- G non-empty $\implies \exists H \subseteq G$ with $\delta(H) \geq \frac{1}{2}\bar{d}(G)$.
- **Havel-Hakimi:** $(d_1, d_2, \cdots, d_n)$ is graphic iff $(d_2 - 1, \cdots, d_{d_1+1} - 1, d_{d_1+2}, \cdots, d_n)$ is graphic.
- **Erdos-Gallai:** non-increasing sequence $(d_1, d_2, \cdots, d_n)$ is graphic iff $\sum_{i=1}^n d_i$ is even and $\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{d_i, k\}$ for all k.
- G contains a cycle $\implies 2\text{diam}(G) + 1 \geq \text{girth}(G)$.
- $\delta(G) \geq k-1 \implies G$ contains a subgraph isomorphic to any order-k tree.
- **Matching:** $\nu(G) \leq \frac{|V(G)|}{2}$; **Vtx cover:** $\tau(G) \geq \nu(G)$.
- **Clique:** $\omega(G)$; **Independence:** $\alpha(G) = |V(G)| - \tau(G)$.
- **Kneser graph:** $KG(n,k)$, vertices are $\mathcal{P}([n])$ and edges exist between disjoint subsets.
- # pairwise intersecting subsets in $KG(n,k) \leq C_{k-1}^{n-1}$.
- Matching is maximum iff no augmenting path.
- **König's Thm:** $\tau(G) = \nu(G)$ for bipartite G.
- **Hall's Thm:** bipartite G contains a matching perfect to A iff $\forall S \subseteq A$, $|N_G(S)| \geq |S|$.
- d-regular bipartite graphs can be decomposed into d perfect matchings.
- $\alpha\big(L(G)\big) = \nu(G)$.
- **Tutte's Thm:** G has a perfect matching iff $\forall U \subseteq V(G)$, $o(G-U) \leq |U|$.
- **Petersen's Thm:** connected 3-regular graph with no cut edge contains a perfect matching.
- **Connectivity:** $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.
- $\kappa(G) \geq 2 \implies G$ contains a cycle.
- **Mader's Thm:** $\bar{d}(G) \geq 4k \implies G$ contains a k-connected subgraph.
- **Block:** maximal connected cut-vertex-free subgraph. Distinct blocks share at most 1 vertex and no edge is in different blocks.
- **Block graph:** bipartite on cut vertices and blocks and tree for connected G.
- **Ear decomposition:** G is 2-connected iff it can be constructed by appending H-paths to a cycle.
- **Menger's Thm:** min size of separating sets for (A,B) = max # of vertex-disjoint A-B paths.
  1. Local: min # vtxs separating a, b = max # internally vertex-disjoint a-b paths; min # edges separating a, b = max # edge-disjoint a-b.
  2. Global: k-connected iff $\forall u, v$, $\exists k$ internally vertex-disjoint u-v paths. k-edge-connected iff $\forall u, v$, $\exists k$ edge-disjoint u-v paths.
- **Colour:** $\omega(G), \frac{v(G)}{\alpha(G)} \leq \chi(G) \leq \frac{1}{2} + \sqrt{2e(G) + \frac{1}{4}}, \Delta(G) - 1, \delta(H) + 1 \leq 1 + \max_{H \subseteq G} \delta(H)$; $\chi'(G) = \chi\big(L(G)\big)$.
- k-critical $\implies \delta(G) \geq k-1$.
- **Brook's Thm**: connected graph not isomorphic to K_n or odd cycle: $\chi(G) \leq \Delta(G)$.
- **k-constructible**: $G \cong K_k$; $\exists k$-constructible H with $x, y \in V(H)$ and $xy \notin E(H)$ s.t. $G \cong (H + xy)/xy$; $\exists k$-constructible H_1, H_2 with $V(H_1) \cap V(H_2) = \{x\}$ and $y_1x \in E(H_1), y_2x \in E(H_2)$ s.t. $G \cong H_1 \cup H_2 - y_1x - y_2x + y_1y_2$.
- **Hajós's Thm:** $\chi(G) \geq k$ iff $\exists k$-constructible subgraph.
- k-critical $\implies k$-constructible.
- $\chi'(K_n) \geq \begin{cases} n-1 & \text{if } n \text{ even} \\ n & \text{otherwise} \end{cases}, \Delta(G)$.
- $\chi'(G) \geq \Delta(G)$, bipartite $\chi'(G) = \Delta(G)$ (König).
- **Vizing's Thm:** (I)$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ (II).
- **Perfect graph:** $\chi = \omega$ for all induced subgraphs. E.g. $K_n, \overline{K_n}$, (for bip) non-empty G, $\overline{G}$, $L(G)$.
- **Weak perfect graph:** perfect iff complement is perfect.
- **Strong perfect graph:** perfect iff no odd hole nor anti-hole.
- **Euler's Formula:** $n - m + \ell = 2$.
- Planar graph of order ≥ 3 has $\leq \big(3v(G) - 6\big)$ edges.
- **Fáry's Thm:** planar G can be draw with straight lines.
- H **topological minor** of G if G has a subdivision of H.
- H is **minor** of G if it can be obtained by deleting edges, deleting vertices or contracting edges.
- Minor with $\Delta \geq 3$ is topological minor.
- **Branching sets:** partition $\mathcal{V} := \{V_1, V_2, \cdots, V_n\}$ s.t. $\forall V_x \in \mathcal{V}$,
  1. $x \in V(G)$,
  2. $G[V_x]$ is connected, and
  3. $\exists V_y \in \mathcal{V} - \{V_x\}$ s.t. $\exists uv \in E(G)$ with $u \in V_x$ and $v \in V_y$.
- **Wagner's Thm:** planar iff K_5 and $K_{3,3}$ are not minor.
