Contents

1 The Real Numbers

1.1 Fields

Definition 1.1.1 ▶ Field

A set F with two binary operations, namely addition and multiplication, is called a **field** if it satisfies the following axioms:

1. $\forall a, b \in F, a +_F b = b +_F a$.
2. $\forall a, b, c \in F, (a +_F b) +_F c = a +_F (b +_F c)$.
3. $\exists 0_F \in F$ such that $\forall a \in F, 0_F +_F a = a +_F 0_F = a$.
4. $\forall a \in F, \exists a' \in F$ such that $a +_F a' = 0_F$.
5. $\forall a, b \in F, a \cdot_F b = b \cdot a$.
6. $\forall a, b, c \in F, (a \cdot_F b) \cdot c = a \cdot_F (b \cdot_F c)$.
7. $\forall a, b, c \in F, a \cdot_F (b +_F c) = a \cdot_F b +_F a \cdot_F b$ and $(a +_F b) \cdot_F c = a \cdot_F c +_F b \cdot_F c$.
8. $\exists 1_F \in F$ such that $\forall a \in F, 1_F \cdot_F a = a \cdot_F 1_F = a$.
9. $\forall a \in F, \exists a' \in F$ such that $a \cdot_F a' = 1_F$.

If we denote addition by "$+_F$" and multiplication by "$\cdot_F$" or "$\times_F$", then we can denote the field over F by $(F, +_F, \cdot_F)$ or $(F, +_F, \times_F)$.

Among the commonly used number sets, one may check that $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ are fields, while $\mathbb{N}$ and $\mathbb{Z}$ are not.

1.1.1 Ordered Fields

Definition 1.1.2 ▶ Total Order

A **total order** on a set X is a binary relation $\leq$ over X such that for all $a, b, c \in X$:

1. $a \leq a$ (reflexive).
2. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitive).
3. $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetric).
4. either $a \leq b$ or $b \leq a$ (strongly connected).

Definition 1.1.3 ▸ Strict Total Order

A **strict total order** on a set X is a binary relation $<$ over X such that for all $a, b, c \in X$:

1. $a \not< a$ (irreflexive).
2. $a < b$ implies $b < a$ (asymmetric).
3. $a < b$ and $b < c$ implies $a < c$ (transitive).
4. if $a \neq b$, then either $a < b$ or $b < a$ (connected).

It is easy to see that the real numbers form the ordered fields $(\mathbb{R}, +, \times, \leq)$ and $(\mathbb{R}, +, \times, <)$. Note that this means $\mathbb{R}$ satisfies trichotomy. If we choose any $x \in \mathbb{R}$, then exactly one of $x = 0$, $x > 0$ and $x < 0$ is true. Therefore, we can define that if $x \in \mathbb{R}$ and $x > 0$, then x is said to be positive. This leads to the following axiomatic results:

1. If a and b are both positive, then $a + b$ is positive;
2. If a and b are both positive, then ab is positive;
3. For any $a \in \mathbb{R}$, either $a = 0$, a is positive, or $-a$ is positive.

Note that $a < b$ if and only if $b - a$ is positive. So the trichotomy of $\mathbb{R}$ guarantees that for any $a, b \in \mathbb{R}$, either $a = b$, $a < b$ or $b < a$ (i.e., $a > b$).

1.2 Properties of $\mathbb{R}$

We can derive a few obvious minor results based on the field properties of $\mathbb{R}$:

1. If $a, b \in \mathbb{R}$, then $-ab + ab = 0$;
2. For all $a \in \mathbb{R}$ with $a \neq 0$, $a^2 > 0$;
3. If $a \in \mathbb{R}$ is such that $0 \leq a < \epsilon$ for all $\epsilon \in \mathbb{R}^+$, then $a = 0$;
4. If $a < b$, then $a + c < b + c$ for all $c \in \mathbb{R}$.
5. If $a < b$, then $ac < bc$ for all $c \in \mathbb{R}^+$ and $ac > bc$ for all $c \in \mathbb{R}^-$.
6. For all $a \in \mathbb{R}$, $a^2 > 0$.

We may consider the following interesting proposition:

Proposition 1.2.1

If $a \in \mathbb{R}$ is such that $0 \leq a < \epsilon$ for all $\epsilon \in \mathbb{R}^+$, then $a = 0$.

