Contents

1 The Real Numbers

1.1 Fields

Definition 1.1.1 ▶ Field

A set F with two binary operations, namely addition and multiplication, is called a **field** if it satisfies the following axioms:

1. $\forall a, b \in F, a +_F b = b +_F a$.
2. $\forall a, b, c \in F, (a +_F b) +_F c = a +_F (b +_F c)$.
3. $\exists 0_F \in F$ such that $\forall a \in F, 0_F +_F a = a +_F 0_F = a$.
4. $\forall a \in F, \exists a' \in F$ such that $a +_F a' = 0_F$.
5. $\forall a, b \in F, a \cdot_F b = b \cdot a$.
6. $\forall a, b, c \in F, (a \cdot_F b) \cdot c = a \cdot_F (b \cdot_F c)$.
7. $\forall a, b, c \in F, a \cdot_F (b +_F c) = a \cdot_F b +_F a \cdot_F b$ and $(a +_F b) \cdot_F c = a \cdot_F c +_F b \cdot_F c$.
8. $\exists 1_F \in F$ such that $\forall a \in F, 1_F \cdot_F a = a \cdot_F 1_F = a$.
9. $\forall a \in F, \exists a' \in F$ such that $a \cdot_F a' = 1_F$.

If we denote addition by "$+_F$" and multiplication by "$\cdot_F$" or "$\times_F$", then we can denote the field over F by $(F, +_F, \cdot_F)$ or $(F, +_F, \times_F)$.

Among the commonly used number sets, one may check that $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ are fields, while $\mathbb{N}$ and $\mathbb{Z}$ are not.

1.1.1 Ordered Fields

Definition 1.1.2 ▶ Total Order

A **total order** on a set X is a binary relation $\leq$ over X such that for all $a, b, c \in X$:

1. $a \leq a$ (reflexive).
2. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitive).
3. $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetric).
4. either $a \leq b$ or $b \leq a$ (strongly connected).

Definition 1.1.3 ▶ Strict Total Order

A **strict total order** on a set X is a binary relation $<$ over X such that for all $a, b, c \in X$:

1. $a \not< a$ (irreflexive).
2. $a < b$ implies $b < a$ (asymmetric).
3. $a < b$ and $b < c$ implies $a < c$ (transitive).
4. if $a \neq b$, then either $a < b$ or $b < a$ (connected).

It is easy to see that the real numbers form the ordered fields $(\mathbb{R}, +, \times, \leq)$ and $(\mathbb{R}, +, \times, <)$. Note that this means $\mathbb{R}$ satisfies trichotomy. If we choose any $x \in \mathbb{R}$, then exactly one of $x = 0$, $x > 0$ and $x < 0$ is true. Therefore, we can define that if $x \in \mathbb{R}$ and $x > 0$, then x is said to be positive. This leads to the following axiomatic results:

1. If a and b are both positive, then $a + b$ is positive;
2. If a and b are both positive, then ab is positive;
3. For any $a \in \mathbb{R}$, either $a = 0$, a is positive, or $-a$ is positive.

Note that $a < b$ if and only if $b - a$ is positive. So the trichotomy of $\mathbb{R}$ guarantees that for any $a, b \in \mathbb{R}$, either $a = b$, $a < b$ or $b < a$ (i.e., $a > b$).

1.2 Properties of $\mathbb{R}$

We can derive a few obvious minor results based on the field properties of $\mathbb{R}$:

1. If $a, b \in \mathbb{R}$, then $-ab + ab = 0$;
2. For all $a \in \mathbb{R}$ with $a \neq 0$, $a^2 > 0$;
3. If $a \in \mathbb{R}$ is such that $0 \leq a < \epsilon$ for all $\epsilon \in \mathbb{R}^+$, then $a = 0$;
4. If $a < b$, then $a + c < b + c$ for all $c \in \mathbb{R}$.
5. If $a < b$, then $ac < bc$ for all $c \in \mathbb{R}^+$ and $ac > bc$ for all $c \in \mathbb{R}^-$.
6. For all $a \in \mathbb{R}$, $a^2 > 0$.

We may consider the following interesting proposition:

Proposition 1.2.1

If $a \in \mathbb{R}$ is such that $0 \leq a < \epsilon$ for all $\epsilon \in \mathbb{R}^+$, then $a = 0$.

Proof. Suppose on contrary that $a > 0$, then we can take $\epsilon_0 = \frac{a}{2}$. Note that $\epsilon_0 \in \mathbb{R}^+$ but $\epsilon_0 < a$, which is a contradiction. So $a = 0$. $\square$

The above essentially asserts that **a non-negative real number is strictly less than any positive real number if and only if it is** 0.

The properties of $\mathbb{R}$ also enables us to manipulate inequalities based on the following trivial results:

1. If $ab > 0$, then a and b are either both positive or both negative;
2. If $ab < 0$, then exactly one of them is positive and exactly one of them is negative.

We shall introduce a few well-known inequalities.

Theorem 1.2.2 ▸ Bernoulli's Inequality

If $x > -1$, then $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

Proof. The case where $n = 0$ is trivial.

Suppose that $(1+x)^k \geq 1 + kx$ for some $k \in \mathbb{N}$, consider

$$\begin{aligned}(1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) \\ &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x.\end{aligned}$$

Therefore, $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$. □

Theorem 1.2.3 ▸ AM-GM-HM Inequality

Let $n \in \mathbb{N}^+$ and let $a_1, a_2, \cdots, a_n$ be positive real numbers, then

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \leq \left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} a_i}{n}.$$

1.2.1 Absolute Value

Given any real number x, intuitively we sense that x possesses a certain "distance" from 0. This distance can be formalised as follows:

Definition 1.2.4 ▸ Absolute Value

Let $x \in \mathbb{R}$, the **absolute value** of x is defined as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}.$$

We have some trivial properties about the absolute value:

1. For all $a, b \in \mathbb{R}$, $|ab| = |a||b|$;
2. For all $a \in \mathbb{R}$, $|a|^2 = a^2$;
3. If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$ for all $a \in \mathbb{R}$;
4. For all $a \in \mathbb{R}$, $-|a| \leq a \leq |a|$.

Using these basic properties, we can prove the following results:

Theorem 1.2.5 ▸ Triangle Inequality

For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

Corollary 1.2.6 ▸ Extended Triangle Inequality

For all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$ *and* $|a - b| \leq |a| + |b|$.

Corollary 1.2.7 ▸ Generalised Triangle Inequality

For all $a_1, a_2, \cdots, a_n \in \mathbb{R}$,

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|.$$

Analogously, if $|x|$ represents the "distance" between x and 0, then by a simple translation we can see that $|x - a|$ represents the "distance" between x and a. Thus, we can have the following definition:

Definition 1.2.8 ▸ Neighbourhood

Let $a \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^+$. The **$\epsilon$-neighbourhood** of a is defined to be the set

$$V_\epsilon(a) := \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

Note that $x \in V_\epsilon(a)$ if and only if $-\epsilon < x - a < \epsilon$ or $a - \epsilon < x < a + \epsilon$. Which leads to the following interesting result:

Proposition 1.2.9

For any $a \in \mathbb{R}$, if $x \in V_\epsilon(a)$ for all $\epsilon \in \mathbb{R}^+$, then $x = a$.

