Ch5_Arbitrary_Coordinates

Defining differentials using arbitrary coordinate

The main property of arbitrary coordinate differ from euclidean standard is

- basis are not necessarily orthogonal
- basis have non-1 magnitude
- basis change magnitude and direction depending on their position

Object	Math
Coordinate Basis	e_w, e_u
Differential in direction of basis d	$dx^\mu \to \{dw, du\}$
Differential in s	$ds = dw\, e_w + du\, e_u = dx^\mu e_\mu$
Vector	$A = A^\mu e_\mu$
Metric Tensor	$g_{uw} = e_u \cdot e_w$

- ⚖️ The Metric Tensor can be derived:

$$ds^2 = d\vec{s} \cdot d\vec{s} = (du\, e_u + dw\, e_w) \cdot (du\, e_u + dw\, e_w)$$

$$= du^2 e_u \cdot e_u + du\,dw\, e_u \cdot e_w + dw\,du\, e_w \cdot e_u + \cdots$$

$$= dx^\mu dx^\nu e_\mu \cdot e_\nu \equiv g_{\mu\nu} dx^\mu dx^\nu$$

$$\hat{g} = \begin{pmatrix} g_{uu} & g_{uw} \\ g_{wu} & g_{ww} \end{pmatrix} = \begin{pmatrix} e_u \cdot e_u & e_u \cdot e_w \\ e_w \cdot e_u & e_w \cdot e_w \end{pmatrix}$$

Transformation between arbitrary coordinates

Transforming from old coordinate u, w to new coordinate p, q .

Old : u, w $\qquad dp = \frac{\partial p}{\partial u} du + \frac{\partial p}{\partial w} dw$

New : p, q $\qquad dq = \cdots$

$$\Rightarrow dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

$$\Rightarrow A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$$

Properties

- ⓘ Some properties: need to prove
- Identity matrix given by doing transform then inverse transform.

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}} = \delta^{\mu}_{\nu}$$

- The other given by invariance of $ds^2 = g'_{\mu\nu}dx'^{\mu}dx'^{\nu} = g_{\alpha\beta}dx^{\alpha}dx^{\beta}$ in both coordinate systems, it states

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}g_{\alpha\beta} \quad \text{and} \quad g_{\alpha\beta} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}g'_{\mu\nu}$$