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The Real Numbers

1.1 Fields

Definition 1.1.1 ▸ Field

A set F with two binary operations, namely addition and multiplication, is called a **field** if it satisfies the following axioms:

1. $\forall a, b \in F, a +_F b = b +_F a$.
2. $\forall a, b, c \in F, (a +_F b) +_F c = a +_F (b +_F c)$.
3. $\exists 0_F \in F$ such that $\forall a \in F, 0_F +_F a = a +_F 0_F = a$.
4. $\forall a \in F, \exists a' \in F$ such that $a +_F a' = 0_F$.
5. $\forall a, b \in F, a \cdot_F b = b \cdot a$.
6. $\forall a, b, c \in F, (a \cdot_F b) \cdot c = a \cdot_F (b \cdot_F c)$.
7. $\forall a, b, c \in F, a \cdot_F (b +_F c) = a \cdot_F b +_F a \cdot_F b$ and $(a +_F b) \cdot_F c = a \cdot_F c +_F b \cdot_F c$.
8. $\exists 1_F \in F$ such that $\forall a \in F, 1_F \cdot_F a = a \cdot_F 1_F = a$.
9. $\forall a \in F, \exists a' \in F$ such that $a \cdot_F a' = 1_F$.

If we denote addition by "$+_F$" and multiplication by "$\cdot_F$" or "$\times_F$", then we can denote the field over F by $(F, +_F, \cdot_F)$ or $(F, +_F, \times_F)$.

Among the commonly used number sets, one may check that $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ are fields, while $\mathbb{N}$ and $\mathbb{Z}$ are not.

1.1.1 Ordered Fields

Definition 1.1.2 ▸ Total Order

A **total order** on a set X is a binary relation $\leq$ over X such that for all $a, b, c \in X$:

1. $a \leq a$ (reflexive).
2. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitive).
3. $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetric).
4. either $a \leq b$ or $b \leq a$ (strongly connected).

Definition 1.1.3 ▸ Strict Total Order

A **strict total order** on a set X is a binary relation $<$ over X such that for all $a, b, c \in X$:

1. $a \nless a$ (irreflexive).
2. $a < b$ implies $b < a$ (asymmetric).
3. $a < b$ and $b < c$ implies $a < c$ (transitive).
4. if $a \neq b$, then either $a < b$ or $b < a$ (connected).

It is easy to see that the real numbers form the ordered fields $(\mathbb{R}, +, \times, \leq)$ and $(\mathbb{R}, +, \times, <)$. Note that this means $\mathbb{R}$ satisfies trichotomy. If we choose any $x \in \mathbb{R}$, then exactly one of $x = 0$, $x > 0$ and $x < 0$ is true. Therefore, we can define that if $x \in \mathbb{R}$ and $x > 0$, then x is said to be positive. This leads to the following axiomatic results:

1. If a and b are both positive, then $a + b$ is positive;
2. If a and b are both positive, then ab is positive;
3. For any $a \in \mathbb{R}$, either $a = 0$, a is positive, or $-a$ is positive.

Note that $a < b$ if and only if $b - a$ is positive. So the trichotomy of $\mathbb{R}$ guarantees that for any $a, b \in \mathbb{R}$, either $a = b$, $a < b$ or $b < a$ (i.e., $a > b$).

1.2 Properties of $\mathbb{R}$

We can derive a few obvious minor results based on the field properties of $\mathbb{R}$:

1. If $a, b \in \mathbb{R}$, then $-ab + ab = 0$;
2. For all $a \in \mathbb{R}$ with $a \neq 0$, $a^2 > 0$;
3. If $a \in \mathbb{R}$ is such that $0 \leq a < \epsilon$ for all $\epsilon \in \mathbb{R}^+$, then $a = 0$;
4. If $a < b$, then $a + c < b + c$ for all $c \in \mathbb{R}$.
5. If $a < b$, then $ac < bc$ for all $c \in \mathbb{R}^+$ and $ac > bc$ for all $c \in \mathbb{R}^-$.
6. For all $a \in \mathbb{R}$, $a^2 > 0$.

We may consider the following interesting proposition:

Proposition 1.2.1

If $a \in \mathbb{R}$ is such that $0 \leq a < \epsilon$ for all $\epsilon \in \mathbb{R}^+$, then $a = 0$.

