Context Free Languages: Expressionally

μ-regular expressions

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Outline

Motivation

- Classify sets of words (*languages*)
- Show language membership is computable/efficient
- Describe languages

Motivation

- Regular expressions 😇

$$0x[0\text{-}9, a\text{-}f][0\text{-}9, a\text{-}f]^* + [1\text{-}9][0\text{-}9]^*$$

- Context Free Gammars 😬

$$S \to H \mid D$$

$$\begin{aligned} A &\to 0 \mid 1 \mid \cdots \mid f \\ H' &\to AH' \mid \epsilon \\ H &\to 0xAH' \end{aligned}$$

$$\begin{aligned} B &\to 0 \mid 1 \mid \cdots \mid 9 \\ B' &\to 1 \mid 1 \mid \cdots \mid 9 \\ D' &\to BD' \mid \epsilon \\ D &\to B'D' \end{aligned}$$

Notation

- S^* is set of finite *words* on S — ie finite sequences of elements in S
- ϵ: empty word
- uv: concatenation of u and v
- letters: $x, y, z \in \Sigma$
- words: $u, v, w \in \Sigma^*$
- regular expressions: A, B, C
- Variables: X, Y, Z

Regular Expressions

Define set R_Σ of regular expressions over Σ inductively:

$$\frac{}{\emptyset \in R_\Sigma} \qquad \frac{}{\epsilon \in R_\Sigma} \qquad \frac{x \in \Sigma}{x \in R_\Sigma} \qquad \frac{A \in R_\Sigma \qquad B \in R_\Sigma}{AB \in R_\Sigma}$$

$$\frac{A \in R_\Sigma \qquad B \in R_\Sigma}{A + B \in R_\Sigma} \qquad \frac{A \in R_\Sigma}{A^* \in R_\Sigma}$$

Regular Expressions

Example

$$0 + 1(0 + 1)^*00$$

Regular Expressions
Example

$$0 + 1(0 + 1)^*00$$

Multiples of 4 (in binary)

Regular Expressions
Semantics

- Consider $A \in R_\Sigma$
- We define the *language* of A: $L(A) \subseteq \Sigma^*$

Regular Expressions

Semantics

- Consider $A \in R_\Sigma$
- We define the *language* of A: $L(A) \subseteq \Sigma^*$
- By induction!

Regular Expressions

Semantics

- $L(x) = \{x\}$ for $x \in \Sigma$
- $L(\epsilon) = \{\epsilon\}$
- $L(\emptyset) = \emptyset$
- $L(AB) = L(A)L(B) = \{uv \mid u \in L(A), v \in L(B)\}$
- $L(A + B) = L(A) \cup L(B)$
- $L(A^*) = \{u_1 \dots u_n \mid n \in \mathbb{N}, u_1, \dots, u_n \in L(A)\}$

Regular Expressions

Not an Example — Balanced Brackets

$$L_1 := \{\langle\rangle, \langle\rangle\langle\rangle, \langle\langle\rangle\rangle, \langle\langle\rangle\rangle\langle\rangle, \langle\rangle\langle\rangle\langle\rangle, \ldots\}$$

Regular Expressions

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Want $A \in R_{\{\langle,\rangle\}}$ that denotes L_1:

- Each $w \in L_1$ is some of form $\langle u_1\rangle\langle u_2\rangle \ldots \langle u_n\rangle$ for $u_i \in L_1$

Regular Expressions

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- $A = (\langle B\rangle)^*$ for some B
- B denotes balanced brackets

Regular Expressions

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- $A = (\langle B\rangle)^*$ for some B
- B denotes balanced brackets
- So $B = A$
- $A = (\langle A\rangle)^*$ 😱

μ-Regular Expressions

- Recursion!
- Set of variables $\mathcal{V}$

$$\frac{X \in \mathcal{V}}{X \in R_\Sigma} \qquad\qquad \frac{X \in \mathcal{V} \qquad A \in R_\Sigma}{\mu X.A \in R_\Sigma}$$

