Contents

1 Topology

1.1 Topological Spaces

Definition 1.1.1 ▶ Topology

A **topology** on a set X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ such that
- $\varnothing, X \in \mathcal{T}$;
- for any index set I, if $\{X_i : i \in I\} \subseteq \mathcal{P}(\mathcal{T})$, then $\bigcup_{i \in I} X_i \in \mathcal{T}$;
- for any $X_1, X_2, \cdots, X_n \in \mathcal{T}$, $\bigcap_{i=1}^{n} X_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is said to be a **topological space**. A subset $Y \subseteq X$ is **open** if $Y \in \mathcal{T}$.

Remark. For any set X, we define $\{\varnothing, X\}$ as the *trivial topology* on X, $\mathcal{P}(X)$ as the *discrete topology*, and $\{X \backslash U : U \subseteq X \text{ is finite}\} \cup \{\varnothing\}$ as the *co-finite topology*.

The set $\{(-\alpha, \alpha) : \alpha > 0\} \cup \{\mathbb{R}, \varnothing\}$ defines a topology on $\mathbb{R}$. This example also demonstrates why it is crucial to only consider closure under finite intersections when defining a topology, because

$$\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = [-1, 1] \notin \mathcal{T}.$$

We now seek a systematic method to generate a topology given any set. The idea here is to make use of a *cover*.

Definition 1.1.2 ▶ Basis

A **basis** for a topology on X is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that
- for any $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$;
- for any $x \in X$ and $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

It may be useful to see a basis as a cover of a set with the second additional property as stated in the above definition. Notice that the first property of the basis $\mathcal{B}$ is basically saying that

$$X \subseteq \bigcup \mathcal{B},$$

i.e., $\mathcal{B}$ is a cover of X.

Given any basis $\mathcal{B}$ for some topology on X, a set generated by $\mathcal{B}$ can be defined as

$$\mathcal{T} := \{U \subseteq X : \text{ for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\}$$

We will show that $\mathcal{T}$ is a topology on X. First, it is clear that $\varnothing, X \in \mathcal{T}$.

Let I be an index set and $\{X_i : i \in I\} \subseteq \mathcal{P}(\mathcal{T})$ be any collection of subsets of X. Notice that for any $x \in \bigcup_{i \in I} X_i$, there exists some $j \in I$ such that $x \in X_j \subseteq \mathcal{T}$. According to our construction, this means that there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq X_j \subseteq \mathcal{T}$. Therefore, $\bigcup_{i \in I} X_i \subseteq \mathcal{T}$ as desired.

To prove that $\mathcal{T}$ is closed under finite intersection, we consider the following lemma:

Lemma 1.1.3 ▶ Finite Intersection of Elements in Basis Is Covered

Let $\mathcal{B}$ be a basis for a topology on X and $B_1, B_2, \cdots, B_n \in \mathcal{B}$, then for any $x \in \bigcap_{i=1}^n B_i$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcap_{i=1}^n B_i$.

Proof. The case where $n = 1$ is trivial by taking $B = B_1$. Suppose that there is some integer $k \geq 1$ such that for any $B_1, B_2, \cdots, B_k \in \mathcal{B}$ and any $x \in \bigcap_{i=1}^k B_i$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcap_{i=1}^k B_i$. Take any $B_{k+1} \in \mathcal{B}$. It is clear that for any $x \in \bigcap_{i=1}^{k+1} B_i$, there exists some $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^{k} B_i.$$

Notice that $x \in B_{k+1} \in \mathcal{B}$, so we know that $x \in B \cap B_{k+1}$. By Definition 1.1.2, this means that there exists some $B' \in \mathcal{B}$ such that

$$x \in B' \subseteq B \cap B_{k+1} \subseteq \bigcap_{i=1}^{k+1} B_i.$$

$\square$

Now, suppose $X_1, X_2, \cdots, X_n \in \mathcal{T}$ are finitely many subsets of X. Take any $x \in \bigcap_{i=1}^n X_i$. It is clear that $x \in X_i$ for each $i = 1, 2, \cdots, n$. Therefore, for each $i = 1, 2, \cdots, n$, there exists some $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq X_i$. By Lemma 1.1.3, this means that there exists some set $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^n B_i \subseteq \bigcap_{i=1}^n X_i.$$

Therefore, $\bigcap_{i=1}^n X_i \in \mathcal{T}$. So this set $\mathcal{T}$ generated by $\mathcal{B}$ is indeed a topology on X.

The following proposition further shows that the topology generated by a basis $\mathcal{B}$ is the set of all possible unions of elements in $\mathcal{B}$:

Proposition 1.1.4 ▶ Equivalent Construction of Topologies Generated from Bases

Let X be any set. If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X, then

$$\mathcal{T} = \left\{\bigcup_{A \in \mathcal{V}} A : \mathcal{V} \in \mathcal{P}(\mathcal{B})\right\}.$$

Proof. Denote

$$\mathcal{T}_{\mathcal{B}} := \{U \subseteq X : \text{ for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\}.$$

It suffices to prove that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. Take any $T \in \mathcal{T}$, then there exists some $V \in \mathcal{P}(\mathcal{B})$ such that $T = \bigcup_{A \in \mathcal{V}} A$. This means that for every $t \in T$, there exists some $B_t \in \mathcal{V}$ such that $t \in B_t \subseteq T$. Therefore, $T \in \mathcal{T}_{\mathcal{B}}$. Conversely, for any $S \in \mathcal{T}_{\mathcal{B}}$, there exists some $B_s \in \mathcal{B}$ for each $s \in S$ such that $s \in B_s$. Denote $U := \{B_s : s \in S\} \in \mathcal{P}(\mathcal{B})$, then it is clear that $S \subseteq \bigcup_{B \in U} B$. Since $B_s \subseteq S$ for each $s \in S$, we have $\bigcup_{B \in U} B \subseteq S$, which implies that $S = \bigcup_{B \in U} B$. This means that $S \in \mathcal{T}$. Therefore, $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$, which means that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. □

Next, we define a special topology in Euclidean spaces using open balls.

Definition 1.1.5 ▶ Standard Topology

For any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and any $r > 0$. Denote the Euclidean open ball centred at $\boldsymbol{x}$ with radius r by

$$B_r(\boldsymbol{x}) := \left\{\boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n : \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} < r\right\}$$

The **standard topology** on $\mathbb{R}^n$ is the set generated by the basis

$$\mathcal{B} := \{B_r(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n, r \in \mathbb{R}^+\}.$$

It may be helpful to actually show that this set $\mathcal{B}$ is indeed a basis of a topology on $\mathbb{R}^n$. The fact that $\mathcal{B}$ is a cover for $\mathbb{R}^n$ is trivial enough. Take any $\boldsymbol{x} \in \mathbb{R}^n$ and balls $B_\alpha(\boldsymbol{x}_1), B_\beta(\boldsymbol{x}_2) \in \mathcal{B}$ such that $\boldsymbol{x} \in B_\alpha(\boldsymbol{x}_1) \cap B_\beta(\boldsymbol{x}_2)$ (the existence of these 2 balls is again trivial enough). Take

$$r = \min\{\alpha - \|\boldsymbol{x} - \boldsymbol{x}_1\|, \beta - \|\boldsymbol{x} - \boldsymbol{x}_2\|\}.$$

Clearly, $r > 0$ and $\boldsymbol{x} \in B_r(\boldsymbol{x})$, so we are done.

