- Half-space: $H = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq b\}$.
- Polyhedral set: **finite** union of half-spaces.
- Convert to standard form:
  1. $\boldsymbol{a}_{\boldsymbol{i}}^{\mathrm{T}}\boldsymbol{x} \leq b_i \longrightarrow \boldsymbol{a}_{\boldsymbol{i}}^{\mathrm{T}}\boldsymbol{x} + s_i = b_i$ for $s_i \geq 0$.
  2. $x_i \leq 0 \longrightarrow -x_i^- = x_i$ s.t. $-x_i^- \geq 0$.
  3. Free variable $x_i = x_i^+ - x_i^-$ for $x_i^+, x_i^- \geq 0$.
- $\max f_i(\boldsymbol{x})$ is convex if f_i's are convex.
- **Extreme Point:** A point $\boldsymbol{x}^* \in P$ is said to be an **extreme point** if whenever there are $\boldsymbol{y}, \boldsymbol{z} \in P$ with $\boldsymbol{x}^* = \lambda\boldsymbol{y} + (1-\lambda)\boldsymbol{z} = \boldsymbol{x}^*$ for some $\lambda \in (0,1)$, we have $\boldsymbol{y} = \boldsymbol{z} = \boldsymbol{x}^*$.
- **Vertex:** A point $\boldsymbol{x}^* \in P$ is said to be a **vertex** if there exists some $\boldsymbol{c}$ such that $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^* > \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{y}$ for all $\boldsymbol{y} \in P - \{\boldsymbol{x}^*\}$.
- **BFS:** $\boldsymbol{x}^* \in P$ is said to be a BFS if there are n linearly independent constraints which are active at $\boldsymbol{x}^*$. The number of linearly independent active constraints at $\boldsymbol{x}^*$ is called the **rank** of $\boldsymbol{x}^*$.
- **Basic solution:** $\boldsymbol{x}^* \in \mathbb{R}^n$ is a basic solution iff
  - $\boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{b}$, and
  - There exists an index set $B \subseteq \{1, 2, \cdots, n\}$ such that the set $\{\boldsymbol{A}_i : i \in B\}$ is linearly independent and $x_j^* = 0$ for all $j \notin B$.
- For non-empty polyhedron P, the followings are equivalent:
  1. P does not contain any straight line.
  2. P has a basic feasible solution.
  3. P has n linearly independent constraints.
- For every BFS, $\boldsymbol{x}_N = \boldsymbol{0}$. If $\boldsymbol{x}_B$ contains zero entries, then $\boldsymbol{x}$ is degenerate.
- Feasible direction to adjacent BFS: $\boldsymbol{d}^j = \left(\boldsymbol{d}_B^j, \boldsymbol{d}_N^j\right)$ for some $j \in N$, such that $\boldsymbol{d}_N^j = \boldsymbol{e}_j$ and $\boldsymbol{d}_B^j = -\boldsymbol{A}_B^{-1}\boldsymbol{A}_j$.
- Every feasible direction can be expressed as a linear combination of $\boldsymbol{d}^j$'s.
- **Reduced cost:** $\bar{c}_j = c_j - \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{A}_j$.
- Step size to adjacent BFS: $\bar{\theta}_j = \min\left\{-\frac{x_i}{d_i^j} : i \in B, d_i^j < 0\right\}$
- If $\bar{\boldsymbol{c}} \geq \boldsymbol{0}$, then $\boldsymbol{x}^*$ is optimal; if $\boldsymbol{x}^*$ is optimal and non-degenerate, then $\bar{\boldsymbol{c}} \geq \boldsymbol{0}$.
- **Simplex method:**
  1. $\boldsymbol{x}_0$: any basic feasible solution.
  2. At the k-th iteration, choose a basis B_k.
  3. Let $N_k := \{1, 2, \cdots, n\} - B_k$. For each $j \in N_k$, compute the reduced cost
  $$\bar{c}_j = c_j - \boldsymbol{c}_{B_k}^{\mathrm{T}}\boldsymbol{A}_{B_k}^{-1}\boldsymbol{A}_j.$$
  4. If $\bar{c}_j \geq 0$ for all $j \in N_k$, $\boldsymbol{x}_k$ is an optimal solution.
  5. Otherwise:
     (a) Take some $j \in N_k$ such that $\bar{c}_j < 0$. $(\boldsymbol{x}_k)_j$ is called an **entering variable**.
     (b) Compute $\boldsymbol{d}_{B_k}^j = -\boldsymbol{A}_{B_k}^{-1}\boldsymbol{A}_j$.
     (c) If $\boldsymbol{d}_{B_k}^j \geq \boldsymbol{0}$, the problem is **unbounded**.
     (d) Otherwise:
        i. Take $\bar{\theta}_j = \min\left\{-\frac{x_i}{d_i^j} : i \in B, d_i^j < 0\right\} = \frac{x_\ell}{d_\ell^j}$. $(\boldsymbol{x}_k)_\ell$ is called a **leaving variable**.
        ii. Update $B_{k+1} := (B - \{\ell\}) \cup \{j\}$.
        iii. Update $\boldsymbol{x}_{k+1}$ by
        $$(\boldsymbol{x}_{k+1})_i = \begin{cases} (\boldsymbol{x}_k)_i + \bar{\theta}_j \boldsymbol{d}_i^j & \text{if } i \in B - \{\ell\} \\ \bar{\theta}_j & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$
- **Tableau implementation:**