μ-Regular Expressions

Example

$$\mu S.(\langle S \rangle)^*$$

μ-Regular Expressions

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Balanced brackets 🤑

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$$\mu E.[1-9][0-9]^* + (E - E) + (E \times E) + (E \div E)$$

μ-Regular Expressions

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$$\mu E.[1-9][0-9]^* + (E - E) + (E \times E) + (E \div E)$$

Arithmetic

μ-Regular Expressions

Recursion

$$\mu X.A$$

- A ‘is’ a function of X
- $\mu X.A$ finds *fixed point* of A (w.r.t. X)

$$\mu X.A \equiv A[X \mapsto \mu X.A]$$

μ-Regular Expressions

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- $\mu X.A$ finds *fixed point* of A (w.r.t. X)

$$\mu X.A \equiv A[X \mapsto \mu X.A]$$

- e.g. $\mu X.\epsilon + \langle X \rangle \equiv \epsilon + \langle(\mu X.\epsilon + \langle X \rangle)\rangle$

μ-Regular Expressions

Example

$$A^* \equiv \epsilon + AA^*$$

μ-Regular Expressions
Example

$$A^* \equiv \epsilon + AA^*$$

- Let $A^* := \mu X.\epsilon + AX$ (for some *fresh* X)

$$\begin{aligned} A^* &= \mu X.\epsilon + AX \\ &\equiv \epsilon + A(\mu X.\epsilon + AX) \\ &= \epsilon + AA^* \end{aligned}$$

μ-Regular Expressions

Variable binding

- Assumption: variable names are unique
  - Bad: $\mu X.\mu X.X$
  - Good: $\mu X.\mu Y.X$

μ-Regular Expressions

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μ-Regular Expressions

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- Assumption: variable names are unique
  - Bad: $\mu X.\mu X.X$
  - Good: $\mu X.\mu Y.X$
- $\mathrm{FV}(A) =$ free variables of A:

$$\begin{aligned}
\mathrm{FV}(X) &= \{X\} && (X \in \mathcal{V}) \\
\mathrm{FV}(\mu X.A) &= \mathrm{FV}(A) \setminus \{X\} \\
\mathrm{FV}(AB) &= \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(A + B) &= \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(\epsilon) = \mathrm{FV}(\emptyset) = \mathrm{FV}(x) &= \emptyset && (x \in \Sigma)
\end{aligned}$$

μ-Regular Expressions

Semantics

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μ-Regular Expressions

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- For $X \in \mathcal{V}$, $L(X) = \dots$
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- $L^\sigma(X) = \sigma(X)$

μ-Regular Expressions

Semantics

- For $X \in \mathcal{V}$, $L(X) = \dots$
- Add *context*: map $\sigma : \mathrm{FV}(A) \to \Sigma^*$
- $L^\sigma(X) = \sigma(X)$
- $L^\sigma(A + B) = L^\sigma(A) \cup L^\sigma(B)$
- $L^\sigma(AB) = L^\sigma(A)L^\sigma(B)$

μ-Regular Expressions

- $L^{\sigma}(\mu X.A) = \ldots$
- $L^{\sigma}(A)$?? No: σ not defined on X 😱

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$$f_A(M) \coloneqq L^{\sigma \cup \{X \mapsto M\}}(A)$$

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$$f_A(M) \coloneqq L^{\sigma \cup \{X \mapsto M\}}(A)$$

- Fixed point of f_A: N such that $f_A(N) = N$
- Does fixed point exist? 🧐
- Which one do we pick? 🤔

μ-Regular Expressions

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- Fixed point of f_A: N such that $f_A(N) = N$
- Does fixed point exist? 🧐
- Which one do we pick? 🤔
- Pick the *least fixed point*

