数值分析

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# Homework 3

## 1 Theoretical Questions

### 1.1 Assignment.I

$$p(0) = 0, \quad p(1) = 1, \quad p'(1) = -3, \quad p''(1) = 6$$

构造差分表为:

| | | | | |
|---|---|---|---|---|
| 0 | 0 | | | |
| 1 | 1 | 1 | | |
| 1 | 1 | -3 | -4 | |
| 1 | 1 | -3 | 3 | 7 |

$$p(x) = x - 4x(x-1) + 7x(x-1)^2 = 7x^3 - 18x^2 + 12x$$

而 $s''(0) \neq 0$，因此 $p(x)$ 不是自然三次样条。

### 1.2 Assignment.II

(a) 因为 $s \in \mathbb{S}_2^1$，在构造 $p_1(x)$ 时，只有 $p_1(x_1) = f_1$ 和 $p_1(x_2) = f_2$ 两个条件，而想构造 2 次多项式 $p_1(x)$，还需要 1 个条件。

(b) 通过构造差分表可以求出:

$$p_i(x) = f_i + m_i(x - x_i) + \frac{f_{i+1} - f_i + m_i(x_{i+1} - x_i)}{(x_{i+1} - x_i)^2}(x - x_i)^2$$

(c) 通过差分表构造出 $p_i(x)$ 为:

$$p_i(x) = f_i + m_i(x - x_i) + \frac{f[x_i, x_{i+1}] - m_i}{x_{i+1} - x_i}(x - x_i)^2$$

由 $p'_{i-1}(x_i) = p'_i(x_i)$ 化简得到:

$$m_{i-1} + m_i = 2f[x_{i-1}, x_i], \quad i = 2, 3, \cdots, n$$

联立后求解线性方程组即可求出 $m_2, m_3, \cdots, m_{n-1}$.

### 1.3 Assignment.III

由 Lemma4.3 和 Lemma4.5 有以下二式:

$$\frac{1}{2}s_1'(-1) + 2s_1'(0) + \frac{1}{2}s_2'(1) = \frac{3}{2}\left[s_2(1) - 1 - c\right] + \frac{3}{2}c$$
$$\frac{1}{2}s_1''(-1) + 2s_1''(0) + \frac{1}{2}s_2''(1) = 3\left(s_2(1) - 1 - c\right)$$

得出: $s_2(1) = 6c + 1$, $s_2'(1) = 6c$, 结合 $s_2(0) = s_1(0) = 1 + c$, $s_2''(1) = 0$ 得:

$$s_2(x) = 1 + c + 5cx + cx(x-1) - cx(x-1)^2 = -cx^3 + 3cx^2 + 3cx + c + 1$$

若 $s(1) = -1$, 则 $c = -\frac{1}{3}$.

### 1.4 Assignment.IV

(a) 首先有 $f_1 = 0$, $f_2 = 1$, $f_3 = 0$, 接着由 $M_1 = M_3 = 0$ 求出 $M_2 = -3$. 于是由 Lemma4.5 构造出 $s(x)$ 为:

$$s(x) = \begin{cases} -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & x \in [-1, 0] \\ \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & x \in [0, 1] \end{cases}$$

(b) 首先我们有:

$$\int_{-1}^{1} \left[s''(x)\right]^2 dx = 6$$

(i) 此时, $g(x) = x + 1 - x(x+1) = -x^2 + 1$, 那么 $\int_{-1}^{1} \left[g''(x)\right]^2 dx = 8$, 故而有

$$\int_{-1}^{1} \left[s''(x)\right]^2 dx < \int_{-1}^{1} \left[g''(x)\right]^2 dx$$

(ii) 此时, $g(x) = f(x) = cos(\frac{\pi}{2}x)$, 那么 $\int_{-1}^{1} \left[g''(x)\right]^2 dx = \frac{\pi^4}{16} > 6$, 因此

$$\int_{-1}^{1} \left[s''(x)\right]^2 dx < \int_{-1}^{1} \left[g''(x)\right]^2 dx$$

