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Topology

1.1 Topological Spaces

Definition 1.1.1 ► Topology

A **topology** on a set X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ such that

- $\emptyset, X \in \mathcal{T}$;
- for any index set I , if $\{X_i : i \in I\} \subseteq \mathcal{T}$, then $\bigcup_{i \in I} X_i \in \mathcal{T}$;
- for any $X_1, X_2, \dots, X_n \in \mathcal{T}$, $\bigcap_{i=1}^n X_i \in \mathcal{T}$.

The pair (X, \mathcal{T}) is said to be a **topological space**. A subset $Y \subseteq X$ is **open** if $Y \in \mathcal{T}$.

Remark. For any set X , we define $\{\emptyset, X\}$ as the *trivial topology* on X , $\mathcal{P}(X)$ as the *discrete topology*, and $\{X \setminus U : U \subseteq X \text{ is finite}\} \cup \{\emptyset\}$ as the *co-finite topology*.

The set $\{(-\alpha, \alpha) : \alpha > 0\} \cup \{\mathbb{R}, \emptyset\}$ defines a topology on \mathbb{R} . This example also demonstrates why it is crucial to only consider closure under finite intersections when defining a topology, because

$$\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = [-1, 1] \notin \mathcal{T}.$$

We now seek a systematic method to generate a topology given any set. The idea here is to make use of a *cover*.

Definition 1.1.2 ► Basis

A **basis** for a topology on X is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that

- for any $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$;
- for any $x \in X$ and $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

It may be useful to see a basis as a cover of a set with the second additional property as stated in the above definition. Notice that the first property of the basis \mathcal{B} is basically saying that

$$X \subseteq \bigcup \mathcal{B},$$

i.e., \mathcal{B} is a cover of X .

Given any basis \mathcal{B} for some topology on X , a set generated by \mathcal{B} can be defined as

$$\mathcal{T} := \{U \subseteq X : \text{for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\}$$

We will show that \mathcal{T} is a topology on X . First, it is clear that $\emptyset, X \in \mathcal{T}$.

Let I be an index set and $\{X_i : i \in I\} \subseteq \mathcal{P}(X)$ be any collection of subsets of X . Notice that for any $x \in \bigcup_{i \in I} X_i$, there exists some $j \in I$ such that $x \in X_j \subseteq \mathcal{T}$. According to our construction, this means that there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq X_j \subseteq \mathcal{T}$. Therefore, $\bigcup_{i \in I} X_i \subseteq \mathcal{T}$ as desired.

To prove that \mathcal{T} is closed under finite intersection, we consider the following lemma:

Lemma 1.1.3 ▶ Finite Intersection of Elements in Basis Is Covered

Let \mathcal{B} be a basis for a topology on X and $B_1, B_2, \dots, B_n \in \mathcal{B}$, then for any $x \in \bigcap_{i=1}^n B_i$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcap_{i=1}^n B_i$.

Proof. The case where $n = 1$ is trivial by taking $B = B_1$. Suppose that there is some integer $k \geq 1$ such that for any $B_1, B_2, \dots, B_k \in \mathcal{B}$ and any $x \in \bigcap_{i=1}^k B_i$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcap_{i=1}^k B_i$. Take any $B_{k+1} \in \mathcal{B}$. It is clear that for any $x \in \bigcap_{i=1}^{k+1} B_i$, there exists some $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^k B_i.$$

Notice that $x \in B_{k+1} \in \mathcal{B}$, so we know that $x \in B \cap B_{k+1}$. By Definition 1.1.2, this means that there exists some $B' \in \mathcal{B}$ such that

$$x \in B' \subseteq B \cap B_{k+1} \subseteq \bigcap_{i=1}^{k+1} B_i.$$

□

Now, suppose $X_1, X_2, \dots, X_n \in \mathcal{T}$ are finitely many subsets of X . Take any $x \in \bigcap_{i=1}^n X_i$. It is clear that $x \in X_i$ for each $i = 1, 2, \dots, n$. Therefore, for each $i = 1, 2, \dots, n$, there exists some $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq X_i$. By Lemma 1.1.3, this means that there exists some set $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^n B_i \subseteq \bigcap_{i=1}^n X_i.$$

Therefore, $\bigcap_{i=1}^n X_i \in \mathcal{T}$. So this set \mathcal{T} generated by \mathcal{B} is indeed a topology on X .