Proof. Suppose on contrary that $a > 0$, then we can take $\epsilon_0 = \frac{a}{2}$. Note that $\epsilon_0 \in \mathbb{R}^+$ but $\epsilon_0 < a$, which is a contradiction. So $a = 0$. □

The above essentially asserts that **a non-negative real number is strictly less than any positive real number if and only if it is** 0.

The properties of $\mathbb{R}$ also enables us to manipulate inequalities based on the following trivial results:

1. If $ab > 0$, then a and b are either both positive or both negative;
2. If $ab < 0$, then exactly one of them is positive and exactly one of them is negative.

We shall introduce a few well-known inequalities.

Theorem 1.2.2 ▸ Bernoulli's Inequality

If $x > -1$, then $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

Proof. The case where $n = 0$ is trivial.

Suppose that $(1+x)^k \geq 1 + kx$ for some $k \in \mathbb{N}$, consider

$$\begin{aligned}(1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) \\ &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x.\end{aligned}$$

Therefore, $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$. □

Theorem 1.2.3 ▸ AM-GM-HM Inequality

Let $n \in \mathbb{N}^+$ and let $a_1, a_2, \cdots, a_n$ be positive real numbers, then

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \leq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} a_i}{n}.$$

1.2.1 Absolute Value

Given any real number x, intuitively we sense that x possesses a certain "distance" from 0. This distance can be formalised as follows:

Definition 1.2.4 ▸ Absolute Value

Let $x \in \mathbb{R}$, the **absolute value** of x is defined as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}.$$

We have some trivial properties about the absolute value:

1. For all $a, b \in \mathbb{R}$, $|ab| = |a||b|$;
2. For all $a \in \mathbb{R}$, $|a|^2 = a^2$;
3. If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$ for all $a \in \mathbb{R}$;
4. For all $a \in \mathbb{R}$, $-|a| \leq a \leq |a|$.

Using these basic properties, we can prove the following results:

Theorem 1.2.5 ▸ Triangle Inequality

For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

Corollary 1.2.6 ▸ Extended Triangle Inequality

For all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$ *and* $|a - b| \leq |a| + |b|$.

Corollary 1.2.7 ▸ Generalised Triangle Inequality

For all $a_1, a_2, \cdots, a_n \in \mathbb{R}$,

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|.$$

Analogously, if $|x|$ represents the "distance" between x and 0, then by a simple translation we can see that $|x - a|$ represents the "distance" between x and a. Thus, we can have the following definition:

Definition 1.2.8 ▸ Neighbourhood

Let $a \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^+$. The **$\epsilon$-neighbourhood** of a is defined to be the set

$$V_\epsilon(a) := \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

Note that $x \in V_\epsilon(a)$ if and only if $-\epsilon < x - a < \epsilon$ or $a - \epsilon < x < a + \epsilon$. Which leads to the following interesting result:

Proposition 1.2.9

For any $a \in \mathbb{R}$, if $x \in V_\epsilon(a)$ for all $\epsilon \in \mathbb{R}^+$, then $x = a$.

Proof. Note that this essentially means that $|x - a| < \epsilon$ for all $\epsilon \in \mathbb{R}^+$. By Proposition 1.2.1, we have $|x - a| = 0$ and therefore $x = a$. □

1.2.2 The Completeness Property of $\mathbb{R}$

Intuitively, there are no "gaps" among the real numbers, i.e., if you take any two real numbers, between them there is nothing else than other real numbers. Therefore, we say that $\mathbb{R}$ is *complete*. This is in contrast with $\mathbb{Q}$ where there are gaps in between any two rational numbers (because there always exists some irrational numbers in between).

In this section, we probe into how the completeness of $\mathbb{R}$ can be established, and how the real numbers themselves can be constructed. To do that, we first establish the notion of *boundedness*.

Definition 1.2.10 ▶ Boundedness

Let $S \subseteq \mathbb{R}$. We say that S is:

- **bounded above** if there exists some $u \in R$ (known as the **upper bound** of S) such that $u \geq s$ for all $s \in S$;
- **bounded below** if there exists some $v \in R$ (known as the **lower bound** of S) such that $v \leq s$ for all $s \in S$;
- **bounded** if S has both an upper bound and a lower bound;
- **unbounded** either if S has no upper bound or if S has no lower bound;

Remark. Note that S is bounded if and only if there is some $M \geq 0$ such that $|s| \leq M$ for all $s \in S$.

