The effect of the gravitational redshift on subwavelength atomic arrays

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Arrays of two-level atoms in the single-excitation manifold can emit light collectively when the interatomic distance is lower than the transition wavelength, giving raise to the so called superradiance and subradiance phenomena. Although subradiant states have been widely studied due to their arbitrarily small decay rate, here I look forward to leveraging their narrow linewidth for metrology purposes. Since these states are thought to be very fragile, one would expect any tiny external signal to alter the system, and therefore it could be used for precise detection of this external signal. Specifically, in this manuscript, I present an overview of the effect of the gravitational redshift in an atomic array, centering my attention in the subradiant modes.

I. INTRODUCTION

In this manuscript, I provide a description of the effect of the gravitational redshift on ordered arrays. The main results are summarized as follows:

In Sec. II, I consider a ring of atoms, which is the only real system with a known analytical solution, and calculate analytically the eigenvalues and eigenmodes of the effective non-Hermitian Hamiltonian under the action of a weak redshift using perturbation theory.

Sec. III treats the case of a finite 1D array, firstly, I come up with some analytical but approximated expressions analogous to the ones in the previous section for the perturbed eigenvalues and eigenvectors. For that, I apply the expression for the eigenmodes of the finite array. The eigenvalues and eigenmodes are also computed numerically. Finally, I consider the preparation of a horizontal 1D array in its most subradiant state, then turn it until it is parallel to the vertical and study numerically the decay rate.

In Sec. IV, I consider the case of a finite 2D array and proceed with the same numerical calculations as in section (2) and compare the sensitivity of the 1D and 2D arrays with the same amount of atoms.

II. RING

Imagine an array of two-level atoms in free-space separated by a distance shorter than the transition wavelength. In all numerical calculations I will take an interatomic distance of $a = 0.2\lambda_0$. Without the gravitational redshift, this system has been well studied in the last years. Under the Born-Markov approximation and tracing out the photonic degrees of freedom, the system can be described with a dissipative "spin" model and its master equation is[1]:

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}[\mathcal{H}, \hat{\rho}] + \mathcal{L}[\hat{\rho}] \tag{1}$$

where $\hat{\rho}$ is the atomic density matrix, and the Hamiltonian and Lindblad operators read

$$\mathcal{H} = \hbar\omega_0 \sum_{i=1}^{N} \hat{\sigma}^i_{ee} + \hbar \sum_{i,j=1}^{N} J^{ij} \hat{\sigma}^i_{eg} \hat{\sigma}^j_{ge} \tag{2a}$$

$$\mathcal{L}[\hat{\rho}] = \sum_{i,j=1}^{N} \frac{\Gamma^{ij}}{2} (2\hat{\sigma}^j_{ge}\hat{\rho}\hat{\sigma}^i_{eg} - \hat{\sigma}^i_{eg}\hat{\sigma}^j_{ge}\hat{\rho} - \hat{\rho}\hat{\sigma}^i_{eg}\hat{\sigma}^j_{ge}) \tag{2b}$$

In the above expressions, ω_0 is the transition frequency, $\hat{\sigma}^i_{eg}$ and $\hat{\sigma}^i_{ge}$ are respectively the creation and lowering operators of the ith atom, $\hat{\sigma}^i_{ee} = \hat{\sigma}^i_{eg}\hat{\sigma}^i_{ge}$, and the rates for coherent and dissipative interactions between atoms i and j are given by

$$J^{ij} = -\frac{\mu_0\omega_0^2}{\hbar} \mathcal{P}^* \cdot \mathrm{Re}\mathbf{G}(\mathbf{r}_i, \mathbf{r}_j, \omega_0) \cdot \mathcal{P} \tag{3a}$$

$$\Gamma^{ij} = \frac{2\mu_0\omega_0^2}{\hbar} \mathcal{P}^* \cdot \mathrm{Im}\mathbf{G}(\mathbf{r}_i, \mathbf{r}_j, \omega_0) \cdot \mathcal{P} \tag{3b}$$

where $\mathcal{P}$ is the dipole moment operator, which is assumed to be the same for all atoms, and $\mathbf{G}(\mathbf{r}_i, \mathbf{r}_j, \omega_0)$ is the dyadic Green tensor in free-space.

