

# Context Free Languages: Expressionally

Zoe Stafford

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We assume there is some finite alphabet  $\Sigma$  and countably infinite set of variables  $\mathcal{V}$ .

We start by defining the set  $R_\Sigma$  of  $\mu$ -regular expressions on  $\Sigma$  by induction:

$$\begin{array}{ccc} \frac{}{\emptyset, \epsilon \in R_\Sigma} & \frac{}{x \in R_\Sigma} (x \in \Sigma) & \frac{A \in R_\Sigma \quad B \in R_\Sigma}{AB \in R_\Sigma} \\[10pt] \frac{A \in R_\Sigma \quad B \in R_\Sigma}{A + B \in R_\Sigma} & \frac{A \in R_\Sigma}{\mu X.A \in R_\Sigma} (X \in \mathcal{V}) & \frac{}{X \in R_\Sigma} (X \in \mathcal{V}) \end{array}$$

Throughout this note we assume that each  $\mu$  binds a fresh variable. We define the set of free variables  $\text{FV}(A)$  of some  $\mu$ -regular expression  $A$  by induction on  $A$ :

$$\begin{array}{ll} \text{FV}(\epsilon) = \emptyset & \\ \text{FV}(\emptyset) = \emptyset & \\ \text{FV}(x) = \emptyset & (\text{for } x \in \Sigma) \\ \text{FV}(AB) = \text{FV}(A) \cup \text{FV}(B) & \\ \text{FV}(A + B) = \text{FV}(A) \cup \text{FV}(B) & \\ \text{FV}(X) = \{X\} & (\text{for } X \in \mathcal{V}) \\ \text{FV}(\mu X.A) = \text{FV}(A) \setminus \{X\} & \end{array}$$

An environment for a  $\mu$ -regular expression  $A$  is a map  $\gamma : \text{FV}(A) \rightarrow \mathcal{P}(\Sigma^*)$ . We may implicitly weaken environments with domain  $D$  to environments with domain  $D'$  for some  $D' \subseteq D$ .

We now define the *language*  $L^\gamma(A)$  of a  $\mu$ -regular expression  $A$  with respect to some environment  $\gamma$  of  $A$  by induction on  $A$ :

$$\begin{aligned}
L^\gamma(\emptyset) &= \emptyset \\
L^\gamma(\epsilon) &= \{\epsilon\} \\
L^\gamma(x) &= \{x\} & (\text{for } x \in \Sigma) \\
L^\gamma(AB) &= \{uv \mid u \in L^\gamma(A), v \in L^\gamma(B)\} \\
L^\gamma(A + B) &= L^\gamma(A) \cup L^\gamma(B) \\
L^\gamma(X) &= \gamma(X) & (\text{for } X \in \mathcal{V})
\end{aligned}$$

It remains to define  $L^\gamma(\mu X.A)$ . We want to define  $L^\gamma(\mu X.A)$  as the *least fixed point* of the function  $f := M \mapsto L^{\gamma \cup \{X \mapsto M\}}(A)$ , however it is not a priori the case that  $f$  has a least fixed point, since  $f$  may have no fixed points, or many incomparable minimal fixed points.

**Definition 1.** For a partial order  $(A, \leq)$  and function  $f : A \rightarrow A$ ,  $x \in A$  is a *fixed point* of  $f$  when  $f(x) = x$ . The *least fixed point* of  $f$  is the minimum of the set  $\{y \in A \mid f(y) = y\}$  if this is defined.

**Definition 2.** For poset  $(D, \leq)$ , a directed subset of  $D$  is a set  $C \subseteq D$  such that every pair of elements  $a, b \in C$  have an upper bound  $c \in C$  with  $a \leq c$  and  $b \leq c$ .

**Definition 3.** For posets  $(C, \leq_C)$  and  $(D, \leq_D)$ , a function  $f : C \rightarrow D$  is *Scott-continuous* when  $f$  preserves all directed supremums, ie for any directed subset  $A \subseteq C$  where  $\sup A$  is defined,  $f(\sup A) = \sup\{f(x) \mid x \in A\}$ .

**Definition 4.** A partial order  $\leq$  on  $D$  is a directed-complete partial order when every directed subset of  $D$  has a supremum.

**Theorem 1** (Kleene Fixed-Point theorem). *Let  $(D, \leq)$  be a directed-complete partial order with a least element  $\perp$ . Then any Scott-continuous function  $f : D \rightarrow D$  has a fixed point  $\mu(f) = \sup\{\perp, f(\perp), f(f(\perp)), \dots\}$*

*Proof.* We first note that any Scott-continuous function is monotonic: for  $a \leq b \in D$ ,  $\sup\{a, b\} = b$  and  $\{a, b\}$  is a directed subset of  $D$  (since  $a \leq b$  and  $b \leq b$ ), so  $f(b) = f(\sup\{a, b\}) = \sup\{f(a), f(b)\}$  so  $f(a) \leq f(b)$ .

