Contents

1 Differential Equations

1.1 Initial Value Problems

Definition 1.1.1 ▸ Differential Equation

A **differential equation** is an equation relating derivatives of a differentiable function. The **order** of a differential equation is the order of the highest order derivative.

Remark. A differential equation is said to be an **ordinary** differential equation (ODE) if it contains only a single independent variable.

A *solution* to a differential equation is naturally a function f which satisfies the given equation in its derivatives. Intuitively, if a function $y = f(x)$ is a solution, then $y = f(x) + c$ for any constant c will also be a solution.

Definition 1.1.2 ▸ Solutions to a Differential Equation

A function $y = f(x)+c$ where c is an arbitrary constant is called a **general solution** to a differential equation if it satisfies the equation. A **particular solution** $y_p = f(x)+b$ is obtained by fixing a value for the constant c.

Differential equations are solved by integration. Notice that in general, we always obtain a family of solutions to an ODE due to the arbitrary constant terms. However, a unique solution may be fixed if an *initial value* is provided, i.e., an additional constraint that $f(x_0) = a$.

Definition 1.1.3 ▸ Initial Value Problem

An **initial value problem** is in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad \text{such that } \left. y\right|_{x=x_0} = y_0.$$

For ODEs, we can view f in 1.1.3 as a function from Ω to $\mathbb{R}^n$, where $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n$, and therefore the initial condition is equivalent to saying that the point (x_0, y_0) is in Ω.

An initial value problem involving higher orders of derivatives can always be reduced to the above standard form via the obvious manner.

1.2 Existence and Uniqueness Theorems

Intuitively, for any initial value problem, its solution should be unique if it exists. We will prove this rigorously in this section. Recall from real analysis the following notion:

Definition 1.2.1 ▶ Lipschitz Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is said to be **Lipschitz continuous** if there exists some positive real constant K such that

$$d_Y\big(f(x_1), f(x_2)\big) \leq K d_X(x_1, x_2)$$

for any $x_1, x_2 \in X$.

Note that in Euclidean spaces, the Lipschitz condition can be written as

$$\|f(x_1) - f(x_2)\| \leq K\|x_1 - x_2\|.$$

We shall now prove that given Lipschitz continuity, the uniqueness of solutions to initial value problems is guaranteed.

Theorem 1.2.2 ▶ The Uniqueness Theorem

Let $y : \mathbb{R} \to \mathbb{R}^n$ be a function. Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\big(x, y(x)\big), \qquad \textit{such that } y(x_0) = y_0.$$

If f is locally Lipschitz continuous in y at y_0, then the solution to the initial value problem is unique if it exists.

Proof. Since f is locally Lipschitz continuous at y_0, there exists some $\delta > 0$ such that for any $y_1, y_2 \in V_\delta(y_0)$, there exists some $K > 0$ such that

$$\Big\|f\big(x_1, y_1\big) - f\big(x_2, y_2\big)\Big\| \leq K\|y_1 - y_2\|$$

for all $x_1, x_2 \in \mathbb{R}$. Suppose on contrary that there exists two distinct solutions α_1, α_2 such that

$$\alpha_1(x_0) = \alpha_2(x_0) = y_0.$$

Note that y is continuous. Without loss of generality, there exists some $\epsilon > 0$ such that $K\epsilon \leq \frac{1}{2}$ and $\alpha_1(x) \neq \alpha_2(x) \in V_\delta(y_0)$ whenever $x \in [x_0, \epsilon]$. For any $t \in [x_0, \epsilon]$, consider

$$\begin{aligned}
\|\alpha_1(t) - \alpha_2(t)\| &= \left\| \int_{x_0}^{t} f\big(x, \alpha_1(x)\big)\,\mathrm{d}x - \int_{0}^{t} f\big(x, \alpha_2(x)\big)\,\mathrm{d}x \right\| \\
&\le \int_{x_0}^{t} \left\| f\big(x, \alpha_1(x)\big) - f\big(x, \alpha_2(x)\big) \right\| \mathrm{d}x \\
&\le K \int_{x_0}^{t} \|\alpha_1(x) - \alpha_2(x)\|\,\mathrm{d}x \\
&\le K \int_{x_0}^{\epsilon} \max_{x \in [0,\epsilon]} \|\alpha_1(x) - \alpha_2(x)\|\,\mathrm{d}x \\
&= K\epsilon \max_{x \in [x_0,\epsilon]} \|\alpha_1(x) - \alpha_2(x)\| \\
&\le \frac{1}{2} \max_{x \in [x_0,\epsilon]} \|\alpha_1(x) - \alpha_2(x)\|.
\end{aligned}$$

This implies that $\|\alpha_1(t) - \alpha_2(t)\| = 0$ for all $t \in [x_0, \epsilon]$, which is a contradiction. □

We shall introduce the notion of a *contraction* in order to generalise our results later for all metric spaces.

