Ω if and only there exists $f, g : \Omega \to \mathbb{C}$ holomorphic, $h \not\equiv 0$, s.t., $f = \frac{g}{h}$.*

Definition: (f_n) is a sequence of meromorphic functions in Ω. We say that $\sum\limits_{n=1}^{\infty} f_n$ converges locally uniformly (normally) on compact sets if $\forall K \subset \Omega$ if $\forall K \subset \Omega$ is compact

- $\exists n_K$ such that f_n ha s no poles in K for $n \geq n_K$.
- $\sum\limits_{n \geq n_K} f_n$ converges uniformly (normally) on K.

Remark: $U \subset \Omega$ is open, $\bar{U}$ compact, then

$$\sum_{n=1}^{\infty} = \sum_{n=1}^{n_U - 1} f_n + \sum_{n \geq n_U} f_n.$$

Then all poles comes from $\sum_{n=1}^{n_U-1} f_n$, in particular, poles of $\sum f_n$ is a subset the union of poles of f_n, $n \geq 1$. So $\sum f_n$ is meromorphic in Ω.

10.2 Infinite Products

Definition: let (f_n) be a sequence of holomorphic (meromorphic) functions in Ω. We say $\prod_{n=1}^{\infty} f_n$ converges normally on compact subsets of Ω if $\sum_{n\geq 1}(f_n - 1)$ converges normally on compact subsets of Ω. We say the prod converges normally in Ω, if $\sum_{n\geq 1}(f_n-1)$ converges normally on Ω.
Note: it is clear from the definition that a necessary condition for convergences of products is that f_n converge to 1 uniformly on compact subsets.

Theorem 10.7 *Assume $\prod\limits_{n\geq 1} f_n$ converges locally normally, then $\prod_{n\geq 1} f_n$ is a holomorphic (meromorphic) function, i.e., $\prod\limits_{n=1}^{N} f_n$ converges to a holomorphic (meromorphic) function on Ω when $N \to \infty$.*
Proof: Observe that the convergence of $\prod\limits_{n\geq n_0} f_n$ implies the convergence of $\prod\limits_{n\geq 1} f_n$ for any $n_0 \geq 1$. We can work on a subdomain of Ω and consider $\prod\limits_{n\geq n_0} f_n$ for a suitable n_0. So we can reduce the problem to the case where all f_n are holomorphic on Ω and the series $\sum\limits_{n\geq 1}(f_n-1)$ converges uniformly on Ω. Thus WLOG, we can also assume that $|f_n - 1| \leq \frac{1}{2}$ for every n.

Consider the principal branch of the logarithmic function $\operatorname{Log} : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$. Observe that for $|w-1| \leq \frac{1}{2}$, we have $|\operatorname{Log} w| \leq A|w-1|$ for some constant $A > 0$.
We can obtain this property by using Schwarz's Lemma, or the fact that $(w-1)^{-1}\operatorname{Log} w$ is holomorphic in $\mathbb{C}\setminus\mathbb{R}_-$ (removable singularity at $w = 1$).
Hence, we have

$$\prod_{1\leq n\leq N} f_n = \exp\left[\sum_{1\leq n\leq N} \operatorname{Log} f_n\right]$$

and

$$\sup_{\Omega} |\operatorname{Log} f_n| \leq A \sup_{\Omega} |f_n - 1|.$$

This and the normal convergence of $\sum(f_n - 1)$ imply that

$$\sum_{1\leq n\leq N} \operatorname{Log} f_n$$

converges normally and hence uniformly on Ω to a holomorphic function. We then deduce that $\prod\limits_{1\leq n\leq N} f_n$ converges uniformly to a holomorphic function. $\square$

Remark: if $f_n \equiv 0$ for some n, then $\prod f_n = 0$, so in general we assume $f_n \not\equiv 0$, $\forall n$.
Remark: in the proof $\prod\limits_{n=1}^{\infty} f_n = e^{\sum \log f_n}$ has no poles or zeros (after reducing Ω and remove a finite number of factors). Then we deduce that the zeros and poles of $\prod f_n$ is a subset of the union of zeros and poles of f_n, $n \in \mathbb{Z}$. If f_n is holomorphic $\forall n$, then there will be no poles, in this case we have the zeros of $\prod f_n$ = the union of zeros of f_n.
Remark: in general $\operatorname{ord}(\prod f_n, a) = \sum \operatorname{ord}(f_n, a)$. And as $f_n(a) \to 1$, so $\operatorname{ord}(f_n, a) = 0$ for n big enough. Hence the right hand side is a finite sum and will always be well-defined.

Theorem 10.8 (Differentiation Theorem) *Suppose (f_n) is a sequence of holomorphic (meromorphic) function in Ω that converges locally normally in Ω. Then*

$$\frac{\left(\prod\limits_{n=1}^{N} f_n\right)'}{\prod\limits_{n=1}^{N} f_n} = \sum_{n=1}^{N} \frac{f_n'}{f_n}.$$

And

$$\frac{\left(\prod\limits_{n=1}^{\infty} f_n\right)'}{\prod\limits_{n=1}^{\infty} f_n} = \sum_{n=1}^{\infty} \frac{f_n'}{f_n}.$$

In particular, $\sum \frac{f_n'}{f_n}$ is a series of meromorphic functions that converges locally normally in Ω.

Proposition 10.9 *If $\prod\limits_{n=1}^{\infty} f_n$ converges normally in Ω, then*

1. *for every bijection $\tau : \mathbb{N}^+ \to \mathbb{N}^+$, the product $\prod\limits_{n=1}^{\infty} f_{\tau(n)}$ converges normally in Ω;*
2. *Every subproduct $\prod\limits_{k\geq 0} f_{n_k}$ converges normally in Ω.*
3. *Let p_j denote $\prod\limits_{n\geq j} f_n$, then p_j is well-defined and converges compactly in Ω to 1.*

Corollary 10.9.1 *If $\mathbb{N}^+ = \bigcup\limits_{k=1}^{\infty} N_k$ is a (finite or infinite) partition of $\mathbb{N}^+$ into pointwise disjoint subsets $N_1, \cdots, N_k, \cdots$. And $\prod\limits_{n=1}^{\infty} f_n$ converges normally in $\Omega \subset \mathbb{C}$. Then every product $\prod\limits_{v\in N_k} f_v$ converges normally in Ω, with*

$$f = \prod_{n=1}^{\infty} f_n = \prod_{k=1}^{\infty} \left(\prod_{v\in N_k} f_v \right).$$

Proposition 10.10 *Suppose (f_n) and (g_n) are sequences of holomorphic (meromorphic) functions in Ω. If $\prod\limits_{n=1}^{\infty} f_n$ and $\prod\limits_{n=1}^{\infty} g_n$ converges locally normally on Ω, then so does $\prod\limits_{n=1}^{\infty} f_n g_n$. Moreover,*

$$\prod_{n=1}^{\infty} f_n g_n = \left(\prod_{n=1}^{\infty} f_n\right) \cdot \left(\prod_{n=1}^{\infty} g_n\right).$$

Proof: Note

$$\sup |f_n g_n - 1| = \sup |f_n(g_n - 1) + f_n - 1| \leq \sup |f_n||g_n - 1| + |f_n - 1|.$$

As n is large enough, we have $|f_n|$ is bounded by some constant M, hence we conclude $\prod\limits_{n=1}^{\infty} f_n g_n$ converges locally normally on Ω. And clearly

$$\prod_{n=1}^{\infty} f_n g_n = e^{\sum \log f_n g_n} = e^{\sum (\log f_n + \log g_n)} = e^{\sum \log f_n + \sum \log g_n} = \left(\prod_{n=1}^{\infty} f_n\right) \cdot \left(\prod_{n=1}^{\infty} g_n\right).$$

$\square$

Definition: we define the Euler's constant by

$$\gamma = \lim_{n\to\infty}\left[-\log n + \sum_{k=1}^{n}\frac{1}{k}\right] > 0.$$

Definition: we define the Gamma function,

$$\Gamma(z) = z^{-1}e^{-\gamma z}\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^{-1}e^{z/k}\right].$$

