Contents

Non-linear Programming Problems

1.1   Basic Terminology and Notations

Definition 1.1.1 ▶ General Non-linear Programming (NLP) Problems

Define the function $f : \mathbf{R}^n \to \mathbf{R}$. Let $\boldsymbol{x} \in \mathbf{R}^n$ be a vector, then a general NLP problem aims to **optimise** (i.e. maximise or minimise) $f(\boldsymbol{x})$ subject to the constraint $\boldsymbol{x} \in S \subseteq \mathbf{R}^n$, where

- f is known as the **objective function**;
- S is known as the **feasible set**;
- A solution (point) $\boldsymbol{x} \in S$ is known as a **feasible solution (point)**. Otherwise, it is known as an **infeasible solution(point)**.

Remark. Note that to maximise $f(\mathbf{x})$ is equivalent to minimising $-f(\mathbf{x})$, so it suffices to only study minimisation problems.

The word "optimal", however, can be ambiguous due to its qualitative nature. Thus, we shall define what it means to be optimal quantitatively with more rigorous terms.

Definition 1.1.2 ▶ Optimal Solution

Consider a minimisation problem subject to constraint $\boldsymbol{x} \in S \subseteq \boldsymbol{R}^n$ whose objective function is $f(\boldsymbol{x})$. A feasible solution $\boldsymbol{x}^*$ is called an **optimal solution** if $f(\boldsymbol{x}^*) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in S$. We can write

$$\boldsymbol{x}^* = \operatorname*{argmin}_{\boldsymbol{x} \in S} f(\boldsymbol{x}).$$

$f(\boldsymbol{x}^*)$ is then known as the **optimal value**.

Remark. For maximisation problems, we can write

$$\boldsymbol{x}^* = \operatorname*{argmax}_{\boldsymbol{x} \in S} f(\boldsymbol{x})$$

Note that not all optimisation problems have an optimal solution. We shall still expect to encounter problems for which no optimal solution nor value exists.

> **Definition 1.1.3 ▶ Unboundedness**
>
> Consider a minimisation problem subject to constraint $x \in S \subseteq R^n$ whose objective
> function is $f(x)$. The objective value is said to be **unbounded** if for all $K \in \mathbf{R}$, there
> exists some $x \in S$ such that $f(x) < K$.

1.2 Unconstrained Non-linear Programs

To introduce the notion of an unconstrained NLP, we shall first define the openness of a
set.

> **Definition 1.2.1 ▶ Open Set**
>
> Let $S \subseteq R^n$ be a set. S is called **open** if for all $x \in S$ there exists $\epsilon > 0$ such that the
> ball
> $$B(x, \epsilon) \coloneqq \{y \in R^n : \ \|y - x\| < \epsilon\}$$
> is a subset of S.

> **Definition 1.2.2 ▶ Unconstrained NLP**
>
> An **unconstrained** NLP is an NLP whose feasible set $\mathcal{X}$ is an **open** subset of $\mathbf{R}^n$.

1.3 Constrained Non-linear Programs

Similarly, to introduce the notion of a constrained NLP, we shall first define the closed-ness
of a set.

> **Definition 1.3.1 ▶ Closed Set**
>
> Let $S \subseteq R^n$ be a non-empty set. S is said to be **closed** if for all convergent sequences
> $\{x_i\}_{i=1}^{\infty}$ with $x_i \in S$ for $i = 1, 2, \cdots$, the limit $\lim_{i \to \infty} x_i \in S$.

The empty set and Euclidean spaces $\mathbb{R}^n$ are both open and closed.

> *Remark.* Note that a set which is not open may not necessarily be closed. However,
> a set is open if and only if its complement is closed.

Theorem 1.3.2 ▶ Intersection of Closed Sets

If C_1 and C_2 are both closed, then $C_1 \cap C_2$ is closed.

Proof. The case where $C_1 \cap C_2 = \varnothing$ is trivial. If $C_1 \cap C_2 \neq \varnothing$, let $\{x_i\}_{i=1}^{\infty}$ be an arbitrary convergent sequence in $C_1 \cap C_2$. Since $\{x_i\}_{i=1}^{\infty} \in C_1$ which is closed, we have $\lim_{i \to \infty} x_i \in C_1$. Similarly, $\lim_{i \to \infty} x_i \in C_2$. Therefore, $\lim_{i \to \infty} x_i \in C_1 \cap C_2$.

Therefore, $C_1 \cap C_2$ is closed. □

We then follow up by introducing three important closed sets.

Theorem 1.3.3

Let $g : \mathbf{R}^n \to \mathbf{R}$ be a continuous function, then the sets

$$S_1 = \{x \in \mathbf{R}^n : g(x) \leq 0\},$$
$$S_2 = \{x \in \mathbf{R}^n : g(x) \geq 0\},$$
$$S_3 = \{x \in \mathbf{R}^n : g(x) = 0\}$$

are closed.

Proof. Consider S_1. Let $S_1' = \mathbf{R}^n - S_1$, then

$$S_1' = \{x \in \mathbf{R}^n : g(x) > 0\}.$$

□

Definition 1.3.4 ▶ Constrained NLP

A **constrained** NLP is an NLP whose feasible set

$$S := \left\{ x \in \mathbf{R}^n : g_i(x) = 0, i = 1, 2, \cdots, p, h_j(x) \leq 0, j = 1, 2, \cdots, q \right\}$$

is **closed**, where each of the g_i's is known as an equality constraint and each of the h_j's is known as an inequality constraint.