Contents

1 Topology **2**

1.1 Topological Spaces 2

1.2 Metric Spaces 6

1.3 Subspace Topologies 12

1

Topology

1.1 Topological Spaces

Definition 1.1.1 ▶ Topology

A **topology** on a set X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ such that

- $\varnothing, X \in \mathcal{T}$;
- for any index set I, if $\{X_i : i \in I\} \subseteq \mathcal{P}(\mathcal{T})$, then $\bigcup_{i\in I} X_i \in \mathcal{T}$;
- for any $X_1, X_2, \cdots, X_n \in \mathcal{T}$, $\bigcap_{i=1}^{n} X_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is said to be a **topological space**. A subset $Y \subseteq X$ is **open** if $Y \in \mathcal{T}$.

Remark. For any set X, we define $\{\varnothing, X\}$ as the *trivial topology* on X, $\mathcal{P}(X)$ as the *discrete topology*, and $\{X \backslash U : U \subseteq X \text{ is finite}\} \cup \{\varnothing\}$ as the *co-finite topology*.

The set $\{(-\alpha, \alpha) : \alpha > 0\} \cup \{\mathbb{R}, \varnothing\}$ defines a topology on $\mathbb{R}$. This example also demonstrates why it is crucial to only consider closure under finite intersections when defining a topology, because

$$\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = [-1, 1] \notin \mathcal{T}.$$

We now seek a systematic method to generate a topology given any set. The idea here is to make use of a *cover*.

Definition 1.1.2 ▶ Basis

A **basis** for a topology on X is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that

- for any $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$;
- for any $x \in X$ and $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

It may be useful to see a basis as a cover of a set with the second additional property as stated in the above definition. Notice that the first property of the basis $\mathcal{B}$ is basically saying that

$$X \subseteq \bigcup \mathcal{B},$$

i.e., $\mathcal{B}$ is a cover of X.

Given any basis $\mathcal{B}$ for some topology on X, a set generated by $\mathcal{B}$ can be defined as

$$\mathcal{T} := \{U \subseteq X : \text{ for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\}$$

We will show that $\mathcal{T}$ is a topology on X. First, it is clear that $\varnothing, X \in \mathcal{T}$.

Let I be an index set and $\{X_i : i \in I\} \subseteq \mathcal{P}(\mathcal{T})$ be any collection of subsets of X. Notice that for any $x \in \bigcup_{i \in I} X_i$, there exists some $j \in I$ such that $x \in X_j \subseteq \mathcal{T}$. According to our construction, this means that there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq X_j \subseteq \mathcal{T}$. Therefore, $\bigcup_{i \in I} X_i \subseteq \mathcal{T}$ as desired.

To prove that $\mathcal{T}$ is closed under finite intersection, we consider the following lemma:

Lemma 1.1.3 ▸ Finite Intersection of Elements in Basis Is Covered

Let $\mathcal{B}$ be a basis for a topology on X and $B_1, B_2, \cdots, B_n \in \mathcal{B}$, then for any $x \in \bigcap_{i=1}^n B_i$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcap_{i=1}^n B_i$.

Proof. The case where $n = 1$ is trivial by taking $B = B_1$. Suppose that there is some integer $k \geq 1$ such that for any $B_1, B_2, \cdots, B_k \in \mathcal{B}$ and any $x \in \bigcap_{i=1}^k B_i$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcap_{i=1}^k B_i$. Take any $B_{k+1} \in \mathcal{B}$. It is clear that for any $x \in \bigcap_{i=1}^{k+1} B_i$, there exists some $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^{k} B_i.$$

