Context Free Languages: Expressionally

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1 μ-Regular Expressions

We assume there is some finite alphabet Σ and countably infinite set of variables $\mathcal{V}$.

We start by defining the set R_Σ of μ-regular expressions on Σ by induction:

$$\frac{}{\emptyset, \epsilon \in R_\Sigma} \qquad \frac{}{x \in R_\Sigma}(x \in \Sigma) \qquad \frac{A \in R_\Sigma \quad B \in R_\Sigma}{AB \in R_\Sigma}$$

$$\frac{A \in R_\Sigma \quad B \in R_\Sigma}{A + B \in R_\Sigma} \qquad \frac{A \in R_\Sigma}{\mu X.A \in R_\Sigma}(X \in \mathcal{V}) \qquad \frac{}{X \in R_\Sigma}(X \in \mathcal{V})$$

Throughout this note we assume that each μ binds a fresh variable. We define the set of free variables $\mathrm{FV}(A)$ of some μ-regular expression A by induction on A:

$$\begin{aligned}
\mathrm{FV}(\epsilon) &= \emptyset \\
\mathrm{FV}(\emptyset) &= \emptyset \\
\mathrm{FV}(x) &= \emptyset && \text{(for } x \in \Sigma\text{)} \\
\mathrm{FV}(AB) &= \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(A + B) &= \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(X) &= \{X\} && \text{(for } X \in \mathcal{V}\text{)} \\
\mathrm{FV}(\mu X.A) &= \mathrm{FV}(A) \setminus \{X\}
\end{aligned}$$

An environment for a μ-regular expression A is a map $\gamma : \mathrm{FV}(A) \to \mathcal{P}(\Sigma^*)$. We may implicitly weaken environments with domain D to environments with domain D' for some $D' \subseteq D$.

We now define the *language* $L^\gamma(A)$ of a μ-regular expression A with respect to some environment γ of A by induction on A:

$$
\begin{aligned}
L^\gamma(\emptyset) &= \emptyset \\
L^\gamma(\epsilon) &= \{\epsilon\} \\
L^\gamma(x) &= \{x\} && \text{(for } x \in \Sigma\text{)} \\
L^\gamma(AB) &= \{uv \mid u \in L^\gamma(A), v \in L^\gamma(B)\} \\
L^\gamma(A+B) &= L^\gamma(A) \cup L^\gamma(B) \\
L^\gamma(X) &= \gamma(X) && \text{(for } X \in \mathcal{V}\text{)}
\end{aligned}
$$

It remains to define $L^\gamma(\mu X.A)$. We want to define $L^\gamma(\mu X.A)$ as the *least fixed point* of the function $f \coloneqq M \mapsto L^{\gamma \cup \{X \mapsto M\}}(A)$, however it is not a priori the case that f has a least fixed point, since f may have no fixed points, or many incomparable minimal fixed points.

Definition 1. For a partial order $(A, \leq)$ and function $f : A \to A$, $x \in A$ is a *fixed point* of f when $f(x) = x$. The *least fixed point* of f is the minimum of the set $\{y \in A \mid f(y) = y\}$ if this is defined.

Definition 2. For poset $(D, \leq)$, a directed subset of D is a set $C \subseteq D$ such that every pair of elements $a, b \in C$ have an upper bound $c \in C$ with $a \leq c$ and $b \leq c$.

Definition 3. For posets $(C, \leq_C)$ and $(D, \leq_D)$, a function $f : C \to D$ is *Scott-continuous* when f preserves all directed supremums, ie for any directed subset $A \subseteq C$ where $\sup A$ is defined, $f(\sup A) = \sup\{f(x) \mid x \in A\}$.

Definition 4. A partial order $\leq$ on D is a directed-complete partial order when every directed subset of D has a supremum.

Theorem 1 (Kleene Fixed-Point theorem)**.** *Let $(D, \leq)$ be a directed-complete partial order with a least element $\bot$. Then any Scott-continuous function $f : D \to D$ has a fixed point $\mu(f) = \sup\{\bot, f(\bot), f(f(\bot)), \dots\}$*

Proof. We first note that any Scott-continuous function is monotonic: for $a \leq b \in D$, $\sup\{a, b\} = b$ and $\{a, b\}$ is a directed subset of D (since $a \leq b$ and $b \leq b$), so $f(b) = f(\sup\{a, b\}) = \sup\{f(a), f(b)\}$ so $f(a) \leq f(b)$.