Basic	$\boldsymbol{x}$	Solution
$\bar{\boldsymbol{c}}$	$\boldsymbol{c}^T - \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{A}$	$-\boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{b}$
$\boldsymbol{x}_B$	$\boldsymbol{A}_B^{-1}\boldsymbol{A}$	$\boldsymbol{A}_B^{-1}\boldsymbol{b}$

Basic	x_1	$\cdots$	x_n	Solution
$\bar{\boldsymbol{c}}$	$\bar{c}_1$	$\cdots$	$\bar{c}_n$	$-\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_B$
$x_{B(1)}$ $\vdots$ $x_{B(m)}$	$\boldsymbol{A}_B^{-1}\boldsymbol{A}_1$	$\cdots$	$\boldsymbol{A}_B^{-1}\boldsymbol{A}_n$	$\boldsymbol{A}_B^{-1}\boldsymbol{b}$

Basic	$\boldsymbol{x}_B$	$\boldsymbol{x}_N$	Solution
$\bar{\boldsymbol{c}}$	$\boldsymbol{0}$	$\boldsymbol{c}_N^T - \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{A}_N$	$-\boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{b}$
$\boldsymbol{x}_B$	$\boldsymbol{I}$	$\boldsymbol{A}_B^{-1}\boldsymbol{A}_N$	$\boldsymbol{A}_B^{-1}\boldsymbol{b}$