Fixed points

- The *least fixed point* of f is minimum element of

$$\{N \mid f(N) = N\}$$

  - Written $\mu(f)$ or $\mu(x \mapsto f(x))$
- Kleene fixed point theorem describes least fixed point

Fixed points

Notation

- For function $f : A \to B$ and subset $D \subseteq A$, the image of D is

$$f[D] \coloneqq \{f(x) \mid x \in D\}$$

- For partial order D and subset $A \subseteq D$, an *upper bound* of A is an $x \in D$ with $\forall y \in A. y \leq x$
- The *supremum* $\sup A$ is the least upper bound of A
  - Not defined for all A!

$$f^n(x) = \underbrace{f(f(\ldots f}_{n \text{ times}}(n) \ldots))$$

Fixed points

Partial orders

- Let $\leq$ be a partial order on D
- A *directed subset* of D is non-empty subset $E \subseteq D$ s.t. every $a, b \in E$ have an upper bound $c \in E$ with $a \leq c$ and $b \leq c$
- D is a *directed-complete partial order* (*dcpo*) when every directed subset of D has a supremum (least upper bound)

Fixed points

Scott-continuity

- Let D be partial order
- Function $f : D \to D$ is *Scott-continuous* when f preserves directed supremums
- ie for any directed subset A of D, $f(\sup D) = \sup(f[D])$

Fixed points

Kleene fixed point theorem

- Let D be dcpo with least element $\perp$
- Let $f : D \to D$ be Scott-continuous
- Then the least fixed point of f is $\sup\{\perp, f(\perp), f(f(\perp)), \dots\}$

Fixed points

Kleene fixed point theorem - proof

- Let $M = \{f^n(\perp) \mid n \in \mathbb{N}\}$
- $\sup M$ exists since M is directed subset
- Need to show:
  1. $\sup M$ is a fixed point
  2. $\sup M$ is the least fixed point

Fixed points

Kleene fixed point theorem - proof (1)

$$
\begin{aligned}
f(\sup M) &= \sup(f[M]) \\
&= \sup\{f^n(\perp) \mid n \in \mathbb{N}, n \geq 1\} \\
&= \sup(M \setminus \{\perp\}) \\
&= \sup M
\end{aligned}
$$

Fixed points

Kleene fixed point theorem - proof (2)

- For any fixed point N of f, we need $\sup M \leq N$
- Claim: $f^n(\perp) \leq N$ for all n — proof by induction:
- case $n = 0$: $f^0(\perp) = \perp \leq N$
- case $n = k + 1$:
  - by IH $f^k(\perp) \leq N$
  - Lemma: f is monotonic
  - So $f^{k+1} \leq f(N) = N$
- N is upper bound of M
- $\sup M \leq N$ $\square$

Fixed points

Kleen fixed point theorem - proof

- Lemma: if f is Scott-continuous then f is monotonic
- For any $a \leq b$:

$$f(b) = f(\sup\{a, b\}) = \sup\{f(a), f(b)\}$$

- So $f(a) \leq f(b)$

Fixed points

In context

- We care about preorder $\subseteq$ on $\mathcal{P}(\Sigma^*)$
- For subset $D \subseteq \mathcal{P}(\Sigma^*)$:

$$\sup D = \bigcup_{d \in D} d$$

- Least element of $\mathcal{P}(\Sigma^*)$ is $\emptyset$

μ-Regular Expressions

Scott-continuity

- We want least fixed point of $M \mapsto L^{\sigma \cup \{X \mapsto M\}}(A)$
- So show it is Scott-continuous

μ-Regular Expressions
Scott-continuity

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- By induction on A!