We define $M \coloneqq \sup\{f^n(\bot) \mid n \in \mathbb{N}\}$ (well-defined since $(D, \leq)$ is directed-complete) and show that M is the least fixed point of f:

First we show that M is a fixed point of f:

$$
\begin{aligned}
f(M) &= f(\sup\{f^n(\bot) \mid n \in \mathbb{N}\}) \\
&= \sup\{f^{n+1}(\bot) \mid n \in \mathbb{N}\} && \text{(f Scott-continuous)} \\
&= \sup(\{f^{n+1}(\bot) \mid n \in \mathbb{N}\} \cup \{\bot\}) && (*) \\
&= \sup\{f^n(\bot \mid n \in \mathbb{N}\}) = M
\end{aligned}
$$

(*) holds since $\bot \leq f^{n+1}(\bot)$ for all n so it does not effect the supremum.

It remains to show that $M \leq N$ for any fixed point N. We show that $f^n(\bot) \leq N$ for all $n \in \mathbb{N}$ by induction on n:

- case $n = 0$: $f^0(\bot) = \bot \leq N$
- case $n = k+1$: By IH, $f^k(\bot) \leq N$ and since f is monotonic, $f^{k+1}(\bot) \leq f(N) = N$.

Hence N is an upper bound of $\{f^n(\bot) \mid n \in \mathbb{N}\}$ and so $M = \sup\{f^n(\bot) \mid n \in \mathbb{N}\} \leq N$. $\square$

The poset we are dealing with is $(\mathcal{P}(\Sigma^*), \subseteq)$, which has a least element $\emptyset$. Any subset $A \subseteq \mathcal{P}(\Sigma^*)$ has a supremum $\sup A = \bigcup_{a \in A} a$, so $(\mathcal{P}(\Sigma^*), \leq)$ is a directed-complete partial order.

Lemma 1. *$G \mapsto L^{\sigma \cup \{X \mapsto G\}}(A)$ is Scott-continuous.*

Proof. We define $f_{\sigma,X,A}(G) \coloneqq L^{\sigma \cup \{X \mapsto G\}}(A)$. We may omit σ and X when it is clear from context, ie $f_A(G)$ shall denotes $f_{\sigma,X,A}(G)$ when there is a unique obvious choice for σ and X.

We procede by induction on A with IH

$$\forall A. \forall \sigma : \mathrm{FV}(A) \to \mathcal{P}(\Sigma^*). \forall X. f_{\sigma,X,A} \text{ is Scott-continuous}$$

- Case $A = \emptyset$ or $A = \epsilon$ or $A = a$ for $a \in \Sigma$ or $A = Y$ for $Y \neq X$: Then f_A is the constant function, so let v be the common value of $f_A(G)$ for all G. For any directed subset $D \subseteq \mathcal{P}(\Sigma^*)$, $f_A(\sup D) = v = \sup\{v\} = \sup(f_A[D])$ as required.
- Case $A = X$: Then f_A is the identity function, so for any directed subset D, $f_A(\sup D) = \sup D = \sup(f_A[D])$.
- Case $A = BC$: By IH, f_B and f_C are Scott-continuous. For any directed subset D, we have $f_B(\sup D) = \sup(f_B[D])$ and $f_C(\sup D) = \sup(f_C[D])$.

  $$\begin{aligned} f_{BC}(\sup D) &= f_B(\sup D) f_C(\sup D) \\ &= \sup(f_B[D]) \sup(f_C[D]) \end{aligned}$$

  For any $G \in f_{BC}[D]$, we have $G = f_{BC}(G')$ for some $G' \in D$, so $G = f_B(G') f_C(G') \subseteq \sup(f_B[D]) \sup(f_C[D])$. Hence $\sup(f_B(G')) \sup(f_C(G'))$ is an upper-bound for $f_{BC}[D]$.

  We now show that $\sup(f_B[D]) \sup(f_C[D])$ is the least upper-bound of $f_{BC}[D]$, so for any upper-bound U of $f_{BC}[D]$ we need to show $\sup(f_B[D]) \sup(f_C[D]) \subseteq U$.

  For any $w \in \sup(f_B[D]) \sup(f_C[D])$ we have $w = uv$ for $u \in \sup(f_B[D])$ and $v \in \sup(f_C[D])$, so $u \in f_B(G)$ and $v \in f_C(F)$ for some $G, F \in D$.