  At every iteration, swap entering with leaving variables, normalise the pivot row, and do EROs to restore $\boldsymbol{I}$.
- **Two-Phase method:**
  1. Manipulate constraint s.t. $\boldsymbol{b} \geq \boldsymbol{0}$.
  2. Construct the auxiliary linear program
  $$\min_{\boldsymbol{y} \in \mathbb{R}^m} \sum_{i=1}^m y_i \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0}.$$
  3. Run simplex method with $(\boldsymbol{0}, \boldsymbol{b})$ to obtain its optimal solution $(\boldsymbol{y}^*, \boldsymbol{x}^*)$ and optimal value v^*.
  4. If $v^* > 0$, the original feasible region is $\varnothing$.
  5. Otherwise, $v^* = 0$, $\boldsymbol{x}^*$ is an initial BFS.
- **Big-M method:**
  1. Manipulate constraint s.t. $\boldsymbol{b} \geq \boldsymbol{0}$.
  2. Augment the problem:
  $$\min \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + M\sum_{i=1}^m y_i \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0}.$$
  3. Run simplex method with $(\boldsymbol{0}, \boldsymbol{b})$.
  4. If an optimal solution $(\boldsymbol{y}^*, \boldsymbol{x}^*)$ exists with $\boldsymbol{y}^* = \boldsymbol{0}$, $\boldsymbol{x}^*$ is an optimal solution.
  5. Otherwise, the original problem has empty feasible set or is unbounded.
- Special cases:
  1. Multiple leaving variables: degenerate BFS.
  2. Multiple optimal solutions iff $\bar{c}_j = 0$ at an optimum.
     (a) If $\bar{c}_j = 0$ but $\boldsymbol{d}^j \geq \boldsymbol{0}$, the optimal set is $\left\{\boldsymbol{x}^* + \theta\boldsymbol{d}^j : \theta \geq 0\right\}$.
     (b) If multiple optimal BFSs exist, the optimal set is $\operatorname{conv}\left\{\boldsymbol{x}^1, \boldsymbol{x}^2, \cdots, \boldsymbol{x}^k\right\}$.
  3. $\bar{c}_j < 0$ but $\boldsymbol{d}^j \geq \boldsymbol{0}$: unbounded.
  4. Empty feasible set: detect with two-phase method or Big-M method.
  5. **Dual problem:** $\max_{\boldsymbol{p} \in \mathbb{R}^n} \boldsymbol{p}^{\mathrm{T}}\boldsymbol{b}$ s.t. $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{A} \leq \boldsymbol{c}^{\mathrm{T}}$.
  6. Dual of the dual is the primal.
  7. **Weak duality:** $\sup \boldsymbol{p}^{\mathrm{T}}\boldsymbol{b} \leq \inf \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$.
  8. If $(\boldsymbol{p}^*)^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*$, then both are optimal solutions.
  9. (P) is unbounded iff (D) is infeasible. (P) is infeasible iff (D) is unbounded.
  10. **Strong duality:** If primal and dual are feasible, then $(\boldsymbol{p}^*)^{\mathrm{T}}\boldsymbol{b} = \inf \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*$. The dual optimal solution $\boldsymbol{p}^* = \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B$.

11. **Complementary slackness:** Let (P) be a primal linear program with constraints $\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \leq b_i$, $\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \geq b_i$ or $\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} = b_i$. Let $\boldsymbol{x}$ and $\boldsymbol{p}$ be feasible solutions to (P) and the dual problem (D) respectively, then $\boldsymbol{x}$ and $\boldsymbol{p}$ are optimal iff

$$p_i\left(\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} - b_i\right) = 0, \qquad \left(c_j - \boldsymbol{p}^{\mathrm{T}}\boldsymbol{A}_j\right)x_j = 0$$

for all i, j. A feasible $\boldsymbol{x}$ is optimal iff there is a feasible dual solution $\boldsymbol{p}$ satisfying complementary slackness conditions.

- **Dual simplex method:**
  1. Manipulate the constraints and add slack variables so that the right-most portion of $\boldsymbol{A}$ becomes an identity matrix.
  2. Run simplex method on the transformed problem, while maintaining $\bar{\boldsymbol{c}} \geq \boldsymbol{0}$.
     (a) At the k-th iteration, if there is no negative basic variable, then an optimal primal solution has been found.
     (b) Otherwise, select some $x_\ell < 0$ as the leaving variable.
        i. If $d_\ell^j \geq 0$ for all $j \in N_k$, then the primal problem is infeasible.
        ii. Otherwise, take

$$i = \operatorname{argmin}_{j\in N_k}\left\{\frac{\bar{c}_j}{\left|d_\ell^j\right|} : d_\ell^j < 0\right\}$$

        as the index of the entering variable.
  3. Terminate when we have obtained $\boldsymbol{x}_B \geq \boldsymbol{0}$.