Now, we discuss the analogue of the subset relation in topologies.

Definition 1.1.6 ▸ Fineness and Coarseness

Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on some set X. We say that $\mathcal{T}$ is **finer** than $\mathcal{T}'$, or equivalently, that $\mathcal{T}'$ is **coarser** than $\mathcal{T}$, if $\mathcal{T}' \subseteq \mathcal{T}$.

Observe that any topology of X must be a subset of $\mathcal{P}(X)$, which is the discrete topology on X, so the discrete topology is the finest topology on a set.

Remark. For any basis $\mathcal{B}$ for a topology on X, the topology generated by $\mathcal{B}$ is the coarsest topology containing $\mathcal{B}$.

The above remark is easy to verify. Let $\mathcal{T}$ be any topology on X with $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}_{\mathcal{B}}$ be the topology generated by $\mathcal{B}$. For any $T \in \mathcal{T}_{\mathcal{B}}$, by Proposition 1.1.4, there exists some $\mathcal{V} \subseteq \mathcal{B}$ such that $T = \bigcup_{A \in \mathcal{V}} A$. Note that $A \in \mathcal{T}$ for all $A \in \mathcal{V}$, so by Definition 1.1.1, $T \in \mathcal{T}$ and so $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$ as desired.

This motivates us to consider fineness in terms of bases.

Proposition 1.1.7 ▸ Fineness in Terms of Bases

Let $\mathcal{B}$ and $\mathcal{B}'$ generate topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for every $B \in \mathcal{B}$ and any $x \in B$, there exists some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B$.

Proof. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$, then $\mathcal{T} \subseteq \mathcal{T}'$. Take any $B \in \mathcal{B}$, then by Proposition 1.1.4, $B \in \mathcal{T}$, which means that $B \in \mathcal{T}'$. Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$, by Definition 1.1.2 for any $x \in B$, there exists some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B$.

Suppose conversely that for every $B \in \mathcal{B}$ and any $x \in B$, there is some $B_x \in \mathcal{B}'$ such that $x \in B' \subseteq B$. Take any $T \in \mathcal{T}$, for each $x \in T$, by Definition 1.1.2 there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq T$, and so we can find some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B \subseteq T$, so $T \in \mathcal{T}'$. Therefore, $\mathcal{T} \subseteq \mathcal{T}'$ and so $\mathcal{T}'$ is finer than $\mathcal{T}$. □

Recall that every basis of a topology on X is an open cover of X consisting only of subsets of X. Therefore, the union of the elements in the basis is essentially X itself. This motivates us to propose another way to generate a topology on a set.

Definition 1.1.8 ▸ Sub-basis

A **sub-basis** of X is a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ such that $\bigcup_{A \in \mathcal{S}} A = X$.

Remark. Every basis is a sub-basis.

For an arbitrary set X, let $\mathcal{S}$ be a sub-basis and denote the collection of all finite subsets of $\mathcal{P}(\mathcal{S})$ as $\mathcal{F}_{\mathcal{S}}$. Define

$$\mathcal{U}_{\mathcal{S}} := \left\{\bigcap_{A\in F} A : F \in \mathcal{F}_{\mathcal{S}}\right\}$$

to be the collection of all finite intersections of sets in $\mathcal{S}$. The topology generated by a sub-basis of X is given by

$$\mathcal{T} := \left\{\bigcup_{A\in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{U}_{\mathcal{S}}\right\}.$$

We shall show that $\mathcal{T}$ is indeed a topology on X by considering the following proposition:

Proposition 1.1.9 ▸ Finite Intersections of Sets in a Sub-basis Form a Basis

Let $\mathcal{S}$ be a sub-basis for a set X and let $\mathcal{U}_{\mathcal{S}}$ be the set of all finite intersections of sets in $\mathcal{S}$, then $\mathcal{U}_{\mathcal{S}}$ is a basis of a topology on X.

Proof. Take any $x \in X$. By Definition 1.1.8, we have $x \in \bigcup_{A\in\mathcal{S}} A$. Therefore, there exists some $A \in \mathcal{S} \subseteq \mathcal{P}(X)$ such that $x \in A$. For any $x \in X$ and $B_1, B_2 \in \mathcal{U}_{\mathcal{S}}$ such that $x \in B_1 \cap B_2$, notice that $B_1 \cap B_2$ is a finite intersection of sets in $\mathcal{S}$, so $B_1 \cap B_2 \in \mathcal{U}_{\mathcal{S}}$. Therefore, by Definition 1.1.2, $\mathcal{U}_{\mathcal{S}}$ is a basis. ☐

With Propositions 1.1.9 and 1.1.4, it is clear that $\mathcal{T}$ as constructed above is a topology on X.

1.2 Metric Spaces

Definition 1.2.1 ▸ Metric

A **metric** on a set S is a function $d : S \times S \to \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, y) = 0$ if and only if $x = y$ (definiteness);
- $d(x, y) = d(x, y)$ for all $x, y \in S$ (symmetry);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

Remark. A metric is sometimes also called a *distance function*.

A metric generalises the notion of distance in Euclidean spaces. We can weaken the above axioms to arrive at the following definition:

Definition 1.2.2 ▸ Pseudo-metric

A **pseudo-metric** on a set S is a function $d: S \times S \to \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, x) = 0$ for all $x \in S$;
- $d(x, y) = d(x, y)$ for all $x, y \in S$ (symmetry);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

The key difference between a pseudo-metric and a metric is that a pseudo-metric only requires that every element is at 0 distance away from itself, whereas a metric requires that every element is **the only element** that is at 0 distance away from itself.

By dropping the requirement on symmetry, we obtain the following definition:

Definition 1.2.3 ▸ Quasi-metric

A **quasi-metric** on a set S is a function $d: S \times S \to \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, y) = 0$ if and only if $x = y$ (definiteness);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

We equip a set with a metric to generalise the Euclidean spaces.

Definition 1.2.4 ▸ Metirc Space

A **metric space** (S, d) is a set S together with a metric d on S.

The most basic example of a metric is the *discrete metric* defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

over any set X, which essentially is just a characteristic function.

