Contents

1 Sets and Classes

1.1 Classes

Russell's Paradox states the following:

Russell's Paradox

Let X be the set of all sets which do not contain themselves, i.e.,

$$X = \{S : S \notin S\}.$$

Now consider X. If $X \in X$, it means that X contains itself and should not be a member of X, i.e., $X \in X \implies X \notin X$. If $X \notin X$, it means that X does not contain itself and therefore should be a member of X, i.e. $X \notin X \implies X \in X$. Hence, we have a paradox and such a set X does not exist.

However, in some cases it is still useful to consider the "set" of all sets for practical reasons. Therefore, we introduce the notion of a *class* to avoid Russell's Paradox.

Definition 1.1.1 ▸ Class

Let ϕ be some formula and $\boldsymbol{u}$ be a vector, the collection

$$\mathbb{C} = \{X : \phi(X, \boldsymbol{u})\}$$

is called a **class** of all sets satisfying $\phi(X, \boldsymbol{u})$, where $\mathbb{C}$ is said to be **definable** from $\boldsymbol{u}$. Equivalently, we say that

$$X \in \mathbb{C} \iff \phi(X, \boldsymbol{u}).$$

In particular, if $\mathbb{C} = \{X : \phi(X)\}$, i.e., ϕ only has one free variable, then we say that $\mathbb{C}$ is **definable**.

Remark. It is easy to see that every set X is a class given by $\{x : x \in X\}$.

Intuitively, two classes are equal if they contain exactly the same members. We are able to give the following rigorous version of the notion of equality:

Definition 1.1.2 ▸ Equality between Classes

Let $\mathbb{C} = \{X : \phi(X, \boldsymbol{u})\}$ and $\mathbb{D} = \{X : \psi(X, \boldsymbol{v})\}$, we say that $\mathbb{C} = \mathbb{D}$ if for all X,

$$\phi(X, \boldsymbol{u}) \iff \psi(X, \boldsymbol{u}).$$

There are clearly two types of classes — the ones which are also sets and the ones which are not. Formally, this is put as follows:

Definition 1.1.3 ▸ Proper Class

A class $\mathbb{C}$ is said to be a **proper class** if $\mathbb{C} \neq X$ for all sets X.

Like sets, we can define subclasses:

Definition 1.1.4 ▸ Subclass

Let $\mathbb{A}$ and $\mathbb{B}$ be classes. We say that $\mathbb{A}$ is a **subclass** of $\mathbb{B}$ if every member of $\mathbb{A}$ is also a member of $\mathbb{B}$, i.e.,

$$\mathbb{A} \subseteq \mathbb{B} \iff (X \in \mathbb{A} \implies X \in \mathbb{B}).$$

We shall also define the operations applicable to classes:

Definition 1.1.5 ▸ Intersection, Union and Difference

Let $\mathbb{A}$ and $\mathbb{B}$ be classes. The **intersection**, **union** and **difference** between $\mathbb{A}$ and $\mathbb{B}$ are given by

$$\begin{aligned}
\mathbb{A} \cap \mathbb{B} &\coloneqq \{X : X \in \mathbb{A} \wedge X \in \mathbb{B}\}, \\
\mathbb{A} \cup \mathbb{B} &\coloneqq \{X : X \in \mathbb{A} \vee X \in \mathbb{B}\}, \\
\mathbb{A} - \mathbb{B} &\coloneqq \{X : X \in \mathbb{A} \wedge X \notin \mathbb{B}\}
\end{aligned}$$

respectively.

Finally, we shall introduce the universal class:

Definition 1.1.6 ▸ Universal Class

The **universal class** is the class of all sets, denoted by

$$V \coloneqq \{X : X = X\}.$$

Remark. It is easy to prove that the universal class is **unique**.

2 Axiomatic Set Theory

2.1 Axioms of Zermelo-Fraenkel (ZF)

In Naïve Set Theory, we define a set as "a collection of mathematical objects which satisfy certain definable properties". However, such a definition is problematic (e.g. it leads to the Russell's Paradox). Thus, instead of viewing a set as a clearly defined mathematical object, we can think a set as an object entirely defined by a set of axioms to which it complies. In this sense, we avoid paradoxes by making the notion of a set undefined but only specify rigorously the axioms a set must satisfy.