Proof. Suppose on contrary that $a > 0$, then we can take $\epsilon_0 = \frac{a}{2}$. Note that $\epsilon_0 \in \mathbb{R}^+$ but $\epsilon_0 < a$, which is a contradiction. So $a = 0$. □

The above essentially asserts that **a non-negative real number is strictly less than any positive real number if and only if it is** 0.

The properties of $\mathbb{R}$ also enables us to manipulate inequalities based on the following trivial results:

1. If $ab > 0$, then a and b are either both positive or both negative;
2. If $ab < 0$, then exactly one of them is positive and exactly one of them is negative.

We shall introduce a few well-known inequalities.

Theorem 1.2.2 ▶ Bernoulli's Inequality

If $x > -1$, then $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

Proof. The case where $n = 0$ is trivial.

Suppose that $(1+x)^k \geq 1 + kx$ for some $k \in \mathbb{N}$, consider

$$\begin{aligned}(1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) \\ &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x.\end{aligned}$$

Therefore, $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$. □

Theorem 1.2.3 ▶ AM-GM-HM Inequality

Let $n \in \mathbb{N}^+$ and let $a_1, a_2, \cdots, a_n$ be positive real numbers, then

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \leq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} a_i}{n}.$$

1.2.1 Absolute Value

Given any real number x, intuitively we sense that x possesses a certain "distance" from 0. This distance can be formalised as follows:

Definition 1.2.4 ▸ Absolute Value

Let $x \in \mathbb{R}$, the **absolute value** of x is defined as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}.$$

We have some trivial properties about the absolute value:

1. For all $a, b \in \mathbb{R}$, $|ab| = |a||b|$;
2. For all $a \in \mathbb{R}$, $|a|^2 = a^2$;
3. If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$ for all $a \in \mathbb{R}$;
4. For all $a \in \mathbb{R}$, $-|a| \leq a \leq |a|$.

Using these basic properties, we can prove the following results:

Theorem 1.2.5 ▸ Triangle Inequality

For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

Corollary 1.2.6 ▸ Extended Triangle Inequality

For all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$ *and* $|a - b| \leq |a| + |b|$.

Corollary 1.2.7 ▸ Generalised Triangle Inequality

For all $a_1, a_2, \cdots, a_n \in \mathbb{R}$,

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|.$$

Analogously, if $|x|$ represents the "distance" between x and 0, then by a simple translation we can see that $|x - a|$ represents the "distance" between x and a. Thus, we can have the following definition:

Definition 1.2.8 ▸ Neighbourhood

Let $a \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^+$. The **$\epsilon$-neighbourhood** of a is defined to be the set

$$V_\epsilon(a) := \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

Note that $x \in V_\epsilon(a)$ if and only if $-\epsilon < x - a < \epsilon$ or $a - \epsilon < x < a + \epsilon$. Which leads to the following interesting result:

Proposition 1.2.9

For any $a \in \mathbb{R}$, if $x \in V_\epsilon(a)$ for all $\epsilon \in \mathbb{R}^+$, then $x = a$.

Proof. Note that this essentially means that $|x - a| < \epsilon$ for all $\epsilon \in \mathbb{R}^+$. By Proposition 1.2.1, we have $|x - a| = 0$ and therefore $x = a$. □

1.2.2 The Completeness Property of $\mathbb{R}$

Intuitively, there are no "gaps" among the real numbers, i.e., if you take any two real numbers, between them there is nothing else than other real numbers. Therefore, we say that $\mathbb{R}$ is *complete*. This is in contrast with $\mathbb{Q}$ where there are gaps in between any two rational numbers (because there always exists some irrational numbers in between).

In this section, we probe into how the completeness of $\mathbb{R}$ can be established, and how the real numbers themselves can be constructed. To do that, we first establish the notion of *boundedness*.

Definition 1.2.10 ▶ Boundedness

Let $S \subseteq \mathbb{R}$. We say that S is:

- **bounded above** if there exists some $u \in R$ (known as the **upper bound** of S) such that $u \geq s$ for all $s \in S$;
- **bounded below** if there exists some $v \in R$ (known as the **lower bound** of S) such that $v \leq s$ for all $s \in S$;
- **bounded** if S has both an upper bound and a lower bound;
- **unbounded** either if S has no upper bound or if S has no lower bound;

Remark. Note that S is bounded if and only if there is some $M \geq 0$ such that $|s| \leq M$ for all $s \in S$.

