Topology

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This note is a summary of Oxford math course A5 Topology and part of B3.5 Topology and Groups.

1 Basics of Topological Spaces

1.1 Topological spaces

axioms of *topological space*.   consists of a non-empty set X with a family of its subsets $\mathcal{T}$ satisfying:

1. (T1) $X, \emptyset \in \mathcal{T}$.

2. (T2) For any $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$ (finite intersection).

3. (T3) For any $U_i \in \mathcal{T}, i \in I$, we have $\bigcup_{i \in I} U_i \in \mathcal{T}$ (any union).

then $(X, \mathcal{T})$ is called a topological space. Sets in $\mathcal{T}$ are all assumed to be open.

open sets in metric space X form a topology $\mathcal{T}_d$.   (only need to satisfy T1 T2 T3.)

examples:

- ***discrete* spaces.** the set containing all subsets of X.

- ***indiscrete* topology.** the trivial topology on X: $\mathcal{T} = \{X, \emptyset\}$.

- ***co-finite* topology.** consists of $\emptyset$ with every $U \subseteq X$ s.t. $X \setminus U$ is finite.

- ***co-countable* topology.** consists of $\emptyset, X$ with every $U \subseteq X$ s.t. $X \setminus U$ is countable.

***metrizable* topological space.**   arises from metric space, i.e. there is at least one metric d on X s.t. $\mathcal{T} = \mathcal{T}_d$ (exists a metric space that induces topology $\mathcal{T}$).

topologically *equivalent* metrics.   they give rise to the same topology (e.g d_1, d_2, d_∞: standard topology on $\mathbb{R}^n$).

***coarser*.** $\mathcal{T}_1$ is coarser if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ given they are on the same X. (e.g. the indiscrete topology is coarser than the discrete topology on X.)

1.2 Openness, Closedness, Closure

***closedness*.** $V \subseteq X$ is closed in X if $X \setminus V$ is open in X (i.e. $X \setminus V \in \mathcal{T}$).

axioms of closeness.   Let X be topological space then (from De Morgan laws):

1. (C1) $X, \emptyset$ closed in X.

2. (C2) V_1, V_2 closed in X, then $V_1 \cup V_2$ closed in X (finite union).

3. (C3) V_i closed in X for all $i \in I$, then $\bigcap_{i \in I} V_i$ closed in X (any intersection).

sequence convergence.

- **convergent sequences.** $(x_n) \to x \in X$. the definition with $\forall\ U$ open containing x, $\exists\ N, x_n \in U\ \forall\ n \geq N$. (in metric space U is open ball $B(x, \epsilon)$).

- **uniqueness of the limit is not guaranteed in topology.** In trivial space, any sequence converges to any point in X. In co-finite topology, any pairwise distinct sequence converges to any point as any $X \setminus U$ is finite and x_n must fall into U for large n.

- **any sequence in closed set F converges to point in F.** The converse may not be true (consider co-countable topology: $V = X \setminus \{x_n : x_n \neq l\}$ is open).

continuity.

- **definition.** $f : X \to Y$, $U \in \mathcal{T}_Y$ implies $f^{-1}(U) \in \mathcal{T}_X$. (preimage preserves openness). (equivalence: $f^{-1}(V)$ is closed in X whenever V is closed in Y).

- **transitivity (composition of cts map is cts).**

- *homeomorphism.* $f : X \to Y$. f and f^{-1} are both cts. X, Y are homeomorphic.

- **continuity preserves sequence convergence.** ($f(x_n) \to f(x)$ as f is cts.)

to show a set is open: (equivalence)

- U is open.

- $\forall x \in U$, $\exists$ an open set U_x containing x s.t. $U_x \subseteq U$.

closure.

- **definition.** $\overline{A} = \bigcap\limits_{F\ \text{closed}, A \subseteq F} F$. (compare this with the out measure in real analysis).

- **equivalent definition.** $\overline{A} = \{x \in X : \forall\ \text{open U} \subseteq \text{X with x} \in \text{U s.t. U} \cap \text{A} \neq \emptyset\}$ (compare this with the definition of boundary in metric space).

- **closure** contains **isolated points** and **accumulation points**.

accumulation point. $\forall$ open set U with $x \in U$, $(U \setminus \{x\}) \cap A \neq \emptyset$.

$f : X \to Y$ **is cts iff $f(\overline{A}) \subseteq \overline{f(A)}$ $\forall A \subseteq X$.**

interior. just compare with closure.

relations between closure and interior.

