MATH 20E PROJECT

The Vector Calculus application in Ampère's circuital law

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1 Introduction to the Line Integral

Vector Calculus and its applications in the world have proven to be very important to understanding the laws of nature in a mathematical context. One such application involves the line integral, denoted as;

$$\int_C F \cdot dS = \int_a^b F(c(t)) \cdot c'(t)dt$$

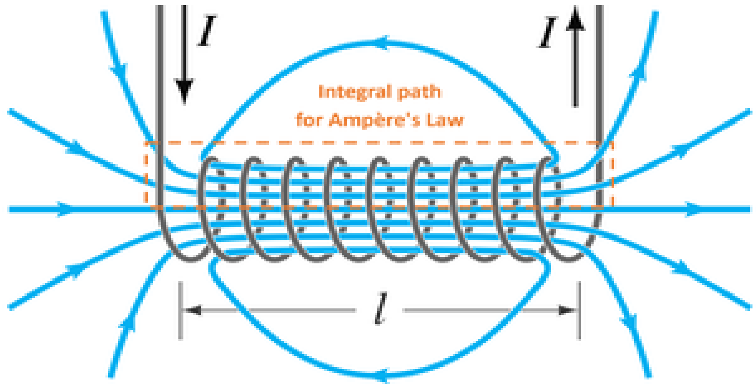
Where F is a vector field on $\mathbb{R}^3$ that is continuous on the C^1 path $c : [a,b] \to \mathbb{R}^3$. Thus we would subsequently read the above formula as being the **line integral** of **F** along **c**. We are all used to seeing abstract high-level descriptions of formulas all throughout math, but what really does a line integral tell us? How is it any different from a regular integral that helps us find the area under a curve? Normal integrals that we observed in Calculus 1 and 2 tell us about finding total area under curves. With line integrals, the importance shifts from under curves to the curve itself. Such that the line integral becomes *the* function that is evaluated. In other words, a line integral is best thought as a measurement of the total effect along a curve. More commonly, we see line integrals as a way to quantify *work done* in physics in which

$$W = \int_L F(s) \cdot dS$$

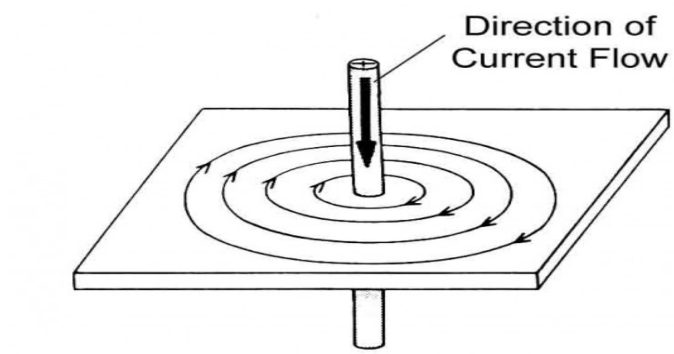
This integral would generally find the total work done on some arbitrary object that is moving through a field (regardless whether the field is scalar or vector) **F** along the path **s**.

2 Using Line Integrals in classical electromagnetism

There are no doubt many applications of the line integral, but the one this project will be focusing on is its application in Ampère's circuital law in classical electromagnetism. At the fundamental level, Ampère's circuital law is able to mathematically define a relationship between an electrical current and the magnetic field that is created by it.



When electricity is conducted through a wire, a current is created. This current is able to produce a magnetic field around the wire that is able to attract or repel other objects. The direction of the current is important to what direction the magnetic field coincides with.



This magnetic field is much stronger when it is close to the conductive wire, and weaker the longer the radius of the magnetic field is. Whats interesting to note is that the magnetic strength is uniform at any radius r from the wire itself. Such that the strength of the magnetic field is the same around the circle. This in essence is what Ampère's circuital law attempts to explain to us. However, there are a few *conditions* so-to-speak, that system must follow in order to successfully apply the law.

