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Graph Structures

1.1 Multigraph

1.1.1 Terminologies

Intuitively, we would describe a *graph* as a collection of nodes (or vertices) plus some lines (or edges) joining some nodes together. This can be rigorously defined as follows:

Definition 1.1.1 ▶ Multigraph

A **multigraph** G consists of a non-empty finite set of vertices denoted by $V(G)$ and a finite set of edges denoted by $E(G)$. $|V(G)|$ is known as the **order** of G, denoted by $v(G)$ and $|E(G)|$ is known as the **size** of G, denoted by $e(G)$.

In particular, if $v(G) = m$ and $e(G) = n$, we say that G is an **(m, n)-graph**.

G is said to be **trivial** if $v(G) = 1$ and **non-trivial** otherwise.

Note that in a multigraph, by default there can be a plural number of edges between any two vertices. We would define the notion of *simple graph* as a special multigraph.

Definition 1.1.2 ▶ Simple Graph

A multigraph G is said to be **simple** if there is at most one edge between any two distinct vertices.

Remark. Note that if G is simple and undirected, then

$$E(G) \subseteq \left\{(v_j, v_i) : v_j, v_i \in V(G), j \geq i\right\}.$$

Notice that if (v_i, v_i) is an edge, it connects a vertex to itself. This is known as a *loop*.

In a graph, it is important to know the layout of the vertices and edges. For this purpose, we define the notion of *adjacency* in a multigraph.

Definition 1.1.3 ▸ Adjacency and Neighbourhood

Let $v_i, v_j \in V(G)$, we say that they are **adjacent** if $v_i v_j \in E(G)$. Alternatively, we say that v_i and v_j are **neighbours** to each other. The edge $v_i v_j$ is said to be **incident** with v_i and v_j. Two edges e and f are said to be **adjacent** if there exists some $v \in V(G)$ such that both e and f are incident with v.

The set of all neighbours to some $v_i \in V(G)$ is called the **neighbourhood set** of v_i, denoted by $N_G(v_i)$. In particular, the set $N_G[v_i] \coloneqq N_G(v_i) \cup \{v_i\}$ is known as the **closed neighbourhood set** of v_i.

Alternatively, one may write $v \sim u$ if v and u are adjacent vertices. We can further add on to the definition by discussing the size of the neighbourhood of a vertex.

Definition 1.1.4 ▸ Degree

The **degree** of v, denoted by $d_G(v)$ is defined as the number of edges incident to v. If $d_G(v)$ is even (repectively, odd), then we say that v is an even (respectively, odd) vertex. If $d_G(v) = 0$, we say that v is **isolated**; if $d_G(v) = 1$, we say that v is an **end** vertex.

In particular, we define

$$\Delta(G) = \max_{v \in G} d_G(v), \qquad \delta(G) = \min_{v \in G} d_G(v).$$

In particular, we can denote the average degree of a graph G by

$$\bar{d}(G) = \frac{\sum_{v \in V(G)} d_G(v)}{v(G)}.$$

Remark. Note that a high maximal degree does not imply a high average degree. A classic counter example is a complete graph plus an isolated vertex.

With the notion of degree established, we can now define a *regular graph*.

Definition 1.1.5 ▸ Regular Graph

A **regular graph** is a graph in which every vertex has the same degree. In particular, if $d_G(v) = k$ for all $v \in V(G)$, G is known as a **k-regular graph**.

Remark. A graph G is regular if and only if $\Delta(G) = \delta(G)$.

Since a graph essentially consists of two sets, it is natural to consider the notion of graph complementation. We now proceed to introducing the notion of *complement*.

Definition 1.1.6 ▸ Complement

Let G be a graph of order n, the **complement** of G, denoted by $\overline{G}$, is the graph of order n where

$$V\left(\overline{G}\right) = V(G), \qquad E\left(\overline{G}\right) = \{(u,v) : (u,v) \notin E(G)\}.$$

1.1.2 Handshaking Lemma

In this section, we discuss the following interesting question:

> 15 students went to a party. During the party some of them shook hands with each other. At the end of the party, the number of handshakes made by each student was recorded and it was reported that the sum was 39. Was this possible?

