Caroll_Ch2.2_Manifold

Choosing the right mathematical model for GR

We need a mathematical framework for spacetime that accurately **maps our knowledge in physics to math language**. Here are the physical requirements and the math solution for them:

- **Globally curved** (So that we model gravity by curvature):
  - Solution: Manifold
- **Locally flat** (GR is locally SR, Einstein Equivalence Principle)
  - Solution: Manifold is locally euclidean
- **Locally defined vector and vector space** (Locally defined vectors for momentum, velocity, mass, and other 4-vectors, and space for them to live in)
  - Solution: Tangent space with partial differential operators as vectors. It requires manifold to be differentiable
- **Seamless transition between patches** (A way of bringing vector from one patch to another)
  - Solution: Charts and atlas

2.2 A manifold

- A manifold is n dimension if the local euclidean space is n dimensional.
- Define Mapping

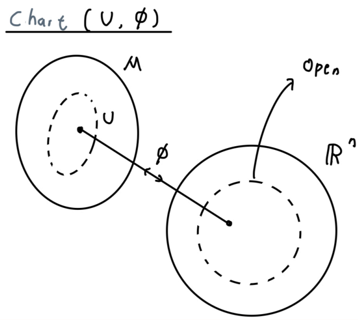
Examples of manifold:

- n-dim euclidean space
- n-dim sphere S^n, defined as the set of points of the same distance to the origin in $n+1$ euclidean space
- donuts and donuts with more than 1 holes
- The $SO(n)$ group has 1-1 correspondence to S^1 using θ

Construction of a manifold

Review on math terms

- We first define a **open ball** as a open set in the shape of a ball:
- We define: a set V is **open** is there exist a open ball for any point y inside V, there exist a open ball that is also in that set.
- We define a **chart** of U, a open subset of M, that is equipped with a mapping ϕ that maps to a open subset of $\mathbb{R}^n$.



- We then define a C^∞ **atlas** as an indexed collection of charts $\{U_\alpha, \phi_\alpha\}$, that satisfy the following condition:
  - The union of all U_α is M. In other words, $\{U_\alpha\}$ cover M
  - Element of the intersection $U_\alpha \cap U_\beta$ can be mapped to both charts $\phi_\alpha(U_\alpha)$ and $\phi_\beta(U_\beta)$. The charts are smoothly sewn together such that the map $(\phi_\alpha \circ \phi_\beta^{-1})$ takes elements from $\phi_\beta(U_\alpha \cap U_\beta)$ **onto** $\phi_\alpha(U_\alpha \cap U_\beta)$. These mapping also must be C^∞

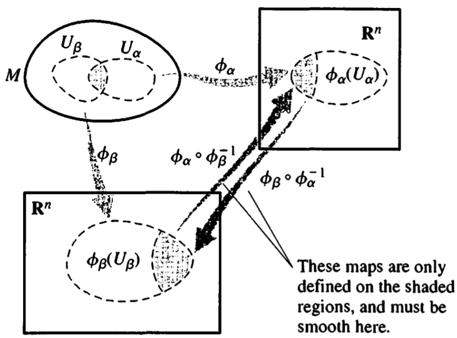
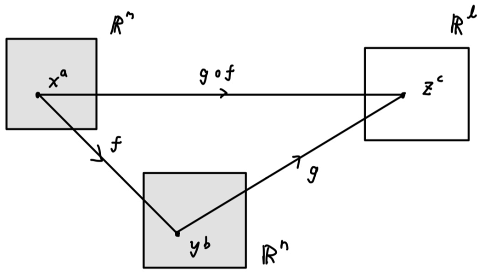


FIGURE 2.14 Overlapping coordinate charts.

- Then, **we define manifold** by the set M with a maximal atlas. (the maximal requirement is to ensure two equivalent spaces with different atlas counts as equivalent manifold.)
- Any n-manifold can be embedded in $\mathbb{R}^{2n}$ (Whitney's embedding thm)

Derivative chain rule

Suppose two functions f and g illustrated by the graph below.



To take the derivative $\frac{\partial z^c}{\partial x^a}$ means to quantify the rate of change of $c'th$ component of z with respect to the change in $a'th$ component of x .

- The notation $\frac{\partial}{\partial x^a}(g \circ f)^c$ and $\frac{\partial z^c}{\partial x^a}$ are equivalent, although the first is more rigorous.
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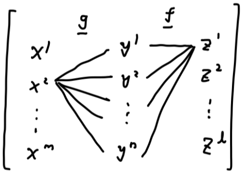
The chain rule dictates how to perform the derivative: (The upper index on function suggest the n-tuple component.)

$$\frac{\partial}{\partial x^a}(g \circ f)^c = \sum_b \frac{\partial f^b}{\partial x^a} \frac{\partial g^c}{\partial y^b}$$

For example: taking the derivative $\frac{\partial z^1}{\partial x^2}$:



$$\frac{\partial z^1}{\partial x^2} = \sum_{b}^{n} \frac{\partial z^1}{\partial y^b} \frac{\partial y^b}{\partial x^2}$$

$$= \frac{\partial z^1}{\partial y^1} \frac{\partial y^1}{\partial x^2} + \frac{\partial z^1}{\partial y^2} \frac{\partial y^2}{\partial x^2} + \cdots + \frac{\partial z^1}{\partial y^n} \frac{\partial y^n}{\partial x^2}$$