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1 Permutations and Combinations

1.1 Basic Counting Principles

An important motivation to study combinatorics is to count the **number of ways** in which an event may occur. Intuitively, we have two approaches to count.

The first approach is to categorise the event into **non-overlapping cases**. This means that we break an event into mutually exclusive sub-events, after which we can count the number of ways for each sub-event to occur. The agregate of these counts is the total number of ways for the original event to occur.

Those familiar with basic set theory may consider E to be the set containing all distinct ways for an event to occur. By breaking up the event, we essentially establish a **partition** of E, so that the sum of cardinalities of all the elements in that partition equals the cardinality of E.

This motivates us to write the following principle using set notations.

Theorem 1.1.1 ▸ Addition Principle (AP)

Let $k \in \mathbb{N}^+$ and let $A_1, A_2, \cdots, A_k$ be k finite sets which are pairwise disjoint, i.e. $A_i \cap A_j = \varnothing$ whenever $i \neq j$, then

$$\left| \bigcup_{i=1}^{k} A_i \right| = \sum_{i=1}^{k} |A_i|.$$

Proof. The case where $k = 1$ is trivial.
Suppose that when $k = n$, we have

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|$$

for any n finite sets which are pairwise disjoint. Let A_{n+1} be an arbitrary finite set which is disjoint with any of the A_i's from the n sets. So we have:

$$\begin{aligned}
\left|\bigcup_{i=1}^{n+1} A_i\right| &= \left|\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right| \\
&= \left|\bigcup_{i=1}^{n} A_i\right| + |A_{n+1}| - \left|\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1}\right| \\
&= \left(\sum_{i=1}^{n} |A_i|\right) + |A_{n+1}| - |\varnothing| \\
&= \sum_{i=1}^{n+1} |A_i|.
\end{aligned}$$

Therefore, the original statement holds for all $k \in \mathbb{N}^+$. □

In more casual language, this means that if an event E_k has n_k distinct ways to occur, then there is $\sum_{i=1}^{k} n_k$ ways for at least one of the events $E_1, E_2, \cdots, E_k$ to occur, provided that E_i and E_j can never occur concurrently whenever $i \neq j$.

Given an event E, the other approach to count the number of ways for it to occur is to break E up internally into **non-overlapping stages**.

With set notations, we can write the i-th stage for E to occur as e_i, and so a way for E to occur can be represented by an ordered tuple $(e_1, e_2, \cdots, e_k)$, where k is the total number of stages to undergo for E to occur.

Let E_i denote the set of all distinct ways to undergo the i-th stage of E, then it is easy to see that E is just the **Cartesian product** of all the E_i's. Hence, we derive the following principle:

Theorem 1.1.2 ▸ Multiplication Principle (MP)

Let $k \in \mathbb{N}^+$ and let $A_1, A_2, \cdots, A_k$ be k pairwise disjoint finite sets, then

$$\left|\prod_{i=1}^{k} A_i\right| = \prod_{i=1}^{k} |A_i|.$$

Proof. The case where $k = 1$ is trivial.

Suppose that when $k = n$, we have

$$\left|\prod_{i=1}^{n} A_i\right| = \prod_{i=1}^{n} |A_i|$$

for any n finite sets which are pairwise disjoint. Let A_{n+1} be an arbitrary finite set which is disjoint with any of the A_i's from the n sets. Take $a_i, a_j \in A_{n+1}$. Note that for all $\boldsymbol{a} \in \prod_{i=1}^{n} A_i$, $(\boldsymbol{a}, a_i) \neq (\boldsymbol{a}, a_j)$ whenever $a_i \neq a_j$. This means that

$$\begin{aligned}
\left|\prod_{i=1}^{n+1} A_i\right| &= \left|\prod_{i=1}^{n} A_i \times A_{n+1}\right| \\
&= \left|\prod_{i=1}^{n} A_i\right| |A_{n+1}| \\
&= \left(\prod_{i=1}^{n} |A_i|\right) |A_{n+1}| \\
&= \prod_{i=1}^{n+1} |A_i|
\end{aligned}$$

Therefore, the original statement holds for all $k \in \mathbb{N}^+$. □

In more casual language, this means that if an event E requires k stages to be undergone before it occurs and the i-th stage has n_i ways to complete, then there is $\prod_{i=1}^{k} n_k$ ways for E to occur, provided that no two different stages complete concurrently.