The following proposition further shows that the topology generated by a basis \mathcal{B} is the set

of all possible unions of elements in \mathcal{B} :

Proposition 1.1.4 ▶ Equivalent Construction of Topologies Generated from Bases

Let X be any set. If \mathcal{B} is a basis for a topology \mathcal{T} on X , then

$$\mathcal{T} = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \in \mathcal{P}(\mathcal{B}) \right\}.$$

Proof. Denote

$$\mathcal{T}_{\mathcal{B}} := \{U \subseteq X : \text{for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\}.$$

It suffices to prove that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. Take any $T \in \mathcal{T}$, then there exists some $\mathcal{V} \in \mathcal{P}(\mathcal{B})$ such that $T = \bigcup_{A \in \mathcal{V}} A$. This means that for every $t \in T$, there exists some $B_t \in \mathcal{V}$ such that $t \in B_t \subseteq T$. Therefore, $T \in \mathcal{T}_{\mathcal{B}}$. Conversely, for any $S \in \mathcal{T}_{\mathcal{B}}$, there exists some $B_s \in \mathcal{B}$ for each $s \in S$ such that $s \in B_s$. Denote $\mathcal{U} := \{B_s : s \in S\} \in \mathcal{P}(\mathcal{B})$, then it is clear that $S \subseteq \bigcup_{B \in \mathcal{U}} B$. Since $B_s \subseteq S$ for each $s \in S$, we have $\bigcup_{B \in \mathcal{U}} B \subseteq S$, which implies that $S = \bigcup_{B \in \mathcal{U}} B$. This means that $S \in \mathcal{T}$. Therefore, $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$, which means that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. \square

Next, we define a special topology in Euclidean spaces using open balls.

Definition 1.1.5 ▶ Standard Topology

For any $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and any $r > 0$. Denote the Euclidean open ball centred at \mathbf{x} with radius r by

$$B_r(\mathbf{x}) := \left\{ \mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n : \sqrt{\sum_{i=1}^n (x_i - y_i)^2} < r \right\}$$

The **standard topology** on \mathbb{R}^n is the set generated by the basis

$$\mathcal{B} := \{B_r(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n, r \in \mathbb{R}^+\}.$$

It may be helpful to actually show that this set \mathcal{B} is indeed a basis of a topology on \mathbb{R}^n . The fact that \mathcal{B} is a cover for \mathbb{R}^n is trivial enough. Take any $\mathbf{x} \in \mathbb{R}^n$ and balls $B_{\alpha}(\mathbf{x}_1), B_{\beta}(\mathbf{x}_2) \in \mathcal{B}$ such that $\mathbf{x} \in B_{\alpha}(\mathbf{x}_1) \cap B_{\beta}(\mathbf{x}_2)$ (the existence of these 2 balls is again trivial enough). Take

$$r = \min \{\alpha - \|\mathbf{x} - \mathbf{x}_1\|, \beta - \|\mathbf{x} - \mathbf{x}_2\|\}.$$

Clearly, $r > 0$ and $\mathbf{x} \in B_r(\mathbf{x})$, so we are done.

Now, we discuss the analogue of the subset relation in topologies.

Definition 1.1.6 ► Fineness and Coarseness

Let \mathcal{T} and \mathcal{T}' be topologies on some set X . We say that \mathcal{T} is **finer** than \mathcal{T}' , or equivalently, that \mathcal{T}' is **coarser** than \mathcal{T} , if $\mathcal{T}' \subseteq \mathcal{T}$.

Observe that any topology of X must be a subset of $\mathcal{P}(X)$, which is the discrete topology on X , so the discrete topology is the finest topology on a set.

Remark. For any basis \mathcal{B} for a topology on X , the topology generated by \mathcal{B} is the coarsest topology containing \mathcal{B} .

The above remark is easy to verify. Let \mathcal{T} be any topology on X with $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}_{\mathcal{B}}$ be the topology generated by \mathcal{B} . For any $T \in \mathcal{T}_{\mathcal{B}}$, by Proposition 1.1.4, there exists some $V \subseteq \mathcal{B}$ such that $T = \bigcup_{A \in V} A$. Note that $A \in \mathcal{T}$ for all $A \in \mathcal{V}$, so by Definition 1.1.1, $T \in \mathcal{T}$ and so $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$ as desired.

This motivates us to consider fineness in terms of bases.

Proposition 1.1.7 ► Fineness in Terms of Bases

Let \mathcal{B} and \mathcal{B}' generate topologies \mathcal{T} and \mathcal{T}' respectively on X . \mathcal{T}' is finer than \mathcal{T} if and only if for every $B \in \mathcal{B}$ and any $x \in B$, there exists some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B$.

Proof. Suppose that \mathcal{T}' is finer than \mathcal{T} , then $\mathcal{T} \subseteq \mathcal{T}'$. Take any $B \in \mathcal{B}$, then by Proposition 1.1.4, $B \in \mathcal{T}$, which means that $B \in \mathcal{T}'$. Since \mathcal{B}' is a basis for \mathcal{T}' , by Definition 1.1.2 for any $x \in B$, there exists some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B$.