- $\overline{X \setminus A} = X \setminus \mathring{A}$

- $X \mathring{\setminus} A = X \setminus \overline{A}$

- *boundary.* $\partial A = \overline{A} \setminus \mathring{A} = \overline{A} \cap \overline{X \setminus A}$.

1.3 Separation Axioms

separation axioms:

*First: for any two distinct cts $a \neq b$, there exists an open set U containing a but not b.

- X **satisfies first separation axioms iff singletons are closed.**

*Second: *Hausdorff* space. X is Hausdorff if given two distinct pts in $x, y \in X$, there exist disjoint open sets U, V with $x \in U, y \in V$.

- metric space is Hausdorff.

- in-discrete space, infinite space with co-finite topology are not Hausdorff (but satisfy the first axiom).

- in Hausdorff space, a singleton is closed.

consequences.

- **preimage of injective cts map preserves both separation axioms.** $f : X \to Y$ injective and cts. If Y satisfies the separation axiom, so is X. The proof uses the property that the preimage of cts map preserves openness.

- **homeomorphism conducts separation axioms.** If X, Y are homeomorphic, then X satisfies s.a. 1 or 2 iff Y satisfies s.a. 1 or 2.

- **Hausdorff guarantees the uniqueness of sequence convergence.** However, s.a. 1 is not strong enough (e.g. co-finite topology).

1.4   Subspaces, Basis, Product spaces, Disjoint unions

subspace. top. space $(X, \mathcal{T})$, and $A \subseteq X$ nonempty. Induced subspace $\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}$. Note we don't have do define A is open or not. Actually, the criterion for open set in A has been loosened. (T1)(T2)(T3) are easily checked. Some properties:

- the inclusion map $i : A \to X$ is cts

- W is closed in $\mathcal{T}_A$ iff there is F closed in $\mathcal{T}$ s.t. $W = A \cap F$.

- suppose $B \subseteq A$. closure $\overline{B}^A = A \cap \overline{B}^X$, interior $\mathring{B}^X \subseteq \mathring{B}^A$ (may be strict, but if A is open, then =).

***basis* for a topology.** idea: use the minimal amount of data to recover the whole space. useful in topology induced by metric, and product spaces. (e.g., open balls is a basis for $\mathcal{T}_d$).

- **definition for basis.** $\mathcal{B}$ is a basis for $(X, \mathcal{T})$ if:

  1. $\mathcal{B} \subseteq \mathcal{T}$ (every set in $\mathcal{B}$ is open in X)
  2. every set in $\mathcal{T}$ can be expressed as a union of sets in $\mathcal{B}$.

- **open set criterion** via basis. A set U is open in $(X, \mathcal{T})$ iff for every $x \in U$, there exists $B \in \mathcal{B}$ s.t. $x \in B \subseteq U$.

- **cts map criterion** via basis. X, Y, Let $\mathcal{B}$ be a basis for Y. A map $f : X \to Y$ is cts iff for every $B \in \mathcal{B}$, its inverse image $f^{-1}(B)$ is open in X.

- **construction of topology via basis.** Let X be a set and $\mathcal{B}$ be a family of subset of X s.t.

  1. (B1) X is a union of sets in $\mathcal{B}$
  2. (B2) for any $B, B' \in \mathcal{B}$, $B \cap B'$ can be expressed as a union of sets in $\mathcal{B}$.

  then we can recover (T1)(T2)(T3).

product spaces.

- defn 1: all unions from $\mathcal{B}_{X\times Y}$ forms a topology $\mathcal{T}_{X\times Y}$, i.e., $(X \times Y, \mathcal{T}_{X\times Y})$.

- defn 2: $\mathcal{T}_{X\times Y}$ is the coarsest topology making $p_X(x,y) = x, p_Y(x,y) = y$ cts.

- defn 3: the product space $X \times Y$ is the only topology for which $f : Z \to X$ and $g : Z \to Y$ both cts for all possible Z.

- via basis. $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$: $\mathcal{B}_{X\times Y} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ satisfies (B1)(B2).

- **open set criterion.** $W \subseteq X \times Y$ is open iff for any $(x,y) \in W$, there exists $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$ s.t. $(x,y) \in U \times V \subseteq W$.

- **closed set.** product of closed set is closed in the product space.

- in $\mathbb{R}^n$, the product topology coincides with the standard topology ($\mathrm{d}_{1/2/\infty}$). as they have the same open sets.