- **Kuratowski's Thm:** planar iff contain no subdivision of K_5 nor $K_{3,3}$.
- **Hanani-Tutte Thm:** planar iff has drawing where all pairs of edges cross even number of times.
- **Geometric dual:** $|V(G^*)| = |\mathcal{F}(G)|, |E(G^*)| = |E(G)|, |\mathcal{F}(G^*)| = |V(G)|$.
- $S \subseteq E(G)$ induces a cycle iff S^* induces a min edge cut.
- **Abstract dual:** $\exists$bijection $f\colon E(G) \to E(G^*)$ s.t. $\forall S \subseteq E(G)$, S induces a cycle iff $S^* := f(S)$ is a min edge cut.
- **Whitney's Thm:** A graph is planar if and only if it has an abstract dual.
- **Four-colour Thm:** planar $\chi \leq 4$.
- **Grötzsch's Thm:** triangle-free planar $\chi \leq 3$.
- **Eulerian circuit:** uses all edges exactly once.
- Connected multigraph is Eulerian iff every vtx has an even degree.
- $2k$ vertices with odd degrees $\implies$ min # trails to decompose G is $\max\{k, 1\}$.
- K_n, $K_{m,n}$, Q^d and platonic graphs are Hamiltonian. Petersen is not, but is hypo-Hamiltonian.
- G is Hamiltonian $\implies$ # CCs in $G - X \leq |X|$.
- **Dirac's Thm:** G with order ≥ 3 and $\delta(G) \geq \frac{n}{2}$ is Ham.
- n-vertex Hamiltonian graph contains a path of length $\geq \min\{n-1, 2\delta(G)\}$.
- graph without any path of length k, then $e(G) \leq \frac{k-1}{2}n$.
- **Ore's Thm:** graph of order ≥ 3 is Hamiltonian if $\forall u \not\sim v$, $d_G(u) + d_G(v) \geq n$.

- **Hamiltonian closure:** connect all non-adjacent vertices with degree sum $\geq n$.
- Hamiltonian iff closure is Hamiltonian.
- **Chvátal-Erdös Thm:** G with order ≥ 3 is Hamiltonian if $\kappa(G) \geq \alpha(G)$.
- 2-connected k-regular graphs of order $\leq 3k$ are Ham.
- **Robinson-Wormald Thm:** almost all regular graphs are Hamiltonian.
- **Smith's Thm:** d-regular graph where d is odd, $\forall e$, $\exists$ even # Hamiltonian cycles containing e.
- Every cubic Hamiltonian graph contains at least 3 Hamiltonian cycles.
- **Hamiltonian sequence:** all degree sequences element-wise at least $\boldsymbol{a}$ produce Hamiltonian graphs.
- **Chvátal's Thm:** if $n \geq 3$, a non-decreasing integer sequence $(a_1, a_2, \cdots, a_n)$ is Hamiltonian iff $\forall i < \frac{n}{2}$, $a_i \leq i \implies a_{n-i} \geq n - i$.
- An integer sequence $\{a_i\}_{i=1}^n \coloneqq (a_1, a_2, \cdots, a_n)$ is path-Hamiltonian iff for every $i \leq \frac{n}{2}$, $a_i < i$ implies $a_{n+1-i} \geq n - i$.
- **Vertex-transitive graph:** $\forall v_1, v_2 \in V(G)$, $\exists$ isomorphism $f\colon V(G) \to V(G)$ s.t. $f(v_1) = v_2$.
- **Babai's Thm:** Every connected vertex-transitive graph G contains a cycle of length at least $\sqrt{3n}$.
- **Oriented graph:** digraph with no bi-directional edges.
- $\sum_{v \in V(D)} d^+(v) = \sum_{v \in V(D)} d^-(v) = e(D)$
- Eulerian iff $d^+(v) = d^-(v)$ for all v.
- **Kernel:** $S \subseteq V(D)$ s.t. $\forall u, v \in S$, $uv, vu \notin E(D)$ and $\forall w \in V(D) - S$, $\exists w' \in S$ with $ww' \in E(D)$.
- Directed odd cycles have no kernel.
- **Richardson's Thm:** directed-odd-cycle-free graph has a kernel.
- Every tournament has a Hamiltonian path. Acyclic tournament is called transitive.
- A digraph is acyclic iff it has a topological ordering.
- A tournament contains a unique Hamiltonian path iff it's transitive.
- A non-transitive tournament contains a directed triangle.
- **King:** a vertex in a tournament s.t. every other vertex can reach it by a path of length at most 2.
- **Landau's Thm:** Every tournament has a king.
- **Quasi-kernel:** a set of "king"s.
- **Chvátal-Lovász Thm:** Every digraph has a quasi-kernel.
- K_n: $\nu = \left\lfloor \frac{n}{2} \right\rfloor, \tau = n-1, \alpha = 1, \kappa = \kappa' = n-1, \chi = n$.
- C_n: $\nu = \alpha = \left\lfloor \frac{n}{2} \right\rfloor, \tau = \left\lceil \frac{n}{2} \right\rceil, \kappa = \kappa' = 2, \chi = 2$ if $n|2$ else 3.
- P_n: $\nu = \alpha = \left\lfloor \frac{n+1}{2} \right\rfloor, \tau = \left\lceil \frac{n}{2} \right\rceil, \kappa = \kappa' = 1$.
- $K_{m,n}$: $\nu = \tau = \kappa = \kappa' = \min\{m, n\}, \alpha = \max\{m, n\}$.
- Q^d: $\nu = \tau = 2^{d-1}, \alpha = 2^{d-1}, \kappa = \kappa' = d$.
- Petersen graph: $\nu = 5, \tau = 6, \alpha = 4, \chi = 3$.