Theorem 10.11 *Γ is meromorphic in $\mathbb{C}$ and holomorphic in $\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. Γ has no zeros, and has simple poles at $-n\in\mathbb{Z}_{\leq 0}$. In addition $\mathrm{Res}(\Gamma,-n)=\frac{(-1)^n}{n!}$, $\Gamma(z+1)=z\Gamma(z)$.*

Theorem 10.12 *Let*

$$P(z)=\sum_{n\in Z}\frac{1}{(z-n)^2},\quad S(z)=z\prod_{1}^{\infty}(1-\frac{z^2}{n^2}),\quad Z(z)=\frac{1}{z}+\sum_{1}^{\infty}\frac{2z}{z^2-n^2}.$$

Then

$$P(z)=-Z'(z),\ \text{ and } Z(z)=\frac{S'(z)}{S(z)}$$

and

$$P(z)=\frac{\pi^2}{\sin^2(\pi z)},\quad Z(z)=\pi\cot(\pi z),\quad S(z)=\frac{\sin(\pi z)}{\pi}.$$

Proof: The normal convergence of the series and the relationships between P,S,Z. Now if we assume $P=\frac{\pi^2}{\sin^2(\pi z)}$, we show $Z(z)=\pi\cot(\pi z)$ and $S(z)=\pi\sin(\pi z)$. We have

$$(Z-\pi\cot(\pi z))' = -P+\pi^2\cdot\frac{1}{\sin^2(\pi z)}=0.$$

So $Z=\pi\cot(\pi z)+c$ in $\mathbb{C}\setminus\mathbb{Z}$. Observe that Z and $\pi\cot(\pi z)$ are odd functions. Then we conclude that $c=0$, i.e., $Z(z)=\pi\cot(\pi z)$.

Next consider $h(z)=\frac{S(z)}{\sin(\pi z)}$, we have

$$\frac{h'}{h}=\frac{S'}{S}-\frac{\sin(\pi z)'}{\sin(\pi z)}=Z-\frac{\pi\cos(\pi z)}{\sin(\pi z)}=0.$$

So $\frac{h'}{h} \equiv$ const. on $\mathbb{C} \setminus \mathbb{Z}$. Next, note

$$\begin{aligned} h(z) &= \frac{S(z)}{\sin(\pi z)} \\ &= \frac{z(1+o(1))}{\pi z + o(z)} \\ &= \frac{1}{\pi} + o(1) \\ \Rightarrow \lim_{z\to 0} h(z) &= \frac{1}{\pi} \end{aligned}$$

Hence $\frac{S}{\sin(\pi z)} = \frac{1}{\pi}$, i.e., $S(z) = \frac{\sin(\pi z)}{z}$.

Lastly we prove $P(z) = \frac{\pi^2}{\sin^2(\pi z)}$. Consider $f(z) = P(z) - \frac{\pi^2}{\sin^2(\pi z)}$. WTS that $f \equiv 0$. Note f is holomorphic in $C \setminus \mathbb{Z}$, f is even because both $P(z)$ and $\frac{\pi^2}{\sin^2(\pi z)}$ are even. Next f is periodic with period 1, as this is the case for P and $\frac{\pi^2}{\sin^2(\pi z)}$. Near the point $z = 0$, we have

$$\begin{aligned} f(z) &= P(z) - \frac{\pi^2}{\sin^2(\pi z)} \\ &= \frac{1}{z^2} + O(1) - \frac{\pi^2}{[\pi z + O(z^3)]^2} \\ &= \frac{1}{z^2} + O(1) - \frac{1}{z^2[1+O(z^2)]^2} \\ &= \frac{1}{z^2} + O(1) - \frac{1}{z^2}[1+O(z^2)] \\ &= O(1). \end{aligned}$$

Thus 0 is removable. Since f is 1-periodic, then for all $n \in \mathbb{Z}$, n is a removable singularity for f. So f is entire. We show f is constant, it suffices to show f is bounded in $\{0 \le x \le 1\}$ as f is 1 periodic. We compute the limit of $\sup_{0\le x\le 1} |f|$ as $|y| \to \infty$.

$$\begin{aligned} P(z) &= \sum_{n\in\mathbb{Z}} \frac{1}{(x+iy-n)^2} \\ |P(z)| &\le \sum_{n\in\mathbb{Z}} \frac{1}{|x+iy-n|^2} \\ &= \sum_{n\in\mathbb{Z}} \frac{1}{(x-n)^2+y^2} \\ &= \frac{1}{x^2+y^2} + \frac{1}{(x-1)^2+y^2} + \frac{1}{(x+1)^3+y^2} + \sum_{|n|\ge 2} \frac{1}{|x-n|^2+y^2} \\ &\le \frac{3}{y^2} + 2\sum_{k=1}^{\infty} \frac{1}{k^2+y^2} \\ &\le \frac{5}{y^2} + \int_1^{\infty} \frac{1}{t^2+y^2} dt \\ &\to 0 \;\text{ as } y \to \infty. \end{aligned}$$

So f is bounded, hence it is a constant by Liouville's Theorem. Further $f \equiv 0$, as the limit of $|f|$ as $y \to 0$ is 0. Hence $P(z) = \frac{\pi^2}{\sin^2(\pi z)}$ as desired. $\square$

11 Holomorphic Functions in Higher Dimensions

11.1 Holomorphic Functions in $\mathbb{C}^n$

Definition: let $\Omega \subset \mathbb{C}^n$ be a domain, $f : \Omega \to \mathbb{C}$ is continuous. We say that f is holomorphic if it is holomorphic in each variable. That is if we fix $n-1$ components of $\mathbf{z} = (z_1, \cdots, z_n)$, then $z_i \mapsto f(z_1, z_2, \cdots, z_n)$ is holomorphic in z_i. (See Hartog's Theorem).

Definition: let $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{C}^n$, $\mathbf{r} = (r_1, r_2, \cdots, r_n) \in \mathbb{R}^n_{>0}$, then we define a polydisk of radius $\mathbf{r}$ and centered at $\mathbf{a}$ by

$$\mathbb{D}(\mathbf{a}, \mathbf{r}) = \{z = (z_1, \cdots, z_k) : |z_k - a_k| < r_k, \ \forall k\}.$$

Note

$$\mathbb{D}(\mathbf{a}, \mathbf{r}) = \mathbb{D}(a_1, r_1) \times \mathbb{D}(a_2, r_2) \times \cdots \times \mathbb{D}(a_n, r_n).$$

Definition: let $\mathbf{I} = (i_1, i_2, \cdots, i_n)$, then we denote $\mathbf{z}^{\mathbf{I}}$ to be $z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}$.

Theorem 11.1 *$f : \Omega \to \mathbb{C}$, then f is holomorphic if and only if one of the following is true:*

1. *$f \in \mathscr{C}^1$, and df is $\mathbb{C}$ linear at every point. I.e., $df_{\mathbf{a}}(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2) = \lambda_1 df_{\mathbf{a}}(\mathbf{v}_1) + \lambda_2 df_{\mathbf{a}}(\mathbf{v}_2)$, $\lambda_1, \lambda_2 \in \mathbb{C}, \mathbf{a} \in \mathbb{C}^n$. (For a general function in $\mathbb{C}^n$, df is $\mathbb{R}$ linear, it is $\mathbb{C}$ linear if and only if f is holomorphic).*

2. *$f \in \mathscr{C}^1$, and $\frac{\partial f}{\partial \bar{z}_i} \equiv 0 \ \forall i$.*

3. *$\forall a \in \Omega$, $\exists$ a power series such that*
$$f(z) = \sum_{\mathbf{I} \in \mathbb{N}^n} c_{\mathbf{I}} (\mathbf{z} - \mathbf{a})^{\mathbf{I}}$$
for $\mathbf{z}$ near $\mathbf{a}$, and the sequence converges normally near $\mathbf{a}$.

4. *$\forall \overline{\mathbb{D}(\mathbf{a}, \mathbf{r})} \subset \Omega$,*
$$f(z) = \frac{1}{(2i\pi)^n} \int_{|\xi - a_1| = r_1} \cdots \int_{|\xi_n - a_n| = r_n} \frac{f(\xi) d\xi_1 \cdots d\xi_n}{(\xi_1 - z_1)(\xi_2 - z_2) \cdots (\xi_n - z_n)}.$$
In particular, $f \in \mathscr{C}^\infty$.