Recall that in an inner product space (V, g) over some field $\mathbb{F}$, we can define the length of any $\boldsymbol{v} \in V$ as

$$\|\boldsymbol{v}\| = \sqrt{g(\boldsymbol{v}, \boldsymbol{v})}.$$

This length function induces a metric over V given by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|.$$

In the Euclidean space $\mathbb{R}^n$, a usual definition for distance is

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{n} (y_i - x_i)^2\right]^{\frac{1}{2}}.$$

Note that $(\mathbb{R}^n, d_2)$ is a metric space, where d_2 is known as the *Euclidean distance*. In general, we can prove that for any $p \in \mathbb{N}^+$,

$$d_p(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{n} \|y_i - x_i\|^p\right]^{\frac{1}{p}}$$

is a metric over $\mathbb{F}^n$ for any inner product space $(\mathbb{F}^n, g)$ where $\mathbb{F}$ is a field, known as the *L^p-norm*. Furthermore, notice that

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p \leq \sum_{i=1}^{n} \|y_i - x_i\|^p \leq n \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p.$$

Taking the p-th root on all three parts, we have

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\| \leq \left[\sum_{i=1}^{n} \|y_i - x_i\|^p\right]^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

By Squeeze Theorem, this allows us to define

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \lim_{p \to \infty} d_p(\boldsymbol{x}, \boldsymbol{y}) = \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

$d_\infty(\boldsymbol{x}, \boldsymbol{y})$ can be alternatively written as $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty$, which is known as the *infinite norm*.

The p-adic numbers can be defined from the following lemma:

Lemma 1.2.5 ▸ p-adic Numbers

Let p be any prime number. For all $x \in \mathbb{Q} \setminus \{0\}$, there exists a unique $k \in \mathbb{Z}$ such that

$$x = \frac{p^k r}{s}, \qquad r, s \in \mathbb{Z}$$

with $p \nmid r, s$ and $s \neq 0$.

The *p-adic norm* is defined as

$$|x|_p = \begin{cases} p^{-k} & \text{if } x = \frac{p^k r}{s} \\ 0 & \text{if } x = 0 \end{cases},$$

which induces a metric over $\mathbb{Q}$ defined by

$$d(x,y) = |x-y|_p.$$

We can show that the p-adic metric satisfies

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}$$

for all $x, y, z \in \mathbb{Q}$. Such a metric is known as an *ultra-metric*.

Given any metric space, the metric will induce a distance between subsets of the space.

Definition 1.2.6 ▸ Distance between Subsets

Let (X,d) be a metric space and $A, B \subseteq X$ be non-empty. The **distance** between A and B is defined as

$$d(A,B) \coloneqq \inf\{d(x,y) : (x,y) \in A \times B\}.$$

Additionally, we may wish to define a measure for the size of a subset in a metric space.

Definition 1.2.7 ▸ Diameter

Let (X,d) be a metric space. The **diameter** of a set $A \subseteq X$ is defined as

$$\operatorname{diam}(A) \coloneqq \sup\{d(x,y) : (x,y) \in A \times A\}.$$

The set A is **bounded** if $\operatorname{diam}(A)$ is finite.

The name "diameter" is not a coincidence with the diameter of a graph. Specifically, if we consider a graph $G = (V,E)$, the pair (V,d) forms a metric space with $d(u,v)$ being the usual distance between two vertices in G defined as the size of the shortest u-v path in G. It is clear that d is indeed a metric.

Now, let us consider the subgraph $H \subseteq G$ induced by any $U \subseteq V$ and check the eccentricity for H, i.e.,

$$\epsilon(u) = \max\{d_H(u,u') : u' \in U\} \qquad \text{for all } u \in U.$$

Now, the diameters for H can be computed as

$$\begin{aligned}\operatorname{diam}(H) &= \max\{\epsilon(u) : u \in U\} \\ &= \sup\{d_H(u,u') : (u,u') \in U \times U\},\end{aligned}$$

and this obviously agrees with Definition 1.2.7!

Recall that in Definition 1.1.5, we use Euclidean open balls to construct a basis for a topology on $\mathbb{R}^n$. We can generalise this idea in any metric space.

Proposition 1.2.8 ▸ Metric Induces a Basis

Let (X, d) be a metric space. Define

$$B_r(x) := \{y \in X : d(x, y) < r\},$$

then collection

$$\mathcal{B}_d := \{B_r(x) : x \in X, r \in \mathbb{R}^+\}$$

is a basis for a topology on X.

Proof. Notice that for any $x \in X$, we have $x \in B_1(x) \in \mathcal{B}_d$. Let $B_p(x_1), B_q(x_2) \in \mathcal{B}_d$ be such that $x \in B_p(x) \cap B_q(x)$. Take $k = \min\{p - d(x, x_1), q - d(x, x_2)\}$, then clearly $k > 0$ and we can find $B_k(x) \subseteq B_p(x) \cap B_q(x)$ such that $x \in B_k(x) \in \mathcal{B}_d$. Therefore, $\mathcal{B}_d$ is a basis for a topology on X. □

Since we can obtain a basis from a metric, it follows naturally that we can generate a topology using this induced basis.

Definition 1.2.9 ▸ Metrisable Topology

Let (X, d) be a metric space. A topology $\mathcal{T}$ on X is **metrisable**, or **induced** by d, if it is generated by $\mathcal{B}_d$

We can verify that the discrete topology $\mathcal{P}(X)$ is induced by the discrete metric. Let the discrete metric on X be χ, then it is easy to see that

$$B_r(x) = \begin{cases} \{x\} & \text{if } 0 < r \leq 1 \\ X & \text{if } r > 1 \end{cases}.$$

Therefore,

$$\mathcal{B}_\chi = \{X\} \cup \{\{x\} : x \in X\}.$$

Let $\mathcal{T}_\chi$ be the topology on X generated by $\mathcal{B}_\chi$, then it suffices to prove that $\mathcal{P}(X) \subseteq \mathcal{T}_\chi$. Take any $U \in \mathcal{P}(X)$, then for any $u \in U$, we have $u \in \{u\} \subseteq U$. Clearly $\{u\} \in \mathcal{B}_\chi$, so $\mathcal{T}_\chi = \mathcal{P}(X)$ is the discrete topology indeed.

In particular, for Euclidean spaces, the following result extends Definition 1.1.5:

Proposition 1.2.10 ▸ Every L^p-metric Generates the Standard Topology

Let $\mathcal{T}$ be the standard topology on $\mathbb{R}^n$, then $\mathcal{T}$ is induced by any L^p-metric d_p.

