Contents

1

Topology

1.1 Topological Spaces

Definition 1.1.1 ▸ Topology

A **topology** on a set X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ such that
- $\varnothing, X \in \mathcal{T}$;
- for any index set I, if $\{X_i : i \in I\} \subseteq \mathcal{P}(\mathcal{T})$, then $\bigcup_{i\in I} X_i \in \mathcal{T}$;
- for any $X_1, X_2, \cdots, X_n \in \mathcal{T}$, $\bigcap_{i=1}^{n} X_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is said to be a **topological space**. A subset $Y \subseteq X$ is **open** if $Y \in \mathcal{T}$.

Remark. For any set X, we define $\{\varnothing, X\}$ as the *trivial topology* on X, $\mathcal{P}(X)$ as the *discrete topology*, and $\{X \backslash U : U \subseteq X \text{ is finite}\}$ as the *co-finite topology*.

The set $\{(-\alpha, \alpha) : \alpha > 0\} \cup \{\mathbb{R}, \varnothing\}$ defines a topology on $\mathbb{R}$. This example also demonstrates why it is crucial to only consider closure under finite intersections when defining a topology, because

$$\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = [-1, 1] \notin \mathcal{T}.$$

We now seek a systematic method to generate a topology given any set. The idea here is to make use of a *cover*.

Definition 1.1.2 ▸ Basis

A **basis** for a topology on X is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that
- for any $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$;
- for any $x \in X$ and $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

It may be useful to see a basis as a cover of a set with the second additional property as stated in the above definition. Notice that the first property of the basis $\mathcal{B}$ is basically saying that

$$X \subseteq \bigcup \mathcal{B},$$

i.e., $\mathcal{B}$ is a cover of X.

Given any basis $\mathcal{B}$ for some topology on X, a set generated by $\mathcal{B}$ can be defined as

$$\mathcal{T} := \{U \subseteq X : \text{ for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\}$$

We will show that $\mathcal{T}$ is a topology on X. First, it is clear that $\varnothing, X \in \mathcal{T}$.

Let I be an index set and $\{X_i : i \in I\} \subseteq \mathcal{P}(\mathcal{T})$ be any collection of subsets of X. Notice that for any $x \in \bigcup_{i \in I} X_i$, there exists some $j \in I$ such that $x \in X_j \subseteq \mathcal{T}$. According to our construction, this means that there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq X_j \subseteq \mathcal{T}$. Therefore, $\bigcup_{i \in I} X_i \subseteq \mathcal{T}$ as desired.

To prove that $\mathcal{T}$ is closed under finite intersection, we consider the following lemma:

Lemma 1.1.3 ▶ Finite Intersection of Elements in Basis Is Covered

Let $\mathcal{B}$ be a basis for a topology on X and $B_1, B_2, \cdots, B_n \in \mathcal{B}$, then for any $x \in \bigcap_{i=1}^n B_i$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcap_{i=1}^n B_i$.

Proof. The case where $n = 1$ is trivial by taking $B = B_1$. Suppose that there is some integer $k \geq 1$ such that for any $B_1, B_2, \cdots, B_k \in \mathcal{B}$ and any $x \in \bigcap_{i=1}^k B_i$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcap_{i=1}^k B_i$. Take any $B_{k+1} \in \mathcal{B}$. It is clear that for any $x \in \bigcap_{i=1}^{k+1} B_i$, there exists some $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^{k} B_i.$$

Notice that $x \in B_{k+1} \in \mathcal{B}$, so we know that $x \in B \cap B_{k+1}$. By Definition 1.1.2, this means that there exists some $B' \in \mathcal{B}$ such that

$$x \in B' \subseteq B \cap B_{k+1} \subseteq \bigcap_{i=1}^{k+1} B_i.$$

□

Now, suppose $X_1, X_2, \cdots, X_n \in \mathcal{T}$ are finitely many subsets of X. Take any $x \in \bigcap_{i=1}^n X_i$. It is clear that $x \in X_i$ for each $i = 1, 2, \cdots, n$. Therefore, for each $i = 1, 2, \cdots, n$, there exists some $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq X_i$. By Lemma 1.1.3, this means that there exists some set $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^{n} B_i \subseteq \bigcap_{i=1}^{n} X_i.$$

Therefore, $\bigcap_{i=1}^n X_i \in \mathcal{T}$. So this set $\mathcal{T}$ generated by $\mathcal{B}$ is indeed a topology on X.