Proof. Note that this essentially means that $|x - a| < \epsilon$ for all $\epsilon \in \mathbb{R}^+$. By Proposition 1.2.1, we have $|x - a| = 0$ and therefore $x = a$. □

1.2.2 The Completeness Property of $\mathbb{R}$

Intuitively, there are no "gaps" among the real numbers, i.e., if you take any two real numbers, between them there is nothing else than other real numbers. Therefore, we say that $\mathbb{R}$ is *complete*. This is in contrast with $\mathbb{Q}$ where there are gaps in between any two rational numbers (because there always exists some irrational numbers in between).

In this section, we probe into how the completeness of $\mathbb{R}$ can be established, and how the real numbers themselves can be constructed. To do that, we first establish the notion of *boundedness*.

Definition 1.2.10 ▶ Boundedness

Let $S \subseteq \mathbb{R}$. We say that S is:

- **bounded above** if there exists some $u \in R$ (known as the **upper bound** of S) such that $u \geq s$ for all $s \in S$;
- **bounded below** if there exists some $v \in R$ (known as the **lower bound** of S) such that $v \leq s$ for all $s \in S$;
- **bounded** if S has both an upper bound and a lower bound;
- **unbounded** either if S has no upper bound or if S has no lower bound;

Remark. Note that S is bounded if and only if there is some $M \geq 0$ such that $|s| \leq M$ for all $s \in S$.

2

Sequences and Series

2.1 Sequences

Informally, a sequence is a list of enuerable numbers. This means that we can view a sequence as a mapping from an interval of $\mathbb{N}^+$ to $\mathbb{R}$. In this course, we will mainly focus on infinite sequences.

Definition 2.1.1 ▸ Sequence

A **sequence** in $\mathbb{R}$ is a real-valued function $X : \mathbb{N}^+ \to \mathbb{R}$. Where $X(n)$ is called the n-th **term** of the sequence.

By convention, we denote $X(n)$ by x_n, and the sequence X by (x_n) or $(x_n : n \in \mathbb{N}^+)$.

Alternatively, a sequence (x_n) may be defined in the following manner: first, we define the value of x_1. Secondly, we define a mapping $(x_1, x_2, \cdots, x_n) \mapsto x_{n+1}$. Sequences defined in this way are said to be **inductively** and **recursively** defined.

2.1.1 Limits of Sequences

As n becomes very large, a sequence may exhibit certain limiting behaviour.

Definition 2.1.2 ▸ Convergence of Sequences

A sequence (x_n) defined in $\mathbb{R}$ is said to be **convergent** to x if for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that whenever $n \geq N$, $|x_n - x| < \epsilon$. x is known as the **limit** of (x_n), denoted as

$$\lim_{n\to\infty} x_n = x.$$

A sequence which is not convergent is said to be *divergent*.

Intuitively, a sequence can not converge to different values concurrently. This idea can be formulated formally as follows:

Theorem 2.1.3 ▶ Uniqueness of Limits

If (x_n) converges, then its limit is unique.

Proof. Suppose that x and x' are both limits of (x_n). For all $\epsilon > 0$, there exists $N_1 \in \mathbb{N}^+$ such that

$$|x_n - x| < \frac{\epsilon}{2}$$

whenever $n \geq N_1$ and there exists $N_2 \in \mathbb{N}^+$ such that

$$|x_n - x'| < \frac{\epsilon}{2}$$

whenever $n \geq N_2$. Take $N = \max\{N_1, N_2\}$, then for all $n \geq N$,

$$\begin{aligned} |x - x'| &= |x - x_n + x_n - x'| \\ &\leq |x_n - x| + |x_n - x'| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

for all $\epsilon > 0$. By Proposition 1.2.1, $x - x' = 0$, i.e., $x = x'$. This means that $\lim_{n\to\infty} x_n$ is unique. □

Given sequences (x_n) and (y_n), we can form new sequences by applying arithmetic operations onto them, and we can relate the limits of these new sequences with the limits of (x_n) and (y_n).

Theorem 2.1.4 ▶ Limit Laws

If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

1. $\lim_{n\to\infty}(x_n + y_n) = x + y$.
2. $\lim_{n\to\infty}(x_n y_n) = xy$.
3. $\lim_{n\to\infty}(cx_n) = cx$ *for all* $c \in \mathbb{R}$.
4. $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$.

In some cases, it may not be easy to prove the existence of limit or compute it directly for a sequence. Thus, the following may be useful:

Theorem 2.1.5 ▶ Squeeze Theorem

Let (x_n), (y_n) and (z_n) be sequences such thatr $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}^+$. If (x_n)

and (z_n) both converge and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell$, then (y_n) converges and

$$\lim_{n\to\infty} y_n = \ell.$$

Proof. Let $\epsilon > 0$ be an arbitrary real number. Note that there exists $N \in \mathbb{N}^+$ such that for all $n \geq N$,

$$|x_n - \ell| < \epsilon, \qquad |z_n - \ell| < \epsilon.$$

Therefore,

$$-\epsilon < x_n - \ell \leq y_n - \ell \leq z_n - \ell < \epsilon,$$

which implies that

$$|y_n - \ell| < \epsilon$$

for all $n \geq N$. Therefore, $\lim_{n\to\infty} y_n = \ell$. □

Notice that the limit of a sequence essentially "bounds" the sequence. This motivates us to investigate the relation between a convergent sequence and the its bounds.

Definition 2.1.6 ▸ Boundedness of Sequences

A sequence (x_n) in $\mathbb{R}$ is **bounded** if there exists some $M \in \mathbb{R}^+$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}^+$.

Remark. Note that (x_n) is bounded if and only if the set $\{x_n : n \in \mathbb{N}^+\}$ is bounded.

Theorem 2.1.7 ▸ Boundedness of Convergent Sequences

A sequence (x_n) is bounded if it is convergent.

Proof. Let $\lim_{n\to\infty} x_n = x$. Note that there exists some $N \in \mathbb{N}^+$ such that whenever $n \geq N$, $|x_n - x| < 1$. Consider

$$\begin{aligned} |x_n| &= |x_n - x + x| \\ &\leq |x_n - x| + |x| \\ &< 1 + |x|. \end{aligned}$$

Let

$$M := \sup\left\{|x_1|, |x_2|, \cdots, |x_{N-1}|, 1 + |x|\right\},$$

then $|x_n| \leq M$ for all $n \in \mathbb{N}^+$, and so (x_n) is bounded. □

The contrapositive statement to the above theorem concludes that **any unbounded se-**

quence must be divergent. However, note that the converse of Theorem 2.1.7 is not true in general! As a counter example, consider $x_n = (-1)^n$.

Proposition 2.1.8

Let (x_n) be a convergent sequence. If $x_n \geq 0$ for all $n \in \mathbb{N}^+$, then $\lim_{n\to\infty} x_n \geq 0$.

Proof. Let $\lim_{n\to\infty} x_n = x$. Suppose on contrary $x < 0$, then $-x > 0$. Therefore, there exists some $N \in \mathbb{N}^+$ such that whenever $n \geq N$, $|x_n - x| < -x$. This means that for all $n \geq N$,

$$x_n < x - x = 0.$$

However, this is a contradiction, so $\lim_{n\to\infty} x_n = x > 0$. □

Here are two simple corollaries from Proposition 2.1.8, the proofs of which are left to the reader as an exercise.