- **Sensitivity analysis:**
  1. Optimality condition: $\bar{\boldsymbol{c}} = \boldsymbol{c}^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{A} \geq \boldsymbol{0}$; Feasibility condition: $\boldsymbol{A}_B^{-1}\boldsymbol{b} \geq \boldsymbol{0}$.
  2. $\boldsymbol{b} + \delta\boldsymbol{e}_i$: $\boldsymbol{x}_B^* + \delta\left(\boldsymbol{A}_B^{-1}\boldsymbol{e}_i\right) \geq \boldsymbol{0}$. New optimal value: $\boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\left(\boldsymbol{b} + \delta\boldsymbol{e}_i\right) = \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{b} + \delta p_i^*$. p_i^* is the **marginal cost**.
  3. $\boldsymbol{c} + \delta\boldsymbol{e}_i$: If i is non-basic, optimal iff $\delta \geq -\bar{c}_j$. If i is basic, for each $j \in N$ we need $c_i - \left(\boldsymbol{c}_B + \delta\boldsymbol{e}_j\right)^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{A}_i = \bar{c}_j - \delta\boldsymbol{e}_j^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{A}_i \geq 0$. Define $\bar{a}_{i,j} = \boldsymbol{e}_j^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{A}_i$, then $\boldsymbol{x}^*$ is still optimal iff

$$\max_{\bar{a}_{i,j}<0}\frac{\bar{c}_j}{\bar{a}_{i,j}} \leq \delta \leq \min_{\bar{a}_{i,j}>0}\frac{\bar{c}_j}{\bar{a}_{i,j}}.$$

  4. $a_{ij} + \delta$ for some $j \in N$. $\boldsymbol{x}^*$ is still feasible iff $\bar{c}_j' = \bar{c}_j - \delta p_i \geq 0$.
  5. New variable x_{n+1}: $(\boldsymbol{x}^*, 0)$ is a BFS, so it is optimal iff $\bar{c}_{n+1} = c_{n+1} - \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{A}_{n+1} \geq 0$. Otherwise, x_{n-1} is entering variable so we run simplex again.
  6. New constraint $\boldsymbol{a}_{m+1}^{\mathrm{T}}\boldsymbol{x} \leq b_{m+1}$ with induced new slack variable x_{n+1}: nothing to do if $\boldsymbol{x}^*$ satisfies the constraint. Otherwise add x_{n+1} as new basic variable, and $x_{n+1} < 0$ (x_{n+1} $\boldsymbol{a}_B^{\mathrm{T}}$ $\boldsymbol{a}_N^{\mathrm{T}}$ 1 b_{m+1} in optimal tableau). Run dual simplex method to obtain new solution.

- Flow balance constraint:

$$\sum_{j\in O(i)} x_{ij} - \sum_{k\in I(i)} x_{ki} = b_i \quad \forall i \in V.$$

- Row sums of node-arc incidence matrix and supply vector are zero.

- If $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A}$ is a 0-1 matrix such that the 1's in each column appear consecutively, then

$$\boldsymbol{A}' = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix}\boldsymbol{A}$$

is a node-arc incidence matrix.

- **SSSP:** $\min \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ s.t. $\boldsymbol{Ax} = \boldsymbol{e}_s - \boldsymbol{e}_t, \boldsymbol{x} \geq \boldsymbol{0}$. Dual: $\max p_s - p_t$ s.t. $p_i - p_j \leq c_{(i,j)}$ $\forall (i,j) \in E(G)$. We can set $p_t = 0$ when solving the dual.

- **Three Jug Puzzle:**
  - $(i,j) \to (b,j)$: $1 \to 2$;
  - $(i,j) \to (0,j)$: $2 \to 1$;
  - $(i,j) \to (i,c)$: $1 \to 3$;
  - $(i,j) \to (i,0)$: $3 \to 1$;
  - $(i,j) \to (\min\{0, i+j-c\}, \max\{i+j, c\})$: $2 \to 3$;
  - $(i,j) \to (\max\{i+j, b\}, \min\{0, i+j-b\})$: $3 \to 2$.

