Ch2_Boxes

2.2.1 - Between Inertial frames

The goal is to prove that frame S' is inertial iff it moves at a constant velocity relative to inertial frame S.

- To prove 'if S' moves at constant v wrt S, it is inertial' If S is inertial, an object stays at rest is always at rest in S. If S' moves at constant velocity wrt S, then the object is moving at some velocity and will stay moving in that velocity. So S' satisfy 1st law.
- To prove the reverse (if S' is inertial, it moves at constant v wrt S), the logic goes: Taken the two frames to be inertial as given. Hypothesize that there is mutual acceleration between two frames. An object stays at rest in 1 frame would have to be accelerating viewed from the other from. This suggest the given is false. Thus, the hypothesis is false. There cannot be a mutual acceleration between the two frames. Thus, the two frames can only move with relative velocity.

2.2.2 - Unit conversion

① $g = 9.8\ m s^{-2}$

$$= 9.8 \frac{m}{(3\times10^{8})^{2}\ m^{2}}$$

$$= 1.09 \times 10^{-16}\ m^{-1}$$

$$= 1/(9.17\times10^{15}\ m) \approx 1\ ly^{-1}$$

② $G = 6.77 \times 10^{-11}\ m^{3} kg^{-1} s^{-2}$

$$= 6.77\times10^{-11}\ m^{3} kg^{-1}\ (3\times10^{8})^{-2}\ m^{-2}$$

$$= 7.52 \times 10^{-28}\ m\ kg^{-1}$$

$$= 7.52 \times 10^{-28}\ m\ kg^{-1}\ \frac{1.989\ kg}{M_{\odot}} \times 10^{30}$$

$$= 1496\ m\ M_{\odot}^{-1}$$

2.3.1 - Derive Lorentz Transform

The first step is to verify that relativistic effect (e.g. length contraction) only take affect in the direction of motion. This can be done by proof by contradiction:

- Say a train on a track moving in x direction. Assume that Lorentz contraction happens in some y direction (say the direction of width)
- An observer on the ground observe that as the train speed up, it shrink in width, so the train derail and the wheel fell in between the tracks.
- Another observer on the train observe on the the rail contracts in width, the train derail and the wheel both fell out of the track.
- The two things cannot happen together, thus relativistic effect in y and z directions are not legit. In fact, they have to agree on any distance measured in yz plane ($y = y',\ \ z = z'$) ...... Solving the

system of equation:

$$t_E - x_E = \gamma(1-\beta)(t'_E - x'_E) \quad ①$$

$$t_E + x_E = \gamma(1+\beta)(t'_E + x'_E) \quad ②$$

① + ② :

$$2t_E = \gamma\ (2-\beta+\beta)\, t'_E + \gamma\ (-1+\beta+1+\beta\ )\, x'_E$$

$$\boxed{t_E = \gamma t'_E + \gamma\beta x'_E}$$

-① + ②

$$2x_E = \gamma(\beta-1)(t'_E - x'_E) + \gamma(1+\beta)(t'_E + x'_E)$$

$$= \gamma(\beta-1+1+\beta)\, t'_E + \gamma(-\beta+1+1+\beta)\, x'_E$$

$$\boxed{x_E = \gamma\beta t'_E + \gamma x'_E}$$

2.3.2 - Inverse Lorentz transformation

Inverse transformation is just the same matrix with β replaced by $-\beta$. One can prove there inverse relationship by multiplying the two matrix and see

$$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2 - \beta^2\gamma^2 & 0 \\ 0 & -\beta^2\gamma^2 + \gamma^2 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2\gamma^{-2} & 0 \\ 0 & \gamma^2\gamma^{-2} \end{pmatrix} = I$$

if the result is Identity.

2.5.1 - Invariance of interval shown

- Logic: Direct show $LT()$ This is to show invariance interval can be inferred from Lorentz transformation:

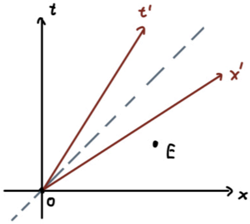
$$-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -\gamma^2 \Delta t'^2 - \gamma^2 \beta^2 \Delta x'^2 + \gamma^2 \beta^2 \Delta t'^2 + \gamma^2 \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

$$= \Delta t'^2 [\gamma^2 (\beta^2 - 1)] + \Delta x'^2 [\gamma^2 (1 - \beta^2)] + \Delta y'^2 + \Delta z'^2$$

$$-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

2.6.1 - Frame dependence of time ordering of events

Choosing a suitable lorentz transformation to prove that order of event can be different for space-like event.