Corollary 2.1.9

If (x_n) and (y_n) are convergent sequences with $x_n \geq y_n$ for all $n \in \mathbb{N}^+$, then

$$\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n.$$

Corollary 2.1.10

If (x_n) is a convergent sequence with $a \leq x_n \leq b$ for all $n \in \mathbb{N}^+$, then

$$a \leq \lim_{n\to\infty} x_n \leq b.$$

In casual languages, we may be tempted to describe the limit of a sequence as "a value to which the terms can get as close as possible, but which is never surpassed". This intuition gives us an idea to prove convergence for a bounded sequence.

Definition 2.1.11 ▸ Monotone Sequences

Let (x_n) be a sequence. (x_n) is said to be **increasing** if $x_i \geq x_j$ whenever $i \geq j$, and **decreasing** if $x_i \leq x_j$ whenever $i \geq j$. A sequence is said to be **monotone** if it is either increasing or decreasing.

Note that an increasing sequence is the same as an non-decreasing sequence and vice versa. Recall that we have stated that the converse of Theorem 2.1.7 is not true in general, but if we impose an additional constraint on the monotonicity of the bounded sequence, we will get a stronger condition.

Theorem 2.1.12 ▶ Monotone Convergence Theorem

Let (x_n) be a monotone sequence in $\mathbb{R}$, then (x_n) converges if and only if it is bounded.

Proof. Suppose (x_n) is convergent, then it follows from Theorem 2.1.7 that it is bounded.

Suppose conversely that (x_n) is bounded. Without loss of generality, assume that (x_n) is increasing, so (x_n) has an upper bound. Let $\sup(x_n) = x$ and let $\epsilon > 0$ be an arbitrary real number. Note that $x - \epsilon$ is not an upper bound for (x_n), so there exists some $x_N \in (x - \epsilon, x]$, which means that $0 \leq x - x_N < \epsilon$. Since (x_n) is increasing, for all $n \geq N$, we have $x \geq x_n \geq x_N$, so

$$0 \leq x - x_n \leq x - x_N < \epsilon.$$

Therefore, $|x - x_n| < \epsilon$ for all $\epsilon > 0$ whenever $n \geq N$, and so $\lim_{n\to\infty} x_n = x$. □

A classic application of Theorem 2.1.12 is an approximation of $\sqrt{2}$.

Example 2.1.13 ▶ Mesopotamian Approximation of $\sqrt{2}$

Define (x_n) such that $x_1 = 2$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$. Show that $\lim_{n\to\infty} x_n = \sqrt{2}$.

2.1.2 Subsequences

Recall that the sequence $x_n = (-1)^n$ is divergent. However, suppose we were to take all the odd terms from (x_n) to form a new sequence, and to take all the even terms to form another new sequence. One would realise that both new sequences are convergent. This motivates us to study "a part" of a sequence as a new sequence.

Definition 2.1.14 ▶ Subsequence

Let (x_n) be a sequence in $\mathbb{R}$ and let

$$n_1 < n_2 < n_3 < \cdots < n_k < \cdots$$

be an infinite sequence of strictly increasing positive integers, then the sequence (x_{n_k}) is called a **subsequence** of (x_n).

Remark. Note that a sequence is always a subsequence of itself.

Intuitively, if a sequence is convergent, then any of its subsequences should be convergent, too.

Theorem 2.1.15 ▸ Convergence of Subsequences

Let (x_n) be a convergent sequence with $\lim_{n\to\infty} x_n = x$, then for any subsequence (x_{n_k}),

$$\lim_{n_k\to\infty} x_{n_k} = \lim_{k\to\infty} x_{n_k} = x.$$

Proof. Note that for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq N$. Observe that $n_k \geq k$, so whenever $k \geq N$, we have $n_k \geq N$, and so

$$\left|x_{n_k} - x\right| < \epsilon$$

whenever $k \geq N$, i.e.,

$$\lim_{n_k\to\infty} x_{n_k} = \lim_{k\to\infty} x_{n_k} = x.$$

□

Theorems 2.1.15 and 2.1.3 give rise to the following corollary. The proof is left to the reader as an exercise.

Corollary 2.1.16

Let (x_n) be a sequence, then (x_n) is divergent if there exists two subsequences (x_{n_k}) and (x_{n_h}) such that

$$\lim_{k\to\infty} x_{n_k} \neq \lim_{h\to\infty} x_{n_h}.$$

We may apply Theorem 2.1.12 with respect to subsequences. Let us first introduce the notion of *peak points*.

Definition 2.1.17 ▸ Peak Point

Let (x_n) be a sequence in $\mathbb{R}$, x_m is called a **peak** if for all $n \in \mathbb{N}$ with $n > m$, $x_m \geq x_n$.

Next, we shall prove that one can find a monotone subsequence from every sequence.

Theorem 2.1.18 ▸ Existence of Monotone Subsequences

Every infinite sequence has an infinite monotone subsequence.

Proof. Let (x_n) be any sequence in $\mathbb{R}$. We consider the following cases:

Case 1. (x_n) has infinitely many peak points.

This means that there exists infinitely many $m_1, m_2, \cdots \in \mathbb{N}$ such that $m_j > m_i$ whenever $j > i$. Therefore, the subsequence (x_{m_n}) is a monotone decreasing sequence.

Case 2. (x_n) has finitely many peak points.
This means that there exists $m_1, m_2, \cdots, m_k \in \mathbb{N}$ such that $x_{m_1}, x_{m_2}, \cdots, x_{m_k}$ are all the peak points of (x_n). Take $N = m_k + 1$, then for all $n_i \geq N$, since x_{n_i} is not a peak point, there exists some $n_{i+1} > n_i$ such that $x_{n_{i+1}} > x_{n_i}$. Therefore, (x_{n_i}) is an increasing sequence. □

With these preparations done, we state the following theorem:

Theorem 2.1.19 ▸ Bolzano-Weierstrass Theorem (simplified ver.)

Every bounded sequence has a convergent subsequence.

Proof. Let (x_n) be a bounded sequence, then by Theorem 2.1.18 we can find some subsequence (x_{n_k}) which is monotone. Note that (x_{n_k}) is also bounded, so by Theorem 2.1.12 it is convergent. □

2.1.3 Cauchy Criterion

Intuitively, if a sequence is convergent, then over a large interval of $\mathbb{N}$, the change in values of its terms will become smaller and smaller. Correspondingly, this means that the adjacent terms of the sequence will get closer and closer as n becomes large.

Definition 2.1.20 ▸ Cauchy Sequence

A sequence (x_n) is said to be a **Cauchy sequence** if for every $\epsilon > 0$, there exists some $H \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ with $n, m \geq H$, $|x_n - x_m| < \epsilon$.

We can make use of the Cauchy sequence to test for convergence.

Theorem 2.1.21 ▸ Cauchy Convergence Criterion

A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.

Proof. Let (x_n) be a sequence in $\mathbb{R}$. Suppose that (x_n) converges to x, then for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that whenever $n > N$, $|x_n - x| < \frac{\epsilon}{2}$. Therefore, for all $m, n > N$, we have

$$\begin{aligned} |x_m - x_n| &= |x_m - x - x_n + x| \\ &\leq |x_m - x| + |x_n - x| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

and so (x_n) is a Cauchy sequence.

Suppose conversely that (x_n) is a Cauchy sequence on $\mathbb{R}$. We consider the following lemma:

Lemma 2.1.22 ▸ Boundedness of Cauchy Sequences

A Cauchy sequence in $\mathbb{R}$ is bounded.