Notice that $x \in B_{k+1} \in \mathcal{B}$, so we know that $x \in B \cap B_{k+1}$. By Definition 1.1.2, this means that there exists some $B' \in \mathcal{B}$ such that

$$x \in B' \subseteq B \cap B_{k+1} \subseteq \bigcap_{i=1}^{k+1} B_i.$$

□

Now, suppose $X_1, X_2, \cdots, X_n \in \mathcal{T}$ are finitely many subsets of X. Take any $x \in \bigcap_{i=1}^n X_i$. It is clear that $x \in X_i$ for each $i = 1, 2, \cdots, n$. Therefore, for each $i = 1, 2, \cdots, n$, there exists some $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq X_i$. By Lemma 1.1.3, this means that there exists some set $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^{n} B_i \subseteq \bigcap_{i=1}^{n} X_i.$$

Therefore, $\bigcap_{i=1}^n X_i \in \mathcal{T}$. So this set $\mathcal{T}$ generated by $\mathcal{B}$ is indeed a topology on X.

The following proposition further shows that the topology generated by a basis $\mathcal{B}$ is the set of all possible unions of elements in $\mathcal{B}$:

Proposition 1.1.4 ▸ Equivalent Construction of Topologies Generated from Bases

Let X be any set. If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X, then

$$\mathcal{T} = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \in \mathcal{P}(\mathcal{B}) \right\}.$$

Proof. Denote

$$\mathcal{T}_{\mathcal{B}} \coloneqq \{U \subseteq X : \text{ for any } u \in U, \text{ there exists some } B \in \mathcal{B} \text{ such that } u \in B \subseteq U\}.$$

It suffices to prove that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. Take any $T \in \mathcal{T}$, then there exists some $V \in \mathcal{P}(\mathcal{B})$ such that $T = \bigcup_{A \in \mathcal{V}} A$. This means that for every $t \in T$, there exists some $B_t \in \mathcal{V}$ such that $t \in B_t \subseteq T$. Therefore, $T \in \mathcal{T}_{\mathcal{B}}$. Conversely, for any $S \in \mathcal{T}_{\mathcal{B}}$, there exists some $B_s \in \mathcal{B}$ for each $s \in S$ such that $s \in B_s$. Denote $U \coloneqq \{B_s : s \in S\} \in \mathcal{P}(\mathcal{B})$, then it is clear that $S \subseteq \bigcup_{B \in U} B$. Since $B_s \subseteq S$ for each $s \in S$, we have $\bigcup_{B \in U} B \subseteq S$, which implies that $S = \bigcup_{B \in U} B$. This means that $S \in \mathcal{T}$. Therefore, $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$, which means that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. □

Next, we define a special topology in Euclidean spaces using open balls.

Definition 1.1.5 ▸ Standard Topology

For any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and any $r > 0$. Denote the Euclidean open ball centred at $\boldsymbol{x}$ with radius r by

$$B_r(\boldsymbol{x}) \coloneqq \left\{ \boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n : \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} < r \right\}$$

The **standard topology** on $\mathbb{R}^n$ is the set generated by the basis

$$\mathcal{B} \coloneqq \{B_r(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n, r \in \mathbb{R}^+\}.$$

It may be helpful to actually show that this set $\mathcal{B}$ is indeed a basis of a topology on $\mathbb{R}^n$. The fact that $\mathcal{B}$ is a cover for $\mathbb{R}^n$ is trivial enough. Take any $\boldsymbol{x} \in \mathbb{R}^n$ and balls $B_\alpha(\boldsymbol{x}_1), B_\beta(\boldsymbol{x}_2) \in \mathcal{B}$ such that $\boldsymbol{x} \in B_\alpha(\boldsymbol{x}_1) \cap B_\beta(\boldsymbol{x}_2)$ (the existence of these 2 balls is again trivial enough). Take

$$r = \min\{\alpha - \|\boldsymbol{x} - \boldsymbol{x}_1\|, \beta - \|\boldsymbol{x} - \boldsymbol{x}_2\|\}.$$

Clearly, $r > 0$ and $\boldsymbol{x} \in B_r(\boldsymbol{x})$, so we are done.

Now, we discuss the analogue of the subset relation in topologies.

