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# Topology

## 1.1 Topological Spaces

### Definition 1.1.1 ► Topology

A **topology** on a set  $X$  is a collection  $\mathcal{T} \subseteq \mathcal{P}(X)$  such that

- $\emptyset, X \in \mathcal{T}$ ;
- for any index set  $I$ , if  $\{X_i : i \in I\} \subseteq \mathcal{T}$ , then  $\bigcup_{i \in I} X_i \in \mathcal{T}$ ;
- for any  $X_1, X_2, \dots, X_n \in \mathcal{T}$ ,  $\bigcap_{i=1}^n X_i \in \mathcal{T}$ .

The pair  $(X, \mathcal{T})$  is said to be a **topological space**. A subset  $Y \subseteq X$  is **open** if  $Y \in \mathcal{T}$ .

*Remark.* For any set  $X$ , we define  $\{\emptyset, X\}$  as the *trivial topology* on  $X$ ,  $\mathcal{P}(X)$  as the *discrete topology*, and  $\{X \setminus U : U \subseteq X \text{ is finite}\} \cup \{\emptyset\}$  as the *co-finite topology*.

The set  $\{(-\alpha, \alpha) : \alpha > 0\} \cup \{\mathbb{R}, \emptyset\}$  defines a topology on  $\mathbb{R}$ . This example also demonstrates why it is crucial to only consider closure under finite intersections when defining a topology, because

$$\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = [-1, 1] \notin \mathcal{T}.$$

We now seek a systematic method to generate a topology given any set. The idea here is to make use of a *cover*.

### Definition 1.1.2 ► Basis

A **basis** for a topology on  $X$  is a collection  $\mathcal{B} \subseteq \mathcal{P}(X)$  such that

- for any  $x \in X$ , there exists some  $B \in \mathcal{B}$  such that  $x \in B$ ;
- for any  $x \in X$  and  $B_1, B_2 \in \mathcal{B}$  with  $x \in B_1 \cap B_2$ , there exists some  $B \in \mathcal{B}$  such that  $x \in B \subseteq B_1 \cap B_2$ .

It may be useful to see a basis as a cover of a set with the second additional property as stated in the above definition. Notice that the first property of the basis  $\mathcal{B}$  is basically saying that

$$X \subseteq \bigcup \mathcal{B},$$

i.e.,  $\mathcal{B}$  is a cover of  $X$ .

Given any basis  $\mathcal{B}$  for some topology on  $X$ , a set generated by  $\mathcal{B}$  can be defined as

$$\mathcal{T} := \{U \subseteq X : \text{for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\}$$

We will show that  $\mathcal{T}$  is a topology on  $X$ . First, it is clear that  $\emptyset, X \in \mathcal{T}$ .

Let  $I$  be an index set and  $\{X_i : i \in I\} \subseteq \mathcal{P}(X)$  be any collection of subsets of  $X$ . Notice that for any  $x \in \bigcup_{i \in I} X_i$ , there exists some  $j \in I$  such that  $x \in X_j \subseteq \mathcal{T}$ . According to our construction, this means that there exists some  $B \in \mathcal{B}$  such that  $x \in B \subseteq X_j \subseteq \mathcal{T}$ . Therefore,  $\bigcup_{i \in I} X_i \subseteq \mathcal{T}$  as desired.

To prove that  $\mathcal{T}$  is closed under finite intersection, we consider the following lemma:

**Lemma 1.1.3 ▶ Finite Intersection of Elements in Basis Is Covered**

*Let  $\mathcal{B}$  be a basis for a topology on  $X$  and  $B_1, B_2, \dots, B_n \in \mathcal{B}$ , then for any  $x \in \bigcap_{i=1}^n B_i$ , there exists some  $B \in \mathcal{B}$  such that  $x \in B \subseteq \bigcap_{i=1}^n B_i$ .*