μ-Regular Expressions

Scott-continuity

- We want least fixed point of $M \mapsto L^{\sigma \cup \{X \mapsto M\}}(A)$
- So show it is Scott-continuous
- By induction on A!
- We need a stronger inductive hypothesis:
  - For μ-RE $A \in R_\Sigma$
  - and variable $X \in \mathcal{V}$
  - and context σ : $(\mathrm{FV}(A) \setminus \{X\}) \to \mathcal{P}(\Sigma^*)$
  - Let $f_{\sigma,X,A}(M) \coloneqq L^{\sigma \cup \{X \mapsto M\}}(A)$
  - Then $f_{\sigma,X,A}$ is Scott-continuous
- If σ and X are obvious, write f_A for $f_{\sigma,X,A}$

μ-Regular Expressions
Scott-continuity

- Cases:

$$\begin{aligned} A &= \emptyset & f_\emptyset(M) &= \emptyset \\ A &= \epsilon & f_\epsilon(M) &= \{\epsilon\} \\ A &= x \in \Sigma & f_x(M) &= \{x\} \\ A &= Y \neq X & f_{\sigma,X,Y}(M) &= \sigma(Y) \end{aligned}$$

- f_A is constant (taking value G)
- So $f_A(\sup D) = G = \sup\{G\} = \sup(f_A[D])$ for all D

μ-Regular Expressions
Scott-continuity

- Case $A = X$:
- $f_{\sigma,X,X}(M) = L^{\sigma \cup \{X \mapsto M\}}(X) = M$

$$f_{\sigma,X,X}(\sup D) = \sup D = \sup(f_{\sigma,X,X}[D])$$

μ-Regular Expressions

Scott-continuity — Concatenation

- case $A = BC$ for μ-REs B and C
- For any directed supset D:

$$\begin{aligned} f_{BC}(\sup D) &= f_B(\sup D) f_C(\sup D) \\ &= \sup(f_B[D]) \sup(f_C[D]) \qquad \text{(IH)} \end{aligned}$$

- Goal: $\sup(f_{BC}[D]) = \sup(f_B[D]) \sup(f_C[D])$
- So we need to show
  1. $\sup(f_B[D]) \sup(f_C[D])$ is an upper bound of $f_{BC}[D]$
  2. $\sup(f_B[D]) \sup(f_C[D]) \leq U$ any other upper bound U

μ-Regular Expressions

Scott-continuity — Concatenation (1)

Goal: $\sup(f_B[D])\sup(f_C[D])$ is an upper bound of $f_{BC}[D]$

- For any $G \in f_{BC}[D]$:
- $G = f_{BC}(G')$ for some $G' \in D$
- So $G = f_B(G')f_C(G')$
- $f_B(G') \in \sup(f_B[D])$ and $f_C(G') \in \sup(f_C[D])$
- So $G \in \sup(f_B[D])\sup(f_C[D])$

μ-Regular Expressions

Scott-continuity — Concatenation (2)

Goal: $\sup(f_B[D])\sup(f_C[D]) \subseteq U$ for upper bound U of $f_{BC}[D]$

- For any $w \in \sup(f_B[D])\sup(f_C[D])$
- $w = uv$ for $u \in \sup(f_B[D])$ and $v \in \sup(f_C[D])$
- So $u \in f_B(G)$ and $v \in f_C(F)$ for some $G, F \in D$
- D is directed, so exists $H \in D$ with $G, F \subseteq H$
- f_B and f_C are Scott-continuous and hence monotonic
- So $u \in f_B(H)$ and $v \in f_C(H)$
- So $w = uv \in f_{BC}(H) \subseteq U$

- Case $A = B + C$ is similar to case $A = BC$
- See supplementary material for full proof (link at end)

μ-Regular Expressions

Scott-continuity – $\mu Y.B$

- Case $A = \mu Y.B$
- Goal: $f_{\sigma,X,\mu Y.B}(\sup D) = \sup(f_{\sigma,X,\mu Y.B}[D])$
- By IH, $f_{\sigma',Y,B}$ is Scott-continuous, so

$$L^{\sigma'}(\mu Y.B) = \mu(f_{\sigma',Y,B}) = \sup\{f^n_{\sigma',Y,B}(\emptyset) \mid n \in \mathbb{N}\}$$