2.1.1 Extensionality

Axiom 2.1.1 ▸ Extensionality

Let X and Y be sets, then $X = Y$ if for all u, $u \in X$ if and only if $u \in Y$.

An immediate result from Axiom 2.1.1 is that there exists a set X such that $X = X$, i.e. every set equals itself. Moreover, we can also prove the following:

Theorem 2.1.2 ▸ The Empty Set

The set which has no elements is unique.

Proof. Let X be a set with no elements. Note that this means that for all u, $u \notin X$.

Let Y be another set. Note that the statement $u \in X \implies u \in Y$ is vacuously true. Suppose that Y has no elements, then similarly for all u, the statement $u \in Y \implies u \in X$ is also vacuously true.

Therefore, for all u, we have proven that $u \in X$ if and only if $u \in Y$. By Axiom 2.1.1, this means that $X = Y$, i.e. the set with no elements is unique. □

This set with no elements is known as the **empty set**, denoted by $\varnothing$.

2.1.2 Pairing

Axiom 2.1.3 ▸ Pairing

For all u and v, there exists a set X such that for all z, $z \in X$ if and only if $z = u$ or $z = v$.

Remark. Note that Axiom 2.1.3 essentially says that given any sets u and v, there exists a set whose elements are exactly u and v.

This allows us to formally define the notion of a *pair* as follows:

Definition 2.1.4 ▸ Pair

For all a, b, the **pair** $\{a, b\}$ is defined to be the set C such that for all x, $x \in C$ if and only if $x = a$ or $x = b$.

Remark. In particular, we can define the **singleton** $\{a\}$ to be the pair $\{a, a\}$.

Furthermore, given any a and b, we can prove by Extensionality that the the pair $\{a, b\}$ is unique:

Theorem 2.1.5 ▸ Uniqueness of Pairs

For all a, b, the pair $\{a, b\}$ is unique.

Proof. Let $C := \{a, b\}$ and $D := \{a, b\}$. Suppose $x \in C$, then $x = a$ or $x = b$, which means $x \in D$. Similarly, suppose $y \in D$, we can prove that $y \in C$. Therefore, for all x, we have $x \in C$ if and only if $x \in D$. By Axiom 2.1.1, this means that $C = D$, i.e., the pair $\{a, b\}$ is unique. $\square$

We can further define the notion of an *ordered pair*:

Definition 2.1.6 ▸ Ordered Pair

For all a and b, the **ordered pair** (a, b) is defined to be the set $\{\{a\}, \{a, b\}\}$.

Again, one can use Extensionality to prove that such an ordered pair is always unique and that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. The notions of pair and ordered pair can be extended to ordered and un-ordered n-tuples, which will have similar properties as we have proven as above. Recursively, we can write the following definition:

Definition 2.1.7 ▶ Ordered n-tuple

The ***n*-tuple** is defined as

$$(a_1, a_2, \cdots, a_n) = ((a_1, a_2, \cdots, a_{n-1}), a_n).$$

By Extensionality, we can similarly prove that two ordered n-tuples $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ if and only if $a_i = b_i$ for $i = 1, 2, \cdots, n$.

2.1.3 Separtion

Axiom 2.1.8 ▶ Axiom Schema of Separation

If P is a property with parameter p, then for all X and p there exists a set

$$Y := \{u \in X : P(u, p)\}.$$

The above axiom justifies our set-builder notation

$$\{x : \varphi(x, \boldsymbol{p})\},$$

where φ is some formula and $\boldsymbol{p}$ is an ordered n-tuple of parameters.

Alternatively, we can write Axiom Schema 2.1.8 in the following form:

Let $\mathbb{C} = \{u : \varphi(u, \boldsymbol{p})\}$ be a class, then for all sets X there exists a set Y such that $\mathbb{C} \cap X = Y$.