Definition 1.1.6 ▸ Fineness and Coarseness

Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on some set X. We say that $\mathcal{T}$ is **finer** than $\mathcal{T}'$, or equivalently, that $\mathcal{T}'$ is **coarser** than $\mathcal{T}$, if $\mathcal{T}' \subseteq \mathcal{T}$.

Observe that any topology of X must be a subset of $\mathcal{P}(X)$, which is the discrete topology on X, so the discrete topology is the finest topology on a set.

> *Remark.* For any basis $\mathcal{B}$ for a topology on X, the topology generated by $\mathcal{B}$ is the coarsest topology containing $\mathcal{B}$.

The above remark is easy to verify. Let $\mathcal{T}$ be any topology on X with $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}_{\mathcal{B}}$ be the topology generated by $\mathcal{B}$. For any $T \in \mathcal{T}_{\mathcal{B}}$, by Proposition 1.1.4, there exists some $V \subseteq \mathcal{B}$ such that $T = \bigcup_{A \in \mathcal{V}} A$. Note that $A \in \mathcal{T}$ for all $A \in \mathcal{V}$, so by Definition 1.1.1, $T \in \mathcal{T}$ and so $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$ as desired.

This motivates us to consider fineness in terms of bases.

Proposition 1.1.7 ▸ Fineness in Terms of Bases

Let $\mathcal{B}$ and $\mathcal{B}'$ generate topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for every $B \in \mathcal{B}$ and any $x \in B$, there exists some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B$.

Proof. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$, then $\mathcal{T} \subseteq \mathcal{T}'$. Take any $B \in \mathcal{B}$, then by Proposition 1.1.4, $B \in \mathcal{T}$, which means that $B \in \mathcal{T}'$. Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$, by Definition 1.1.2 for any $x \in B$, there exists some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B$.

Suppose conversely that for every $B \in \mathcal{B}$ and any $x \in B$, there is some $B_x \in \mathcal{B}'$ such that $x \in B' \subseteq B$. Take any $T \in \mathcal{T}$, for each $x \in T$, by Definition 1.1.2 there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq T$, and so we can find some $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq B \subseteq T$, so $T \in \mathcal{T}'$. Therefore, $\mathcal{T} \subseteq \mathcal{T}'$ and so $\mathcal{T}'$ is finer than $\mathcal{T}$. □

Recall that every basis of a topology on X is an open cover of X consisting only of subsets of X. Therefore, the union of the elements in the basis is essentially X itself. This motivates us to propose another way to generate a topology on a set.

Definition 1.1.8 ▸ Sub-basis

A **sub-basis** of X is a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ such that $\bigcup_{A \in \mathcal{S}} A = X$.

Remark. Every basis is a sub-basis.

For an arbitrary set X, let $\mathcal{S}$ be a sub-basis and denote the collection of all finite subsets of $\mathcal{P}(\mathcal{S})$ as $\mathcal{F}_\mathcal{S}$. Define

$$\mathcal{U}_\mathcal{S} := \left\{ \bigcap_{A \in F} A : F \in \mathcal{F}_\mathcal{S} \right\}$$

to be the collection of all finite intersections of sets in $\mathcal{S}$. The topology generated by a sub-basis of X is given by

$$\mathcal{T} := \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{U}_\mathcal{S} \right\}.$$

We shall show that $\mathcal{T}$ is indeed a topology on X by considering the following proposition:

Proposition 1.1.9 ▶ Finite Intersections of Sets in a Sub-basis Form a Basis

Let $\mathcal{S}$ be a sub-basis for a set X and let $\mathcal{U}_\mathcal{S}$ be the set of all finite intersections of sets in $\mathcal{S}$, then $\mathcal{U}_\mathcal{S}$ is a basis of a topology on X.