- **separation axioms.** separation axioms preserve in $X \times Y$.

disjoint unions. $X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$. It contains $(x, 0), x \in X$ and $(y, 1), y \in Y$. A set U is open in $X \sqcup Y$ iff $U \cap X \times \{0\}$ and $U \cap Y \times \{1\}$ open in X and Y.

1.5 Connectedness, Path-connectedness

connectedness. X is disconnected if there are open non-empty subsets U, V s.t. $\overline{U} \cap V = U \cap \overline{V} = \emptyset$, $U \cup V = X$. X is connected if it is not disconnected.

- **equivalent defn for connectedness.**
  - X is connected.
  - the only both open and closed subsets of X are X and $\emptyset$.
  - any cts map $\phi : X \to \{0, 1\}$ with the discrete topology is const.

- **show sth is connected?** Let U, V open in X s.t. $A \subseteq U \cup V$ and $A \cap U \cap V = \emptyset$, wts $U \cap A$ or $V \cap A$ is empty.

- **subspace.** a subset A is connected if the induced subspace topological space $X \cap A$ is connected. $\emptyset$ is assumed to be connected.

- **cts image of a connected set is connected.**

- **sunflower lemma.** (there are two versions.)

- **product space.** $X \times Y$ is connected iff X and Y are connected. (prove by sunflower lemma).

- **cnt and closure.** connectedness of A and $A \subseteq B \subseteq \overline{A}$ implies connectedness of B.

path.

- **defn of path** x and y: $p : [0, 1] \to X$ cts with $p(0) = x$ and $p(1) = y$.

- **path-connected.** any two points in X are connected by a path in X.

- **path-connected implies connected.** the converse is not true: there is an e.g. in metric space.

- **the cts image of path-cnt set is path-cnt.**

2 Compact Spaces

2.1 Compactness.

compact defn.

- cover, open cover, subcover, finite subcover.

- *compact.* X is compact if any open cover of X has a finite subcover.

  - **equivalent def.** $i \in I$, $V_i \subseteq X$ closed. If $\bigcap_{j \in J} V_j \neq \emptyset$ for any finite $J \subseteq I$, then $\bigcap_{i \in I} V_i \neq \emptyset$.

- **subset.** the subset is compact if the subset is compact when endowing the subspace topology.

- **product space criterion.** $X \times Y$ compact iff both X and Y compact.

$\cup$ **and** $\cap$**.**

- **finite** $\cup$ of compact subsets is compact.

- In **Hausdorff space**, any $\cap$ of compact set is compact. (in non-Hausdorff space, consider singleton.)

compact v.s. closed.

- closed implies compact: any **closed** subset of **compact space** is compact. (proof consider defn of compact via closed set, and defn of compact subspace). the converse is not true: consider singleton in X is compact but may not closed. Need to add Hausdorff:

- compact and Hausdorff implies closed: compact subset of a **Hausdorff** space is closed. (proof idea: we can separate point in and out of K using disjoint open sets, which shows $X \setminus K$ open.)

compact in metric spaces.

- **Heine-Borel theorem in** $\mathbb{R}$**.** Any closed bounded interval $[a, b]$ in $\mathbb{R}$ is compact. What's the topology on $\mathbb{R}$? Euclidean measure?

- compact implies closed and bounded. for reverse direction, see general Heine-Borel.

- **general Heine-Borel:** closed and bounded implies compact in $\mathbb{R}^n$: any closed bounded subset of $(\mathbb{R}^n, \| \cdot \|_i)$, $i \in \{1, 2, \infty\}$, is compact. (note this may fail in some other metric spaces e.g $(0, 1]$).

compact v.s. continuity.

- **cts mapping preserves compactness.** In metric space, closedness and boundedness are preserved by cts mapping. (however, inverse mapping may not cts: need Hausdorff, see below (condition for f^{-1} to be cts.)

- homeomorphism. X **compact**, Y **Hausdorff**, $f : X \to Y$ **cts bijective**. Then f is homeomorphic (f^{-1} also cts). (If f is only injective, then substitute Y by $f(X)$).

- **extreme value thm.** In compact space X, cts $f : X \to \mathbb{R}$ is bounded, and attains max and min in X.

2.2 Sequential compact

defn of *seq compact.* X seq compact if every seq in X has a subseq that cvg to pt in X.