Proof. For any $p \in \mathbb{N}^+$, notice that

$$\max_{i\in\mathbb{N}^+, i\leq n} \|y_i - x_i\|^p \leq \sum_{i=1}^{n} \|y_i - x_i\|^p \leq n \max_{i\in\mathbb{N}^+, i\leq n} \|y_i - x_i\|^p.$$

Taking the p-th root yields

$$\max_{i\in\mathbb{N}^+, i\leq n} \|y_i - x_i\| \leq \left[\sum_{i=1}^{n} \|y_i - x_i\|^p\right]^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max_{i\in\mathbb{N}^+, i\leq n} \|y_i - x_i\|.$$

This means that

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_p(\boldsymbol{x}, \boldsymbol{y}) \leq n^{\frac{1}{p}} d_\infty(\boldsymbol{x}, \boldsymbol{y}).$$

Let $\mathcal{T}_0$ and $\mathcal{T}_p$ be topologies on $\mathbb{R}^n$ generated by $\mathcal{B}_{d_\infty}$ and $\mathcal{B}_{d_p}$ respectively. Take any $T \in \mathcal{T}_p$, then for any $\boldsymbol{t} \in T$, there is some $B_r(\boldsymbol{t}') \in \mathcal{B}_{d_p}$ such that $\boldsymbol{t} \in B_r(\boldsymbol{t}') \subseteq T$. Take some $\ell = \frac{1}{2}|r - d_p(\boldsymbol{t}, \boldsymbol{t}')|$, then we have found $B_\ell(\boldsymbol{t}) \in \mathcal{B}_{d_p}$ such that

$$\boldsymbol{t} \in B_\ell(\boldsymbol{t}) \subseteq B_r(\boldsymbol{t}') \subseteq T.$$

Take $k = \frac{1}{2}\ell n^{-\frac{1}{p}}$ and consider

$$B_k(\boldsymbol{t}) \coloneqq \{\boldsymbol{y} \in \mathbb{R}^n : d_\infty(\boldsymbol{t}, \boldsymbol{y}) < k\} \in \mathcal{B}_{d_\infty}.$$

Notice that for each $\boldsymbol{y} \in B_k(\boldsymbol{t})$, we have

$$d_p(\boldsymbol{t}, \boldsymbol{y}) \leq n^{\frac{1}{p}} d_\infty(\boldsymbol{t}, \boldsymbol{y}) < \ell,$$

so $\boldsymbol{t} \in B_k(\boldsymbol{t}) \subseteq B_\ell(\boldsymbol{t}) \subseteq T$. This implies that $T \in \mathcal{T}_0$ and so $\mathcal{T}_p \subseteq \mathcal{T}_0$. By a similar argument, one may check that $\mathcal{T}_0 \subseteq \mathcal{T}_p$. Therefore, $\mathcal{T}_0 = \mathcal{T}_p$ for any $p \in \mathbb{N}^+$. Note that by Definition 1.1.5, $\mathcal{T}$ is generated by $\mathcal{B}_{d_2}$, which means that $\mathcal{T} = \mathcal{T}_2 = \mathcal{T}_0 = \mathcal{T}_p$ for any $p \in \mathbb{N}^+$. Therefore, $\mathcal{T}$ is induce by any L^p-metric d_p. □

The fact that

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_p(\boldsymbol{x}, \boldsymbol{y}) \leq n^{\frac{1}{p}} d_\infty(\boldsymbol{x}, \boldsymbol{y})$$

means that all L^p-metrics are equivalent over the same space.

1.3 Subspace Topologies

Definition 1.3.1 ▶ Subspace Topology

Let $(Y, \mathcal{T}_Y)$ be a topological space and $X \subseteq Y$ be some subset. The collection

$$\mathcal{T}_X := \{U \cap X : U \in \mathcal{T}_Y\}$$

is the **subspace topology** on X.

We may check that $\mathcal{T}_X$ defined as such is indeed a topology on X. First, by taking $U = \varnothing$ and $U = Y$ respectively, we know that $\varnothing, X \in \mathcal{T}_X$. For any $U \in \mathcal{T}_Y$, we have $Y \backslash U \in \mathcal{T}_Y$ and so

$$X \backslash (U \cap X) = (Y \backslash U) \cap X \in \mathcal{T}_X.$$

For any $\mathcal{V} \subseteq \mathcal{T}_X$, we define a subset $\mathcal{U}_\mathcal{V} \subseteq \mathcal{T}_Y$ such that for each $V \in \mathcal{V}$ there is a unique $U_V \in \mathcal{U}_\mathcal{V}$ such that $V = U_V \cap X$. Then,

$$\begin{aligned} \bigcup_{A \in \mathcal{V}} A &= \bigcup_{B \in \mathcal{U}_\mathcal{V}} (B \cap X) \\ &= \left(\bigcup_{B \in \mathcal{U}_\mathcal{V}} B \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

Let $X_1, X_2, \cdots, X_n \in \mathcal{T}_X$ and define $X_i = U_i \cap X$ where $U_i \in \mathcal{T}_Y$ for $i = 1, 2, \cdots, n$, then

$$\begin{aligned} \bigcap_{i=1}^n X_i &= \bigcap_{i=1}^n (U_i \cap X) \\ &= \left(\bigcap_{i=1}^n U_i \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

So $\mathcal{T}_X$ is really a topology on X. Intuitively, the following holds:

Proposition 1.3.2 ▶ Basis for a Subspace

Let $(Y, \mathcal{T}_Y)$ be a topological space and $\mathcal{T}_X$ be the subspace topology on some $X \subseteq Y$. If $\mathcal{B}_Y$ is a basis of $\mathcal{T}_Y$, then

$$\mathcal{B}_X := \{B \cap X : B \in \mathcal{B}_Y\}$$

is a basis of $\mathcal{T}_X$.

Proof. We first prove that $\mathcal{B}_X$ is a basis. Take any $x \in X \subseteq Y$. Note that there exists some $B \in \mathcal{B}_Y$ such that $x \in B$. Take $B \cap X \in \mathcal{B}_X$, then $x \in B \cap X$. For any $B_1, B_2 \in \mathcal{B}_X$ with $x \in B_1 \cap B_2$, we write $B_1 := B_1' \cap X$ and $B_2 := B_2' \cap X$ where $B_1', B_2' \in \mathcal{B}_Y$, then we have $x \in B_1' \cap B_2'$. This means that there is some $B \in \mathcal{B}_Y$ such that $x \in B \subseteq B_1' \cap B_2'$. Write $B' := B \cap X \in \mathcal{B}_X$, then for each $b \in B'$, we know that $b \in B_1' \cap B_2'$ and $b \in X$, which implies that $b \in B_1 \cap B_2$. Therefore, $x \in B' \subseteq B_1 \cap B_2$. This means that $\mathcal{B}_X$ is a basis of a topology on X.