Definition: let $\Omega \subset \mathbb{C}^n$ be a domain, $f : \Omega \to \mathbb{C}^m$, write $\mathbf{f} = (f_1, f_2, \cdots, f_m)$, where $f_k : \Omega \to \mathbb{C}$. Then we say that $\mathbf{f}$ is holomorphic if f_k is holomorphic $\forall k$, that is $\mathbf{f}$ is linear, and $d\mathbf{f}_{\mathbf{a}}$ (a complex matrix in $m \times n$) is $\mathbb{C}-$ linear.

Remark, in this case $d\mathbf{f}_{\mathbf{a}} : \mathbb{C}^n \to \mathbb{C}^m$ is linear and given by

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial z_n}(\mathbf{a}) \\ & \ddots & \\ \frac{\partial f_m}{\partial z_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial z_n}(\mathbf{a}) \end{pmatrix}.$$

Theorem 11.2 *If $\Omega_1 \subset \mathbb{C}^n$, $\Omega_2 \subset \mathbb{C}^m$, $f : \Omega_1 \to \Omega_2$ is holomorphic and $g : \Omega_2 \to \mathbb{C}^l$ is holomorphic. Then $g \circ f : \Omega_1 \to \mathbb{C}^l$ is holomorphic. And we have the chain rule:*

$$d(g \circ f)_{\mathbf{a}} = (dg_{f(\mathbf{a})}) \circ (df)_{\mathbf{a}}.$$

Notation: let us use $(\mathbb{C}^n, 0)$ denote a neighborhood of 0 in $\mathbb{C}^n$.

Theorem 11.3 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic map, and assume $df(0)$ is invertible. Then $\exists g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ holomorphic, such that $g \circ f : (C^n, 0) \to (\mathbb{C}^n, 0)$ is the identity map, same for $f \circ g$ and $(df)_{\mathbf{a}} \circ (dg)_{f^{-1}(\mathbf{a})} = 1$.*

Theorem 11.4 (Implicit Function/Map) *$f : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ holomorphic, $n \geq m$. Assume rank $df(0)$ is maximal, i.e. equal to m. Then there exists $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ holomorphic and invertible, such that $f(g(z_1, z_2, \cdots, z_n)) = (z_1, z_2, \cdots, z_m)$.*

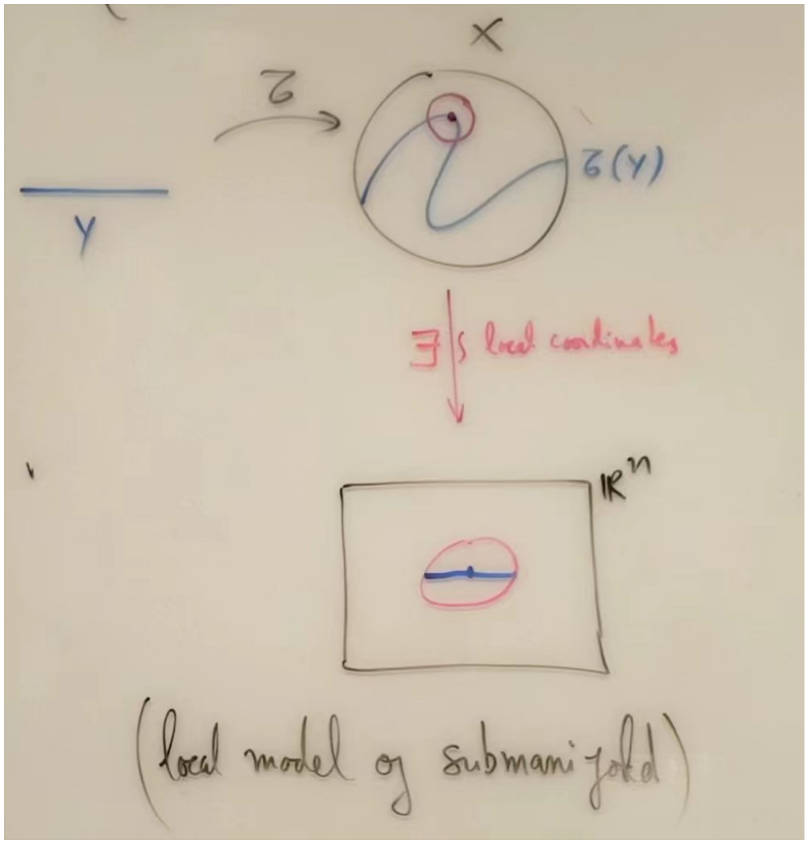
Remark: there is no open mapping theorem in higher dimension.
Counter example: $f : \mathbb{C}^2 \to \mathbb{C}^2$, $(z_1, z_2) \mapsto (z_1, z_1 z_2)$. We can check that $f(\mathbb{C}^2) = \mathbb{C}^* \times \mathbb{C} \cup \{0\}$. f^{-1} blows up to the point 0 to the line $\{z_1 = 0\}$.

11.2 Manifold

Definition: X satisfies the second axiom of countability if there exists a countable basis of open sets.
Definition: Suppose X is Hausdorff and satisfies the second axiom of countability. We call atlas (of real dimension n) to be an open cover $(U_j)_{j \in J}$ together with homeomorphism $\phi_j : U_j \to \Omega_j$ with $\Omega_j \subset \mathbb{R}^n$ open such that $\forall j, k$ the transition map, $\phi_j \circ \phi_k^{-1} : \phi_k(U_j \cap U_k) \to \phi_j(U_j \cap U_k)$, $x \mapsto \phi_j \circ \phi_k^{-1}(x)$, is $\mathscr{C}^\infty$.
Definition: we say 2 atlas to be equivalent if their union is also an atlas
Definition: any equivalence class of atlas of X is called a manifold ($\mathscr{C}^\infty$ manifold of dimension n).
Remark: For each manifold, we have a maximal atlas (union of all atlas in the class).
Remark: We can define $\mathscr{C}^k$ manifolds instead of $\mathscr{C}^\infty$ by loosening the restriction on the transitions.
Remark: Each homomorphism allows us to talk about local coordinates for the set U_j. Whenever two open sets intersects, then we have a change of coordinate map given by $\phi_j \circ \phi_k^{-1}$.
Remark: Open sets in $\mathbb{R}^n$ are local models of manifolds.
Definition: (U_j, ϕ_j) is called a chart of X.

Proposition 11.5 *If X, Y is a manifold, then $X \times Y$ is also a manifold.*

Definition: X, U are manifold of dimension n, m respectively, with n smaller than m. Let $\tau : Y \to X$ be a map, that is $\mathscr{C}^\infty$, injective, proper of maximal rank at every point. Then we say $\tau(Y)$ is a submanifold of dimension m, codimension $n - m$ of X. If $m = n - 1$, we say that $\tau(Y)$ is a (smooth) hypersurface of X. Here $\mathscr{C}^\infty$ is in terms of local coordinates, so technically $\mathbb{R}^n \to \mathbb{R}^m$. This is independent of atlas, as whenever we have a transition (change of coordinate), it is $\mathscr{C}^\infty$.
Definition: $\tau : Y \to X$ is proper if $\tau^{-1}(K)$ is compact $\forall k \subset X$ compact.
Remark: $\tau : Y \to X$ as above, ($\tau(Y)$ is a submanifold)