Note that if we represent each student as a vertex, then we can use $V(G) = \{v_1, v_2, \cdots, v_{15}\}$ to construct a graph, in which an edge $v_i v_j \in E(G)$ if and only if students i and j shook hands. As such, $d_G(v_i)$ is the number of persons student i shook hands with.

Note that if we sum up $d_G(v_i)$ for all the vertices, every edge will be counted exactly twice! This means that

$$\sum_{i=1}^{15} d_G(v_i) = 39$$

is impossible since the left-hand side must be even. In fact, by the above reasoning, we see that in any graph, the sum of degrees of its vertices must be even.

Lemma 1.1.7 ▸ Handshaking Lemma

If G is a graph of order n and size m, then

$$\sum_{i=1}^{n} d_G(v_i) = 2m.$$

> *Remark.* It can be easily deduced from the above lemma that in any graph, the number of vertices with odd degrees must be even.

Relating to Definition 1.1.4, we see that the average degree of a graph G can be computed as

$$\bar{d}(G) = \frac{2e(G)}{v(G)}.$$

As an extension of Lemma 1.1.7, the minimum size of any graph is obviously 0, and the maximum size of a simple graph occurs when there is an edge between any two vertices.

Definition 1.1.8 ▸ Empty and Complete Graphs

Let G be a simple graph of order n. G is said to be an **empty graph** or **null graph**, denoted by 0_n, if $e(G) = 0$, and a **complete graph**, denoted by K_n if for all $u, v \in V(G)$, we have $(u, v) \in E(G)$.

1.1.3 Subgraph

Since a multigraph is just two sets, we can define a "subset" relation between graphs.

Definition 1.1.9 ▸ Subgraph

Let G, H be graphs, then H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In particular, a subgraph H is a **proper subgraph** of G if $V(H) \neq V(G)$ or $E(H) \neq E(G)$.

Note that we a graph can possibly be reproduced by connecting its vertices correctly, so a subgraph containing all vertices of a graph "spans" the original graph.

Definition 1.1.10 ▸ Spanning Subgraph

Let H be a subgraph of G. H is called a **spanning subgraph** of G if $V(H) = V(G)$.

Note that by definition, a spanning subgraph retains all vertices of the original graph. Therefore, a way to quickly generate a subgraph of a given graph G is to keep the vertex set and delete some edges from the edge set. We denote such a graph by $H = G - F$ for some $F \subset E(G)$. Observe that

$$V(H) = V(G), \qquad E(H) = E(G) - F.$$

On the other hand, by deleting some vertices together with edges incident to them, we can produce a subgraph from any given graph.

Definition 1.1.11 ▸ Induced Subgraph

Let H be a subgraph of G. H is called a **induced subgraph** of G if

$$E(H) = \{uv \in E(G) : u, v \in V(H)\}.$$

Let $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $[S]$.

Alternatively, let $F \subseteq E(G)$. Define

$$V' := \bigcup \{\{u, v\} : uv \in F\},$$

then (V', F) is the subgraph of G induced by F, denoted by $G[F]$.

Intuitively, an induced subgraph pf G consists of a selected subset of the vertices of G together with all edges in G connecting any vertices in this subset.

Using the idea of deletion, we can quickly generate an induced subgraph H of G as follows: first, set $V(H) = V(G) - A$ for some $A \subseteq V(G)$, i.e., remove some vertices; then, we will remove all edges from $E(G)$ which are incident to some vertices in A, i.e., set

$$E(H) = \{e \in E(G) : \ e \text{ is not incident to any } v \in A\}.$$

Then, $H = \big(V(H), E(H)\big)$ is an induced subgraph of G. In fact, we can denote H as

$$H = G - A = [V(G) - A].$$

This leads to the following proposition:

Proposition 1.1.12

Let G be a graph. If $A \subseteq V(G)$, then $G - A = [V(G) - A]$. Suppose H is a subgraph of G, then H is an induced subgraph of G if and only if $H = G - \big(V(G) - V(H)\big)$.