Analogously, the dynamics of the system in the absence of a driving field can be described with an effective non-Hermitian Hamiltonian $\mathcal{H} = \hbar\omega_0 \sum_{i=1}^{N} \hat{\sigma}^i_{ee} + \mathcal{H}_{\text{eff}}$

$$\mathcal{H}_{\text{eff}} = \sum_{i,j=1}^{N} \left(J^{ij} - \frac{i}{2}\Gamma^{ij} \right) \hat{\sigma}^i_{eg} \hat{\sigma}^j_{ge} \tag{4}$$

In the case of a ring, the eigenmodes and eigenvalues of the effective Hamiltonian are [3]:

$$\hat{\tilde{\sigma}}^k_{eg} = \frac{1}{\sqrt{N}} \sum_{j=1} e^{ik\theta_j} \hat{\sigma}^j_{eg} \tag{5a}$$

$$E^{(0)}_k = \frac{1}{N} \sum_{j,l=1}^{N} e^{ik(\theta_l - \theta_j)} \left(J^{jl} - \frac{i}{2}\Gamma^{jl} \right) \tag{5b}$$

where $k = 0, \pm1, \pm2, \ldots, \lceil \pm(N-1)/2 \rceil$ is the (integer) angular momentum of the mode. $\theta_i = \frac{2\pi}{N} i \quad i = 0, 1, \ldots, N-1$ is the angle of the ith atom.

Next, we are going to add the gravitational field in our study. Gravitational redshift is a phenomenon predicted by Einstein's General Relativity Theory that consists on the loss of energy of an electromagnetic wave when it travels out of a gravitational field, which is also related to the concept of time dilation.

The transition energy between the levels of a two-level atom is also affected by the gravitational redshift. We will observe a lower transition frequency when the atom moves towards a stronger gravitational field.

Gravitational redshift has already been measured in atomic clocks [2]. In a classical test, the frequency difference due to the redshift is $\frac{\Delta\omega_0}{\omega_0} = \frac{\Delta U}{c^2}$ where U is the gravitational potential and c the speed of light in vacuum. To first order, on Earth, $\Delta U = gh + O(h^2)$, so $\frac{\Delta\omega_0}{\omega_0} \approx \frac{gh}{c^2}$ [4]. The perturbed Hamiltonian, with redshift, is $\mathcal{H} = \mathcal{H}_0 + \lambda V$ with $V = \sum_{j=0}^{N-1} h_j |e_j\rangle\langle e_j|$ and $\lambda = \frac{g}{c^2}$, where $|e_j\rangle$ is the state with the jth excited. We shall denote now $|k^{(0)}\rangle = \hat{\tilde{\sigma}}^k_{eg}|g\rangle$, with $|g\rangle$ the ground state, the eigenvalues of $\mathcal{H}_0$. Taking the positions of the atoms as $\left(R\cos\left(\frac{2\pi}{N}i\right), R\sin\left(\frac{2\pi}{N}i\right)\right) \quad i = 0, 1, \cdots, N-1$, with the first axis being the y or vertical, one can express the perturbation in that basis with a discrete Fourier transform:

$$V = \sum_{k,k'} |k^{(0)}\rangle\langle k'^{(0)}| \frac{R}{2} \left(\delta[k-k'+1] + \delta[k-k'-1]\right) \tag{6}$$

where R is the radius of the ring, and δ is the Kronecker delta. In this expression, we can see a certain mode $|k^{(0)}\rangle$ only couples to the adjacent ones $|k-1^{(0)}\rangle$ and $|k+1^{(0)}\rangle$.

If we consider a weak gravitational redshift, we can apply perturbation theory to calculate the eigenvalues and eigenmodes of the perturbed Hamiltonian. I consider a second order approximation of the eigenvalues:

$$E_k = E_k^{(0)} + \lambda E_k^{(1)} + \lambda^2 E_k^{(2)} + O(\lambda^3) \tag{7a}$$

$$E_k^{(1)} = \langle k^{(0)}|V|k^{(0)}\rangle \tag{7b}$$

$$E_k^{(2)} = \sum_{k'\neq k} \frac{|\langle k'^{(0)}|V|k^{(0)}\rangle|^2}{E_k - E_{k'}} \tag{7c}$$

The first order approximation of the eigenvalues would be

Plugging equation (6) in one gets

$$E_k = E_k^{(0)} + \lambda^2 \frac{R^2}{4} \left[\frac{1}{E_k - E_{k-1}} + \frac{1}{E_k - E_{k+1}}\right] \tag{8}$$

The first order coefficient vanishes because we are considering a ring centered at the origin, at height zero. One would have to center the array at a certain height to have first order perturbation, which we will see in the analytical results for the 1D array. Anyway, if the perturbation is of second-order it will be rather small, although it will increase with the number of atoms since the differences between energies E_k, E_{k-1}, E_{k+1} will decrease.