We define  $M := \sup\{f^n(\perp) \mid n \in \mathbb{N}\}$  (well-defined since  $(D, \leq)$  is directed-complete) and show that  $M$  is the least fixed point of  $f$ :

First we show that  $M$  is a fixed point of  $f$ :

$$\begin{aligned}
f(M) &= f(\sup\{f^n(\perp) \mid n \in \mathbb{N}\}) \\
&= \sup\{f^{n+1}(\perp) \mid n \in \mathbb{N}\} & (f \text{ Scott-continuous}) \\
&= \sup(\{f^{n+1}(\perp) \mid n \in \mathbb{N}\} \cup \{\perp\}) & (*) \\
&= \sup\{f^n(\perp) \mid n \in \mathbb{N}\} = M
\end{aligned}$$

(\*) holds since  $\perp \leq f^{n+1}(\perp)$  for all  $n$  so it does not effect the supremum.

It remains to show that  $M \leq N$  for any fixed point  $N$ . We show that  $f^n(\perp) \leq N$  for all  $n \in \mathbb{N}$  by induction on  $n$ :

- case  $n = 0$ :  $f^0(\perp) = \perp \leq N$
- case  $n = k + 1$ : By IH,  $f^k(\perp) \leq N$  and since  $f$  is monotonic,  $f^{k+1}(\perp) \leq f(N) = N$ .

Hence  $N$  is an upper bound of  $\{f^n(\perp) \mid n \in \mathbb{N}\}$  and so  $M = \sup\{f^n(\perp) \mid n \in \mathbb{N}\} \leq N$ .  $\square$

The poset we are dealing with is  $(\mathcal{P}(\Sigma^*), \subseteq)$ , which has a least element  $\emptyset$ . Any subset  $A \subseteq \mathcal{P}(\Sigma^*)$  has a supremum  $\sup A = \bigcup_{a \in A} a$ , so  $(\mathcal{P}(\Sigma^*), \subseteq)$  is a directed-complete partial order.

**Lemma 1.**  $G \mapsto L^{\sigma \cup \{X \mapsto G\}}(A)$  is Scott-continuous.

*Proof.* We define  $f_{\sigma, X, A}(G) := L^{\sigma \cup \{X \mapsto G\}}(A)$ . We may omit  $\sigma$  and  $X$  when it is clear from context, ie  $f_A(G)$  shall denote  $f_{\sigma, X, A}(G)$  when there is a unique obvious choice for  $\sigma$  and  $X$ .

We proceed by induction on  $A$  with IH

$$\forall A. \forall \sigma : \text{FV}(A) \rightarrow \mathcal{P}(\Sigma^*). \forall X. f_{\sigma, X, A} \text{ is Scott-continuous}$$

- Case  $A = \emptyset$  or  $A = \epsilon$  or  $A = a$  for  $a \in \Sigma$  or  $A = Y$  for  $Y \neq X$ : Then  $f_A$  is the constant function, so let  $v$  be the common value of  $f_A(G)$  for all  $G$ . For any directed subset  $D \subseteq \mathcal{P}(\Sigma^*)$ ,  $f_A(\sup D) = v = \sup\{v\} = \sup(f_A[D])$  as required.
- Case  $A = X$ : Then  $f_A$  is the identity function, so for any directed subset  $D$ ,  $f_A(\sup D) = \sup D = \sup(f_A[D])$ .
- Case  $A = BC$ : By IH,  $f_B$  and  $f_C$  are Scott-continuous. For any directed subset  $D$ , we have  $f_B(\sup D) = \sup(f_B[D])$  and  $f_C(\sup D) = \sup(f_C[D])$ .

$$\begin{aligned} f_{BC}(\sup D) &= f_B(\sup D) f_C(\sup D) \\ &= \sup(f_B[D]) \sup(f_C[D]) \end{aligned}$$

For any  $G \in f_{BC}[D]$ , we have  $G = f_{BC}(G')$  for some  $G' \in D$ , so  $G = f_B(G') f_C(G') \subseteq \sup(f_B[D]) \sup(f_C[D])$ . Hence  $\sup(f_B[D]) \sup(f_C[D])$  is an upper-bound for  $f_{BC}[D]$ .

We now show that  $\sup(f_B[D]) \sup(f_C[D])$  is the least upper-bound of  $f_{BC}[D]$ , so for any upper-bound  $U$  of  $f_{BC}[D]$  we need to show  $\sup(f_B[D]) \sup(f_C[D]) \subseteq U$ .

For any  $w \in \sup(f_B[D]) \sup(f_C[D])$  we have  $w = uv$  for  $u \in \sup(f_B[D])$  and  $v \in \sup(f_C[D])$ , so  $u \in f_B(G)$  and  $v \in f_C(F)$  for some  $G, F \in D$ .