## 1.5 Assignment.V

(a)

$$B_i^2(x) = \frac{x - t_{i-1}}{t_{i+n} - t_{i-1}}\hat{B}_i(x) + \frac{t_{i+n+1} - x}{t_{i+n+1} - t_i}\hat{B}_{i+1}(x)$$
$$= \begin{cases} \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & x \in (t_{i-1}, t_i] \\ \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} & x \in (t_i, t_{i+1}] \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & x \in (t_{i+1}, t_{i+2}] \\ 0 & otherwise \end{cases}$$

(b) 由 Thm4.35：

$$\frac{d}{dx}B_i^2(x) = \frac{2B_i^1(x)}{t_{i+1} - t_{i-1}} - \frac{2B_{i+1}^1(x)}{t_{i+2} - t_i}$$

因为右式两项都在 $t_i, t_{i+1}$ 上连续，则 $\frac{d}{dx}B_i^2(x)$ 在 $t_i, t_{i+1}$ 上连续。

(c) 当 $x \in (t_{i-1}, t_i]$ 时，上式右端前半项不为 0, 后半项为 0, 则 $x$ 不是零点。同理，$x \in (t_{i+1}, t_{i+2}]$ 时也是如此。则 $x^* \in [t_i, t_{i+1})$，且

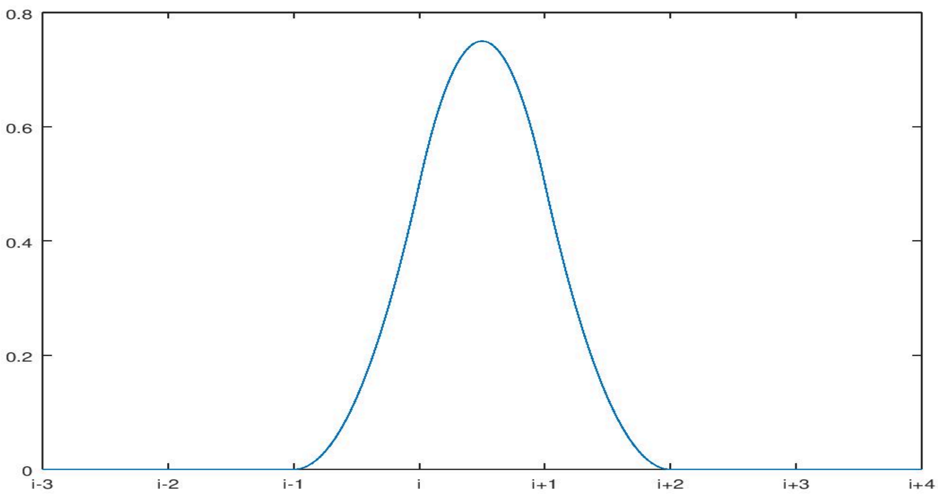
$$x^* = \frac{t_{i+1}t_{i+2} - t_i t_{i-1}}{t_{i+2} + t_{i+1} - t_i - t_{i-1}}$$

(d) 首先 $B_i^2(x) \geqslant 0$，又因为 $\frac{d}{dx}B_i^2(x)$ 只有 $x^*$ 一个零点，且 $x < x^*$ 时 $B_i^2(x)$ 递增，$x > x^*$ 时 $B_i^2(x)$ 递减，则

$$\max B_i^2(x) = B_i^2(x^*) < 1$$

于是，$B_i^2(x) \in [0, 1)$.

(e) 绘制 $B_1^2(x)$ 图像如下：



### 1.6 Assignment.VI

$$
\begin{aligned}
&(t_{i+2}-t-i-1)[t_{i-1},t_i,t_{i+1},t_{i+2}](t-x)_+^2 \\
=&\frac{\frac{(t_{i+1}-x)_+^2-(t_i-x)_+^2}{t_{i+1}-t_i}-\frac{(t_i-x)_+^2-(t_{i-1}-x)_+^2}{t_i-t_{i-1}}}{t_{i+1}-t_{i-1}}-\frac{\frac{(t_{i+2}-x)_+^2-(t_{i+1}-x)_+^2}{t_{i+2}-t_{i+1}}-\frac{(t_{i+1}-x)_+^2-(t_i-x)_+^2}{t_{i+1}-t_i}}{t_{i+2}-t_i} \\
=&\begin{cases}
\frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & x\in(t_{i-1},t_i] \\
\frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}+\frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} & x\in(t_i,t_{i+1}] \\
\frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & x\in(t_{i+1},t_{i+2}] \\
0 & otherwise
\end{cases} \\
=&B_i^2(x)
\end{aligned}
$$