We then prove that $\mathcal{T}_X$ is generated by $\mathcal{B}_X$. Let $\mathcal{T}$ be the topology generated by $\mathcal{B}_X$. By Proposition 1.1.4, we have

$$\mathcal{T} = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{B}_X \right\}.$$

Similarly, we can write

$$\mathcal{T}_Y = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{B}_Y \right\}.$$

Take any $T \in \mathcal{T}_X$, then there exists some $\mathcal{V} \subseteq \mathcal{B}_Y$ such that

$$\begin{aligned} T &= \left(\bigcup_{A \in \mathcal{V}} A \right) \cap X \\ &= \bigcup_{A \in \mathcal{V}} A \cap X \\ &\in \mathcal{T}. \end{aligned}$$

Therefore, $\mathcal{T}_X \subseteq \mathcal{T}$. Conversely, take any $T' \in \mathcal{T}$, there exists some $\mathcal{U} \subseteq \mathcal{B}_Y$ such that

$$\begin{aligned} T' &= \bigcup_{B \in \mathcal{U}} (B \cap X) \\ &= \left(\bigcup_{B \in \mathcal{U}} B \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

Therefore, $\mathcal{T} \subseteq \mathcal{T}_X$ and so $\mathcal{T}_X = \mathcal{T}$. □

The following result shows that open sets in subspaces remain open in the superspace:

Proposition 1.3.3 ▸ Superspace Preserve Open Sets

Let $(Y, \mathcal{T}_Y)$ be a topological space. If $X \subseteq Y$ is open in Y and $U \subseteq X$ is open in X, then U is open in Y.

Proof. Let $\mathcal{T}_X$ be the subspace topology on X. Since U is open in X, we have $U \in \mathcal{T}_X$. By Definition 1.3.1, there exists some $V \in \mathcal{T}_Y$ such that $U = V \cap X$. However, $U \subseteq X$, so $U = V \in \mathcal{T}_Y$, which means that U is open in Y. □

We can do a similar manipulation with metric spaces and induce a metric on a subspace.

Definition 1.3.4 ▸ Subspace Metric

Let (X, d) be a metric space. The **subspace metric** of some $A \subseteq X$ is the restriction of d to A, denoted as

$$d_A(x, y) = d(x, y), \qquad \text{for all } x, y \in A.$$

Naturally, the following result is true:

Proposition 1.3.5 ▸ Subspace Metric Induces Subspace Topology

Let (X, d) be a metric space. The topology induced by the subspace metric d_A on some subspace $A \subseteq X$ is the subspace topology on A.

Proof. Let $\mathcal{T}_d$ and $\mathcal{T}_{d_A}$ be topologies induced by d on X with basis $\mathcal{B}_d$ and by d_A on A with basis $\mathcal{B}_{d_A}$ respectively. Let $\mathcal{T}_A$ be the subspace topology on A with basis $\mathcal{B}_A$. Take any $B_A \in \mathcal{B}_A$, then there exists $B_r(x) \in \mathcal{B}_d$ such that $B_A = B_r(x) \cap A$. For any $y \in B_A$, consider the ball

$$B_{r'}(y) := \{z \in A : d_A(z, y) < r'\} \in \mathcal{B}_{d_A}.$$

Note that $y \in B_{r'}(y)$, so by Proposition 1.1.7, we have $\mathcal{T}_A \subseteq \mathcal{T}_{d_A}$. Conversely, for any $B_r(x) \in \mathcal{B}_{d_A}$, there exists some $B_{r'}(x) \in \mathcal{B}_d$ such that $B_r(x) \subseteq B_{r'}(x)$. Notice that $B_r(x) \subseteq A$, so for any $y \in B_r(x)$, we have $y \in B_{r'}(x) \cap A \in \mathcal{B}_A$. Therefore, by Proposition 1.1.7, $\mathcal{T}_{d_A} \subseteq \mathcal{T}_A$ and so $\mathcal{T}_{d_A} = \mathcal{T}_A$. □

1.4 Closed Sets

Definition 1.4.1 ▸ Closed Set

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is **closed** if $X \backslash A \in \mathcal{T}$.

A set might be open and closed simultaneously. For example, every set X is both open and closed in itself.

Proposition 1.4.2 ▸ Arbitrary Intersection and Finite Union of Closed Sets Are Closed

Let $(X, \mathcal{T})$ be a topological space, then

1. *if $\mathcal{G} := \{G_\alpha : \alpha \in I\}$ is a family of closed set in X with respect to some index set I, then $\bigcap_{\alpha\in I} G_\alpha$ is closed in X;*
2. *if $G_1, G_2, \cdots, G_n$ are closed in X, then $\bigcup_{i=1}^n G_i$ is closed in X.*

Proof. Notice that

$$X\backslash \bigcap_{\alpha\in I} G_\alpha = \bigcup_{\alpha\in G} X\backslash G_\alpha.$$

Since $X\backslash G_\alpha$ is open in X for all $\alpha \in I$, this means that $X\backslash \bigcap_{\alpha\in I} G_\alpha$ is open in X, and so $\bigcap_{\alpha\in I} G_\alpha$ is closed in X. Notice also that

$$X\backslash \bigcup_{i=1}^n G_i = \bigcap_{i=1}^n X\backslash G_i.$$

By a similar argument $\bigcup_{i=1}^n G_i$ is closed in X. □

The following proposition justifies the fact that intersecting a closed set with a subspace produces a closed set in that subspace:

Proposition 1.4.3 ▸ Closed Sets in Subspace Topology

Let $Y \subseteq X$, then $A \subseteq Y$ is closed in Y if and only if there exists some closed set $G \subseteq X$ such that $A = G \cap Y$.

Proof. Suppose that A is closed in Y, then $Y\backslash A$ is open in Y. Therefore, there exists some open set $B \subseteq X$ such that $Y\backslash A = B \cap Y$. Take $G := X\backslash B$, then

$$G \cap Y = A.$$

Suppose conversely that there exists some closed set $G \subseteq X$ such that $A = G \cap Y$. Consider

$$Y\backslash (G \cap Y) = (X\backslash G) \cap Y.$$

Notice that $X\backslash G$ is open in X, so $Y \cap A$ is open in Y, i.e., A is closed in Y. □

The following result is analogous to Proposition 1.3.3:

Proposition 1.4.4 ▸ Superspace Preserves Closed Sets

If $Y \subseteq X$ is closed in X and $A \subseteq Y$ is closed in Y, then A is closed in X.

Proof. Consider $X\backslash A = X\backslash Y \cup Y\backslash A$. Since Y is closed in X, this means that $X\backslash Y$ is open in X. Note also that $Y\backslash A$ is open in Y. By Proposition 1.3.3, $Y\backslash A$ is open in X. Therefore, $X\backslash A$ is open in X and so A is closed in X. □

Closed sets help define the notion "interior of a set".