The following proposition further shows that the topology generated by a basis $\mathcal{B}$ is the set of all possible unions of elements in $\mathcal{B}$:

Proposition 1.1.4 ▶ Equivalent Construction of Topologies Generated from Bases

Let X be any set. If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X, then

$$\mathcal{T} = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \in \mathcal{P}(\mathcal{B}) \right\}.$$

Proof. Denote

$$\mathcal{T}_{\mathcal{B}} \coloneqq \{U \subseteq X : \text{ for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\}.$$

It suffices to prove that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. Take any $T \in \mathcal{T}$, then there exists some $V \in \mathcal{P}(\mathcal{B})$ such that $T = \bigcup_{A \in \mathcal{V}} A$. This means that for every $t \in T$, there exists some $B_t \in \mathcal{V}$ such that $t \in B_t \subseteq T$. Therefore, $T \in \mathcal{T}_{\mathcal{B}}$. Conversely, for any $S \in \mathcal{T}_{\mathcal{B}}$, there exists some $B_s \in \mathcal{B}$ for each $s \in S$ such that $s \in B_s$. Denote $U \coloneqq \{B_s : s \in S\} \in \mathcal{P}(\mathcal{B})$, then it is clear that $S \subseteq \bigcup_{B \in U} B$. Since $B_s \subseteq S$ for each $s \in S$, we have $\bigcup_{B \in U} B \subseteq S$, which implies that $S = \bigcup_{B \in U} B$. This means that $S \in \mathcal{T}$. Therefore, $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$, which means that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. □

Next, we define a special topology in Euclidean spaces using open balls.

Definition 1.1.5 ▶ Standard Topology

For any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and any $r > 0$. Denote the Euclidean open ball centred at $\boldsymbol{x}$ with radius r by

$$B_r(\boldsymbol{x}) \coloneqq \left\{ \boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n : \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} < r \right\}$$

The **standard topology** on $\mathbb{R}^n$ is the set generated by the basis

$$\mathcal{B} \coloneqq \{B_r(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n, r \in \mathbb{R}^+\}.$$

It may be helpful to actually show that this set $\mathcal{B}$ is indeed a basis of a topology on $\mathbb{R}^n$. The fact that $\mathcal{B}$ is a cover for $\mathbb{R}^n$ is trivial enough. Take any $\boldsymbol{x} \in \mathbb{R}^n$ and balls $B_\alpha(\boldsymbol{x}_1), B_\beta(\boldsymbol{x}_2) \in \mathcal{B}$ such that $\boldsymbol{x} \in B_\alpha(\boldsymbol{x}_1) \cap B_\beta(\boldsymbol{x}_2)$ (the existence of these 2 balls is again trivial enough). Take

$$r = \min\{\alpha - \|\boldsymbol{x} - \boldsymbol{x}_1\|, \beta - \|\boldsymbol{x} - \boldsymbol{x}_2\|\}.$$

Clearly, $r > 0$ and $\boldsymbol{x} \in B_r(\boldsymbol{x})$, so we are done.

Now, we discuss the analogue of the subset relation in topologies.

Definition 1.1.6 ▸ Fineness and Coarseness

Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on some set X. We say that $\mathcal{T}$ is **finer** than $\mathcal{T}'$, or equivalently, that $\mathcal{T}'$ is **coarser** than $\mathcal{T}$, if $\mathcal{T}' \subseteq \mathcal{T}$.

Observe that any topology of X must be a subset of $\mathcal{P}(X)$, which is the discrete topology on X, so the discrete topology is the finest topology on a set.

Remark. For any basis $\mathcal{B}$ for a topology on X, the topology generated by $\mathcal{B}$ is the coarsest topology containing $\mathcal{B}$.