2

Sequences and Series

2.1 Sequences

Informally, a sequence is a list of enuerable numbers. This means that we can view a sequence as a mapping from an interval of $\mathbb{N}^+$ to $\mathbb{R}$. In this course, we will mainly focus on infinite sequences.

Definition 2.1.1 ▸ Sequence

A **sequence** in $\mathbb{R}$ is a real-valued function $X : \mathbb{N}^+ \to \mathbb{R}$. Where $X(n)$ is called the n-th **term** of the sequence.

By convention, we denote $X(n)$ by x_n, and the sequence X by (x_n) or $(x_n : n \in \mathbb{N}^+)$.

Alternatively, a sequence (x_n) may be defined in the following manner: first, we define the value of x_1. Secondly, we define a mapping $(x_1, x_2, \cdots, x_n) \mapsto x_{n+1}$. Sequences defined in this way are said to be **inductively** and **recursively** defined.

2.1.1 Limits of Sequences

As n becomes very large, a sequence may exhibit certain limiting behaviour.

Definition 2.1.2 ▸ Convergence of Sequences

A sequence (x_n) defined in $\mathbb{R}$ is said to be **convergent** to x if for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that whenever $n \geq N$, $|x_n - x| < \epsilon$. x is known as the **limit** of (x_n), denoted as

$$\lim_{n\to\infty} x_n = x.$$

A sequence which is not convergent is said to be *divergent*.

Intuitively, a sequence can not converge to different values concurrently. This idea can be formulated formally as follows:

Theorem 2.1.3 ▸ Uniqueness of Limits

If (x_n) converges, then its limit is unique.

Proof. Suppose that x and x' are both limits of (x_n). For all $\epsilon > 0$, there exists $N_1 \in \mathbb{N}^+$ such that
$$|x_n - x| < \frac{\epsilon}{2}$$
whenever $n \geq N_1$ and there exists $N_2 \in \mathbb{N}^+$ such that
$$|x_n - x'| < \frac{\epsilon}{2}$$
whenever $n \geq N_2$. Take $N = \max\{N_1, N_2\}$, then for all $n \geq N$,
$$\begin{aligned} |x - x'| &= |x - x_n + x_n - x'| \\ &\leq |x_n - x| + |x_n - x'| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$
for all $\epsilon > 0$. By Proposition 1.2.1, $x - x' = 0$, i.e., $x = x'$. This means that $\lim_{n\to\infty} x_n$ is unique. □

Given sequences (x_n) and (y_n), we can form new sequences by applying arithmetic operations onto them, and we can relate the limits of these new sequences with the limits of (x_n) and (y_n).

Theorem 2.1.4 ▸ Limit Laws

If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

1. $\lim_{n\to\infty}(x_n + y_n) = x + y$.
2. $\lim_{n\to\infty}(x_n y_n) = xy$.
3. $\lim_{n\to\infty}(cx_n) = cx$ *for all* $c \in \mathbb{R}$.
4. $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$.

In some cases, it may not be easy to prove the existence of limit or compute it directly for a sequence. Thus, the following may be useful:

Theorem 2.1.5 ▸ Squeeze Theorem

Let (x_n), (y_n) and (z_n) be sequences such thatr $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}^+$. If (x_n)

and (z_n) both converge and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell$, then (y_n) converges and

$$\lim_{n\to\infty} y_n = \ell.$$

Proof. Let $\epsilon > 0$ be an arbitrary real number. Note that there exists $N \in \mathbb{N}^+$ such that for all $n \geq N$,

$$|x_n - \ell| < \epsilon, \qquad |z_n - \ell| < \epsilon.$$

Therefore,

$$-\epsilon < x_n - \ell \leq y_n - \ell \leq z_n - \ell < \epsilon,$$

which implies that

$$|y_n - \ell| < \epsilon$$

for all $n \geq N$. Therefore, $\lim_{n\to\infty} y_n = \ell$. □

Notice that the limit of a sequence essentially "bounds" the sequence. This motivates us to investigate the relation between a convergent sequence and the its bounds.

Definition 2.1.6 ▸ Boundedness of Sequences

A sequence (x_n) in $\mathbb{R}$ is **bounded** if there exists some $M \in \mathbb{R}^+$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}^+$.

Remark. Note that (x_n) is bounded if and only if the set $\{x_n : n \in \mathbb{N}^+\}$ is bounded.