Proof. Let (x_n) be a Cauchy sequence, then by Definition 2.1.20, there exists some $H \in \mathbb{N}$ such that for all natural numbers $n \geq H$, $|x_n - x_H| < 1$. By Corollary 1.2.6, we have

$$||x_n| - |x_H|| \leq |x_n - x_H| < 1,$$

and so $|x_n| < |x_H| + 1$. Take

$$m = \max\{|x_1|, |x_2|, \cdots, |x_H|, |x_H| + 1\},$$

then $|x_n| < m$ for all $n \in \mathbb{N}^+$. □

Therefore, by Theorem 2.1.19 there exists a subsequence (x_{m_n}) which converges to some $x \in \mathbb{R}$. Thus, there exists some $M \in \mathbb{N}$ such that $|x_{m_n} - x| < \frac{\epsilon}{2}$ for all $\epsilon > 0$ whenever $m_n > M$. By Definition 2.1.20, there exists some $N \in \mathbb{N}$ such that $|x_n - x_{m_n}| < \frac{\epsilon}{2}$ for all $\epsilon > 0$ and for all $n, m_n > N$. Take $K = \max\{M, N\}$, then whenever $n > K$, there is some $m_n > K$ such that

$$\begin{aligned} |x_n - x| &= |x_n - x_{m_n} + x_{m_n} - x| \\ &\leq |x_n - x_{m_n}| + |x_{m_n} - x| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} x_n = x$. $\square$

3

Functions

3.1 Limits of Functions

As a priliminary concept, we shall introduce the notion of a *cluster point*.

Definition 3.1.1 ▸ Cluster Point

Let $A \subseteq \mathbb{R}$. A point c is called a **cluster point** of A if for all $\delta > 0$, there exists at least one $x \in A$ such that $0 < |x - c| < \delta$, i.e., $\big(V_\delta(c) - \{c\}\big) \cap A \neq \varnothing$ for all $\delta > 0$.

Intuitively, this means that the elements of a set A is **densely distributed** around the cluster point, which motivates the following alternative definition:

Theorem 3.1.2 ▸ Alternative Definition of Cluster Points

Let $A \subseteq \mathbb{R}$, then $c \in \mathbb{R}$ is a cluster point of A if and only if there exists a sequence (a_n) in A such that $\lim_{n\to\infty} a_n = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

Proof. Suppose that c is a cluster point of A. Fix any $n \in \mathbb{N}^+$, then there exists some $a_n \in A$ such that $0 < |a_n - c| < \frac{1}{n}$. This means we can obtain a sequence (a_n) with $a_n \neq c$ for all $n \in \mathbb{N}^+$.

For any $\epsilon > 0$, note that there exists some $N \in \mathbb{N}^+$ such that $0 < \frac{1}{N} < \epsilon$. Therefore, for all $n \geq N$, we have $|a_n - c| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$, which means that $\lim_{n\to\infty} a_n = c$.

Conversely, suppose that there is a sequence (a_n) in A with $a_n \neq c$ for all $n \in \mathbb{N}^+$ and $\lim_{n\to\infty} a_n = c$. For any $\delta > 0$, there is some $N \in \mathbb{N}^+$ such that for all $n \geq N$, $0 < |a_n - c| < \delta$. Note that $a_n \in A$, so c is a cluster point of A. □

Analogously to the limit of sequences, we may describe the limiting behaviour of a function as the follows:

> As x gets arbitrarily close to some point c, the function value $f(x)$ can get as close to a constant L as possible.

The notion of x being "arbitrarily close to c" can be precisely captured by a cluster point.

The above intuition is formulated formally as follows:

Definition 3.1.3 ▶ Limit of Functions

Let f be a function over some $A \subseteq \mathbb{R}$. If c is a cluster point of A, then $L \in \mathbb{R}$ is called the **limit** of f at c if for all $\epsilon > 0$, there exists some $\delta \in \mathbb{R}$ such that whenever $0 < |x - c| < \delta$, $|f(x) - L| < \epsilon$.

It is worth noting that a cluster point c of A may not belong to A itself. Therefore, f can have a limit at c even if it is not defined at c. Conversely, even if f is defined at c, the value $f(c)$ is unrelated to $\lim_{x \to c} f(x)$ in general.

Similar to sequences and series, the limit of a function is unique.

Theorem 3.1.4 ▶ Uniqueness of Limit of Functions

Let f be a function over some $A \subseteq \mathbb{R}$ and c be a cluster point of A. If $\lim_{x \to c} f(x)$ exists, then it is unique.

Proof. Suppose there are $L, L' \in \mathbb{R}$ to which f converges at c, then for any $\epsilon > 0$, there exists δ_1 such that whenever $0 < |x - c| < \delta_1$, $|f(x) - L| < \frac{\epsilon}{2}$ and there exists δ_2 such that whenever $0 < |x - c| < \delta_2$, $|f(x) - L| < \frac{\epsilon}{2}$.

Take $\delta = \min\{\delta_1, \delta_2\}$, then whenever $0 < |x - c| < \delta$, we have

$$\begin{aligned} |L - L'| &= |f(x) - L' - f(x) + L| \\ &\leq |f(x) - L'| + |f(x) - L| \\ &< \epsilon \end{aligned}$$

for all $\epsilon > 0$. Therefore, $L = L'$, i.e., $\lim_{x \to c} f(x)$ is unique. □

We now proceed to discussing the boundedness of functions.

Theorem 3.1.5 ▶ Boundedness of Functions

Let f be a function over some $A \subseteq \mathbb{R}$. If $\lim_{x \to c} f(x)$ exists for some cluster point c of A, then f is bounded in some neighbourhood of c.

Proof. Let $\lim_{x \to c} f(x) = L$. Note that there exists some $\delta > 0$ such that whenever

$x \in V_\delta(c)$, $|f(x) - L| < 1$. Consider

$$\begin{aligned} |f(x)| &= |f(x) - L + L| \\ &\leq |f(x) + L| + |L| \\ &< 1 + |L|. \end{aligned}$$

Take

$$M = \begin{cases} 1 + |L| & \text{if } c \notin A \\ \max\{f(c), 1 + |L|\} & \text{if } c \in A \end{cases},$$

then $|f(x)| < M$ for all $x \in V_\delta(c)$, i.e., f is bounded in $V_\delta(c)$. □

Now suppose $\lim_{x\to c} f(x) = L$, then it is easy to visualise that the value of $f(x)$ approaches L indefinitely. Therefore, if we "choose" infinitely many values of f near c, we naturally obtain a convergent sequence.

Theorem 3.1.6 ▶ Sequential Criterion of Limits

Let f be a function over some $A \subseteq \mathbb{R}$ and c be a cluster point of A. $\lim_{x\to c} f(x) = L$ if and only if for every sequence (x_n) in A that converges to c, the sequence $\big(f(x_n)\big)$ converges to L.

Proof. Suppose $\lim_{x\to c} f(x) = L$, then for all $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$.

Let (x_n) be a sequence with $\lim_{n\to\infty} x_n = c$, then there exists some $N \in \mathbb{N}$ such that whenever $n \geq N$, $|x_n - c| < \delta$. Therefore, $|f(x_n) - L| < \epsilon$ for all $n \geq N$, and so $\lim_{n\to\infty} f(x_n) = L$.

We shall prove the converse by considering its contrapositive. If f converges to $L' \neq L$, then by the previous argument every sequence (x_n) converging to c is such that $\big(f(x_n)\big)$ converges to $L' \neq L$.

