- **Topology**: $\mathcal{T} \subseteq \mathcal{P}(X)$ s.t.
  - $\varnothing, X \in \mathcal{T}$;
  - closed under **arbitrary union** and **finite intersection**.
- **Co-finite topology**: set of **complements of finite subsets**.
- **Basis**:
  - $\forall x \in X, \exists B \in \mathcal{B}$ s.t. $x \in B \iff X \subseteq \bigcup_{B \in \mathcal{B}} B$.
  - $\forall x \in X$ and $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq B_1 \cap B_2$.
- **Topology generated by basis**: set of all unions of sets in $\mathcal{B}$.
- $\mathcal{T}_1 \subseteq \mathcal{T}_2 \iff \mathcal{T}_2$ is finer.
- Topology generated by $\mathcal{B}$ is the coarsest topology containing $\mathcal{B}$.
- $\mathcal{T}_1 \subseteq \mathcal{T}_2 \iff \forall B \in \mathcal{B}_1$ and $\forall x \in B$, $\exists B_x \in \mathcal{B}_2$ s.t. $x \in B_x \subseteq B$.
- **sub-basis**: $\bigcup_{S \in \mathcal{S}} S = X$ and every basis is a sub-basis.
- All finite intersections of sets in a sub-basis is a basis.
- **Metric**: **positive, definite, symmetric, $\triangle$-inequality**.
- **Pseudo-metric**: $d(x,x) = 0$ but can be not definite.
- **Quasi-metric**: can be not symmetric.
- **Norm**: **positive, definite, $\triangle$-inequality**, $\|\lambda x\| = |\lambda| \|x\|$.
- **Distance between sets**: smallest pointwise distance.
- **Diameter of set**: greatest pointwise distance.
- **Metrisable** topology: induced with open balls.
- L^p**-metric**: generates the standard topology on $\mathbb{R}$.

$$\max \|y_i - x_i\| \leq \left[\sum_{i=1}^{n} \|y_i - x_i\|^p\right]^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max \|y_i - x_i\|.$$

- Metrics are equivalent iff $c_1 d \leq d' \leq c_2 d$.
- **Subspace topology**: $\{U \cap X : U \text{ is open}\}$. Basis is analogous.
- Open sets in open subspace is open in superspace.
- **Subspace metric**: restriction to subspace. Induces subspace topology with respect to metrisable topology.
- If $Y \subseteq X$, then A is closed in $Y \iff \exists G$ closed in X s.t. $A = G \cap Y$.
- Closed sets in closed subspace is closed in superspace.
- **Interior** $\mathring{A}$: union of all open subsets of A.
- **Closure** $\overline{A}$: intersection of all closed supersets of A. **Smallest closed superset of** A.
- **Boundary** $\partial A = \overline{A} \setminus \mathring{A}$.
- **Limit point**: $(A \setminus \{x\}) \cap U \neq \varnothing$ for any open U.
- $x \in \overline{A} \iff \forall$ open neighbourhood U of x, $U \cap A \neq \varnothing$;
- $\overline{A} = A \cup A'$, i.e., closure is the set plus all its limit point.
- Limit may not be a limit point.
- **Continuity**: U open $\implies f^{-1}(U)$ open, equivalent to $f^{-1}(S)$ is open $\forall S$ in sub-basis.
- TFAE:
  1. f is continuous;
  2. for all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$;
  3. for any closed set $B \subseteq Y$, $f^{-1}(B)$ is closed in X;
  4. $\forall x \in X$ and any open $V \subseteq Y$ with $f(x) \in V$, there is an open set $U \subseteq X$ s.t. $x \in U$ and $f(U) \subseteq V$.
- **Pasting lemma**: if $X = A \cup B$ for closed A, B and $f(x) = g(x)$ for all $x \in A \cap B$, then