*Proof.* The case where  $n = 1$  is trivial by taking  $B = B_1$ . Suppose that there is some integer  $k \geq 1$  such that for any  $B_1, B_2, \dots, B_k \in \mathcal{B}$  and any  $x \in \bigcap_{i=1}^k B_i$ , there exists some  $B \in \mathcal{B}$  such that  $x \in B \subseteq \bigcap_{i=1}^k B_i$ . Take any  $B_{k+1} \in \mathcal{B}$ . It is clear that for any  $x \in \bigcap_{i=1}^{k+1} B_i$ , there exists some  $B \in \mathcal{B}$  such that

$$x \in B \subseteq \bigcap_{i=1}^k B_i.$$

Notice that  $x \in B_{k+1} \in \mathcal{B}$ , so we know that  $x \in B \cap B_{k+1}$ . By Definition 1.1.2, this means that there exists some  $B' \in \mathcal{B}$  such that

$$x \in B' \subseteq B \cap B_{k+1} \subseteq \bigcap_{i=1}^{k+1} B_i.$$

□

Now, suppose  $X_1, X_2, \dots, X_n \in \mathcal{T}$  are finitely many subsets of  $X$ . Take any  $x \in \bigcap_{i=1}^n X_i$ . It is clear that  $x \in X_i$  for each  $i = 1, 2, \dots, n$ . Therefore, for each  $i = 1, 2, \dots, n$ , there exists some  $B_i \in \mathcal{B}$  such that  $x \in B_i \subseteq X_i$ . By Lemma 1.1.3, this means that there exists some set  $B \in \mathcal{B}$  such that

$$x \in B \subseteq \bigcap_{i=1}^n B_i \subseteq \bigcap_{i=1}^n X_i.$$

Therefore,  $\bigcap_{i=1}^n X_i \in \mathcal{T}$ . So this set  $\mathcal{T}$  generated by  $\mathcal{B}$  is indeed a topology on  $X$ .

The following proposition further shows that the topology generated by a basis  $\mathcal{B}$  is the set

of all possible unions of elements in  $\mathcal{B}$ :

**Proposition 1.1.4 ▶ Equivalent Construction of Topologies Generated from Bases**

Let  $X$  be any set. If  $\mathcal{B}$  is a basis for a topology  $\mathcal{T}$  on  $X$ , then

$$\mathcal{T} = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \in \mathcal{P}(\mathcal{B}) \right\}.$$

*Proof.* Denote

$$\mathcal{T}_{\mathcal{B}} := \{U \subseteq X : \text{for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\}.$$

It suffices to prove that  $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ . Take any  $T \in \mathcal{T}$ , then there exists some  $\mathcal{V} \in \mathcal{P}(\mathcal{B})$  such that  $T = \bigcup_{A \in \mathcal{V}} A$ . This means that for every  $t \in T$ , there exists some  $B_t \in \mathcal{V}$  such that  $t \in B_t \subseteq T$ . Therefore,  $T \in \mathcal{T}_{\mathcal{B}}$ . Conversely, for any  $S \in \mathcal{T}_{\mathcal{B}}$ , there exists some  $B_s \in \mathcal{B}$  for each  $s \in S$  such that  $s \in B_s$ . Denote  $\mathcal{U} := \{B_s : s \in S\} \in \mathcal{P}(\mathcal{B})$ , then it is clear that  $S \subseteq \bigcup_{B \in \mathcal{U}} B$ . Since  $B_s \subseteq S$  for each  $s \in S$ , we have  $\bigcup_{B \in \mathcal{U}} B \subseteq S$ , which implies that  $S = \bigcup_{B \in \mathcal{U}} B$ . This means that  $S \in \mathcal{T}$ . Therefore,  $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{B}}$  and  $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$ , which means that  $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ .  $\square$

Next, we define a special topology in Euclidean spaces using open balls.

**Definition 1.1.5 ▶ Standard Topology**

For any  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  and any  $r > 0$ . Denote the Euclidean open ball centred at  $\mathbf{x}$  with radius  $r$  by

$$B_r(\mathbf{x}) := \left\{ \mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n : \sqrt{\sum_{i=1}^n (x_i - y_i)^2} < r \right\}$$

The **standard topology** on  $\mathbb{R}^n$  is the set generated by the basis

$$\mathcal{B} := \{B_r(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n, r \in \mathbb{R}^+\}.$$

It may be helpful to actually show that this set  $\mathcal{B}$  is indeed a basis of a topology on  $\mathbb{R}^n$ . The fact that  $\mathcal{B}$  is a cover for  $\mathbb{R}^n$  is trivial enough. Take any  $\mathbf{x} \in \mathbb{R}^n$  and balls  $B_{\alpha}(\mathbf{x}_1), B_{\beta}(\mathbf{x}_2) \in \mathcal{B}$  such that  $\mathbf{x} \in B_{\alpha}(\mathbf{x}_1) \cap B_{\beta}(\mathbf{x}_2)$  (the existence of these 2 balls is again trivial enough). Take

$$r = \min \{\alpha - \|\mathbf{x} - \mathbf{x}_1\|, \beta - \|\mathbf{x} - \mathbf{x}_2\|\}.$$

Clearly,  $r > 0$  and  $\mathbf{x} \in B_r(\mathbf{x})$ , so we are done.

