Contents

1 Sets and Classes

1.1 Classes

Russell's Paradox states the following:

Russell's Paradox

Let X be the set of all sets which do not contain themselves, i.e.,

$$X = \{S : S \notin S\}.$$

Now consider X. If $X \in X$, it means that X contains itself and should not be a member of X, i.e., $X \in X \implies X \notin X$. If $X \notin X$, it means that X does not contain itself and therefore should be a member of X, i.e. $X \notin X \implies X \in X$. Hence, we have a paradox and such a set X does not exist.

However, in some cases it is still useful to consider the "set" of all sets for practical reasons. Therefore, we introduce the notion of a *class* to avoid Russell's Paradox.

Definition 1.1.1 ▸ Class

Let ϕ be some formula and $\boldsymbol{u}$ be a vector, the collection

$$\mathbb{C} = \{X : \phi(X, \boldsymbol{u})\}$$

is called a **class** of all sets satisfying $\phi(X, \boldsymbol{u})$, where $\mathbb{C}$ is said to be **definable** from $\boldsymbol{u}$. Equivalently, we say that

$$X \in \mathbb{C} \iff \phi(X, \boldsymbol{u}).$$

In particular, if $\mathbb{C} = \{X : \phi(X)\}$, i.e., ϕ only has one free variable, then we say that $\mathbb{C}$ is **definable**.

Remark. It is easy to see that every set X is a class given by $\{x : x \in X\}$.

Intuitively, two classes are equal if they contain exactly the same members. We are able to give the following rigorous version of the notion of equality:

Definition 1.1.2 ▸ Equality between Classes

Let $\mathbb{C} = \{X : \phi(X, \boldsymbol{u})\}$ and $\mathbb{D} = \{X : \psi(X, \boldsymbol{v})\}$, we say that $\mathbb{C} = \mathbb{D}$ if for all X,

$$\phi(X, \boldsymbol{u}) \iff \psi(X, \boldsymbol{u}).$$

There are clearly two types of classes — the ones which are also sets and the ones which are not. Formally, this is put as follows:

Definition 1.1.3 ▸ Proper Class

A class $\mathbb{C}$ is said to be a **proper class** if $\mathbb{C} \neq X$ for all sets X.

Like sets, we can define subclasses:

Definition 1.1.4 ▸ Subclass

Let $\mathbb{A}$ and $\mathbb{B}$ be classes. We say that $\mathbb{A}$ is a **subclass** of $\mathbb{B}$ if every member of $\mathbb{A}$ is also a member of $\mathbb{B}$, i.e.,

$$\mathbb{A} \subseteq \mathbb{B} \iff (X \in \mathbb{A} \implies X \in \mathbb{B}).$$

We shall also define the operations applicable to classes:

Definition 1.1.5 ▸ Intersection, Union and Difference

Let $\mathbb{A}$ and $\mathbb{B}$ be classes. The **intersection**, **union** and **difference** between $\mathbb{A}$ and $\mathbb{B}$ are given by

$$\begin{aligned} \mathbb{A} \cap \mathbb{B} &:= \{X : X \in \mathbb{A} \wedge X \in \mathbb{B}\}, \\ \mathbb{A} \cup \mathbb{B} &:= \{X : X \in \mathbb{A} \vee X \in \mathbb{B}\}, \\ \mathbb{A} - \mathbb{B} &:= \{X : X \in \mathbb{A} \wedge X \notin \mathbb{B}\} \end{aligned}$$

respectively.

Finally, we shall introduce the universal class:

Definition 1.1.6 ▸ Universal Class

The **universal class** is the class of all sets, denoted by

$$V := \{X : X = X\}.$$

Remark. It is easy to prove that the universal class is **unique**.

2

Axiomatic Set Theory

2.1 Axioms of Zermelo-Fraenkel (ZF)

In Naïve Set Theory, we define a set as "a collection of mathematical objects which satisfy certain definable properties". However, such a definition is problematic (e.g. it leads to the Russell's Paradox). Thus, instead of viewing a set as a clearly defined mathematical object, we can think a set as an object entirely defined by a set of axioms to which it complies. In this sense, we avoid paradoxes by making the notion of a set undefined but only specify rigorously the axioms a set must satisfy. The following sections discuss each of the axioms in ZF set theory.

