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# Graph Structures

## 1.1 Multigraphs

### 1.1.1 Multigraphs

Intuitively, we would describe a *graph* as a collection of nodes (or vertices) plus some lines (or edges) joining some nodes together. This can be rigorously defined as follows:

#### Definition 1.1.1 ► Multigraph

A **multigraph**  $G$  consists of a non-empty finite set of vertices denoted by  $V(G)$  and a finite set of edges denoted by  $E(G)$ .  $|V(G)|$  is known as the **order** of  $G$ , denoted by  $v(G)$  and  $|E(G)|$  is known as the **size** of  $G$ , denoted by  $e(G)$ .

In particular, if  $v(G) = m$  and  $e(G) = n$ , we say that  $G$  is an  **$(m, n)$ -graph**.

$G$  is said to be **trivial** if  $v(G) = 1$  and **non-trivial** otherwise.

Note that in a multigraph, by default there can be a plural number of edges between any two vertices. We would define the notion of *simple graph* as a special multigraph.

#### Definition 1.1.2 ► Simple Graph

A multigraph  $G$  is said to be **simple** if there is at most one edge between any two distinct vertices.

*Remark.* Note that if  $G$  is simple and undirected, then

$$E(G) \subseteq \{(v_j, v_i) : v_j, v_i \in V(G), j \geq i\}.$$

Notice that if  $(v_i, v_i)$  is an edge, it connects a vertex to itself. This is known as a *loop*.

In a graph, it is important to know the layout of the vertices and edges. For this purpose, we define the notion of *adjacency* in a multigraph.

**Definition 1.1.3 ▶ Adjacency and Neighbourhood**

Let  $v_i, v_j \in V(G)$ , we say that they are **adjacent** if  $v_i v_j \in E(G)$ . Alternatively, we say that  $v_i$  and  $v_j$  are **neighbours** to each other. The edge  $v_i v_j$  is said to be **incident** with  $v_i$  and  $v_j$ . Two edges  $e$  and  $f$  are said to be **adjacent** if there exists some  $v \in V(G)$  such that both  $e$  and  $f$  are incident with  $v$ .

The set of all neighbours to some  $v_i \in V(G)$  is called the **neighbourhood set** of  $v_i$ , denoted by  $N_G(v_i)$ . In particular, the set  $N_G[v_i] := N_G(v_i) \cup \{v_i\}$  is known as the **closed neighbourhood set** of  $v_i$ .

Alternatively, one may write  $v \sim u$  if  $v$  and  $u$  are adjacent vertices. We can further add on to the definition by discussing the size of the neighbourhood of a vertex.

**Definition 1.1.4 ▶ Degree**

The **degree** of  $v$ , denoted by  $d_G(v)$  is defined as the number of edges incident to  $v$ . If  $d_G(v)$  is even (repectively, odd), then we say that  $v$  is an even (respectively, odd) vertex. If  $d_G(v) = 0$ , we say that  $v$  is **isolated**; if  $d_G(v) = 1$ , we say that  $v$  is an **end** vertex.

In particular, we define

$$\Delta(G) = \max_{v \in G} d_G(v), \quad \delta(G) = \min_{v \in G} d_G(v).$$

In particular, we can denote the average degree of a graph  $G$  by

$$\bar{d}(G) = \frac{\sum_{v \in V(G)} d_G(v)}{v(G)}.$$

*Remark.* Note that a high maximal degree does not imply a high average degree. A classic counter example is a complete graph plus an isolated vertex.

With the notion of degree established, we can now define a *regular graph*.

**Definition 1.1.5 ▶ Regular Graph**

A **regular graph** is a graph in which every vertex has the same degree. In particular, if  $d_G(v) = k$  for all  $v \in V(G)$ ,  $G$  is known as a  **$k$ -regular graph**.

*Remark.* A graph  $G$  is regular if and only if  $\Delta(G) = \delta(G)$ .

Since a graph essentially consists of two sets, it is natural to consider the notion of graph

complementation. We now proceed to introducing the notion of *complement*.

### Definition 1.1.6 ► Complement

Let  $G$  be a graph of order  $n$ , the **complement** of  $G$ , denoted by  $\overline{G}$ , is the graph of order  $n$  where

$$V(\overline{G}) = V(G), \quad E(\overline{G}) = \{(u, v) : (u, v) \notin E(G)\}.$$

We discuss the following interesting question:

15 students went to a party. During the party some of them shook hands with each other. At the end of the party, the number of handshakes made by each student was recorded and it was reported that the sum was 39. Was this possible?

Note that if we represent each student as a vertex, then we can use  $V(G) = \{v_1, v_2, \dots, v_{15}\}$  to construct a graph, in which an edge  $v_i v_j \in E(G)$  if and only if students  $i$  and  $j$  shook hands. As such,  $d_G(v_i)$  is the number of persons student  $i$  shook hands with.

Note that if we sum up  $d_G(v_i)$  for all the vertices, every edge will be counted exactly twice! This means that

$$\sum_{i=1}^{15} d_G(v_i) = 39$$

is impossible since the left-hand side must be even. In fact, by the above reasoning, we see that in any graph, the sum of degrees of its vertices must be even.

### Lemma 1.1.7 ► Handshaking Lemma

If  $G$  is a graph of order  $n$  and size  $m$ , then

$$\sum_{i=1}^n d_G(v_i) = 2m.$$

*Remark.* It can be easily deduced from the above lemma that in any graph, the number of vertices with odd degrees must be even.

Relating to Definition 1.1.4, we see that the average degree of a graph  $G$  can be computed as

$$\bar{d}(G) = \frac{2e(G)}{v(G)}.$$

As an extension of Lemma 1.1.7, the minimum size of any graph is obviously 0, and the maximum size of a simple graph occurs when there is an edge between any two vertices.

**Definition 1.1.8 ▶ Empty and Complete Graphs**

Let  $G$  be a simple graph of order  $n$ .  $G$  is said to be an **empty graph** or **null graph**, denoted by  $0_n$ , if  $e(G) = 0$ , and a **complete graph**, denoted by  $K_n$  if for all  $u, v \in V(G)$ , we have  $(u, v) \in E(G)$ .

**1.1.2 Subgraphs**

Since a multigraph is just two sets, we can define a “subset” relation between graphs.

**Definition 1.1.9 ▶ Subgraph**

Let  $G, H$  be graphs, then  $H$  is a **subgraph** of  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . In particular, a subgraph  $H$  is a **proper subgraph** of  $G$  if  $V(H) \neq V(G)$  or  $E(H) \neq E(G)$ .

Note that we a graph can possibly be reproduced by connecting its vertices correctly, so a subgraph containing all vertices of a graph “spans” the original graph.

**Definition 1.1.10 ▶ Spanning Subgraph**

Let  $H$  be a subgraph of  $G$ .  $H$  is called a **spanning subgraph** of  $G$  if  $V(H) = V(G)$ .

Note that by definition, a spanning subgraph retains all vertices of the original graph. Therefore, a way to quickly generate a subgraph of a given graph  $G$  is to keep the vertex set and delete some edges from the edge set. We denote such a graph by  $H = G - F$  for some  $F \subset E(G)$ . Observe that

$$V(H) = V(G), \quad E(H) = E(G) - F.$$

On the other hand, by deleting some vertices together with edges incident to them, we can produce a subgraph from any given graph.

**Definition 1.1.11 ▶ Induced Subgraph**

Let  $H$  be a subgraph of  $G$ .  $H$  is called a **induced subgraph** of  $G$  if

$$E(H) = \{uv \in E(G) : u, v \in V(H)\}.$$

Let  $S \subseteq V(G)$ , the subgraph of  $G$  induced by  $S$  is denoted by  $[S]$ .

Alternatively, let  $F \subseteq E(G)$ . Define

$$V' := \bigcup \{\{u, v\} : uv \in F\},$$

then  $(V', F)$  is the subgraph of  $G$  induced by  $F$ , denoted by  $G[F]$ .

Intuitively, an induced subgraph of  $G$  consists of a selected subset of the vertices of  $G$  together with all edges in  $G$  connecting any vertices in this subset.

Using the idea of deletion, we can quickly generate an induced subgraph  $H$  of  $G$  as follows: first, set  $V(H) = V(G) - A$  for some  $A \subseteq V(G)$ , i.e., remove some vertices; then, we will remove all edges from  $E(G)$  which are incident to some vertices in  $A$ , i.e., set

$$E(H) = \{e \in E(G) : e \text{ is not incident to any } v \in A\}.$$

Then,  $H = (V(H), E(H))$  is an induced subgraph of  $G$ . In fact, we can denote  $H$  as

$$H = G - A = [V(G) - A].$$

This leads to the following proposition:

### Proposition 1.1.12

*Let  $G$  be a graph. If  $A \subseteq V(G)$ , then  $G - A = [V(G) - A]$ . Suppose  $H$  is a subgraph of  $G$ , then  $H$  is an induced subgraph of  $G$  if and only if  $H = G - (V(G) - V(H))$ .*

We would like to consider the relations between the properties of a graph to those of its subgraphs. In particular, it follows from intuition that if a graph has a high average degree, then we can naturally find a subgraph with a high minimal degree.

This seems trivial from intuition, as we can always keep deleting vertices with the minimal degree from a graph until we cannot find any vertex whose degree is less than  $\frac{1}{2}\bar{d}(G)$ , but there is more to that — for instance, how do we know if we would not end up deleting all vertices from the original graph? Thus, to prove our claim is essentially asking to prove the correctness of this greedy deletion, which we shall do in the following proposition.

### Proposition 1.1.13

*Every non-empty graph  $G$  has a subgraph  $H$  such that  $\delta(H) \geq \frac{1}{2}\bar{d}(G)$ .*

*Proof.* Define  $H_0 = G$  and  $H_{i+1} = H_i - v_i$  where  $v_i$  is a vertex in  $H_i$  whose degree is the smallest. Note that we can always find this  $v_i$  for any  $H_i$  with  $V(H_i) \neq \emptyset$

because  $V(H_i)$  is a finite well-ordered set. We will repeatedly perform the deletion until we obtain some  $H_k$  with  $d_{H_k}(v_k) \geq \frac{1}{2}\bar{d}(G)$ . We shall proceed to proving that this algorithm always terminates with  $V(H_k) \neq \emptyset$ .