The first order perturbation of the eigenmodes is

$$|k\rangle = |k^{(0)}\rangle + \lambda \sum_{k'\neq k} \frac{\langle k'^{(0)}|V|k^{(0)}\rangle}{E_k^{(0)} - E_k'^{(0)}} |k'^{(0)}\rangle \tag{9}$$

giving

$$|k\rangle = |k^{(0)}\rangle + \lambda \frac{R}{2} \left[\frac{|k-1^{(0)}\rangle}{E_k - E_{k-1}} + \frac{|k+1^{(0)}\rangle}{E_k - E_{k+1}}\right] \tag{10}$$

III. 1D ARRAY

Imagine we have a 1D finite array of N atoms parallel to the vertical axis, the y axis, so that the lowest atom is at a height y_0 and the height of the ith atom is $y_i = y_0 + a\cdot i \quad i = 0, 1, \ldots, N-1$. I intend to do the same formalism as in section II, although there are no analytical expressions for the 1D finite array. It is possible to get a closed formula if one makes the approximation of applying the formula for the eigenmodes of the infinite case[1]:

$$\hat{\tilde{\sigma}}^k_{eg} = \frac{1}{\sqrt{N}} \sum_k e^{-iky_j} \hat{\sigma}^j_{eg} \tag{11}$$

Using this expression, without any further approximation, one gets the following results

$$\langle k^{(0)}|V|k^{(0)}\rangle = \frac{1}{N} \sum_{j=0}^{N-1} y_j \tag{12a}$$

$$\langle k'^{(0)}|V|k^{(0)}\rangle = \frac{e^{i(k-k')y_0}}{1-e^{i(k-k')a}} \left[-ae^{i(k-k')aN} + \frac{1-e^{i(k-k')aN}}{N} \left(y_0 + \frac{a}{e^{-i(k-k')a}-1}\right)\right] \tag{12b}$$

$$E_k = E_k^{(0)} + \lambda \frac{1}{N} \sum_{j=0}^{N-1} y_j \tag{13}$$

which depends on the mean height of all atoms. This is coherent with the results found for the ring, since when the ring was centered at the origin there was no first order perturbation. Regarding the second order perturbation, the coupling between the modes $|k^{(0)}\rangle$ and $|k'^{(0)}\rangle$ is inversely proportional to the difference between their energies $E_k^{(0)} - E_{k'}^{(0)}$. In the particular case of parallel polarization, one can see in Fig. 1 that the energy plateaus around the middle $k = 0$ and the edge $k = \pm\frac{\pi}{a}$ of the first Brillouin zone. For this reason, we would expect the perturbation to be stronger for the modes in those areas.

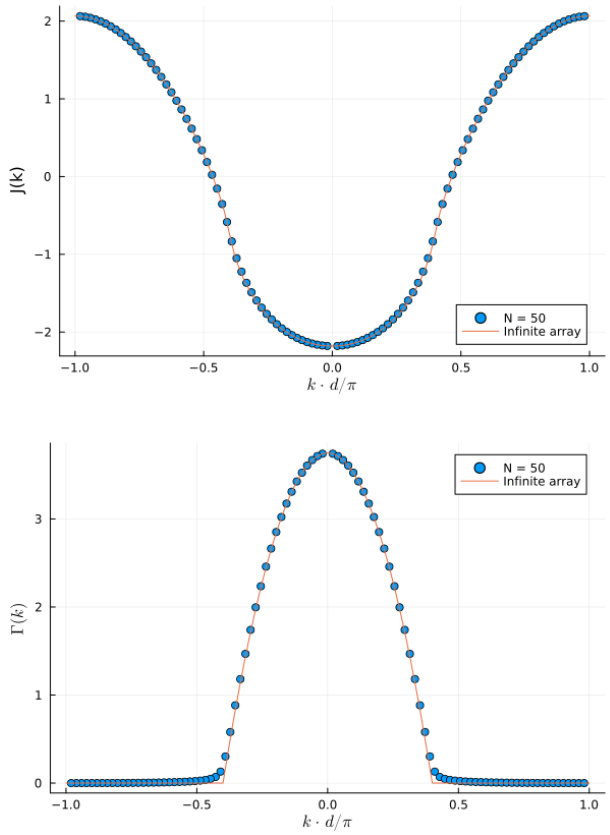

FIG. 1: (a) Energy shifts divided by the spontaneous decay rate γ_0 and (b) Decay rates divided by the spontaneous decay rate for a 1D array of 50 atoms and parallel polarization, compared with the analytical result for an infinite array.

