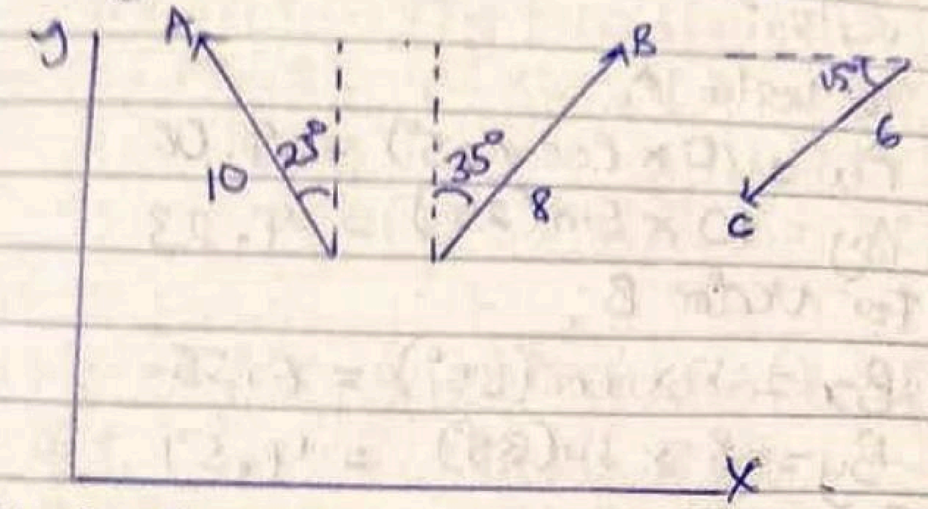
Using vectors, A, B and C evaluate



(a) A . B × C

(b) A × (B × C)

(c) (B . C) A

Solution

To evaluate the expression using vectors A, B and C, we need to first convert each vector into its Cartesian form (x, y components), and then perform the necessary operation.

Given

A = 10 magnitude 25°

B = 8 magnitude 35

C = 6 magnitude 15°

Let's find the Cartesian components of each vector:

For vector A:

$A_x = 10 \times \cos(25^\circ) = 9.06$

$A_y = 10 \times \sin(25^\circ) = 4.23$

For vector B:

$B_x = 8 \times \cos(35^\circ) = 6.55$

$B_y = 8 \times \sin(35^\circ) = 4.59$

For vector C

$C_x = 6 \times \cos(15^\circ) = 5.80$

$C_y = 6 \times \sin(15^\circ) = 1.55$

Now, let's calculate each expression.

(a) $A.(B \times C)$

find the cross product $B \times C$

$B \times C = (B_x \times C_y - B_y \times C_x)$

take the dot product of A with the result

$A.(B \times C) = A_x \times (B_x \times C_y - B_y \times C_x) + A_y (B_y$
$C_x - B_x \times C_y)$

$9.06 \times (6.55 \times 1.55 - 4.59 \times 5.80) + 4.$
$(4.59 \times 5.80) - 6.55 \times 1.55)$

$= 9.06(10.15 - 26.62) + 4.23(26.62 - 10.15)$
$= 9.06(-16.47) + 4.23(16.62)$
$= -149.22 + 70.30$
$= -78.92$

(b) $A \times (B \times C)$:
This expression means A crossed with (B) crossed with (C), find B crossed with C
$B \times C = (B_x \times C_y - B_y \times C_x)$
Take the cross product of A with the result:
$A \times (B \times C) = (A_y \times (B_x \times C_y - B_y \times C_x)) - (A_x \times (B_y \times C_x - B_x \times C_y))$

$= 4.23 \times (6.55 \times 1.55 - 4.59 \times 5.80)) - (9.06 \times (4.59 \times 5.80 - 6.55 \times 1.55))$

$= 4.23 \times (10.15 - 26.62) - 9.06(26.62 - 10.15)$
$= 4.23 \times (-16.47) - 9.06(16.62)$
$= -69.67 - 150.58$
$= -220.25$

$$\tan\theta = \frac{350(\sqrt{3}/2 - 1/2)}{300}$$

$$\theta = \arctan\left(\frac{350(\sqrt{3}/2 - 1/2)}{300}\right)$$

$$\theta = 27.75^\circ$$

So, the direction the pilot should fly to travel directly from A to B is approximately 27.75° north of east.