2

Sequences and Series

2.1 Sequences

Definition 2.1.1 ▸ Sequence

A **sequence** in $\mathbb{R}$ is a real-valued function $X : \mathbb{N}^+ \to \mathbb{R}$. Where $X(n)$ is called the n-th **term** of the sequence.

Definition 2.1.2 ▸ Convergence of Sequences

A sequence (x_n) in $\mathbb{R}$ is said to be **convergent** to x if for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that whenever $n > N$, $|x_n - x| < \epsilon$. x is known as the **limit** of (x_n), denoted as

$$\lim_{n\to\infty} x_n = x.$$

2.1.1 Subsequences

Definition 2.1.3 ▸ Peak Point

Let (x_n) be a sequence in $\mathbb{R}$, x_m is called a **peak** if for all $n \in \mathbb{N}$ with $n > m$, $x_m \geq x_n$.

Theorem 2.1.4 ▸ Existence of Monotone Subsequences

Every infinite sequence has an infinite monotone subsequence.

Proof. Let (x_n) be any sequence in $\mathbb{R}$. We consider the following cases:

Case 1. (x_n) has infinitely many peak points, so there exists infinitely many $m_1, m_2, \cdots \in \mathbb{N}$ such that $m_j > m_i$ whenever $j > i$. Therefore, the subsequence (x_{m_n}) is a monotone decreasing sequence.

Case 2. (x_n) has finitely many peak points, so there exists $m_1, m_2, \cdots, m_k \in \mathbb{N}$. □

Theorem 2.1.5 ▸ Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Definition 2.1.6 ▸ Cauchy Sequence

A sequence (x_n) is said to be a **Cauchy sequence** if for every $\epsilon > 0$, there exists some $H \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ with $n, m \geq H$, $|x_n - x_m| < \epsilon$.

Theorem 2.1.7 ▸ Cauchy Convergence Criterion

A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.

Proof. Let (x_n) be a sequence in $\mathbb{R}$. Suppose that (x_n) converges to x, then for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that whenever $n > N$, $|x_n - x| < \frac{\epsilon}{2}$. Therefore, for all $m, n > N$, we have

$$\begin{aligned} |x_m - x_n| &= |x_m - x - x_n + x| \\ &\leq |x_m - x| + |x_n - x| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

and so (x_n) is a Cauchy sequence.

Suppose conversely that (x_n) is a Cauchy sequence on $\mathbb{R}$. We consider the following lemma:

Lemma 2.1.8 ▸ Boundedness of Cauchy Sequences

A Cauchy sequence in $\mathbb{R}$ is bounded.

Proof. Let (x_n) be a Cauchy sequence, then by Definition 2.1.6 there is some $H \in \mathbb{N}$ such that for all natural numbers $n \geq H$, $|x_n - x_H| < 1$. By Corollary 1.2.6, we have

$$||x_n| - |x_H|| \leq |x_n - x_H| < 1,$$

and so $|x_n| < |x_H| + 1$. Take

$$m = \max\{|x_1|, |x_2|, \cdots, |x_H|, |x_H| + 1\},$$

then $|x_n| < m$ for all $n \in \mathbb{N}^+$. □

Therefore, by Theorem 2.1.5 there exists a subsequence (x_{m_n}) which converges to some $x \in \mathbb{R}$. Thus there exists some $M \in \mathbb{N}$ such that whenever $m_n > M$, $|x_{m_n} - x| < \frac{\epsilon}{2}$ for all $\epsilon > 0$. By Definition 2.1.6, there exists some $N \in \mathbb{N}$ such that $|x_n - x_{m_n}| < \frac{\epsilon}{2}$ for all $\epsilon > 0$ and for all $n, m_n > N$. Take $K = \max\{M, N\}$, then whenever $n > K$, there is some $m_n > K$ such that

$$\begin{aligned}
|x_n - x| &= |x_n - x_{m_n} + x_{m_n} - x| \\
&\leq |x_n - x_{m_n}| + |x_{m_n} - x| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} x_n = x$. $\square$