Bolzano-Weierstrass. in **compact** topological space X, every infinite subset has accumulation pts. (but don't know subsets are seq compact)

in metric spaces:

- **compact $\iff$ seq compact.** (pf for $\impliedby$ uses ε-net)

- **compact implies complete.** because of seq compact.

Lebesgue's number lemma.

- lemma: For compact metric space X, for any open cover $\mathcal{U}$, there exists $\varepsilon > 0$ s.t. every subset with diameter $d \le \varepsilon$ is contained in some set in $\mathcal{U}$. (ε is called a Lebesgue number for open cover $\mathcal{U}$.)

- ε-*net*: the subset $N \subseteq X$ with open balls cover X, i.e. $\bigcup\limits_{x \in N} B(x, \varepsilon) \supseteq X$.

- metric space X seq compact, then there exists finite ε-net for X (so we can approximate metric space using finite sets).

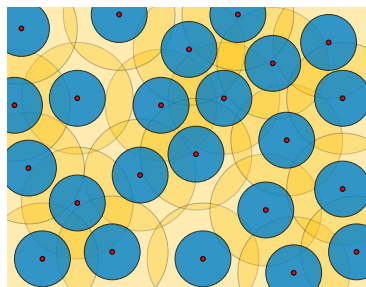Figure 1: ε-net: there exists $\varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq U$ for some $U \in \mathcal{U}$

3   Quotient Spaces

3.1   Basics.

construction.

- **motivation.** we want to glue points in the space together.

- **equivalence relation.** $[x] = \{y \in X : y \mathcal{R} x\}$. (all points in $[x]$ are glued together)

- **quotient space of X with respect to $\mathcal{R}$.** $X/\mathcal{R}$, the set containing equivalence classes $[x]$. collapsing map: $p : X \to X/\mathcal{R}$ assigns $x \in X$ to each equivalence class.

quotient **topology.**   i.e. we define open sets on quotient space.

- **open sets.** $\tilde{\mathcal{T}}$ contains subsets $\tilde{U} \in X/\mathcal{R}$ s.t. $p^{-1}(\tilde{U}) \in \mathcal{T}$. Then $\tilde{\mathcal{T}}$ is a topology for $X/\mathcal{R}$, the quotient topology. (note by this definition, p is cts as all open set in quotient space are mapped from open set.)

- **closed set.** $\tilde{V}$ closed in $X/\mathcal{R}$ iff $p^{-1}(\tilde{V})$ closed in X. proof uses cts of p.

- **continuity.** $g : X/\mathcal{R} \to Z$ cts iff $f : g \circ p : X \to Z$ cts.

homeomorphism. $f : X \to Y$ **surjective** cts, X **compact** and Y **Hausdorff**, then $X/\mathcal{R}$ and Y are homeomorphic, where $\mathcal{R}$ is defined later by $\{f^{-1}(y) : y \in Y\}$.

- examples: $[0, 1] \to S^1$; $[0, 1] \times [0, 1] \to \mathbb{T}^2$; $\mathbb{D}^2 \to S^2$.

transitive closure of $\mathcal{R}$. $\overline{\mathcal{R}}$ given by $x \overline{\mathcal{R}} x'$ iff there is a finite sequence of points s.t. $x = x_1 \mathcal{R} x_2 \mathcal{R} ... \mathcal{R} x_{n-1} \mathcal{R} x_n = x'$. $\overline{\mathcal{R}}$ defines an equivalence class. (think about the torus)
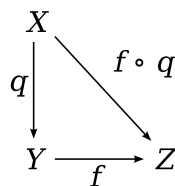
Figure 2: universal property of quotient topology: quotient map and quotient space: where $Y = X/\mathcal{R}$

3.2 Separation axioms.

***saturated* set.** $A \subseteq X$ is saturated w.r.t. $\mathcal{R}$ if it is a union of equivalence classes.

separation axioms.

- X Hausdorff but $X/\mathcal{R}$ may be non-Hausdorff. e.g. $\mathbb{R}/\mathbb{Q}$

- $X/\mathcal{R}$ staisfies **first s.a.** iff every $[x] \subseteq X$ is closed. (note the first s.a. iff every singleton is closed.)