- **Dynamic Lot Sizing:** $c_{(i,j)} = K_i + c_i x_i + \sum_{k=i}^{j-1} h_k I_k$ where $I_k = x_i - \sum_{r=i}^{k} d_r = \sum_{r=k+1}^{j-1} d_r$, so

$$c_{(i,j)} = K_i + c_i\sum_{k=i}^{j-1} d_k + \sum_{k=i}^{j-2}\left(h_k\sum_{r=k+1}^{j-1} d_r\right).$$

- **Max flow:** $\min v$ s.t. $\boldsymbol{Ax} = (\boldsymbol{e}_s - \boldsymbol{e}_t)v, \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{u}$. If there are multiple sources/destinations, add artificial single source and destination with uncapacitated arcs.

- **Min cut:** $\min \boldsymbol{u}^{\mathrm{T}}\boldsymbol{z}$ s.t. $\boldsymbol{d}^{\mathrm{T}}\boldsymbol{y} = 1, \boldsymbol{z} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{0}, \boldsymbol{z} \geq \boldsymbol{0}$. Here $\boldsymbol{u}$ is the capacities, $\boldsymbol{y}, \boldsymbol{z}$ are boolean vectors. $\boldsymbol{y}$ indicates the section a vertex is in, $\boldsymbol{z}$ indicates if an edge is taken. It is dual to max flow.

- Truncated node-arc incidence matrix $\widetilde{A}$: delete last row from $\boldsymbol{A}$. An edge set B is a basis of $\widetilde{A}$ iff it induces a spanning tree. Dual vector $\boldsymbol{p}^{\mathrm{T}} = \boldsymbol{c}_B^{\mathrm{T}}\widetilde{A}_B^{-1}$.

- **Network simplex method:**
  1. Take a spanning tree $T_0 \subseteq G$ and find a basis B from $E(T_0)$ and a feasible tree solution $\boldsymbol{x}_0$.
  2. At the k-th iteration, compute $\boldsymbol{p}^{\mathrm{T}} = \boldsymbol{c}_B^{\mathrm{T}}\widetilde{\boldsymbol{A}}_B^{-1}$.
  3. Solve $p_n = 0, p_i - p_j = c_{(i,j)}$ for all $(i,j) \in B$.
  4. Reduced cost $\bar{c}_{(i,j)} = c_{(i,j)} - (p_{i_k} - p_{j_k})$.
  5. If $\bar{c}_{(i,j)} \geq 0$ for all $(i,j) \in E(G)$, then $\boldsymbol{x}_k$ is optimal.
  6. Otherwise, choose $(i,j) \notin B$ with $\bar{c}_{(i,j)} < 0$.
  7. Add (i,j) to produce a cycle, $i \to j$ is forward.
     (a) Unbounded if no backward arc.
     (b) Otherwise, $\theta^* := x_{pq} = \min_{(k,\ell)\in C_b} x_{k\ell}$.
  8. Update $\boldsymbol{x}_k$ to $\boldsymbol{x}_{k+1}$ by

$$\widehat{x_{k\ell}} = \begin{cases} x_{k\ell} + \theta^* & \text{if } (k,\ell) \in C_f \\ x_{k\ell} - \theta^* & \text{if } (k,\ell) \in C_b \\ x_{k\ell} & \text{otherwise} \end{cases}.$$

  9. Update $T_{k+1} = \big(T - (p,q)\big) \cup (i,j)$.

- **Two-phase network simplex method:**
  1. Connect all $i \to n$, set $c_{(i,j)} = 0$ and $c_{(i,n)} = 1$.
  2. Use star rooted at n as initial basis to remove all extra arcs. Take the resultant basis to run network simplex method.

- If G is a weakly connected network, then $\widetilde{A}_B^{-1}$ is integer for all B.

- Let G be an n-vertex weakly connected network. Consider a network flow problem (P) such that an optimal solution exists. If the supply vector $\boldsymbol{b}$ consists of purely integer entries, then there is an integer optimal solution; if the cost vector $\boldsymbol{c}$ consists of purely integer entries, then there is an integer dual optimal solution.