### 1.7 Assignment.VII

由 Thm4.35：对 $\forall n\geqslant 1$，

$$
\begin{aligned}
0=B_i^{n+1}(t_{i+n+1})-B_i^{n+1}(t_{i-1})&=\int_{t_{i-1}}^{t_{i+n+1}}\left[\frac{d}{dx}B_i^{n+1}(x)\right]dx \\
&=\frac{n}{t_{i+n}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n+1}}B_i^n(x)dx-\frac{n}{t_{i+n+1}-t_i}\int_{t_{i-1}}^{t_{i+n+1}}B_{i+1}^n(x)dx \quad \cdots(*)
\end{aligned}
$$

注意到，$x\in[t_{i+n},t_{i+n+1}]$ 时，$B_i^n(x)=0$，且 $x\in[t_{i-1},t_i]$ 时，$B_{i+1}^n(x)=0$. 因此，

$$
\begin{aligned}
(*)=\frac{n}{t_{i+n}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n}}B_i^n(x)dx-\frac{n}{t_{i+n+1}-t_i}\int_{t_i}^{t_{i+n+1}}B_{i+1}^n(x)dx=0 \\
\Longrightarrow \frac{1}{t_{i+n}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n}}B_i^n(x)dx=\frac{1}{t_{i+n+1}-t_i}\int_{t_i}^{t_{i+n+1}}B_{i+1}^n(x)dx
\end{aligned}
$$

于是，$B_i^n(x)$ 与 $B_{i+1}^n(x)$ 的 scaled integral 相等，即 $B_i^n(x)$ 的 scaled integral 与 i 无关。

### 1.8 Assignment.VIII

(a) 构造差分表：

| $x_i$ | $x_i^4$ | | |
|---|---|---|---|
| $x_{i+1}$ | $x_{i+1}^4$ | $(x_{i+1}+x_i)(x_{i+1}^2+x_i^2)$ | |
| $x_{i+2}$ | $x_{i+2}^4$ | $(x_{i+2}+x_{i+1})(x_{i+2}^2+x_{i+1}^2)$ | $[x_i,x_{i+1},x_{i+2}]x^4$ |

于是推出：

$$
\begin{aligned}
[x_i, x_{i+1}, x_{i+2}]x^4 &= \frac{(x_{i+2}+x_{i+1})(x_{i+2}^2+x_{i+1}^2)-(x_{i+1}+x_i)(x_{i+1}^2+x_i^2)}{x_{i+2}-x_i} \\
&= \tau_2(x_i, x_{i+1}, x_{i+2})
\end{aligned}
$$

(b) 由 Lemma4.46：

$$
\begin{aligned}
&(x_{n+1}-x_1)\tau_k(x_1,\cdots,x_n,x_{n+1}) \\
=&\tau_{k+1}(x_1,\cdots,x_n,x_{n+1})-\tau_k(x_1,\cdots,x_n)-x_1\tau_k(x_1,\cdots,x_n,x_{n+1}) \\
=&\tau_{k+1}(x_1,\cdots,x_n,x_{n+1})+x_1\tau_k(x_1,\cdots,x_n,x_{n+1})-\tau_{k+1}(x_1,\cdots,x_n)-x_1\tau_k(x_1,\cdots,x_n,x_{n+1}) \\
=&\tau_{k+1}(x_2,\cdots,x_n,x_{n+1})-\tau_{k+1}(x_1,\cdots,x_n)
\end{aligned}
$$