The above remark is easy to verify. Let $\mathcal{T}$ be any topology on X with $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}_{\mathcal{B}}$ be the topology generated by $\mathcal{B}$. For any $T \in \mathcal{T}_{\mathcal{B}}$, by Proposition 1.1.4, there exists some $\mathcal{V} \subseteq \mathcal{B}$ such that $T = \bigcup_{A \in \mathcal{V}} A$. Note that $A \in \mathcal{T}$ for all $A \in \mathcal{V}$, so by Definition 1.1.1, $T \in \mathcal{T}$ and so $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$ as desired.

This motivates us to consider fineness in terms of bases.

Proposition 1.1.7 ▸ Fineness in Terms of Bases

Let $\mathcal{B}$ and $\mathcal{B}'$ generate topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on X, then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for every $B \in \mathcal{B}$, there exists some $B' \in \mathcal{B}'$ such that for any $x \in B$, we have $x \in B' \subseteq B$.

Proof. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$, then $\mathcal{T} \subseteq \mathcal{T}'$. Take any $B \in \mathcal{B}$, then by Proposition 1.1.4, $B \in \mathcal{T}$, which means that $B \in \mathcal{T}'$. Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$, by Definition 1.1.2 for any $x \in B$, there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Suppose conversely that for every $B \in \mathcal{B}$, there exists some $B' \in \mathcal{B}'$ such that for any $x \in B$, we have $x \in B' \subseteq B$. Take any $T \in \mathcal{T}$, for each $x \in T$, by Definition 1.1.2 there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq T$. Notice that there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B \subseteq T$, so $T \in \mathcal{T}'$. Therefore, $\mathcal{T} \subseteq \mathcal{T}'$ and so $\mathcal{T}'$ is finer than $\mathcal{T}$. □

Recall that every basis of a topology on X is an open cover of X consisting only of subsets of X. Therefore, the union of the elements in the basis is essentially X itself. This motivates us to propose another way to generate a topology on a set.

Definition 1.1.8 ▸ Sub-basis

A **sub-basis** of X is a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ such that $\bigcup_{A \in \mathcal{S}} A = X$.

Remark. Every basis is a sub-basis.

For an arbitrary set X, let $\mathcal{S}$ be a sub-basis and denote the collection of all finite subsets of $\mathcal{S}$ as $\mathcal{F}_\mathcal{S}$. Define

$$\mathcal{U}_\mathcal{S} := \left\{\bigcap_{A\in F} A : F \in \mathcal{F}_\mathcal{S}\right\}$$

to be the collection of all finite intersections of sets in $\mathcal{S}$. The topology generated by a sub-basis of X is given by

$$\mathcal{T} := \left\{\bigcup_{A\in\mathcal{V}} A : \mathcal{V} \subseteq \mathcal{U}_\mathcal{S}\right\}.$$

We shall show that $\mathcal{T}$ is indeed a topology on X by considering the following proposition:

Proposition 1.1.9 ▸ Finite Intersections of Sets in a Sub-basis Form a Basis

Let $\mathcal{S}$ be a sub-basis for a set X and let $\mathcal{U}_\mathcal{S}$ be the set of all finite intersections of sets in $\mathcal{S}$, then $\mathcal{U}_\mathcal{S}$ is a basis of a topology on X.

Proof. Take any $x \in X$. By Definition 1.1.8, we have $x \in \bigcup_{A\in\mathcal{S}} A$. Therefore, there exists some $A \in \mathcal{S} \subseteq \mathcal{P}(X)$ such that $x \in A$. For any $x \in X$ and $B_1, B_2 \in \mathcal{U}_\mathcal{S}$ such that $x \in B_1 \cap B_2$, notice that $B_1 \cap B_2$ is a finite intersection of sets in $\mathcal{S}$, so $B_1 \cap B_2 \in \mathcal{U}_\mathcal{S}$. Therefore, by Definition 1.1.2, $\mathcal{U}_\mathcal{S}$ is a basis. □

With Propositions 1.1.9 and 1.1.4, it is clear that $\mathcal{T}$ as constructed above is a topology on X.