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The Real Numbers

1.1 Fields

Definition 1.1.1 ► Field

A set F with two binary operations, namely addition and multiplication, is called a **field** if it satisfies the following axioms:

1. $\forall a, b \in F, a +_F b = b +_F a$.
2. $\forall a, b, c \in F, (a +_F b) +_F c = a +_F (b +_F c)$.
3. $\exists 0_F \in F$ such that $\forall a \in F, 0_F +_F a = a +_F 0_F = a$.
4. $\forall a \in F, \exists a' \in F$ such that $a +_F a' = 0_F$.
5. $\forall a, b \in F, a \cdot_F b = b \cdot_F a$.
6. $\forall a, b, c \in F, (a \cdot_F b) \cdot_F c = a \cdot_F (b \cdot_F c)$.
7. $\forall a, b, c \in F, a \cdot_F (b +_F c) = a \cdot_F b +_F a \cdot_F c$ and $(a +_F b) \cdot_F c = a \cdot_F c +_F b \cdot_F c$.
8. $\exists 1_F \in F$ such that $\forall a \in F, 1_F \cdot_F a = a \cdot_F 1_F = a$.
9. $\forall a \in F, \exists a' \in F$ such that $a \cdot_F a' = 1_F$.

If we denote addition by “ $+_F$ ” and multiplication by “ \cdot_F ” or “ \times_F ”, then we can denote the field over F by $(F, +_F, \cdot_F)$ or $(F, +_F, \times_F)$.

Among the commonly used number sets, one may check that \mathbb{R} , \mathbb{Q} and \mathbb{C} are fields, while \mathbb{N} and \mathbb{Z} are not.

1.1.1 Ordered Fields

Definition 1.1.2 ► Total Order

A **total order** on a set X is a binary relation \leq over X such that for all $a, b, c \in X$:

1. $a \leq a$ (reflexive).
2. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitive).
3. $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetric).
4. either $a \leq b$ or $b \leq a$ (strongly connected).

Definition 1.1.3 ▶ Strict Total Order

A **strict total order** on a set X is a binary relation $<$ over X such that for all $a, b, c \in X$:

1. $a \not< a$ (irreflexive).
2. $a < b$ implies $b < a$ (asymmetric).
3. $a < b$ and $b < c$ implies $a < c$ (transitive).
4. if $a \neq b$, then either $a < b$ or $b < a$ (connected).

It is easy to see that the real numbers form the ordered fields $(\mathbb{R}, +, \times, \leq)$ and $(\mathbb{R}, +, \times, <)$. Note that this means \mathbb{R} satisfies trichotomy. If we choose any $x \in \mathbb{R}$, then exactly one of $x = 0$, $x > 0$ and $x < 0$ is true. Therefore, we can define that if $x \in \mathbb{R}$ and $x > 0$, then x is said to be positive. This leads to the following axiomatic results:

1. If a and b are both positive, then $a + b$ is positive;
2. If a and b are both positive, then ab is positive;
3. For any $a \in \mathbb{R}$, either $a = 0$, a is positive, or $-a$ is positive.

Note that $a < b$ if and only if $b - a$ is positive. So the trichotomy of \mathbb{R} guarantees that for any $a, b \in \mathbb{R}$, either $a = b$, $a < b$ or $b < a$ (i.e., $a > b$).

1.2 Properties of \mathbb{R}

We can derive a few obvious minor results based on the field properties of \mathbb{R} :

1. If $a, b \in \mathbb{R}$, then $-ab + ab = 0$;
2. For all $a \in \mathbb{R}$ with $a \neq 0$, $a^2 > 0$;
3. If $a \in \mathbb{R}$ is such that $0 \leq a < \epsilon$ for all $\epsilon \in \mathbb{R}^+$, then $a = 0$;
4. If $a < b$, then $a + c < b + c$ for all $c \in \mathbb{R}$.
5. If $a < b$, then $ac < bc$ for all $c \in \mathbb{R}^+$ and $ac > bc$ for all $c \in \mathbb{R}^-$.
6. For all $a \in \mathbb{R}$, $a^2 > 0$.

We may consider the following interesting proposition:

Proposition 1.2.1

If $a \in \mathbb{R}$ is such that $0 \leq a < \epsilon$ for all $\epsilon \in \mathbb{R}^+$, then $a = 0$.

Proof. Suppose on contrary that $a > 0$, then we can take $\epsilon_0 = \frac{a}{2}$. Note that $\epsilon_0 \in \mathbb{R}^+$ but $\epsilon_0 < a$, which is a contradiction. So $a = 0$. □

The above essentially asserts that a **non-negative real number is strictly less than any positive real number if and only if it is 0**.