We would like to consider the relations between the properties of a graph to those of its subgraphs. In particular, it follows from intuition that if a graph has a high average degree, then we can naturally find a subgraph with a high minimal degree.

This seems trivial from intuition, as we can always keep deleting vertices with the minimal degree from a graph until we cannot find any vertex whose degree is less than $\frac{1}{2}\bar{d}(G)$, but there is more to that — for instance, how do we know if we would not end up deleting all vertices from the original graph? Thus, to prove our claim is essentially asking to prove the correctness of this greedy deletion, which we shall do in the following proposition.

Proposition 1.1.13

Every non-empty graph G has a subgraph H such that $\delta(H) \geq \frac{1}{2}\bar{d}(G)$.

Proof. Define $H_0 = G$ and $H_{i+1} = H_i - v_i$ where v_i is a vertex in H_i whose degree is the smallest. Note that we can always find this v_i for any H_i with $V(H_i) \neq \varnothing$

because $V(H_i)$ is a finite well-ordered set. We will repeatedly perform the deletion until we obtain some H_k with $d_{H_k}(v_k) \geq \frac{1}{2}\bar{d}(G)$. We shall proceed to proving that this algorithm always terminates with $V(H_k) \neq \varnothing$.

Suppose on contrary that $V(H_k) = \varnothing$, then we have performed the deletion for $v(G)$ times. Suppose we have deleted N edges in total, then clearly,

$$N < v(G)\frac{1}{2}\bar{d}(G) = v(G)\frac{e(G)}{v(G)} = e(G),$$

which means $e(H_k) = e(G) - N > 0$. However, this is impossible since $v(H_k) = 0$, which is a contradiction. □

1.1.4 Graph Isomorphism

Given two graphs G and H, are they the same graph? This seemingly innocent question proves to be extremely hard to answer. Two graphs can look drastically different but be structurally identical in reality. For example, we could shift around the vertices of a graph without changing any edge to alter the shape of the graph dramatically. Therefore, to compare the structures of graphs, we require some rigorous definition.

Definition 1.1.14 ▶ Graph Isomorphism

Two graphs G and H are said to be **isomorphic**, denoted by $G \cong H$, if there exists a bijection $f : V(G) \to V(H)$ such that

$$uv \in E(G) \quad \text{if and only if } f(u)f(v) \in E(H).$$

Such a bijection f is said to **preserve adjacency**, i.e., if u and v are neighbours in G, then their images are also neighbours in H. In particular, it is possible to map a graph to its own complement via an isomorphism.

Definition 1.1.15 ▶ Self-Complementary Graph

A graph G is said to be **self-complementary** if $G \cong \overline{G}$.

It turns out that a self-complementary graph satisfies some special properties.

Proposition 1.1.16 ▶ Order of Self-Complementray Graphs

If G is a self-complementary graph of order n, then $n = 4k$ or $n = 4k+1$ for some $k \in \mathbb{Z}^+$.

Proof. Since G is self-complementary, $G \cong \overline{G}$ and so $e(G) = e\left(\overline{G}\right)$. Note that

$$e(G) + e\left(\overline{G}\right) = \binom{n}{2} = \frac{n(n-1)}{2},$$

so we have $e(G) = \frac{n(n-1)}{4}$. Since $e(G) \in \mathbb{Z}^+$, either n or $n-1$ is divisible by 4, so $n = 4k$ or $n = 4k + 1$ for some $k \in \mathbb{Z}^+$. □

It is very hard to determine whether two specific graphs are isomorphic, but there are some considerations in the general case. For example, some trivial conclusions include:

- Two graphs with different orders cannot be isomorphic.
- Two graphs with different sizes cannot be isomorphic.
- Two graphs with different numbers of components cannot be isomorphic.
- If the numbers of vertices with degree k are different in two graphs, they cannot be isomorphic.