Proof. Take any $x \in X$. By Definition 1.1.8, we have $x \in \bigcup_{A \in \mathcal{S}} A$. Therefore, there exists some $A \in \mathcal{S} \subseteq \mathcal{P}(X)$ such that $x \in A$. For any $x \in X$ and $B_1, B_2 \in \mathcal{U}_\mathcal{S}$ such that $x \in B_1 \cap B_2$, notice that $B_1 \cap B_2$ is a finite intersection of sets in $\mathcal{S}$, so $B_1 \cap B_2 \in \mathcal{U}_\mathcal{S}$. Therefore, by Definition 1.1.2, $\mathcal{U}_\mathcal{S}$ is a basis. □

With Propositions 1.1.9 and 1.1.4, it is clear that $\mathcal{T}$ as constructed above is a topology on X.

1.2 Metric Spaces

Definition 1.2.1 ▶ Metric

A **metric** on a set S is a function $d : S \times S \to \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, y) = 0$ if and only if $x = y$ (definiteness);
- $d(x, y) = d(x, y)$ for all $x, y \in S$ (symmetry);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

Remark. A metric is sometimes also called a *distance function*.

A metric generalises the notion of distance in Euclidean spaces. We can weaken the above axioms to arrive at the following definition:

Definition 1.2.2 ▸ Pseudo-metric

A **pseudo-metric** on a set S is a function $d : S \times S \to \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, x) = 0$ for all $x \in S$;
- $d(x, y) = d(x, y)$ for all $x, y \in S$ (symmetry);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

The key difference between a pseudo-metric and a metric is that a pseudo-metric only requires that every element is at 0 distance away from itself, whereas a metric requires that every element is **the only element** that is at 0 distance away from itself.

By dropping the requirement on symmetry, we obtain the following definition:

Definition 1.2.3 ▸ Quasi-metric

A **quasi-metric** on a set S is a function $d : S \times S \to \mathbb{R}$ such that:

- $d(x, y) \geq 0$ for all $x, y \in S$ (positivity);
- $d(x, y) = 0$ if and only if $x = y$ (definiteness);
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in S$ (triangular inequality).

We equip a set with a metric to generalise the Euclidean spaces.

Definition 1.2.4 ▸ Metirc Space

A **metric space** (S, d) is a set S together with a metric d on S.

The most basic example of a metric is the *discrete metric* defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

over any set X, which essentially is just a characteristic function.

Recall that in an inner product space (V, g) over some field $\mathbb{F}$, we can define the length of any $\boldsymbol{v} \in V$ as

$$\|\boldsymbol{v}\| = \sqrt{g(\boldsymbol{v}, \boldsymbol{v})}.$$

This length function induces a metric over V given by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|.$$

In the Euclidean space $\mathbb{R}^n$, a usual definition for distance is

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{n} (y_i - x_i)^2\right]^{\frac{1}{2}}.$$

Note that $(\mathbb{R}^n, d_2)$ is a metric space, where d_2 is known as the *Euclidean distance*. In general, we can prove that for any $p \in \mathbb{N}^+$,

$$d_p(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{n} \|y_i - x_i\|^p\right]^{\frac{1}{p}}$$

is a metric over $\mathbb{F}^n$ for any inner product space $(\mathbb{F}^n, g)$ where $\mathbb{F}$ is a field, known as the L^p*-norm*. Furthermore, notice that

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p \leq \sum_{i=1}^{n} \|y_i - x_i\|^p \leq n \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p.$$

Taking the p-th root on all three parts, we have

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\| \leq \left[\sum_{i=1}^{n} \|y_i - x_i\|^p\right]^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

By Squeeze Theorem, this allows us to define

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \lim_{p \to \infty} d_p(\boldsymbol{x}, \boldsymbol{y}) = \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

$d_\infty(\boldsymbol{x}, \boldsymbol{y})$ can be alternatively written as $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty$, which is known as the *infinite norm*.

The p-adic numbers can be defined from the following lemma:

Lemma 1.2.5 ▶ p-adic Numbers

Let p be any prime number. For all $x \in \mathbb{Q} \setminus \{0\}$, there exists a unique $k \in \mathbb{Z}$ such that

$$x = \frac{p^k r}{s}, \qquad r, s \in \mathbb{Z}$$

with $p \nmid r, s$ and $s \neq 0$.