Consequently, the intersection and the difference between two sets is a set, which can be defined as

$$X \cap Y := \{u \in X : u \in Y\} \quad \text{and} \quad X - Y := \{u \in X : u \notin Y\}.$$

Suppose that there exists some set X such that $X = X$, we can use Separtion to define the empty set as

$$\varnothing := \{u : u \neq u\}.$$

We shall define other notions related to Separation Axioms:

Definition 2.1.9 ▶ Disjoint

Two sets X and Y are called **disjoint** if $X \cap Y = \varnothing$.

Definition 2.1.10 ▸ Unary Intersection

Let $\mathbb{C}$ be a non-empty class of sets, we define the **unary intersection** of $\mathbb{C}$ to be

$$\bigcap \mathbb{C} := \{u : u \in X \text{ for all } X \in C\}.$$

Note that the unary intersection helps us define the intersection of two sets as

$$X \cap Y = \bigcap \{X, Y\}.$$

2.1.4 Union

Axiom 2.1.11 ▸ Axiom of Union

For all X, there exists a set $Y = \bigcup X$ whose elements are all the elements of all elements of X, i.e.

$$Y := \{u \in U : U \in X\}.$$

Remark. We often call $\bigcup X$ the **unary union** of X.

The unary union defines the union of two sets as

$$X \cup Y = \bigcup \{X, Y\}.$$

One can prove that union between sets is **associative**. In general, we can also see that

$$\{a_1, a_2, \cdots, a_n\} = \bigcup_{i=1}^{n} \{a_i\}.$$

In addition, we can also define the notion of *symmetric difference*:

Definition 2.1.12 ▸ Symmetric Difference

The **symmetric difference** between two sets X and Y is defined as

$$X \triangle Y := \{u : u \in X \cup Y, u \notin X \cap Y\} = (X - Y) \cup (Y - X).$$

2.1.5 Power Set

Axiom 2.1.13 ▸ Axiom of Power Set

For all X, there exists a set $Y = \mathcal{P}(X)$, known as the ***power set*** *of X, such that*

$$Y \coloneqq \{U : U \subseteq X\}.$$

This allows us to define the notion of the *Cartesian product* (or simply the *product*) of two sets:

Definition 2.1.14 ▸ Cartesian Product

Let X and Y be sets. The **Cartesian product** of X and Y is defined as the set

$$X \times Y \coloneqq \{(x, y) : x \in X, y \in Y\}.$$

Remark. Note that $X \times Y$ is a set because $X \times Y \subseteq \mathcal{P}(X \cup Y)$.

Similar to how we defined n-tuples, we can also define Cartesian products of countably many sets recursively.

Definition 2.1.15 ▸ Cartesian Product of Countably Many Sets

et $n \in \mathbb{N}^+$ and let X be a set, we define

$$X^n \coloneqq \prod_{i=1}^{n} X = \left(\prod_{i=1}^{n-1} X\right) \times X.$$

Axiom 2.1.16 ▸ Axiom of Infinity

There exists an infinite set.

Axiom 2.1.17 ▸ Axiom Schema of Replacement

If a class F is a function, then for all X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.

Axiom 2.1.18 ▸ Axiom of Regularity

For every non-empty set X, there exists some $Y \in X$ such that $Y \cap X = \varnothing$.

Remark. Axiom 2.1.18 is sometimes known as the **Axiom of Foundation**. A direct result from it is that for all sets X, there exists some $x \in X$ such that $x \nsubseteq X$.

Furthermore, we can use Axiom 2.1.18 to prove the following seemingly trivial result:

Theorem 2.1.19

There is no set A such that $A \in A$.

Proof. If $A = \varnothing$, it is immediate that $A \notin A$ by definition.

Suppose that there exists a non-empty set A such that $A \in A$. Note that $A \in \{A\}$, so

$$A \cap \{A\} = A.$$

However, by Axiom 2.1.18, since A is the only member of $\{A\}$, we have

$$A \cap \{A\} = \varnothing,$$

which is a contraction. Therefore, there exists no set A such that $A \in A$. □

Additionally, we also introduce the Axiom of Choice:

Axiom 2.1.20 ▸ Axiom of Choice

*For every X with $\varnothing \notin X$, there exists a **choice function***

$$f : X \to \bigcup X$$

such that for all $S \in X$, we have $f(S) \in S$.

Remark. Essentially, the choice function maps every set which is a member of some family of sets to one and only one element in that set.