Notice that by Definition 1.1.14, we can define a function g such that $uv \notin E(G)$ if and only if $g(u)g(v) \notin E(H)$ and relate this function to the complement graphs of G and H. Here we introduce a way to determine isomorphism by considering complement graphs, the proof of which is left to the reader as an exercise.

Theorem 1.1.17 ▸ Complementation Preserves Isomorphism

Let G and H be two graphs of the same order. $G \cong H$ if and only if $\overline{G} \cong \overline{H}$.

Now let us think the reverse: if it is not easy to prove isomorphism, can we find a way to quickly determine that two graphs are not isomorphic? Here we present the necessary conditions for isomorphism:

Theorem 1.1.18 ▸ Necessary Conditions for Isomorphism

If $G \cong H$, then

1. *G and H must have the same order and size.*
2. *$\delta(G) = \delta(H)$ and $\Delta(G) = \Delta(H)$.*
3. *The number of vertices with degree i in G and H is the same for all $i \in \mathbb{N}$.*

The first two conditions are easy to observe. For the third condition, we introduce a tool known as *degree sequences*.

Definition 1.1.19 ▶ Degree Sequence

Let G be a graph of order n. If we label its vertices by $v_1, v_2, \cdots, v_n$ such that

$$d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n),$$

then the non-increasing sequence $\big(d(v_n)\big)$ is known as the **degree sequence** of G.

Now we consider the following question:

> Let (d_n) be a non-increasing sequence, is there some graph whose degree sequence is (d_n)?

We will make use of the following definition:

Definition 1.1.20 ▶ Graphic Sequence

Let (d_n) be a sequence of non-negative integers at most $n-1$. (d_n) is said to be **graphic** if there exists a graph G whose degree sequence is (d_n).

To determine whether a sequence is graphic by eye power is difficult. Fortunately, we have the following recursive approach:

Theorem 1.1.21 ▶ Havel-Hakimi Algorithm

Let (d_n) be a non-increasing sequence of non-negative integers at most $n-1$. Define

$$d_m^* = \begin{cases} d_{m+1} - 1 & \text{if } 1 \leq m \leq d_1 \\ d_{m+1} & \text{if } d_1 + 1 \leq m \leq n-1 \end{cases}.$$

(d_n) is graphic if and only if (d_m^*) is graphic.

Proof. Suppose (d_n) is graphic,we consider the following lemma:

Lemma 1.1.22

For any graphic sequence (d_n), there is some graph G of order n with $d_G(v_i) = d_i$ for $i = 1, 2, \cdots, n$ such that v_1 is adjacent to v_j for $j = 2, 3, \cdots, d_1 + 1$.

Proof. Suppose on contrary there is no such a graph G, then for any graph G with degree sequence (d_n), v_1 is adjacent to at most $d_1 - 1$ vertices in

$$A := \{v_j : j = 2, 3, \cdots, d_1 + 1\}.$$

Thus, there exists some $v_j \in A$ such that $v_1 v_j \notin E(G)$. However, $d_G(v_1) = d_1$, so there exists some $v_k \in V(G) - A$ with $v_k \neq v_1$ such that $v_1 v_k \in E(G)$.

Notice that since $v_k \notin A \cup \{v_1\}$, $k > j$ and so $d_G(v_k) = d_k \leq d_j = d_G(v_j)$. Therefore, v_j has at least as many neighbours as v_k, which means there must exists some v_r with $v_j v_r \in E(G)$ such that $v_k v_r \notin E(G)$.

