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Permutations and Combinations

1.1 Basic Counting Principles

An important motivation to study combinatorics is to count the **number of ways** in which an event may occur. Intuitively, we have two approaches to count.

The first approach is to categorise the event into **non-overlapping cases**. This means that we break an event into mutually exclusive sub-events, after which we can count the number of ways for each sub-event to occur. The agregate of these counts is the total number of ways for the original event to occur.

Those familiar with basic set theory may consider E to be the set containing all distinct ways for an event to occur. By breaking up the event, we essentially establish a **partition** of E, so that the sum of cardinalities of all the elements in that partition equals the cardinality of E.

This motivates us to write the following principle using set notations.

Theorem 1.1.1 ▸ Addition Principle (AP)

Let $k \in \mathbb{N}^+$ and let $A_1, A_2, \cdots, A_k$ be k finite sets which are pairwise disjoint, i.e. $A_i \cap A_j = \varnothing$ whenever $i \neq j$, then

$$\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{i=1}^{k} |A_i|.$$

Proof. The case where $k = 1$ is trivial.
Suppose that when $k = n$, we have

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

for any n finite sets which are pairwise disjoint. Let A_{n+1} be an arbitrary finite set which is disjoint with any of the A_i's from the n sets. So we have:

$$
\begin{aligned}
\left|\bigcup_{i=1}^{n+1} A_i\right| &= \left|\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right| \\
&= \left|\bigcup_{i=1}^{n} A_i\right| + |A_{n+1}| - \left|\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1}\right| \\
&= \left(\sum_{i=1}^{n} |A_i|\right) + |A_{n+1}| - |\varnothing| \\
&= \sum_{i=1}^{n+1} |A_i|.
\end{aligned}
$$

Therefore, the original statement holds for all $k \in \mathbb{N}^+$. □

In more casual language, this means that if an event E_k has n_k distinct ways to occur, then there is $\sum_{i=1}^{k} n_k$ ways for at least one of the events $E_1, E_2, \cdots, E_k$ to occur, provided that E_i and E_j can never occur concurrently whenever $i \neq j$.

Given an event E, the other approach to count the number of ways for it to occur is to break E up internally into **non-overlapping stages**.

With set notations, we can write the i-th stage for E to occur as e_i, and so a way for E to occur can be represented by an ordered tuple $(e_1, e_2, \cdots, e_k)$, where k is the total number of stages to undergo for E to occur.

Let E_i denote the set of all distinct ways to undergo the i-th stage of E, then it is easy to see that E is just the **Cartesian product** of all the E_i's. Hence, we derive the following principle:

Theorem 1.1.2 ▸ Multiplication Principle (MP)

Let $k \in \mathbb{N}^+$ and let $A_1, A_2, \cdots, A_k$ be k pairwise disjoint finite sets, then

$$\left|\prod_{i=1}^{k} A_i\right| = \prod_{i=1}^{k} |A_i|.$$

Proof. The case where $k = 1$ is trivial.
Suppose that when $k = n$, we have

$$\left|\prod_{i=1}^{n} A_i\right| = \prod_{i=1}^{n} |A_i|$$

for any n finite sets which are pairwise disjoint. Let A_{n+1} be an arbitrary finite set which is disjoint with any of the A_i's from the n sets. Take $a_i, a_j \in A_{n+1}$. Note that for all $\boldsymbol{a} \in \prod_{i=1}^{n} A_i$, $(\boldsymbol{a}, a_i) \neq (\boldsymbol{a}, a_j)$ whenever $a_i \neq a_j$. This means that

$$\begin{aligned}\left|\prod_{i=1}^{n+1} A_i\right| &= \left|\prod_{i=1}^{n} A_i \times A_{n+1}\right| \\ &= \left|\prod_{i=1}^{n} A_i\right| |A_{n+1}| \\ &= \left(\prod_{i=1}^{n} |A_i|\right) |A_{n+1}| \\ &= \prod_{i=1}^{n+1} |A_i|\end{aligned}$$

Therefore, the original statement holds for all $k \in \mathbb{N}^+$. $\square$

In more casual language, this means that if an event E requires k stages to be undergone before it occurs and the i-th stage has n_i ways to complete, then there is $\prod_{i=1}^{k} n_k$ ways for E to occur, provided that no two different stages complete concurrently.

1.2 Permutations

A fundamental problem in combinatorics is described as follows: given a set S, how many ways are there to arrange r elements in S, i.e. how many **distinct sequences** can be formed using the elements in S without repetition? The process of selecting elements from S and arranging them as a sequence is known as **permutation**.

Note that forming a sequence using r elements from a set S is an event consisting of r stages, as we need to select an element for each of the r terms of the sequence. Suppose S has n elements. For the first term of the sequence, we can choose any of the elements in S, so there is n ways to do it. For the second term, since we cannot repeat the elements, we are left with $n - 1$ choices.

Continue choosing elements in this way, we realise that if we choose the terms sequentially, when we reach the k-th term we will be left with $n - k + 1$ options as the previous $(k - 1)$ terms have taken away $(k - 1)$ elements. By Theorem 1.1.2, we know that the number of sequences which can be formed is given by $\prod_{i=1}^{r}(n - r + i)$.

Definition 1.2.1 ▶ Permutations

Let A be a finite set such that $|A| = n$, an r-permutation of A is a way to arrange r elements of A, denoted as P_r^n and given by

$$P_r^n = \prod_{i=1}^{r}(n - r + i) = \frac{n!}{(n-r)!}.$$