Now, we discuss the analogue of the subset relation in topologies.

### Definition 1.1.6 ► Fineness and Coarseness

Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on some set  $X$ . We say that  $\mathcal{T}$  is **finer** than  $\mathcal{T}'$ , or equivalently, that  $\mathcal{T}'$  is **coarser** than  $\mathcal{T}$ , if  $\mathcal{T}' \subseteq \mathcal{T}$ .

Observe that any topology of  $X$  must be a subset of  $\mathcal{P}(X)$ , which is the discrete topology on  $X$ , so the discrete topology is the finest topology on a set.

*Remark.* For any basis  $\mathcal{B}$  for a topology on  $X$ , the topology generated by  $\mathcal{B}$  is the coarsest topology containing  $\mathcal{B}$ .

The above remark is easy to verify. Let  $\mathcal{T}$  be any topology on  $X$  with  $\mathcal{B} \subseteq \mathcal{T}$  and  $\mathcal{T}_{\mathcal{B}}$  be the topology generated by  $\mathcal{B}$ . For any  $T \in \mathcal{T}_{\mathcal{B}}$ , by Proposition 1.1.4, there exists some  $V \subseteq \mathcal{B}$  such that  $T = \bigcup_{A \in V} A$ . Note that  $A \in \mathcal{T}$  for all  $A \in \mathcal{V}$ , so by Definition 1.1.1,  $T \in \mathcal{T}$  and so  $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$  as desired.

This motivates us to consider fineness in terms of bases.

### Proposition 1.1.7 ► Fineness in Terms of Bases

Let  $\mathcal{B}$  and  $\mathcal{B}'$  generate topologies  $\mathcal{T}$  and  $\mathcal{T}'$  respectively on  $X$ .  $\mathcal{T}'$  is finer than  $\mathcal{T}$  if and only if for every  $B \in \mathcal{B}$  and any  $x \in B$ , there exists some  $B_x \in \mathcal{B}'$  such that  $x \in B_x \subseteq B$ .

*Proof.* Suppose that  $\mathcal{T}'$  is finer than  $\mathcal{T}$ , then  $\mathcal{T} \subseteq \mathcal{T}'$ . Take any  $B \in \mathcal{B}$ , then by Proposition 1.1.4,  $B \in \mathcal{T}$ , which means that  $B \in \mathcal{T}'$ . Since  $\mathcal{B}'$  is a basis for  $\mathcal{T}'$ , by Definition 1.1.2 for any  $x \in B$ , there exists some  $B_x \in \mathcal{B}'$  such that  $x \in B_x \subseteq B$ .

Suppose conversely that for every  $B \in \mathcal{B}$  and any  $x \in B$ , there is some  $B_x \in \mathcal{B}'$  such that  $x \in B_x \subseteq B$ . Take any  $T \in \mathcal{T}$ , for each  $x \in T$ , by Definition 1.1.2 there exists some  $B \in \mathcal{B}$  such that  $x \in B \subseteq T$ , and so we can find some  $B_x \in \mathcal{B}'$  such that  $x \in B_x \subseteq B \subseteq T$ , so  $T \in \mathcal{T}'$ . Therefore,  $\mathcal{T} \subseteq \mathcal{T}'$  and so  $\mathcal{T}'$  is finer than  $\mathcal{T}$ .  $\square$

Recall that every basis of a topology on  $X$  is an open cover of  $X$  consisting only of subsets of  $X$ . Therefore, the union of the elements in the basis is essentially  $X$  itself. This motivates us to propose another way to generate a topology on a set.

### Definition 1.1.8 ► Sub-basis

A **sub-basis** of  $X$  is a collection  $\mathcal{S} \subseteq \mathcal{P}(X)$  such that  $\bigcup_{A \in \mathcal{S}} A = X$ .

*Remark.* Every basis is a sub-basis.