Suppose conversely that for every $B \in \mathcal{B}$ and any $x \in B$, there is some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B$. Take any $T \in \mathcal{T}$, for each $x \in T$, by Definition 1.1.2 there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq T$, and so we can find some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B \subseteq T$, so $T \in \mathcal{T}'$. Therefore, $\mathcal{T} \subseteq \mathcal{T}'$ and so \mathcal{T}' is finer than \mathcal{T} . \square

Recall that every basis of a topology on X is an open cover of X consisting only of subsets of X . Therefore, the union of the elements in the basis is essentially X itself. This motivates us to propose another way to generate a topology on a set.

Definition 1.1.8 ► Sub-basis

A **sub-basis** of X is a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ such that $\bigcup_{A \in \mathcal{S}} A = X$.

Remark. Every basis is a sub-basis.

For an arbitrary set X , let \mathcal{S} be a sub-basis and denote the collection of all finite subsets of $\mathcal{P}(\mathcal{S})$ as $\mathcal{F}_{\mathcal{S}}$. Define

$$\mathcal{U}_{\mathcal{S}} := \left\{ \bigcap_{A \in F} A : F \in \mathcal{F}_{\mathcal{S}} \right\}$$

to be the collection of all finite intersections of sets in \mathcal{S} . The topology generated by a sub-basis of X is given by

$$\mathcal{T} := \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{U}_{\mathcal{S}} \right\}.$$

We shall show that \mathcal{T} is indeed a topology on X by considering the following proposition:

Proposition 1.1.9 ▶ Finite Intersections of Sets in a Sub-basis Form a Basis

Let \mathcal{S} be a sub-basis for a set X and let $\mathcal{U}_{\mathcal{S}}$ be the set of all finite intersections of sets in \mathcal{S} , then $\mathcal{U}_{\mathcal{S}}$ is a basis of a topology on X .

Proof. Take any $x \in X$. By Definition 1.1.8, we have $x \in \bigcup_{A \in \mathcal{S}} A$. Therefore, there exists some $A \in \mathcal{S} \subseteq \mathcal{P}(X)$ such that $x \in A$. For any $x \in X$ and $B_1, B_2 \in \mathcal{U}_{\mathcal{S}}$ such that $x \in B_1 \cap B_2$, notice that $B_1 \cap B_2$ is a finite intersection of sets in \mathcal{S} , so $B_1 \cap B_2 \in \mathcal{U}_{\mathcal{S}}$. Therefore, by Definition 1.1.2, $\mathcal{U}_{\mathcal{S}}$ is a basis. \square

With Propositions 1.1.9 and 1.1.4, it is clear that \mathcal{T} as constructed above is a topology on X .

1.2 Metric Spaces

Definition 1.2.1 ▶ Metric

A **metric** on a set S is a function $d : S \times S \rightarrow \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, y) = 0$ if and only if $x = y$ (definiteness);
- $d(x, y) = d(y, x)$ for all $x, y \in S$ (symmetry);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

Remark. A metric is sometimes also called a *distance function*.

A metric generalises the notion of distance in Euclidean spaces. We can weaken the above axioms to arrive at the following definition:

Definition 1.2.2 ▶ Pseudo-metric

A **pseudo-metric** on a set S is a function $d : S \times S \rightarrow \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, x) = 0$ for all $x \in S$;
- $d(x, y) = d(y, x)$ for all $x, y \in S$ (symmetry);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

The key difference between a pseudo-metric and a metric is that a pseudo-metric only requires that every element is at 0 distance away from itself, whereas a metric requires that every element is **the only element** that is at 0 distance away from itself.

By dropping the requirement on symmetry, we obtain the following definition:

Definition 1.2.3 ▶ Quasi-metric

A **quasi-metric** on a set S is a function $d : S \times S \rightarrow \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, y) = 0$ if and only if $x = y$ (definiteness);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

We equip a set with a metric to generalise the Euclidean spaces.

Definition 1.2.4 ▶ Metric Space

A **metric space** (S, d) is a set S together with a metric d on S .

The most basic example of a metric is the *discrete metric* defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

over any set X , which essentially is just a characteristic function.

Recall that in an inner product space (V, g) over some field \mathbb{F} , we can define the length of any $\mathbf{v} \in V$ as

$$\|\mathbf{v}\| = \sqrt{g(\mathbf{v}, \mathbf{v})}.$$

This length function induces a metric over V given by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

In the Euclidean space \mathbb{R}^n , a usual definition for distance is

$$d_2(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^n (y_i - x_i)^2 \right]^{\frac{1}{2}}.$$

Note that (\mathbb{R}^n, d_2) is a metric space, where d_2 is known as the *Euclidean distance*. In general, we can prove that for any $p \in \mathbb{N}^+$,

$$d_p(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^n \|y_i - x_i\|^p \right]^{\frac{1}{p}}$$

is a metric over \mathbb{F}^n for any inner product space (\mathbb{F}^n, g) where \mathbb{F} is a field, known as the L^p -norm. Furthermore, notice that

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p \leq \sum_{i=1}^n \|y_i - x_i\|^p \leq n \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p.$$

Taking the p -th root on all three parts, we have

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\| \leq \left[\sum_{i=1}^n \|y_i - x_i\|^p \right]^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

By Squeeze Theorem, this allows us to define

$$d_\infty(\mathbf{x}, \mathbf{y}) = \lim_{p \rightarrow \infty} d_p(\mathbf{x}, \mathbf{y}) = \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

$d_\infty(\mathbf{x}, \mathbf{y})$ can be alternatively written as $\|\mathbf{x} - \mathbf{y}\|_\infty$, which is known as the *infinite norm*.