2.2 Extensionality

Axiom 2.2.1 ▸ Extensionality

Let X and Y be sets, then $X = Y$ if for all u, $u \in X$ if and only if $u \in Y$.

An immediate result from Axiom 2.2.1 is that there exists a set X such that $X = X$, i.e. every set equals itself. Moreover, we can also prove the following:

Theorem 2.2.2 ▸ The Empty Set

The set which has no elements is unique.

Proof. Let X be a set with no elements. Note that this means that for all u, $u \notin X$.

Let Y be another set. Note that the statement $u \in X \implies u \in Y$ is vacuously true. Suppose that Y has no elements, then similarly for all u, the statement $u \in Y \implies u \in X$ is also vacuously true.

Therefore, for all u, we have proven that $u \in X$ if and only if $u \in Y$. By Axiom 2.2.1, this means that $X = Y$, i.e. the set with no elements is unique. $\square$

This set with no elements is known as the **empty set**, denoted by $\varnothing$.

2.3 Pairing

Axiom 2.3.1 ▸ Pairing

For all u and v, there exists a set X such that for all z, $z \in X$ if and only if $z = u$ or $z = v$.

Remark. Note that Axiom 2.3.1 essentially says that given any sets u and v, there exists a set whose elements are exactly u and v.

This allows us to formally define the notion of a *pair* as follows:

Definition 2.3.2 ▸ Pair

For all a, b, the **pair** $\{a, b\}$ is defined to be the set C such that for all x, $x \in C$ if and only if $x = a$ or $x = b$.

Remark. In particular, we can define the **singleton** $\{a\}$ to be the pair $\{a, a\}$.

Furthermore, given any a and b, we can prove by Extensionality that the the pair $\{a, b\}$ is unique:

Theorem 2.3.3 ▸ Uniqueness of Pairs

For all a, b, the pair $\{a, b\}$ is unique.

Proof. Let $C := \{a, b\}$ and $D := \{a, b\}$. Suppose $x \in C$, then $x = a$ or $x = b$, which means $x \in D$. Similarly, suppose $y \in D$, we can prove that $y \in C$. Therefore, for all x, we have $x \in C$ if and only if $x \in D$. By Axiom 2.2.1, this means that $C = D$, i.e., the pair $\{a, b\}$ is unique. □

We can further define the notion of an *ordered pair*:

Definition 2.3.4 ▸ Ordered Pair

For all a and b, the **ordered pair** (a, b) is defined to be the set $\{\{a\}, \{a, b\}\}$.

Again, one can use Extensionality to prove that such an ordered pair is always unique and that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. The notions of pair and ordered pair can be extended to ordered and un-ordered n-tuples, which will have similar properties as we have proven as above. Recursively, we can write the following definition:

Definition 2.3.5 ▶ Ordered n-tuple

The **n-tuple** is defined as

$$(a_1, a_2, \cdots, a_n) = ((a_1, a_2, \cdots, a_{n-1}), a_n).$$

By Extensionality, we can similarly prove that two ordered n-tuples $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ if and only if $a_i = b_i$ for $i = 1, 2, \cdots, n$.

2.4 Separtion

Axiom 2.4.1 ▶ Axiom Schema of Separation

If P is a property with parameter p, then for all X and p there exists a set

$$Y \coloneqq \{u \in X : P(u, p)\}.$$

The above axiom justifies our set-builder notation

$$\{x : \varphi(x, \boldsymbol{p})\},$$

where φ is some formula and $\boldsymbol{p}$ is an ordered n-tuple of parameters.

Alternatively, we can write Axiom Schema 2.4.1 in the following form:

Let $\mathbb{C} = \{u : \varphi(u, \boldsymbol{p})\}$ be a class, then for all sets X there exists a set Y such that $\mathbb{C} \cap X = Y$.

Consequently, the intersection and the difference between two sets is a set, which can be defined as

$$X \cap Y \coloneqq \{u \in X : u \in Y\} \quad \text{and} \quad X - Y \coloneqq \{u \in X : u \notin Y\}.$$

Suppose that there exists some set X such that $X = X$, we can use Separtion to define the empty set as

$$\varnothing \coloneqq \{u : u \neq u\}.$$

We shall define other notions related to Separation Axioms:

Definition 2.4.2 ▶ Disjoint

Two sets X and Y are called **disjoint** if $X \cap Y = \varnothing$.