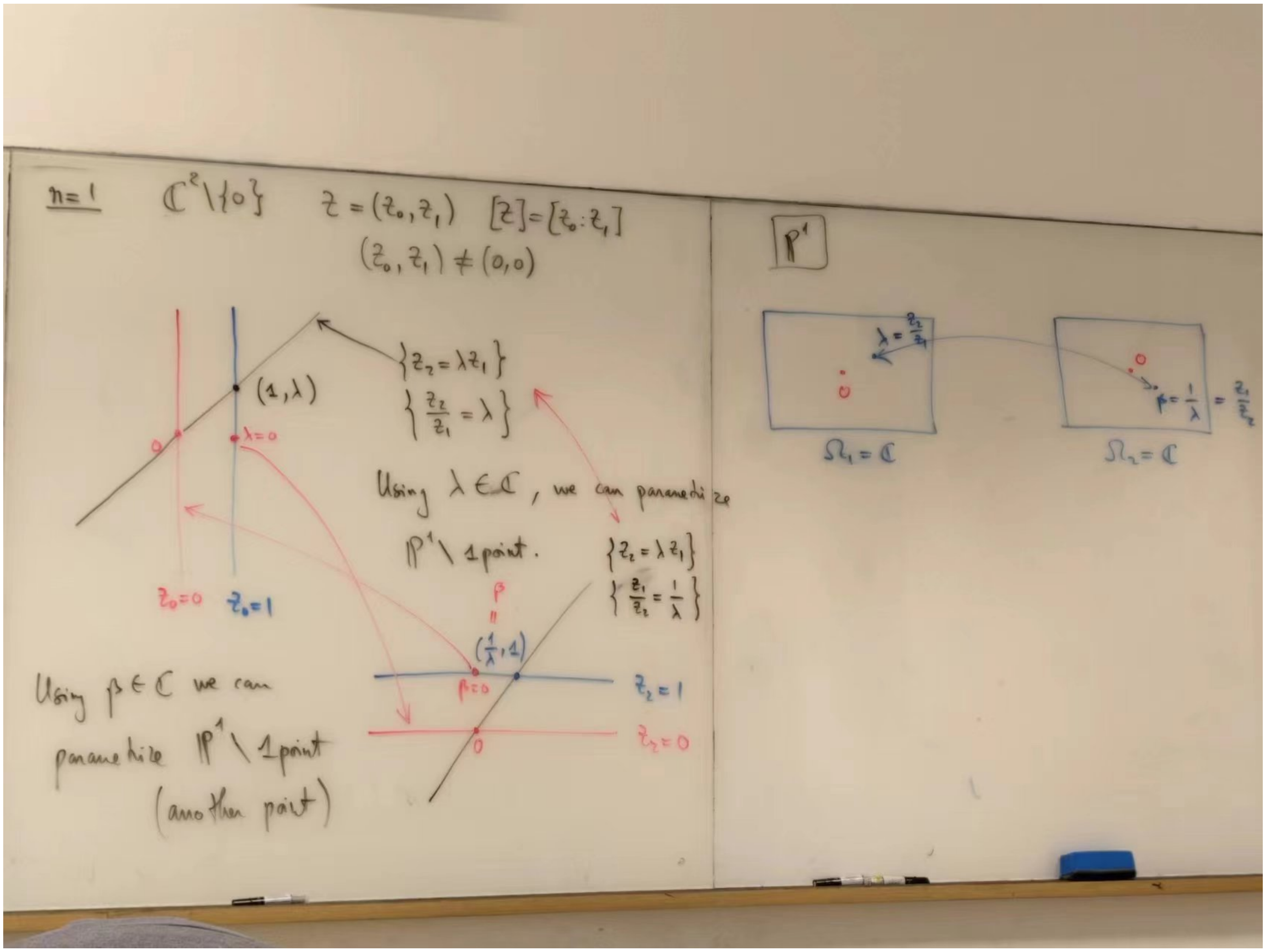
Example: f =identity: $X \to X$. $\Gamma = \triangle$ diagonal of X^2. Then Γ is a submanifold of X^2. In this case, $A = \pi_2(\pi_1^{-1}(A) \cap \Delta) \subset X$.
Example: Torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Locally we can map $\mathbb{R}^n \to \mathbb{T}^n$, by $x \mapsto x/\sim$.

Definition: complex manifold and complex submanifold can be defined in the same way using open sets of $\mathbb{C}^n$ and holomorphic transitions.
Caution:Take $\tau \in \mathbb{C} \setminus \mathbb{R}$, and consider $\mathbb{Z} + \tau\mathbb{Z}$ which can be considered as a lattice of $\mathbb{C}$. $\mathbb{T} = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ is a torus, which is also an elliptic curve. $\exists \mathscr{C}^\infty$ diffeomorphism: $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \to \mathbb{R}^2/\mathbb{Z}^2$ (That is if we consider them to be real manifolds, then all such torus are the same). But in general, if $\tau_1 \neq \tau_2$, $\mathbb{C}/\mathbb{Z} + \tau_1\mathbb{Z}$ is not holomorphically isomorphic to $\mathbb{C}/\mathbb{Z} + \tau_2\mathbb{Z}$ (That is if we consider them as complex manifold, they are different).

Definition: a Riemann surface is a complex manifold of (complex) dimension 1. Sometimes, we also call it a complex curve.
Definition: a complex surface is a complex manifold of dimension 2.
Definition: a complex hypersurface is a complex submanifold of codimension 1.
Example: Riemann Sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, ($\mathbb{P}^1$, $\mathbb{CP}^1$, $\mathbb{P}^1(\mathbb{C})$) is a Riemann Surface.
Example: Projective Spaces $\mathbb{P}^n$, ($\mathbb{CP}^n, \mathbb{P}^n(\mathbb{C})$). Note if $n = 1$, then we get a Riemann sphere. Consider the set of family of all complex lines through 0 in $\mathbb{C}^{n+1}$, then $\mathbb{P}^n$ is the parameter space of these lines. I.e., consider $\mathbb{C}^{n+1} \setminus \{0\}$, we define an equivalence relation $\sim$ by $z \sim z'$ if $z = \lambda z'$ for some $\lambda \in \mathbb{C}^*$. Then $\mathbb{P}^n$ is defined to be $(\mathbb{C}^{n+1} \setminus \{0\})/\sim$. There is a canonical projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ induced by this equivalence relation. If $z = (z_0, z_1, \cdots, z_n)$, then

$$\pi(z) = [z] = [z_0 : z_1 : \cdots : z_n] = [\lambda z_0 : \lambda z_1 : \cdots : \lambda z_n](\lambda \in \mathbb{C}^*).$$

When $n = 1$, $\mathbb{C}^2 \setminus \{0\}$, $z = (z_0, z_1)$, $[z] = [z_0 : z_1]$, $(z_0, z_1) \neq (0, 0)$. Suppose $z_1 = \lambda z_0$, then $\frac{z_1}{z_0} = \lambda$, and $(1, \lambda)$ will be the intersection of the line $z_1 = \lambda z_0$ and $z_0 = 1$; $(\frac{1}{\lambda}, 1)$ will be the intersection of the line $z_1 = \lambda z_0$ and $z_0 = 1$.



Remark: $\mathbb{P}^n$ is a compact complex manifold of dimension n.

Theorem 11.6 (Maximum Modulus Principle) *Let $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function, Ω is a domain and $w = (w_1, \cdots, w_n) \in \Omega$ such that w is a local maximum for $|f|$. Then there exists a polydisc $D = D_1 \times \cdots \times D_n$ around w such that f is constant on D.*

Proof: Consider $f_1(z) = (z, w_2, \cdots, w_n)$. This achieves its maximum at $z = w_1$, and is therefore a constant by one-dimensional maximum principle. Since f_1 is constant, if w_1' is near w_1, then

$$f(w_1', w_2, \cdots, w_n) = f_1(w_1') = f_1(w_1) = f(w_1, \cdots, w_n).$$

Thus f also achieves its maximum at $f(w_1', w_2, \cdots, w_n)$. Hence, $f_2(z) = (w_1', z, w_3, \cdots, w_n)$ achieves its maximum at $z = w_2$, so f_2 is constant by one-dimensional maximum principle.
Together this shows that $f(z_1, z_2, w_3, \cdots, w_n)$ is constant in a neighborhood of $(w_1, w_2) \times \{w_3\} \times \cdots \times \{w_n\}$. Now we have the freedom to adjust both w_1 and w_2 a little bitm and we can finish the proof by induction. $\square$

Corollary 11.6.1 *Suppose $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic and Ω is a domain and $|f|$ obtains its maximum at some point w in Ω, then f is constant.*

Proof: Then $|f(w)|$ is a local maximum. So f is a constant in a neighbourhood of w. Then in that neighbourhood, every point becomes a local maximum for $|f|$. Then we can keep expanding the domain in which f is constant, until it fills up the entire domain Ω. $\square$

Proposition 11.7 *X is compact (connected) complex manifold. $f : X \to \mathbb{C}$ is holomorphic, then $f \equiv$ constant. Hence $\forall f : X \to \mathbb{C}^n$ holomorphic should be constant as well.*