The *p-adic norm* is defined as

$$|x|_p = \begin{cases} p^{-k} & \text{if } x = \frac{p^k r}{s} \\ 0 & \text{if } x = 0 \end{cases},$$

which induces a metric over $\mathbb{Q}$ defined by

$$d(x, y) = |x - y|_p.$$

We can show that the p-adic metric satisfies

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

for all $x, y, z \in \mathbb{Q}$. Such a metric is known as an *ultra-metric*.

Given any metric space, the metric will induce a distance between subsets of the space.

Definition 1.2.6 ▸ Distance between Subsets

Let (X, d) be a metric space and $A, B \subseteq X$ be non-empty. The **distance** between A and B is defined as

$$d(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

Additionally, we may wish to define a measure for the size of a subset in a metric space.

Definition 1.2.7 ▸ Diameter

Let (X, d) be a metric space. The **diameter** of a set $A \subseteq X$ is defined as

$$\operatorname{diam}(A) := \sup\{d(x, y) : (x, y) \in A \times A\}.$$

The set A is **bounded** if $\operatorname{diam}(A)$ is finite.

The name "diameter" is not a coincidence with the diameter of a graph. Specifically, if we consider a graph $G = (V, E)$, the pair (V, d) forms a metric space with $d(u, v)$ being the usual distance between two vertices in G defined as the size of the shortest u-v path in G. It is clear that d is indeed a metric.

Now, let us consider the subgraph $H \subseteq G$ induced by any $U \subseteq V$ and check the eccentricity for H, i.e.,

$$\epsilon(u) = \max\{d_H(u, u') : u' \in U\} \qquad \text{for all } u \in U.$$

Now, the diameters for H can be computed as

$$\begin{aligned}\operatorname{diam}(H) &= \max\{\epsilon(u) : u \in U\} \\ &= \sup\{d_H(u, u') : (u, u') \in U \times U\},\end{aligned}$$

and this obviously agrees with Definition 1.2.7!

Recall that in Definition 1.1.5, we use Euclidean open balls to construct a basis for a topology on $\mathbb{R}^n$. We can generalise this idea in any metric space.

Proposition 1.2.8 ▶ Metric Induces a Basis

Let (X, d) be a metric space. Define

$$B_r(x) := \{y \in X : d(x, y) < r\},$$

then collection

$$\mathcal{B}_d := \{B_r(x) : x \in X, r \in \mathbb{R}^+\}$$

is a basis for a topology on X.

Proof. Notice that for any $x \in X$, we have $x \in B_1(x) \in \mathcal{B}_d$. Let $B_p(x_1), B_q(x_2) \in \mathcal{B}_d$ be such that $x \in B_p(x) \cap B_q(x)$. Take $k = \min\{p - d(x, x_1), q - d(x, x_2)\}$, then clearly $k > 0$ and we can find $B_k(x) \subseteq B_p(x) \cap B_q(x)$ such that $x \in B_k(x) \in \mathcal{B}_d$. Therefore, $\mathcal{B}_d$ is a basis for a topology on X. □

Since we can obtain a basis from a metric, it follows naturally that we can generate a topology using this induced basis.

Definition 1.2.9 ▶ Metrisable Topology

Let (X, d) be a metric space. A topology $\mathcal{T}$ on X is **metrisable**, or **induced** by d, if it is generated by $\mathcal{B}_d$

We can verify that the discrete topology $\mathcal{P}(X)$ is induced by the discrete metric. Let the discrete metric on X be χ, then it is easy to see that

$$B_r(x) = \begin{cases} \{x\} & \text{if } 0 < r \leq 1 \\ X & \text{if } r > 1 \end{cases}.$$

Therefore,

$$\mathcal{B}_\chi = \{X\} \cup \big\{\{x\} : x \in X\big\}.$$

Let $\mathcal{T}_\chi$ be the topology on X generated by $\mathcal{B}_\chi$, then it suffices to prove that $\mathcal{P}(X) \subseteq \mathcal{T}_\chi$. Take any $U \in \mathcal{P}(X)$, then for any $u \in U$, we have $u \in \{u\} \subseteq U$. Clearly $\{u\} \in \mathcal{B}_\chi$, so $\mathcal{T}_\chi = \mathcal{P}(X)$ is the discrete topology indeed.