(b) To find the distance we use the magnitude of the resultant vector, we simply, calculate the leng the resultant vector, we simply, calculate the length of the resultant vector. We add the magnitudes of all component

Magnitude of resultant vector =

$$\sqrt{(300)^2 + \left(350\times\frac{\sqrt{3}}{2}\right) + \left(350\times\frac{1}{2}\right)^2 + (150)^2}$$

Magnitude of resultant vector =

$$\sqrt{900 + 306250 + 61250 + 22500}$$

$$= \sqrt{48500}$$

Magnitude of resultant vector = 692.82km

∴ The pilot will fly approximately 692.82km to the travel directly from A to B

1.13 If $A = 3\hat{i} - \hat{j} + 2\hat{k}$ and $B = 2\hat{i} + 3\hat{j} + k$, find $A \times B$, $a = QP = 2\hat{i} - \hat{j} - 6k$

$b = QR = \hat{i} - 4\hat{j} - 5k = \frac{1}{2}\sqrt{4-6}$

Solution

$A = 3i - j + 2k$

$B = 2i + 3j + k$

The cross product of two vector in three-dimensional space can be calculated using the determinate of a 3×3 matrix formed by the unit vectors i, j and k and the compounds of the vector A and B. The components of the resulting vector are given by.

$$A \times B = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Substituting the values of A and B

$$A \times B = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

Expanding the determinant

$A \times B = i(-1\times1 - 3(2)) - j(3(1) - 2(2)) + k(3(3) - (-2))$

$= -7i - j(-1) + k(9+2)$

$= -7i + j + 11k$

1.14 Find the area of the triangle with vertices $P(2,3,5)$, $Q(4,2,-1)$, $R(3,6,4)$.

Solution

Let the vectors PQ and PR be as $\vec{v}$ and $\vec{w}$ respectively

First, let find the vectors $\vec{v}$ and $\vec{w}$ using the coordinate of the given points

Vector $\vec{v}$ (PQ)

$\vec{v} = (4-2, 2-3, -1-5)$

$= (2, 1, -6)$

Let take the dot product of $\vec{A}$ with $-6i + 2j - k$;

$\vec{A} \cdot (-6i + 2j - k)$

$= (3)(-6) + (-1)(2) - (0)(-1)$

$= -18 - 2 - 0$

$= -20$

The absolute value of -20 is 20

$V = 20$ cubic units

∴ The volume of the parallelepiped is $|20|$ which is 20 cubic units

1.16 What is the value of each of the angles whose side are 95, 150, 190 in length.

Solution

To find the angle of a triangle when the lengths of its side are known; we can use the law of Cosines, which states

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Where a, b, c are the lengths of the sides of the triangle and C is the angle opposite side c

$a = 95$, $b = 150$, $c = 190$

A_x and A_y is x and y component of vector A
B_x and B_y is x and y component of vector B
Given data;

- Vector A points 75.0° north of east
- Vector B points due west, which means its x-component is negative and y component is zero
- Vector C points due south and it has a magnitude of 185 metres

Let set up the equation
for the x-component : $A_x + B_x = 0$ --- (1)
y-component : $A_y + B_y + C_y = 0$ --- (2)

$A_x = A \cos 75^\circ$
$A_y = A \sin 75$
$B_x = -B$
$B_y = 0$
$C_y = -185$ (since it points due south)

Substituting the values into the equation

$A \cos 75 - B = 0$ --- (1)
$A \sin 75 + 0 - 185 = 0$ --- (2)

Ax and Ay is x and y component of vector A
Bx and By is x and y component of vector B
Given data;

- Vector A points 75.0° north of east
- Vector B points due west, which means its x-component is negative and y component is zero
- Vector C points due south and it has a magnitude of 185 metres

Let set up the equation
for the x-component: $A_x + B_x = 0$ --- ①
y-component: $A_y + B_y + C_y = 0$ --- ②

$A_x = A\cos 75^\circ$
$A_y = A\sin 75$
$B_x = -B$
$B_y = 0$
$C_y = -185$ (since it points due south)

Substituting the values into the equation

$A\cos 75 - B = 0$ --- ①
$A\sin 75 + 0 - 185 = 0$ --- ②

$= i(-12-40) - j(24-18) + k(40-(-9))$
$= -52i - 6j + 49k$

$|a \times b| = \sqrt{(-52)^2 + (-6)^2 + (49)^2}$
$= \sqrt{2704 + 36 + 2401}$
$= \sqrt{5141} = 71.7$

$$c = \frac{-52i - 6j + 49k}{71.7}$$

$$= -\frac{52i}{71.7} - \frac{6j}{71.7} + \frac{49k}{71.7}$$

$= -0.725i - 0.084j + 0.683k$
$= -0.73\hat{i} - 0.084j + 0.68\hat{k}$

1.19 If $A = 6\hat{i} - 8\hat{j}$, $B = -8\hat{i} + 3j$ and $C = 26\hat{i} + 19j$ determine a and b since that $aA + bB + C = 0$

Solution

To solve for a and b given the equation $aA + bB + C = 0$, where $A = 6i - 8j$, $B = -8i + 3j$ and $C = 26i + 19j$, you can equate the components of the vector separately.