3 The Calculus of Ampère's circuital law

1. Ampère's circuital law can be applied to any closed loop, not just a circle. However we tend to use a circle cause it is much easier to explain

2. The current in the closed loop *must* be constant

The Vector Calculus text book on page 372 gives us a simple form of Ampère's circuital law, in terms of $\mathbf{H}$, which just takes into account the free current in amperes per meter, i.e $A \cdot m^{-1}$ While the magnetic fields $\mathbf{B}$ and $\mathbf{H}$ both can accurately account for their respective fields, they do so in different ways. At the end of the day, this project is based around Vector Calculus and its application in electromagnetism, so there won't be much detail as to why $\mathbf{B}$ / $\mathbf{H}$ are different, it's just important to note that two equivalent mathematical equations of Ampère's circuital law stem from them. The two integral forms of Ampère's circuital law are as follows;

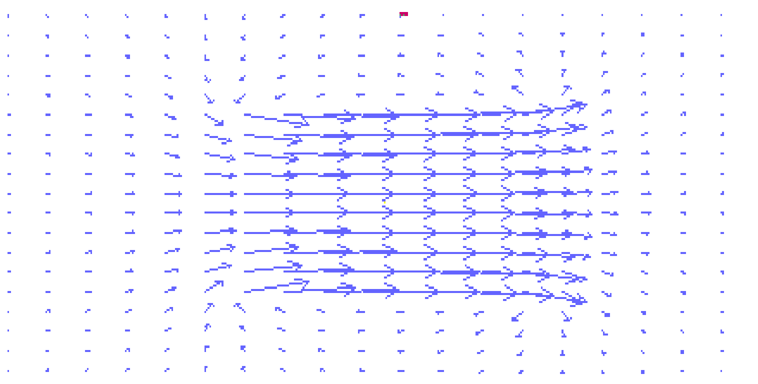
$$\oint_C B \cdot dl = \mu_0 \iint_S J \cdot dS$$

$$\oint_C H \cdot dl = \iint_S J_f \cdot dS$$

As stated before, since the textbook talks about the magnetic field **H**, the paper will opt to discuss the first equation, in terms of **B**, just to be contrarian.

$$\oint_C B \cdot dl = \mu_0 \iint_S J \cdot dS$$

1. **B**, known as the magnetic flux density, in Vector Calculus, the magnetic flux density field would be considered a solenodial vector field



2. $\oint_C$ is known to be a line integral that represents a closed loop

3. dl which is known to us as the differential, or the collection of infinitesimal fragments of the line integral added together.

This is all equal to: [This part of the multi-part-equation will not be covered rigorously]

1. μ_0, this is interesting as its known as a magnetic constant (or the permeability of a vacuum), it is given as roughly, $4\pi 10^{-7} N/A^2$ (Newtons / Amperes squared)

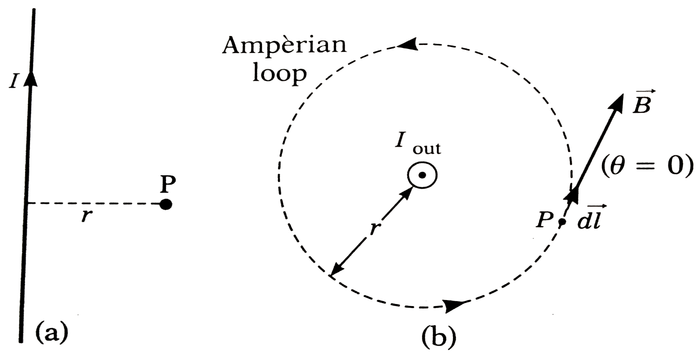
2. $\iint_S$ is the 2-dimensional surface integral over C.

3. J is the total current density (this is what differentiates **B** from **H**)

4. dS is the area of the surface S

4 An introductory example of Ampère's circuital law

As again, this is just an introductory explanation to Ampère's circuital law, because of this, we will only worry about $\oint_C B \cdot dl$ which can very much be applied to a straight wire.



The figure above describes how the Amperian loop is relative to the magnetic field of the wire. Generally, because the angle θ is parallel universally to the wire, θ always ends up being the 0 such that $cos(0) = 1$. Such in practical use, θ can very much be disregarded as $1 \cdot \alpha = \alpha$. Also note that the magnetic field **B** is constant all around the wire. Understanding the rules of integration, **B** can be moved to the front. In the example given above, Ampère's circuital law becomes very easy to compute as

$$\oint_C B \cdot dl$$

of a straight wire with a constant magnetic field is simply $B \cdot 2 \cdot \pi \cdot r$, **B** being the magnetic field and $2\pi r$ as the circumference of a circle. Again, note that the closed loop of the line integral does not need to be a circle, but for simplicity's sake, we can assume that it is.
In summation, what Ampère's circuital law tells us that there is a very defined relationship between the magnetic field of a closed loop and the electric current that passes through the wire(or loop).

5 Sources

Wiki Page
Youtube video
Educational website about the law