Definition 2.4.3 ▸ Unary Intersection

Let $\mathbb{C}$ be a non-empty class of sets, we define the **unary intersection** of $\mathbb{C}$ to be

$$\bigcap \mathbb{C} := \{u : u \in X \text{ for all } X \in C\}.$$

Note that the unary intersection helps us define the intersection of two sets as

$$X \cap Y = \bigcap \{X, Y\}.$$

2.5 Union

Axiom 2.5.1 ▸ Axiom of Union

For all X, there exists a set $Y = \bigcup X$ whose elements are all the elements of all elements of X, i.e.

$$Y := \{u \in U : U \in X\}.$$

Remark. We often call $\bigcup X$ the **unary union** of X.

The unary union defines the union of two sets as

$$X \cup Y = \bigcup \{X, Y\}.$$

One can prove that union between sets is **associative**. In general, we can also see that

$$\{a_1, a_2, \cdots, a_n\} = \bigcup_{i=1}^{n} \{a_i\}.$$

In addition, we can also define the notion of *symmetric difference*:

Definition 2.5.2 ▸ Symmetric Difference

The **symmetric difference** between two sets X and Y is defined as

$$X \triangle Y := \{u : u \in X \cup Y, u \notin X \cap Y\} = (X - Y) \cup (Y - X).$$

2.6 Power Set

Axiom 2.6.1 ▸ Axiom of Power Set

For all X, there exists a set $Y = \mathcal{P}(X)$, known as the **power set** *of X, such that*

$$Y := \{U : U \subseteq X\}.$$

This allows us to define the notion of the *Cartesian product* (or simply the *product*) of two sets:

Definition 2.6.2 ▸ Cartesian Product

Let X and Y be sets. The **Cartesian product** of X and Y is defined as the set

$$X \times Y := \{(x, y) : x \in X, y \in Y\}.$$

Remark. Note that $X \times Y$ is a set because $X \times Y \subseteq \mathcal{P}(X \cup Y)$.

The above offers a new way to define n-tuples, as we can define Cartesian products of countably many sets recursively.

Definition 2.6.3 ▸ Cartesian Product of Countably Many Sets

et $n \in \mathbb{N}^+$ and let X be a set, we define

$$X^n := \prod_{i=1}^{n} X = \left(\prod_{i=1}^{n-1} X\right) \times X.$$

2.6.1 Relations

Colloquially, we may want to express the idea that a collection of n objects are related by some rules. Observe that such a *relation* between n objects can be precisely abstracted as an ordered n-tuple, which motivates the following definition:

Definition 2.6.4 ▸ Relation

An **n-ary relation** R is a set of n-tuples. We say that R is an n-ary relation on X if $R \subseteq X^n$. Conventionally, to say that $x_1, x_2, \cdots, x_n$ are related by the rules defined by R, we use the notation $R(x_1, x_2, \cdots, x_n)$. Note that this notation is equivalent to

$$(x_1, x_2, \cdots, x_n) \in R.$$

Remark. In the case where R is a binary relation, we can also use the notation xRy to express that $(x, y) \in R$.

If R is a binary relation, then we define the *domain* of R to be

$$\mathrm{dom}(R) = \{u : \exists v \text{s.t.} (u, v) \in R\},$$

and the *range* of R to be

$$\mathrm{ran}(R) = \{v : \exists u \text{s.t.} (u, v) \in R\}.$$

Note that

$$\mathrm{dom}(R) \subseteq \bigcup\left(\bigcup R\right) \quad \text{and} \quad \mathrm{ran}(R) \subseteq \bigcup\left(\bigcup R\right),$$

so the domain and range of a relation are sets. Additionally, we define the *field* of R to be the set

$$\mathrm{field}(R) = \mathrm{dom}(R) \cup \mathrm{ran}(R).$$

2.6.2 Functions

Given a binary relation R, we can see R as a **mapping** which corresponds each $u \in \mathrm{dom}(R)$ with some $v \in \mathrm{ran}(R)$. From this, we are able to derive the following definition for a *function*:

Definition 2.6.5 ▸ Function

Let X be a set. A binary relation f on X is a **function** if $(x, y) \in f$ and $(x, z) \in f$ implies that $y = z$, i.e. for all $x \in X$ there exists a unique y such that $(x, y) \in f$. This unique y is called the **value** of f at x. We may use the notations

$$y = f(x) \quad \text{or} \quad f : x \mapsto y$$

to express that $(x, y) \in f$.