In particular, for Euclidean spaces, the following result extends Definition 1.1.5:

Proposition 1.2.10 ▸ Every L^p-metric Generates the Standard Topology

Let $\mathcal{T}$ be the standard topology on $\mathbb{R}^n$, then $\mathcal{T}$ is induced by any L^p-metric d_p.

Proof. For any $p \in \mathbb{N}^+$, notice that

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p \leq \sum_{i=1}^{n} \|y_i - x_i\|^p \leq n \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|^p.$$

Taking the p-th root yields

$$\max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\| \leq \left[\sum_{i=1}^{n} \|y_i - x_i\|^p\right]^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max_{i \in \mathbb{N}^+, i \leq n} \|y_i - x_i\|.$$

This means that

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_p(\boldsymbol{x}, \boldsymbol{y}) \leq n^{\frac{1}{p}} d_\infty(\boldsymbol{x}, \boldsymbol{y}).$$

Let $\mathcal{T}_0$ and $\mathcal{T}_p$ be topologies on $\mathbb{R}^n$ generated by $\mathcal{B}_{d_\infty}$ and $\mathcal{B}_{d_p}$ respectively. Take any $T \in \mathcal{T}_p$, then for any $\boldsymbol{t} \in T$, there is some $B_r(\boldsymbol{t}') \in \mathcal{B}_{d_p}$ such that $\boldsymbol{t} \in B_r(\boldsymbol{t}') \subseteq T$. Take some $\ell = \frac{1}{2}\left|r - d_p(\boldsymbol{t}, \boldsymbol{t}')\right|$, then we have found $B_\ell(\boldsymbol{t}) \in \mathcal{B}_{d_p}$ such that

$$\boldsymbol{t} \in B_\ell(\boldsymbol{t}) \subseteq B_r(\boldsymbol{t}') \subseteq T.$$

Take $k = \frac{1}{2}\ell n^{-\frac{1}{p}}$ and consider

$$B_k(\boldsymbol{t}) := \{\boldsymbol{y} \in \mathbb{R}^n : d_\infty(\boldsymbol{t}, \boldsymbol{y}) < k\} \in \mathcal{B}_{d_\infty}.$$

Notice that for each $\boldsymbol{y} \in B_k(\boldsymbol{t})$, we have

$$d_p(\boldsymbol{t}, \boldsymbol{y}) \leq n^{\frac{1}{p}} d_\infty(\boldsymbol{t}, \boldsymbol{y}) < \ell,$$

so $\boldsymbol{t} \in B_k(\boldsymbol{t}) \subseteq B_\ell(\boldsymbol{t}) \subseteq T$. This implies that $T \in \mathcal{T}_0$ and so $\mathcal{T}_p \subseteq \mathcal{T}_0$. By a similar argument, one may check that $\mathcal{T}_0 \subseteq \mathcal{T}_p$. Therefore, $\mathcal{T}_0 = \mathcal{T}_p$ for any $p \in \mathbb{N}^+$. Note that by Definition 1.1.5, $\mathcal{T}$ is generated by $\mathcal{B}_{d_2}$, which means that $\mathcal{T} = \mathcal{T}_2 = \mathcal{T}_0 = \mathcal{T}_p$ for any $p \in \mathbb{N}^+$. Therefore, $\mathcal{T}$ is induce by any L^p-metric d_p. □

The fact that

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_p(\boldsymbol{x}, \boldsymbol{y}) \leq n^{\frac{1}{p}} d_\infty(\boldsymbol{x}, \boldsymbol{y})$$

means that all L^p-metrics are equivalent over the same space.

