- **Cayley-Hamilton Thm:** $\chi_T(T) = 0$ for all linear operator T.
- $T^k = -\frac{1}{c_k}\sum_{i=0}^{k-1} c_i T^i$.
- $T^{-1} = T^{-1} \circ \mathrm{id}_V = T^{-1} \circ \left(-\frac{1}{c_0}\sum_{i=1}^{k} c_i T^i\right)$.
- Rotation matrix:

$$\begin{bmatrix}\cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i\end{bmatrix} = \begin{bmatrix}1 & -1 \\ -\mathrm{i} & -\mathrm{i}\end{bmatrix}\begin{bmatrix}\mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta}\end{bmatrix}\begin{bmatrix}1 & -1 \\ -\mathrm{i} & -\mathrm{i}\end{bmatrix}^{-1}.$$

- Euclidean operators:
  1. a series of stretching and/or crushing transformations if it has n distinct real eigenvalues;
  2. a series of rotations followed by a series of stretching and/or crushing transformations if it has n distinct eigenvalues not all real;
  3. a series of shearing transformations followed by a series of stretching and/or crushing transformations if it has less than n distinct real eigenvalues;
  4. a series of shearing transformations, followed by a series of rotations, and then followed by another series of stretching and/or crushing transformations if it has less than n distinct eigenvalues not all real.
- Bilinear forms: $b = \sum_{i=1}^{n}\sum_{j=1}^{n} b\left(\boldsymbol{z}_i, \boldsymbol{z}_j\right)\zeta^i \otimes \zeta^j$.
- For every bilinear form b on V, there is $b^{\#}\colon V \to \widehat{V}$ such that $b^{\#}(\boldsymbol{v})(\boldsymbol{u}) = b(\boldsymbol{u}, \boldsymbol{v})$.
- i-th component of $b^{\#}(\boldsymbol{v})$: $b^{\#}(\boldsymbol{v})(\boldsymbol{z}_i) = b(\boldsymbol{z}_i, \boldsymbol{v}) = \sum_{j=1}^{n} b_{ij} v_j$.
- Change-of-basis:
  1. In general, $\boldsymbol{P}$ is the matrix for $z^{-1} \circ y$, and $M_y(T) = \boldsymbol{P}^{-1} M_z(T) \boldsymbol{P}$.
  2. Bilinear form: $M_y(b) = \boldsymbol{P}^{\mathrm{T}} M_z(b) \boldsymbol{P}$.
  3. Change between orthonormal bases: $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{P} = \boldsymbol{I}$. Bilinear form has $\boldsymbol{I}$ under orthonormal basis.
  4. Sesquilinear form: $M_y(b) = \boldsymbol{P}^{\mathrm{T}} M_z(b) \overline{\boldsymbol{P}}$.
  5. Unitary matrix: $\overline{\boldsymbol{P}^{\mathrm{T}}}\boldsymbol{P} = \boldsymbol{I}$. **Any unitary matrix forms an orthonormal basis!**
- **Orthogonal decomposition:** $\boldsymbol{u} = \frac{g(\boldsymbol{u},\boldsymbol{v})}{|\boldsymbol{v}|^2}\boldsymbol{v} + \boldsymbol{u} - \frac{g(\boldsymbol{u},\boldsymbol{v})}{|\boldsymbol{v}|^2}\boldsymbol{v}$.
- **Cauchy-Schwarz Inequality:** $g(\boldsymbol{u}, \boldsymbol{v}) \le |\boldsymbol{u}|\,|\boldsymbol{v}|$.
- **Angle:** $\cos\theta = \frac{g(\boldsymbol{u},\boldsymbol{v})}{|\boldsymbol{u}||\boldsymbol{v}|}$.
- **Orthonormal basis:** $g(\boldsymbol{u}, \boldsymbol{v}) = z^{-1}(\boldsymbol{u}) \cdot z^{-1}(\boldsymbol{v})$.
- **Gram-Schmidt:**

$$\boldsymbol{z}_1^+ = \boldsymbol{z}_1, \qquad \boldsymbol{z}_k^+ \coloneqq \frac{\boldsymbol{z}_k - \sum_{i=1}^{k-1} g\left(\boldsymbol{z}_k, \boldsymbol{z}_i^+\right)\boldsymbol{z}_i^+}{\left|\boldsymbol{z}_k - \sum_{i=1}^{k-1} g\left(\boldsymbol{z}_k, \boldsymbol{z}_i^+\right)\boldsymbol{z}_i^+\right|}.$$