Definition 1.4.5 ▸ Interior, Closure and Boundary

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The **interior** of A is

$$\mathring{A} := \bigcup_{\substack{U \in \mathcal{T} \\ U \subseteq A}} A \cap U.$$

The **closure** of A is

$$\overline{A} := \bigcap_{\substack{X\backslash G \in \mathcal{T} \\ A \subseteq G}} G.$$

The **boundary** of A is

$$\partial A = \overline{A} \backslash \mathring{A}.$$

We interpret the above definition as follows: $\mathring{A}$ is the union of all open sets contained by A. Moreover, $\overline{A}$ is the smallest closed set in X which contains A. To see this, let $C \subseteq X$ be any closed set in X containing A. Take any $a \in \overline{A}$, then since a is contained by all closed sets containing A, it is clear that $a \in C$, which implies that $\overline{A} \subseteq C$.

Remark.

1. $\mathring{A} \subseteq A \subseteq \overline{A}$.
2. $\mathring{A} = A$ if and only if A is open in X.
3. $\overline{A} = A$ if and only if A is closed in X.

We would like to discuss the properties of closure. The following definition is useful:

Definition 1.4.6 ▸ Limit Point

Let $(X, \mathcal{T})$ be a topological space. For any $A \subseteq X$, a point $x \in X$ is a **limit point** of A if for every open set $U \subseteq X$ containing x,

$$(A \backslash \{x\}) \cap U \neq \varnothing.$$

Now, we propose two properties for the closure:

Proposition 1.4.7 ▸ Properties of Closure

Let $(X, \mathcal{T})$ be a topological space. For any $A \subseteq X$,

1. *$x \in \overline{A}$ if and only if for any open set $U \subseteq X$ containing x, $U \cap A \neq \varnothing$;*
2. *if A' is the set of limit points of A, then $\overline{A} = A \cup A'$.*

Proof. We will prove the first statement by considering the contrapositive, i.e., we prove that $x \in X \backslash \overline{A}$ if and only if there exists some open set $U \subseteq X$ containing x such that $U \cap A = \varnothing$. The "if" direction is trivial because $X \backslash \overline{A}$ is open in X such that $\left(X \backslash \overline{A}\right) \cap A = \varnothing$. Take $U \subseteq X$ to be an open set in X with $x \in U$ and $U \cap A = \varnothing$. This means that $x \notin A \subseteq \overline{A}$. Therefore, $x \in X \backslash \overline{A}$.

Take any $a \in A'$, then for every open set $U \subseteq X$ with $a \in U$, we have

$$U \cap A \supseteq U \cap (A \backslash \{a\}) \neq \varnothing.$$

Therefore, $A \cup A' \subseteq \overline{A}$. Take any $x \in \overline{A}$, we shall prove that if $x \notin A'$, then $x \in A$. Since x is not a limit point of A, there exists some open set $V \subseteq X$ containing x such that $(A \backslash \{x\}) \cap V = \varnothing$. However, $x \in \overline{A}$ implies that $V \cap A \neq \varnothing$, so $x \in A$. □

The notion of limit points also leads to the definition of convergence. Before that, we shall define the notion of *neighbourhood*.

Definition 1.4.8 ▸ Neighbourhood

Let $(X, \mathcal{T})$ be a topological space. An open set $U \subseteq X$ is called a **neighbourhood** of some $x \in X$ if $x \in U$.

Intuitively, we think of the statement $x_i \to x$ as the fact that no matter how small a neighbourhood we choose for x, there is always a consecutive infinite subsequence of the x_i's which falls in this neighbourhood.

Definition 1.4.9 ▸ Convergence

A sequence $\{x_i\}_{i=1}^{\infty}$ of points in a topological space $(X, \mathcal{T})$ **converges** to $x \in X$ if for any neighbourhood $U \subseteq X$ containing x, there exists some $N \in \mathbb{N}^+$ such that $x_k \in U$ for all $k > N$, denoted as $x_i \to x$. x is said to be the **limit** of $\{x_i\}_{i=1}^{\infty}$.

It is important to distinguish between limit and limit points. For example, consider the constant sequence $\{1\}_{i=1}^{\infty}$. Clearly, $x_i \to 1$ but one may check that 1 is not a limit point for this sequence.

In a metric space, we can make use of the metric to describe convergence in a more quantitative way.

Theorem 1.4.10 ▸ Convergence in Metric Spaces

Let (X, d) be a metric space. A sequence $\{x_i\}_{i=1}^{\infty}$ in X converges to x if and only if for every $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that $d(x_i, x) < \epsilon$ for all $i > N$.

Proof. Suppose that $x_i \to x$ as $i \to \infty$. For all $\epsilon > 0$, take the open ball $B_\epsilon(x) \subseteq X$. Clearly, $B_\epsilon(x)$ is a neighbourhood of x. By Definition 1.4.9, there exists some $N \in \mathbb{N}^+$ such that $x_i \in B_\epsilon(x)$ for all $i > N$, i.e., $d(x_i, x) < \epsilon$ for all $i > N$. Conversely, suppose that for every $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that $d(x_i, x) < \epsilon$ for all $i > N$. Let $U \subseteq X$ be any neighbourhood containing x. Note that U is open in X, so by Theorem 1.1.4, there exists some open ball $B_r(x) \subseteq U$ such that $x \in B_r(x)$. Therefore, there exists some $M \in \mathbb{N}^+$ such that $d(x_i, x) < r$, i.e., $x_i \in B_r(x) \subseteq U$, for all $i > M$. Therefore, $x_i \to x$. □

1.5 Continuity

Definition 1.5.1 ▸ Continuous Map

Let X and Y be topological spaces. A map $f : X \to Y$ is **continuous** if for any open set $U \subseteq Y$, the pre-image $f^{-1}(U)$ is open in X.

Suppose $\mathcal{T}_X$ and $\mathcal{T}_Y$ are topologies on X and Y respectively. The above definition basically says that for all $U \in \mathcal{T}_Y$, we have $f^{-1}(U) \in \mathcal{T}_X$. The following proposition gives an equivalent definition for continuity in terms of sub-bases.

Proposition 1.5.2 ▸ Equivalent Definition of Continuity

If $\mathcal{S}$ is a sub-basis for a topology on some set Y, then for any topological space X, a map $f : X \to Y$ is continuous if and only if $f^{-1}(S)$ is open in X for any $S \in \mathcal{S}$.