Proof: f is holomorphic in terms of local coordinates, then $f \in \mathscr{C}^0$. Since X is compact, then $|f|$ admits a maximum. However, f is an open map, since it is locally $f_\Omega : \Omega \to \mathbb{C}$, where $\Omega \subset \mathbb{C}$, and f_Ω is holomorphic. So the maximum of $|f|$ is obtained iff f is a constant.
Now if $f : X \to \mathbb{C}^n$ is holomorphic, and X is compact. Then for each $f_1, \cdots, f_n$, we have $|f_i|$ is obtained somewhere (since X is compact). As X is locally $\mathbb{C}^m$ and connected, then we know this can only happen iff f_i is a constant by Maximum Modulus Principle. So f is a constant. $\square$

Definition: a biholomorphic function τ from two complex manifold U and V, is one that is injective, holomorphic, whose inverse $\tau^{-1} : f(U) \to U$ is also holomorphic, i.e., τ need to be proper and its derivative is of maximum rank at every point. If there is a biholomorphism $\phi : U \to V$, then we say U and V are biholomorphic.
Definition: X is a complex manifold. We say X is a Stein manifold if it is biholomorphic to some submanifhold of some $\mathbb{C}^N$.
Remark: if X is compact, $\dim X \geq 1$, then X is not Stein. Remark: there exists many holomorphic function on Stein Manifolds. As there exists many holomorphic functions $f : \mathbb{C}^N \to \mathbb{C}$, and $f \circ \tau : X \to \mathbb{C}$ is holomorphic.
Remark: on $\mathbb{P}^n$, there are many meromorphic functions. Example: when $N = 1$, $z = (z_0, z_1)$, $[z] = [z_0 : z_1]$, Consider $f([z]) = \frac{z_1}{z_0} = \frac{\lambda z_1}{\lambda z_0}$, which is well-defined. Then f is meromorphic, but it is not holomorphic as $f([0 : 1]) = \infty$.
Remark: any meromorphic function on $\mathbb{P}^n$ has the form

$$f = \frac{P(z)}{Q(z)},$$

where P, Q are homogeneous polynomials of same degree, i.e., the sum of monomials with the same degree.

Definition: X is projective manifold if it is biholomorphic to some submanifold of some $\mathbb{P}^N$.
Remark: if X is compact, then there exists many meromorphic functions from X to $\mathbb{C}$.

Theorem 11.8 *For all submanifold $X \subseteq \mathbb{P}^N$, there exists homogeneous polynomials $P_1, \cdots, P_m$ such that $X = \{P_1 = P_2 = \cdots = P_m = 0\}$, i.e., X is the set of common zeros for $P_1, P_2, \cdots, P_m$.*

Definition: if P is a homogeneous polynomial of degree 1, then $\{P = 0\}$ is a hyperplane in $\mathbb{P}^n$.

Example: let Γ be a lattice in $\mathbb{C}^n$, Take $\mathbb{C}^n/\Gamma$ be a torus of dimension n. $\forall n \geq 2$, $\mathbb{C}^n/\Gamma$ for "generic" Γ is not projective. "The most interesting ones" are projective, all of them are Kählen manifold.
Not compact manifold can be classified into Stein ($\approx$ submanifold of $\mathbb{C}^N$) and non-Stein (e.g. product of Stein manifold and a compact manifold).

Compact manifold can be classified in several ways, such as: {projective} $\subsetneq$ {Kählen compact} $\subsetneq$ {all compact} or {projective} $\subsetneq$ {Kählen compact} $\subsetneq$ {Fujiki class}.
Example (Hopf): Consider $\mathbb{C}^n \setminus \{0\}$, and define an equivalence relation, $z \sim z'$ iff $z = 2^k z'$ for some $k \in \mathbb{Z}$. Then we consider $(\mathbb{C}^n \setminus \{0\})/\sim$. If $n = 1$, then we get a torus, if $n = 2$, then we get a Hopf surface, which is compact but not Kählen (Fujiki).
Remark: in dimension 1, any non-compact Riemann surface is Stein. Any compact Riemann surface is projective. Classification in $2D$ is "almost" complete. In $3D$, many things we know. In higher dimensions, things get more complicated.

Theorem 11.9 *Suppose $Z \subset X$ is a complex submanifold, $\dim z = m$, $\dim x = n$. Let $a \in Z$, then $\exists$ local coordinates $z = (z_1, \cdots, z_n)$ of X centered at a, such that near a, z is given by $\{z_{m+1} = \cdots = z_n = 0\}$. That is there is a chart (U, ϕ) of X containing a such that $\phi(a) = 0$ and $Z \cap U$ is the inverse image of ϕ of the linear subspace $\{z_{m+1} = \cdots = z_n = 0\}$.*

Remark: submanifolds are local graphs of holomorphic maps.

11.3 Analytical Sets

Definition: let X be a complex manifold, $Z \subset X$ closed. We say Z is an analytic subset of X if $\forall a \in X$, $\exists$ a neighbourhood U of a and a family $\{f_i\}_{i \in I}$ of holomorphic functions on U such that

$$Z \cap U = \{f_i = 0 \, : \, \forall i \in I\}$$

I.e., locally Z is the set of common zeros of a family of holomorphic function $\{f_i\}_{i \in I}$.
Remark: even if we assume that I is finite, we still have an equivalent definition. I.e., the set of analytic subset defined by both definition are equal.

Definition: assume there does not exists analytic subsets Z_1, Z_2 such that $Z_1 \cup Z_2 = Z$, $Z_1 \neq Z$ and $Z_2 \neq Z$, then we say that Z is irreducible.
Remark: any connected submanifold is an irreducible analytic subset.

Theorem 11.10 *Any analytic subset of X is a locally finite ($\forall$ compact subset $K \subset X$, exists only finitely many irreducible components which intersect K) union of irreducible analytic subset.*

Theorem 11.11 *Suppose X is a complex manifold, $Z \subset X$ is a analytic subset. Then $\exists$ a smallest analytic subset of X (denoted by $\mathrm{Sing}(Z)$) such that*

1. *$\mathrm{Sing}(Z) \subset Z$, $\mathrm{Sing}(Z) \setminus neqZ$ (maybe empty);*

2. *the irreducible component of $Z \setminus Sing(Z)$ are submanifolds of $X \setminus \mathrm{Sing}(Z)$.*

Definition: we define $\mathrm{Sing}(Z)$, the set of singularities of the analytic subset Z as in the theorem, we define the regular part of Z to be $\mathrm{Reg}(Z) = Z \setminus \mathrm{Sing}(Z)$.

Definition: the dimension of Z is the maximum dimension of components of $\mathrm{Reg}(Z)$.
Definition: we say Z is of pure dimension m if all components of $\mathrm{Reg}(Z)$ are of dimension m.

Theorem 11.12 *If Z is an analytic subset of a complex manifold X, then either $\mathrm{Sing}(Z) = \emptyset$ or $\dim \mathrm{Sing}(Z) < \dim(Z)$.*

Remark: given an analytic subset Z, we can decompose it into $\mathrm{Sing}(Z)$ and $\mathrm{Reg}(Z)$. Since $\mathrm{Sing}(\mathrm{Reg}(Z)) = \emptyset$, and we can decompose $\mathrm{Sing}(Z)$ into $\mathrm{Sing}(\mathrm{Sing}(Z))$ and $\mathrm{Reg}(\mathrm{Sing}(Z))$, and so on. This process will eventually ends as every time, the dimension of the singular part will decrease.