Since D is a directed subset, there is some $H \in D$ with $G \subseteq H$ and $F \subseteq H$. Since f_B and f_C are monotonic, $f_B(G) \subseteq f_B(H)$ and $f_C(G) \subseteq f_C(H)$ so $w = uv \in f_B(H)f_C(H) = f_{BC}(H)$. Since U is an upper bound of $f_{BC}[D]$ and $w \in f_{BC}(H)$, $w \in U$, so $\sup(f_B[D])\sup(f_C[D])$ is the least upper bound of $f_{BC}[D]$:

$$\begin{aligned} f_{BC}(\sup D) &= \sup(f_B[D])\sup(f_C[D]) \\ &= \sup(f_{BC}[D]) \end{aligned}$$

- Case $A = B + C$: For any directed subset D:

$$\begin{aligned} f_{B+C}(\sup D) &= f_B(\sup D) \cup f_C(\sup D) \\ &= \sup(f_B[D]) \cup \sup(f_C[D]) && \text{(IH)} \end{aligned}$$

  For any $G \in f_{B+C}[D]$ we have $G = f_{B+C}(G') = f_B(G') \cup f_C(G')$ for some $G' \in D$. Also $f_B(G') \subseteq \sup(f_B[D])$ and $f_C(G') \subseteq \sup(f_C[D])$ so $f_B(G') \cup f_C(G') \subseteq \sup(f_B[D]) \cup \sup(f_C[D])$, so $\sup(f_B[D]) \cup \sup(f_C[D])$ is an upper bound of $f_{B+C}[D]$.

  We now show $\sup(f_B[D]) \cup \sup(f_C[D])$ is the least upper bound. For any upper bound U of $f_{B+C}[D]$, we want to show $\sup(f_B[D]) \cup \sup(f_C[D]) \subseteq U$. For any $w \in \sup(f_B[D]) \cup \sup(f_C[D])$, we have either $w \in \sup(f_B[D])$ or $w \in \sup(f_C[D])$.

  - If $w \in \sup(f_B[D])$ then $w \in f_B(G)$ for some $G \in D$, so $w \in f_B(G) \cup f_C(G) \subseteq U$.
  - If $w \in \sup(f_C[D])$ then $w \in f_C(G)$ for some $G \in D$ so $w \in f_B(G) \cup f_C(G) \subseteq U$.

  Hence $\sup(f_B[D]) \cup \sup(f_C[D]) \subseteq U$ so $\sup(f_{B+C}[D]) = \sup(f_B[D]) \cup \sup(f_C[D])$ so

$$\begin{aligned} f_{B+C}(\sup D) &= \sup(f_B[D]) \cup \sup(f_C[D]) \\ &= \sup(f_{B+C}[D]) \end{aligned}$$

- Case $A = \mu Y.B$: By IH, $f_{\sigma \cup \{X \mapsto G\}, Y, B}$ is Scott-continuous for any σ, so by Theorem 1, the least fixed point of $f_{\sigma \cup \{X \mapsto G\}, Y, B}$ is:

$$\mu(f_{\sigma \cup \{X \mapsto G\}, Y, B}) = \sup\{f^n_{\sigma \cup \{X \mapsto G\}, Y, B}(\emptyset) \mid n \in \mathbb{N}\}$$

  For all directed subsets D, we want to show:

$$f_{\sigma, X, \mu Y.B}(\sup D) = \sup(f_{\sigma, X, \mu Y.B}[D])$$

Starting from the left-hand side we have:

$$\begin{aligned}
f_{\sigma,X,\mu Y.B}(\sup D) &= \mu(G \mapsto L^{\sigma\cup\{X\mapsto \sup D, Y\mapsto G\}}(B)) \\
&= \mu(f_{\sigma\cup\{X\mapsto \sup D\},Y,B}) \\
&= \sup\{f^n_{\sigma\cup\{X\mapsto \sup D\},Y,B}(\emptyset) \mid n \in \mathbb{N}\}
\end{aligned}$$

Starting from the right-hand side we have:

$$\begin{aligned}
f_{\sigma,X,\mu Y.B}[D] &= \{f_{\sigma,X,\mu Y.B}(G) \mid G \in D\} \\
&= \{\mu(F \mapsto L^{\sigma\cup\{X\mapsto G, Y\mapsto F\}}(B)) \mid G \in D\} \\
&= \{\mu(f_{\sigma\cup\{X\mapsto G\},Y,B}) \mid G \in D\} \\
&= \{\sup\{f^n_{\sigma\cup\{X\mapsto G\},Y,B}(\emptyset) \mid n \in \mathbb{N}\} \mid G \in D\}
\end{aligned}$$

Let $M_n \coloneqq \{f^n_{\sigma\cup\{X\mapsto G\},Y,B}(\emptyset) \mid G \in D\}$.