Define a graph G' by

$$V(G') = V(G), \qquad E(G') = E(G) - v_j v_r - v_1 v_k + v_j v_k + v_1 v_j.$$

Note that $d_{G'}(v_i) = d_G(v_i)$ for all $i = 1, 2, \cdots, n$, so (d_n) is also a degree sequence for G'. However, now v_1 in G' is adjacent to all vertices in A, which is a contradiction. $\square$

By Lemma 1.1.22, we can choose some graph G whose degree sequence is (d_n) such that v_1 is adjacent to d_1 vertices $v_2, v_3, \cdots, v_{d_1+1}$. Let $H = G - v_1$, then for any $u_i \in H$,

$$d_H(u_i) = \begin{cases} d_G(v_{i+1}) - 1 & \text{if } 1 \leq i \leq d_1 \\ d_G(v_{i+1}) & \text{if } d_1 + 1 \leq i \leq n - 1 \end{cases}.$$

Therefore, (d_m^*) is graphic.

Conversely, suppose (d_m^*) is graphic, then there is some graph H of order $n - 1$ such that $d_H(u_i) = d_i^*$ for $i = 1, 2, \cdots, d_1$. Construct a graph $G = H + v_1$ such that $v_i = u_{i-1}$ for $i = 2, 3, \cdots, n$ and v_1 is adjacent to $v_2, v_3, \cdots, v_{d_1+1}$. Therefore,

$$d_G(v_i) = \begin{cases} d_1 & \text{if } i = 1 \\ d^*_{i-1} + 1 & \text{if } 2 \leq i \leq d_1 + 1 \\ d^*_i & \text{if } d_1 + 2 \leq i \leq n \end{cases} .$$

Note that $d_G(v_i) = d_i$, so (d_n) is graphic. □

By Theorem 1.1.21, we can reduce a sequence recursively until we reach some obvious case. This obvious case is graphic if and only if the original sequence is graphic. We can also build the graph corresponding to the original sequence from a simple graphic sequence.

Alternatively, the following is another algorithmic way to determine whether a sequence is graphic.

Theorem 1.1.23 ▸ Erdos-Gallai Algorithm

A non-increasing sequence $(d_1, d_2, \cdots, d_n)$ is graphic if and only if $\sum_{i=1}^n d_i$ is even and for all $k = 1, 2, \cdots, n$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}.$$

Proposition 1.1.24

Let $\boldsymbol{M}$ and $\boldsymbol{A}$ be the incidence matrix and adjacency matrix of a simple graph G respectively, then

$$\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix} + \boldsymbol{A}.$$

1.2 Graph Traversal

1.2.1 Paths and Cycles

We are often interested in the "connectedness" in a graph. In particular, we would wish to study how to traverse a graph from some vertices to others.

Definition 1.2.1 ▸ Walk, Trail, Path

Let G be a graph and $x, y \in V(G)$, then an ***x*-*y* walk** is an alternating sequence

$$W := v_0 e_1 v_1 e_2 v_2 \cdots v_{n-1} e_n v_n$$

where $v_i \in V(G)$, $e_i = v_{i-1}v_i \in E(G)$, $x = v_0$ and $y = v_n$. In particular, W is **open** if $v_0 \neq v_n$ and **closed** otherwise. The **length** of a walk is defined as the number of edges in it.

If $e_i \neq e_j$ whenever $i \neq j$, then W is called a **trail**. A closed trail is called a **circuit**. If $v_i \neq v_j$ whenever $i \neq j$, then W is called a **path**. A closed path is called a **cycle**.

Remark. All paths are trails and all trails are walks, but the converses are not true.

In more informal terms, a trail is a walk with no repeated edge and a path is a trail with no repeated vertex.

Note that a path is essentially a graph. We denote a path of order n by P_n and a cycle of order n by C_n. We also have the following relevant definitions with regard to cycles:

Definition 1.2.2 ▸ Girth and Circumference

Let G be a graph. The **girth** of G is the size of the shortest non-trivial cycle in G and the **circumference** of G is the size of the longest non-trivial cycle in G.