- $X/\mathcal{R}$ is **Hausdorff** iff any two distinct equivalence classes are contained in two **disjoint open saturated** sets in X. (proof is not hard)

real n-D projective space $\mathbb{R}P^n$.   the quotient space of $(\mathbb{R}^{n+1} \setminus \{0\})/\mathcal{R}$ where $x\mathcal{R}y$ iff $\exists \lambda \neq 0$ s.t. $x = \lambda y$. (i.e. it contains lines through 0, and is Hausdorff and compact). e.g. project 3d space onto 2d sphere surface.

3.3 Quotient maps.

motivation.   how to prove a quotient space is homeomorphic to another space. here the quotient map may be abstract. maybe we don't know the explicit equivalence relation $\mathcal{R}$.

second def for quotient topology. $p : X \to X/\mathcal{R}$ be $x \to [x]$, then $X/\mathcal{R}$ is the finest topology on it making p cts.

open mapping. $f : X \to Y$ s.t. $U \in X$ open then $f(U) \in Y$ open. (cts map may not be open map e.g. quadratic function in $\mathbb{R}$, collapsing map may not be open map e.g. Möbius band)

***quotient map* $p : X \to Y$**   if:

1. p surjective

2. every $U \subseteq Y$, U open iff $p^{-1}(U)$ open.

collapsing map is a quotient map.

relation between quotient map and quotient space.   define $q : X \to Y$ first, then define $\mathcal{R}$ on X corresp. to partition $\{q^{-1}(y) : y \in Y\}$. If X compact and Y Hausdorff, then $X/\mathcal{R}$ and Y are homeomorphic. (show by defining bijection $g([x]) = q(x)$)

- $\mathbb{R}/\mathbb{Z} \cong S^1$ (consider $f : \mathbb{R} \to S^1$)

- $\mathbb{R}^n/\mathbb{Z}^n \cong S^1 \times \ldots \times S^1$ (n-d torus $\mathbb{T}^n$).

4 Simplicial Complexes

4.1 Definitions.

motivation.   build complicated spaces from simplicies such as triangles (of different dimensions).

standard n-*simplex*. $\Delta^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ and } \sum_i x_i = 1\}$

- *face* of Δ^n. for $A \subseteq \{1, ..., n+1\}$, the face is $\{(x_1, ..., x_{n+1}) \in \Sigma^n : x + i = 0 \ \forall \ i \notin A\}$

- *inside* of Δ^n is the set of $(x_1, ..., x_{n+1})$ with $x_i > 0 \ \forall \ i$. particularly, inside of Σ^0 is Σ^0.

- *face inclusion*: $m < n$, $\Delta^m \to \Delta^n$ injective (send to higher dimension)

abstract simplicial complex. (V, Σ). V: set of vertices. Σ: simplices, non-empty finite subsets of V s.t. 1) singletons in Σ. 2) $\sigma \in \Sigma$ implies any non-empty subset of σ in Σ. (V, Σ) finite if V finite.

topological *realisation* $|K|$ of $K = (V, \Sigma)$. 1) obtain disjoint union from V. 2) glue them via equivalence relation. Call the final product simplicial complex.

triangulation of space X. complex K with a choice of homeomorphism $|K| \to X$. (e.g. tetrahedron T is a triangulation of 2-sphere as $T \to S^2$ by radial projection)

simplicial circle. Σ contains 0-simplices and 1-simplicies, where 1-simplicies are $\{v_i, v_{i+1}\}_{i=1}^{n-1}$ and $\{v_n, v_1\}$.

4.2 substructures of a simplicial complex.

realisation $|K|$ is the union of the insides of its simplices. i.e. there is a bijection between $|K|$ and the disjoint union of the insides of its simplices.

subcomplex.

- **defn.** $K' = (V', \Sigma')$ with $V' \subseteq V$ and $\Sigma' \subseteq \Sigma$.

- **closedness.** $|K'|$ is closed in $|K|$.

- *span.* the subcomplex spanned by V' consists of all simplicies in Σ that have vertices in V'.

link. $v \in V$ a vertex. $\mathrm{lk}(v)$ is the subcomplex containing vertices $\{w \in V \setminus \{v\} : \{v, w\} \in \Sigma\} \subseteq V$ and simplices σ s.t. $v \notin \sigma$ and $\sigma \cup \{v\} \in \Sigma$.

star. $\mathrm{st}(v) := \bigcup \{\mathrm{inside}(\sigma) : \sigma \in \Sigma \text{ and } v \in \sigma\}$. (picture see lec note.)

