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1 Sets and Classes

1.1 Classes

Russell's Paradox states the following:

Russell's Paradox

Let X be the set of all sets which do not contain themselves, i.e.,

$$X = \{S : S \notin S\}.$$

Now consider X. If $X \in X$, it means that X contains itself and should not be a member of X, i.e., $X \in X \implies X \notin X$. If $X \notin X$, it means that X does not contain itself and therefore should be a member of X, i.e. $X \notin X \implies X \in X$. Hence, we have a paradox and such a set X does not exist.

However, in some cases it is still useful to consider the "set" of all sets for practical reasons. Therefore, we introduce the notion of a *class* to avoid Russell's Paradox.

Definition 1.1.1 ▸ Class

Let ϕ be some formula and $\boldsymbol{u}$ be a vector, the collection

$$\mathbb{C} = \{X : \phi(X, \boldsymbol{u})\}$$

is called a **class** of all sets satisfying $\phi(X, \boldsymbol{u})$, where $\mathbb{C}$ is said to be **definable** from $\boldsymbol{u}$. Equivalently, we say that

$$X \in \mathbb{C} \iff \phi(X, \boldsymbol{u}).$$

In particular, if $\mathbb{C} = \{X : \phi(X)\}$, i.e., ϕ only has one free variable, then we say that $\mathbb{C}$ is **definable**.

Remark. It is easy to see that every set X is a class given by $\{x : x \in X\}$.

Intuitively, two classes are equal if they contain exactly the same members. We are able to give the following rigorous version of the notion of equality:

Definition 1.1.2 ▶ Equality between Classes

Let $\mathbb{C} = \{X : \phi(X, \boldsymbol{u})\}$ and $\mathbb{D} = \{X : \psi(X, \boldsymbol{v})\}$, we say that $\mathbb{C} = \mathbb{D}$ if for all X,

$$\phi(X, \boldsymbol{u}) \iff \psi(X, \boldsymbol{u}).$$

There are clearly two types of classes — the ones which are also sets and the ones which are not. Formally, this is put as follows:

Definition 1.1.3 ▶ Proper Class

A class $\mathbb{C}$ is said to be a **proper class** if $\mathbb{C} \neq X$ for all sets X.

Like sets, we can define subclasses:

Definition 1.1.4 ▶ Subclass

Let $\mathbb{A}$ and $\mathbb{B}$ be classes. We say that $\mathbb{A}$ is a **subclass** of $\mathbb{B}$ if every member of $\mathbb{A}$ is also a member of $\mathbb{B}$, i.e.,

$$\mathbb{A} \subseteq \mathbb{B} \iff (X \in \mathbb{A} \implies X \in \mathbb{B}).$$

We shall also define the operations applicable to classes:

Definition 1.1.5 ▶ Intersection, Union and Difference

Let $\mathbb{A}$ and $\mathbb{B}$ be classes. The **intersection**, **union** and **difference** between $\mathbb{A}$ and $\mathbb{B}$ are given by

$$\begin{aligned}\mathbb{A} \cap \mathbb{B} &:= \{X : X \in \mathbb{A} \wedge X \in \mathbb{B}\},\\ \mathbb{A} \cup \mathbb{B} &:= \{X : X \in \mathbb{A} \vee X \in \mathbb{B}\},\\ \mathbb{A} - \mathbb{B} &:= \{X : X \in \mathbb{A} \wedge X \notin \mathbb{B}\}\end{aligned}$$

respectively.

Finally, we shall introduce the universal class:

Definition 1.1.6 ▶ Universal Class

The **universal class** is the class of all sets, denoted by

$$V := \{X : X = X\}.$$

Remark. It is easy to prove that the universal class is **unique**.

2

Axiomatic Set Theory

2.1 Axioms of Zermelo-Fraenkel (ZF)

In Naïve Set Theory, we define a set as "a collection of mathematical objects which satisfy certain definable properties". However, such a definition is problematic (e.g. it leads to the Russell's Paradox). Thus, instead of viewing a set as a clearly defined mathematical object, we can think a set as an object entirely defined by a set of axioms to which it complies. In this sense, we avoid paradoxes by making the notion of a set undefined but only specify rigorously the axioms a set must satisfy.

Axiom 2.1.1 ▸ Existence

There exists a set X such that $X = X$.

Axiom 2.1.2 ▸ Extensionality

Let X and Y be sets, then $X = Y$ if for all u, $u \in X$ if and only if $u \in Y$.

Axiom 2.1.3 ▸ Pairing

For all u and v, there exists a set X such that for all z, $z \in X$ if and only if $z = u$ or $z = v$.

Remark. Note that Axiom 2.1.3 essentially says that given any sets u and v, there exists a set whose elements are exactly u and v.