The p -adic numbers can be defined from the following lemma:

Lemma 1.2.5 ► p -adic Numbers

Let p be any prime number. For all $x \in \mathbb{Q} \setminus \{0\}$, there exists a unique $k \in \mathbb{Z}$ such that

$$x = \frac{p^k r}{s}, \quad r, s \in \mathbb{Z}$$

with $p \nmid r, s$ and $s \neq 0$.

The p -adic norm is defined as

$$|x|_p = \begin{cases} p^{-k} & \text{if } x = \frac{p^k r}{s}, \\ 0 & \text{if } x = 0 \end{cases},$$

which induces a metric over \mathbb{Q} defined by

$$d(x, y) = |x - y|_p.$$

We can show that the p -adic metric satisfies

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

for all $x, y, z \in \mathbb{Q}$. Such a metric is known as an *ultra-metric*.

Given any metric space, the metric will induce a distance between subsets of the space.

Definition 1.2.6 ► Distance between Subsets

Let (X, d) be a metric space and $A, B \subseteq X$ be non-empty. The **distance** between A and B is defined as

$$d(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

Additionally, we may wish to define a measure for the size of a subset in a metric space.

Definition 1.2.7 ► Diameter

Let (X, d) be a metric space. The **diameter** of a set $A \subseteq X$ is defined as

$$\text{diam}(A) := \sup\{d(x, y) : (x, y) \in A \times A\}.$$

The set A is **bounded** if $\text{diam}(A)$ is finite.

The name “diameter” is not a coincidence with the diameter of a graph. Specifically, if we consider a graph $G = (V, E)$, the pair (V, d) forms a metric space with $d(u, v)$ being the usual distance between two vertices in G defined as the size of the shortest u - v path in G . It is clear that d is indeed a metric.

Now, let us consider the subgraph $H \subseteq G$ induced by any $U \subseteq V$ and check the eccentricity for H , i.e.,

$$\epsilon(u) = \max\{d_H(u, u') : u' \in U\} \quad \text{for all } u \in U.$$

Now, the diameters for H can be computed as

$$\begin{aligned} \text{diam}(H) &= \max\{\epsilon(u) : u \in U\} \\ &= \sup\{d_H(u, u') : (u, u') \in U \times U\}, \end{aligned}$$

and this obviously agrees with Definition 1.2.7!

Recall that in Definition 1.1.5, we use Euclidean open balls to construct a basis for a topology on \mathbb{R}^n . We can generalise this idea in any metric space.

Proposition 1.2.8 ► Metric Induces a Basis

Let (X, d) be a metric space. Define

$$B_r(x) := \{y \in X : d(x, y) < r\},$$

then collection

$$\mathcal{B}_d := \{B_r(x) : x \in X, r \in \mathbb{R}^+\}$$

is a basis for a topology on X .

Proof. Notice that for any $x \in X$, we have $x \in B_1(x) \in \mathcal{B}_d$. Let $B_p(x_1), B_q(x_2) \in \mathcal{B}_d$ be such that $x \in B_p(x_1) \cap B_q(x_2)$. Take $k = \min\{p - d(x, x_1), q - d(x, x_2)\}$, then clearly $k > 0$ and we can find $B_k(x) \subseteq B_p(x_1) \cap B_q(x_2)$ such that $x \in B_k(x) \in \mathcal{B}_d$. Therefore, \mathcal{B}_d is a basis for a topology on X . \square

Since we can obtain a basis from a metric, it follows naturally that we can generate a topology using this induced basis.

Definition 1.2.9 ► Metrisable Topology

Let (X, d) be a metric space. A topology \mathcal{T} on X is **metrisable**, or **induced** by d , if it is generated by \mathcal{B}_d

We can verify that the discrete topology $\mathcal{P}(X)$ is induced by the discrete metric. Let the discrete metric on X be χ , then it is easy to see that

$$B_r(x) = \begin{cases} \{x\} & \text{if } 0 < r \leq 1 \\ X & \text{if } r > 1 \end{cases}.$$

Therefore,

$$\mathcal{B}_\chi = \{X\} \cup \{\{x\} : x \in X\}.$$

Let \mathcal{T}_χ be the topology on X generated by \mathcal{B}_χ , then it suffices to prove that $\mathcal{P}(X) \subseteq \mathcal{T}_\chi$. Take any $U \in \mathcal{P}(X)$, then for any $u \in U$, we have $u \in \{u\} \subseteq U$. Clearly $\{u\} \in \mathcal{B}_\chi$, so $\mathcal{T}_\chi = \mathcal{P}(X)$ is the discrete topology indeed.