If G is acyclic, we define the girth of G to be ∞ and the circumference to be 0. In some special graphs, we realise that we can traverse through every vertex and return to the starting point in a single traversal. Such graphs are said to be *Hamiltonian*.

Definition 1.2.3 ▸ Hamiltonian Graph

A graph G is **Hamiltonian** if it contains a cycle of size $v(G)$.

A common tool in proofs of graph theory is the **extremal approach**, which can be demonstrated in the proposition below.

Proposition 1.2.4

If a graph G contains a u-v walk of length k, then G contains a u-v path of length at most k.

Proof. Let S be the set of all u-v walks in G, Note that $S \neq \varnothing$ since there is a u-v walk of length k. Let P be a walk of the shortest length. We claim that P must be a path.

Suppose P is not a path, then there exists some vertices $w_1 = w_2$ in P. Suppose Q is obtained by removing the w_1-w_2 walk from P, then Q is a walk. However, Q is shorter than P which is not possible. Therefore, P must be a path.

Note that the length of P is at most k, so G contains a u-v path of length at most k. □

1.2.2 Connected Components

An important question we are interested in with graphs is the reachability of a vertex, i.e., given two vertices u and v, we want to know whether we can reach one vertex from another. Intuitively, we can traverse between two vertices if there is a path between them. From here we define the notion of a *connected graph*.

Definition 1.2.5 ▸ Connected Graph

Let G be a graph. Two vertices u and v are **connected** if there is a path between them. G is said to be **connected** if for any $u, v \in V(G)$, there exists a path from u to v. For any $u \in V(G)$, we denote the set of all vertices connected to u (inclusive of u) as $c(u)$.

Intuitively, we of course always can find a connected subgraph of a connected graph, but we can in fact produce a much stronger result.

Proposition 1.2.6

For every connected graph G, $V(G)$ can be labelled as $\{v_1, v_2, \cdots, v_n\}$ such that the graph $G_i := [\{v_1, v_2, \cdots, v_i\}]$ is connected for every $i = 1, 2, \cdots, n$.

Intuitively, if a graph has many edges, we would believe that it is easy to find a long path in it. Note that the notion of "having many edges" can be related to the minimal degree of a graph. More formally, the following result is true:

Proposition 1.2.7

Every graph G with $\delta(G) \geq 2$ contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$.

Proof. Let $P = x_0x_1 \cdots x_k$ be the longest path in G. Suppose that there exists some $v \in N(x_k)$ such that $v \notin V(P)$, then we can find a longer path which is not possible. Therefore, $N(x_k) \subseteq V(P)$. Note that $|N(x_k)| \geq \delta(G)$, so $k \geq \delta(G)$ and so G contains a $P_{\delta(G)} \subseteq P$. □

Note that connectedness is an equivalence relation. Let P be a partition formed by the equivalence classes of G under the connectedness relation, then for any $u, v \in V(G)$, u and v are in the same equivalence class if and only if u and v are connected. Any $c(u) \in P$ induces a subgraph $[c(u)]$.

Definition 1.2.8 ▸ Connected Component

Let G be a graph and R be a relation such that for any $u, v \in V(G)$, uRv if and only if u and v are connected. Let

$$C := V(G)/R$$

be the quotient set, then any $c(u) \in C$ is known as a (connected) **component** of G.

The number of components of G is denoted by $\omega(G)$.

Alternatively, a connected component is a maximally connected subgraph, that is, a connected component of G is a subgraph $H \subseteq G$ such that for all $e \in E(G)$, $H + e$ is not connected and for all $v \in V(G) - V(H)$, $[V(H) \cup \{v\}]$ is not connected.

It can be easily seen that G is connected if and only if $\omega(G) = 1$.

We will now establish a relationship between connectedness and complementation.

Theorem 1.2.9 ▸ Connectedness of Complement

If G is disconnected, then $\overline{G}$ is connected.