Theorem 2.1.7 ▸ Boundedness of Convergent Sequences

A sequence (x_n) is bounded if it is convergent.

Proof. Let $\lim_{n\to\infty} x_n = x$. Note that there exists some $N \in \mathbb{N}^+$ such that whenever $n \geq N$, $|x_n - x| < 1$. Consider

$$\begin{aligned} |x_n| &= |x_n - x + x| \\ &\leq |x_n - x| + |x| \\ &< 1 + |x|. \end{aligned}$$

Let

$$M := \sup\left\{|x_1|, |x_2|, \cdots, |x_{N-1}|, 1 + |x|\right\},$$

then $|x_n| \leq M$ for all $n \in \mathbb{N}^+$, and so (x_n) is bounded. □

The contrapositive statement to the above theorem concludes that **any unbounded se-**

quence must be divergent. However, note that the converse of Theorem 2.1.7 is not true in general! As a counter example, consider $x_n = (-1)^n$.

Proposition 2.1.8

Let (x_n) be a convergent sequence. If $x_n \geq 0$ for all $n \in \mathbb{N}^+$, then $\lim_{n\to\infty} x_n \geq 0$.

Proof. Let $\lim_{n\to\infty} x_n = x$. Suppose on contrary $x < 0$, then $-x > 0$. Therefore, there exists some $N \in \mathbb{N}^+$ such that whenever $n \geq N$, $|x_n - x| < -x$. This means that for all $n \geq N$,

$$x_n < x - x = 0.$$

However, this is a contradiction, so $\lim_{n\to\infty} x_n = x > 0$. □

Here are two simple corollaries from Proposition 2.1.8, the proofs of which are left to the reader as an exercise.

Corollary 2.1.9

If (x_n) and (y_n) are convergent sequences with $x_n \geq y_n$ for all $n \in \mathbb{N}^+$, then

$$\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n.$$

Corollary 2.1.10

If (x_n) is a convergent sequence with $a \leq x_n \leq b$ for all $n \in \mathbb{N}^+$, then

$$a \leq \lim_{n\to\infty} x_n \leq b.$$

In casual languages, we may be tempted to describe the limit of a sequence as "a value to which the terms can get as close as possible, but which is never surpassed". This intuition gives us an idea to prove convergence for a bounded sequence.

Definition 2.1.11 ▸ Monotone Sequences

Let (x_n) be a sequence. (x_n) is said to be **increasing** if $x_i \geq x_j$ whenever $i \geq j$, and **decreasing** if $x_i \leq x_j$ whenever $i \geq j$. A sequence is said to be **monotone** if it is either increasing or decreasing.

Note that an increasing sequence is the same as an non-decreasing sequence and vice versa. Recall that we have stated that the converse of Theorem 2.1.7 is not true in general, but if we impose an additional constraint on the monotonicity of the bounded sequence, we will get a stronger condition.

Theorem 2.1.12 ▸ Monotone Convergence Theorem

Let (x_n) be a monotone sequence in $\mathbb{R}$, then (x_n) converges if and only if it is bounded.

Proof. Suppose (x_n) is convergent, then it follows from Theorem 2.1.7 that it is bounded.

Suppose conversely that (x_n) is bounded. Without loss of generality, assume that (x_n) is increasing, so (x_n) has an upper bound. Let $\sup(x_n) = x$ and let $\epsilon > 0$ be an arbitrary real number. Note that $x - \epsilon$ is not an upper bound for (x_n), so there exists some $x_N \in (x - \epsilon, x]$, which means that $0 \leq x - x_N < \epsilon$. Since (x_n) is increasing, for all $n \geq N$, we have $x \geq x_n \geq x_N$, so

$$0 \leq x - x_n \leq x - x_N < \epsilon.$$

Therefore, $|x - x_n| < \epsilon$ for all $\epsilon > 0$ whenever $n \geq N$, and so $\lim_{n\to\infty} x_n = x$. □

A classic application of Theorem 2.1.12 is an approximation of $\sqrt{2}$.

Example 2.1.13 ▸ Mesopotamian Approximation of $\sqrt{2}$

Define (x_n) such that $x_1 = 2$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$. Show that $\lim_{n\to\infty} x_n = \sqrt{2}$.

2.1.2 Subsequences

Recall that the sequence $x_n = (-1)^n$ is divergent. However, suppose we were to take all the odd terms from (x_n) to form a new sequence, and to take all the even terms to form another new sequence. One would realise that both new sequences are convergent. This motivates us to study "a part" of a sequence as a new sequence.