In particular, for Euclidean spaces, the following result extends Definition 1.1.5:

Proposition 1.2.10 ▶ Every L^p -metric Generates the Standard Topology

Let \mathcal{T} be the standard topology on \mathbb{R}^n , then \mathcal{T} is induced by any L^p -metric d_p .

Proof. For any $p \in \mathbb{N}^+$, notice that

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p \leq \sum_{i=1}^n \|y_i - x_i\|^p \leq n \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p.$$

Taking the p -th root yields

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\| \leq \left[\sum_{i=1}^n \|y_i - x_i\|^p \right]^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

This means that

$$d_\infty(\mathbf{x}, \mathbf{y}) \leq d_p(\mathbf{x}, \mathbf{y}) \leq n^{\frac{1}{p}} d_\infty(\mathbf{x}, \mathbf{y}).$$

Let \mathcal{T}_0 and \mathcal{T}_p be topologies on \mathbb{R}^n generated by \mathcal{B}_{d_∞} and \mathcal{B}_{d_p} respectively. Take any $T \in \mathcal{T}_p$, then for any $\mathbf{t} \in T$, there is some $B_r(\mathbf{t}') \in \mathcal{B}_{d_p}$ such that $\mathbf{t} \in B_r(\mathbf{t}') \subseteq T$. Take some $\ell = \frac{1}{2}|r - d_p(\mathbf{t}, \mathbf{t}')|$, then we have found $B_\ell(\mathbf{t}) \in \mathcal{B}_{d_p}$ such that

$$\mathbf{t} \in B_\ell(\mathbf{t}) \subseteq B_r(\mathbf{t}') \subseteq T.$$

Take $k = \frac{1}{2}\ell n^{-\frac{1}{p}}$ and consider

$$B_k(\mathbf{t}) := \{\mathbf{y} \in \mathbb{R}^n : d_\infty(\mathbf{t}, \mathbf{y}) < k\} \in \mathcal{B}_{d_\infty}.$$

Notice that for each $\mathbf{y} \in B_k(\mathbf{t})$, we have

$$d_p(\mathbf{t}, \mathbf{y}) \leq n^{\frac{1}{p}} d_\infty(\mathbf{t}, \mathbf{y}) < \ell,$$

so $\mathbf{t} \in B_k(\mathbf{t}) \subseteq B_\ell(\mathbf{t}) \subseteq T$. This implies that $T \in \mathcal{T}_0$ and so $\mathcal{T}_p \subseteq \mathcal{T}_0$. By a similar argument, one may check that $\mathcal{T}_0 \subseteq \mathcal{T}_p$. Therefore, $\mathcal{T}_0 = \mathcal{T}_p$ for any $p \in \mathbb{N}^+$. Note that by Definition 1.1.5, \mathcal{T} is generated by \mathcal{B}_{d_2} , which means that $\mathcal{T} = \mathcal{T}_2 = \mathcal{T}_0 = \mathcal{T}_p$ for any $p \in \mathbb{N}^+$. Therefore, \mathcal{T} is induced by any L^p -metric d_p . \square

The fact that

$$d_\infty(\mathbf{x}, \mathbf{y}) \leq d_p(\mathbf{x}, \mathbf{y}) \leq n^{\frac{1}{p}} d_\infty(\mathbf{x}, \mathbf{y})$$

means that all L^p -metrics are equivalent over the same space.

1.3 Subspace Topologies

Definition 1.3.1 ► Subspace Topology

Let (Y, \mathcal{T}_Y) be a topological space and $X \subseteq Y$ be some subset. The collection

$$\mathcal{T}_X := \{U \cap X : U \in \mathcal{T}_Y\}$$

is the **subspace topology** on X .

We may check that \mathcal{T}_X defined as such is indeed a topology on X . First, by taking $U = \emptyset$ and $U = Y$ respectively, we know that $\emptyset, X \in \mathcal{T}_X$. For any $U \in \mathcal{T}_Y$, we have $Y \setminus U \in \mathcal{T}_Y$ and so

$$X \setminus (U \cap X) = (Y \setminus U) \cap X \in \mathcal{T}_X.$$

For any $\mathcal{V} \subseteq \mathcal{T}_X$, we define a subset $\mathcal{U}_{\mathcal{V}} \subseteq \mathcal{T}_Y$ such that for each $V \in \mathcal{V}$ there is a unique $U_V \in \mathcal{U}_{\mathcal{V}}$ such that $V = U_V \cap X$. Then,

$$\begin{aligned} \bigcup_{A \in \mathcal{V}} A &= \bigcup_{B \in \mathcal{U}_{\mathcal{V}}} (B \cap X) \\ &= \left(\bigcup_{B \in \mathcal{U}_{\mathcal{V}}} B \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