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

  is continuous if $f\colon A \to Y$ and $g\colon B \to Y$ are.
- **Pull-back** topology: $\left\{f^{-1}(U) : U \text{ is open}\right\}$ is the coarsest topology ensuring continuous f.
- **Uniform continuity**: $\forall \epsilon > 0$, there exists some $\delta > 0$ s.t. $d_X(x,y) < \delta \implies d_Y\big(f(x), f(y)\big) < \epsilon$.
- f is uniformly continuous $\iff \forall \{x_i\}_i^\infty, \{y_i\}_i^\infty$ s.t. $\lim_{i\to\infty} d_X(x_i, y_i) = 0$, $\lim_{i\to\infty} d_Y\big(f(x_i), f(y_i)\big) = 0$.
- $\{f_n\}$ **converges pointwisely**: $\forall x$, $f_n(x) to f(x)$.
- $\{f_n\}$ **converges uniformly**: $\forall \epsilon > 0$, $\exists N \in \mathbb{N}^+$ s.t. $\forall n \geq N, \forall x \in X$, $d\big(f_n(x), f(x)\big) < \epsilon$.
- Limit of uniformly convergent sequence is continuous.
- **Projection** $\pi_{X_\beta} \coloneqq \boldsymbol{x} \mapsto x_\beta$, $\pi_{X_\beta}^{-1}(U)$ is all vectors whose β-th component is in U.
- **Product** topology is generated by the sub-basis of all pre-images of all projections.
- **Box** topology is generated by the basis of all products of open sets.
- Box topology and product topology are equal only for finite product.
- Product topology is the coarsest topology to ensure continuous projection.
- $f(y) = \big(f_\alpha(y)\big)_{\alpha\in\Lambda}$ is continuous iff f_α's are continuous.
- Subspace topology of product topology equals product topology of subspace topologies.
- Standard topology on $\mathbb{R}^n$ is the product topology by standard topologies on $\mathbb{R}^{m_i}$'s.
- Product of basis is the basis for product topology.
- $d_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^n d_{X_i}(x_i, y_i)$ and $d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max d_{X_i}(x_i, y_i)$ both induce the product topology.
- $\rho(x,y) \coloneqq \frac{d(x,y)}{1+d(x,y)}$ is a metric with $\operatorname{diam}(X) < 1$. ρ and d generate the same topology.
- $d(\boldsymbol{x}, \boldsymbol{y}) \coloneqq \sup\left\{\frac{\rho_i(x_i, y_i)}{i} : i \in \mathbb{Z}^+\right\}$ is a metric inducing the infinite product topology.
- **Quotient map**: surjective and U is open $\iff f^{-1}(U)$ is open.
- **Open map**: continuous map from open set to open set.
- Surjective continuous + open or closed = quotient map.
- Quotient, open and closed maps are preserved under $\circ$.
- **Saturated**: pre-image under a surjective continuous map. $A = f^{-1}\big(f(A)\big)$.

- Surjective continuous f is a quotient map iff $f(A)$ is open (closed) in Y whenever A is saturated open (closed).
- Restriction of quotient map to a saturated set is a quotient map.
- **Quotient topology**: unique topology on co-domain to ensure quotient map.
- T_1: $\forall x \neq y$, $\exists$ open U containing only x.
- T_2 (**Hausdorff**): $\forall x \neq y$, $\exists$ disjoint neighbourhoods.
- Co-finite topology is T_1 and T_2 if X is finite.
- $T_1 \iff \{x\}$ is closed $\forall x \in X$.
- Metric spaces are T_2 so all finite subsets are closed.
- **Countable basis**: countable $\mathcal{B}$ s.t. $\forall$ open Y containing x, $\exists B \in \mathcal{B}, B \subseteq Y$. **First countable** if every x has a countable basis.
- Uncountable co-finite have no countable basis.
- $\exists$ nested countable basis $B_1 \subseteq B_2 \subseteq \cdots$.
- Limit $x \in \overline{A}$, if X is first countable, then $x \in \overline{A}$ is a limit.
- If f is continuous, then for any sequence $f(x_i) \to f(x)$. The converse is true if X is first countable.
- Closed subspace of compact space is compact.
- Subset of co-finite space is compact but closed iff it's finite.
- Compact subspace of Hausdorff space is closed.
- Continuous f maps compact set to compact set.
- **Tube lemma**: If Y is compact and $N \subseteq X \times Y$ is open and contains $\{x_0\} \times Y$, then $\exists W \supseteq \{x_0\}$ open s.t. $W \times Y \subseteq N$.
- Cartesian product of compact spaces is compact.
- **Finite intersection property**: $\mathcal{G} \subseteq \mathcal{P}(X)$ s.t. finite intersections of sets in $\mathcal{G}$ are non-empty.
- X is compact iff for any collection of closed sets $\mathcal{G}$ with the finite intersection property, we have $\bigcap_{G \in \mathcal{G}} G \neq \varnothing$.
- x is **isolated** $\iff \{x\}$ is open.
- If $U \neq \varnothing$ is open in a Hausdorff space and $x \in X$ is not isolated, then $\exists$ non-empty open $V \subseteq U$ s.t. $x \notin \overline{V}$.
- Non-empty Hausdorff space is uncountable if it has no isolated point.
- **Limit point compact**: every infinite $Y \subseteq X$ has a limit point in X. Limit point compact $\not\Longrightarrow$ compact but compact $\Longrightarrow$ limit point compact.
- **Sequentially compact**: every sequence has a convergent subsequence. Sequentially compact $\Longrightarrow$ limit point compact but limit point compact $\not\Longrightarrow$ sequentially compact.
- $\mathcal{U}$: open cover for a metric space X. $\delta > 0$ is a **Lebesgue number** for $\mathcal{U}$ if $\forall S \subseteq X$ with $\operatorname{diam}(S) < \delta$, $\exists U \in \mathcal{U}$ s.t. $S \subseteq U$.
- Every open cover of sequentially compact metric space has a Lebesgue number.
- **Totally bounded**: $\forall \epsilon > 0$, $\exists$ finite cover of X by $B_\epsilon(x_i)$.
- Every sequentially compact metrisable topological space is totally bounded.
- If X is a metrisable topological space, TFAE:
  1. X is compact;
  2. X is limit point compact;
  3. X is sequentially compact.
- Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \to Y$ be continuous. If X is compact, then f is uniformly continuous.