For an arbitrary set  $X$ , let  $\mathcal{S}$  be a sub-basis and denote the collection of all finite subsets of  $\mathcal{S}$  as  $\mathcal{F}_{\mathcal{S}}$ . Define

$$\mathcal{U}_{\mathcal{S}} := \left\{ \bigcap_{A \in F} A : F \in \mathcal{F}_{\mathcal{S}} \right\}$$

to be the collection of all finite intersections of sets in  $\mathcal{S}$ . The topology generated by a sub-basis of  $X$  is given by

$$\mathcal{T} := \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{U}_{\mathcal{S}} \right\}.$$

We shall show that  $\mathcal{T}$  is indeed a topology on  $X$  by considering the following proposition:

**Proposition 1.1.9 ▶ Finite Intersections of Sets in a Sub-basis Form a Basis**

Let  $\mathcal{S}$  be a sub-basis for a set  $X$  and let  $\mathcal{U}_{\mathcal{S}}$  be the set of all finite intersections of sets in  $\mathcal{S}$ , then  $\mathcal{U}_{\mathcal{S}}$  is a basis of a topology on  $X$ .

*Proof.* Take any  $x \in X$ . By Definition 1.1.8, we have  $x \in \bigcup_{A \in \mathcal{S}} A$ . Therefore, there exists some  $A \in \mathcal{S} \subseteq \mathcal{P}(X)$  such that  $x \in A$ . For any  $x \in X$  and  $B_1, B_2 \in \mathcal{U}_{\mathcal{S}}$  such that  $x \in B_1 \cap B_2$ , notice that  $B_1 \cap B_2$  is a finite intersection of sets in  $\mathcal{S}$ , so  $B_1 \cap B_2 \in \mathcal{U}_{\mathcal{S}}$ . Therefore, by Definition 1.1.2,  $\mathcal{U}_{\mathcal{S}}$  is a basis.  $\square$

With Propositions 1.1.9 and 1.1.4, it is clear that  $\mathcal{T}$  as constructed above is a topology on  $X$ .

## 1.2 Metric Spaces

**Definition 1.2.1 ▶ Metric**

A **metric** on a set  $S$  is a function  $d : S \times S \rightarrow \mathbb{R}$  such that:

- $d(x, y) \geq 0$  for all  $x, y \in S$  (positivity);
- $d(x, y) = 0$  if and only if  $x = y$  (definiteness);
- $d(x, y) = d(y, x)$  for all  $x, y \in S$  (symmetry);
- $d(x, y) \leq d(x, z) + d(y, z)$  for all  $x, y, z \in S$  (triangular inequality).

*Remark.* A metric is sometimes also called a *distance function*.

A metric generalises the notion of distance in Euclidean spaces. We can weaken the above axioms to arrive at the following definition:

**Definition 1.2.2 ▶ Pseudo-metric**

A **pseudo-metric** on a set  $S$  is a function  $d : S \times S \rightarrow \mathbb{R}$  such that:

- $d(x, y) \geq 0$  for all  $x, y \in S$  (positivity);
- $d(x, x) = 0$  for all  $x \in S$ ;
- $d(x, y) = d(y, x)$  for all  $x, y \in S$  (symmetry);
- $d(x, y) \leq d(x, z) + d(y, z)$  for all  $x, y, z \in S$  (triangular inequality).

The key difference between a pseudo-metric and a metric is that a pseudo-metric only requires that every element is at 0 distance away from itself, whereas a metric requires that every element is **the only element** that is at 0 distance away from itself.

By dropping the requirement on symmetry, we obtain the following definition:

**Definition 1.2.3 ▶ Quasi-metric**

A **quasi-metric** on a set  $S$  is a function  $d : S \times S \rightarrow \mathbb{R}$  such that:

- $d(x, y) \geq 0$  for all  $x, y \in S$  (positivity);
- $d(x, y) = 0$  if and only if  $x = y$  (definiteness);
- $d(x, y) \leq d(x, z) + d(y, z)$  for all  $x, y, z \in S$  (triangular inequality).

We equip a set with a metric to generalise the Euclidean spaces.

**Definition 1.2.4 ▶ Metric Space**

A **metric space**  $(S, d)$  is a set  $S$  together with a metric  $d$  on  $S$ .

The most basic example of a metric is the *discrete metric* defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

over any set  $X$ , which essentially is just a characteristic function.