Space-like: $\Delta s^2 > 0$

Suppose $E: (t, x)$, $y = z = 0$, $t > 0$

$\Rightarrow \Delta s^2 = -t^2 + x^2 > 0$

$\Rightarrow x > t$

Lorentz :

$$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix},$$

Let $\beta = 1 - \delta$, whr $\delta \ll 1$

$$\Rightarrow \begin{pmatrix} \gamma & -\gamma(1-\delta) \\ -\gamma(1-\delta) & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$= \begin{pmatrix} \gamma t - \gamma(1-\delta)x \\ -\gamma(1-\delta)t + \gamma x \end{pmatrix} = \begin{pmatrix} \gamma t - \gamma x + \gamma\delta x \\ -\gamma t + \gamma\delta t + \gamma x \end{pmatrix}$$

$$= \gamma \begin{pmatrix} t - x + \delta x \\ x - t + \delta t \end{pmatrix} = \begin{pmatrix} t' \\ x' \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} t' \approx \gamma(t-x) < 0 \\ x' \approx \gamma(x-t) > 0 \end{matrix}\right\} \text{for } \delta \ll 1$$

$\Rightarrow$ $t' < 0$ in this frame, time ordering is reversed

For time-like and light-light events, the same logic can be used to analyze:

Case 2: $\Delta s^2 < 0 \rightarrow t > x$

$$\Rightarrow \begin{array}{l} t' \approx \gamma(t-x) > 0 \\ x' \approx \gamma(x-t) < 0 \end{array} \Rightarrow \text{time ordering is preserved}$$

Case 3: $\Delta s^2 = 0 \rightarrow t = x$

$$\Rightarrow t' = \gamma 0 + \delta \gamma x \rightarrow x$$

$$x' \rightarrow t$$

$\Rightarrow$ Stays light-like.

2.7.1 - Proper time along a path

$$d\tau = dt' = \sqrt{-ds^2}$$

$$= \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$

$$= \sqrt{dt^2\left(1 - \frac{dx^2}{dt^2} - \frac{dy^2}{dt^2} - \frac{dz^2}{dt^2}\right)}$$

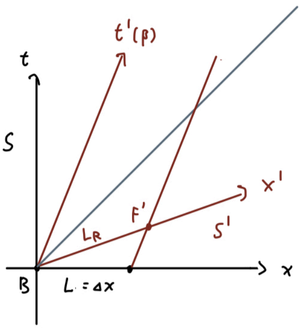
$$= dt\sqrt{1 - (v_x^2 + v_y^2 + v_z^2)}$$

$$= dt\sqrt{1 - v^2}$$

Another way of doing the same is Lorentz transform on a differential of proper interval $(d\tau, 0, 0, 0)$, which gives $\Lambda(d\tau, 0, 0, 0) = (\gamma d\tau, \beta\gamma d\tau, 0, 0)$. Which gives $dt = \gamma d\tau$. here dx would be the relative speed β times dt, which gives the relative displacement between two frames during $d\tau$.

2.8.1 - Length contraction derived from Lorentz

Lorentz transform together with spacetime diagram gives the Lorentz contraction formula



$$[F']_{S'} : (0, L_R)$$

$$[F']_S : \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ L_R \end{pmatrix}$$

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \beta\gamma L_R \\ \gamma L_R \end{pmatrix}$$

This gives the spacetime coordinate in S frame. From the spacetime diagram, the length L would simply be the x-axis of t' axis in frame S. Using the slope of t' axis and coordinate F', one can find coordinate of F.

$$F : (t = 0,\ x)$$

$$\Rightarrow [F]_S = [F']_S - \beta\gamma L_R \overset{\text{slope}}{\begin{pmatrix} 1 \\ \beta \end{pmatrix}}$$

$$= \begin{pmatrix} \beta\gamma L_R \\ \gamma L_R \end{pmatrix} - \begin{pmatrix} \beta\gamma L_R \\ \beta^2\gamma L_R \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \gamma(1-\beta^2) L_R \end{pmatrix}$$

$$\Rightarrow L = \sqrt{1-\beta^2}\, L_R = \frac{1}{\gamma} L_R$$

2.9.1 - Velocity addition

A particle moving in x direction is viewed from S and S'. S' moves in x direction relative to S with a velocity β. Express the particle velocity in x direction for S' in terms of velocity in S.

The process can be done by Lorentz transforming the differentials.

$$V_x' \equiv \frac{\partial x'}{\partial t'}$$

$$= \frac{-\gamma\beta\, dt + dx\,\gamma}{\gamma\, dt - \gamma\beta\, dx}$$

$$= \frac{dx - \beta dt}{dt - \beta dx}$$

$$= \frac{dx}{dt}\left(\frac{1-\beta\frac{dt}{dx}}{1-\beta\frac{dx}{dt}}\right)$$

$$= V_x\left(\frac{1-\beta/V_x}{1-\beta V_x}\right) = \frac{V_x - \beta}{1-\beta V_x}$$