Suppose on contrary that  $V(H_k) = \emptyset$ , then we have performed the deletion for  $v(G)$  times. Suppose we have deleted  $N$  edges in total, then clearly,

$$N < v(G)\frac{1}{2}\bar{d}(G) = v(G)\frac{e(G)}{v(G)} = e(G),$$

which means  $e(H_k) = e(G) - N > 0$ . However, this is impossible since  $v(H_k) = 0$ , which is a contradiction.  $\square$

### 1.1.3 Graph Isomorphism

Given two graphs  $G$  and  $H$ , are they the same graph? This seemingly innocent question proves to be extremely hard to answer. Two graphs can look drastically different but be structurally identical in reality. For example, we could shift around the vertices of a graph without changing any edge to alter the shape of the graph dramatically. Therefore, to compare the structures of graphs, we require some rigorous definition.

#### Definition 1.1.14 ► Graph Isomorphism

Two graphs  $G$  and  $H$  are said to be **isomorphic**, denoted by  $G \cong H$ , if there exists a bijection  $f : V(G) \rightarrow V(H)$  such that

$$uv \in E(G) \quad \text{if and only if} \quad f(u)f(v) \in E(H).$$

*Remark.* One may check that graph isomorphism is obviously an equivalence relation.

Such a bijection  $f$  is said to **preserve adjacency**, i.e., if  $u$  and  $v$  are neighbours in  $G$ , then their images are also neighbours in  $H$ . In particular, it is possible to map a graph to its own complement via an isomorphism.

#### Definition 1.1.15 ► Self-Complementary Graph

A graph  $G$  is said to be **self-complementary** if  $G \cong \bar{G}$ .

It turns out that a self-complementary graph satisfies some special properties.

**Proposition 1.1.16 ▶ Order of Self-Complementary Graphs**

If  $G$  is a self-complementary graph of order  $n$ , then  $n = 4k$  or  $n = 4k+1$  for some  $k \in \mathbb{Z}^+$ .

*Proof.* Since  $G$  is self-complementary,  $G \cong \overline{G}$  and so  $e(G) = e(\overline{G})$ . Note that

$$e(G) + e(\overline{G}) = \binom{n}{2} = \frac{n(n-1)}{2},$$

so we have  $e(G) = \frac{n(n-1)}{4}$ . Since  $e(G) \in \mathbb{Z}^+$ , either  $n$  or  $n-1$  is divisible by 4, so  $n = 4k$  or  $n = 4k + 1$  for some  $k \in \mathbb{Z}^+$ .  $\square$

It is very hard to determine whether two specific graphs are isomorphic, but there are some considerations in the general case. For example, some trivial conclusions include:

- Two graphs with different orders cannot be isomorphic.
- Two graphs with different sizes cannot be isomorphic.
- Two graphs with different numbers of components cannot be isomorphic.
- If the numbers of vertices with degree  $k$  are different in two graphs, they cannot be isomorphic.

Notice that by Definition 1.1.14, we can define a function  $g$  such that  $uv \notin E(G)$  if and only if  $g(u)g(v) \notin E(H)$  and relate this function to the complement graphs of  $G$  and  $H$ . Here we introduce a way to determine isomorphism by considering complement graphs, the proof of which is left to the reader as an exercise.

**Theorem 1.1.17 ▶ Complementation Preserves Isomorphism**

Let  $G$  and  $H$  be two graphs of the same order.  $G \cong H$  if and only if  $\overline{G} \cong \overline{H}$ .

Now let us think the reverse: if it is not easy to prove isomorphism, can we find a way to quickly determine that two graphs are not isomorphic? Here we present the necessary conditions for isomorphism:

**Theorem 1.1.18 ▶ Necessary Conditions for Isomorphism**

If  $G \cong H$ , then

1.  $G$  and  $H$  must have the same order and size.
2.  $\delta(G) = \delta(H)$  and  $\Delta(G) = \Delta(H)$ .
3. The number of vertices with degree  $i$  in  $G$  and  $H$  is the same for all  $i \in \mathbb{N}$ .



The first two conditions are easy to observe. For the third condition, we introduce a tool known as *degree sequences*.

### Definition 1.1.19 ► Degree Sequence

Let  $G$  be a graph of order  $n$ . If we label its vertices by  $v_1, v_2, \dots, v_n$  such that

$$d(v_1) \geq d(v_2) \geq \dots \geq d(v_n),$$

then the non-increasing sequence  $(d(v_n))$  is known as the **degree sequence** of  $G$ .

Now we consider the following question:

Let  $(d_n)$  be a non-increasing sequence, is there some graph whose degree sequence is  $(d_n)$ ?

We will make use of the following definition:

### Definition 1.1.20 ► Graphic Sequence

Let  $(d_n)$  be a sequence of non-negative integers at most  $n-1$ .  $(d_n)$  is said to be **graphic** if there exists a graph  $G$  whose degree sequence is  $(d_n)$ .

To determine whether a sequence is graphic by eye power is difficult. Fortunately, we have the following recursive algorithm:

### Theorem 1.1.21 ► Havel-Hakimi Algorithm

Let  $(d_n)$  be a non-increasing sequence of non-negative integers at most  $n-1$ . Define

$$d_m^* = \begin{cases} d_{m+1} - 1 & \text{if } 1 \leq m \leq d_1 \\ d_{m+1} & \text{if } d_1 + 1 \leq m \leq n-1 \end{cases}.$$

$(d_n)$  is graphic if and only if  $(d_m^*)$  is graphic.

*Proof.* Suppose  $(d_n)$  is graphic, we consider the following lemma:

**Lemma 1.1.22**

For any graphic sequence  $(d_n)$ , there is some graph  $G$  of order  $n$  with  $d_G(v_i) = d_i$  for  $i = 1, 2, \dots, n$  such that  $v_1$  is adjacent to  $v_j$  for  $j = 2, 3, \dots, d_1 + 1$ .

*Proof.* Suppose on contrary there is no such a graph  $G$ , then for any graph  $G$  with degree sequence  $(d_n)$ ,  $v_1$  is adjacent to at most  $d_1 - 1$  vertices in

$$A := \{v_j : j = 2, 3, \dots, d_1 + 1\}.$$

Thus, there exists some  $v_j \in A$  such that  $v_1 v_j \notin E(G)$ . However,  $d_G(v_1) = d_1$ , so there exists some  $v_k \in V(G) - A$  with  $v_k \neq v_1$  such that  $v_1 v_k \in E(G)$ .

Notice that since  $v_k \notin A \cup \{v_1\}$ ,  $k > j$  and so  $d_G(v_k) = d_k \leq d_j = d_G(v_j)$ . Therefore,  $v_j$  has at least as many neighbours as  $v_k$ , which means there must exist some  $v_r$  with  $v_j v_r \in E(G)$  such that  $v_k v_r \notin E(G)$ .

Define a graph  $G'$  by

$$V(G') = V(G), \quad E(G') = E(G) - v_j v_r - v_1 v_k + v_j v_k + v_1 v_j.$$

Note that  $d_{G'}(v_i) = d_G(v_i)$  for all  $i = 1, 2, \dots, n$ , so  $(d_n)$  is also a degree sequence for  $G'$ . However, now  $v_1$  in  $G'$  is adjacent to all vertices in  $A$ , which is a contradiction.  $\square$

By Lemma 1.1.22, we can choose some graph  $G$  whose degree sequence is  $(d_n)$  such that  $v_1$  is adjacent to  $d_1$  vertices  $v_2, v_3, \dots, v_{d_1+1}$ . Let  $H = G - v_1$ , then for any  $u_i \in H$ ,

$$d_H(u_i) = \begin{cases} d_G(v_{i+1}) - 1 & \text{if } 1 \leq i \leq d_1 \\ d_G(v_{i+1}) & \text{if } d_1 + 1 \leq i \leq n - 1 \end{cases}.$$

Therefore,  $(d_m^*)$  is graphic.

Conversely, suppose  $(d_m^*)$  is graphic, then there is some graph  $H$  of order  $n - 1$  such that  $d_H(u_i) = d_i^*$  for  $i = 1, 2, \dots, d_1$ . Construct a graph  $G = H + v_1$  such

that  $v_i = u_{i-1}$  for  $i = 2, 3, \dots, n$  and  $v_1$  is adjacent to  $v_2, v_3, \dots, v_{d_1+1}$ . Therefore,

$$d_G(v_i) = \begin{cases} d_1 & \text{if } i = 1 \\ d_{i-1}^* + 1 & \text{if } 2 \leq i \leq d_1 + 1 \\ d_i^* & \text{if } d_1 + 2 \leq i \leq n \end{cases}$$

Note that  $d_G(v_i) = d_i$ , so  $(d_n)$  is graphic. □

By Theorem 1.1.21, we can reduce a sequence recursively until we reach some obvious case. This obvious case is graphic if and only if the original sequence is graphic. We can also build the graph corresponding to the original sequence from a simple graphic sequence.

Alternatively, the following is another algorithmic way to determine whether a sequence is graphic.

### Theorem 1.1.23 ► Erdos-Gallai Algorithm

A non-increasing sequence  $(d_1, d_2, \dots, d_n)$  is graphic if and only if  $\sum_{i=1}^n d_i$  is even and for all  $k = 1, 2, \dots, n$ ,

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{d_i, k\}.$$

We will not give the full proof but only briefly justify the “only if” direction. Let  $G$  be any graph. Define  $d_i = d_G(v_i)$  where  $v_i \in V(G)$  for  $i = 1, 2, \dots, n$ , then  $(d_1, d_2, \dots, d_n)$  is graphic. Note that  $\sum_{i=1}^n d_i$  is obviously even by Lemma 1.1.7.

Take some  $k \in \mathbb{Z}^+$  with  $k \leq n$  and let  $S = \{v_1, v_2, \dots, v_k\}$ , then by Lemma 1.1.7 again,

$$\begin{aligned} \sum_{i=1}^k d_i &= 2e(G[S]) + e_G(S, V(G) - S) \\ &\leq k(k-1) + e_G(S, V(G) - S), \end{aligned}$$

where  $e_G(S, V(G) - S)$  is the number of edges in  $G$  which connect a vertex in  $S$  to a vertex in  $V(G) - S$ . Notice that

$$\begin{aligned} e_G(S, V(G) - S) &= \sum_{i=k+1}^n |\{u \in S : uv_i \in E(G)\}| \\ &\leq \sum_{i=k+1}^n \min\{k, d_i\}, \end{aligned}$$

and so it follows that

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{d_i, k\}.$$

*Remark.* Both Theorems 1.1.21 and 1.1.23 are  $\mathcal{O}(n^2)$  algorithms.