If f diverges at c, then there exists some $\epsilon_0 > 0$ such that for all $\delta > 0$, there is some x_0 with $0 < |x - c| < \delta$ such that $|f(x_0) - L| \geq \epsilon_0$. Therefore, for all $n \in \mathbb{N}$, we can take some $x_n \in V_{1/n}(c)$ such that $|f(x_n) - L| \geq \epsilon_0$. Either way, there exists a sequence (x_n) converging to c such that $\big(f(x_n)\big)$ does not converge to L. □

Let $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. By using Theorem 3.1.6, we obtain the following limit laws:

1. $\lim_{x\to c}\big(f(x)+g(x)\big) = L + M$.
2. $\lim_{x\to c}\big(f(x)g(x)\big) = LM$.
3. $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$.

Additionally, Theorem 2.1.5 also applies for limits of functions.

3.1.1 One-Sided Limits

In constrast to limits of sequences, for a function, x can approach the cluster point c from different directions. This may result in different limiting behaviours.

Definition 3.1.7 ▶ One-Sided Limit

Let f be a function over some $A \subseteq \mathbb{R}$ and c be a cluster point of A. If for all $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$, then $L = \lim_{x\to c^+} f(x)$ is called the **right-hand limit** of f at c.

If for all $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $-\delta < x - c < 0$, then $L = \lim_{x\to c^-} f(x)$ is called the **left-hand limit** of f at c.

By restricting the values of x_n to either $(-\infty, c)$ or (c, ∞), we obtain the one-sided version of Theorem 3.1.6. One can check that the limit laws still hold for one-sided limits. Note that for f to have a limit at c, its limits from both sides must be consistent.

Theorem 3.1.8 ▶ One-Sided Limits and Limit

$\lim_{x\to c} f(x)$ exists if and only if both $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exists and are equal.

3.2 Continuity

Previously we have stated that the existence and value of $\lim_{x\to c} f(x)$ is unrelated to $f(c)$. However, when it is indeed that case that $f(c) = \lim_{x\to c} f(x)$, it is intuitive to think that f has no gaps around c, which motivates us to define a *continuous function*.

Definition 3.2.1 ▶ Continuity

Let f be a function over some $A \subseteq \mathbb{R}$. f is said to be **continuous** at $c \in A$ if for all $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Otherwise, f is **discontinuous** at c. If f is continuous at all $x \in A$, we say that f is continuous on A.

Specially, we consider the situation where c is not a cluster point of A, i.e., c is "isolated". In this case f is continuous at c trivially, but such continuity is of less interest.

If we wish to prove a function f is continuous over a closed interval $[a, b]$, then other than proving $\lim_{x\to c} f(x) = f(c)$ for all $c \in (a, b)$, we also need to make sure $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

We can make use of Theorem 3.1.6 to prove continuity, the proof of which is left to the reader as an exercise.

Theorem 3.2.2 ▸ Sequential Criterion of Continuity

Let f be a function over some $A \subseteq \mathbb{R}$. f is continuous at $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $\big(f(x_n)\big)$ converges to $f(c)$.

By negating the above, we derive a simple corollary:

Corollary 3.2.3 ▸ Discontinuous Criterion

Let f be a function over some $A \subseteq \mathbb{R}$. f is discontinuous at $c \in A$ if and only if there is a sequence (x_n) in A that converges to c such that the sequence $\big(f(x_n)\big)$ does not converge to $f(c)$.

If f and g are continuous at c, using limit laws, we can easily prove the following results:

1. $f \pm g$ is continuous at c.
2. fg is continuous at c.
3. kf is continuous at c for all $k \in \mathbb{R}$.
4. $\frac{f}{g}$ is continuous at c if $g(c) \neq 0$.

In particular, we also have the following conclusion:

Theorem 3.2.4 ▸ Continuity of Composite Functions

Let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ be functions with $f(A) \subseteq B$. If f is continuous at c ad g is continuous at $b = f(c)$, then $g \circ f$ is continuous at c.

Proof. Since g is continuous at b, for all $\epsilon > 0$ there exists some $\delta > 0$ such that $|g(x) - g(b)| < \epsilon$ whenever $|x - b| < \delta$.

Since f is continuous at c, there is some $\gamma > 0$ such that $|f(x) - f(c)| < \delta$ whenever $|x - c| < \gamma$. Therefore, for all $\epsilon > 0$, whenever $|x - c| < \gamma$, we have $|g\big(f(x)\big) - g\big(f(c)\big)| < \epsilon$, and so $g \circ f$ is continuous at c. □

Recall that in Theorem 3.1.5, we obtain some boundedness properties using the limiting behaviour of a function. For a continuous function, we can derive some more specific conclusion.

Theorem 3.2.5 ▶ Boundedness Theorem

If f is continuous on $[a, b]$, then f is bounded on $[a, b]$.

Proof. Suppose on contrary f is unbounded on $[a, b]$, then for all $n \in \mathbb{N}$, there is some $x_n \in [a, b]$ such that $|f(x_n)| > n$. Note that by this way we obtain a sequence (x_n) bounded on $[a, b]$. By Theorem 2.1.19, this means that (x_n) has a convergent subsequence (x_{n_k}).

Let $\lim_{k\to\infty} x_{n_k} = c$, then $c \in [a, b]$. Since f is continuous on $[a, b]$, by Theorem 3.2.2, $\left(f\left(x_{n_k}\right)\right)$ converges to $f(c)$. However,

$$\left|f\left(x_{n_k}\right)\right| > n_k \geq k$$

for all $k \in \mathbb{N}$, which means that $\left(f\left(x_{n_k}\right)\right)$ is divergent. This is a contradiction. □

Intuitively, if f is bounded on $[a, b]$, it cannot increase nor decrease indefinitely in $[a, b]$ and so it must attain a maximum and a minimum value somewhere in the interval.

Theorem 3.2.6 ▶ Maximum-Minimum Theorem

If f is continuous on $[a, b]$, then there exists some $x^* \in [a, b]$ such that $f(x^*) \geq f(x)$ for all $x \in [a, b]$ and there exists some $x_* \in [a, b]$ such that $f(x_*) \leq f(x)$ for all $x \in [a, b]$.

Proof. It suffices to prove the existence of x^*. The existence of x_* can be proved similarly by considering symmetry. By Theorem 3.2.5, $\sup f([a, b])$ exists. Let $M = \sup f([a, b])$, then for all $n \in \mathbb{N}$, there exists some $x_n \in [a, b]$ such that

$$M - \frac{1}{n} < f(x_n) \leq M.$$

By Theorem 2.1.5, $\lim_{n\to\infty} f(x_n) = M$. Notice that (x_n) is bounded in $[a, b]$, so by Theorem 2.1.19, it has a subsequence $\left(x_{n_k}\right)$ that converges to $x^* \in [a, b]$. Note that f is continuous on $[a, b]$, so by Theorem 3.2.2,

$$\lim_{k\to\infty} f\left(x_{n_k}\right) = f(x^*).$$

This implies that $\lim_{n\to\infty} f(x_n) = f(x^*)$, i.e., $x^* = M$. Therefore, $f(x^*) \geq f(x)$ for all $x \in [a, b]$ and is an absolute maximum. □

3.2.1 Bisection Method

Continuity of a function is a very useful property which can be exploited in root approximation. Informally, if $f(a) < 0$ and $f(b) > 0$ and f is continuous on $[a, b]$, then for f to "reach" $f(b)$ from $f(a)$, it has to cross over some point at which it evaluates to 0.