- the star of v is open containing v. show that $|K| \setminus \mathrm{st}(v)$ is subcomplex of K, thus closed.

edge path. sequence $(v_0, ..., v_n)$ s.t. for all i, $\{v_i, v_{i+1}\} \in \Sigma$, the simplices of K.

connectedness, compactness and s.a. of K.

- **connectedness of K.** equivalence:

  - any two vertices in K can be joined by an edge path.
  - $|K|$ is connected.
  - $|K|$ is path-connected.

- **compactness.** realisation of finite simplicial complex is compact.

- **Hausdorff.**

  - **standardlize.** *standarlized simplicial complex.* for any simplicial complex $K = (V, \Sigma)$, there is cts injection $f : |K| \to \mathbb{R}^n$.
  - **metrizable.** realisation of finite simplicial complex is metrizable. and thus is **Hausdorff**.

4.3 Simplicial map.

This part had been discussed before, but we are adding something here.

referring to a point in $|K|$. $x \in |K|$ lies in the inside of a unieuq simplex $\sigma = (v_0, .., v_n)$, expressed $x = \sum_{i=0}^{n} \lambda_i v_i$ for unique λ's.

simplicial map. from (V_1, Σ_1) to (V_2, Σ_2), $f : V_1 \to V_2$ s.t. $\forall \sigma_1 \in \Sigma_1$, $f(\sigma_1) = \sigma_2$ for some $\sigma_2 \in \Sigma_2$ (No need to be injective). it is a *simplicial isomorphism* if it has inverse.

- **construct cts map.** $|f| : |K_1| \to |K_2|$ is defined by extending $|f|$ from $V(|K_1|)$ (vertices of $|K_1|$) to each simplex using unique affine extension.

subdivision **of** K. e.g. refinement on triangularizations.

5   Surfaces.

5.1   defns.

n-**dimensional** ***manifold.*** a Hausdorff topological space M on which every point lies in an open set that is homeomorphic to an open set in $\mathbb{R}^n$. e.g. S^n, $\mathbb{T}$, $\mathbb{RP}^n$.

surface. a 2-manifold.

side identifications.

polygon with complete set of side identifications (*Abbr.* PCSI) is a closed surface. (closedness is clear, we mainly verify the condition for a surface i.e. open set.)

two lists of surfaces: M_g **and** N_h.

- M_0: $xx^{-1}yy^{-1}$. 2-sphere

- M_g: $x_1 y_1 x_1^{-1} y_1^{-1} ... x_g y_g x^{-1} y_g^{-1}$. g-holed torus.

- N_1: $xxyy^-1$. the projective plane.

- N_h: $x_1 x_1 x_2 x_2 ... x_h x_h$. h-crosscaps.

adding handles and crosscaps.

- adding a *handle*. $Axyx^{-1}y^{-1}$. the surface M_g is obtained from a 2-sphere by adding g handles.

- adding a *crosscap*. Axx. the surface N_h is obtained from 2-sphere by adding h crosscaps.

$\mathbb{RP}^2$ **construction from Möbius band.**

closed combinatorial surface (*Abbr.* **CCS**). connected finite simplicial complex K s.t. for every vertex v of K, the link of c is a simplicial circle. then

- every simplex of K has dim 0,1,2.

- every 1-simplex of K is a face of exactly two 2-simplices.

- every point of $|K|$ lies in a 2-simplex.

- $|K|$ is a surface.

5.2   Classification of surfaces.

any PCSI is homeomorphic to a CCS. this indicates we can use **PCSI** to construct **CCS** (not all), which is homeomorphic to manifolds M_g and N_h (by the classification theorems).

the classification theorems. every **CCS** is homeomorphic to one of the manifolds M_g ($g \geq 0$) or N_h ($h \geq 1$). (this is because every **CCS** is from $2n$-gons, and we can simplify every $2n$-gon into M_g or N_h). lemmas used:

- realization of **CCS** can be obtained from $2n$-gons ($n \geq 2$).

- cyclically permute letters in A or replace A with A^{-1} do not change the resulting surface.

- $xAxB$, $xxA^{-1}B$, $A^{-1}xxB$ give the same surface. (handle two x, no matter where are the standard pair x's, we can move them to the front, then the remaining word will have the tail only contain reversed pairs.)

- $xABx^{-1}C$ and $xBAx^{-1}C$ give the same surface. (between x and x^{-1} we can rearrange cylinderically)

- $Exxyzy^{-1}z^{-1}F$ and $ExxyyzzF$ give the same surface. (transform reversed pairs into standard pairs.)