Since  $D$  is a directed subset, there is some  $H \in D$  with  $G \subseteq H$  and  $F \subseteq H$ . Since  $f_B$  and  $f_C$  are monotonic,  $f_B(G) \subseteq f_B(H)$  and  $f_C(G) \subseteq f_C(H)$  so  $w = uv \in f_B(H)f_C(H) = f_{BC}(H)$ . Since  $U$  is an upper bound of  $f_{BC}[D]$  and  $w \in f_{BC}(H)$ ,  $w \in U$ , so  $\sup(f_B[D])\sup(f_C[D])$  is the least upper bound of  $f_{BC}[D]$ :

$$\begin{aligned} f_{BC}(\sup D) &= \sup(f_B[D])\sup(f_C[D]) \\ &= \sup(f_{BC}[D]) \end{aligned}$$

- Case  $A = B + C$ : For any directed subset  $D$ :

$$\begin{aligned} f_{B+C}(\sup D) &= f_B(\sup D) \cup f_C(\sup D) \\ &= \sup(f_B[D]) \cup \sup(f_C[D]) \end{aligned} \quad (\text{IH})$$

For any  $G \in f_{B+C}[D]$  we have  $G = f_{B+C}(G') = f_B(G') \cup f_C(G')$  for some  $G' \in D$ . Also  $f_B(G') \subseteq \sup(f_B[D])$  and  $f_C(G') \subseteq \sup(f_C[D])$  so  $f_B(G') \cup f_C(G') \subseteq \sup(f_B[D]) \cup \sup(f_C[D])$ , so  $\sup(f_B[D]) \cup \sup(f_C[D])$  is an upper bound of  $f_{B+C}[D]$ .

We now show  $\sup(f_B[D]) \cup \sup(f_C[D])$  is the least upper bound. For any upper bound  $U$  of  $f_{B+C}[D]$ , we want to show  $\sup(f_B[D]) \cup \sup(f_C[D]) \subseteq U$ . For any  $w \in \sup(f_B[D]) \cup \sup(f_C[D])$ , we have either  $w \in \sup(f_B[D])$  or  $w \in \sup(f_C[D])$ .

- If  $w \in \sup(f_B[D])$  then  $w \in f_B(G)$  for some  $G \in D$ , so  $w \in f_B(G) \cup f_C(G) \subseteq U$ .
- If  $w \in \sup(f_C[D])$  then  $w \in f_C(G)$  for some  $G \in D$  so  $w \in f_B(G) \cup f_C(G) \subseteq U$ .

Hence  $\sup(f_B[D]) \cup \sup(f_C[D]) \subseteq U$  so  $\sup(f_{B+C}[D]) = \sup(f_B[D]) \cup \sup(f_C[D])$  so

$$\begin{aligned} f_{B+C}(\sup D) &= \sup(f_B[D]) \cup \sup(f_C[D]) \\ &= \sup(f_{B+C}[D]) \end{aligned}$$

- Case  $A = \mu Y.B$ : By IH,  $f_{\sigma \cup \{X \mapsto G\}, Y, B}$  is Scott-continuous for any  $\sigma$ , so by Theorem 1, the least fixed point of  $f_{\sigma \cup \{X \mapsto G\}, Y, B}$  is:

$$\mu(f_{\sigma \cup \{X \mapsto G\}, Y, B}) = \sup\{f_{\sigma \cup \{X \mapsto G\}, Y, B}^n(\emptyset) \mid n \in \mathbb{N}\}$$

For all directed subsets  $D$ , we want to show:

$$f_{\sigma, X, \mu Y.B}(\sup D) = \sup(f_{\sigma, X, \mu Y.B}[D])$$

Starting from the left-hand side we have:

$$\begin{aligned} f_{\sigma, X, \mu Y.B}(\sup D) &= \mu(G \mapsto L^{\sigma \cup \{X \mapsto \sup D, Y \mapsto G\}}(B)) \\ &= \mu(f_{\sigma \cup \{X \mapsto \sup D\}, Y, B}) \\ &= \sup\{f_{\sigma \cup \{X \mapsto \sup D\}, Y, B}^n(\emptyset) \mid n \in \mathbb{N}\} \end{aligned}$$

Starting from the right-hand side we have:

$$\begin{aligned} f_{\sigma, X, \mu Y.B}[D] &= \{f_{\sigma, X, \mu Y.B}(G) \mid G \in D\} \\ &= \{\mu(F \mapsto L^{\sigma \cup \{X \mapsto G, Y \mapsto F\}}(B)) \mid G \in D\} \\ &= \{\mu(f_{\sigma \cup \{X \mapsto G\}, Y, B}) \mid G \in D\} \\ &= \{\sup\{f_{\sigma \cup \{X \mapsto G\}, Y, B}^n(\emptyset) \mid n \in \mathbb{N}\} \mid G \in D\} \end{aligned}$$

Let  $M_n := \{f_{\sigma \cup \{X \mapsto G\}, Y, B}^n(\emptyset) \mid G \in D\}$ .