Theorem 3.2.7 ▶ Location of Roots Theorem

If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists some $c \in (a, b)$ such that $f(c) = 0$.

Proof. Without loss of generality, assume $f(a) < 0 < f(b)$. Take $a_1 = a$, $b_1 = b$ and let $I_1 = [a_1, b_1]$. Let $m_n = \frac{a_n+b_n}{2} \in (a, b)$. If $f(m_1) = 0$, then we are done. Otherwise, define recursively that

$$I_{n+1} = \begin{cases} [a_n, m_n] & \text{if } f(m_n) > 0 \\ [m_n, b_n] & \text{if } f(m_n) < 0 \end{cases}.$$

This way, we obtain a sequence of nested intervals (I_n). Note that

$$b_n - a_n = \frac{b-a}{2^{n-1}},$$

so $\lim_{n\to\infty}(b_n - a_n) = 0$. Note that (a_n) is a monotone increasing sequence and (b_n) is a monotone decreasing sequence, so $\sup(a_n) = \inf(b_n)$. Therefore,

$$\bigcap_{n=1}^{\infty}[a_n, b_n] = \{c\} \subseteq (a, b).$$

Note that both (a_n) and (b_n) are bounded. Since f is continuous at c, by Theorems 2.1.12 and 3.2.2,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(c).$$

Notice that $f(a_n) < 0$ for all $n \in \mathbb{N}$, so $f(c) = \lim_{n\to\infty} f(a_n) \leq 0$. Similarly, $f(c) \geq 0$. Therefore, $f(c) = 0$. □

In particular, the construction of nested intervals I_n's is known as the *bisection algorithm*. Theorem 3.2.7 helps approximate the roots for $f(x) = 0$. Clearly, this means we can determine the existence and location of roots for $f(x) = b$ in a similar fashion.

Theorem 3.2.8 ▸ Intermediate Value Theorem

Let f be continuous over some interval I. If $a, b \in I$ are such that $f(a) \leq f(b)$, then for all $k \in [f(a), f(b)]$, there is some $c \in I$ with $f(c) = k$.

Proof. Without loss of generality, assume $a \leq b$. If $k = f(a)$ or $k = f(b)$, by taking $c = a$ or $c = b$ respectively we are done. If $f(a) < k < f(b)$, consider $g(x) = f(x) - k$. Note that g is continuous over $[a, b] \subseteq I$ and $g(a)g(b) < 0$, so by Theorem 3.2.7, there is some $c \in (a, b) \subseteq I$ such that $g(c) = f(c) - k = 0$, i.e., $f(c) = k$. □

We can generalise the above further.

Theorem 3.2.9 ▸ Preservation of Closed Intervals

If f is continuous on $[a, b]$, then

$$f([a,b]) = \big[\inf f([a,b]), \sup f([a,b])\big].$$

Proof. Note that $f([a,b]) \subseteq \big[\inf f([a,b]), \sup f([a,b])\big]$. By Theorem 3.2.6, there exists $x^*, x_* \in [a,b]$ such that $f(x^*) = \sup f([a,b])$ and $f(x_*) = \inf f([a,b])$. Take any $k \in [f(x_*), f(x^*)]$, by Theorem 3.2.8, there is some $c \in [a,b]$ such that $f(c) = k$. This means $k \in f([a,b])$ and so

$$\big[\inf f([a,b]), \sup f([a,b])\big] = [f(x_*), f(x^*)] \subseteq f([a,b]).$$

Therefore, $f([a,b]) = \big[\inf f([a,b]), \sup f([a,b])\big]$. □

3.2.2 Uniform Continuity

To determine continuity of a function f at a point c, we need to find a $\delta > 0$ for each $\epsilon > 0$. Note that our choice of δ here might depend on both c and ϵ. However, for certain functions, we can "generate" an appropriate δ based solely on the ϵ given, i.e., we can find some $\delta = g(\epsilon)$ which is **independent** of the value of c. Such functions are said to be *uniformly continuous*.

Definition 3.2.10 ▸ Uniform Continuity

Let f be a function over some $A \subseteq \mathbb{R}$. f is said to be **uniformly continuous** if for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x, y \in A$,

$$|f(x) - f(y)| < \epsilon$$

whenever $|x - y| < \delta$.

Notice that in Definition 3.2.10, the choice of δ is independent of x and y.

Theorem 3.2.2 has a version for uniform continuity as well.

Theorem 3.2.11 ▸ Sequential Criterion of Uniform Continuity

Let f be a function over some $A \subseteq \mathbb{R}$. f is uniformly continuous on A if and only if for any two convergent sequences (x_n) and (y_n) in A such that $\lim_{n\to\infty}(x_n - y_n) = 0$,

$$\lim_{n\to\infty}\big(f(x_n) - f(y_n)\big) = 0.$$

Proof. Suppose f is uniformly continuous on A, then for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

for all $x, y \in A$ with $|x - y| < \delta$. Let (x_n) and (y_n) be two convergent sequences in A with $\lim_{n\to\infty}(x_n - y_n) = 0$, then there is some $N \in \mathbb{N}$ such that

$$|x_n - y_n| < \delta$$

whenever $n \geq N$. Therefore, for all $\epsilon > 0$, whenever $n \geq N$,

$$|f(x_n) - f(y_n)| < \epsilon,$$

and so $\lim_{n\to\infty}\big(f(x_n) - f(y_n)\big) = 0$.

We shall prove the converse by considering its contrapositive. Suppose f is not uniformly continuous on A, then there is some $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists some $x, y \in A$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon_0$.

This means that for all $n \in \mathbb{N}$, we can take some $x_n, y_n \in A$ with $0 \leq |x_n - y_n| < \frac{1}{n}$, thus obtaining two sequences (x_n) and (y_n). Note that $\lim_{n\to\infty}(x_n - y_n) = 0$, but

$$|f(x_n) - f(y_n)| \geq \epsilon_0,$$

which means $\big(f(x_n) - f(y_n)\big)$ does not converge to 0. □

The negations of Definition 3.2.10 and Theorem 3.2.11 can be very useful when proving by contrapositive. For example, we can consider the following result:

Theorem 3.2.12 ▸ Uniform Continuity Theorem

If a function f is continuous on a closed and bounded interval $[a, b]$, then it is uniformly continuous on $[a, b]$.

Proof. Suppose on contrary that f is not uniformly continuous on $[a, b]$, then there is some $\epsilon_0 > 0$ such that for all $n \in \mathbb{N}$, we can find $x_n, y_n \in [a, b]$ with $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$. Note that (x_n) is bounded, so by Theorem 2.1.19, it has a subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = c$. Similarly, (y_n) has a convergent subsequence (y_{n_k}). Therefore,

$$\begin{aligned}\lim_{k\to\infty} y_{n_k} &= \lim_{k\to\infty}\left[x_{n_k} - (x_{n_k} - y_{n_k})\right]\\ &= c - \lim_{n\to\infty}(x_n - y_n)\\ &= c.\end{aligned}$$

By Theorem 3.2.2, since f is continuous at c,

$$\lim_{k\to\infty} f\left(x_{n_k}\right) = \lim_{k\to\infty} f\left(y_{n_k}\right) = f(c).$$

This implies $\lim_{k\to\infty}\left(f\left(x_{n_k}\right) - f\left(y_{n_k}\right)\right) = 0$, but

$$\left|f\left(x_{n_k}\right) - f\left(x_{n_k}\right)\right| \geq \epsilon_0 > 0,$$

which is a contradiction. □

While the negations of Definition 3.2.10 and Theorem 3.2.11 aids in many proofs, proving uniform continuity directly with them are difficult. Thus, we introduce another tool to prove uniform continuity.

