Weiestrass Theorem: S is compact $\implies f$ has a global max and a global min in S.

If f is convex, then $\{\boldsymbol{x}\colon f(\boldsymbol{x}) \le a\}$ is convex.

Epigraph E_f is convex $\iff$ f is convex.

Directional derivative at $\boldsymbol{x}$ along $\boldsymbol{d}$:

$$\nabla f(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{d} = \lim_{\lambda\to 0}\frac{f(\boldsymbol{x}+\lambda\boldsymbol{d}) - f(\boldsymbol{x})}{\lambda}.$$

f is convex if and only if $f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y}-\boldsymbol{x}) \le f(\boldsymbol{y})$.

f is a convex and continuously differentiable function, then $\boldsymbol{x}^*$ is a global minimiser $\iff$ $\nabla f(\boldsymbol{x}^*)^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{x}^*)$.

Eigenvalue Test: If $\boldsymbol{A}$ is a **symmetric real** matrix, then $\boldsymbol{A}$ is positive semidefinite $\iff$ all eigenvalues of $\boldsymbol{A}$ are non-negative.

If $\boldsymbol{A}$ is a **symmetric** matrix, then $\boldsymbol{A}$ is positive definite $\iff$ $\Delta_k < 0$ and negative definite $\iff$ $(-1)^k\Delta_k > 0$.

Taylor's Theorem: If f has continuous 2nd order partial derivatives and if the set

$$[\boldsymbol{x},\boldsymbol{y}] \coloneqq \{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}\colon \lambda\in[0,1]\}$$

is in the interior of D_f, then $\exists \boldsymbol{z}\in[\boldsymbol{x},\boldsymbol{y}]$ s.t.

$$f(\boldsymbol{y}) = f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y}-\boldsymbol{x}) + \frac{1}{2}(\boldsymbol{y}-\boldsymbol{x})^{\mathrm{T}}H_f(\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{x}).$$

H_f is semidefinite $\iff$ f is convex/concave; H_f is definite $\implies$ f is strictly convex/concave; H_f is indefinite $\implies$ f is neither convex nor concave.

Coercive Function: $\lim_{\|\boldsymbol{x}\|\to\infty} f(\boldsymbol{x}) = \infty$.

$\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\| \le \sqrt{2}\,\|\boldsymbol{x}\|_\infty$, where $\|\boldsymbol{x}\|_\infty = \max\{|x_i|\}$.

$\nabla f(\boldsymbol{x}^*) = 0$ and $H_f(\boldsymbol{x}^*)$ is positive definite $\implies$ $\boldsymbol{x}^*$ is a **strict** local minimiser.

If f is convex, then a local minimiser of f is a global minimiser. If f is strictly convex, then it has a unique global minimiser.

If f is convex, then any stationary point of f is a global minimiser.

$q(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ is a quadratic function where $\boldsymbol{Q}$ is symmetric.

If q is defined over a convex set, then $\boldsymbol{x}^*$ is a global minimiser $\iff$ $\boldsymbol{Q}\boldsymbol{x}^* = -\boldsymbol{c}$.

Bisection Method:

$$[a_{k+1}, b_{k+1}] = \begin{cases} \left[a_k, \frac{a_k+b_k}{2}\right], & \text{if } f\left(\frac{a_k+b_k}{2}\right) f(a_k) < 0 \\ \left[\frac{a_k+b_k}{2}, b_k\right], & \text{if } f\left(\frac{a_k+b_k}{2}\right) f(a_k) > 0 \end{cases}$$

Take $x_k = \frac{a_k+b_k}{2}$. At termination, $|x^* - x_k| \le \frac{|a_k-b_k|}{2} \le \epsilon$, so we need

$$k = \left\lceil \frac{\log\left(\frac{b_1-a_1}{\epsilon}\right)}{\log 2} \right\rceil$$

Newton's Method: $x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$ until $|f'(x_k)| < \epsilon$.

Golden Section Method:

Set $[a_0, b_0] = [a, b]$ and take $\alpha = \frac{\sqrt{5}-1}{2}$. Compute

$$\begin{aligned} \lambda_0 &= b - \alpha(b-a) \\ \mu_0 &= a + \alpha(b-a). \end{aligned}$$

If $f(\lambda_k) > f(\mu_k)$, then

$$\begin{aligned} a_{k+1} &= \lambda_k & b_{k+1} &= b_k \\ \lambda_{k+1} &= \mu_k & \mu_{k+1} &= \lambda_k + \alpha(b_k - \lambda_k). \end{aligned}$$