Theorem 11.13 (Hironoka) *Let $Z \subset X$ be an analytic subset. Then $\exists \hat{X}$ complex manifold and a holomorphic map $\pi : \hat{X} \to X$ such that*

1. *$\pi : \hat{X} \setminus \pi^{-1}(\mathrm{Sing}\, Z) \to X \setminus \mathrm{Sing}(Z)$ is biholomorphic.*
2. *The closure of $\pi^{-1}(\mathrm{Reg}(Z))$ is a regular analytic (the singular part is $\emptyset$) subset of $\hat{X}$.*

11.4 Differential Forms

Let $\Omega \subset \mathbb{R}^n$ be a domain, $a \in \Omega$, $f : \Omega \to \mathbb{R}$, $f \in \mathscr{C}^\infty$. Then $df(a) : \mathbb{R}^n \to \mathbb{R}$ is a linear map,

$$x = (x_1, \cdots, x_n) \mapsto \frac{\partial f}{\partial x_1}(a) \cdot x_1 + \cdots + \frac{\partial f}{\partial x_n}(a) \cdot x_n.$$

Consider when $f(x) = x_1 : \mathbb{R}^n \to \mathbb{R}$. $df(a) : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto x_1$, which is independent of the value of a. Then we can denote $dx_1 : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto x_1$. Then we can write

$$df(a) = \frac{\partial f}{\partial x_1}(a) dx_1 + \cdots + \frac{\partial f}{\partial x_n}(a) dx_n.$$

Or we can write

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

Definition: a differential 1-form or a differential form of degree 1 on $\Omega \subset \mathbb{R}^n$ is an expression

$$\alpha = \alpha_1 dx_1 + \cdots + \alpha_n dx_n,$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are functions on Ω.
Remark: If $\alpha_1, \cdots, \alpha_n$ are $\mathscr{C}^n$, then α is an $\mathscr{C}^n$ 1-form. If $\alpha_1, \cdots, \alpha_n$ are smooth, then α is known as a smooth $1-$form.
Remark: $\forall x \in \Omega$, $\alpha(x) : \mathbb{R}^n \to \mathbb{R}^n$ is linear, α maps to a family of linear form $\mathbb{R}^n \to \mathbb{R}^n$ parametrized by $x \in \Omega$.

Definition: the wedge product or exterior product is the operation satisfying:

- $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

- $dx_i \wedge dx_i = 0$.

Notation: $I = (i_1, i_2, \cdots, i_p)$, $1 \le i_k \le n$, $d_{X_I} = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$.
Remark: If $i_s = i_r$ for some $r \neq s$, then $dx_I = 0$.
Remark: $dx_I = \pm dx_J$, where J is a permutation of I, and $(j_1, \cdots, j_p)$ satisfies $J_1 \le j_2 \le \cdots \le j_p$.

Definition: a $p-$form is any expression

$$\alpha = \sum_{|I|=P} \alpha_I dx_I, \quad \alpha_I : \Omega \to \mathbb{R}$$

Remark: if α is a p-form, then we can write

$$\alpha = \sum_{\substack{I=(i_1,i_2,\cdots,i_p) \\ i_1<\cdots<i_p}} \alpha_I dx_I$$

in a unique way.
Remark: we can see α as a function on Ω with value in the vector space spanned by these dx_I. By linearity, we can extend the wedge product to forms of any degree:

$$dx_{(i_1,\cdots,i_p)} \wedge dx_{(j_1,\cdots,j_q)} = dx_{(i_1,\cdots,i_p,j_1,\cdots,j_q)}.$$

Remark: any $p-$form with $p > n$ should be 0 by PHP. And any $n-$form is given by $\alpha = h(x)dx_1 \wedge \cdots \wedge dx_n$. A zero form is a function.

Definition: operator d can be extended to forms:

$$d(f dx_I) = df \wedge dx_I,$$

where df is the normal differential operator:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

And by linearity, we can extend to all forms by

$$\alpha = \sum \alpha dx_I, \quad d\alpha = \sum d\alpha_I \wedge dx_I.$$

Proposition 11.14 (Leibnitz) *Suppose α is a p-form, β is a $q-$form, then*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

Theorem 11.15 *$d \circ d = 0$. If α is a smooth $p-$form,then $d(d\alpha) = 0$.*

Definition: we say that a p-form α is closed if $d\alpha = 0$. For $p \ge 1$, we say that α is exact if $\exists$ a $(p-1)$ form β, such that $\alpha = d\beta$.

Proposition 11.16 *If α is exact, then α is closed.*

Remark: given α, consider the equation $d\beta = \alpha$, β unknown. Then the necessary condition to have a solution is that α is closed.
Remark: this is a topological property, if there is a diffeomorphism between Ω and a ball (with the trivial topology), then if $d\beta = \alpha$, α closed, α is smooth, then it has a solution.
Remark: deRham cohomology:

$$H^P_{dR}(\Omega, \mathbb{R}) = \frac{\{\text{closed p-forms}\}}{\{\text{exact p-forms}\}}.$$

In many cases, the dimensions is finite.
Definition: Let $\Omega, \Omega' \subset \mathbb{R}^n$ be two domains, and $F : \Omega' \to \Omega$, $F(x') = (F_1(x'), F_2(x'), \cdots, F_n(x'))$. If α is a p-form on Ω, write $\alpha = \sum\limits_{|I|=p} \alpha_I dx_I$. We define the pull back operator by

$$F^*(\alpha) = \sum_{|I|=p} \alpha_I(F(x'))dF_{i_1}(x') \wedge \cdots \wedge dF_{i_p}(x').$$

So $F^*(\alpha)$ is a p-form on Ω'.
When $p = 0$, i.e., h is a function, then $F^*(h) = h \circ F$.

Proposition 11.17 *$(F \circ G)^* = G^* \circ F^*$.*

Particular case, if F is a diffeomorphism, then $\exists F^{-1}$, so $(F^{-1})^* \circ F^* = id$ and $(F^*) \circ (F^{-1})^* = id$. Then we can use F as a change of coordinate. Then we can define forms on manifolds: local coordinates + change of coordinates.

Definition: suppose $\Omega \subset \mathbb{R}^n$ and α is an $n-$form on Ω, i.e., $\alpha = hdx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$, then we define the integral

$$\int_\Omega \alpha = \int_\Omega hdx \ \ \text{(Lebesgue integral)}.$$

11.5 Complex Forms

Let $\Omega \subset \mathbb{C}^n$, $z = (z_1, z_2, \cdots, z_n)$, $z_k = x_k + iy_k$, $\bar{z}_k = x_k - iy_k$. Then

$$dz_k = dx_k + idy_k \ \text{ and } \ d\bar{z}_k = dx_k - idy_k,$$

$$dx_k = \frac{dz_k + d\bar{z}_k}{2} \ \text{ and } \ dy_k = \frac{dz_k - d\bar{z}_k}{2i}.$$

Definition: α is a $(p, q)-$form if

$$\alpha = \sum_{\substack{|I|=p \\ |J|=q}} \alpha_{I,J}\, dz_I \wedge d\bar{z}_J.$$

I.e., $\alpha$is a $(p + q)$-form.
Remark: if α is a 2-form, then $\alpha = \alpha_{2,0} + \alpha_{1,1} + \alpha_{0,2}$, where $\alpha_{2,0}$ is a (2,0)-form, $\alpha_{1,1}$ is a (1,1)-form and $\alpha_{0,2}$ is a (0,2)-form.

Definition: we define the operators ∂,$\bar{\partial}$ by

$$\partial = \sum \frac{\partial}{\partial z_k} dz_k$$
$$\bar{\partial} = \sum \frac{\partial}{\partial \bar{z}_k} d\bar{z}_k.$$

Proposition 11.18 *$d = \partial + \bar{\partial}$, $\partial \circ \partial = 0$, $\bar{\partial} \circ \bar{\partial} = 0$, and $\partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0$.*

11.6 Currents (Dual of Forms)

Idea: $(E, \|\cdot\|_E)$ is a normed vector space,

$$E^* = \{L : E \to \mathbb{R} \,:\, \text{linear bounded} = \text{continuous}\}.$$

We know bounded linear operators are continuous, i.e., $|L(v)| \leq \text{const} \cdot \|v\|_E$.

Notation: let $\Omega \subset \mathbb{R}^n$ be a domain, we write

$$\mathfrak{D}^p(\Omega) = \{\text{all smooth p-forms } \alpha \text{ with compact support in } \Omega\}.$$

$\alpha = \sum \alpha_I dx_I$, α_I are smooth functions with compact support, $I = (i_1, i_2, \cdots, i_p)$, $i_1 < i_2 < \cdots < i_p$.