Definition 3.2.13 ▸ Lipschitz Function

Let f be a function over some $A \subseteq \mathbb{R}$. f is said to be a **Lipschitz Function** (or satisfy the **Lipschitz Condition**) if there exists some $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in A$.

Remark. Note that we can re-write the Lipschitz Condition as

$$\frac{|f(x) - f(y)|}{|x - y|} \leq K$$

for all $x \neq y$. Note that the limit as $y \to x$ of the left-hand side is the *first order derivative* of f, so the above expression suggests that $f'(x)$ is bounded.

It can be easily seen that by taking $\delta = \frac{\epsilon}{K}$ for each $\epsilon > 0$, a Lipschitz function can be proven to be uniformly continuous on A.

Theorem 3.2.14 ▸ Uniform Continuity of Lipschitz Functions

If $f : A \to \mathbb{R}$ is a Lipschitz function, then it is uniformly continuous on A.

Lastly, we shall state the relation between uniform continuity and Cauchy sequences.

Theorem 3.2.15 ▸ Preservation of Cauchy Sequences

If f is uniformly continuous on $A \subseteq \mathbb{R}$ and (x_n) is a Cauchy sequence on A, then $\big(f(x_n)\big)$ is also a Cauchy sequence.

Proof. Since f is uniformly continuous on A, for all $\epsilon > 0$ there is some $\delta > 0$ such that for all $x, y \in A$,

$$|f(x) - f(y)| < \epsilon$$

whenever $|x - y| < \delta$. Since (x_n) is Cauchy, there is some $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|x_m - x_n| < \delta$. Therefore,

$$|f(x_m) - f(y_n)| < \epsilon$$

whenever $m, n \geq N$. Therefore, $\big(f(x_n)\big)$ is a Cauchy sequence. □

Using the preservation of Cauchy sequences, we can prove the following result:

Theorem 3.2.16 ▸ Continuous Extension Theorem

A function f is uniformly continuous on (a, b) if and only if it can be defined at a and b such that the extended function is continuous on $[a, b]$.

Proof. The leftward direction is obvious by Theorem 3.2.12. Suppose f is uniformly continuous on (a, b). Let (x_n) be a sequence in (a, b) that converges to a. By Theorem 2.1.21, (x_n) is Cauchy, and so $\big(f(x_n)\big)$ is Cauchy by Theorem 3.2.15.

Define $f(a) = \lim_{n\to\infty} f(x_n)$, and let (y_n) be a sequence in (a, b) that converges to a. Since f is uniformly continuous on (a, b), by Theorem 3.2.11,

$$\lim_{n\to\infty} \big(f(x_n) - f(y_n)\big) = 0,$$

and so

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \big[f(x_n) - \big(f(x_n) - f(y_n)\big)\big] = f(a).$$

Note that $y_n > a$ for all $n \in \mathbb{N}$, so $\lim_{x\to a^+} f(x) = f(a)$. Similarly, defining $f(b) = \lim_{n\to\infty} f(z_n)$ for some sequence (z_n) that converges to b, we can prove that $\lim_{x\to b^-} f(x) = f(b)$. Therefore, the extended function is continuous on $[a, b]$. □

3.2.3 Discontinuity

A discontinuous function can take a number of forms, one of which is a sudden change of value at a point c. Such functions can be vividly depicted as a continuous function "ripped apart" at $x = c$. Intuitively, the left-hand and right-hand limits of f would still exist despite discontinuity.

Theorem 3.2.17 ▶ One-Sided Limit for Monotone Functions

et f be increasing on $I \subseteq \mathbb{R}$. If $c \in I$ is not an end point, then

$$\lim_{x\to c^-} f(x) = \sup\{f(x) : x \in I, x < c\}$$
$$\lim_{x\to c^+} f(x) = \inf\{f(x) : x \in I, x > c\}.$$

Proof. Without loss of generality, it suffices to prove for the left-hand limit. Note that $S := \{f(x) : x \in I, x < c\}$ is non-empty and $f(c)$ is its upper bound. Therefore, $\sup S$ exists. Denote this supremum by L, then for all $\epsilon > 0$, $L - \epsilon$ is not an upper bound of S, which mean that there is some $y \in I$ with $y < c$ such that $L - \epsilon < f(y) \leq L$.

Take $\delta = c - y > 0$, then whenever $-\delta < x - c < 0$, we have $y < x < c$. Since f is increasing on I, this implies that

$$L - \epsilon < f(y) < f(x) \leq L < L + \epsilon.$$

Therefore, $|f(x) - L| < \epsilon$ and so $\lim_{x\to c^-} = \sup\{f(x) : x \in I, x < c\}$. □

Note that c is the only discontinuous point of f, therefore, f is continuous if the "gap" at c were to be closed.

Corollary 3.2.18

Let f be increasing on $I \subseteq \mathbb{R}$. If $c \in I$ is not an end point, then the followings are equivalent:

1. *f is continuous at c.*
2. $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$.
3. $\sup\{f(x) : x \in I, x < c\} = \inf\{f(x) : x \in I, x > c\} = f(c)$.

Such discontinuity as described above is known as a *jump discontinuity* at c. The following definition gives a measure for "how discontinuous" f is at c:

Definition 3.2.19 ▶ Jump

Let f be increasing on $[a, b]$. The **jump** of f at c is defined as

$$j_f(c) = \begin{cases} \lim_{x\to a^+} f(x) - f(a) & \text{if } c = a \\ f(b) - \lim_{x\to b^-} f(x) & \text{if } c = b \\ \lim_{x\to c^+} f(x) - \lim_{x\to c^-} f(x) & \text{otherwise} \end{cases}.$$

It can be immediately seen that f is continuous at c if and only if $j_f(c) = 0$.

Next, let us consider an interesting question:

> How many jump discontinuities can a monotone function have over its domain?

The answer might be surprising: the number of jump discontinuities of a monotone function is at most countably infinite!

Theorem 3.2.20 ▶ Discontinuous Points of Monotone Functions

Let f be a monotone function over $I \subseteq \mathbb{R}$, then f has countably many points of discontinuity.

Lastly, we shall state a theorem regarding inverse functions.

Theorem 3.2.21 ▶ Continuous Inverse Theorem

Let $f : I \to \mathbb{R}$ be a strictly monotone and continuous function, then $f^{-1} : f(I) \to I$ is also a strictly monotone and continuous function.

4 Topology and Metric Spaces

4.1 Metric Space

So far we have been dealing with sequences and functions which are real-valued. However, other abstract objects might also exhibit limiting bahaviours. Note that when we define the notion of limits, we focus on the behaviour of an object as it “gets arbitrarily close” to a certain point. We would like to abstract the concept of closeness.

Definition 4.1.1 ▸ Metric

A **metric** on a set S is a function $d : S \times S \to \mathbb{R}$ that satisfies the following:

1. $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
2. $d(x, y) = 0$ if and only if $x = y$ (definiteness);
3. $d(x, y) = d(y, x)$ for all $x, y \in S$ (symmetry);
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in S$ (triangular inequality).