Recall that in an inner product space  $(V, g)$  over some field  $\mathbb{F}$ , we can define the length of any  $\mathbf{v} \in V$  as

$$\|\mathbf{v}\| = \sqrt{g(\mathbf{v}, \mathbf{v})}.$$

This length function induces a metric over  $V$  given by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

In the Euclidean space  $\mathbb{R}^n$ , a usual definition for distance is

$$d_2(\mathbf{x}, \mathbf{y}) = \left[ \sum_{i=1}^n (y_i - x_i)^2 \right]^{\frac{1}{2}}.$$

Note that  $(\mathbb{R}^n, d_2)$  is a metric space, where  $d_2$  is known as the *Euclidean distance*. In general, we can prove that for any  $p \in \mathbb{N}^+$ ,

$$d_p(\mathbf{x}, \mathbf{y}) = \left[ \sum_{i=1}^n \|y_i - x_i\|^p \right]^{\frac{1}{p}}$$

is a metric over  $\mathbb{F}^n$  for any inner product space  $(\mathbb{F}^n, g)$  where  $\mathbb{F}$  is a field, known as the  *$L^p$ -norm*. Furthermore, notice that

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p \leq \sum_{i=1}^n \|y_i - x_i\|^p \leq n \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p.$$

Taking the  $p$ -th root on all three parts, we have

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\| \leq \left[ \sum_{i=1}^n \|y_i - x_i\|^p \right]^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

By Squeeze Theorem, this allows us to define

$$d_\infty(\mathbf{x}, \mathbf{y}) = \lim_{p \rightarrow \infty} d_p(\mathbf{x}, \mathbf{y}) = \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

$d_\infty(\mathbf{x}, \mathbf{y})$  can be alternatively written as  $\|\mathbf{x} - \mathbf{y}\|_\infty$ , which is known as the *infinite norm*.

The  $p$ -adic numbers can be defined from the following lemma:

#### Lemma 1.2.5 ► $p$ -adic Numbers

Let  $p$  be any prime number. For all  $x \in \mathbb{Q} \setminus \{0\}$ , there exists a unique  $k \in \mathbb{Z}$  such that

$$x = \frac{p^k r}{s}, \quad r, s \in \mathbb{Z}$$

with  $p \nmid r, s$  and  $s \neq 0$ .

The  $p$ -adic norm is defined as

$$|x|_p = \begin{cases} p^{-k} & \text{if } x = \frac{p^k r}{s}, \\ 0 & \text{if } x = 0 \end{cases},$$



which induces a metric over  $\mathbb{Q}$  defined by

$$d(x, y) = |x - y|_p.$$

We can show that the  $p$ -adic metric satisfies

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

for all  $x, y, z \in \mathbb{Q}$ . Such a metric is known as an *ultra-metric*.

Given any metric space, the metric will induce a distance between subsets of the space.

#### Definition 1.2.6 ► Distance between Subsets

Let  $(X, d)$  be a metric space and  $A, B \subseteq X$  be non-empty. The **distance** between  $A$  and  $B$  is defined as

$$d(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

Additionally, we may wish to define a measure for the size of a subset in a metric space.

#### Definition 1.2.7 ► Diameter

Let  $(X, d)$  be a metric space. The **diameter** of a set  $A \subseteq X$  is defined as

$$\text{diam}(A) := \sup\{d(x, y) : (x, y) \in A \times A\}.$$

The set  $A$  is **bounded** if  $\text{diam}(A)$  is finite.

The name “diameter” is not a coincidence with the diameter of a graph. Specifically, if we consider a graph  $G = (V, E)$ , the pair  $(V, d)$  forms a metric space with  $d(u, v)$  being the usual distance between two vertices in  $G$  defined as the size of the shortest  $u$ - $v$  path in  $G$ . It is clear that  $d$  is indeed a metric.

Now, let us consider the subgraph  $H \subseteq G$  induced by any  $U \subseteq V$  and check the eccentricity for  $H$ , i.e.,

$$\epsilon(u) = \max\{d_H(u, u') : u' \in U\} \quad \text{for all } u \in U.$$

Now, the diameters for  $H$  can be computed as

$$\begin{aligned} \text{diam}(H) &= \max\{\epsilon(u) : u \in U\} \\ &= \sup\{d_H(u, u') : (u, u') \in U \times U\}, \end{aligned}$$

and this obviously agrees with Definition 1.2.7!

Recall that in Definition 1.1.5, we use Euclidean open balls to construct a basis for a topology on  $\mathbb{R}^n$ . We can generalise this idea in any metric space.

### Proposition 1.2.8 ► Metric Induces a Basis

Let  $(X, d)$  be a metric space. Define

$$B_r(x) := \{y \in X : d(x, y) < r\},$$

then collection

$$\mathcal{B}_d := \{B_r(x) : x \in X, r \in \mathbb{R}^+\}$$

is a basis for a topology on  $X$ .

