P & C:
$P_r^{n+1} = P_r^n + rP_{r-1}^n$.

Circular permutation: $Q_r^n = \frac{P_r^n}{r}$.

$C_r^{n+1} = C_{r-1}^n + C_r^n$.

$H_r^n = C_r^{r+n-1}$.

Arrange r distinct objects around n identical circles such that no circle is empty: $s(r,n) = s(r-1,n-1) + (r-1)s(r-1,n)$.

$s(r, r-1) = C_2^r$.

Binomial & Multinomial:
$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$.
$\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$.
$\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}$.

Vandermonde's Identity:

$$\sum_{i=0}^{r}\left[\binom{m}{i}\binom{n}{r-i}\right] = \binom{m+n}{r}.$$

Chu Shih-Chieh Identity:

$$\sum_{i=0}^{n-r}\binom{r+i}{r} = \binom{n+1}{r+1}$$

$$\sum_{i=0}^{k}\binom{r+i}{i} = \binom{r+k+1}{k}.$$

Multinomial coefficient:

$$\binom{n}{n_1, n_2, \cdots, n_m} = \frac{n!}{\prod_{i=1}^m n_i!}.$$

Pigeonhole Principle: If at least $kn+1$ objects are distributed into n distinct sets, then there exists a set with at least $k+1$ objects.

Generalised PP: If at least $\sum_{i=1}^n k_i + (n-1)$ distinct objects are distributed into n distinct sets, then there exists at least one set (i-th) with at least k_i objects.

Ramsey Numbers: $R(p,q) \le R(p-1,q) + R(p,q-1)$. (Bound lowered by 1 if both on RHS are even.)

$R(2,q) = q$, $R(1,q) = 1$.

Distribution Problems:
Distinct into distinct:

- Each box at most 1: P_r^n.
- Each box any number of objects: n^r.
- Each box any number of objects with internal ordering: $\frac{(n-1+r)!}{(n-1)!}$.

Identical into distinct:

- Each box any number of objects: $H_r^n = C_r^{n+r-1}$.
- No box empty: $H_{r-n}^n = C_{r-n}^{r-1}$.

Distinct into identical:

- No box empty: $S(r,n) = S(r-1,n-1) + nS(r-1,n)$

Number of partitions of A with $|A| = n$: $\sum_{i=1}^n S(n,i)$.

Number of surjective mapping from $[1,r] \cap \mathbb{N}$ to $[1,n] \cap \mathbb{N}$: $F(r,n) = \sum_{k=0}^n (-1)^k C_k^n (n-k)^r$.

$S(r,n) = \frac{1}{n!}F(r,n)$.

$D(n,r,k) = \frac{C_k^r}{(n-r)!}\sum_{i=0}^{r-k}(-1)^i C_i^{r-k}(n-k-i)!$.

$D_n = D(n,n,0) = n!\sum_{i=0}^n \frac{(-1)^i}{i!}$.

$\varphi(n) = n\prod_{i=1}^k \left(1 - \frac{1}{p_i}\right)$.

Identicial into Identical

Number of partitions of n into k parts equals the number of partition of k where the largest part has size k.

$\sum_{k=1}^m p(n,k) = p(n+m,m)$.

GPIE:

$$E(m) = \sum_{k=m}^{q}(-1)^{k-m}C_m^k\omega(k).$$

OGF:

- $\binom{\alpha}{r} = \frac{\prod_{i=0}^{r-1}(\alpha - i)}{r!}$.
- $(1 \pm x)^\alpha = \sum_{r=0}^\infty \binom{\alpha}{r}(\pm x)^r$.
- $\frac{1}{1-kx}$ generates $(1, k, k^2, \cdots)$.
- $(1-x)^{-n} = \sum_{i=0}^\infty \binom{n-1+i}{i} x^i$.
- $\alpha A(x) + \beta B(x)$ generates $\alpha a_r + \beta b_r$.
- $A(x)B(x)$ generates $\sum_{i=0}^r a_i b_{r-i}$.
- $x^m A(x)$ translates a_i to a_{i+m}.
- $A(kx)$ generates $k^r a_r$.
- $(1-x)A(x)$ generates $c_r = a_r - a_{r-1}$.
- $\frac{A(x)}{1-x}$ generates $c_r = \sum_{i=0}^r a_i$.
- $A'(x)$ generates $(r+1)a_{r+1}$.
- $xA'(x)$ generates ra_r.
- r-combination of multi-set: $\prod_{i=1}^k \left(\sum_{j=0}^{n_i} x^j\right)$.
- r-partition of n of size at most k: $\frac{1}{\prod_{i=1}^k (1-x^i)}$.