Remark. A metric is sometimes also called a **distance function**.

Definition 4.1.2 ▸ Metirc Space

A **metric space** (S, d) is a set S together with a metric d on S.

In the Euclidean space $\mathbb{R}^n$, a usual definition for distance is

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{n}(y_i - x_i)^2\right]^{\frac{1}{2}}.$$

Note that $(\mathbb{R}^n, d_2)$ is a metric space, where d_2 is known as the *Euclidean distance*. In general, we can prove that for any $p \in \mathbb{N}^+$,

$$d_p(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{n}|y_i - x_i|^p\right]^{\frac{1}{p}}$$

is a metric over $\mathbb{R}^n$. Furthermore, notice that

$$\max_{i\in\mathbb{N}^+,i\leq n}|y_i - x_i|^p \leq \sum_{i=1}^{n}|y_i - x_i|^p \leq n \max_{i\in\mathbb{N}^+,i\leq n}|y_i - x_i|^p.$$

Taking the p-th root on all three parts, we have

$$\max_{i\in\mathbb{N}^+,i\leq n}|y_i - x_i| \leq \left[\sum_{i=1}^{n}|y_i - x_i|^p\right]^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max_{i\in\mathbb{N}^+,i\leq n}|y_i - x_i|.$$

By Theorem 2.1.5, this allows us to define

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \lim_{p\to\infty} d_p(\boldsymbol{x}, \boldsymbol{y}) = \max_{i\in\mathbb{N}^+,i\leq n}|y_i - x_i|.$$

$d_\infty(\boldsymbol{x}, \boldsymbol{y})$ can be alternatively written as $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty$, which is known as the *infinite norm*.

Using metric spaces, we can extend and generalise many notions defined in previous chapters from $\mathbb{R}$ to any set.

Definition 4.1.3 ▶ Neighbourhood

Let (S, d) be a metric space. For any $\epsilon > 0$, the **ϵ-neighbourhood** of $x_0 \in S$ is the set

$$V_\epsilon(x_0) := \{x \in S : d(x_0, x) < \epsilon\}.$$

Generally, a set U is called a **neighbourhood** of $x \in S$ if there exists some $V_\epsilon(x) \subseteq U$.

Now, we are able to define convergence of a sequence over any general set.

Definition 4.1.4 ▶ Convergence

Let (x_n) be a sequence over a metric space (S, d). (x_n) is said to be **convergent** to $x \in S$ if for all $\epsilon > 0$, there exists some $K \in \mathbb{N}$ such that whenever $n \geq K$, $d(x_n, x) < \epsilon$.

Similarly, we can re-state the definitions and theorems regarding limits and continuity in terms of metric spaces.

4.2 Compact Sets

In real numbers, we can define the boundedness and openness of an interval. These ideas can be extended to a metric space. In particular, we would like to investigate a set which can be informally stated as "closed and bounded". We will define openness first.

Definition 4.2.1 ▶ Open and Closed Sets

Let (S, d) be a metric space. S is called an **open set** if and only if for all $x \in S$, S is a neighbourhood of x. A set whose complement is open is said to be **closed**.

Remark. Note that $\mathbb{R}$ and $\varnothing$ are both closed and open.

We have the following important properties:

Theorem 4.2.2 ▶ Open Set Properties

The union of open sets is open and the intersection of finitely many open sets is open.

Theorem 4.2.3 ▶ Closed Set Properties

The intersection of closed sets is closed and the union of finitely many closed sets is closed.

It turns out that openness of a set has important relations to the convergence of sequences defined over the set.

Theorem 4.2.4 ▶ Characterisation of Closed Sets

A set F is closed if and only if for every convergent sequence (x_n) in F, $\lim_{n\to\infty} x_n \in F$.

Theorem 4.2.5 ▶ Alternative Characterisation of Closed Sets

A set F is closed if and only if it contains all of its cluster points.

We can use open sets to test the continuity of a function.

Theorem 4.2.6 ▶ Global Continuity Theorem

Let (A, d_A) and (B, d_B) be metric spaces. $f : A \to B$ is continuous on A if and only if for every open set $G \subseteq B$, there exists an open set H such that $H \cap A = f^{-1}(G)$.

In real numbers, the above theorem gives the following corollary:

Corollary 4.2.7 ▶ Global Continuity Theorem on $\mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$ is continuous over $\mathbb{R}$ if and only if for every open set $G \subseteq \mathbb{R}$, $f^{-1}(G)$ is open.

A closed and bounded set is the pre-requisite of many important theorems. Before we formally define it, we introduce the notion of an *open cover*.

Definition 4.2.8 ▶ Open Cover

Let (S, d) be a metric space. For $A \subseteq S$, an **open cover** of A is a collection

$$\mathcal{G} := \left\{G \in \mathcal{P}(S) : G \text{ is open}, A \subseteq \bigcup G\right\}.$$

If $\mathcal{G}' \subseteq \mathcal{G}$ and $\mathcal{G}'$ is an open cover of A, then $\mathcal{G}'$ is a **subcover** of $\mathcal{G}$.

Indeed, an open cover is just a collection of open sets which "covers up" a set A. Intuitively, A has a boundary if it can be "covered" with finitely many such open sets.

Definition 4.2.9 ▶ Compact Set

Let (S, d) be a metric space. A set $K \subseteq S$ is said to be **compact** if every open cover $\mathcal{G}$ of K has a finite subcover.

In other words, Definition 4.2.9 states that if $\mathcal{G}$ is any open cover of K, then K is compact if there are finitely many open sets $G_1, G_2, \cdots, G_n \in \mathcal{G}$ whose union is K.

Next, we reveal the equivalence between compactness and closed-and-boundedness.

Definition 4.2.10 ▶ Bounded Set

Let (S, d) be a metric space. A set $A \subseteq S$ is **bounded** if there exist some $M \in \mathbb{R}$ and $x \in S$ such that $A \subseteq V_M(x)$.

It can be seen that a bounded set is confined within a particular open set. Naturally, this means we only need finitely many open sets for their union to contain this set.

Theorem 4.2.11 ▶ Heine-Borel Theorem

Let (S, d) be a metric space. $K \subseteq S$ is compact if and only if it is closed and bounded.

Consider a compact set K and a function f whose domain contains K. This means that f would map K to another set. Intuitively, this image should be compact as well provided that $f(K)$ has no "gaps".

Theorem 4.2.12 ▶ Preservation of Compactness

Let (S, d) be a metric space and $f : S \to \mathbb{R}$. If $K \subseteq S$ is compact and f is continuous, then $f(K)$ is compact.

The notion of having no gaps in a set means that the set is connected, which can be defined as follows:

Definition 4.2.13 ▸ Connected Set

A set U is **disconnected** if it has an open cover $\{A, B\}$ such that $A \cap U$, $B \cap U$ are both non-empty but disjoint. Otherwise, U is **connected**.

An interval in $\mathbb{R}$ is trivially connected. However, we would like to state that any connected subset of $\mathbb{R}$ must also be an interval.

Proposition 4.2.14 ▸ Connectedness and Intervals

A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Lastly, we state without proof the analogue of the Intermediate Value Theorem in terms of connected sets.

Theorem 4.2.15 ▸ Generalised Intermediate Value Theorem

Let $f : S \to \mathbb{R}$ be a function. If $E \subseteq S$ is connected, then $f(E)$ is connected.