Proof. Suppose that f is continuous. Note that any $S \in \mathcal{S}$ is open in Y, so by Definition 1.5.1, $f^{-1}(S)$ is open in X. Suppose conversely that $f^{-1}(S)$ is open in X for any $S \in \mathcal{S}$. Take any open set $U \subseteq Y$. By Propositions 1.3.2 and 1.1.4, there exists finite subsets $\mathcal{U}_i \subseteq \mathcal{P}(\mathcal{S})$ where $i \in I$ for some index set I such that

$$U = \bigcup_{i \in I} \left(\bigcap_{S \in \mathcal{U}_i} S \right).$$

Therefore,

$$f^{-1}(U) = \bigcup_{i \in I} \left(\bigcap_{S \in \mathcal{U}_i} f^{-1}(S) \right),$$

which is clearly open in X. Therefore, f is continuous. □

A trivial example for continuous maps is the *constant map* $f : X \to Y$ such that $f(x) = y_0$ for some fixed $y_0 \in Y$. This is simply because

$$f^{-1}(U) = \begin{cases} X & \text{if } y_0 \in U \\ \varnothing & \text{otherwise} \end{cases}.$$

The following result should be very intuitive:

Proposition 1.5.3 ▸ Composition Preserves Continuity

Let X, Y, Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then $g \circ f$ is continuous.

It is also clear that for any topological space X, the *inclusion map* $f : A \to X$ for any $A \subseteq X$ such that $f(a) = a$ is continuous. Analogously, if $f : X \to Y$ is continuous, then the restriction $f|_A : A \to Y$ for any subspace $A \subseteq X$ is also continuous.

Proposition 1.5.4 ▸ Properties of Continuous Maps

Let X and Y be topological spaces. For any map $f : X \to Y$, the followings are equivalent:

1. *f is continuous;*
2. *for all $A \subseteq X$, $f\left(\overline{A}\right) \subseteq \overline{f(A)}$;*
3. *for any closed set $B \subseteq Y$, $f^{-1}(B)$ is closed in X;*
4. *for any $x \in X$ and any open set $V \subseteq Y$ with $f(x) \in V$, there exists an open set $U \subseteq X$ such that $x \in U$ and $f(U) \subseteq V$.*

Proof. Suppose that f is continuous. Note that $f(A) \subseteq \overline{f(A)}$, so $A \subseteq f^{-1}\left(\overline{f(A)}\right)$. Since $\overline{f(A)}$ is closed in Y, so by Definition 1.5.1,

$$f^{-1}\left(Y \backslash \overline{f(A)}\right) = X \backslash f^{-1}\left(\overline{f(A)}\right)$$

is open in X. Therefore, $f^{-1}\left(\overline{f(A)}\right)$ is closed in X. By Proposition 1.4.7, $\overline{A} = A \cup A'$

where A' is the set of limit points of A. We claim that $A' \subseteq f^{-1}\left(\overline{f(A)}\right)$. Suppose on contrary that there exists some $a \in A' \backslash f^{-1}\left(\overline{f(A)}\right)$. Since $X \backslash f^{-1}\left(\overline{f(A)}\right)$ is open, by Definition 1.4.6,

$$(A \backslash \{a\}) \cap \left(X \backslash f^{-1}\left(\overline{f(A)}\right)\right) \neq \varnothing,$$

which is a contradiction because $A \backslash \{a\} \subseteq f^{-1}\left(\overline{f(A)}\right)$. Therefore, $\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$, which means that $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.

Suppose that $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for any $A \subseteq X$. For any closed set $B \subseteq Y$, we have $B = \overline{B}$. Notice that

$$f\left(\overline{f^{-1}(B)}\right) \subseteq \overline{f(f^{-1}(B))} = B = f\left(f^{-1}(B)\right),$$

so $\overline{f^{-1}(B)} \subseteq f^{-1}(B)$. This implies that $\overline{f^{-1}(B)} \subseteq f^{-1}(B)$, and so $f^{-1}(B)$ is closed in X.

Suppose that $f^{-1}(B)$ is closed in X for any closed set $b \subseteq Y$. Take any $x \in X$ and any open set $V \subseteq Y$ with $f(x) \in V$. Since $Y \backslash V$ is closed in Y, $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is closed in X. Therefore, $f^{-1}(V)$ is open in X. It is clear that $x \in f^{-1}(V)$ and $f\left(f^{-1}(V)\right) \subseteq V$. $\square$

Next, we introduce a lemma which specifies a methodology to construct a continuous map from two different continuous maps.

Lemma 1.5.5 ▸ Pasting Lemma

Let X and Y be topological spaces such that $X = A \cup B$ for some closed sets A and B. If $f : A \to Y$ and $g : B \to Y$ are continuous and $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}.$$

Proof. Let $U \subseteq Y$ be any open set. Then it is clear that $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$. Since f and g are continuous, both $f^{-1}(U)$ and $g^{-1}(U)$ are open in X, so it follows that $h^{-1}(U)$ is open in X. Therefore, h is continuous. $\square$

Observe that the continuity of a function $f : X \to Y$ actually depends on our choice of topologies on X and Y. On the other hand, this also means that every function f could induce a topology on X such that it is continuous.

Definition 1.5.6 ▶ Pull-Back Topology

Let $\mathcal{T}_Y$ be a topology on Y and let $f : X \to Y$. The **pull-back topology** on X is defined as

$$\mathcal{T}_X \coloneqq \left\{f^{-1}(U) : U \in \mathcal{T}_Y\right\}.$$

Note that the pull-back topology is the coarsest topology on X such that f is a continuous map. To verify this, let $\mathcal{T}$ be any topology on X such that f is continuous. Take any $T \in \mathcal{T}_X$, then there exists some $U \in \mathcal{T}_Y$ such that $T = f^{-1}(U)$. However, this means that $T \in \mathcal{T}$ since f is continuous with respect to $\mathcal{T}$. This shows that $\mathcal{T}_X \subseteq \mathcal{T}$.

Definition 1.5.7 ▶ Uniform Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is **uniformly continuous** on X if for all $\epsilon > 0$, there exists some $\delta > 0$ such that $d_Y\big(f(x), f(y)\big) < \epsilon$ whenever $d_X(x, y) < \delta$.

Essentially, uniform continuity describes a phenomenon where the choice of δ is irrelevant to the point in the function's domain.

We wish to use the following proposition to characterise all uniformly continuous functions:

Proposition 1.5.8 ▶ Uniform Continuity Characterisation

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is uniformly continuous if and only if for any sequences $\{x_i\}_i^\infty$ and $\{y_i\}_i^\infty$ in X such that $\lim_{i\to\infty} d_X(x_i, y_i) = 0$, we have $\lim_{i\to\infty} d_Y\big(f(x_i), f(y_i)\big) = 0$.