μ-Regular Expressions

Scott-continuity – $\mu Y.B$

- Lemma: $h_n(G) \coloneqq f^n_{\sigma\cup\{X\mapsto G\},Y,B}(\emptyset)$ is Scott-continuous

$$\begin{aligned}
\sup(f_{\sigma,X,\mu Y.B}[D]) &= \sup(L^{\sigma\cup\{X\mapsto G\}}(\mu Y.B) \mid G \in D) \\
&= \sup\{\sup\{f^n_{\sigma\cup\{X\mapsto G\},Y,B}(\emptyset) \mid n \in \mathbb{N}\} \mid G \in D\} \\
&= \sup\{\sup\{h_n(G) \mid G \in D\} \mid n \in \mathbb{N}\} \\
&= \sup\{f^n_{\sigma\cup\{X\mapsto \sup D\},Y,B}(\emptyset) \mid n \in \mathbb{N}\} \\
&= L^{\sigma\cup\{X\mapsto \sup D\}}(\mu Y.B) \\
&= f_{\sigma,X,\mu Y.B}(\sup D)
\end{aligned}$$

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&= \sup\{f^n_{\sigma \cup \{X \mapsto \sup D\}, Y, B}(\emptyset) \mid n \in \mathbb{N}\} \\
&= L^{\sigma \cup \{X \mapsto \sup D\}}(\mu Y.B) \\
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&= L^{\sigma \cup \{X \mapsto \sup D\}}(\mu Y.B) \\
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&= L^{\sigma \cup \{X \mapsto \sup D\}}(\mu Y.B) \\
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μ-Regular Expressions

Scott-continuity – $\mu Y.B$

- Lemma: $h_n(G) \coloneqq f^n_{\sigma\cup\{X\mapsto G\},Y,B}(\emptyset)$ is Scott-continuous

μ-Regular Expressions

Scott-continuity – $\mu Y.B$

- Lemma: $h_n(G) \coloneqq f^n_{\sigma \cup \{X \mapsto G\}, Y, B}(\emptyset)$ is Scott-continuous
- Proof by induction on n
- Case $n = 0$:

$$h_n(G) = f^0_{\sigma \cup \{X \mapsto G\}, Y, B}(\emptyset) = \emptyset$$

μ-Regular Expressions

Scott-continuity – $\mu Y.B$

- Lemma: $h_n(G) \coloneqq f^n_{\sigma \cup \{X \mapsto G\}, Y, B}(\emptyset)$ is Scott-continuous

μ-Regular Expressions

Scott-continuity – $\mu Y.B$

- Lemma: $h_n(G) \coloneqq f^n_{\sigma \cup \{X \mapsto G\}, Y, B}(\emptyset)$ is Scott-continuous
- Case $n = k + 1$:
- By IH, h_k is Scott-continuous

$$N_1 \coloneqq h_{k+1}[D] = \{f_{\sigma \cup \{X \mapsto G\}, Y, B}(h_k(G)) \mid G \in D\}$$
$$N_2 \coloneqq \{f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup(h_k[D])) \mid G \in D\}$$

- Want to show $\sup N_1 = \sup N_2$
- Claim: N_1 and N_2 have the same upper bounds

μ-Regular Expressions

Scott-continuity – $\mu Y.B$

$$N_1 = \{f_{\sigma \cup \{X \mapsto G\}, Y, B}(h_k(G)) \mid G \in D\}$$
$$N_2 = \{f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup(h_k[D])) \mid G \in D\}$$

- Let U be an upper bound of N_1
- For all $A \in N_2$, we want $A \subseteq U$ ie $w \in A \implies w \in U$

$$\begin{aligned} A &= f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup(h_k[D])) && \text{(for some } G \in D\text{)} \\ &= \sup(f_{\sigma \cup \{X \mapsto G\}, Y, B}[h_k[D]]) \\ &= \bigcup_{F \in D} f_{\sigma \cup \{X \mapsto G\}, Y, B}(h_k(F)) \end{aligned}$$