Often times, it is not straight-forward to count directly due to the presence of restrictions. We shall consider the following:

Let E be the set of all possible ways for an event to occur. Let p be some predicate representing some restriction and let $E(p)$ denote the set of all possible ways for the event to occur while p holds. Note that:

$$E(p) \cup E(\neg p) = E \quad \text{and} \quad E(p) \cap E(\neg p) = \varnothing,$$

i.e. $\{E(p), E(\neg p)\}$ is a partition of E. Therefore, to count the number of ways for the event to occur while p holds, it suffices to compute $E(\neg p)$, i.e. find the number of ways for the event to occur while p does not hold.

Theorem 1.1.3 ▸ Principle of Complementation

Let U be a set and let $E \subseteq U$, then

$$|E| = |U| - |U - E|.$$

1.2 Permutations

A fundamental problem in combinatorics is described as follows: given a set S, how many ways are there to arrange r elements in S, i.e. how many **distinct sequences** can be formed using the elements in S without repetition? The process of selecting elements from S and arranging them as a sequence is known as **permutation**.

Note that forming a sequence using r elements from a set S is an event consisting of r stages, as we need to select an element for each of the r terms of the sequence. Suppose S has n elements. For the first term of the sequence, we can choose any of the elements in S, so there is n ways to do it. For the second term, since we cannot repeat the elements, we are left with $n - 1$ choices.

Continue choosing elements in this way, we realise that if we choose the terms sequentially, when we reach the k-th term we will be left with $n - k + 1$ options as the previous $(k - 1)$ terms have taken away $(k - 1)$ elements. By Theorem 1.1.2, we know that the number of sequences which can be formed is given by $\prod_{i=1}^{r}(n - r + i)$.

Definition 1.2.1 ▸ Permutations

Let A be a finite set such that $|A| = n$, an r-permutation of A is a way to arrange r elements of A, denoted as P_r^n and given by

$$P_r^n = \prod_{i=1}^{r}(n - r + i) = \frac{n!}{(n - r)!}.$$

With some algebraic manipulations, it is easy to derive the following formula, which we, however, will prove in a combinatorial manner.

Theorem 1.2.2

Let $n, r \in \mathbb{N}$ with $r \leq n$, then $P_r^{n+1} = P_r^n + rP_{r-1}^n$.

Proof. Let $S = \{x \in \mathbb{N}^+ : x \leq n + 1\}$ represent $(n + 1)$ distinct objects. Consider a permutation of S:

If $n + 1$ is not inside the permutation, this is equivalent to an r-permutation of $S - \{n + 1\}$, so there are P_r^n such permutations.

If $n + 1$ is inside the permutation, it means we need to first find an $(r - 1)$-permuation of $S - \{n + 1\}$, which has P_{r-1}^n ways to do. After that, we need to insert $n + 1$ into each of these $(r - 1)$-permutations. Note that for each of such permutations, there are r positions into which we can place $n+1$. Therefore, the total number of r-permuations of S derived in this manner is rP^n_{r-1}.

Therefore, there are $P^n_r + rP^n_{r-1}$ r-permutations of S, i.e. $P^{n+1}_r = P^n_r + rP^n_{r-1}$. □

1.2.1 Circular Permutations

Consider arranging n distinct objects around a circle. If the slots around the circle are uniquely labelled, this is exactly the same as permutations along a straight line.

However, if the slots are identical, i.e. we are arranging n distinct objects around a circle with identical slots, only the **relative positions** of the objects matter.

Let $\boldsymbol{x}_i$ be an arbitrary straight-line permutations of the n objects and let $\boldsymbol{y}_i$ be the corresponding circular permutation of the n objects.

Note that if we translate every element in $\boldsymbol{x}_i$ by k positions, this will result in a different straight-line permutation $\boldsymbol{x}_j$ but does not change the corresponding circular permutation because the relative positions of the objects remain unchanged.