Remark. If $\mathrm{dom}(f) = X^n$, we also say that f is an **n-nary function** on X.

We denote a function f from X to Y by

$$f : X \to Y,$$

where $\mathrm{dom}(f) = X$ and $\mathrm{ran}(f) \subseteq Y$. The set of all functions from X to Y is denoted as Y^X, which is a set because

$$Y^X \subseteq \mathcal{P}(X \times Y).$$

If $\operatorname{ran}(f) = Y$, we say that f is *onto* Y or that f is *surjective*. A function f is *one-to-one* or *injective* if

$$f(x) = f(y) \implies x = y.$$

Additionally, we may call the function $f : X^n \to X$ an *n-nary operation* on X.

We may also define new functions from some existing function(s).

Definition 2.6.6 ▸ Restriction

Let f be a function. The **restriction** of f to a set X is defined to be the function

$$f|_X \coloneqq \{(x, y) \in f : x \in X\}.$$

Definition 2.6.7 ▸ Extension

Let f, g be functions. g is called an **extension** of f if $f \subseteq g$, i.e.,

$$\operatorname{dom}(f) \subseteq \operatorname{dom}(g) \quad \text{and} \quad g(x) = f(x) \quad \text{for all } x \in \operatorname{dom}(f).$$

Definition 2.6.8 ▸ Composition

Let f and g be functions such that $\operatorname{ran}(g) \subseteq \operatorname{dom}(f)$. The **composition** of f and g is the function denoted by $f \circ g$ with $\operatorname{dom}(f \circ g) = \operatorname{dom}(g)$ such that

$$(f \circ g)(x) = f(g(x)) \quad \text{for all } x \in \operatorname{dom}(g).$$

Note that a function provides a mapping from one set to another set, and so we can define the notion of an *image*.

Definition 2.6.9 ▸ Image and Inverse Image

Let f be a function and X be a set. The **image** of X by f is the set

$$\{y : \exists x \in X \text{ s.t. } y = f(x)\},$$

denoted by $f[X]$. The **inverse image** of X by f is the set

$$\{x : f(x) \in X\},$$

denoted by $f^{-1}[X]$.

Remark. Trivially, if $X \cap \operatorname{dom}(f) = \varnothing$, then $f[X] = \varnothing$.

For injections, we can also define their *inverses*.

Definition 2.6.10 ▸ Inverse

Let f be an injective function, then we denote the **inverse** of f by f^{-1}, which is defined by

$$f^{-1}(x) = y \quad \text{if and only if } x = f(y).$$

The above definitions for functions can be applied similarly with respect to classes.

Axiom 2.6.11 ▸ Axiom of Infinity

There exists an infinite set.

Axiom 2.6.12 ▸ Axiom Schema of Replacement

If a class F is a function, then for all X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.

Axiom 2.6.13 ▸ Axiom of Regularity

For every non-empty set X, there exists some $Y \in X$ such that $Y \cap X = \varnothing$.

Remark. Axiom 2.6.13 is sometimes known as the **Axiom of Foundation**. A direct result from it is that for all sets X, there exists some $x \in X$ such that $x \not\subseteq X$.

Furthermore, we can use Axiom 2.6.13 to prove the following seemingly trivial result:

Theorem 2.6.14

There is no set A such that $A \in A$.

Proof. If $A = \varnothing$, it is immediate that $A \notin A$ by definition.

Suppose that there exists a non-empty set A such that $A \in A$. Note that $A \in \{A\}$, so

$$A \cap \{A\} = A.$$

However, by Axiom 2.6.13, since A is the only member of $\{A\}$, we have

$$A \cap \{A\} = \varnothing,$$

which is a contradiction. Therefore, there exists no set A such that $A \in A$. □

Additionally, we also introduce the Axiom of Choice:

Axiom 2.6.15 ▶ Axiom of Choice

For every X with $\varnothing \notin X$, there exists a ***choice function***

$$f : X \to \bigcup X$$

such that for all $S \in X$, we have $f(S) \in S$.

Remark. Essentially, the choice function maps every set which is a member of some family of sets to one and only one element in that set.