Ch3_Boxes

3.1.1 - Invariant of dot product across frame

$$[\underset{\sim}{A}\cdot\underset{\sim}{B}]_{s'} = \underset{\sim}{A}\cdot\underset{\sim}{B} \boxed{= A^T \eta B}$$

$$[\underset{\sim}{A}\cdot\underset{\sim}{B}]_{s} = L\underset{\sim}{A}\cdot L\underset{\sim}{B}$$

$$= (\underset{\sim}{A}^T L^T)\,\eta\, L\,\underset{\sim}{B}$$

$$= \underset{\sim}{A}^T\,(L^T \eta L)\,\underset{\sim}{B}$$

$$\boxed{= A^T \eta\, \underset{\sim}{B}}$$

Aside: $\underset{\sim}{C}\cdot\underset{\sim}{D} = \eta\, \underset{\sim}{C}^T D$.

where $\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Aside: $\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}$

$$= \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} -\gamma & -\beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} -\gamma^2+\beta^2\gamma^2 & -\beta\gamma^2+\beta^2\gamma^2 \\ -\beta\gamma^2+\beta^2\gamma^2 & -\beta^2\gamma^2+\gamma^2 \end{pmatrix}$$

$$= \begin{pmatrix} -\gamma^2(1-\beta^2) & 0 \\ 0 & \gamma^2(1-\beta^2) \end{pmatrix} = \eta$$

3.2.1 - Invariant magnitude of 4-velocity

$$u\cdot u = \begin{pmatrix} dt/d\tau \\ dx/d\tau \\ \vdots \end{pmatrix}\cdot\begin{pmatrix} dt/d\tau \\ \vdots \\ \vdots \end{pmatrix}$$

$$= -\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 + \cdots$$

$$= \frac{-dt^2+dx^2+dy^2+dz^2}{d\tau^2}$$

$$= \left(\frac{ds}{d\tau}\right)^2 \quad \text{Invariance of } ds^2$$

$$= \left(\frac{ds'}{d\tau}\right)^2 \quad ds'^2 = -d\tau^2 + \cancel{dx'^2}^{\,0} + \cancel{dy'^2}^{\,0} + \cancel{dz'^2}^{\,0}$$

$$= -\left(\frac{d\tau}{d\tau}\right)^2 = -1$$

3.3.1 - Taylor expansion

$$(1+(-v^2))^{-1/2} = 1 + (-\tfrac{1}{2})(-v^2) + (-\tfrac{1}{2})(-\tfrac{3}{2})\tfrac{1}{2} v^4 + \cdots$$

$$= 1 + \tfrac{1}{2}v^2 + \tfrac{3}{8}v^4 + \cdots$$

3.4.1 - Velocity addition example

frame S': $\beta = v$

$$V_1 = 0$$

$$V_2' = \frac{V_2 - \beta}{1 - \beta V_2} = \frac{V_2 - V}{1 - V V_2}$$

$$V_2' = \frac{-V - V}{1 + V^2} = \frac{-2V}{1+V^2}$$

$\Rightarrow$ Mass M:

$$V_3 = 0$$

$$V_3' = \frac{V_3 - \beta}{1 - \beta V_3} = \frac{-\beta}{1} = -V$$