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The Real Numbers

1.1 Fields

Definition 1.1.1 ▶ Field

A set F with two binary operations, namely addition and multiplication, is called a **field** if it satisfies the following axioms:

1. $\forall a, b \in F, a + b = b + a$.
2. $\forall a, b, c \in F, (a + b) + c = a + (b + c)$.
3. $\exists 0_F \in F$ such that $\forall a \in F, 0_F + a = a + 0_F = a$.
4. $\forall a \in F, \exists a' \in F$ such that $a + a' = 0_F$.
5. $\forall a, b \in F, a \cdot b = b \cdot a$.
6. $\forall a, b, c \in F, (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
7. $\forall a, b, c \in F, a \cdot (b + c) = a \cdot b + a \cdot b$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
8. $\exists 1_F \in F$ such that $\forall a \in F, 1_F \cdot a = a \cdot 1_F = a$.
9. $\forall a \in F, \exists a' \in F$ such that $a \cdot a' = 1_F$.

If we denote addition by "$+_F$" and multiplication by "$\cdot_F$" or "$\times_F$", then we can denote the field over F by $(F, +_F, \cdot_F)$ or $(F, +_F, \times_F)$.

1.1.1 Ordered Fields

Definition 1.1.2 ▶ Total Order

A **total order** on a set X is a binary relation R over X such that for all $a, b, c \in X$:

1. aRa.
2. aRb and bRc implies aRc.
3. aRb and bRa implies $a = b$.
4. aRb or bRa.

Definition 1.1.3 ▶ Strict Total Order

A **strict total order** on a set X is a binary relation R over X such that for all $a, b, c \in X$:

1. $(a, a) \notin R$.
2. aRb implies $(b, a) \notin R$.

3. aRb and bRc implies aRc.
4. trichotomy is satisfied.

Define a field $(\mathbb{R}, +, \times)$ together with the total order $\leq$ (or the strict total order $<$), then clearly this field is an ordered field, which is known as the real numbers.

Theorem 1.1.4

If $a, b \in \mathbb{R}$, then $-ab + ab = 0$.

Theorem 1.1.5

For all $a \in \mathbb{R}$ with $a \neq 0$, $a^2 > 0$.

Theorem 1.1.6

If $a \in \mathbb{R}$ is such that $0 \leq a < \epsilon$ for all $\epsilon \in \mathbb{R}^+$, then $a = 0$.

Theorem 1.1.7 ▸ Bernoulli's Inequality

If $x > -1$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.