Let $X_1, X_2, \dots, X_n \in \mathcal{T}_X$ and define $X_i = U_i \cap X$ where $U_i \in \mathcal{T}_Y$ for $i = 1, 2, \dots, n$, then

$$\begin{aligned} \bigcap_{i=1}^n X_i &= \bigcap_{i=1}^n (U_i \cap X) \\ &= \left(\bigcap_{i=1}^n U_i \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

So \mathcal{T}_X is really a topology on X . Intuitively, the following holds:

Proposition 1.3.2 ► Basis for a Subspace

Let (Y, \mathcal{T}_Y) be a topological space and \mathcal{T}_X be the subspace topology on some $X \subseteq Y$. If \mathcal{B}_Y is a basis of \mathcal{T}_Y , then

$$\mathcal{B}_X := \{B \cap X : B \in \mathcal{B}_Y\}$$

is a basis of \mathcal{T}_X .

Proof. We first prove that \mathcal{B}_X is a basis. Take any $x \in X \subseteq Y$. Note that there exists some $B \in \mathcal{B}_Y$ such that $x \in B$. Take $B \cap X \in \mathcal{B}_X$, then $x \in B \cap X$. For any $B_1, B_2 \in \mathcal{B}_X$ with $x \in B_1 \cap B_2$, we write $B_1 := B'_1 \cap X$ and $B_2 := B'_2 \cap X$ where $B'_1, B'_2 \in \mathcal{B}_Y$, then we have $x \in B'_1 \cap B'_2$. This means that there is some $B \in \mathcal{B}_Y$ such that $x \in B \subseteq B'_1 \cap B'_2$. Write $B' := B \cap X \in \mathcal{B}_X$, then for each $b \in B'$, we know that $b \in B'_1 \cap B'_2$ and $b \in X$, which implies that $b \in B_1 \cap B_2$. Therefore, $x \in B' \subseteq B_1 \cap B_2$. This means that \mathcal{B}_X is a basis of a topology on X .

We then prove that \mathcal{T}_X is generated by \mathcal{B}_X . Let \mathcal{T} be the topology generated by \mathcal{B}_X . By Proposition 1.1.4, we have

$$\mathcal{T} = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{B}_X \right\}.$$

Similarly, we can write

$$\mathcal{T}_Y = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{B}_Y \right\}.$$

Take any $T \in \mathcal{T}_X$, then there exists some $\mathcal{V} \subseteq \mathcal{B}_Y$ such that

$$\begin{aligned} T &= \left(\bigcup_{A \in \mathcal{V}} A \right) \cap X \\ &= \bigcup_{A \in \mathcal{V}} A \cap X \\ &\in \mathcal{T}. \end{aligned}$$

Therefore, $\mathcal{T}_X \subseteq \mathcal{T}$. Conversely, take any $T' \in \mathcal{T}$, there exists some $\mathcal{U} \subseteq \mathcal{B}_Y$ such that

$$\begin{aligned} T' &= \bigcup_{B \in \mathcal{U}} (B \cap X) \\ &= \left(\bigcup_{B \in \mathcal{U}} B \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

Therefore, $\mathcal{T} \subseteq \mathcal{T}_X$ and so $\mathcal{T}_X = \mathcal{T}$. □

The following result shows that open sets in subspaces remain open in the superspace:

Proposition 1.3.3 ▶ Superspace Preserve Open Sets

Let (Y, \mathcal{T}_Y) be a topological space. If $X \subseteq Y$ is open in Y and $U \subseteq X$ is open in X , then U is open in Y .

Proof. Let \mathcal{T}_X be the subspace topology on X . Since U is open in X , we have $U \in \mathcal{T}_X$. By Definition 1.3.1, there exists some $V \in \mathcal{T}_Y$ such that $U = V \cap X$. However, $U \subseteq X$, so $U = V \in \mathcal{T}_Y$, which means that U is open in Y . \square

We can do a similar manipulation with metric spaces and induce a metric on a subspace.

Definition 1.3.4 ▶ Subspace Metric

Let (X, d) be a metric space. The **subspace metric** of some $A \subseteq X$ is the restriction of d to A , denoted as

$$d_A(x, y) = d(x, y), \quad \text{for all } x, y \in A.$$

Naturally, the following result is true:

Proposition 1.3.5 ▶ Subspace Metric Induces Subspace Topology

Let (X, d) be a metric space. The topology induced by the subspace metric d_A on some subspace $A \subseteq X$ is the subspace topology on A .

