

P & C: $P_r^{n+1} = P_r^n + rP_{r-1}^n$.

Circular permutation: $Q_r^n = \frac{P_r^n}{r}$.

$$C_r^{n+1} = C_{r-1}^n + C_r^n.$$

$$H_r^n = C_r^{r+n-1}.$$

Arrange r distinct objects around n identical circles such that no circle is empty: $s(r, n) = s(r-1, n-1) + (r-1)s(r-1, n)$.

$$s(r, r-1) = C_2^r.$$

Binomial & Multinomial:

$$\binom{n}{r} = \frac{n}{r} \binom{n-1}{r-1}.$$

$$\binom{n}{r} = \frac{n-r+1}{r} \binom{n}{r-1}.$$

$$\binom{n}{m} \binom{m}{r} = \binom{n}{r} \binom{n-r}{m-r}.$$

Vandermonde's Identity:

$$\sum_{i=0}^r \left[\binom{m}{i} \binom{n}{r-i} \right] = \binom{m+n}{r}.$$

Chu Shih-Chieh Identity:

$$\sum_{i=0}^{n-r} \binom{r+i}{r} = \binom{n+1}{r+1}$$

$$\sum_{i=0}^k \binom{r+i}{i} = \binom{r+k+1}{k}.$$

Multinomial coefficient:

$$\binom{n}{n_1, n_2, \dots, n_m} = \frac{n!}{\prod_{i=1}^m n_i!}.$$

Pigeonhole Principle: If at least $kn+1$ objects are distributed into n distinct sets, then there exists a set with at least $k+1$ objects.

Generalised PP: If at least $\sum_{i=1}^n k_i + (n-1)$ distinct

objects are distributed into n distinct sets, then there exists at least one set (i -th) with at least k_i objects.

Ramsey Numbers: $R(p, q) \leq R(p-1, q) + R(p, q-1)$.
(Bound lowered by 1 if both on RHS are even.)

$$R(2, q) = q, R(1, q) = 1.$$

Distribution Problems:

Distinct into distinct:

- Each box at most 1: P_r^n .
- Each box any number of objects: n^r .
- Each box any number of objects with internal ordering: $\frac{(n-1+r)!}{(n-1)!}$.

Identical into distinct:

- Each box any number of objects: $H_r^n = C_r^{r+n-1}$.
- No box empty: $H_{r-n}^n = C_{r-n}^{r-1}$.

Distinct into identical:

- No box empty: $S(r, n) = S(r-1, n-1) + nS(r-1, n)$

Number of partitions of A with $|A| = n$: $\sum_{i=1}^n S(n, i)$.

Number of surjective mapping from $[1, r] \cap \mathbb{N}$ to $[1, n] \cap \mathbb{N}$: $F(r, n) = \sum_{k=0}^n (-1)^k C_k^n (n-k)^r$.

$$S(r, n) = \frac{1}{n!} F(r, n).$$

$$D(n, r, k) = \frac{C_k^r}{(n-r)!} \sum_{i=0}^{r-k} (-1)^i C_i^{r-k} (n-k-i)!.$$

$$D_n = D(n, n, 0) = n! \sum_{i=0}^n \frac{(-1)^i}{i!}.$$

$$\varphi(n) = n \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right).$$

$$\varphi(2n) = \varphi(n) \text{ (} n \text{ odd) or } 2\varphi(n) \text{ (} n \text{ even)}$$

Identical into Identical

- Number of partitions of n into k parts equals the number of partition of k where the largest part has size k .

- r -partition of n of size at most k : $\frac{1}{\prod_{i=1}^k (1-x^i)}$.

- Number of partitions of r into distinct parts is the number of partitions of r into odd parts.

- Number of partitions of r into at most k parts is the number of partitions of r into parts whose size does not exceed k

$$\sum_{k=1}^m p(n, k) = p(n+m, m).$$

GPIE:

$$E(m) = \sum_{k=m}^q (-1)^{k-m} C_m^k \omega(k).$$

OGF (Identical into distinct & identical into identical):

$$\binom{\alpha}{r} = \frac{\prod_{i=0}^{r-1} (\alpha-i)}{r!}.$$

$$(1 \pm x)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} (\pm x)^r.$$

$$\frac{1}{1-kx} \text{ generates } (1, k, k^2, \dots).$$

$$(1-x)^{-n} = \sum_{i=0}^{\infty} \binom{n-1+i}{i} x^i.$$

$$\alpha A(x) + \beta B(x) \text{ generates } \alpha a_r + \beta b_r.$$

$$A(x)B(x) \text{ generates } \sum_{i=0}^r a_i b_{r-i}.$$

$$x^m A(x) \text{ translates } a_i \text{ to } a_{i+m}.$$

$$A(kx) \text{ generates } k^r a_r.$$

$$(1-x)A(x) \text{ generates } c_r = a_r - a_{r-1}.$$

$$\frac{A(x)}{1-x} \text{ generates } c_r = \sum_{i=0}^r a_i.$$

$$A'(x) \text{ generates } (r+1)a_{r+1}.$$

- $xA'(x)$ generates ra_r .
- r -combination of multi-set: $\prod_{i=1}^k \left(\sum_{j=0}^{n_i} x^j \right)$.