## 1.2 Graph Traversal

### 1.2.1 Paths and Cycles

We are often interested in the “connectedness” in a graph. In particular, we would wish to study how to traverse a graph from some vertices to others.

#### Definition 1.2.1 ▶ Walk, Trail, Path

Let  $G$  be a graph and  $x, y \in V(G)$ , then an  **$x$ - $y$  walk** is an alternating sequence

$$W := v_0 e_1 v_1 e_2 v_2 \cdots v_{n-1} e_n v_n$$

where  $v_i \in V(G)$ ,  $e_i = v_{i-1}v_i \in E(G)$ ,  $x = v_0$  and  $y = v_n$ . In particular,  $W$  is **open** if  $v_0 \neq v_n$  and **closed** otherwise. The **length** of a walk is defined as the number of edges in it.

If  $e_i \neq e_j$  whenever  $i \neq j$ , then  $W$  is called a **trail**. A closed trail is called a **circuit**. If  $v_i \neq v_j$  whenever  $i \neq j$ , then  $W$  is called a **path**. A closed path is called a **cycle**.

*Remark.* All paths are trails and all trails are walks, but the converses are not true.

In more informal terms, a trail is a walk with no repeated edge and a path is a trail with no repeated vertex.

Note that a path is essentially a graph. We denote a path of order  $n$  by  $P_n$  and a cycle of order  $n$  by  $C_n$ . It is easy to see that all cycles are circuits and all circuits are all closed walks, but the converses are not true. However, if we have a circuit, we can guarantee that this circuit contains at least one cycle.

Intuitively, all walks can be reduced to a path by removing duplicated edges and vertices repeatedly.

**Proposition 1.2.2 ▶ Any Walk Contains A Path**

If a graph  $G$  contains a  $u$ - $v$  walk of length  $k$ , then  $G$  contains a  $u$ - $v$  path of length at most  $k$ .

*Proof.* Let  $S$  be the set of all  $u$ - $v$  walks in  $G$ . Note that  $S \neq \emptyset$  since there is a  $u$ - $v$  walk of length  $k$ . Let  $P$  be a walk of the shortest length. We claim that  $P$  must be a path.

Suppose  $P$  is not a path, then there exists some vertices  $w_1 = w_2$  in  $P$ . Suppose  $Q$  is obtained by removing the  $w_1$ - $w_2$  walk from  $P$ , then  $Q$  is a walk. However,  $Q$  is shorter than  $P$  which is not possible. Therefore,  $P$  must be a path.

Note that the length of  $P$  is at most  $k$ , so  $G$  contains a  $u$ - $v$  path of length at most  $k$ . □

We also have the following relevant definitions with regard to cycles:

**Definition 1.2.3 ▶ Girth and Circumference**

Let  $G$  be a graph. The **girth** of  $G$  is the size of the shortest non-trivial cycle in  $G$  and the **circumference** of  $G$  is the size of the longest non-trivial cycle in  $G$ .

If  $G$  is acyclic, we define the girth of  $G$  to be  $\infty$  and the circumference to be 0. In some special graphs, we realise that we can traverse through every vertex and return to the starting point in a single traversal. Such graphs are said to be *Hamiltonian*.

**Definition 1.2.4 ▶ Hamiltonian Graph**

A graph  $G$  is **Hamiltonian** if it contains a cycle of size  $v(G)$ .

## 1.2.2 Connected Components

An important question we are interested in with graphs is the reachability of a vertex, i.e., given two vertices  $u$  and  $v$ , we want to know whether we can reach one vertex from another. Intuitively, we can traverse between two vertices if there is a path between them. From here we define the notion of a *connected graph*.

**Definition 1.2.5 ▶ Connected Graph**

Let  $G$  be a graph. Two vertices  $u$  and  $v$  are **connected** if there is a path between them.  $G$  is said to be **connected** if for any  $u, v \in V(G)$ , there exists a path from  $u$  to  $v$ . For any  $u \in V(G)$ , we denote the set of all vertices connected to  $u$  (inclusive of  $u$ ) as  $c(u)$ .

Intuitively, we of course always can find a connected subgraph of a connected graph, but we can in fact produce a much stronger result.

### Proposition 1.2.6 ▶ Labelling of Vertices of Connected Graphs

For every connected graph  $G$ ,  $V(G)$  can be labelled as  $\{v_1, v_2, \dots, v_n\}$  such that the graph  $G_i := [\{v_1, v_2, \dots, v_i\}]$  is connected for every  $i = 1, 2, \dots, n$ .

*Proof.* For  $i = 1$ , we can pick any vertex in  $V(G)$  as  $v_1$  and  $G_1$  is always trivially connected.

Suppose we have fixed up to  $k$  vertices for some  $k = 1, 2, \dots, n-1$ , then  $G_1, G_2, \dots, G_k$  are connected subgraphs of  $G$ . Note that  $v(G) = n > k$ , so there exists some  $v \in V(G) - V(G_k)$  such that  $v$  and  $v_1$  are connected by a path  $P$ . Note that there exists some  $u \in V(P)$  such that  $u \sim v$  for some  $v \in V(G_k)$ , so taking  $v_{k+1} = u$  guarantees that  $G_{k+1}$  is connected.  $\square$

Intuitively, if a graph has many edges, we would believe that it is easy to find a long path in it. Note that the notion of “having many edges” can be related to the minimal degree of a graph. More formally, the following result is true:

### Proposition 1.2.7 ▶ Finding Paths and Cycles in Graphs

Every graph  $G$  with  $\delta(G) \geq 2$  contains a path of length  $\delta(G)$  and a cycle of length at least  $\delta(G) + 1$ .

*Proof.* Let  $P = x_0x_1 \dots x_k$  be the longest path in  $G$ . Note that it suffices to prove that  $k \geq \delta(G)$ . Suppose that there exists some  $v \in N(x_k)$  such that  $v \notin V(P)$ , then we can find a longer path which is not possible. Therefore,  $N(x_k) \subseteq V(P)$ . Note that  $|N(x_k)| \geq \delta(G)$ , so  $k \geq \delta(G)$  and so  $G$  contains a  $P_{\delta(G)} \subseteq P$ .

Let  $i \in \mathbb{N}$  be such that  $x_jx_k \notin E(G)$  for all  $0 \leq j < i$ . Take  $P' = x_ix_{i+1} \dots x_k$ , then  $e(P') \geq \delta(G)$  by the previous argument. However,  $x_i \sim x_k$ , so  $P' + x_ix_k$  is a cycle of length at least  $\delta(G) + 1$  in  $G$ .  $\square$

A classic problem regarding graph traversal in connected graphs is the Königsberg Bridge Problem:

Let  $A, B, C, D$  be 4 distinct vertices such that there are 2 edges between  $A$  and  $B$ , 2 edges between  $A$  and  $C$ , and 1 edge between  $A$  and  $D$ ,  $B$  and  $D$ ,  $C$  and  $D$ . Can all the edges be visited exactly once in a single traversal of the graph produced?

This can be generalised as: given a connected multigraph  $G$ , does there exist a circuit in  $G$

which contains all the edges of  $G$ ? Such multigraphs are said to be *Eulerian*.

### Definition 1.2.8 ► Eulerian Circuit

Let  $G$  be a connected multigraph, an **Eulerian circuit** in  $G$  is a circuit which uses all edges of  $G$  exactly once.

Intuitively, if a graph is Eulerian, then following any Eulerian circuit will visit each of its vertices for an even number of times as there are equally many incoming edges as outgoing edges for every vertex in a circuit. This implies that an Eulerian graph will necessarily force all of its vertices to have even degrees. However, it turns out that actually this is also a sufficient condition for the existence of Eulerian circuits!

### Theorem 1.2.9 ► Necessary and Sufficient Condition for Eulerian Graphs

*A connected multigraph  $G$  is Eulerian if and only if every vertex in  $G$  has an even degree.*

*Proof.* The necessity is trivial, so we will only prove for sufficiency.

Consider the case where  $e(G) = 0$ , then the only connected multigraph is the trivial graph consisting of a singleton vertex, which is Eulerian.

Suppose that for all  $k < m$  where  $m \in \mathbb{Z}^+$ , a connected multigraph  $H$  with  $e(H) = k$  is Eulerian if every vertex in  $H$  has an even degree. Let  $G$  be a connected multigraph with  $e(G) = m > 0$  such that every vertex in  $G$  has an even degree, then  $\delta(G) \geq 2$ . By Proposition 1.2.7, there exists a cycle  $S \subseteq G$  where  $e(S) \geq 3$ .

Let  $G'$  be the multigraph such that  $V(G') = V(G)$  and  $E(G') = E(G) - E(S)$ , then clearly,  $e(G') < m$  and every vertex in  $G'$  has an even degree. Therefore, each connected component of  $G'$  is Eulerian.

Let  $C_1$  be a connected component of  $G'$  and let  $v_1 \in V(C_1) \cap V(S)$ . Note that there exists an Eulerian circuit in  $C_1$  which starts and ends with  $v_1$ . We can traverse along this circuit and then traverse in the clockwise orientation along  $S$  until we reach another vertex in  $S$  which is in some connected component of  $G'$  and repeat the process. Note that  $v_1 \in V(S)$ , so we will eventually traverse back to  $v_1$ , which gives an Eulerian circuit in  $G$ . □

Intuitively, even if a graph  $G$  is not Eulerian, it is possible to traverse all edges in  $G$  with finitely many trails. This is known as a decomposition of  $G$  into trails.

**Proposition 1.2.10 ▶ Minimal Number of Trails to Decompose A Multigraph**

Let  $G$  be a graph with  $2k$  vertices having odd degrees for some  $k \in \mathbb{N}$ , then the minimal number of trails needed to decompose  $G$  is  $\max\{k, 1\}$ .

*Proof.* The case where  $k = 0$  is immediate by Theorem 1.2.9. For  $k \geq 1$ , suppose on contrary that  $G$  can be decomposed with  $k - 1$  trails labelled as  $T_1, T_2, \dots, T_{k-1}$ . Since a trail contains at most 2 vertices with odd degrees, then

$$\left| \left\{ v \in \bigcup_{i=1}^{k-1} T_i : d_G(v) \text{ is odd} \right\} \right| \leq 2(k-1) < 2k,$$

which is a contradiction. Therefore, we need at least  $k$  trails to decompose  $G$ .