Proof. It suffices to prove the "if" direction only because the other direction is trivial from Definition 1.5.7. We shall consider the contrapositive statement. Suppose that there exist sequences $\{x_i\}_i^\infty$ and $\{y_i\}_i^\infty$ with $\lim_{i\to\infty} d_X(x_i, y_i) = 0$ such that $\lim_{i\to\infty} d_Y\big(f(x_i), f(y_i)\big) \neq 0$, then for all $\delta > 0$, there exists some $N \in \mathbb{N}^+$ such that $d_X(x_i, y_i) > 0$ for all $i > N$. However, notice that there exists some $\epsilon > 0$ such that for all $M \in \mathbb{N}^+$, there exists some $m > M$ with $d_Y\big(f(x_m), f(y_m)\big) \geq \epsilon$. This means that for any $\epsilon > 0$, we can find some k such that for all $\delta > 0$, we have $d_X(x_k, y_k) < \delta$ but $d_Y\big(f(x_k), f(y_k)\big) \geq \epsilon$. By Definition 1.5.7, this implies that f is not uniformly continuous. $\square$

Definition 1.5.9 ▸ Point-wise and Uniform Convergence

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of maps from X to a metric space (Y, d). We say that $\{f_n\}_{n=1}^{\infty}$ **converges point-wisely** to $f : X \to Y$ if for any $x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$, and that $\{f_n\}_{n=1}^{\infty}$ **converges uniformly** to $f : X \to Y$ if for any $\epsilon > 0$, there exists some $N \in \mathbb{N}^+$ such that for all $n \geq N$ and any $x \in X$, $d\big(f_n(x), f(x)\big) < \epsilon$.

Proposition 1.5.10 ▸ Uniform Convergence Implies Continuity

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of maps from a metric space (X, d_X) to a metric space (Y, d_Y). If $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f : X \to Y$, then f is continuous.

1.6 Product Space

Definition 1.6.1 ▸ Projection

Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a sequence of non-empty sets, consider the product space $\prod_{\alpha\in\Lambda} X_\alpha$. The **projection** on the β-th factor is

$$\pi_{X_\beta} : \prod_{\alpha\in\Lambda} X_\alpha \to X_\beta$$

such that $\pi_{X_\beta}(\boldsymbol{x}) = x_\beta$.

Note that for any $\beta \in \Lambda$ and any $U \subseteq X_\beta$, we have

$$\pi_{X_\beta}^{-1}(U) = \left\{\boldsymbol{x} \in \prod_{\alpha\in\Lambda} X_\alpha : x_\beta \in U\right\}.$$

Definition 1.6.2 ▸ Product Topology

Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\Lambda}$ be a sequence of non-empty topological spaces. The **product topology** is the topology generated by the sub-basis

$$\mathcal{S} := \left\{\pi_{X_\alpha}^{-1}(U_\alpha) : \alpha \in \Lambda, U_\alpha \in \mathcal{T}_\alpha\right\}.$$

Definition 1.6.3 ▶ Box Topology

Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\Lambda}$ be a sequence of non-empty topological spaces. The **box topology** is the topology generated by the sub-basis

$$\mathcal{B} := \left\{\prod_{\alpha\in\Lambda} U_\alpha : U_\alpha \in \mathcal{T}_\alpha\right\}.$$

Proposition 1.6.4 ▶ Product Topology Guarantees Continuous Projection

Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\Lambda}$ be a sequence of non-empty topological spaces. The product topology on $\prod_{\alpha\in\Lambda} X_\alpha$ is the coarsest topology such that π_{X_α} is continuous for all $\alpha \in \Lambda$.

Proposition 1.6.5 ▶ Continuous of Functions over Product Spaces

Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\Lambda}$ be a sequence of non-empty topological spaces and let Y be a topological space. For any $\alpha \in \Lambda$, define $f_\alpha : Y \to X_\alpha$, then $f : Y \to \prod_{\alpha\in\Lambda} X_\alpha$ defined by

$$f(y) = (f_\alpha(y))_{\alpha\in\Lambda}$$

is continuous if and only if f_α is continuous for all $\alpha \in \Lambda$.

Let $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ such that $f(x) = (x, x, \cdots)$. We claim that f is not continuous with respect to the box topology.

Corollary 1.6.6

Let X be a topological space and let $f, g : X \to \mathbb{R}$ be continuous functions, then the functions $f + g$, $f - g$ and fg are continuous. If $0 \notin g(X)$, then $\frac{f}{g}$ is continuous.

Proposition 1.6.7

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $A \subseteq X$, $B \subseteq Y$ and $\mathcal{T}_{A\times B}$ be the subspace topology on $A \times B$ induced by $\mathcal{T}_X \times \mathcal{T}_Y$, then $\mathcal{T}_{A\times B} = \mathcal{T}_A \times \mathcal{T}_B$.

Proposition 1.6.8

Let $n \in \mathbb{Z}^+$ and $n = \sum_{i=1}^{k} m_i$ with $m_i \in \mathbb{Z}^+$ for $i = 1, 2, \cdots, k$, then

$$\prod_{i=1}^{k} \mathbb{R}^{m_i} = \mathbb{R}^n.$$

Definition 1.6.9 ▶ L^p-Metric

Let $(X_1, d_{X_1}), (X_2, d_{X_2}), \cdots, (X_n, d_{X_n})$ be metric spaces, we define

$$d_1((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)) := \sum_{i=1}^{n} d_{X_i}(x_i, y_i)$$
$$d_\infty((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)) := \max_{1\le i\le n} d_{X_i}(x_i, y_i).$$

Proposition 1.6.10 ▶ Basis for Product Topology

Let $\{(X_i, \mathcal{T}_i)\}_{i=1}^n$ be a sequence of topological spaces and let $\mathcal{B}_i$ be the basis for $\mathcal{T}_i$ for all $i = 1, 2, \cdots, n$, then $\prod_{i=1}^n \mathcal{B}_i$ is a basis for the product topology over $\prod_{i=1}^n X_i$.

Proposition 1.6.11 ▶ Metric for Product Topology

Let $\{(X_i, d_{X_i})\}_{i=1}^n$ be a sequence of topological spaces, then d_1 and d_∞ both induce the product topology over $\prod_{i=1}^n X_i$.

Proposition 1.6.12 ▶ ρ-Metric Induces the Same Topology

Let (X, d) be a metric space and define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x, y) := \frac{d(x, y)}{1 + d(x, y)},$$

then ρ is a metric on X and $\operatorname{diam}_\rho(X) < 1$. Let $\mathcal{T}_\rho$ and $\mathcal{T}_d$ be the topologies on X induced by ρ and d respectively, then $\mathcal{T}_\rho = \mathcal{T}_d$.

Proposition 1.6.13 ▶ Product Topology on Infinite Product

Let $\{(X, d_{X_i})\}_{i=1}^\infty$ be a sequence of metric spaces and define $\rho_i : X_i \times X_i \to \mathbb{R}$ by

$$\rho_i(x, y) := \frac{d_{X_i}(x, y)}{1 + d_{X_i}(x, y)},$$

then $d : \prod_{i=1}^\infty X_i \times \prod_{i=1}^\infty X_i \to \mathbb{R}$ defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) := \sup\left\{\frac{\rho_i(x_i, y_i)}{i} : i \in \mathbb{Z}^+\right\}$$

is a metric inducing the product topology on $\prod_{i=1}^\infty X_i$.