Definition 2.1.14 ▸ Subsequence

Let (x_n) be a sequence in $\mathbb{R}$ and let

$$n_1 < n_2 < n_3 < \cdots < n_k < \cdots$$

be an infinite sequence of strictly increasing positive integers, then the sequence (x_{n_k}) is called a **subsequence** of (x_n).

Remark. Note that a sequence is always a subsequence of itself.

Intuitively, if a sequence is convergent, then any of its subsequences should be convergent, too.

Theorem 2.1.15 ▶ Convergence of Subsequences

Let (x_n) be a convergent sequence with $\lim_{n\to\infty} x_n = x$, then for any subsequence (x_{n_k}),

$$\lim_{n_k\to\infty} x_{n_k} = \lim_{k\to\infty} x_{n_k} = x.$$

Proof. Note that for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq N$. Observe that $n_k \geq k$, so whenever $k \geq N$, we have $n_k \geq N$, and so

$$\left|x_{n_k} - x\right| < \epsilon$$

whenever $k \geq N$, i.e.,

$$\lim_{n_k\to\infty} x_{n_k} = \lim_{k\to\infty} x_{n_k} = x.$$

□

Theorems 2.1.15 and 2.1.3 give rise to the following corollary. The proof is left to the reader as an exercise.

Corollary 2.1.16

Let (x_n) be a sequence, then (x_n) is divergent if there exists two subsequences (x_{n_k}) and (x_{n_h}) such that

$$\lim_{k\to\infty} x_{n_k} \neq \lim_{h\to\infty} x_{n_h}.$$

We may apply Theorem 2.1.12 with respect to subsequences. Let us first introduce the notion of *peak points*.

Definition 2.1.17 ▶ Peak Point

Let (x_n) be a sequence in $\mathbb{R}$, x_m is called a **peak** if for all $n \in \mathbb{N}$ with $n > m$, $x_m \geq x_n$.

Next, we shall prove that one can find a monotone subsequence from every sequence.

Theorem 2.1.18 ▶ Existence of Monotone Subsequences

Every infinite sequence has an infinite monotone subsequence.

Proof. Let (x_n) be any sequence in $\mathbb{R}$. We consider the following cases:

Case 1. (x_n) has infinitely many peak points.

This means that there exists infinitely many $m_1, m_2, \cdots \in \mathbb{N}$ such that $m_j > m_i$ whenever $j > i$. Therefore, the subsequence (x_{m_n}) is a monotone decreasing sequence.

Case 2. (x_n) has finitely many peak points.
This means that there exists $m_1, m_2, \cdots, m_k \in \mathbb{N}$ such that $x_{m_1}, x_{m_2}, \cdots, x_{m_k}$ are all the peak points of (x_n). Take $N = m_k + 1$, then for all $n_i \geq N$, since x_{n_i} is not a peak point, there exists some $n_{i+1} > n_i$ such that $x_{n_{i+1}} > x_{n_i}$. Therefore, (x_{n_i}) is an increasing sequence. □

With these preparations done, we state the following theorem:

Theorem 2.1.19 ▸ Bolzano-Weierstrass Theorem (simplified ver.)

Every bounded sequence has a convergent subsequence.

Proof. Let (x_n) be a bounded sequence, then by Theorem 2.1.18 we can find some subsequence (x_{n_k}) which is monotone. Note that (x_{n_k}) is also bounded, so by Theorem 2.1.12 it is convergent. □

2.1.3 Cauchy Criterion

Intuitively, if a sequence is convergent, then over a large interval of $\mathbb{N}$, the change in values of its terms will become smaller and smaller. Correspondingly, this means that the adjacent terms of the sequence will get closer and closer as n becomes large.

Definition 2.1.20 ▸ Cauchy Sequence

A sequence (x_n) is said to be a **Cauchy sequence** if for every $\epsilon > 0$, there exists some $H \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ with $n, m \geq H$, $|x_n - x_m| < \epsilon$.

We can make use of the Cauchy sequence to test for convergence.

Theorem 2.1.21 ▸ Cauchy Convergence Criterion

A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.