Let  $v_1, v_2, \dots, v_{2k}$  be the vertices in  $G$  with odd degrees. Let  $G'$  be the graph produced by adding the edges  $v_1v_2, v_3v_4, \dots, v_{2k-1}v_{2k}$  to  $G$ , then the vertices in  $G'$  all have even degrees. Therefore, there exists an Eulerian circuit in  $G'$  by Theorem 1.2.9.

Note that for all  $i = 1, 2, \dots, k - 1$ ,  $v_{2i-1}v_{2i}$  and  $v_{2i+1}v_{2i+2}$  are disjoint, so there exists a  $v_{2i}-v_{2i+1}$  trail. Note also that there exists a  $v_{2k}-v_1$  trail. The union of these trails is exactly  $G' - v_1v_2 - v_3v_4 - \dots - v_{2k-1}v_{2k} = G$ .  $\square$

Note that connectedness is an equivalence relation. Let  $P$  be a partition formed by the equivalence classes of  $G$  under the connectedness relation, then for any  $u, v \in V(G)$ ,  $u$  and  $v$  are in the same equivalence class if and only if  $u$  and  $v$  are connected. Any  $c(u) \in P$  induces a subgraph  $[c(u)]$ .

**Definition 1.2.11 ▶ Connected Component**

Let  $G$  be a graph and  $R$  be a relation such that for any  $u, v \in V(G)$ ,  $uRv$  if and only if  $u$  and  $v$  are connected. Let

$$C := V(G)/R$$

be the quotient set, then any  $c(u) \in C$  is known as a (connected) **component** of  $G$ .

The number of components of  $G$  is denoted by  $\omega(G)$ .

Alternatively, a connected component is a maximally connected subgraph, that is, a connected component of  $G$  is a subgraph  $H \subseteq G$  such that for all  $e \in E(G)$ ,  $H + e$  is not connected and for all  $v \in V(G) - V(H)$ ,  $[V(H) \cup \{v\}]$  is not connected.



It can be easily seen that  $G$  is connected if and only if  $\omega(G) = 1$ .

We will now establish a relationship between connectedness and complementation.

### Theorem 1.2.12 ► Connectedness of Complement

*If  $G$  is disconnected, then  $\overline{G}$  is connected.*

*Proof.* Let  $u, v \in V(\overline{G})$  be two arbitrary vertices. If  $uv \notin E(G)$ , then  $uv \in E(\overline{G})$  and so  $u$  and  $v$  are connected in  $\overline{G}$ .

If  $uv \in E(G)$ , since  $G$  is disconnected, there exists some  $w \in V(G)$  such that  $uw, wv \notin E(G)$ . Therefore,  $uw, wv \in E(\overline{G})$ . This means that there is a  $u$ - $v$  path in  $\overline{G}$  and so  $u$  and  $v$  are connected in  $\overline{G}$ .

Therefore,  $\overline{G}$  is connected. □

Intuitively, we can transform a connected graph into a disconnected one by deleting some vertices or edges. It can be easily observed that in certain graphs, deleting one particular vertex or edge will immediately disconnect the graph.

### Definition 1.2.13 ► Cut-Vertex, Bridge

Let  $G$  be a non-trivial graph.  $v \in V(G)$  is called a **cut-vertex** of  $G$  if  $\omega(G - v) > \omega(G)$ .  
 $e \in E(G)$  is called a **bridge** of  $G$  if  $\omega(G - e) > \omega(G)$ .

Note that a graph does not have to be connected to have cut vertices and bridges. Essentially, a cut-vertex (or bridge) is just a vertex (or edge) which disconnects a component upon removal.

Now the next question is how do we identify a cut-vertex or a bridge in a graph? Intuitively, a cut-vertex divides a graph into two portions such that traversal between the two portions has to pass through it.

### Theorem 1.2.14 ► Cut-Vertex Characterisation

*Let  $G$  be a graph.  $v \in V(G)$  is a cut-vertex if and only if there exists  $a, b \in V(G)$  such that  $v$  is in every path between  $a$  and  $b$ .*

*Proof.* Let  $v \in V(G)$  be a cut-vertex and consider  $H = G - v$ , then we can find two non-empty disjoint connected components of  $H$ , say  $H_1$  and  $H_2$ . Now, take  $a \in V(H_1)$  and  $b \in V(H_2)$ , then they are disconnected. However,  $a$  and  $b$  are connected in  $G$ , which means  $v$  is in every  $a$ - $b$  path.

We will prove the converse by contrapositive. Suppose  $v$  is not a cut-vertex of  $G$ , then  $G - v$  is connected. Therefore, for any  $u, w \in V(G)$ , there is a path between them which does not contain  $v$ .  $\square$

A similar argument can be established for bridges. Furthermore, since we have to pass through the bridge when traversing between the two portions, it can be easily seen that we have to reuse the bridge in order to traverse back.

### Theorem 1.2.15 ► Bridge Characterisation

*Let  $G$  be a graph.  $e \in E(G)$  is a bridge if and only if  $e$  is not contained by any cycle in  $G$ .*

*Proof.* We will prove the forward direction by contrapositive. Suppose  $e = xy$  is contained in some cycle in  $G$ , then there is some  $x$ - $y$  path in  $G - e$ . Suppose  $a, b \in V(G)$  are connected in  $G$  via a path containing  $e$ , this implies that there is some  $a$ - $b$  path in  $G - e$  and so  $G - e$  is connected, which means that  $e$  is not a bridge.

We will prove the converse also by contrapositive. Suppose  $e = xy$  is not a bridge, the  $G - e$  is connected and so there is some  $x$ - $y$  path in  $G - e$ . Let this path be  $P$ , then  $P \cup \{e\}$  is a cycle in  $G$ .  $\square$

Combining the two characterisations, we can relate cut-vertices to bridges in the following proposition, the proof of which is left to the reader as an exercise:

### Proposition 1.2.16

*Let  $G$  be a graph. If  $uv \in E(G)$  is a bridge and  $u$  is not an end vertex, then  $u$  is a cut-vertex.*

A direct consequence of this is the following corollary:

### Corollary 1.2.17

*If  $G$  is a graph with order at least 3 and  $G$  contains a bridge, then  $G$  contains a cut-vertex.*

## 1.2.3 Graphs as Metric Spaces

Given a graph  $G$  with  $u, v$  being two connected vertices, we are interested in the notion of distance between them. Intuitively, if there are multiple paths between  $u$  and  $v$ , we would take the length of the shortest one to represent their distance.

**Definition 1.2.18 ▶ Distance, Eccentricity, Diameter**

Let  $G$  be a connected graph and let  $u, v \in V(G)$ . The **distance** between  $u$  and  $v$ , denoted by  $d(u, v)$ , is the length of the shortest path between  $u$  and  $v$ . The **eccentricity** of  $u$  is defined to be

$$e(u) = \max_{v \in V(G)} \{d(u, v)\}.$$

The **diameter** of  $G$  is defined by

$$\text{diam}(G) = \max_{u \in V(G)} \{e(u)\}.$$

The **radius** of  $G$  is defined by

$$\text{rad}(G) = \min_{u \in V(G)} \{e(u)\}.$$

A vertex  $v$  is called a **central** vertex if

$$e(v) = \text{rad}(G).$$

The subgraph induced by the set of central vertices of  $G$  is known as the **centre** of  $G$ .

This also justifies the use of words “diameter” and “radius” in a circle. We can view a circle as a graph consisting of a centre and infinitely many vertices in the circumference, with edges of length  $r$  connecting the centre to the circumference. Indeed, in this definition, the farthest vertices will be any two on the circumference with a distance of  $2r$ , and the nearest vertices will be the centre and any vertex on the circumference with a distance of  $r$ .

Intuitively, having a large diameter means that the graph is sparse, i.e., one has to take a very long path to traverse between two vertices. Now let us consider a connected but sparse graph, the above intuition means that we may expect to find that the cycles in this graph are very large.

**Proposition 1.2.19 ▶ Graphs with High Girth Have High Diameter**

Let  $G$  be a graph with girth  $g(G)$ . If  $G$  contains a cycle, then  $g(G) \leq 2\text{diam}(G) + 1$ .

*Proof.* Suppose on contrary that  $g(G) \geq 2\text{diam}(G) + 2$ . Let  $C \subseteq G$  be the shortest cycle in  $G$ , then we can take  $x, y \in V(C)$  such that  $d_C(x, y)$  is the greatest. Clearly, we have  $d_C(x, y) \geq \text{diam}(G) + 1$ .

By Definition 1.2.18, there exists some path  $P$  between  $x$  and  $y$  with  $e(P) \leq \text{diam}(G)$ . Let  $P' \subseteq C$  be the  $x$ - $y$  path in  $C$ , then  $e(P) < e(P')$ . This means that  $P$  and  $P'$  are

distinct paths and so there is some  $q \in V(P') - V(P)$ .

Since  $P$  and  $P'$  are both  $x$ - $y$  walks,  $|P \cap P'| \geq 2$ . Therefore, there are  $p_1, p_2 \in P \cap P'$  which are connected to  $q$ , which implies that there is a  $p_1$ - $p_2$  path in  $P'$ . Since  $p_1$  and  $p_2$  are connected in  $P$ , we have found 2 different  $p_1$ - $p_2$  paths and so there is some cycle  $Q \subseteq P \cup P'$ . Therefore,

$$\begin{aligned} e(Q) &\leq e(P \cup P') \\ &\leq e(P) + e(P') \\ &\leq \text{diam}(G) + d_C(x, y) \\ &\leq 2d_C(x, y) - 1 \\ &= 2 \left\lfloor \frac{e(C)}{2} \right\rfloor - 1 \\ &< e(C), \end{aligned}$$

which is a contradiction. □

For readers with knowledge in real analysis or topology, it is easy to see that an undirected unweighted connected graph is a metric space with distance between vertices as its metric.

### Theorem 1.2.20 ► Connected Graph as A Metric Space

*If  $G$  is connected, then  $(V(G), d_G)$  is a metric space.*

*Proof.* Clearly, for all  $x, y \in V(G)$ ,  $d_G(x, y) \geq 0 \in V(G)$  with  $d_G(x, y) = 0$  if and only if  $x = y$  and  $d_G(x, y) = d_G(y, x)$ .