I computed numerically the eigenvalues and eigenmodes of a 1D array of 50 atoms parallel to the vertical and centered around height zero, and compared them with the eigenvalues and eigenmodes without gravitational redshift. In Fig. 2, one can see how the eigenvalues and eigenmodes change with the applied redshift. The eigenvalues and eigenmodes are in increasing order of energy shift or, equivalently in the case of parallel polarization, by their momenta. While the relative error of the energy shifts and decay rates is, overall, less than 10^{-3}, the angle between the eigenmodes with and without redshift gets up to 1 radian. This shows that the perturbation affects more the eigenmodes than the eigenvalues, and in Fig. 2(c) we can also see that the modes near $k = 0$ and $k = \pi/a$ change the most, as I had discussed.

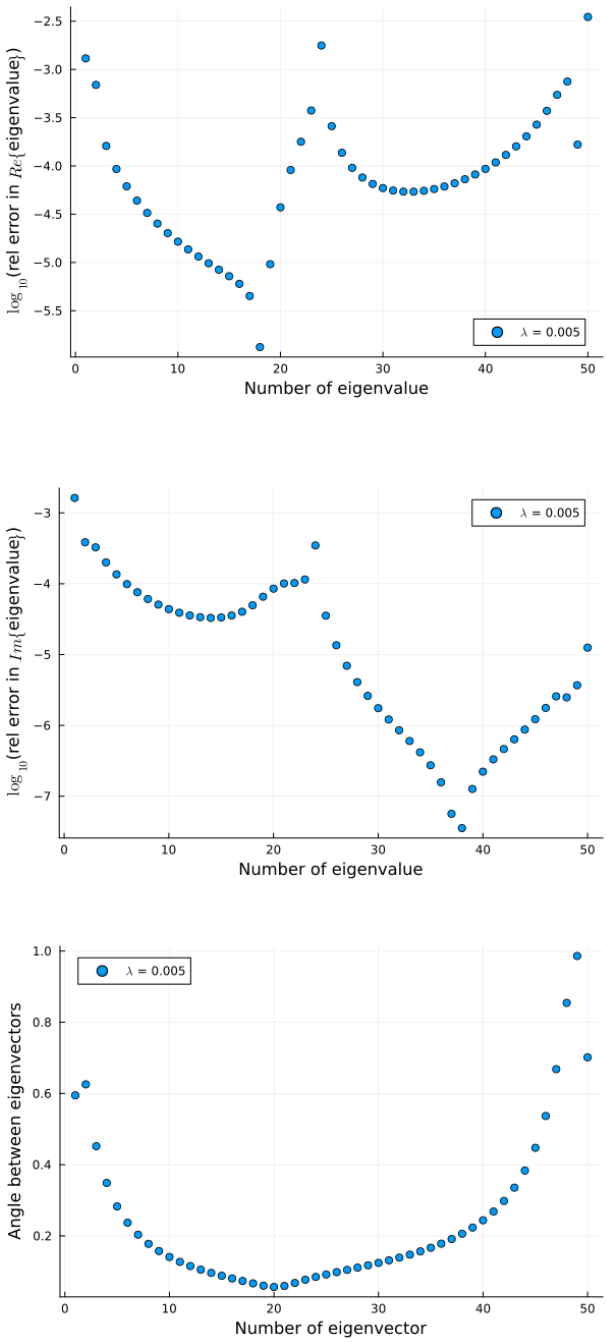

FIG. 2: (a) Relative error between the energy shifts (b) Relative error between the decay rates and (c) Angle, in radians, between the eigenmodes of the atomic array with and without redshift. The array has 50 atoms along the vertical axis with parallel polarization, and $\lambda = 5 \cdot 10^{-3}$ comes from $\mathcal{H} = \mathcal{H}_\prime + \lambda V$.