Proof. Let (x_n) be a sequence in $\mathbb{R}$. Suppose that (x_n) converges to x, then for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that whenever $n > N$, $|x_n - x| < \frac{\epsilon}{2}$. Therefore, for all $m, n > N$, we have

$$\begin{aligned}|x_m - x_n| &= |x_m - x - x_n + x| \\ &\leq |x_m - x| + |x_n - x| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon,\end{aligned}$$

and so (x_n) is a Cauchy sequence.

Suppose conversely that (x_n) is a Cauchy sequence on $\mathbb{R}$. We consider the following lemma:

Lemma 2.1.22 ▸ Boundedness of Cauchy Sequences

A Cauchy sequence in $\mathbb{R}$ is bounded.

Proof. Let (x_n) be a Cauchy sequence, then by Definition 2.1.20, there exists some $H \in \mathbb{N}$ such that for all natural numbers $n \geq H$, $|x_n - x_H| < 1$. By Corollary 1.2.6, we have

$$||x_n| - |x_H|| \leq |x_n - x_H| < 1,$$

and so $|x_n| < |x_H| + 1$. Take

$$m = \max\{|x_1|, |x_2|, \cdots, |x_H|, |x_H| + 1\},$$

then $|x_n| < m$ for all $n \in \mathbb{N}^+$. □

Therefore, by Theorem 2.1.19 there exists a subsequence (x_{m_n}) which converges to some $x \in \mathbb{R}$. Thus, there exists some $M \in \mathbb{N}$ such that $|x_{m_n} - x| < \frac{\epsilon}{2}$ for all $\epsilon > 0$ whenever $m_n > M$. By Definition 2.1.20, there exists some $N \in \mathbb{N}$ such that $|x_n - x_{m_n}| < \frac{\epsilon}{2}$ for all $\epsilon > 0$ and for all $n, m_n > N$. Take $K = \max\{M, N\}$, then whenever $n > K$, there is some $m_n > K$ such that

$$\begin{aligned}|x_n - x| &= |x_n - x_{m_n} + x_{m_n} - x| \\ &\leq |x_n - x_{m_n}| + |x_{m_n} - x| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Therefore, $\lim_{n\to\infty} x_n = x$. $\square$

3 Functions

3.1 Limits of Functions

Definition 3.1.1 ▸ Cluster Point

Let $A \subseteq \mathbb{R}$. A point c is called a **cluster point** of A if for all $\delta > 0$, there exists at least one $x \in A$ such that $0 < |x - c| < \delta$, i.e., $\big(V_\delta(c) - \{c\}\big) \cap A \neq \varnothing$ for all $\delta > 0$.

Intuitively, this means that the elements of a set A is **densely distributed** around the cluster point, which motivates the following alternative definition:

Theorem 3.1.2 ▸ Alternative Definition of Cluster Points

Let $A \subseteq \mathbb{R}$, then $c \in \mathbb{R}$ is a cluster point of A if and only if there exists a sequence (a_n) in A such that $\lim_{n\to\infty} a_n = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

Proof. Suppose that c is a cluster point of A. Fix any $n \in \mathbb{N}^+$, then there exists some $a_n \in A$ such that $0 < |a_n - c| < \frac{1}{n}$. This means we can obtain a sequence (a_n) with $a_n \neq c$ for all $n \in \mathbb{N}^+$.

For any $\epsilon > 0$, note that there exists some $N \in \mathbb{N}^+$ such that $0 < \frac{1}{N} < \epsilon$. Therefore, for all $n \geq N$, we have $|a_n - c| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$, which means that $\lim_{n\to\infty} a_n = c$.

Conversely, suppose that there is a sequence (a_n) in A with $a_n \neq c$ for all $n \in \mathbb{N}^+$ and $\lim_{n\to\infty} a_n = c$. For any $\delta > 0$, there is some $N \in \mathbb{N}^+$ such that for all $n \geq N$, $0 < |a_n - c| < \delta$. Note that $a_n \in A$, so c is a cluster point of A. □

Analogously to the limit of sequences, we may describe the limiting behaviour of a function as the follows:

As x gets arbitrarily close to some point c, the function value $f(x)$ can get as close to a constant L as possible.

The above intuition is formulated formally as follows:

Definition 3.1.3 ▶ Limit of Functions

Let f be a function over some $A \subseteq \mathbb{R}$. If c is a cluster point of A, then $L \in \mathbb{R}$ is called the **limit** of f at c if for all $\epsilon > 0$, there exists some $\delta \in \mathbb{R}$ such that whenever $0 < |x - c| < \delta$, $|f(x) - L| < \epsilon$.