Definition: let $k \in \mathbb{N}$, define $\mathscr{C}^k$ norm:

$$\|\alpha\|_{\mathscr{C}^k} = \sum_{|I| \leq k} \|\alpha_I\|_{\mathscr{C}^k}.$$

where the $\mathscr{C}^k$ norm for functions f is

$$\|f\|_{\mathscr{C}^k} = \sum_{|J| \leq K} \max \left| \frac{\partial^{|J|} f}{\partial x_J} \right|.$$

Definition: a current of degree p (dimension $n-p$) and of order on Ω is a linear form: $T : \mathfrak{D}^{n-p}(\Omega) \to \mathbb{R}$ such that $\forall K \subset \Omega$ compact, $\exists c_K > 0$ with

$$\|T(\alpha)\| \leq c_K \|\alpha\|_{\mathscr{C}^k}$$

$\forall \alpha \in \mathfrak{D}^{n-p}(\Omega)$ with support in K. Notation: $T(\alpha)$ or $\langle T, \alpha \rangle$.
Remark: if order $(T) \leq k$, then we can extent T continuously to the space of $\mathscr{C}^k$ forms of degree $n-p$.
Remark:to "see" T we use test forms α.
Definition: distributions are $n-$currents, i.e., $p = n$. and T acts directly on functions. Distribution of order 0 are known as Radon Measures.
Definition: a distribution T is positive if $T(h) \geq 0$ for all function $\phi \geq 0$.
Note positive distributions is implies it is a positive Radon Measures. In addition, any real Radon measure can be written as the difference of two positive Radon measure. If T is a complex measure, we can write $T = R + is$, R, S are real radon measures.

Example: let $\Omega \subset \mathbb{R}^n$, $a \in \mathbb{R}^n$ be fixed, we define the Dirac mass at a is defined to be $\langle \delta_a, \phi \rangle = \phi(a)$, $\forall$ continuous function ϕ. Note δ_a is a positive measure (it is a distribution of order 0, and $\delta_a(\phi) \geq 0$ if $\phi \geq 0$).
Then we can take linear combination of Dirac mass, to obtain other measures.

Example: $\Omega \subset \mathbb{R}^n$, we define 0$-$current $[\Omega]$, the current of integration on Ω, by

$$[\Omega] : \mathfrak{D}^n \to \mathbb{R}, \quad \alpha \mapsto \int_\Omega \alpha,$$

where α is a smooth $n-$form with compact support. Then $[\Omega]$ is of order 0.

Example: Φ is $\mathscr{C}^0$ $p-$form on Ω. Then Φ defines a $p-$ current of order 0:

$$\Phi : \mathfrak{D}^{n-p}(\Omega) \to \mathbb{R}, \quad \alpha \mapsto \int_\Omega \Phi \wedge \alpha.$$

Exercise: if α is supported by$K$, then $|\int_\Omega \Phi \wedge \alpha| \leq$ const. $\cdot$ volume $(K) \cdot \|\alpha\|_{\mathscr{C}^0}$.

Definition: T is a $p-$current, and γ is a smooth $q-$form. Assume $p+q \leq n$, we define $T \wedge \gamma$ as a $(p+q)-$current, in the following way:

$$\langle T \wedge \gamma, \alpha \rangle = \langle T, \gamma \wedge \alpha \rangle.$$

The order of $T \wedge \gamma$ is smaller or equal to the order of T (less bound). If k is the order of T, we can take γ of class $\mathscr{C}^k$.

Definition: Let T be a p-current of order $\leq k$. We define dT as a $(p+1)-$current of order $\leq k+1$ as follows:

$$\langle dT, \beta \rangle = (-1)^{p+1} \langle T, d\beta \rangle.$$

Example: take $\Omega = \mathbb{R}$, $\chi_{[a,b]}$ is the characteristic function, then $d\chi_{[a,b]} = \delta_b - \delta_a$.

Theorem 11.19 (Green) *let $D \subset \mathbb{R}^2$ be a bounded domain and bD is smooth. Then $[D]$ is a 0-current defined by $\langle [D], \alpha \rangle = \int_D \alpha$; $[bD]$ is a 1-current defined by $\langle [bD], \beta \rangle = \int_{bD} \beta$. Then $d[D] = [bD]$.*

Proof: Let β be any $1-$form, we show $\langle d[D], \beta \rangle = \langle [bD], \beta \rangle$.
β is a one form, so $\beta = Pdx + Qdy$,then

$$\begin{aligned}
\langle d[D], \beta \rangle &= -\langle [D], d\beta \rangle \\
&= -\int_D d\beta \\
&= -\int_D dP \wedge dx + dQ \wedge dy \\
&= -\int_D \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy \\
&= \int_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial X} \right) dx \wedge dy \\
&\overset{\text{Green}}{=} \int_{bD} Pdx + Qdy \\
&= \int_{bD} \beta \\
&= \langle [bD], \beta \rangle
\end{aligned}$$

□

11.7 Manifolds with Boundary

Manifold with smooth boundary are modeled by open sets of the half space $\{x_n \geq 0\} \subset \mathbb{R}^n$.
Manifolds with piecewise smooth boundary can be modeled by open sets of the space $\{x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0\}$.

Consider the Euclidean space $\mathbb{R}^n$ and standard coordinate system $x = (x_1, \cdots, x_n)$. It has a natural orientation. If $h(x)$ is a continuous function with compact support in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} h(x) dx_1 \wedge \cdots \wedge dx_n = \int_{\mathbb{R}^n} h(x) dx.$$

So if $h \geq 0$, these integrals are nonnegative. In other words, we impose that $dx_1 \wedge \cdots \wedge dx_n$ is "positive". Now if we have another orientation using the following coordinates:

$$x_1' = x_2, \;\; x_2' = x_1, \;\; x_3' = x_3, \;\; \cdots \;\; x_n' = x_n.$$

Then in these coordinates, we have

$$dx_1 \wedge \cdots \wedge dx_n = -dx_1' \wedge \cdots \wedge \cdots dx_n'$$

which is negative (for the new coordinates/new orientation).

Definition: a manifold is orientable if it admits an atlas such that all transition maps have positive Jacobian (i.e., the determinant of the Jacobian matrix at each point is positive).
Remark: we can bring the "orientation" of $\mathbb{R}^n$ to X if X is orientable, as the charts can be identified with open

subsets of $\mathbb{R}^n$. And since the Jacobian of the transition is always positive, then the orientation between any two overlapping charts stays the same. That is an $n-$form is positive if it is positive on each chart. So we can define integrals of $n-$forms on X.
Remark: if X is oriented, with bX then bX is also oriented.

Theorem 11.20 (Stoke's Theorem) *Let Z be an oriented submanifold of dimension m with boundary in Ω. Then $d[Z] = (-1)^m[bZ]$ in the sense of currents.*
More specifically: let X be a manifold of dimension n without boundary. Let Z be an oriented submanifold of dimension m with or without boundary of X (in particular, Z is closed in X). Let bZ be the boundary of Z which is a submanifold of dimension $m-1$ without boundary in X (empty when Z has no boundary). Then for every smooth $(m-1)$-form α with compact support in X,

$$\int_{bZ} \alpha = \int_Z d\alpha.$$

Remark: If Z has no boundary then the second integral vanishes. If Z is compact, we don't need to assume that α has compact support.

11.8 Generalized Cauchy Formula

Definition: $u \in L^1_{loc}(\Omega) \Leftrightarrow \int_K |u| < +\infty\ \forall K \subset \Omega$ compact.
Definition: $a \in \mathbb{C}$, $\log|z-a| \in L^1_{loc}$, the function $\log|z-a|$ defines a zero current on $\mathbb{C}$:

$$\langle \log|z-a|, \alpha\rangle = \int_C \log|z-a| \cdot \alpha = \int_K \log|z-a| \cdot h(z) dz \wedge d\bar{z}.$$

where α is a $(1,1)$-form. This is a current of order 0.

Proposition 11.21 $\frac{\partial}{\partial z} \log|z-a| = \frac{dz}{2(z-a)}$, *and* $\frac{\partial}{\partial \bar{z}} \log|z-a| = \frac{d\bar{z}}{2(\bar{z}-\bar{a})}$.

Remark: $\frac{1}{z-a} \in L^1_{loc}$, then $\frac{dz}{z(z-a)}$ is a $(1,0)-$current.