We show  $G \mapsto f_{\sigma \cup \{X \mapsto G\}, Y, B}^n(\emptyset)$  is Scott-continuous for all  $n \in \mathbb{N}$  (ie for any directed subset  $D$ ,  $\sup M_n = f_{\sigma \cup \{X \mapsto \sup D\}, Y, B}^n(\emptyset)$ ) by induction on  $n$ :

- case  $n = 0$ : then both sides are  $\emptyset$
- case  $n = k + 1$ :

$$\begin{aligned} \sup M_{k+1} &= \sup\{f_{\sigma \cup \{X \mapsto G\}, Y, B}(f_{\sigma \cup \{X \mapsto G\}, Y, B}^k(\emptyset)) \mid G \in D\} \\ &= \sup\{f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup M_k) \mid G \in D\} \quad (*) \\ &= \sup\{f_{\sigma \cup \{Y \mapsto \sup M_k\}, X, B}(G) \mid G \in D\} \\ &= f_{\sigma \cup \{Y \mapsto \sup M_k\}, X, B}(\sup D) \\ &= f_{\sigma \cup \{X \mapsto \sup D\}, Y, B}(\sup M_k) \\ &= f_{\sigma \cup \{X \mapsto \sup D\}, Y, B}^{k+1}(\emptyset) \quad (\text{IH}) \end{aligned}$$

We now justify equation (\*) by showing  $N_1$  and  $N_2$  have the same set of upper bounds, where

$$\begin{aligned} N_1 &:= \{f_{\sigma \cup \{X \mapsto G\}, Y, B}(f_{\sigma \cup \{X \mapsto G\}, Y, B}^k(\emptyset)) \mid G \in D\} \\ N_2 &:= \{f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup M_k) \mid G \in D\} \end{aligned}$$

Consider any upper bound  $U$  of  $N_2$ . Then for any  $A \in N_1$ , there is some  $G \in D$  such that:

$$\begin{aligned} A &= f_{\sigma \cup \{X \mapsto G\}, Y, B}(f_{\sigma \cup \{X \mapsto G\}, Y, B}^k(\emptyset)) \\ &\subseteq f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup M_k) \quad (f_{\sigma \cup \{X \mapsto G\}, Y, B} \text{ monotonic}) \end{aligned}$$

and  $f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup M_k) \in N_2$  so  $A \subseteq U$ .

Consider any upper bound  $U$  of  $N_1$ . Then for any  $A \in N_2$ , there is some  $G \in D$  such that:

$$\begin{aligned} A &= f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup M_k) \\ &= \sup(f_{\sigma \cup \{X \mapsto G\}, Y, B}[M_k]) \end{aligned}$$

Since  $f_{\sigma \cup \{X \mapsto G\}, Y, B}$  is Scott-continuous.

For any word  $w \in A$ , we have  $w \in f_{\sigma \cup \{X \mapsto G\}}(P)$  for some  $P \in M_k$ . Then  $P = f_{\sigma \cup \{X \mapsto F\}, Y, B}^n(\emptyset)$  for some  $F \in D$  and since  $D$  is directed, there is some  $H$  with  $G \subseteq H$  and  $F \subseteq H$ . By the monotonicity of Scott-continuous functions we have

$$\begin{aligned} f_{\sigma \cup \{X \mapsto G\}, Y, B}(f_{\sigma \cup \{X \mapsto F\}, Y, B}^n(\emptyset)) \\ \subseteq f_{\sigma \cup \{X \mapsto H\}, Y, B}(f_{\sigma \cup \{X \mapsto H\}, Y, B}^n(\emptyset)) \in N_1 \end{aligned}$$

So  $w \in \sup N_1 \subseteq U$  so  $U$  is an upper bound of  $N_2$

Hence

$$\begin{aligned} \sup(f_{\sigma, X, \mu Y, B}[D]) &= \sup\{\sup\{f_{\sigma \cup \{X \mapsto G\}, Y, B}^n(\emptyset) \mid n \in \mathbb{N}\} \mid G \in D\} \\ &= \sup\{\sup\{f_{\sigma \cup \{X \mapsto G\}, Y, B}^n(\emptyset) \mid G \in D\} \mid n \in \mathbb{N}\} \\ &= \sup\{f_{\sigma \cup \{X \mapsto \sup D\}, Y, B}^n(\emptyset) \mid n \in \mathbb{N}\} \\ &= f_{\sigma, X, \mu Y, B}(\sup D) \end{aligned}$$

□