*Proof.* Notice that for any  $x \in X$ , we have  $x \in B_1(x) \in \mathcal{B}_d$ . Let  $B_p(x_1), B_q(x_2) \in \mathcal{B}_d$  be such that  $x \in B_p(x_1) \cap B_q(x_2)$ . Take  $k = \min\{p - d(x, x_1), q - d(x, x_2)\}$ , then clearly  $k > 0$  and we can find  $B_k(x) \subseteq B_p(x_1) \cap B_q(x_2)$  such that  $x \in B_k(x) \in \mathcal{B}_d$ . Therefore,  $\mathcal{B}_d$  is a basis for a topology on  $X$ .  $\square$

Since we can obtain a basis from a metric, it follows naturally that we can generate a topology using this induced basis.

### Definition 1.2.9 ► Metrisable Topology

Let  $(X, d)$  be a metric space. A topology  $\mathcal{T}$  on  $X$  is **metrisable**, or **induced** by  $d$ , if it is generated by  $\mathcal{B}_d$

We can verify that the discrete topology  $\mathcal{P}(X)$  is induced by the discrete metric. Let the discrete metric on  $X$  be  $\chi$ , then it is easy to see that

$$B_r(x) = \begin{cases} \{x\} & \text{if } 0 < r \leq 1 \\ X & \text{if } r > 1 \end{cases}.$$

Therefore,

$$\mathcal{B}_\chi = \{X\} \cup \{\{x\} : x \in X\}.$$

Let  $\mathcal{T}_\chi$  be the topology on  $X$  generated by  $\mathcal{B}_\chi$ , then it suffices to prove that  $\mathcal{P}(X) \subseteq \mathcal{T}_\chi$ . Take any  $U \in \mathcal{P}(X)$ , then for any  $u \in U$ , we have  $u \in \{u\} \subseteq U$ . Clearly  $\{u\} \in \mathcal{B}_\chi$ , so  $\mathcal{T}_\chi = \mathcal{P}(X)$  is the discrete topology indeed.

In particular, for Euclidean spaces, the following result extends Definition 1.1.5:

**Proposition 1.2.10 ▶ Every  $L^p$ -metric Generates the Standard Topology**

Let  $\mathcal{T}$  be the standard topology on  $\mathbb{R}^n$ , then  $\mathcal{T}$  is induced by any  $L^p$ -metric  $d_p$ .

*Proof.* For any  $p \in \mathbb{N}^+$ , notice that

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p \leq \sum_{i=1}^n \|y_i - x_i\|^p \leq n \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p.$$

Taking the  $p$ -th root yields

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\| \leq \left[ \sum_{i=1}^n \|y_i - x_i\|^p \right]^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

This means that

$$d_\infty(\mathbf{x}, \mathbf{y}) \leq d_p(\mathbf{x}, \mathbf{y}) \leq n^{\frac{1}{p}} d_\infty(\mathbf{x}, \mathbf{y}).$$

Let  $\mathcal{T}_0$  and  $\mathcal{T}_p$  be topologies on  $\mathbb{R}^n$  generated by  $\mathcal{B}_{d_\infty}$  and  $\mathcal{B}_{d_p}$  respectively. Take any  $T \in \mathcal{T}_p$ , then for any  $\mathbf{t} \in T$ , there is some  $B_r(\mathbf{t}') \in \mathcal{B}_{d_p}$  such that  $\mathbf{t} \in B_r(\mathbf{t}') \subseteq T$ . Take some  $\ell = \frac{1}{2}|r - d_p(\mathbf{t}, \mathbf{t}')|$ , then we have found  $B_\ell(\mathbf{t}) \in \mathcal{B}_{d_p}$  such that

$$\mathbf{t} \in B_\ell(\mathbf{t}) \subseteq B_r(\mathbf{t}') \subseteq T.$$

Take  $k = \frac{1}{2}\ell n^{-\frac{1}{p}}$  and consider

$$B_k(\mathbf{t}) := \{\mathbf{y} \in \mathbb{R}^n : d_\infty(\mathbf{t}, \mathbf{y}) < k\} \in \mathcal{B}_{d_\infty}.$$