Proof: We need to show that $\forall$ smooth $(0,1)$-form α with compact support, then

$$\langle \partial \log|z-a|, \alpha\rangle = \langle \frac{dz}{2(z-a)}, \alpha\rangle. \quad (*)$$

By definition,

$$\begin{aligned}\langle \partial \log|z-a|, \alpha\rangle &= -\langle \log|z-a|, \partial\alpha\rangle \\ &= -\int_{\mathbb{C}} \log|z-a| \partial\alpha \\ \langle \frac{dz}{2(z-a)}, \alpha\rangle &= \int_{\mathbb{C}} \frac{dz}{2(z-a)}\alpha\end{aligned}$$

For simplicity, let $a = 0$, $\alpha = h(z)d\bar{z}$. h is a smooth function, with compact support.

$$\partial\alpha = \partial h \wedge d\bar{z} = \frac{\partial h}{\partial z} dz \wedge d\bar{z} = (dh - \bar{\partial} h) \wedge d\bar{z} = dh \wedge d\bar{z} = d(hd\bar{z}).$$

We first show $(*)$ holds in $\mathbb{C}^*$, that is $(*)$ holds for α with compact support in $\mathbb{C}^*$. Claim $\int_{\mathbb{C}^*} d(\log|z|\alpha) = 0$ Let D be $\mathbb{D}(0,R) - \mathbb{D}(0,r)$, where r, R are such that the support of α is in D. Then smooth $\log|z|\alpha$ is smooth and vanish in a neighbourhood of $\mathbb{C} \setminus D$. So

$$\int_{\mathbb{C}^*} d(\log|z|\alpha) = \int_{\mathbb{C}^*} d\beta = \int_D d\beta \overset{Green}{=} \pm \int_{bD} \beta = \pm \int_0 = 0.$$

because on bD, β vanishes. Then

$$\begin{aligned}
\int_{\mathbb{C}^*} \partial(\log|z|\alpha) + \bar{\partial}(\log|z|\alpha) &= 0 \\
\int_{\mathbb{C}^*} \partial(\log|z|\alpha) &= 0 \\
\int_{\mathbb{C}^*} \partial \log|z| \wedge \alpha + \int_{\mathbb{C}^*} \log|z| \wedge \partial\alpha &= 0 \\
\int_{\mathbb{C}^*} \frac{dz}{2z} \wedge \alpha + \int_{\mathbb{C}^*} \log|z| \partial\alpha &= 0
\end{aligned}$$

Step 2: we prove the general case for $(*)$ in $\mathbb{C}$.
Let χ_r be the cut-off function, s.t., $0 \leq \chi_r \leq 1$, $\chi_r \in \mathscr{C}^\infty$, $|\chi_r'| \leq \frac{300}{r}$. Then

$$-\langle \log|z|, \partial\alpha \rangle = -\langle \log|z|, \partial(\chi_r\alpha) \rangle - \langle \log|z|, \partial((1-\chi_r)\alpha) \rangle = -I_1 - I_2.$$

For I_2, $I_2 = \langle \frac{dz}{2z}, (1-\chi_r)\alpha \rangle$, as $r \to 0$, then $I_2 \to \langle \frac{dz}{2z}, \alpha \rangle$.
For I_1, $I_1 = \int_{\mathbb{D}(2,r)} \log|z| (\partial\chi_r \wedge \alpha + \chi_\alpha \wedge \alpha)$. Since $\partial\chi_r \wedge \alpha$ is $O(\frac{1}{r}$ and $\chi_r \wedge \partial\alpha$ is $O(1)$, then

$$|I_1| \lesssim \int_{\mathbb{D}(2r)} |\log|z|| \frac{1}{r} d\mu \approx \frac{1}{2} |\log r| \cdot \frac{r^2}{=} r \log r.$$

Let $r \to 0$, then $I_1 \to 0$. So in conclusion, we have

$$-\langle \log|z|, \partial\alpha \rangle = \langle \frac{dz}{2z}, \alpha \rangle + 0.$$

$\square$

Proposition 11.22 *$\partial\bar{\partial} \log|z-a| = -i\pi\delta_a$, or equivalently $\frac{i\partial\bar{\partial}}{\pi} \log|z-a| = \delta_a$. Define operator $d^c = \frac{1}{2\pi}(\bar{\partial} - \partial) \Rightarrow dd^c = \frac{i\partial\bar{\partial}}{\pi}$. Then $dd^c \log|z-a| = \delta_a$.*

Proof:

$$dd^c = (\partial + \bar{\partial}) \circ \frac{1}{2\pi}(\bar{\partial} - \partial) = \frac{1}{2\pi}(\partial\bar{\partial} + \bar{\partial}\bar{\partial} - \partial\partial - \bar{\partial}\partial) = \frac{\partial\bar{\partial}}{\pi}.$$

Next, for simplicity, take $a = 0$. We show $\frac{i\partial\bar{\partial}}{\pi}\log|z| = \delta_0$. Since $\log|z|$ is harmonic in $\mathbb{C}^*$, then $\partial\bar{\partial}\log|z| = 0$ on $\mathbb{C}^*$. $\forall h$ smooth function with compact support in $\mathbb{C}$, we want

$$\langle\partial\bar{\partial}\log|z|, h\rangle = \langle -i\pi\delta_0, h\rangle = -i\pi h(0).$$

By definition,

$$\begin{aligned}
\langle\partial\bar{\partial}\log|z|, h\rangle &= -\langle\log|z|, \partial\bar{\partial}h\rangle \\
&= -\int_{\mathbb{C}}\log|z|\partial\bar{\partial}h \\
&= \lim_{\epsilon\to 0} -\int_{\mathbb{C}}\frac{1}{2}\log(|z|^2+\epsilon^2)\partial\bar{\partial}h \\
&= \lim_{\epsilon\to 0} -\langle\frac{1}{2}\log(|z|^2+\epsilon^2), \partial\bar{\partial}h\rangle \\
&= \lim_{\epsilon\to 0}\langle\partial\bar{\partial}\frac{1}{2}\log(|z|^2+\epsilon^2), h\rangle \\
\partial\bar{\partial}\frac{1}{2}\log(|z|^2+\epsilon^2) &= \frac{1}{8}\Delta\log(r^2+\epsilon^2)(-2ir dr\wedge d\theta) \\
&= -\frac{i}{4}r\Delta\log(r^2+\epsilon^2)dr\wedge d\theta \\
&= -\frac{i}{4}r\left[\left(\frac{2}{r^2+\epsilon^2} - \frac{4r^2}{(r^2+\epsilon^2)^2}\right) + \frac{2}{r^2+\epsilon^2}\right]dr\wedge d\theta \\
&= -\frac{i}{4}r\left[\frac{4}{r^2+\epsilon^2} - \frac{4r^2}{(r^2+\epsilon^2)^2}\right]dr\wedge d\theta \\
&= -ir\frac{\epsilon^2}{(r^2+\epsilon^2)^2}dr\wedge d\theta.
\end{aligned}$$

Then our integral $= -i\int_{\mathbb{C}} h\frac{\epsilon^2 r}{(r^2+\epsilon^2)^2 dr\wedge d\theta} = -ih(0)\cdot\pi$. □

Theorem 11.23 *Suppose $g \in \mathscr{C}^\infty(\mathbb{C})$. Define*

$$h(w) = -\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{g(z)}{z-w}dz\wedge d\bar{z}.$$

Then $\frac{\partial h}{\partial\bar{z}} = g$. (solution to $\bar{\partial}$-equation).
The solution is unique up to a holomorphic function.

Proof: Let $I = \int_{\mathbb{C}}\frac{g(z)}{z-w}dz\wedge d\bar{z}$, then

$$I = \langle\frac{dz}{z-w}, g d\bar{z}\rangle$$

□

Theorem 11.24 (Generalized Cauchy Formula) *Let $D \subset \mathbb{C}$ be a bounded domain, bD is piecewise $\mathscr{C}^1$. f is a smooth function on $\bar{D}$. Then $\forall w \in D$,*

$$f(w) = \frac{1}{2i\pi}\int_{bD}\frac{f(z)dz}{z-w} - \frac{1}{2i\pi}\int_D\frac{\partial f}{\partial\bar{z}}\frac{1}{z-w}dz\wedge d\bar{z}.$$