接着采用归纳法，对 $n=0$，必然有

$$\tau_m(x_i)=[x_i]x^m$$

现在假设待证式对非负整数 $n<m$ 成立，那么上面的推导表明：

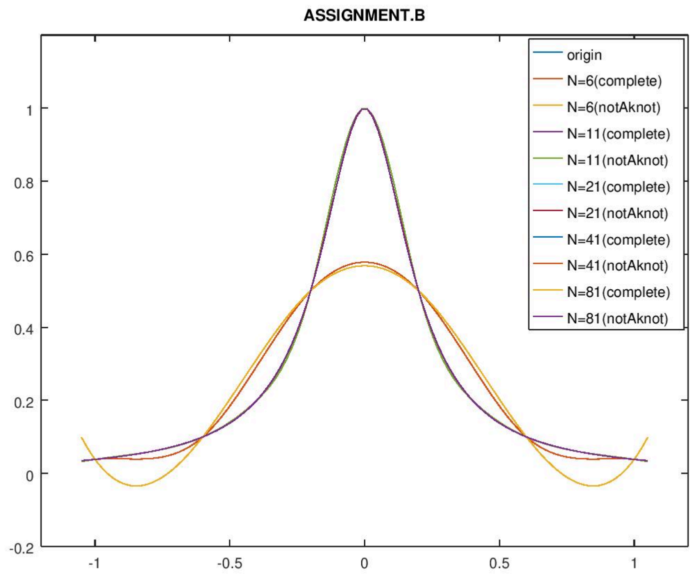
$$
\begin{aligned}
&\tau_{m-n-1}(x_i,\cdots,x_{i+n+1}) \\
=&\frac{\tau_{m-n}(x_{i+1},\cdots,x_{i+n+1})-\tau_{m-n}(x_i,\cdots,x_{i+n})}{x_{i+n+1}-x_i} \\
=&\frac{[x_{i+1},\cdots,x_{i+n+1}]x^m-[x_i,\cdots,x_{i+n}]x^m}{x_{i+n+1}-x_i} \\
=&[x_i,\cdots,x_{i+n+1}]x^m
\end{aligned}
$$

由归纳法，证毕。

# 2 Programming

## 2.1 Assignment.B

(1) 程序输出结果绘制图像如下：



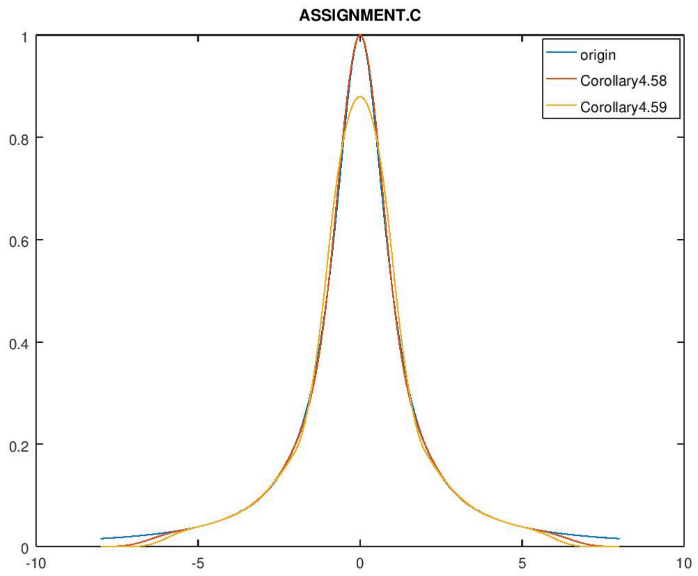

(2) 相应的误差向量的 max-norm 为:

```
When n = 10:
 Error for complete cubic spline: 0.0205289
 Error for notAknot cubic spline: 0.0205334
When n = 20:
 Error for complete cubic spline: 0.00316894
 Error for notAknot cubic spline: 0.00316894
When n = 40:
 Error for complete cubic spline: 0.00012413
 Error for notAknot cubic spline: 0.00012413
When n = 80:
 Error for complete cubic spline: 7.04042e-06
 Error for notAknot cubic spline: 7.04042e-06
```

可以看到收敛速度几乎一致。

## 2.2 Assignment.C & D

(1) 习题 C 的输出图像如下:



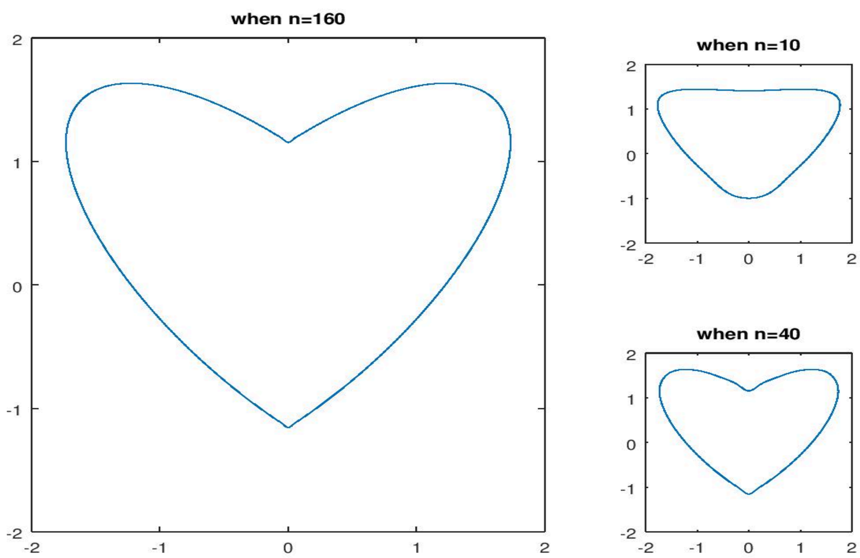

(2) 习题 D 的输出结果见于./output/exercise_CD.txt，有些误差非常接近机器精度是因为，这些点本身就是选取的“knots”，这些点处的误差本应是 0，只是在计算中保留为了机器精度。从图像上看，显然符合 Corollary 4.58 的三次样条更精确。

## 2.3 Assignment.E

在选点时，首先将原方程化为了极坐标方程

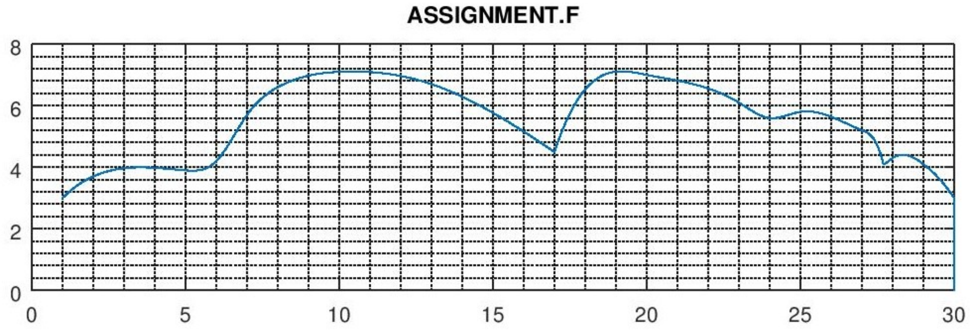
$$r = \sqrt{\frac{3}{(\frac{1}{4}sin\theta - 3|cos\theta|)sin\theta + 2}}$$