Let  $U, V$  be the shortest  $u$ - $w$  path and shortest  $w$ - $v$  path respectively, then  $U + V$  is a  $u$ - $v$  walk with length  $d(u, w) + d(w, v)$ . Since the shortest  $u$ - $v$  path has length  $d(u, v)$ , we have

$$d(u, v) \leq d(u, w) + d(w, v).$$

Therefore,  $(V(G), d_G)$  is a metric space. □

An application of Theorem 1.2.20 allows us to establish the following:

### Theorem 1.2.21 ► Boundedness of Diameter

*If  $G$  is connected, then*

$$\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G).$$

*Proof.*  $\text{rad}(G) \leq \text{diam}(G)$  is immediate from Definition 1.2.18. Take  $u, v \in V(G)$  with  $d(u, v) = \text{diam}(G)$  and take  $w \in V(G)$  such that  $e(w) = \text{rad}(G)$ . Notice that this implies

$$d(u, w) \leq e(w) = \text{rad}(G)$$

$$d(w, v) \leq e(w) = \text{rad}(G).$$

By Theorem ??,

$$\text{diam}(G) = d(u, v) \leq d(u, w) + d(w, v) = 2\text{rad}(G).$$

□

### 1.3 Graphs and Matrices

So far we have been using hand-drawn illustrations to aid us in graph visualisation. However, for large graphs with many vertices and edges, drawing out the graph structures becomes nearly impossible. Therefore, we need to use some algebraic tools to help us represent a large graph.

#### Definition 1.3.1 ► Adjacency Matrix

Let  $G$  be a multigraph with  $v(G) = n$ , then the **adjacency matrix** denoted by  $\mathbf{A}(G)$  is an  $n \times n$  matrix such that  $a_{ij}$  is the number of edges incident to both  $v_i, v_j \in V(G)$ .

*Remark.* For an undirected multigraph,  $\mathbf{A}$  is always symmetric.

Note that an adjacency matrix does not tell us which edges are incident to a particular vertex, so we may consider the following alternative:

#### Definition 1.3.2 ► Incidence Matrix

Let  $G$  be a multigraph with  $v(G) = n$  and  $e(G) = m$ , then the **incidence matrix** denoted by  $\mathbf{M}(G)$  is an  $n \times m$  matrix such that

$$m_{ij} = \begin{cases} 1, & \text{if } e_j \text{ is incident with } v_i \\ 0, & \text{otherwise} \end{cases}.$$

*Remark.* For an undirected multigraph, consider an arbitrary  $k$ -th column of  $\mathbf{M}$ . Note that any column will have exactly 2 non-zero entries. Suppose  $m_{ik} = m_{jk} = 1$ , it means that  $e_k = v_i v_j$ .

For readers who have learnt about basic data structures in computer science, an incidence matrix is just an adjacency list with each list of neighbours converted to a (non-compact) boolean array.

Note that the sum of the  $i$ -th row for both  $\mathbf{A}$  and  $\mathbf{M}$  is the degree of vertex  $i$ .

Since  $\mathbf{A}$  and  $\mathbf{M}$  carries information about adjacencies between vertices, we can use them to draw some conclusions regarding graph traversal.

### Proposition 1.3.3 ► Counting the Number of Walks between Two Vertices

Let  $G$  be a simple graph of order  $n$  with adjacency matrix  $\mathbf{A}$ , then the  $(i, j)$  entry of  $\mathbf{A}^k$  is the number of  $v_i$ - $v_j$  walks of length  $k$ .

*Proof.* The case where  $k = 1$  is trivially true since there is a  $v_i$ - $v_j$  walk of length 1 if and only if  $v_i$  and  $v_j$  are neighbours.

Let the  $(i, j)$  entry of  $\mathbf{A}^m$  be  $m_{ij}$ . Suppose  $m_{ij}$  is the number of  $v_i$ - $v_j$  walks of length  $m$ . Then the  $(i, j)$  entry of  $\mathbf{A}^{m+1}$  is given by

$$\sum_{r=1}^n m_{ir} a_{rj}.$$

Notice that  $m_{ir}$  is the number of  $v_i$ - $v_r$  walks of length  $m$  and  $a_{rj}$  is the number of  $v_r$ - $v_j$  walks of length 1, so by Theorem ??,  $m_{ir} a_{rj}$  is the number of  $v_i$ - $v_j$  walks of length  $n$  that passes through  $v_r$ . Therefore, summing up  $m_{ir} a_{rj}$  for  $r = 1, 2, \dots, n$  gives the total number of  $v_i$ - $v_j$  walks of length  $n + 1$ .  $\square$

Additionally, we have the following corollary:

### Corollary 1.3.4 ► Connectedness Test

Let  $G$  be a graph with  $v(G) \geq 2$  and adjacency matrix  $\mathbf{A}$ . Let

$$\mathbf{B} = \sum_{i=1}^{n-1} \mathbf{A}^i,$$

then  $G$  is connected if and only if  $b_{ij} \neq 0$  for all  $i \neq j$ .

As it turns out, the incidence and adjacency matrices are in fact related by a simple equation.

**Proposition 1.3.5**

Let  $\mathbf{M}$  and  $\mathbf{A}$  be the incidence matrix and adjacency matrix of a simple graph  $G$  respectively, then

$$\mathbf{M}\mathbf{M}^T = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix} + \mathbf{A}.$$

# Trees and Bipartite Graphs

## 2.1 Tree

In computer science and data science, it is often useful to use a tree diagram. Colloquially, we view a tree as a graph consisting of nodes, each of which has exactly one parent node and some children nodes. Here, we shall introduce the rigorous mathematical definition for a tree.

### Definition 2.1.1 ▶ Tree and Forest

A **forest** is an acyclic graph. A **tree** is a connected forest.

In other words, a tree is an **connected acyclic graph**. We also have the following alternative definition for a tree:

### Theorem 2.1.2 ▶ Equivalent Definitions of a Tree

*The followings are equivalent:*

1.  $T$  is a tree.
2. For any  $u, v \in V(T)$ , there exists a unique  $u$ - $v$  path in  $T$ .
3.  $T$  is minimally connected, i.e.,  $T$  is connected but  $T - e$  is disconnected for any  $e \in E(T)$ .
4.  $T$  is maximally acyclic, i.e.,  $T$  is acyclic but  $T + xy$  for any  $x, y \in V(T)$  with  $xy \notin E(T)$  is cyclic.

*Proof.* Let  $T$  be a tree. For any  $u, v \in V(T)$ , there exists a  $u$ - $v$  path  $P \subseteq T$ . Suppose  $P' \subseteq T$  is also a  $u$ - $v$  path, then there exists a non-trivial closed walk between  $u$  and  $v$  which contains a cycle, leading to a contradiction.

Suppose  $T$  is such that any  $u, v \in V(T)$  are connected by a unique path. Suppose on contrary  $T - e$  is connected for some  $e = xy \in E(T)$ , then there exists some  $x$ - $y$  path in  $T$  which does not contain  $xy$ . However,  $xy$  is a different  $x$ - $y$  path, which is a contradiction.

Suppose  $T$  is minimally connected. Note that if  $T$  contains some cycle  $C$ , then



for any  $e \in E(C)$ ,  $T - e$  is connected. So  $T$  being minimally connected implies that  $T$  is acyclic. Take any  $x, y \in V(T)$  with  $xy \notin E(T)$ , then there exists some  $x$ - $y$  path  $P \subseteq T$  which does not contain  $xy$ . Therefore,  $P + xy$  is a cycle, which implies that  $T$  is maximally acyclic.

Suppose  $T$  is maximally acyclic. Suppose  $T$  is not connected, then there exists  $u, v \in V(T)$  which are disconnected. Note that  $T + uv$  is acyclic because  $uv$  is the unique  $u$ - $v$  path. Therefore,  $T$  is not maximally acyclic. The contrapositive shows that  $T$  being maximally acyclic implies that  $T$  is a tree.  $\square$

We first consider the following proposition:

### Proposition 2.1.3 ► Degrees of Vertices in a Tree

Let  $T$  be a tree with  $\Delta(T) = k$ . Let  $n_i$  be the number of vertices in  $T$  of degree  $i$ , then

$$n_1 = 2 + \sum_{i=1}^{k-2} i n_{i+2}.$$

*Proof.* Let the order of  $T$  be  $n$ , then  $n = \sum_{i=1}^k n_i$ . By Lemma 1.1.7, we have

$$\sum_{v \in V(T)} d(v) = 2(n - 1).$$

However, since there are  $n_i$  vertices with degree  $i$ , we have

$$\sum_{v \in V(T)} d(v) = \sum_{i=1}^k i n_i.$$

Therefore,

$$\begin{aligned} 2 \sum_{i=1}^k n_i - 2 &= \sum_{i=1}^k i n_i \\ n_1 &= \sum_{i=3}^k i n_i \\ n_1 &= \sum_{i=1}^{k-2} i n_{i+2}. \end{aligned}$$

$\square$

The following corollary is an immediate consequence of Proposition 2.1.3:

**Corollary 2.1.4 ► Number of End Vertices in a Tree**

*A tree of order at least 2 contains at least 2 end vertices.*

In particular, a tree with exactly 2 end vertices is a doubly linked list. We will now give a name to these vertices in a tree whose degree is 1.

**Definition 2.1.5 ► Leaf**

Let  $G$  be a tree. A **leaf** of  $G$  is a vertex  $v \in V(G)$  with  $d_G(v) = 1$ .

Given any tree, we can fix a vertex as the “root”. We can carry out depth-first search to traverse the tree’s vertices starting from the root. An interesting phenomenon here is that at any point of time, we notice that the vertex which we are currently at is adjacent to exactly one of the vertices which we have visited.