1.3 Subspace Topologies

Definition 1.3.1 ▸ Subspace Topology

Let $(Y, \mathcal{T}_Y)$ be a topological space and $X \subseteq Y$ be some subset. The collection

$$\mathcal{T}_X := \{U \cap X : U \in \mathcal{T}_Y\}$$

is the **subspace topology** on X.

We may check that $\mathcal{T}_X$ defined as such is indeed a topology on X. First, by taking $U = \varnothing$ and $U = Y$ respectively, we know that $\varnothing, X \in \mathcal{T}_X$. For any $U \in \mathcal{T}_Y$, we have $Y \backslash U \in \mathcal{T}_Y$ and so

$$X \backslash (U \cap X) = (Y \backslash U) \cap X \in \mathcal{T}_X.$$

For any $\mathcal{V} \subseteq \mathcal{T}_X$, we define a subset $\mathcal{U}_\mathcal{V} \subseteq \mathcal{T}_Y$ such that for each $V \in \mathcal{V}$ there is a unique $U_V \in \mathcal{U}_\mathcal{V}$ such that $V = U_V \cap X$. Then,

$$\begin{aligned} \bigcup_{A \in \mathcal{V}} A &= \bigcup_{B \in \mathcal{U}_\mathcal{V}} (B \cap X) \\ &= \left(\bigcup_{B \in \mathcal{U}_\mathcal{V}} B \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

Let $X_1, X_2, \cdots, X_n \in \mathcal{T}_X$ and define $X_i = U_i \cap X$ where $U_i \in \mathcal{T}_Y$ for $i = 1, 2, \cdots, n$, then

$$\begin{aligned} \bigcap_{i=1}^{n} X_i &= \bigcap_{i=1}^{n} (U_i \cap X) \\ &= \left(\bigcap_{i=1}^{n} U_i \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

So $\mathcal{T}_X$ is really a topology on X. Intuitively, the following holds:

Proposition 1.3.2 ▸ Basis for a Subspace

Let $(Y, \mathcal{T}_Y)$ be a topological space and $\mathcal{T}_X$ be the subspace topology on some $X \subseteq Y$. If $\mathcal{B}_Y$ is a basis of $\mathcal{T}_Y$, then

$$\mathcal{B}_X := \{B \cap X : B \in \mathcal{B}_Y\}$$

is a basis of $\mathcal{T}_X$.

Proof. We first prove that $\mathcal{B}_X$ is a basis. Take any $x \in X \subseteq Y$. Note that there exists some $B \in \mathcal{B}_Y$ such that $x \in B$. Take $B \cap X \in \mathcal{B}_X$, then $x \in B \cap X$. For any $B_1, B_2 \in \mathcal{B}_X$ with $x \in B_1 \cap B_2$, we write $B_1 := B_1' \cap X$ and $B_2 := B_2' \cap X$ where $B_1', B_2' \in \mathcal{B}_Y$, then we have $x \in B_1' \cap B_2'$. This means that there is some $B \in \mathcal{B}_Y$ such that $x \in B \subseteq B_1' \cap B_2'$. Write $B' := B \cap X \in \mathcal{B}_X$, then for each $b \in B'$, we know that $b \in B_1' \cap B_2'$ and $b \in X$, which implies that $b \in B_1 \cap B_2$. Therefore, $x \in B' \subseteq B_1 \cap B_2$. This means that $\mathcal{B}_X$ is a basis of a topology on X.