Proof. Let \mathcal{T}_d and \mathcal{T}_{d_A} be topologies induced by d on X with basis \mathcal{B}_d and by d_A on A with basis \mathcal{B}_{d_A} respectively. Let \mathcal{T}_A be the subspace topology on A with basis \mathcal{B}_A . Take any $B_A \in \mathcal{B}_A$, then there exists $B_r(x) \in \mathcal{B}_d$ such that $B_A = B_r(x) \cap A$. For any $y \in B_A$, consider the ball

$$B_{r'}(y) := \{z \in A : d_A(z, y) < r'\} \in \mathcal{B}_{d_A}.$$

Note that $y \in B_{r'}(y)$, so by Proposition 1.1.7, we have $\mathcal{T}_A \subseteq \mathcal{T}_{d_A}$. Conversely, for any $B_r(x) \in \mathcal{B}_{d_A}$, there exists some $B_{r'}(x) \in \mathcal{B}_d$ such that $B_r(x) \subseteq B_{r'}(x)$. Notice that $B_r(x) \subseteq A$, so for any $y \in B_r(x)$, we have $y \in B_{r'}(x) \cap A \in \mathcal{B}_A$. Therefore, by Proposition 1.1.7, $\mathcal{T}_{d_A} \subseteq \mathcal{T}_A$ and so $\mathcal{T}_{d_A} = \mathcal{T}_A$. \square

1.4 Closed Sets

Definition 1.4.1 ▶ Closed Set

Let (X, \mathcal{T}) be a topological space. A subset $A \subseteq X$ is **closed** if $X \setminus A \in \mathcal{T}$.

A set might be open and closed simultaneously. For example, every set X is both open and

closed in itself.

Proposition 1.4.2 ▶ Arbitrary Intersection and Finite Union of Closed Sets Are Closed

Let (X, \mathcal{T}) be a topological space, then

1. if $\mathcal{G} := \{G_\alpha : \alpha \in I\}$ is a family of closed set in X with respect to some index set I , then $\bigcap_{\alpha \in I} G_\alpha$ is closed in X ;
2. if G_1, G_2, \dots, G_n are closed in X , then $\bigcup_{i=1}^n G_i$ is closed in X .

Proof. Notice that

$$X \setminus \bigcap_{\alpha \in I} G_\alpha = \bigcup_{\alpha \in I} X \setminus G_\alpha.$$

Since $X \setminus G_\alpha$ is open in X for all $\alpha \in I$, this means that $X \setminus \bigcap_{\alpha \in I} G_\alpha$ is open in X , and so $\bigcap_{\alpha \in I} G_\alpha$ is closed in X . Notice also that

$$X \setminus \bigcup_{i=1}^n G_i = \bigcap_{i=1}^n X \setminus G_i.$$

By a similar argument $\bigcup_{i=1}^n G_i$ is closed in X . □

The following proposition justifies the fact that intersecting a closed set with a subspace produces a closed set in that subspace:

Proposition 1.4.3 ▶ Closed Sets in Subspace Topology

Let $Y \subseteq X$, then $A \subseteq Y$ is closed in Y if and only if there exists some closed set $G \subseteq X$ such that $A = G \cap Y$.

Proof. Suppose that A is closed in Y , then $Y \setminus A$ is open in Y . Therefore, there exists some open set $B \subseteq X$ such that $Y \setminus A = B \cap Y$. Take $G := X \setminus B$, then

$$G \cap Y = A.$$

Suppose conversely that there exists some closed set $G \subseteq X$ such that $A = G \cap Y$. Consider

$$Y \setminus (G \cap Y) = (X \setminus G) \cap Y.$$

Notice that $X \setminus G$ is open in X , so $Y \cap A$ is open in Y , i.e., A is closed in Y . □

The following result is analogous to Proposition 1.3.3:

Proposition 1.4.4 ► Superspace Preserves Closed Sets

If $Y \subseteq X$ is closed in X and $A \subseteq Y$ is closed in Y , then A is closed in X .

Proof. Consider $X \setminus A = X \setminus Y \cup Y \setminus A$. Since Y is closed in X , this means that $X \setminus Y$ is open in X . Note also that $Y \setminus A$ is open in Y . By Proposition 1.3.3, $Y \setminus A$ is open in X . Therefore, $X \setminus A$ is open in X and so A is closed in X . \square

Closed sets help define the notion “interior of a set”.

Definition 1.4.5 ► Interior, Closure and Boundary

Let (X, \mathcal{T}) be a topological space and $A \subseteq X$. The **interior** of A is

$$\overset{\circ}{A} := \bigcup_{\substack{U \in \mathcal{T} \\ U \subseteq A}} A \cap U.$$

The **closure** of A is

$$\bar{A} := \bigcap_{\substack{X \setminus G \in \mathcal{T} \\ A \subseteq G}} G.$$

The **boundary** of A is

$$\partial A = \bar{A} \setminus \overset{\circ}{A}.$$

We interpret the above definition as follows: $\overset{\circ}{A}$ is the union of all open sets contained by A . Moreover, \bar{A} is the smallest closed set in X which contains A . To see this, let $C \subseteq X$ be any closed set in X containing A . Take any $a \in \bar{A}$, then since a is contained by all closed sets containing A , it is clear that $a \in C$, which implies that $\bar{A} \subseteq C$.