**Proposition 2.1.6 ► Re-labelling of Vertices in a Tree**

*Let  $T$  be a tree with order  $n$ , then there exists an ordering of  $V(T)$  such that for any  $v_k \in V(T)$ ,  $v_k$  is adjacent to exactly one of  $v_1, v_2, \dots, v_{k-1}$ .*

*Proof.* By Corollary 2.1.4,  $T$  has a leaf vertex. Let this vertex be  $v_n$ . Note that  $T - v_n \subseteq T$ , so it is acyclic. Let  $T' = T - v_n$ , then  $v_n$  has a unique neighbour in  $V(T')$ . Note that  $T'$  is also a tree, so we can repeat this process to take  $v_{n-1}$ . Continue doing so until we have taken all vertices in  $T$ , then for any  $k = 1, 2, \dots, n$ ,  $v_k$  has a unique neighbour in  $v_1, v_2, \dots, v_k$ . □

In fact, such an ordering of the tree is known as a *topological sort*.

Intuitively, if a connected graph contains cycles, we can always reduce it to a tree by deleting some edges. This leads to the following result:

**Corollary 2.1.7 ► Number of Edges in a Tree**

*A connected graph  $T$  is a tree of order  $n$  if and only if  $e(T) = n - 1$ .*

*Proof.* If  $T$  is a tree, then by Proposition 2.1.6 we can order  $V(T)$  such that for all  $i = 2, 3, \dots, n$ ,  $v_i$  is adjacent to exactly one vertex from  $v_1, v_2, \dots, v_{i-1}$ . Therefore,  $e(T) = n - 1$ .

If  $e(T) = n - 1$ , note that  $T$  contains a tree  $T' \subseteq T$  which has  $n$  vertices. Therefore,  $e(T') = n - 1 = e(T)$ . This implies that  $T = T'$ , so  $T$  is a tree. □

Recall that every graph contains a tree. In fact, this result can be further strengthened.

**Proposition 2.1.8 ▶ Existence of Subgraph Trees**

Let  $T$  be a tree with  $k$  vertices. If  $G$  is a graph with  $\delta(G) \geq k - 1$ , then  $G$  contains a subgraph which is isomorphic to  $T$ .

*Proof.* Fix an ordering on  $V(T)$  such that  $v_i$  has a unique neighbour in  $v_1, v_2, \dots, v_{i-1}$ . Take any vertex in  $G$  to be  $v_1$ . Suppose we have chosen  $v_1, v_2, \dots, v_i$  in  $G$  for some  $i < k$ , for any  $v_j$  among these vertices, at most  $(i - 1)$  of  $v_j$ 's neighbours have been chosen, but

$$d_G(v_j) \geq \delta(G) = k - 1 > i - 1.$$

Therefore, there exists some  $u \in N_G(v_j)$  which is not one of  $v_1, v_2, \dots, v_i$ . Take  $u$  as  $v_{i+1}$  with the edge  $uv_j$ , then  $v_{i+1}$  is adjacent to exactly one of  $v_1, v_2, \dots, v_i$ . This means we can choose all vertices of  $T$  from  $V(G)$ , which means  $T \subseteq G$ .  $\square$

## 2.2 Bipartite Graphs

Consider a tree  $T$ . Take some  $r \in V(T)$  as the root. We define a vertex  $v \in V(T)$  to be at depth  $k$  from the root if there is a  $r$ - $v$  path of length  $k$ . Clearly, if we partition  $V(T)$  into vertices of odd depth and even depth, we can easily see that all edges in  $T$  joins one vertex from the former to one from the latter.

In fact, there are many graphs which can be bi-partitioned in a similar manner, which are known as *bipartite* graphs.

**Definition 2.2.1 ▶ Bipartite Graph**

A graph  $G$  is **bipartite** if  $V(G)$  can be partitioned into two disjoint subsets  $V_1$  and  $V_2$  such that every edge of  $G$  joins a vertex in  $V_1$  to a vertex in  $V_2$ .  $(V_1, V_2)$  is known as a **bipartition** of  $G$ .

Note that the size of a bipartite graph  $G$  with partite sets  $V_1$  and  $V_2$  is just

$$e(G) = \sum_{x \in V_1} d_G(x) = \sum_{y \in V_2} d_G(y).$$

It can be deduced that the maximal size of a bipartite graph is achieved when every vertex in  $V_1$  is connected to every vertex in  $V_2$ . To define this rigorously, we first introduce the notion of *join*.

**Definition 2.2.2 ▶ Join**

Let  $G_1$  and  $G_2$  be two disjoint graphs of order  $n_1$  and  $n_2$  respectively. The **join** of  $G_1$  and  $G_2$ , denoted by  $G_1 + G_2$  is the graph such that

$$V(G_1 + G_2) = V(G_1) \cup V(G_2)$$

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

Informally, we can describe the join of  $G_1$  and  $G_2$  as a graph which contains all the vertices from  $G_1$  and  $G_2$ , all existing edges from  $G_1$  and  $G_2$ , and new edges connecting every vertex in  $G_1$  to every vertex in  $G_2$ . Thus, we can define a complete bipartite graph using a join.

**Definition 2.2.3 ▶ Complete Bipartite Graph**

A **complete bipartite graph** is defined as  $K_{p,q} = 0_p + 0_q$ .

Consider a  $k$ -regular bipartite graph. Intuitively, if every vertex in the first partite set must be adjacent to  $k > 0$  vertices in the second partite set and vice versa, then it is only natural that the two partite sets have equal cardinalities.

**Proposition 2.2.4 ▶ Regular Bipartite Graph**

For all  $k \in \mathbb{Z}^+$ , a  $k$ -regular bipartite graph with bipartition  $(U, V)$  is such that  $|U| = |V|$ .

*Proof.* Note that we have

$$k|U| = e(G) = k|V|.$$

Since  $k > 0$ , this implies  $|U| = |V|$ . □

Sometimes, it is not easy to identify the bipartition in a bipartite graph. Therefore, to quickly determine whether a graph is bipartite, we consider the following theorem:

**2.2.1 Chromatic Number**

Suppose  $G$  is bipartite with bipartition  $(U, V)$ , then we can colour the vertices in  $G$  such that all vertices in  $U$  receives one colour and all those in  $V$  receives another.

**Definition 2.2.5 ▶  $k$ -Colourability**

A graph  $G$  is said to be  **$k$ -colourable** if one can colour  $V(G)$  with  $k$  distinct colours such that every pair of adjacent vertices receive distinct colours.

Clearly, if a graph is  $k$ -colourable, it must be  $(k + i)$ -colourable for all  $i \in \mathbb{N}$ . Therefore, we are interested in the smallest possible  $k$  such that a graph is  $k$ -colourable.

**Definition 2.2.6 ▶ Chromatic Number**

The **chromatic number** of a graph  $G$  is the smallest positive integer  $k$  such that  $G$  is  $k$ -colourable, denoted as  $\chi(G)$ .

Observe that if  $\chi(G) \leq 2$ , it is equivalent to saying that we can find a bipartition of  $V(G)$ , so a bipartite graph can be alternatively defined as a graph  $G$  with  $\chi(G) \leq 2$ .

Furthermore, observe that if  $H \subseteq G$ , then  $\chi(H) \leq \chi(G)$ . Consider a cycle  $C_{2n+1}$  for some  $n \in \mathbb{N}$ . It is clear that  $\chi(C_{2n+1}) = 3$ , so if  $C_{2n+1} \subseteq G$ , then  $G$  cannot be bipartite.

**Theorem 2.2.7 ▶ Bipartite Graph Characterisation**

*A graph is bipartite if and only if it contains no odd cycles.*

*Proof.* The necessity following from the previous argument, so we only prove the sufficiency. Suppose  $G$  contains no odd cycles. Note that there exists some spanning tree  $T \subseteq G$  with root  $u$ . For every  $v \in V(T)$ ,  $v$  is coloured red if  $d_T(u, v)$  is odd, and blue otherwise.

Suppose that there exist some  $w_1, w_2 \in V(G)$  such that both receive the same colour but  $w_1 w_2 \in E(G)$ . Note that this means that there exist a  $w_1$ - $u$  path in  $T$  and a  $w_2$ - $u$  path in  $T$  with the same parity. This implies that there exists a closed walk of odd length in  $G$ , and so  $G$  contains an odd cycle, which is a contradiction. Therefore,  $G$  is bi-colourable and so is bipartite.  $\square$

Similar to bipartite graphs, we can define the notion of an  $r$ -partite graph with chromatic numbers.

**Definition 2.2.8 ▶  $r$ -partite Graph**

A graph  $G$  is an  **$r$ -partite graph** if  $\chi(G) \leq r$ .

# Matchings and Vertex Covers

## 3.1 Matching

In set theory, we define the notion of a function

$$f : X \rightarrow Y$$

as a mapping such that every  $x \in X$  has a unique image in  $Y$ . In a graph-theoretical perspective, we can view  $X$  and  $Y$  as vertex sets and a “mapping” in this case can be seen as a set of disjoint edges joining a vertex from  $X$  to a vertex from  $Y$ .

### Definition 3.1.1 ► Matching

A **matching** in a graph  $G$  is a set  $M \subseteq E(G)$  of vertex-disjoint edges.

The term “vertex-disjoint edges” means that for any  $e, f \in M$ , they are not incident to the same vertex in  $G$ . Here we define two notions which sound similar:

### Definition 3.1.2 ► Maximal and Maximum Matchings

A **maximal matching** is a matching  $M \subseteq E(G)$  such that for all  $e \in E(G) - M$ ,  $M \cup \{e\}$  is not a matching.

A **maximum matching** is a matching  $M \subseteq E(G)$  such that for any matching  $N \subseteq E(G)$ ,  $|N| \leq |M|$ .

It is clear that a maximum matching must be maximal but not the other way round.

### Definition 3.1.3 ► Matching Number

The **matching number** of a graph  $G$  is the size of the maximum matching in  $G$ , denoted as  $\nu(G)$ .

Clearly, for any graph  $G$ ,  $\nu(G) \leq \frac{\nu(G)}{2}$  as the best we can do is to partition  $V(G)$  into disjoint pairs of vertices with respect to  $E(G)$ .

**Definition 3.1.4 ▶ Perfect Matching**

Let  $G$  be a graph. We say that  $G$  has a **perfect matching** if  $\nu(G) = \frac{v(G)}{2}$ .

*Remark.* In particular, if  $G$  is a bipartite graph with partite sets  $A$  and  $B$ , we say that a matching  $M \subseteq G$  is perfect to  $A$  if for any  $a \in A$ ,  $a \in M$ .

Intuitively, if  $G$  has a perfect matching, then we can find a bipartition  $(U, V)$  of  $V(G)$  such that there exists a bijection between  $U$  and  $V$ .