(a) Finding a and b

$(a+b) + (a-b) = (11\hat{i} - \hat{j} + 5\hat{k}) + (-5i - 11j + 9k)$

$2a = 6i - 12j + 14k$

$a = 3i - 6j + 7k$

Substituting a onto one of the original equation to find b;

$a + b = 11i - j + 5k$

$3i - 6j + 7k + b = 11i - j + 5k$

$b = 11\hat{i} - j + 5k - 3i + 6j - 7k$

$b = 8i + 5j - 2k$

So, $a = 3i - 6j + 7k$ and $b = 8i + 5j - 2k$

b. finding the angle included between a and a+b

$$\cos\theta = \frac{a \cdot (a+b)}{\|a\| \, \|a+b\|}$$

$a \cdot (a+b) = (3i - 6j + 7k) \cdot (11i - j + 5k)$

$= (3 \cdot 11) + (-6 \cdot -1) + (7 \cdot 5)$

$= 33 + 6 + 35$

$= 74$

A racetrack consists of two parallel sides with semicircular ends as shown. The vector d connects track position A and B. Express d in polar form

Solution

To express the vector d in polar form, we need to find its magnitude and direct angle

The magnitude of d can be found using the pythagoras theorem since its the hypotenuse of a right triangle formed by the sides of the lengths 600 and 800

$$r = \sqrt{600^2 + 800^2}$$

$$= \sqrt{1000000}$$

$$= \sqrt{1000}$$

Direction angle (θ) can be found using the tangent function

$$\theta = \tan^{-1}\left(\frac{800}{600}\right) = \tan^{-1}(1.33) = 53.66$$

So, in polar form, d can be expressed as

$$d = r < \theta$$

Since a = a, it is positive

$R_y = 2 \times 6.09$

$R_y = 12.1$

c The angle r makes with x-axis

$\Theta = \tan^{-1}\left(\frac{r_y}{r_x}\right)$

$= \tan^{-1}(8.06)$

$= 84^\circ$

Therefore, $R_x = 1.5$

$R_y = 12.1$

$\Theta = 84^\circ$

1.24 Two vectors a and b have components in arbitrary units $a_x = 3.2$, $a_y = 1.6$, $b_x = 0.50$

$b_y = 4.5$

a) find the angle between a and b

b) find the components of a vector c that is perpendicular to a and is in the x-y plane, and has magnitude of 5.0 units

(b.) To find the components of a vector, let use the cross product. Since C lies in the x-y plane and is perpendicular to a, its z component will be zero. The magnitude of C is given as 5 units.
Let denote C as $C = C_x i + C_y j$

The cross product $c = a \times C$ will give us a vector perpendicular to both a and C, which will point in the z-direction.
The magnitude of the cross product

$$|c| = |a| \times |C| \times \sin\theta$$

$$5 = \sqrt{12.8} \times 5 \times \sin\theta$$

$$\sin\theta = \frac{5}{\sqrt{12.8} \times 5}$$

$$\sin\theta = 0.2795$$

$$\theta = \sin^{-1}(0.2795$$

$$= 16.24^\circ$$

$10 \cos 16.24$ and $10 \sin 16.24$

i.e. The component of $C_x = \pm 2.22$ or
$C_y = \pm 4.5$ approximately.