Notice that for each  $\mathbf{y} \in B_k(\mathbf{t})$ , we have

$$d_p(\mathbf{t}, \mathbf{y}) \leq n^{\frac{1}{p}} d_\infty(\mathbf{t}, \mathbf{y}) < \ell,$$

so  $\mathbf{t} \in B_k(\mathbf{t}) \subseteq B_\ell(\mathbf{t}) \subseteq T$ . This implies that  $T \in \mathcal{T}_0$  and so  $\mathcal{T}_p \subseteq \mathcal{T}_0$ . By a similar argument, one may check that  $\mathcal{T}_0 \subseteq \mathcal{T}_p$ . Therefore,  $\mathcal{T}_0 = \mathcal{T}_p$  for any  $p \in \mathbb{N}^+$ . Note that by Definition 1.1.5,  $\mathcal{T}$  is generated by  $\mathcal{B}_{d_2}$ , which means that  $\mathcal{T} = \mathcal{T}_2 = \mathcal{T}_0 = \mathcal{T}_p$  for any  $p \in \mathbb{N}^+$ . Therefore,  $\mathcal{T}$  is induced by any  $L^p$ -metric  $d_p$ .  $\square$

The fact that

$$d_\infty(\mathbf{x}, \mathbf{y}) \leq d_p(\mathbf{x}, \mathbf{y}) \leq n^{\frac{1}{p}} d_\infty(\mathbf{x}, \mathbf{y})$$

means that all  $L^p$ -metrics are equivalent over the same space.

## 1.3 Subspace Topologies

### Definition 1.3.1 ► Subspace Topology

Let  $(Y, \mathcal{T}_Y)$  be a topological space and  $X \subseteq Y$  be some subset. The collection

$$\mathcal{T}_X := \{U \cap X : U \in \mathcal{T}_Y\}$$

is the **subspace topology** on  $X$ .

We may check that  $\mathcal{T}_X$  defined as such is indeed a topology on  $X$ . First, by taking  $U = \emptyset$  and  $U = Y$  respectively, we know that  $\emptyset, X \in \mathcal{T}_X$ . For any  $U \in \mathcal{T}_Y$ , we have  $Y \setminus U \in \mathcal{T}_Y$  and so

$$X \setminus (U \cap X) = (Y \setminus U) \cap X \in \mathcal{T}_X.$$

For any  $\mathcal{V} \subseteq \mathcal{T}_X$ , we define a subset  $\mathcal{U}_{\mathcal{V}} \subseteq \mathcal{T}_Y$  such that for each  $V \in \mathcal{V}$  there is a unique  $U_V \in \mathcal{U}_{\mathcal{V}}$  such that  $V = U_V \cap X$ . Then,

$$\begin{aligned} \bigcup_{A \in \mathcal{V}} A &= \bigcup_{B \in \mathcal{U}_{\mathcal{V}}} (B \cap X) \\ &= \left( \bigcup_{B \in \mathcal{U}_{\mathcal{V}}} B \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

Let  $X_1, X_2, \dots, X_n \in \mathcal{T}_X$  and define  $X_i = U_i \cap X$  where  $U_i \in \mathcal{T}_Y$  for  $i = 1, 2, \dots, n$ , then

$$\begin{aligned} \bigcap_{i=1}^n X_i &= \bigcap_{i=1}^n (U_i \cap X) \\ &= \left( \bigcap_{i=1}^n U_i \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

So  $\mathcal{T}_X$  is really a topology on  $X$ . Intuitively, the following holds:

### Proposition 1.3.2 ► Basis for a Subspace

Let  $(Y, \mathcal{T}_Y)$  be a topological space and  $\mathcal{T}_X$  be the subspace topology on some  $X \subseteq Y$ . If  $\mathcal{B}_Y$  is a basis of  $\mathcal{T}_Y$ , then

$$\mathcal{B}_X := \{B \cap X : B \in \mathcal{B}_Y\}$$

is a basis of  $\mathcal{T}_X$ .

*Proof.* We first prove that  $\mathcal{B}_X$  is a basis. Take any  $x \in X \subseteq Y$ . Note that there exists some  $B \in \mathcal{B}_Y$  such that  $x \in B$ . Take  $B \cap X \in \mathcal{B}_X$ , then  $x \in B \cap X$ . For any  $B_1, B_2 \in \mathcal{B}_X$  with  $x \in B_1 \cap B_2$ , we write  $B_1 := B'_1 \cap X$  and  $B_2 := B'_2 \cap X$  where  $B'_1, B'_2 \in \mathcal{B}_Y$ , then we have  $x \in B'_1 \cap B'_2$ . This means that there is some  $B \in \mathcal{B}_Y$  such that  $x \in B \subseteq B'_1 \cap B'_2$ . Write  $B' := B \cap X \in \mathcal{B}_X$ , then for each  $b \in B'$ , we know that  $b \in B'_1 \cap B'_2$  and  $b \in X$ , which implies that  $b \in B_1 \cap B_2$ . Therefore,  $x \in B' \subseteq B_1 \cap B_2$ . This means that  $\mathcal{B}_X$  is a basis of a topology on  $X$ .