**Definition 3.1.5 ▶ Vertex Covering Number**

Let  $G$  be a graph. A **vertex cover** of  $G$  is defined to be a set  $S \subseteq V(G)$  such that for all  $e \in E(G)$ , there is some  $v \in S$  such that  $v \in e$ . The **vertex covering number** is defined as the size of the smallest vertex cover of  $G$ , denoted as  $\tau(G)$ .

Now let us examine some special graphs:

- Complete graph  $K_n$ :  $\nu(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$ ,  $\tau(K_n) = n - 1$ .
- Cycle  $C_n$ :  $\nu(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$ ,  $\tau(C_n) = \left\lceil \frac{n}{2} \right\rceil$ .
- Path  $P_n$ :  $\nu(P_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$ ,  $\tau(P_n) = \left\lceil \frac{n}{2} \right\rceil$ .
- Complete bipartite graph  $K_{m,n}$ :  $\nu(K_{m,n}) = \tau(K_{m,n}) = \min\{m, n\}$ .
- Hypercube graph  $Q^d$ :  $\nu(Q^d) = \tau(Q^d) = 2^{d-1}$ .
- Petersen graph  $P$ :  $\nu(P) = 5$ ,  $\tau(P) = 6$ .

We may have guessed based on the above observation that  $\tau(G) \geq \nu(G)$  for all graphs.

**Proposition 3.1.6 ▶ Vertex Covering Number Is at Most Matching Number**

For any graph  $G$ ,  $\tau(G) \geq \nu(G)$ .

*Proof.* Fix  $M := \{e_1, e_2, \dots, e_{\nu(G)}\}$  to be a maximum matching in  $G$ . Let  $S \subseteq V(G)$  be any vertex cover of  $G$ . For any  $i = 1, 2, \dots, \nu(G)$ , we have  $|e_i \cap S| \geq 1$ , otherwise  $S$  does not cover  $e_i$ . Therefore,

$$\tau(G) = |S| = \sum_{i=1}^{\nu(G)} |e_i \cap S| \geq \nu(G).$$

□

## 3.2 Clique and Independent Set

Given any graph  $G$ , then theoretically, if we were to find a subgraph of order  $n$ , the one with the greatest size is a complete graph and that with the smallest size is an empty graph.

### Definition 3.2.1 ► Clique

A **clique** in  $G$  is a complete subgraph of  $G$ . The **clique number** of  $G$  is the order of the largest clique in  $G$ , denoted as  $\omega(G)$ .

### Definition 3.2.2 ► Independent Set

The **independent set** of  $G$  is a subset  $I \subseteq V(G)$  such that for all  $u, v \in I$ ,  $uv \notin E(G)$ . The **independence number** of  $G$ , denoted as  $\alpha(G)$ , is the size of the largest independent set in  $G$ .

Intuitively, if a set  $S \subseteq V(G)$  induces an empty subgraph in  $G$ , it will induce a complete subgraph in  $\bar{G}$ . Therefore,  $\alpha(G) = \omega(\bar{G})$ .

We shall examine the independence numbers of some special graphs.

- Complete graph  $K_n$ :  $\alpha(K_n) = 1$ .
- Cycle  $C_n$ :  $\alpha(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$ .
- Path  $P_n$ :  $\alpha(P_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$ .
- Complete bipartite graph  $K_{m,n}$ :  $\alpha(K_{m,n}) = \max\{m, n\}$ .
- Hypercube graph  $Q^d$ :  $\alpha(Q^d) = 2^{d-1}$ .
- Petersen graph  $P$ :  $\alpha(P) = 4$ .

The following theorem discusses the numerical relationship between the independence number of vertex covering number:

### Proposition 3.2.3 ► Sum of Independent Number and Vertex Covering Number

For any graph  $G$  of order  $n$ ,  $\alpha(G) + \tau(G) = n$ .

*Proof.* It suffices to prove that  $S$  is an independent set in  $G$  if and only if  $V(G) - S$  is a vertex cover of  $G$ .

Let  $S \subseteq V(G)$  be an independent set, then for any edge  $e = uv \in E(G)$ , at least one of  $u$  and  $v$  is in  $V(G) - S$  for otherwise there exists  $u, v \in S$  such that  $u \sim v$ , which is not possible. Therefore,  $V(G) - S$  is a vertex cover.



Suppose conversely that  $S \subseteq V(G)$  is such that  $V(G) - S$  is a vertex cover.  $\square$

So far we have been talking a lot about the Petersen graph, but it turns out that the Petersen graph can be generalised into a *Kneser graph*.

### Definition 3.2.4 ► Kneser Graph

The **Kneser graph** is a graph  $KG(n, k)$  such that

$$\begin{aligned} V(KG(n, k)) &= \{S \in \mathcal{P}([n]) : |S| = k\}, \\ E(KG(n, k)) &= \{(S, T) : S \cap T = \emptyset\}. \end{aligned}$$

The vertex set of a Kneser graph is the set of all  $k$ -element subsets of  $\{1, 2, \dots, n\}$  and two vertices are adjacent if and only if their corresponding subsets are disjoint.

We introduce a theorem closely related to the Kneser graph:

### Theorem 3.2.5 ► Erdos-Ko-Rado Theorem

Let  $n \geq 2k$ , then any family

$$\mathcal{F} \subseteq \{S \in \mathcal{P}([n]) : |S| = k\}$$

with  $S \cap T \neq \emptyset$  for all  $S, T \in \mathcal{F}$  is such that  $|\mathcal{F}| \leq \binom{n-1}{k-1}$ .

We can see that  $\mathcal{F}$  is in fact a subset of  $V(KG(n, k))$ , so this is equivalent to saying that the biggest subset of  $V(KG(n, k))$  such that no vertices are adjacent is  $C_{k-1}^{n-1}$ , which is just to prove that  $\alpha(KG(n, k)) = C_{k-1}^{n-1}$ .

Another property of the independence number is related to the notion of a *line graph*.

### Definition 3.2.6 ► Line Graph

Let  $G$  be a graph. The **line graph** of  $G$  is defined as  $L(G)$  with  $V(L(G)) = E(G)$  and

$$E(L(G)) := \{(e, f) \in E(G)^2 : e \neq f, e \cap f \neq \emptyset\}.$$

In other words,  $L(G)$  uses  $E(G)$  as the vertex set and two vertices are adjacent if and only if the respective edges share at least 1 common vertex in  $G$ .

**Proposition 3.2.7 ▶ Independence Number of Line Graphs**

For any graph  $G$ ,  $\nu(G) = \alpha(L(G))$ .

*Proof.* Fix a maximum matching  $M \subseteq G$ , then for every  $e, f \in E(M)$  with  $e \neq f$ , we have  $ef \notin E(L(G))$ , therefore,  $E(M)$  is an independent set in  $L(G)$  and so  $\alpha(L(G)) \geq \nu(G)$ .

Suppose  $\alpha(L(G)) > \nu(G)$ , then there exists some  $e_0 \in V(L(G)) = E(G)$  such that  $e_0$  is vertex-disjoint with all  $e \in E(M)$ . However, this means that we can find a matching with bigger size in  $G$ , which is a contradiction. Therefore,

$$\nu(G) = \alpha(L(G)).$$

□

### 3.3 König's Theorem

Let us take a slight detour and unpack the matching number from a different perspective. Note that while forming a matching in a graph  $G$ , we actually divide  $E(G)$  into two subsets, one of which is included in the matching and the other excluded. Now, let us denote an included edge by letting it have a weight of 1, and assign 0 weight to all other edges which are unselected.

For every  $e \in E(G)$ , let  $x_e$  denote its weight. This way, to find a maximum matching is in fact equivalent to finding an optimal solution for the *integer program*

$$\begin{aligned} & \max \sum_{e \in E(G)} x_e \\ & \text{such that } x_e \in \{0, 1\}, \\ & \sum_{v \in e} x_e \leq 1 \quad \text{for all } v \in V(G), \end{aligned}$$

where the optimal value is just  $\nu(G)$ . Similarly, we can assign 0-1 weights,  $y_v$ , to every  $v \in V(G)$ , then a vertex cover is equivalent to an optimal solution for the integer program

$$\begin{aligned} & \min \sum_{v \in V(G)} y_v \\ & \text{such that } y_v \in \{0, 1\}, \\ & \sum_{v \in e} y_v \geq 1 \quad \text{for all } e \in E(G), \end{aligned}$$

where the optimal value is just  $\tau(G)$ . We can relax both programs into linear programs and it is clear that the 2 programs form a pair of primal and dual problems. Let the optimal values for the 2 linear programs be  $\nu^*(G)$  and  $\tau^*(G)$  respectively. By considering strong duality of linear programs, we can check that

$$\nu(G) \leq \nu^*(G) = \tau^*(G) \leq \tau(G).$$

Here  $\nu^*(G)$  and  $\tau^*(G)$  are known as *fractional matching number* and *fractional vertex covering number* respectively.

It turns out that for bipartite graphs, the above inequality can be further strengthened to strict equality. We shall first discuss a few useful concepts.

Fix some matching  $M$  in a bipartite graph  $G$ . Intuitively, we can find some paths in  $G$  which consisting of alternating edges from  $M$  and  $G - M$ .

### Definition 3.3.1 ► Alternating Path

Let  $G$  be a graph and fix  $M \subseteq G$  to be a matching. An **alternating path** in  $G$  with respect to  $M$  is a path  $v_1v_2 \cdots v_n$  such that  $v_1v_2 \in G - M$ ,  $v_kv_{k+1} \in M$  for all even  $k$  and  $v_kv_{k+1} \in G - M$  for all odd  $k$ .

Intuitively, if an alternating path contains more edges from  $G - M$  than from  $M$ , we can utilise it to extend the matching  $M$  by doing a switch of edges.

### Definition 3.3.2 ► Augmenting Path

An **augmenting path** with respect to a matching  $M$  in  $G$  is an alternating path which ends at some  $v \in V(G) - V(M)$

In general, we can characterise a maximum matching by using augmenting paths.

### Theorem 3.3.3 ► Maximum Matching Characterisation

A matching  $M \subseteq G$  is maximum if and only if there is no augmenting path in  $G$  with respect to  $M$ .