$d = 1000 < 53.06°$

$d = 600.98 + 799.28$

$= 1400.24$

∴ d in polar form as. $1000 < 53.06°$

4.23 Two vector a and b have equal magnitudes of 10units. They are oriented as shown and their vector sum is r. Find the

(a) the x components of r

(b) the magnitude of r

(c) the angle r makes with the x axis

Solution

a To find the x component

$r_x = 10 \cos 30$

$= 10 \times 0.15$

$= 1.5$

b. To find the magnitude of ry

$R_y = 2 \times 10 \sin 60$

$= 20 \sin 60$

$= -6.09$

are perpendicular to each other

Magnitude of A + B = $\sqrt{|A|^2 + |B|^2}$

$= \sqrt{6^2 + \left(\frac{2\sqrt{3}}{3}\right)^2}$

$= \sqrt{36 + (4/3)}$

$= \sqrt{\frac{108}{3} + \frac{4}{3}}$

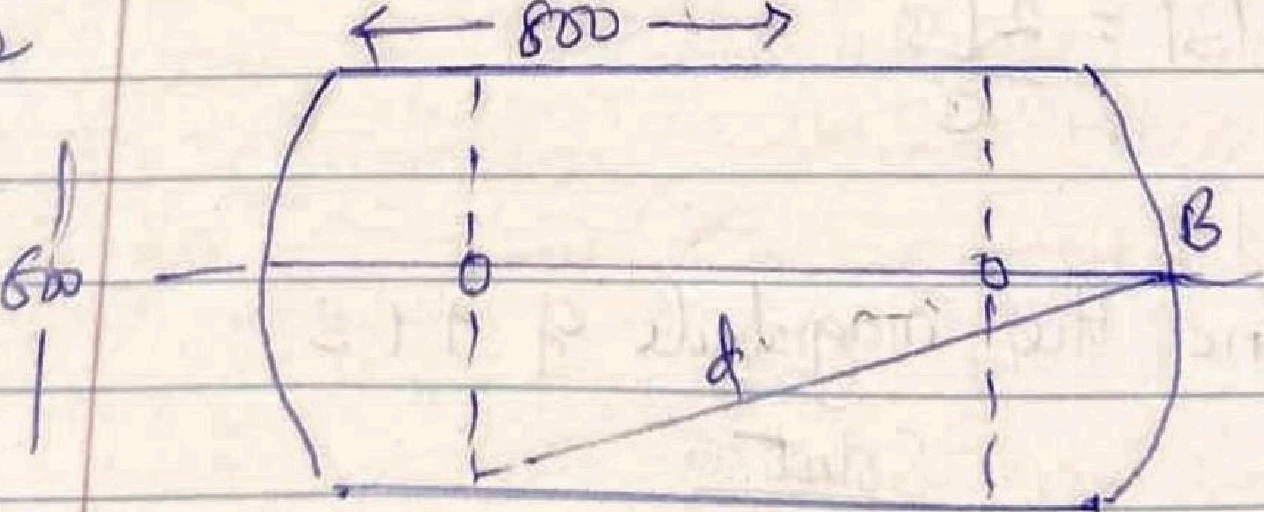
$= \sqrt{\frac{112}{3}}$

$= \sqrt{37.3}$

Magnitude of A + B = 6.11 units

$\therefore$ the magnitude of B is approximately $\frac{2\sqrt{3}}{3}$ unit, and the magnitude of A + B is approximately 6.11 units

1.22



Vector B points due north
The resultant vector (A+B) point 60.0 north of east
(ri) The vertical Component of B, $B_y = |B| \sin 60°$
The vertical component of B equals to magnitude

$|B| = B_y = |B| \sin 60°$

$|B| = |B| \sin 60$

$1 = \sin 60$

$$|B| = \frac{1}{\sin 60}$$

$$= \frac{1}{\sqrt{3}/2}$$

$$|B| = \frac{2}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$|B| = \frac{2\sqrt{3}}{3}$$

Find the magnitude of A+B

<u>Solution</u>

Apply pythagoras theorem because A and B

Lets find the magnitude of a and a+b

$\|a\| = \sqrt{3^2 + (-6)^2 + 7^2} = \sqrt{9+36+49}$

$= \sqrt{94}$

$\|a+b\| = \sqrt{(11)^2 + (-1)^2 + 5^2} = \sqrt{121+1+25}$

$= \sqrt{147}$

$\cos\theta = \dfrac{74}{\sqrt{94} \times \sqrt{147}}$

$= \dfrac{74}{117.54}$

$\cos\theta = 0.6296$

$\theta = \cos^{-1}(0.6296)$

$= 50.97^\circ$

$\therefore \theta = 50.97^\circ$

The angle between a and a+b is 51°

1-21 Vector A has magnitude 6.0 units and points due east, vector B points due north.

(a) What is the magnitude of B if the vector A+B points 60.0 north of east?