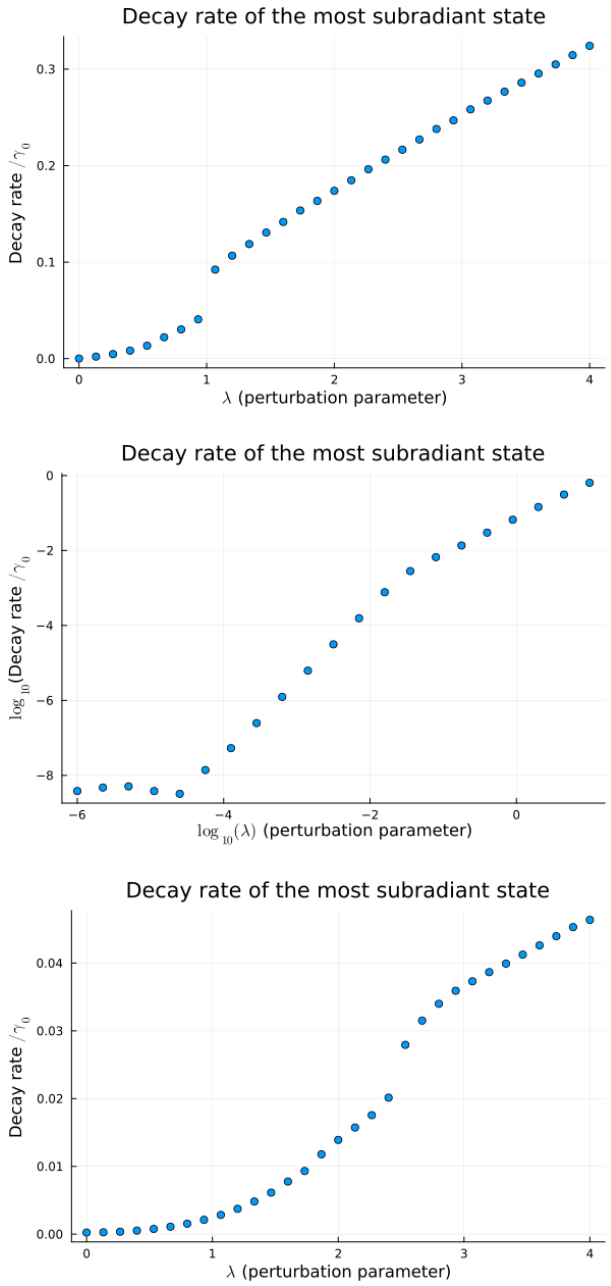

FIG. 3: (a) and (c) illustrate the decay rate of the most subradiant state for different perturbation parameters in the case of a 1D array of 25 atoms and a 2D array of 5×5 atoms respectively, both with parallel polarization. (b) is a similar plot, this time in logarithmic scale, in for a 1D array with 25 atoms and perpendicular polarization.

Next, I shall take another approach to study the gravitational redshift. Suppose we have a horizontal 1D array prepared in the most subradiant state, and then, we quickly turn the array until it is vertical and centered at height zero. I suppose the gravitational redshift is "turned on" immediately at $t = 0$, and analyze the decay rate of the array under these circumstances for different strengths of the gravitational shift. Fig. 3 shows how the redshift effectively couples the most subradiant state to other modes, and hence its decay rate rises with the perturbation parameter λ. In Fig. 3(c) we can even see some phase transitions: for a very small redshift the decay rate seems more or less constant, for an intermediate perturbation parameter the decay rate is quadratic and for higher parameters it is clearly linear. Lastly, comparing the plots for the 1D and 2D arrays with parallel polarization both with 25 atoms, they are quite similar in shape, but the decay rate for the 1D array is almost 2 orders of magnitude larger, even though the decay rate without perturbation of the 1D array is half of the 2D array's. This can be explained because the redshift only affects along one dimension, and since the 1D array spans longer along the vertical direction its perturbation is bigger.

IV. CONCLUSION

To understand the effect of the gravitational redshift on an atomic array, it can be simply seen as a linear space-dependent detuning. That is why it changes not only the energy shifts but also the decay rates and the eigenmodes of the Hamiltonian. I have seen the effect on the Hamiltonian and the dynamics in the case of a ring, 1D and 2D arrays. I got an analytical expression for the ring, showing how each mode couples to the adjacent ones, and similarly for the 1D array each mode couples mainly to the closest ones. The eigenmodes are more affected than the eigenvalues, and for the particular case of the 1D array with parallel polarization, the modes near $k = 0$ and $k = \pi/a$ are the most perturbed.

Despite the fact that I have taken arbitrary values of the parameter λ throughout the manuscript, on Earth, it takes a much more specific and smaller range of values, around $\lambda \approx 8.33 \cdot 10^{-16} m^{-1}$. Consequently, the study done is more useful as a review of the effect of any linear detuning. For future work, experimental setups can be designed to detect space-dependent detunings on a subwavelength atomic array with other figure of merit apart from the decay rate, taking advantage of the narrow linewidth of subradiant states.

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