Caroll_Ch2.3_Vectors_in_Manifold

2.3 Vectors on manifold

To define a vector, there are certain needed elements. **Vectors** live in **vector spaces**, and vectors spaces necessarily associates with a **field** (We say is as 'a vector space V over a field $\mathcal{F}$'). So three elements needs to be defined:

- Vector
- Vector space (tangent space)
- Field that the vectors act on We have to define these things using only the things in the manifold. Notice what we used to define vectors by (length and a direction) doesn't naturally exist in a manifold. So we have to find other things to define these by. (we avoid referring to a higher dimension euclidean space, so that the concept of manifold stays general even for abstract manifolds like Lie groups)Defining Vector in manifold

The idea is to define differentiation operator as the vector. The field they act on is the field of functions. The vector space T_p can be identified with basis: partial differential ∂_μ operators. So we have:

Object on Manifold	**Defined as**
Vector	$\frac{d}{d\lambda}$
Basis vector $\hat{e}_{(\mu)}$	∂_μ
Vector space T_p	$\{\partial_\mu\}$
Field $\mathcal{F}$ associate w/ space	$\{f \in C^\infty : \mathcal{M} \to \mathbb{R}\}$
Chart	$\{x^\mu\}$

The basis vector $\hat{e}_{(\mu)}$ has a μ inside a bracket as it stands for an index. It is the basis for μ 'th coordinate. The index for partial operator ∂_μ means it is the partial with respect to x^μ. (∂_μ is covariant wrt the coordinate system so index is at the bottom)

There are some properties that needs to be satisfied for our definition of vectors space to be mathematically accurate. One can check that these criteria are satisfied

Properties on vectors	**Defined as**
Spectrum decomposition	$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$ ...sum is implied
Vector space axioms	

- ⚖ Spectrum decomposition can be proved: Note that the coefficients of the vector is simply $\frac{dx^\mu}{d\lambda}$

  $\frac{d}{d\lambda}$ acting on f:

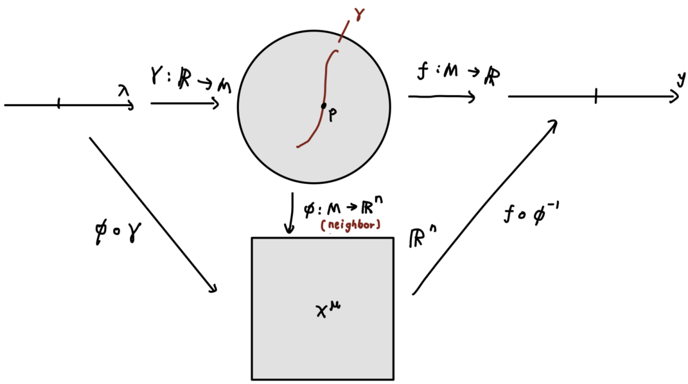
  $$= \frac{d}{d\lambda}(f\circ\gamma) = \frac{d}{d\lambda}(f\circ\phi^{-1})\circ(\phi\circ\gamma)$$

  $$= \sum_\mu \frac{\partial}{\partial x^\mu}(f\circ\phi^{-1})\frac{d}{d\lambda}(\phi\circ\gamma)^\mu$$

  $$= \frac{\partial(f\circ\phi^{-1})}{\partial x^\mu}\frac{d(\phi\circ\gamma)^\mu}{d\lambda} \quad \text{[Implicit Sum over } \mu \text{]}$$

  $$= \frac{d(\phi\circ\gamma)^\mu}{d\lambda}\partial_\mu$$

The relationship can be visualized:



Vector can be transformed using differentials between coordinates like $\frac{\partial x^\mu}{\partial x^{\mu'}}$. Vector fields can be created. They have commutator that is independent of coordinate system.

(1)

vector $V^{\mu}\partial_{\mu} = V^{\mu'}\partial_{\mu'} = V^{\mu'}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\partial x^{\mu}$

- written as $\boxed{V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'}}$ [∂_{μ} are not written explicitly]

  Vector transformation law. encompase arbitary change of coordinate.

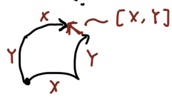
- Cosistant with Lorentz $V^{\mu'} = \Lambda^{\mu'}{}_{\mu} V^{\mu}$

  w/ $x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu}$

(2)

vector field of ∂_{μ} $X : \underset{\text{func}}{\mathcal{F}} \to \underset{\text{dir der of func.}}{\mathcal{F}}$

$$\underbrace{[X, Y](f) = X(Y(f)) - Y(X(f))}_{\text{Independent of coordinate.}}$$



- $[X, Y](fg) = f[X, Y](g) + g[X, Y](f)$

  $[X, Y]^{\mu} = X^{\lambda}\partial_{\lambda}Y^{\mu} - Y^{\lambda}\partial_{\lambda}X^{\mu}$

Vector field and