Definition 1.2.3 ▸ Contraction

Let (X, d) be a metric space. A map $T : X \to X$ is called a **contraction** if there exists some $q \in [0, 1)$ such that

$$d\big(T(x), T(y)\big) \le q d(x, y)$$

for all $x, y \in X$.

It is clear that any contraction is uniformly continuous (the proof is left to the reader as an revision exercise). The following theorem guarantees an *invariant point* under a contraction map:

Theorem 1.2.4 ▸ Banach Fixed Point Theorem

Let (X, d) be a non-empty complete metric space and $T : X \to X$ be a contraction map, then there exists a unique $x^* \in X$ such that $x^* = T(x^*)$.

Proof. By Definition 1.2.3, there exists some $q \in [0, 1)$ such that for all $n \ge 1$,

$$d(x_n, x_{n-1}) \le q^n d(x_1, x_0).$$

Notice that for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$q^n < \frac{\epsilon(1-q)}{d(x_1, x_0)}$$

whenever $n \geq N$. Therefore, for any $m, n \in \mathbb{N}$ with $m > n \geq N$, we have

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=0}^{m-n-1} d(x_{m-i}, x_{m-i-1}) \\ &\leq q^n d(x_1, x_0) \sum_{i=0}^{m-n-1} q^i \\ &\leq q^n d(x_1, x_0) \sum_{i=0}^{\infty} q^i \\ &= \frac{q^n}{1-q} d(x_1, x_0) \\ &< \epsilon. \end{aligned}$$

Therefore, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and so it converges to some $x^* \in X$ because X is complete. Since T is continuous, we have

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T\left(\lim_{n\to\infty} x_{n-1}\right) = T(x^*).$$

Now, suppose on contrary that there exists some $y^* \in X$ with $y^* = T(y^*) \neq x^*$, then

$$d\big(T(x^*), T(y^*)\big) = d(x^*, y^*) > qd(x^*, y^*),$$

which is a contradiction. Therefore, x^* is unique. $\square$

Now let us revisit our initial value problem. Since $\frac{\mathrm{d}y}{\mathrm{d}x} = f\big(x, y(x)\big)$, we have

$$\int_{x_0}^{x} y'(t)\,\mathrm{d}t = \int_{x_0}^{x} f\big(t, y(t)\big)\,\mathrm{d}t.$$

By applying $y(x_0) = y_0$, we have derived a general formula for y as follows:

$$y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big)\,\mathrm{d}t.$$

Now, define a map

$$\Phi : \ (\mathbb{R}^n)^{\mathbb{R}} \to (\mathbb{R}^n)^{\mathbb{R}}$$

whose explicit form is given by

$$\Phi(g)(x) = y_0 + \int_{x_0}^{x} f\big(t, g(t)\big)\,\mathrm{d}t,$$

then clearly, the solution to the initial value problem is exactly the fixed point $y \in (\mathbb{R}^n)^{\mathbb{R}}$ under Φ. To find this fixed point, we apply a similar recursive procedure as the proof of Theorem 1.2.4.

Theorem 1.2.5 ▶ Picard-Lindelöf Theorem

Consider the initial value problem

$$y'(x) = f\big(x, y(x)\big), \qquad y(x_0) = y_0.$$

If $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be a closed rectangle with (x_0, y_0) in the interior of D and $f : D \to R^n$ is continuous in x and Lipschitz continuous in y, then there exists some $\epsilon > 0$ such that the initial value problem has a unique solution on $[x_0 - \epsilon, x_0 + \epsilon]$.

Proof. Without loss of generality, write $D := [x_0 - \epsilon, x_0 + \epsilon] \times [y_0 - \delta, y_0 + \delta]$. Define

$$\mathcal{C} := f \in C^0([x_0 - \epsilon, x_0 + \epsilon], [y_0 - \delta, y_0 + \delta]).$$

Consider the map $d : (\mathbb{R}^n)^{\mathbb{R}}$ such that

$$d(f, g) = \|f(x), g(x)\|_\infty.$$

One may check that $(\mathcal{C}, d)$ is a non-empty complete metric space. Define $\Phi : \mathcal{C} \to \mathcal{C}$ such that

$$\Phi(g)(x) = y_0 + \int_{x_0}^{x} f\big(t, g(t)\big)\,\mathrm{d}t.$$

Define $\phi_0 : [x_0 - \epsilon, x_0 + \epsilon] \to [y_0 - \delta, y_0 + \delta]$ by $\phi_0(x) = y_0$ and $\phi_{k+1} = \Phi(\phi_k)$ for all $k \in \mathbb{N}$. By Theorem 1.2.4, there exists a unique $\alpha \in \mathcal{C}$ such that $\Phi(\alpha) = \alpha$, i.e.,

$$\alpha(x) = y_0 + \int_{x_0}^{x} f\big(t, \alpha(t)\big)\,\mathrm{d}t.$$

Therefore, α is the unique solution to the initial value problem. $\square$

1.3 Separable Differential Equations

Definition 1.3.1 ▸ Separable Differential Equation

A differential equation is **separable** if it can be written in the form

$$f(x)\,\mathrm{d}x = g(y)\,\mathrm{d}y.$$

Remark. The above form is equivalent to $f(x) - g(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Clearly, separable differential equations can be solved by simply

$$\int f(x)\,\mathrm{d}x = \int g(y)\,\mathrm{d}y + c,$$

where c is an arbitrary constant.