再把 $[0, 2\pi]$ 这个区间等分为 n 份，获得对应点的极坐标，再转化为直角坐标即可，绘制出的心形线如下：



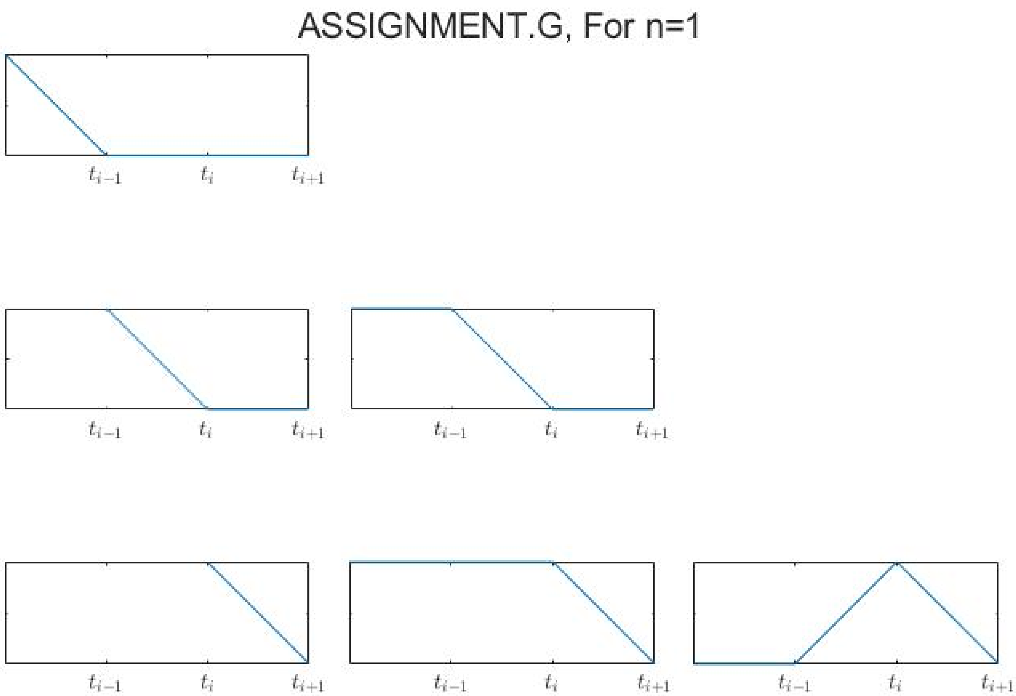
## 2.4 Assignment.F

程序运行结果绘制出的图像如下：

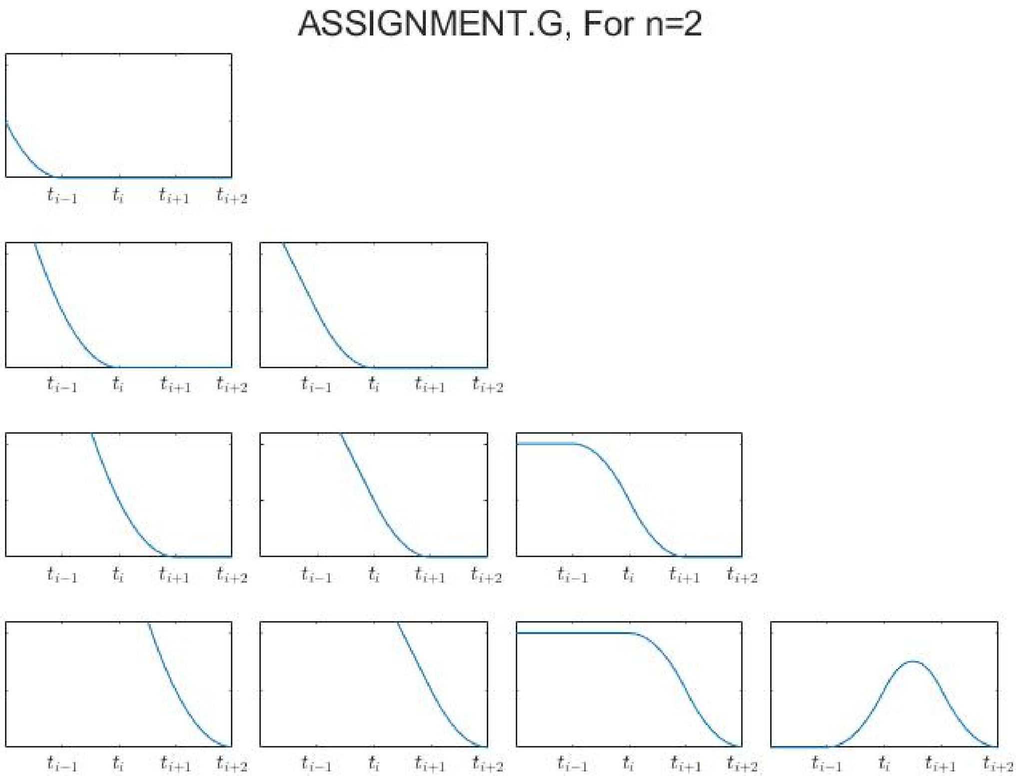



## 2.5 Assignment.G

(1) n=1:




(2) n=2:


ASSIGNMENT.G, For n=2
$t_{i-1}$ $t_i$ $t_{i+1}$ $t_{i+2}$