EGF:

- $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ generates $a_r = 1$.
- $\frac{1}{1-x} = \sum_{i=0}^{\infty} x_i$ generates $a_r = r!$.
- $e^{kx} = \sum_{i=0}^{\infty} \frac{(kx)^i}{i!}$ generates $a_r = k^r$.
- $(1+x)^n = \sum_{i=0}^{n} i! C_r^n \frac{x^i}{i!}$ generates P_r^n.
- r-permutation of multi-set: $\prod_{i=1}^{k} \left(\sum_{j=0}^{n_i} \frac{x^j}{j!} \right)$
- $\frac{e^x + e^{-x}}{2}$: even number of elements.
- $\frac{e^x - e^{-x}}{2}$: odd number of elements.

Particular solutions:

$f(n)$	$a_n^{(p)}$
Ak^n	$\begin{cases} Bk^n & k \text{ not a root} \\ Bn^m k^n & k \text{ has multiplicity } m \end{cases}$
$\sum_{i=0}^{t} p_i n^i$	$\begin{cases} \sum_{i=0}^{t} q_i n^i & 1 \text{ not a root} \\ n^m \sum_{i=0}^{t} p_i n^i & 1 \text{ multiplicity } m \end{cases}$
$An^t k^n$	$\begin{cases} \left(\sum_{i=0}^{t} q_i n^i\right) k^n \\ n^m \left(\sum_{i=0}^{t} p_i n^i\right) k^n \end{cases}$

Graph:

- **Handshaking Lemma**: $\sum d_G(v) = 2e(G)$. The number of vertices with odd degrees is even.
- A subgraph H of G is induced iff $H = G - (V(G) - V(H))$.
- $G \cong H$ iff $\overline{G} \cong \overline{H}$.
- If $G \cong H$, then
  - G and H have same order and size.
  - $\delta(G) = \delta(H)$ and $\Delta(G) = \delta(H)$.
  - Number of vertices with degree i is the same.
- $(d_1, d_2, \cdots, d_n)$ is graphic iff $(d_2 - 1, \cdots, d_{d_1+1} - 1, d_{d_1+2}, \cdots, d_n)$ is graphic.
- Order of self-complementary G is either $4k$ or $4k+1$.
- $\exists u$-v walk of length $k \implies \exists u$-v path of length at most k.
- $\omega(G)$: number of components of G.
- Complement of connected graph is connected.
- v is a cut-vertex iff $\exists a, b$ such that v is in every a-b path.
- e is a bridge iff it is not part of any cycle.
- uv is a bridge and u is not end vertex $\implies u$ is a cut-vertex.
- A graph with order at least 3 which contains a bridge contains a cut-vertex.
- $\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2\operatorname{rad}(G)$.
- Incidence matrix is $v(G)$ rows $e(G)$ columns.
- (i,j) entry of $\boldsymbol{A}^k$ is the number of v_i-v_j walks of length k.
- G is connected if and only if the (i,j) entry of $\sum_{i=1}^{n-1}$ is nonzero for all $i \neq j$.
- Size of bipartite is sum of degrees of any partite set.
- Join: $V(G+H) = V(G) \cup V(H)$, $E(G,H) = E(G) \cup E(H) \cup \{uv \colon u \in V(G), v \in V(H)\}$.
- Bipartite iff no odd cycles.
- T is a tree iff every two vertices are joined by a unique path iff $e(T) = v(T) - 1$.
- If $\Delta(T) = k$, then $n_1 = 2 + \sum_{i=1}^{k-2} i n_{i+2}$.
- A tree of order at least 2 contains at least 2 end vertices.
- The centre of a tree is either K_1 or K_2.
- A graph is connected iff it contains a spanning tree.
- $\tau(G) = \tau(G - e) + \tau(G \circ e)$.
  - $\tau(C_n) = n$.
  - Connected with a cut-vertex or a bridge: $\tau(G) = \tau(G_1)\tau(G_2)$.
  - C_p and C_q sharing a commen edge: $\tau(G) = p + q - 2 + (p-1)(q-1)$.
  - C_p with a duplicated edge: $\tau(G) = 2p - 1$.
  - C_p and C_q sharing a pair of duplicated edges: $\tau(G) = p + q - 2 + 2(p-1)(q-1)$.
  - $\tau(K_n) = n^{n-2}$.
  - $\tau(K_{2,r}) = 2^{r-1} r$.
- **Matrix tree theorem:** $\tau(G)$ is the cofactor of any entry of $\boldsymbol{C} - \boldsymbol{A}$ where $\boldsymbol{C}$ is a diagonal matrix containing the degrees of vertices.