  Matrix representation is upper-triangular.
- **Riesz-Representation:** For any inner product space (V, g), let the mapping $\Gamma\colon V \to \widehat{V}$ be such that

$$\Gamma(\boldsymbol{u})(\boldsymbol{v}) = g(\boldsymbol{v}, \boldsymbol{u}),$$

  then for every $\alpha \in \widehat{V}$, there is a unique $\boldsymbol{u}_\alpha \in V$ such that $\alpha = \Gamma\left(\boldsymbol{u}_\alpha\right)$.
- $\zeta^i = \Gamma(\boldsymbol{z}P_i)$, i-th component of $\boldsymbol{v}$ is $\Gamma(\boldsymbol{z}_i)(\boldsymbol{v})$, i-th component of $\Gamma(\boldsymbol{v})$ is $\Gamma(\boldsymbol{v})(\boldsymbol{z}_i) = \sum_{j=1}^{n} g_{ij} v_j$.
- $\Gamma(\boldsymbol{v}) = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij} v_j \zeta^i$.
- If T is an operator over V, then $b_T \coloneqq \Gamma \circ T$ is a bilinear form over V, $M_z(b_T) = M_z(g) M_z(T)$. If b is a bilinear form over V, then $\Gamma^{-1} \circ b$ is an operator over V.
- Riesz-equivalent: $b(\boldsymbol{u}, \boldsymbol{v}) = (\Gamma \circ T)(\boldsymbol{v})(\boldsymbol{u})$ or $b(\boldsymbol{u}, \boldsymbol{v}) = g\big(\boldsymbol{u}, T(\boldsymbol{v})\big)$, have the (conjugate-)same matrix representation under orthonormal basis.
- Riesz-equivalent operator and sesquilinear form: $M_z(s_T) = M_z(g)\overline{M_z(T)}$.
- **Schur's Triangularisation Thm:** let T be a complex operator with basis y, then there is an orthonormal basis z such that $M_z(T)$ is upper-triangular, where $M_z(T) = \boldsymbol{G}^{-1}\boldsymbol{M}^{-1}\boldsymbol{G}^{-1} M_y(T)\boldsymbol{G}\boldsymbol{M}\boldsymbol{G}$. $\boldsymbol{G}$ is the Gram-Schmidt upper-triangular matrix and $\boldsymbol{M}$ changes to an upper-triangular matrix. Here, $\boldsymbol{M}\boldsymbol{G}$ changes between orthonormal bases and so is unitary.
- **Hermitian:** $s(\boldsymbol{u}, \boldsymbol{v}) = \overline{s(\boldsymbol{v}, \boldsymbol{u})}$.
- If τ is a Riesz-equivalent sesquilinear form to T, then τ is Hermitian iff $g\big(\boldsymbol{u}, T(\boldsymbol{v})\big) = g\big(T(\boldsymbol{u}), \boldsymbol{v}\big)$.
- **Spectral Thm:** Every Hermitian sesquilinear form over a complex inner product space has a real diagonal matrix representation under some orthonormal basis.
- Let T be a Riesz-equivalent operator to a Hermitian sesquilinear form s, then all eigenvalues of T are real.
- Every symmetric bilinear form b over a real inner product space V has a real diagonal matrix representation under some orthonormal basis such that there is a real eigenvector associated to each eigenvalue.
- **Wedge product:** $\alpha \wedge \beta \coloneqq \alpha \otimes \beta - \beta \otimes \alpha$ is a two-form.
- Dimension of m-form space in n-dimensional space: $\binom{n}{m}$.
- n-form space: $\left\{\lambda \bigwedge_{i=1}^{n} \zeta^i : \lambda \in \mathbb{F}\right\}$.
- **Determinant:** For any n-form Ω, $\widehat{T}(\Omega) = \Delta(T)\Omega$. Hermitian operators have real determinant. Non-zero determinant iff bijective.
- **Volume:** $\Theta\colon V^n \to \mathbb{F}$ such that
  1. $\Theta\left(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\right) \ge 0$;
  2. $\Theta\left(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\right) \ne 0$ if and only if $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n$ are linearly independent;
  3. $\Theta\left(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, c\boldsymbol{u}_i, \cdots \boldsymbol{u}_n\right) = c\Theta\left(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\right)$ for any $c \in \mathbb{F}$;
  4. $\Theta\big(\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots, \boldsymbol{u}_i, \cdots, \boldsymbol{u}_j + c\boldsymbol{u}_i, \cdots \boldsymbol{u}_n\big) = \Theta\left(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\right)$ for any $c \in \mathbb{F}$;
  5. $\Theta\left(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\right) = 1$ whenever $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\}$ is an orthonormal basis.
- **Volume form:** Let V be an n-dimensional inner product space over a well-ordered field $\mathbb{F}$. Take any orthonormal basis $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \cdots, \boldsymbol{z}_n\}$ with dual basis $\{\zeta^1, \zeta^2, \cdots, \zeta^n\}$.

$$\Theta \coloneqq |\omega_z| = \left|\bigwedge_{i=1}^{n} \zeta^i\right|.$$

  ω_z is called the **Orientation**.
- $\Theta\left(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\right) = |\boldsymbol{u} \cdot \boldsymbol{v} \times \boldsymbol{w}|$
- For general basis: $\Theta = \sqrt{\Delta(g)}\left|\bigwedge_{i=1}^{n} \eta^i\right|$.