- So $w \in f_{\sigma \cup \{X \mapsto G\}, Y, B}(h_k(F))$

μ-Regular Expressions

Scott-continuity – $\mu Y.B$

$$N_1 = \{f_{\sigma \cup \{X \mapsto G\}, Y, B}(h_k(G)) \mid G \in D\}$$
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- Let U be an upper bound of N_1
- For all $A \in N_2$, we want $A \subseteq U$ ie $w \in A \implies w \in U$
- So $w \in f_{\sigma \cup \{X \mapsto G\}, Y, B}(h_k(F))$
- D directed, so there exists $H \in D$ with $G, F \subseteq H$
- $f_{\sigma', Z, B}$ and h_k Scott-continuous and hence monotonic
- So $w \in f_{\sigma \cup \{X \mapsto H\}, Y, B}(h_k(H)) \in N_1$
- Hence $w \in U$

μ-Regular Expressions

Scott-continuity – $\mu Y.B$

$$N_1 = \{f_{\sigma\cup\{X\mapsto G\},Y,B}(h_k(G)) \mid G \in D\}$$
$$N_2 = \{f_{\sigma\cup\{X\mapsto G\},Y,B}(\sup(h_k[D])) \mid G \in D\}$$

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- So $w \in f_{\sigma\cup\{X\mapsto G\},Y,B}(h_k(F))$
- D directed, so there exists $H \in D$ with $G, F \subseteq H$
- $f_{\sigma',Z,B}$ and h_k Scott-continuous and hence monotonic
- So $w \in f_{\sigma\cup\{X\mapsto H\},Y,B}(h_k(H)) \in N_1$
- Hence $w \in U$

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$$N_2 = \{f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup(h_k[D])) \mid G \in D\}$$

- Now the other way!
- Let U be an upper bound of N_2
- For all $A \in N_1$ we want $A \subseteq U$
- Then $A = f_{\sigma cup \{X \mapsto G\}, Y, B}(h_k(G))$ for some $G \in D$
- $f_{\sigma \cup \{X \mapsto G\}, Y, B}$ is Scott-continuous and hence monotonic
- So $A \subseteq f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup(h_k[D])) \in N_2$
- So $A \subseteq U$

μ-Regular Expressions

Scott-continuity – $\mu Y.B$

$$N_1 = \{f_{\sigma \cup \{X \mapsto G\}, Y, B}(h_k(G)) \mid G \in D\}$$
$$N_2 = \{f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup(h_k[D])) \mid G \in D\}$$

- N_1 and N_2 have the same set of upper bounds
- So $\sup N_1 = \sup N_2$

μ-Regular Expressions

Scott-continuity – $\mu Y.B$

- We can now show that h_{k+1} is Scott-continuous

$$
\begin{aligned}
\sup(h_{k+1}[D]) &= \sup\{f_{\sigma\cup\{X\mapsto G\},Y,B}(h_k(G)) \mid G \in D\} \\
&= \sup\{f_{\sigma\cup\{X\mapsto G\},Y,B}(\sup(h_k[D])) \mid G \in D\} \\
&= \sup\{f_{\sigma\cup\{Y\mapsto \sup(h_k[D])\},X,B}(G) \mid G \in D\} \\
&= f_{\sigma\cup\{Y\mapsto \sup(h_k[D])\},X,B}(\sup G) \\
&= f_{\sigma\cup\{X \sup D\},Y,B}(\sup(h_k[D])) \\
&= h_{k+1}(\sup D) \qquad \square
\end{aligned}
$$

Conclusion

- μ-Regular Expressions extend REs with a least fixed point operator
- Kleene fixed point theorem characterises this least fixed point
- Supplementary material: `https://z-snails.github.io/`