Certain non-separable differential equations can be reduced to a separable form by substitution. Consider

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g\left(\frac{y}{x}\right).$$

By setting $v = \frac{y}{x}$, we have $y = vx$ and so $\frac{\mathrm{d}y}{\mathrm{d}x} = v + \frac{\mathrm{d}v}{\mathrm{d}x}x$. Therefore, the original equation becomes

$$v + \frac{\mathrm{d}v}{\mathrm{d}x}x = g(v),$$

which is separable because

$$\frac{1}{g(v) - v}\,\mathrm{d}v = \frac{1}{x}\,\mathrm{d}x.$$

In general, we have the following result:

Proposition 1.3.2 ▸ Linear Change of Variables

A differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(ax + by + c)$$

where a, b, c are constants can be reduced to a separable form by setting $u = ax + by + c$.

Of course, not all differential equations are separable.

1.4 First Order Linear Ordinary Differential Equations

Definition 1.4.1 ▸ First Order Linear Ordinary Differential Equations

We say that a differential equation is **linear** if it can be written in the form

$$\sum_{i=0}^{n} a_i f^{(i)}(x) = g(x),$$

where g is a function in x and the a_i's are constant coefficients. A **linear first order ordinary differential equation** is an ODE which can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x),$$

where P and Q are functions in x.

Remark. The above form is known as the **standard form** of linear first order ODEs.

To solve a linear first order ODE, consider the function

$$R(x) = \mathrm{e}^{\int P(x)\,\mathrm{d}x}.$$

Note that $R'(x) = R(x)P(x)$, so we have

$$\big(R(x)y\big)' = R(x)P(x)y + R(x)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

However, note that the above is exactly the left-hand side of the standard form multiplied by $R(x)$. Therefore, we have

$$y = \frac{\int R(x)Q(x)\,\mathrm{d}x}{R(x)} + c,$$

where c is an arbitrary constant.

Definition 1.4.2 ▸ Integrating Factor

Consider the linear first order ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x).$$

The **integrating factor** of the ODE is defined as the function

$$R(x) = \mathrm{e}^{\int P(x)\,\mathrm{d}x}.$$

Certain non-linear ODEs can be reduced to a linear form. The most famous example is the *Bernoulli differential equations*.

Definition 1.4.3 ▸ Bernoulli Differential Equations

The **Bernoulli differential equations** are differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n$$

where $n \in \mathbb{R}$.

When $n = 0$ or $n = 1$, the Bernoulli differential equations are linear, but otherwise they are non-linear. However, by setting $z = y^{1-n}$, we have

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x},$$

and so

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)p(x)z = q(x),$$

which is linear.

1.5 Second Order Linear Ordinary Differential Equations

Definition 1.5.1 ▸ Second Order Linear Ordinary Differential Equations

A **second order linear ordinary differential equation** is an ODE which can be written in the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = F(x).$$

If $F(x) = 0$, the ODE is said to be **homogeneous**, and otherwise **non-homogeneous**.

It is clear that the solutions to a homogeneous second order linear ODE form a vector space.

Theorem 1.5.2 ▸ Superposition Principle

Consider a homogeneous second order linear ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0.$$

If y_1 and y_2 are solutions to the ODE on an open interval I, then their linear combinations $\alpha y_1 + \beta y_2$ for any $\alpha, \beta \in \mathbb{C}$ are also solutions to the ODE on I.

The above theorem is trivially true and the proof is left to the reader as a warm-up exercise. Now, for any homogeneous second order linear ODE, let its solution space be S. Since S is a vector space, all there is left to do is to find its basis. Consider the ODE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0.$$

Let $z = \frac{\mathrm{d}y}{\mathrm{d}x}$, then the ODE can be re-written as a system of first order linear ODEs:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = z \\ \frac{\mathrm{d}z}{\mathrm{d}x} = -p(x)z - q(x)y \end{cases}.$$

By Theorem 1.2.5, the solution (y, z) is uniquely determined by the initial value $\big(y(0), z(0)\big)$. This means that

$$S \cong \operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

and so $\dim(S) = 2$. What this means is that to solve any second order linear ODE, we only need to find 2 linearly independent solutions y_1 and y_2, and the solution space will be given by $\operatorname{span}\{y_1, y_2\}$.