We then prove that $\mathcal{T}_X$ is generated by $\mathcal{B}_X$. Let $\mathcal{T}$ be the topology generated by $\mathcal{B}_X$. By Proposition 1.1.4, we have

$$\mathcal{T} = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{B}_X \right\}.$$

Similarly, we can write

$$\mathcal{T}_Y = \left\{ \bigcup_{A \in \mathcal{V}} A : \mathcal{V} \subseteq \mathcal{B}_Y \right\}.$$

Take any $T \in \mathcal{T}_X$, then there exists some $\mathcal{V} \subseteq \mathcal{B}_Y$ such that

$$\begin{aligned} T &= \left(\bigcup_{A \in \mathcal{V}} A \right) \cap X \\ &= \bigcup_{A \in \mathcal{V}} A \cap X \\ &\in \mathcal{T}. \end{aligned}$$

Therefore, $\mathcal{T}_X \subseteq \mathcal{T}$. Conversely, take any $T' \in \mathcal{T}$, there exists some $\mathcal{U} \subseteq \mathcal{B}_Y$ such that

$$\begin{aligned} T' &= \bigcup_{B \in \mathcal{U}} (B \cap X) \\ &= \left(\bigcup_{B \in \mathcal{U}} B \right) \cap X \\ &\in \mathcal{T}_X. \end{aligned}$$

Therefore, $\mathcal{T} \subseteq \mathcal{T}_X$ and so $\mathcal{T}_X = \mathcal{T}$. $\square$

The following result shows that open sets in subspaces remain open in the superspace:

Proposition 1.3.3 ▸ Superspace Preserve Open Sets

Let $(Y, \mathcal{T}_Y)$ be a topological space. If $X \subseteq Y$ is open in Y and $U \subseteq X$ is open in X, then U is open in Y.

Proof. Let $\mathcal{T}_X$ be the subspace topology on X. Since U is open in X, we have $U \in \mathcal{T}_X$. By Definition 1.3.1, there exists some $V \in \mathcal{T}_Y$ such that $U = V \cap X$. However, $U \subseteq X$, so $U = V \in \mathcal{T}_Y$, which means that U is open in Y. □

We can do a similar manipulation with metric spaces and induce a metric on a subspace.

Definition 1.3.4 ▸ Subspace Metric

Let (X, d) be a metric space. The **subspace metric** of some $A \subseteq X$ is the restriction of d to A, denoted as

$$d_A(x, y) = d(x, y), \qquad \text{for all } x, y \in A.$$

Naturally, the following result is true:

Proposition 1.3.5 ▸ Subspace Metric Induces Subspace Topology

Let (X, d) be a metric space. The topology induced by the subspace metric d_A on some subspace $A \subseteq X$ is the subspace topology on A.

Proof. Let $\mathcal{T}_d$ and $\mathcal{T}_{d_A}$ be topologies induced by d on X with basis $\mathcal{B}_d$ and by d_A on A with basis $\mathcal{B}_{d_A}$ respectively. Let $\mathcal{T}_A$ be the subspace topology on A with basis $\mathcal{B}_A$. Take any $B \in \mathcal{B}_{d_A} \subseteq \mathcal{B}_d$, then clearly any $x \in B$ is such that $x \in B \cap A \in \mathcal{B}_A$. Therefore, by Proposition 1.1.7, $\mathcal{T}_{d_A} \subseteq \mathcal{T}_A$. Conversely, take any $B_A \mathcal{B}_A$. For any $x \in B_A$, there exists some $B_1 \in \mathcal{B}_d$ such that $x \in B_1$ and $B_A = B_1 \cap A$. Notice that this implies that $x \in A \in \mathcal{T}_{d_A}$, so we can find some $B_2 \in \mathcal{B}_{d_A} \subseteq \mathcal{B}_d$ such that $x \in B_2$. By Definition 1.1.2, there exists some $B \subseteq B_1 \cap B_2 \in \mathcal{B}_{d_A}$ such that $x \in B$. By Proposition 1.1.7, this means that $\mathcal{T}_A \subseteq \mathcal{T}_{d_A}$. Therefore, $\mathcal{T}_A = \mathcal{T}_{d_A}$. □