Notice that k can take the values $0, 1, 2, \cdots, n-1$, so for the same set of n distinct objects, every circular permutation is mapped to n straight-line permutations.

Definition 1.2.3 ▶ Circular Permutations

Let A be a finite set such that $|A| = n$, a circular r permutation of A is a way to arrange r elements of A around a circular locus, denoted as Q^n_r and given by

$$Q^n_r = \frac{P^n_r}{r} = \frac{n!}{r(n-r)!}.$$

1.2.2 Permutations with Idential Objects

Theorem 1.2.4 ▶ Generalised Formula for Permutations

Let $k \in \mathbb{N}^+$ and let $A_1, A_2, \cdots, A_k$ be k distinct objects, where A_i occurs $n_i > 0$ times for $i = 1, 2, \cdots, k$, then the number of permutations for these k objects are given by

$$\frac{\left(\sum_{i=1}^k n_i\right)!}{\prod_{i=1}^k (n_i!)}.$$

1.3 Combinations

Beside permutations, there are also occasions where we only care about which elements from a particular set are selected instead of the order of selection.

Note that if we want to find a selection of r elements from a set A where the order of selected elements does not matter, it is equivalent to finding a subset of A containing r elements. This motivates us to give the following definition:

Definition 1.3.1 ▸ Combinations

Let A be a finite set such that $|A| = n$, an r-combination of A is a set $B \subseteq A$ with $|B| = r$. The number of combinations of A is given by

$$C_r^n = \frac{P_r^n}{P_r^r} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Remark. Two obvious results:

1. If $r > n$ or $r < 0$, $C_r^n = 0$;
2. $C_r^n = C_{n-r}^n$ (By Theorem 1.1.3).

Similar to permutations, we have the following important identity:

Theorem 1.3.2 ▸ Pascal's Triangle

Let n be an integer with $n \geq 2$ and let r be an integer with $0 \leq r \leq n$, then

$$C_r^{n+1} = C_{r-1}^n + C_r^n.$$

Proof. Let $S = \{x \in \mathbb{N}^+ : x \leq n+1\}$ represent $(n+1)$ distinct objects. Consider an r-combination T of S:

If $n+1 \notin T$, this is equivalent to an r-combination of $S - \{n+1\}$, so there are C_r^n such permutations.

If $n+1 \in T$, it suffices to find an $(r-1)$-combination of $S - \{n+1\}$, which has C_{r-1}^n ways to do.

Therefore, there are $C_r^n + C_{r-1}^n$ r-combinations of S, i.e. $C_r^{n+1} = C_r^n + C_{r-1}^n$. □

1.3.1 Counting Subsets

A useful application of combinations, derived directly from the definition, is to count the number of subsets for a given set which is finite. In other words, given a set A with $|A| = n \in \mathbb{N}$, we wish to find a general formula for $|\mathcal{P}(A)|$.

Let A_i be the set of all subsets of A whose cardinality is i, then clearly

$$|\mathcal{P}(A)| = \sum_{i=0}^{n} |A_i| = \sum_{i=0}^{n} C_i^n.$$

We can expand the above expression algebraically and realise that it simpifies to 2^n. However, in a combinatorial perspective, we are able to prove this result in a more succint manner:

Theorem 1.3.3 ▶ General Formula for $\mathcal{P}(A)$

Let A be a finite set. If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.

Proof. Let S be an arbitrary subset of A. Consider an arbitrary element $a \in A$, then either $a \in S$ or $a \notin S$.

Let $a_i \in A$ for $i = 1, 2, \cdots, n$. For all $S \in \mathcal{P}(A)$, We replace a_i by 1 if $a_i \in S$, and by 0 otherwise. Let B be the set of all binary sequences of length n. It is clear that there exists a bijection between $\mathcal{P}(A)$ and B, and so $|\mathcal{P}(A)| = |B|$.

For each binary sequence of length n, each of its digits is either 0 or 1. By Theorem 1.1.2, this means that there are in total 2^n such binary sequences. Therefore,

$$|\mathcal{P}(A)| = |B| = 2^n.$$

□