(b) Find the magnitude of A+B

Solution

Magnitude of vector A, $|A| = 6.0$ units (point due east)

Substituting the value of a into equation

$-8a + 3b = -19$

$-8(5) + 3b = -19$

$3b = 40 - 19$

$b = \frac{21}{3}$

$= 7$

$\therefore$ The value of a and b are 5 and 7 respectively

1.20 Given two vectors such that $a + b = 11\hat{i} + \hat{j} + 5\hat{k}$ and $\underline{a} - \underline{b} = -5\hat{i} - 11\hat{j} + 9\hat{k}$

(a) Find a and b

(b) Find the angle include between a and a+b

Solution

First, let find the individual component of vector a and b

Data given

$a + b = 11\hat{i} - \hat{j} + 5\hat{k}$

$\underline{a} - \underline{b} = -5\hat{i} - 11\hat{j} + 9\hat{k}$

Let's write the equation with the given vectors

$a(6i-8j) + b(-8i+3j) + 26i + 19j = 0$

For i Component

$6a - 8b + 26 = 0$

For j Component

$-8a + 3b + 19 = 0$

Solve simultaneously to find a and b

$6a - 8b + 2b = 0$

$6a - 8b = -26$

$-8a + 3b + 19 = 0$

$-8a + 3b = -19$

$6a - 8b = -2 \quad \ldots (1) \times 3$

$-8a + 3b = -19 \quad \ldots (2) \times 8$

Using elimination method, let's multiply equation (1) by 3 and ~~explain~~ equation (2) by 8

$18a - 24b = -18$

$+ \; -64a + 24b = -152$

Let add

$-46a = -230$

$+46a = +230$

$a = \frac{230}{46} = 5$

find a unit vector perpendicular to both a and b and normalise the resulting vector

$a = 2i - j + 2k$

$b = 9i + 20j + 12k$

a. Unit vector a:

$|a| = \sqrt{(2)^2 + (-1)^2 + (2)^2} = \sqrt{9} = 3$

$a_{unit} = \frac{a}{|a|} = \frac{2i - j + 2k}{3}$

b Unit vector b:

$|b| = \sqrt{(9)^2 + (20)^2 + (12)^2} = \sqrt{625} = 25$

$b_{unit} = \frac{b}{|b|} = \frac{9i + 20j + 12k}{25}$

c Unit vector C perpendicular to both a and b:

$C = \frac{a \times b}{|a \times b|}$

$a \times b = \begin{vmatrix} i & j & k \\ 2 & -1 & 2 \\ 9 & 20 & 12 \end{vmatrix}$

$= i \begin{vmatrix} -1 & 2 \\ 20 & 12 \end{vmatrix} - j \begin{vmatrix} 2 & 2 \\ 9 & 12 \end{vmatrix} + k \begin{vmatrix} 2 & -1 \\ 9 & 20 \end{vmatrix}$

from equation 1

$A \cdot \cos(75^\circ) = B$

from the equation 2

$A \cdot \sin(75^\circ) = 185$

$A = \frac{185}{\sin 75^\circ} = \frac{185}{0.9659} = 191.53m$

$B = A \cdot \cos 75^\circ$

$= 191.53 \cdot 0.2588$

$= 49.57m$

$\therefore$ The magnitude of A and B are approximately 1.91.53m and 49.57m respectively

1.18 Given $a = 2\hat{i} - \hat{j} + 2\hat{k}$ and $b = 9\hat{i} + 20\hat{j} + 12K$. find $a \wedge b = 19i + 4j - 7K$

(a) a unit vector $\hat{a}$

(b) a unit vector $\hat{b}$

(c) a unit vector $\hat{c}$ perpendicular to both a and b

Solution

To find the unit vectors of a and b, you need to divide each vector by its magnitude. Then,

$a^2 = b^2 + c^2 - 2bc \cos A$

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$= \dfrac{150^2 + 190^2 - 95^2}{2(150)(190)}$

$= \dfrac{49575}{57000}$

$\cos A = 0.8697$

$A = \cos^{-1}(0.8697)$

$A = 29.58^\circ$

$\therefore$ The angle of A, B and C are 29.58°, 51.19° and 99.29° respectively

1.17 Three vectors add together so that the resultant is zero. Vector A point 75.0° north of east. Vector B points due west. Vector C points due south and has magnitude of 185 metres. Find the magnitude of A and B

Solution

To find the magnitudes of vectors A and B, which add up to zero when combined with vector C, we use vector addition.