Remark.

1. $\overset{\circ}{A} \subseteq A \subseteq \bar{A}$.
2. $\overset{\circ}{A} = A$ if and only if A is open in X .
3. $\bar{A} = A$ if and only if A is closed in X .

We would like to discuss the properties of closure. The following definition is useful:

Definition 1.4.6 ► Limit Point

Let (X, \mathcal{T}) be a topological space. For any $A \subseteq X$, a point $x \in X$ is a **limit point** of A if for every open set $U \subseteq X$ containing x ,

$$(A \setminus \{x\}) \cap U \neq \emptyset.$$

Now, we propose two properties for the closure:

Proposition 1.4.7 ► Properties of Closure

Let (X, \mathcal{T}) be a topological space. For any $A \subseteq X$,

1. $x \in \bar{A}$ if and only if for any open set $U \subseteq X$ containing x , $U \cap A \neq \emptyset$;
2. if A' is the set of limit points of A , then $\bar{A} = A \cup A'$.

Proof. We will prove the first statement by considering the contrapositive, i.e., we prove that $x \in X \setminus \bar{A}$ if and only if there exists some open set $U \subseteq X$ containing x such that $U \cap A = \emptyset$. The “if” direction is trivial because $X \setminus \bar{A}$ is open in X such that $(X \setminus \bar{A}) \cap A = \emptyset$. Take $U \subseteq X$ to be an open set in X with $x \in U$ and $U \cap A = \emptyset$. This means that $x \notin A \subseteq \bar{A}$. Therefore, $x \in X \setminus \bar{A}$.

Take any $a \in A'$, then for every open set $U \subseteq X$ with $a \in U$, we have

$$U \cap A \supseteq U \cap (A \setminus \{a\}) \neq \emptyset.$$

Therefore, $A \cup A' \subseteq \bar{A}$. Take any $x \in \bar{A}$, we shall prove that if $x \notin A'$, then $x \in A$. Since x is not a limit point of A , there exists some open set $V \subseteq X$ containing x such that $(A \setminus \{x\}) \cap V = \emptyset$. However, $x \in \bar{A}$ implies that $V \cap A \neq \emptyset$, so $x \in A$. \square

The notion of limit points also leads to the definition of convergence. Before that, we shall define the notion of *neighbourhood*.

Definition 1.4.8 ► Neighbourhood

Let (X, \mathcal{T}) be a topological space. An open set $U \subseteq X$ is called a **neighbourhood** of some $x \in X$ if $x \in U$.

Intuitively, we think of the statement $x_i \rightarrow x$ as the fact that no matter how small a neighbourhood we choose for x , there is always a consecutive infinite subsequence of the x_i 's which falls in this neighbourhood.

Definition 1.4.9 ► Convergence

A sequence $\{x_i\}_{i=1}^{\infty}$ of points in a topological space (X, \mathcal{T}) **converges** to $x \in X$ if for any neighbourhood $U \subseteq X$ containing x , there exists some $N \in \mathbb{N}^+$ such that $x_k \in U$ for all $k > N$, denoted as $x_i \rightarrow x$. x is said to be the **limit** of $\{x_i\}_{i=1}^{\infty}$.

It is important to distinguish between limit and limit points. For example, consider the constant sequence $\{1\}_{i=1}^{\infty}$. Clearly, $x_i \rightarrow 1$ but one may check that 1 is not a limit point for this sequence.

In a metric space, we can make use of the metric to describe convergence in a more quan-

titative way.

Theorem 1.4.10 ► Convergence in Metric Spaces

Let (X, d) be a metric space. A sequence $\{x_i\}_{i=1}^{\infty}$ in X converges to x if and only if for every $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that $d(x_i, x) < \epsilon$ for all $i > N$.

Proof. Suppose that $x_i \rightarrow x$ as $i \rightarrow \infty$. For all $\epsilon > 0$, take the open ball $B_{\epsilon}(x) \subseteq X$. Clearly, $B_{\epsilon}(x)$ is a neighbourhood of x . By Definition 1.4.9, there exists some $N \in \mathbb{N}^+$ such that $x_i \in B_{\epsilon}(x)$ for all $i > N$, i.e., $d(x_i, x) < \epsilon$ for all $i > N$. Conversely, suppose that for every $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that $d(x_i, x) < \epsilon$ for all $i > N$. Let $U \subseteq X$ be any neighbourhood containing x . Note that U is open in X , so by Theorem 1.1.4, there exists some open ball $B_r(x) \subseteq U$ such that $x \in B_r(x)$. Therefore, there exists some $M \in \mathbb{N}^+$ such that $d(x_i, x) < r$, i.e., $x_i \in B_r(x) \subseteq U$, for all $i > M$. Therefore, $x_i \rightarrow x$. □