Proof. Let $u, v \in V\left(\overline{G}\right)$ be two arbitrary vertices. If $uv \notin E(G)$, then $uv \in \left(\overline{G}\right)$ and so u and v are connected in $\overline{G}$.

If $uv \in E(G)$, since G is disconnected, there exists some $w \in V(G)$ such that $uw, wv \notin E(G)$. Therefore, $uw, wv \in E\left(\overline{G}\right)$. This means that there is a u-v path in $\overline{G}$ and so u and v are connected in $\overline{G}$.

Therefore, $\overline{G}$ is connected. □

Intuitively, we can transform a connected graph into a disconnected one by deleting some vertices or edges. It can be easily observed that in certain graphs, deleting one particular vertex or edge will immediately disconnect the graph.

Definition 1.2.10 ▸ Cut-Vertex, Bridge

Let G be a non-trivial graph. $v \in V(G)$ is called a **cut-vertex** of G if $\omega(G - v) > \omega(G)$. $e \in E(G)$ is called a **bridge** of G if $\omega(G - e) > \omega(G)$.

Note that a graph does not have to be connected to have cut vertices and bridges. Essentially, a cut-vertex (or bridge) is just a vertex (or edge) which disconnects a component upon removal.

Now the next question is how do we identify a cut-vertex or a bridge in a graph? Intuitively, a cut-vertex divides a graph into two portions such that traversal between the two portions has to pass through it.

Theorem 1.2.11 ▸ Cut-Vertex Characterisation

Let G be a graph. $v \in V(G)$ is a cut-vertex if and only if there exists $a, b \in V(G)$ such that v is in every path between a and b.

Proof. Let $v \in V(G)$ be a cut-vertex and consider $H = G - v$, then we can find two non-empty disjoint connected components of H, say H_1 and H_2. Now, take $a \in V(H_1)$ and $b \in V_(H_2)$, then they are disconnected. However, a and b are connected in G, which means v is in every a-b path.

We will prove the converse by contrapositive. Suppose v is not a cut-vertex of G, then $G - v$ is connected. Therefore, for any $u, w \in V(G)$, there is a path between them which does not contain v. $\square$

A similar argument can be established for bridges. Furthermore, since we have to pass through the bridge when traversing between the two portions, it can be easily seen that we have to reuse the bridge in order to traverse back.

Theorem 1.2.12 ▸ Bridge Characterisation

Let G be a graph. $e \in E(G)$ is a bridge if and only if e is not contained by any cycle in G.

Proof. We will prove the forward direction by contrapositive. Suppose $e = xy$ is contained in some cycle in G, then there is some x-y path in $G - e$. Suppose $a, b \in V(G)$ are connected in G via a path containing e, this implies that there is some a-b path in $G - e$ and so $G - e$ is connected, which means that e is not a bridge.

We will prove the converse also by contrapositive. Suppose $e = xy$ is not a bridge, the $G - e$ is connected and so there is some x-y path in $G - e$. Let this path be P, then $P \cup \{e\}$ is a cycle in G. □

Combining the two characterisations, we can relate cut-vertices to bridges in the following proposition, the proof of which is left to the reader as an exercise:

Proposition 1.2.13

Let G be a graph. If $uv \in E(G)$ is a bridge and u is not an end vertex, then u is a cut-vertex.

A direct consequence of this is the following corollary:

Corollary 1.2.14

If G is a graph with order at least 3 and G contains a bridge, then G contains a cut-vertex.

1.2.3 Graphs as Metric Spaces

Given a graph G with u, v being two connected vertices, we are interested in the notion of distance between them. Intuitively, if there are multiple paths between u and v, we would take the length of the shortest one to represent their distance.