$$190^2 = 95^2 + 150^2 - 2(95)(150)\cos C$$

$$36100 = 9025 + 22500 - 28500 \cos C$$

$$36100 = 31525 - 28500 \cos C$$

$$\cos C = \frac{36100 - 31525}{-28500}$$

$$\cos C = \frac{-4575}{28500}$$

$$\cos C = -0.1605$$

$$C = \cos^{-1}(-0.1605)$$

$$C = 99.24^\circ$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{95^2 + 190^2 - 150^2}{2(95)(190)}$$

$$= \frac{22625}{36100}$$

$$\cos B = 0.6267$$

$$B = \cos^{-1}(0.6267)$$

$$B = 51.19^\circ$$

1.15 Find the volume of a parallelepiped with sides $A = 3\hat{i} - \hat{j}$, $B = \hat{j} + 2\hat{k}$, $C = \hat{i} + 5\hat{j} + 4\hat{k}$

Solution

Lets denote the side of the parallelepiped as vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$

Data given

$\vec{A} = 3i - j$

$\vec{B} = j + k$

$\vec{C} = i + 5j + 4k$

2(3)

To find the volume of the parallelepiped, we will use the scalar triple product

$\vec{V} = |\vec{A} \cdot (\vec{B} \times \vec{C})|$

First, lets find $\vec{B} \times \vec{C}$

$$\vec{B} \times \vec{C} = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 1 & 5 & 4 \end{vmatrix}$$

$$= i \begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} - j \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} + k \begin{vmatrix} 0 & 1 \\ 1 & 5 \end{vmatrix}$$

$$= i(4-10) - j(0-2) + k(0-1)$$

$$= -6i + 2j - k$$

Vector $\vec{w}$ (PR)

$\vec{w} = (3-2, 6-3, 4-5)$

$= (1, 3, -1)$

Let's find the cross product

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 2 & -1 & -6 \\ 1 & 3 & -1 \end{vmatrix}$$

$= i(-1(-1) - (-6)(3)) - j(2(-1) - 1(-6)) + k(2$

$- 1(-1))$

$= i(1 - (-18)) - j(-2+6) + k(6+1)$

$= 19i - j(4) + 7k$

$= 19i - 4j + 7k$

$= (19, -4, 7)$

The magnitude of the cross product

$\vec{v} \times \vec{w} = \sqrt{19^2 + (-4)^2 + 7^2}$

$= \sqrt{361 + 16 + 49}$

$= \sqrt{426}$

To get the area, let halve the magnitude

$\therefore$ Area $= \frac{1}{2}\sqrt{426}$ square units

Let's add up the vectors

Eastward Component $E = 300$ km east

Northward Component $N = 350 \times \frac{\sqrt{3}}{2}$ km north

Westward Component : $W = 350 \times \frac{1}{2}$ km west

Final northward component : $N' = 150$ km north

Resultant Vector $= E + N - W + N'$

(a) To find the direction of the resultant vector, we calculate the angle it makes with the eastward direction

Using trigonometry, we can calculate θ using the components of the resultant vector

$$\tan(\theta) = \frac{\text{magnitude of N component} - \text{magnitude of N}}{\text{Magnitude of E component}}$$

$$\tan\theta = \frac{350 \times \frac{\sqrt{3}}{2} - 350 \times \frac{1}{2}}{300}$$

1.12 An airplane starting from airport A flies 300km east, then 350km 30° west of north and then 150km north to arrive finally at airport B. Assuming that there was no wind on that day

(a) determine in what direction should the pilot fly to travel directly from A to B?

(b) How far will the pilot fly to travel directly from A to B?

Solution

The airplane flies 300km east from A
Then it flies 350km 30° west of north
Finally, it flies 150Km to arrive at B

Let's calculate the magnitude of the northwar
and westward components

Magnitude of northward component N =

$$350 \times \cos 30 = 350 \times \frac{\sqrt{3}}{2} \text{ km}$$

Magnitude of westward component W

$$= 350 \times \sin 30$$

$$= 350 \times \tfrac{1}{2} \text{ Km}$$

(c) $(B \cdot C) A$:

Find the dot product of B and C

$B \cdot C = (B_x \times C_x) + (B_y \times C_y)$

Multiply the result by A

$(B \cdot C) A = (B \cdot C) \times A$

$(B \cdot C) A = (B_x \times C_y + B_y \times C_y) A_x + (B_x + C_x$

$+ B_y \times C_y) A_y$

$= (6.55 \times 5.80 + 4.59 \times 1.55) 9.06$

$(6.55 \times 5.80 + 4.59 \times 1.55) 4.23$

$= (39.99 + 7.12\cancel{45}) 9.06 +$

$(37.99 + 7.12) 4.23$

$= (45.10) 9.06 + (45.10) 4.23$

$= 408.81 + 190.77$

$= 599.88$