The properties of \mathbb{R} also enables us to manipulate inequalities based on the following trivial results:

1. If $ab > 0$, then a and b are either both positive or both negative;
2. If $ab < 0$, then exactly one of them is positive and exactly one of them is negative.

We shall introduce a few well-known inequalities.

Theorem 1.2.2 ► Bernoulli's Inequality

If $x > -1$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

Proof. The case where $n = 0$ is trivial.

Suppose that $(1 + x)^k \geq 1 + kx$ for some $k \in \mathbb{N}$, consider

$$\begin{aligned}(1 + x)^{k+1} &= (1 + x)(1 + x)^k \\ &\geq (1 + x)(1 + kx) \\ &= 1 + (k + 1)x + kx^2 \\ &\geq 1 + (k + 1)x.\end{aligned}$$

Therefore, $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$. □

Theorem 1.2.3 ► AM-GM-HM Inequality

Let $n \in \mathbb{N}^+$ and let a_1, a_2, \dots, a_n be positive real numbers, then

$$\frac{n}{\sum_{i=1}^n \frac{1}{a_i}} \leq \left(\prod_{i=1}^n a_i \right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^n a_i}{n}.$$

1.2.1 Absolute Value

Given any real number x , intuitively we sense that x possesses a certain “distance” from 0. This distance can be formalised as follows:

Definition 1.2.4 ▶ Absolute Value

Let $x \in \mathbb{R}$, the **absolute value** of x is defined as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}.$$

We have some trivial properties about the absolute value:

1. For all $a, b \in \mathbb{R}$, $|ab| = |a||b|$;
2. For all $a \in \mathbb{R}$, $|a|^2 = a^2$;
3. If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$ for all $a \in \mathbb{R}$;
4. For all $a \in \mathbb{R}$, $-|a| \leq a \leq |a|$.

Using these basic properties, we can prove the following results:

Theorem 1.2.5 ▶ Triangle Inequality

For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

Corollary 1.2.6 ▶ Extended Triangle Inequality

For all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$ and $|a - b| \leq |a| + |b|$.

Corollary 1.2.7 ▶ Generalised Triangle Inequality

For all $a_1, a_2, \dots, a_n \in \mathbb{R}$,

$$\left| \sum_{i=1}^n a_i \right| \leq \sum_{i=1}^n |a_i|.$$

Analogously, if $|x|$ represents the “distance” between x and 0, then by a simple translation we can see that $|x - a|$ represents the “distance” between x and a . Thus, we can have the following definition:

Definition 1.2.8 ▶ Neighbourhood

Let $a \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^+$. The **ϵ -neighbourhood** of a is defined to be the set

$$V_\epsilon(a) := \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

Note that $x \in V_\epsilon(a)$ if and only if $-\epsilon < x - a < \epsilon$ or $a - \epsilon < x < a + \epsilon$. Which leads to the

following interesting result:

Proposition 1.2.9

For any $a \in \mathbb{R}$, if $x \in V_\epsilon(a)$ for all $\epsilon \in \mathbb{R}^+$, then $x = a$.

Proof. Note that this essentially means that $|x - a| < \epsilon$ for all $\epsilon \in \mathbb{R}^+$. By Proposition 1.2.1, we have $|x - a| = 0$ and therefore $x = a$. \square

1.2.2 The Completeness Property of \mathbb{R}

Intuitively, there are no “gaps” among the real numbers, i.e., if you take any two real numbers, between them there is nothing else than other real numbers. Therefore, we say that \mathbb{R} is *complete*. This is in contrast with \mathbb{Q} where there are gaps in between any two rational numbers (because there always exists some irrational numbers in between).

In this section, we probe into how the completeness of \mathbb{R} can be established, and how the real numbers themselves can be constructed. To do that, we first establish the notion of *boundedness*.

Definition 1.2.10 ► Boundedness

Let $S \subseteq \mathbb{R}$. We say that S is:

- **bounded above** if there exists some $u \in \mathbb{R}$ (known as the **upper bound** of S) such that $u \geq s$ for all $s \in S$;
- **bounded below** if there exists some $v \in \mathbb{R}$ (known as the **lower bound** of S) such that $v \leq s$ for all $s \in S$;
- **bounded** if S has both an upper bound and a lower bound;
- **unbounded** either if S has no upper bound or if S has no lower bound;

Remark. Note that S is bounded if and only if there is some $M \geq 0$ such that $|s| \leq M$ for all $s \in S$.