Definition 1.2.15 ▸ Distance, Eccentricity, Diameter

Let G be a connected graph and let $u, v \in V(G)$. The **distance** between u and v, denoted by $d(u, v)$, is the length of the shortest path between u and v. The **eccentricity** of u is defined to be

$$e(u) = \max_{v \in V(G)} \{d(u, v)\}.$$

The **diameter** of G is defined by

$$\operatorname{diam}(G) = \max_{u \in V(G)} \{e(u)\}.$$

The **radius** of G is defined by

$$\operatorname{rad}(G) = \min_{u \in V(G)} \{e(u)\}.$$

A vertex v is called a **central** vertex if

$$e(v) = \operatorname{rad}(v).$$

The subgraph induced by the set of central vertices of G is known as the **centre** of G.

This also justifies the use of words "diameter" and "radius" in a circle. We can view a circle as a graph consisting of a centre and infinitely many vertices in the circumference, with edges of length r connecting the centre to the circumference. Indeed, in this definition, the farthest vertices will be any two on the circumference with a distance of $2r$, and the nearest vertices will be the centre and any vertex on the circumference with a distance of r.

Intuitively, having a large diameter means that the graph is sparse, i.e., one has to take a very long path to traverse between two vertices. Now let us consider a connected but sparse graph, the above intuition means that we may expect to find that the cycles in this graph are very large.

Proposition 1.2.16

Let G be a graph with girth $g(G)$. If G contains a cycle, then $g(G) \leq 2\operatorname{diam}(G) + 1$.

Proof. Suppose on contrary that $g(G) \geq 2\operatorname{diam}(G) + 2$. Let $C \subseteq G$ be the shortest cycle in G, then we can take $x, y \in V(C)$ such that $d_C(x, y)$ is the greatest. Clearly, we have $d_C(x, y) \geq \operatorname{diam}(G) + 1$.

Note that by Definition 1.2.15, there exists some path P between x and y with $e(P) \leq \operatorname{diam}(G)$. Let $P' \subseteq C$ be the x-y path in C, then $e(P) < e(P')$. This means that $Q := P \cup P' - P \cap P' \neq \varnothing$.

Take some $q \in Q$. Since P and P' are both x-y walks, $|P \cap P'| \geq 2$. Therefore, there are $p_1, p_2 \in P \cap P'$ which are connected to q. Since p_1 and p_2 are connected, by Proposition ??, □

For readers with knowledge in real analysis or topology, it is easy to see that an undirected unweighted connected graph is a metric space with distance between vertices as its metric.

Theorem 1.2.17 ▸ Connected Graph as A Metric Space

If G is connected, then $(V(G), d_G)$ is a metric space.

Proof. Clearly, for all $x, y \in V(G)$, $d_G(x, y) \geq 0 \in V(G)$ with $d_G(x, y) = 0$ if and only if $x = y$ and $d_G(x, y) = d_G(y, x)$.
Let U, V be the shortest u-w path and shortest w-v path respectively, then $U + V$ is a u-v walk with length $d(u, w) + d(w, v)$. Since the shortest u-v path has length $d(u, v)$,

we have

$$d(u, v) \leq d(u, w) + d(w, v).$$

Therefore, $(V(G), d_G)$ is a metric space. □

An application of Theorem 1.2.17 allows us to establish the following:

Theorem 1.2.18 ▸ Boundedness of Diameter

If G is connected, then

$$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G).$$

Proof. $\mathrm{rad}(G) \leq \mathrm{diam}(G)$ is immediate from Definition 1.2.15. Take $u, v \in V(G)$ with $d(u, v) = \mathrm{diam}(G)$ and take $w \in V(G)$ such that $e(w) = \mathrm{rad}(G)$. Notice that this implies

$$d(u, w) \leq e(w) = \mathrm{rad}(G)$$
$$d(w, v) \leq e(w) = \mathrm{rad}(G).$$

By Theorem **??**,

$$\mathrm{diam}(G) = d(u, v) \leq d(u, w) + d(w, v) = 2\mathrm{rad}(G).$$

□