EGF (Distinct into distinct):

- $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ generates $a_r = 1$.
- $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$ generates $a_r = r!$.
- $e^{kx} = \sum_{i=0}^{\infty} \frac{(kx)^i}{i!}$ generates $a_r = k^r$.
- $(1+x)^n = \sum_{i=0}^n i! C_r^n \frac{x^i}{i!}$ generates P_r^n .
- r -permutation of multi-set: $\prod_{i=1}^k \left(\sum_{j=0}^{n_i} \frac{x^j}{j!} \right)$
- $\frac{e^x + e^{-x}}{2}$: even number of elements.
- $\frac{e^x - e^{-x}}{2}$: odd number of elements.

Particular solutions:

$f(n)$	$a_n^{(p)}$
Ak^n	$\begin{cases} Bk^n & k \text{ not a root} \\ Bn^m k^n & k \text{ has multiplicity } m \end{cases}$
$\sum_{i=0}^t p_i n^i$	$\begin{cases} \sum_{i=0}^t q_i n^i & 1 \text{ not a root} \\ n^m \sum_{i=0}^t p_i n^i & 1 \text{ multiplicity } m \end{cases}$
$An^t k^n$	$\begin{cases} \left(\sum_{i=0}^t q_i n^i \right) k^n \\ n^m \left(\sum_{i=0}^t p_i n^i \right) k^n \end{cases}$

Graph:

- **Handshaking Lemma:** $\sum d_G(v) = 2e(G)$.
The number of vertices with odd degrees is even.
- A subgraph H of G is induced iff $H = G - (V(G) - V(H))$.
- $G \cong H$ iff $\overline{G} \cong \overline{H}$.
- If $G \cong H$, then
 - G and H have same order and size.

- $\delta(G) = \delta(H)$ and $\Delta(G) = \Delta(H)$.
- Number of vertices with degree i is the same.

- (d_1, d_2, \dots, d_n) is graphic iff $(d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$ is graphic.
- Order of self-complementary G is either $4k$ or $4k + 1$.
- $\exists u-v$ walk of length $k \implies \exists u-v$ path of length at most k .
- $\omega(G)$: number of components of G .
- Complement of connected graph is connected.
- v is a cut-vertex iff $\exists a, b$ such that v is in every $a-b$ path.
- e is a bridge iff it is not part of any cycle.
- uv is a bridge and u is not end vertex $\implies u$ is a cut-vertex.
- A graph with order at least 3 which contains a bridge contains a cut-vertex.
- $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$.
- Incidence matrix is $v(G)$ rows $e(G)$ columns.
- (i, j) entry of \mathbf{A}^k is the number of v_i-v_j walks of length k .
- G is connected if and only if the (i, j) entry of $\sum_{i=1}^{n-1} \mathbf{A}^i$ is nonzero for all $i \neq j$.
- Size of bipartite is sum of degrees of any partite set.
- Join: $V(G + H) = V(G) \cup V(H)$, $E(G, H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.
- Bipartite iff no odd cycles.

- T is a tree iff every two vertices are joined by a unique path iff $e(T) = v(T) - 1$.
- If $\Delta(T) = k$, then $n_1 = 2 + \sum_{i=1}^{k-2} i n_{i+2}$.
- A tree of order at least 2 contains at least 2 end vertices.
- The centre of a tree is either K_1 or K_2 .
- A graph is connected iff it contains a spanning tree.
- $\tau(G) = \tau(G - e) + \tau(G \circ e)$.
 - $\tau(C_n) = n$.
 - Connected with a cut-vertex or a bridge: $\tau(G) = \tau(G_1)\tau(G_2)$.
 - C_p and C_q sharing a common edge: $\tau(G) = p + q - 2 + (p - 1)(q - 1)$.
 - C_p with a duplicated edge: $\tau(G) = 2p - 1$.
 - C_p and C_q sharing a pair of duplicated edges: $\tau(G) = p + q - 2 + 2(p - 1)(q - 1)$.
 - $\tau(K_n) = n^{n-2}$.
 - $\tau(K_{2,r}) = 2^{r-1}r$.
- **Matrix tree theorem:** $\tau(G)$ is the cofactor of any entry of $\mathbf{C} - \mathbf{A}$ where \mathbf{C} is a diagonal matrix containing the degrees of vertices.