We show $G \mapsto f^n_{\sigma\cup\{X\mapsto G\},Y,B}(\emptyset)$ is Scott-continuous for all $n \in \mathbb{N}$ (ie for any directed subset D, $\sup M_n = f^n_{\sigma\cup\{X\mapsto \sup D\},Y,B}(\emptyset)$) by induction on n:

- case $n = 0$: then both sides are $\emptyset$
- case $n = k+1$:

$$\begin{aligned}
\sup M_{k+1} &= \sup\{f_{\sigma\cup\{X\mapsto G\},Y,B}(f^k_{\sigma\cup\{X\mapsto G\},Y,B}(\emptyset)) \mid G \in D\} \\
&= \sup\{f_{\sigma\cup\{X\mapsto G\},Y,B}(\sup M_k) \mid G \in D\} && (*) \\
&= \sup\{f_{\sigma\cup\{Y\mapsto \sup M_k\},X,B}(G) \mid G \in D\} \\
&= f_{\sigma\cup\{Y\mapsto \sup M_k\},X,B}(\sup D) \\
&= f_{\sigma\cup\{X\mapsto \sup D\},Y,B}(\sup M_k) \\
&= f^{k+1}_{\sigma\cup\{X\mapsto \sup D\},Y,B}(\emptyset) && \text{(IH)}
\end{aligned}$$

  We now justify equation (*) by showing N_1 and N_2 have the same set of upper bounds, where

$$\begin{aligned}
N_1 &\coloneqq \{f_{\sigma\cup\{X\mapsto G\},Y,B}(f^k_{\sigma\cup\{X\mapsto G\},Y,B}(\emptyset)) \mid G \in D\} \\
N_2 &\coloneqq \{f_{\sigma\cup\{X\mapsto G\},Y,B}(\sup M_k) \mid G \in D\}
\end{aligned}$$

  Consider any upper bound U of N_2. Then for any $A \in N_1$, there is some $G \in D$ such that:

$$\begin{aligned}
A &= f_{\sigma\cup\{X\mapsto G\},Y,B}(f^k_{\sigma\cup\{X\mapsto G\},Y,B}(\emptyset)) \\
&\subseteq f_{\sigma\cup\{X\mapsto G\},Y,B}(\sup M_k) && (f_{\sigma\cup\{X\mapsto G\},Y,B} \text{ monotonic})
\end{aligned}$$

  and $f_{\sigma\cup\{X\mapsto G\},Y,B}(\sup M_k) \in N_2$ so $A \subseteq U$.

Consider any upper bound U of N_1. Then for any $A \in N_2$, there is some $G \in D$ such that:

$$\begin{aligned} A &= f_{\sigma \cup \{X \mapsto G\}, Y, B}(\sup M_k) \\ &= \sup(f_{\sigma \cup \{X \mapsto G\}, Y, B}[M_k]) \end{aligned}$$

Since $f_{\sigma \cup \{X \mapsto G\}, Y, B}$ is Scott-continuous.

For any word $w \in A$, we have $w \in f_{\sigma \cup \{X \mapsto G\}}(P)$ for some $P \in M_k$. Then $P = f^n_{\sigma \cup \{X \mapsto F\}, Y, B}(\emptyset)$ for some $F \in D$ and since D is directed, there is some H with $G \subseteq H$ and $F \subseteq H$. By the monotonicity of Scott-continuous functions we have

$$\begin{aligned} f_{\sigma \cup \{X \mapsto G\}}(f^n_{\sigma \cup \{X \mapsto F\}, Y, B}(\emptyset)) &\subseteq f_{\sigma \cup \{X \mapsto H\}, Y, B}(f^n_{\sigma \cup \{X \mapsto H\}, Y, B}(\emptyset)) \\ &\in N_1 \end{aligned}$$

So $w \in \sup N_1 \subseteq U$ so U is an upper bound of N_2

Hence

$$\begin{aligned} \sup(f_{\sigma, X, \mu Y.B}[D]) &= \sup\{\sup\{f^n_{\sigma \cup \{X \mapsto G\}, Y, B}(\emptyset) \mid n \in \mathbb{N}\} \mid G \in D\} \\ &= \sup\{\sup\{f^n_{\sigma \cup \{X \mapsto G\}, Y, B}(\emptyset) \mid G \in D\} \mid n \in \mathbb{N}\} \\ &= \sup\{f^n_{\sigma \cup \{X \mapsto \sup D\}, Y, B}(\emptyset) \mid n \in \mathbb{N}\} \\ &= f_{\sigma, X, \mu Y.B}(\sup D) \end{aligned}$$

□