Ch3_Boxes

3.1.1 - Invariant of dot product across frame

$$[\underset{\sim}{A}\cdot\underset{\sim}{B}]_{s'} = \underset{\sim}{A}\cdot\underset{\sim}{B} \boxed{= A^T \eta B}$$

Aside: $\underset{\sim}{C}\cdot\underset{\sim}{D} = \eta\, \underset{\sim}{C}^T D$.

where $\eta = \begin{pmatrix}-1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}$

$$[\underset{\sim}{A}\cdot\underset{\sim}{B}]_{s} = L\underset{\sim}{A}\cdot L\underset{\sim}{B}$$

$$= (\underset{\sim}{A}^T L^T)\, \eta\, L\, \underset{\sim}{B}$$

$$= \underset{\sim}{A}^T (L^T \eta L)\, \underset{\sim}{B}$$

$$\boxed{= A^T \eta\, \underset{\sim}{B}}$$

Aside: $\begin{pmatrix}\gamma & \beta\gamma \\ \beta\gamma & \gamma\end{pmatrix}\begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\gamma & \beta\gamma \\ \beta\gamma & \gamma\end{pmatrix}$

$$= \begin{pmatrix}\gamma & \beta\gamma \\ \beta\gamma & \gamma\end{pmatrix}\begin{pmatrix}-\gamma & -\beta\gamma \\ \beta\gamma & \gamma\end{pmatrix}$$

$$= \begin{pmatrix}-\gamma^2 + \beta^2\gamma^2 & -\beta\gamma^2 + \beta^2\gamma^2 \\ -\beta\gamma^2 + \beta^2\gamma^2 & -\beta^2\gamma^2 + \gamma^2\end{pmatrix}$$

$$= \begin{pmatrix}-\gamma^2(1-\beta^2) & 0 \\ 0 & \gamma^2(1-\beta^2)\end{pmatrix} = \eta$$

3.2.1 - Invariant magnitude of 4-velocity

$$u\cdot u = \begin{pmatrix} dt/d\tau \\ dx/d\tau \\ \vdots \end{pmatrix}\cdot\begin{pmatrix} dt/d\tau \\ \vdots \end{pmatrix}$$

$$= -\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 + \cdots$$

$$= \frac{-dt^2 + dx^2 + dy^2 + dz^2}{d\tau^2}$$

$$= \left(\frac{ds}{d\tau}\right)^2 \quad \text{Invariance of } ds^2$$

$$= \left(\frac{ds'}{d\tau}\right)^2 \quad ds'^2 = -d\tau^2 + \cancelto{0}{dx'^2} + \cancelto{0}{dy'^2} + \cancelto{0}{dz'^2}$$

$$= -\left(\frac{d\tau}{d\tau}\right)^2 = -1$$

3.3.1 - Taylor expansion

$$\left(1+(-v^2)\right)^{-1/2} = 1 + \left(-\tfrac{1}{2}\right)(-v^2) + \left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\tfrac{1}{2}\, v^4 + \cdots$$

$$= 1 + \tfrac{1}{2}v^2 + \tfrac{3}{8}v^4 + \cdots$$

3.4.1 - Velocity addition example

Frame S': $\beta = v$

$$v_1 = 0$$

$$v_2' = \frac{v_2 - \beta}{1 - \beta v_2} = \frac{v_2 - v}{1 - v v_2}$$

$$v_2' = \frac{-v - v}{1 + v^2} = \frac{-2v}{1+v^2}$$

$\Rightarrow$ Mass M:

$$v_3 = 0$$

$$v_3' = \frac{v_3 - \beta}{1 - \beta v_3} = \frac{-\beta}{1} = -v$$

3.4.2 - Conservation of 4-momentum