We then prove that  $\mathcal{T}_X$  is generated by  $\mathcal{B}_X$ . Let  $\mathcal{T}$  be the topology generated by  $\mathcal{B}_X$ . By Proposition 1.1.4, we have

$$\mathcal{T} = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{B}_X \right\}.$$

Similarly, we can write

$$\mathcal{T}_Y = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{B}_Y \right\}.$$

Take any  $T \in \mathcal{T}_X$ , then there exists some  $\mathcal{V} \subseteq \mathcal{B}_Y$  such that

$$\begin{aligned} T &= \left( \bigcup_{A \in \mathcal{V}} A \right) \cap X \\ &= \bigcup_{A \in \mathcal{V}} A \cap X \\ &\in \mathcal{T}. \end{aligned}$$

Therefore,  $\mathcal{T}_X \subseteq \mathcal{T}$ . Conversely, take any  $T' \in \mathcal{T}$ , there exists some  $\mathcal{U} \subseteq \mathcal{B}_Y$  such that

$$\begin{aligned} T' &= \bigcup_{B \in \mathcal{U}} (B \cap X) \\ &= \left( \bigcup_{B \in \mathcal{U}} B \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

Therefore,  $\mathcal{T} \subseteq \mathcal{T}_X$  and so  $\mathcal{T}_X = \mathcal{T}$ . □

The following result shows that open sets in subspaces remain open in the superspace:

**Proposition 1.3.3 ▶ Superspace Preserve Open Sets**

Let  $(Y, \mathcal{T}_Y)$  be a topological space. If  $X \subseteq Y$  is open in  $Y$  and  $U \subseteq X$  is open in  $X$ , then  $U$  is open in  $Y$ .

*Proof.* Let  $\mathcal{T}_X$  be the subspace topology on  $X$ . Since  $U$  is open in  $X$ , we have  $U \in \mathcal{T}_X$ . By Definition 1.3.1, there exists some  $V \in \mathcal{T}_Y$  such that  $U = V \cap X$ . However,  $U \subseteq X$ , so  $U = V \in \mathcal{T}_Y$ , which means that  $U$  is open in  $Y$ .  $\square$

We can do a similar manipulation with metric spaces and induce a metric on a subspace.

**Definition 1.3.4 ▶ Subspace Metric**

Let  $(X, d)$  be a metric space. The **subspace metric** of some  $A \subseteq X$  is the restriction of  $d$  to  $A$ , denoted as

$$d_A(x, y) = d(x, y), \quad \text{for all } x, y \in A.$$

Naturally, the following result is true:

**Proposition 1.3.5 ▶ Subspace Metric Induces Subspace Topology**

Let  $(X, d)$  be a metric space. The topology induced by the subspace metric  $d_A$  on some subspace  $A \subseteq X$  is the subspace topology on  $A$ .

*Proof.* Let  $\mathcal{T}_d$  and  $\mathcal{T}_{d_A}$  be topologies induced by  $d$  on  $X$  with basis  $\mathcal{B}_d$  and by  $d_A$  on  $A$  with basis  $\mathcal{B}_{d_A}$  respectively. Let  $\mathcal{T}_A$  be the subspace topology on  $A$  with basis  $\mathcal{B}_A$ . Take any  $B \in \mathcal{B}_{d_A} \subseteq \mathcal{B}_d$ , then clearly any  $x \in B$  is such that  $x \in B \cap A \in \mathcal{B}_A$ . Therefore, by Proposition 1.1.7,  $\mathcal{T}_{d_A} \subseteq \mathcal{T}_A$ . Conversely, take any  $B_A \in \mathcal{B}_A$ . For any  $x \in B_A$ , there exists some  $B_1 \in \mathcal{B}_d$  such that  $x \in B_1$  and  $B_A = B_1 \cap A$ . Notice that this implies that  $x \in A \in \mathcal{T}_{d_A}$ , so we can find some  $B_2 \in \mathcal{B}_{d_A} \subseteq \mathcal{B}_d$  such that  $x \in B_2$ . By Definition 1.1.2, there exists some  $B \subseteq B_1 \cap B_2 \in \mathcal{B}_{d_A}$  such that  $x \in B$ . By Proposition 1.1.7, this means that  $\mathcal{T}_A \subseteq \mathcal{T}_{d_A}$ . Therefore,  $\mathcal{T}_A = \mathcal{T}_{d_A}$ .  $\square$