*Proof.* Suppose  $M$  is a maximum matching in  $G$ . We will prove that there is no augmenting path in  $G$  with respect to  $M$  by considering the contrapositive statement. Suppose we can find some augmenting path  $P := v_1v_2 \cdots v_n$  in  $G$  with respect to  $M$ , then  $P$  must contain an odd number of edges. Consider the subgraph

$$M' := M - \{v_iv_{i+1} \in P : i \text{ is even} \} + \{v_jv_{j+1} \in P : j \text{ is odd} \}.$$

Note that  $M'$  is clearly a matching and  $e(M') = e(M) + 1$ , so  $M$  is not a maximum matching.

Suppose conversely that there is no augmenting path in  $G$  with respect to  $M$ . We will prove that  $M$  is maximum by considering the contrapositive statement. Suppose that  $M$  is not maximum, then there exists some matching  $M'$  with  $e(M') > e(M)$ . Consider the subgraph  $H := M \triangle M'$ . Note that  $\Delta(H) = 2$  and that every connected component of  $H$  is either a path or a cycle whose edges alternate between  $E(M)$  and  $E(M')$ . Note that  $H$  is bipartite, so every cycle in  $H$  must be even and so contain equal number of edges from  $M$  and  $M'$ . However, since  $e(M') > e(M)$ , we can find a path in  $H$  which begins and ends with edges in  $M' - M$  and whose edges alternate between  $M$  and  $M'$ . This path is an augmenting path with respect to  $M$ .  $\square$

With the above preliminary results established, we are now ready to prove the *König's Theorem*.

### Theorem 3.3.4 ► König's Theorem

If  $G$  is bipartite, then  $\nu(G) = \tau(G)$ .

*Proof.* By Proposition 3.1.6, we already know that  $\nu(G) \leq \tau(G)$ . Therefore, it suffices to prove that  $\nu(G) \geq \tau(G)$ . Fix  $M \subseteq G$  to be a maximum matching. Let  $(A, B)$  be a bipartition of  $G$ , then for every  $e \in E(M)$ , we have  $e = v_A v_B$  for some  $v_A \in A$  and  $v_B \in B$ .

We select a subset of vertices from  $V(M)$  with the following algorithm: for each  $v_A v_B \in E(M)$ , we take  $v_B$  if there exists some alternating path starting at a vertex in  $A$  ends at  $v_B$ . Otherwise, we take  $v_A$ . Let the vertex set produced this way be  $U$ . We claim that  $U$  is a vertex cover for  $G$ .

Take any edge  $ab \in E(G)$ . If  $ab \in E(M)$ , then  $a \in U$  or  $b \in U$ . Otherwise, since  $M$  is a maximum matching in  $G$  which is bipartite, at least one of  $a$  and  $b$  is matched.

If  $b \in V(M)$ , consider  $a$ . If  $a \notin V(M)$ , then  $ab$  is an alternating path and so  $b \in U$ . Otherwise,  $a \in V(M)$  which reduces the case to the other case.

If  $a \in V(M)$  and  $a \in U$  then we are done. Otherwise,  $a \notin U$ . Suppose  $ab' \in E(M)$ , then  $b' \in U$ . This means that we can find some alternating path  $P$  with

respect to  $M$  which ends at  $b'$ . If  $b \in P$ , then there is some alternating path with respect to  $M$  which ends at  $b$ . Otherwise, note that  $a \notin P$  because  $a$  is matched, so  $P + ab$  is an alternating path which ends at  $b$  with respect to  $M$ . Since  $M$  is maximum, by Theorem 3.3.3, this alternating path is not augmenting, which means that  $b \in V(M)$ . Therefore,  $b \in U$ .

Therefore,  $U$  is a vertex cover for  $G$ . Note that  $|U| = \nu(G)$ , so  $\tau(G) \leq \nu(G)$ . This means that  $\nu(G) = \tau(G)$ .  $\square$

We can generalise Theorem 3.3.4 to *fractional vertex covers*.

#### Definition 3.3.5 ▶ Fractional Vertex Cover

The **fractional vertex cover** of a non-negatively weighted graph  $G$  is defined as the function

$$c : V(G) \rightarrow \mathbb{R}_0^+$$

such that  $c(v) + c(u) \geq w(uv)$  for all  $u, v \in V(G)$ , where  $w : E(G) \rightarrow \mathbb{R}_0^+$  is the weight function.

We shall state the generalised theorem without proof.

#### Theorem 3.3.6 ▶ Egerváry's Theorem

*For every bipartite  $G$ , the minimum weight of a fractional vertex cover equals the weight of the matching with the maximum weight.*

## 3.4 Hall's Marriage Theorem

Suppose  $G$  is a bipartite graph with bipartition  $(A, B)$ . In the case where  $|A| \neq |B|$  we cannot get any matching with size greater than  $\min\{|A|, |B|\}$  for the obvious reason. Therefore, the best we can hope is to construct a matching **perfect to one of the partite sets**.

Suppose  $S \subseteq A$  is a set of vertices in  $G$ . We denote the union of the neighbourhoods of all vertices in  $S$  by  $N_G(S)$ . Apparently, if we can find some  $S$  such that  $|N_G(S)| < |S|$ , it means that we cannot find a matching perfect to  $A$  because we simply cannot match every vertex in  $S$ .

This means that for  $G$  to contain a matching perfect to  $A$ , it necessarily needs to satisfy  $|S| \leq |N_G(S)|$  for every  $S \subseteq A$ . However, it turns out that amazingly, this necessary condition alone is also sufficient to ascertain the existence of a matching perfect to  $A$ .

**Theorem 3.4.1 ► Hall's Theorem**

Let  $G$  be a bipartite graph  $G$  with bipartition  $(A, B)$ .  $G$  contains a matching perfect to  $A$  if and only if for every  $S \subseteq A$ ,

$$\left| \bigcup_{v \in S} N_G(v) \right| \geq |S|.$$

*Proof.* Suppose  $M \subseteq G$  is a matching perfect to  $A$ . Let  $S \subseteq A$  be any subset. Clearly, for each  $a \in S$ , there exists a unique  $b \in B$  such that  $ab \in E(M)$  and if  $ab, a'b' \in E(M)$  for  $a \neq a'$ , we have  $b \neq b'$ . Therefore,  $|N_G(S)| \geq |S|$ .

Suppose that for every  $S \subseteq A$ ,  $|N_G(S)| \geq |S|$ , we take any matching  $M \subseteq G$  such that  $A - V(M) \neq \emptyset$ . Take  $(a_0, b_1, a_1, \dots)$  to be a finite sequence of pair-wise distinct vertices with greatest length such that  $a_0$  is unmatched,  $b_i$  is adjacent to some  $a_{f(i)} \in \{a_0, a_1, \dots, a_{i-1}\}$  in  $G$  and  $b_i a_i \in M$  for all  $i \geq 1$ .

We claim that this sequence does not end in  $A$ . Suppose on contrary that the sequence ends at some  $a_{n-1} \in A$ . Take  $T = \{a_0, a_1, \dots, a_{n-1}\}$ , then  $|N_G(T)| \geq n$ . Therefore, there exists some  $b_n \in N_G(T) - \{b_1, b_2, \dots, b_{n-1}\}$ , which is a contradiction because we can now extend the length of the sequence.

Let  $b_k$  be the end of the sequence, then the path  $P := a_0 b_1 a_{f(1)} b_2 a_{f(2)} \dots a_{f(k-1)} b_k$  is an alternating path. If  $b_k$  is matched to some  $a \in A \cap V(M)$ , we know that  $a \notin \{a_1, a_2, \dots, a_{k-1}\}$  because that would imply  $b_k = b_i$  for some  $i \in \{1, 2, \dots, k-1\}$ , which is a contradiction. Since  $a_0$  is unmatched, we have find a longer sequence satisfying our construction, which is not possible. Therefore,  $b_k$  must be unmatched. This means that  $P$  is an augmenting path and so by Theorem 3.3.3  $M$  is not a maximum matching. This means that the maximum matching in  $G$  must be perfect to  $A$ .  $\square$

**Theorem 3.4.2 ► Tutte's Theorem**

A graph  $G$  has a perfect matching if and only if for every  $U \subseteq V(G)$ , the number of connected components with odd orders in  $G - U$  is at most  $|U|$ .

**Theorem 3.4.3 ► Petersen's Theorem**

Every connected 3-regular graph with no cut edge contains a perfect matching.

*Proof.* Let  $U \subseteq V(G)$  be non-empty and consider  $H := G - u$ . If  $H$  contains no odd-order component, then we are done. Otherwise, fix some odd-order component  $C \subseteq H$ . Consider

$$\sum_{v \in V(C)} d_G(v) = 3|C| = 2e(G[V(C)]) + e_G(V(C), U).$$

Note that the above sum is odd, so  $e_G(V(C), U)$  is odd. Since  $G$  contains no cut-edge,  $e_G(V(C), U) \geq 3$ . Let  $p$  be the number of odd components in  $G - u$  and  $M$  be the number of edges between  $U$  and  $V(G) - U$ , then  $M \geq 3p$ . However,  $M \leq 3|U|$  because  $G$  is 3-regular, so  $p \leq |U|$ . By Theorem 3.4.2,  $G$  contains a perfect matching.  $\square$

#### Definition 3.4.4 ► Hypergraph

Let  $V$  be a vertex set, a **hypergraph** on  $V$  is a graph whose edge set is a family

$$E := \{X : X \in \mathcal{P}(V)\}.$$

If  $|X| = k$  for all  $X \in E$ , we say that the hypergraph is  $k$ -uniform.

# Connectivity

## 4.1 Connectivity

### Definition 4.1.1 ► Separating Set

Let  $G$  be a connected graph.  $S \subseteq V(G)$  is a **separating set** (or **vertex cut**) of  $G$  if  $G - S$  is disconnected. If  $S = \{s\}$  is a singleton, then  $s$  is known as a **cut vertex**.

### Definition 4.1.2 ► Vertex Connectivity

The **vertex connectivity** of a graph  $G$ , denoted by  $\kappa(G)$  is the cardinality of the smallest subset  $S \subseteq V(G)$  such that  $G - S$  is either disconnected or a trivial graph.  $G$  is said to be  **$k$ -connected** if  $\kappa(G) \geq k$ .

Intuitively,  $\kappa(G) \leq v(G) - 1$  for any graph  $G$ , and a  $k$ -connected graph must be  $(k - 1)$ -connected. Our “old-fashioned” notion of a connected graph is simply a 1-connected graph.