- $Axx^{-1}B$ and AB give the same surface, provided AB has at least 4 sides. (reduce xx^{-1}.)

- suppose polygon P has at least 6 sides with complete side identifications, and there is more than one equivalence class of vertices, then there is another polygon P' with fewer sides result in a homeomorphic surface. (reduce equivalence classes.)

distinguishing the surface. none of the surfaces M_g ($g \geq 0$) or N_h ($h \geq 1$) are homeomorphic. proof appears later.

Euler characteristic **of** K. K a finite simplicial complex, let n_i be the number of i-simplices for each i. then the Euler characteristic of K is:

$$\chi(K) = \sum_i (-1)^i n_i.$$

invariance of Euler characteristic. two finite simplicial complexes K and K' have homeomorphic topological realisations, then $\chi(K) = \chi(K')$. particularly,

$$\chi(M_g) = 2 - 2g, \quad \chi(N_h) = 2 - h.$$

6 Constructing Spaces

6.1 Cayley graphs of groups.

countable graph. $\Gamma = (V, E)$ with function δ sending each edge e to subset of V with 1 or 2 pts (endpoints of an edge). then can construct 1d topological spaces.

orientation on graph. choice of function $\iota : E \to V$ and $\tau : E \to V$ s.t. for each $e \in E$, $\delta(e) = \{\iota(e), \tau(e)\}$, the initial and terminal vertices of edge e. say the edge running from initial to terminal vertices.

associated Cayley graph of group G. oriented graph with vertex set G, and edge set $G \times S$, i.e. each edge corresp a pair (g, s). ι and τ are $G \times S \to G$ and respectively $(g, s) \mapsto g$ and $(g, s) \mapsto gs$. so edge associated (g, s) run from g to gs.

- e.g. $\mathbb{Z}$ by $\{2, 3\}$, $\mathbb{Z}^2$ by $(1, 0)$ and $(0, 1)$.

- geometric group theory.

prop. any two points in a Cayley graph can be joined by a path.

6.2 Simplicial complexes.

This part had been discussed before, but we are adding something here.

referring to a point in $|K|$. $x \in |K|$ lies in the inside of a unieuq simplex $\sigma = (v_0, .., v_n)$, expressed $x = \sum_{i=0}^{n} \lambda_i v_i$ for unique λ's.

simplicial map. from (V_1, Σ_1) to (V_2, Σ_2), $f : V_1 \to V_2$ s.t. $\forall \sigma_1 \in \Sigma_1$, $f(\sigma_1) = \sigma_2$ for some $\sigma_2 \in \Sigma_2$ (No need to be injective). it is a *simplicial isomorphism* if it has inverse.

- **construct cts map.** $|f| : |K_1| \to |K_2|$ is defined by extending $|f|$ from $V(|K_1|)$ (vertices of $|K_1|$) to each simplex using unique affine extension.

subdivision of K. e.g. refinement on triangularizations.

6.3 Cell complexes.

motivation. simplicial complexes is useful, but sometimes awkward because some simple spaces require many simplices that are not efficient (such as torus).

attaching an n-cell. $f : S^{n-1} \to X$, then the space obtained by attaching an *n-cell* to X along f is defined as $X \sqcup D^n$ s.t. $\forall x \in X$, $f^{-1}(x)$ and x are identified to a point. denote the space by $X \cup_f D^n$.

- **intuition.** map the boundary of a disk (S^{n-1}) to some strange shape $(f(S^{n-1}) \subseteq X)$, while deform the remaining part $(D^n$, disjoint from $X)$ of the disk according to the boundary.

- for picture, see lecture note.

finite cell complex. is a space X decomposed as $K^0 \subset K^1 \subset ... \subset K^n = X$. where K^0 is finite set of pts, and K^i is obtained from K^{i-1} by attaching a finite collection of i-cells.

- a finite graph is by attaching 1-cells (lines) to 0-cells (pts).

- any finite simplicial complex is a finite cell complex. just let each n-simplex be an n-cell.

- torus consists one 0-cell, two 1-cells (graph as skeleton), and one 2-cell (surface to envelop). the direction on the graph corresponds to direction on the polygon. (picture see lecture note)

7 Homotopy

7.1 Basics.

definition. the *homotopy* between two maps $f : X \to Y$ and $g : X \to Y$ is the cts map $H : X \times I \to Y$ s.t. $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. say f and g are homotopic, write $f \simeq g$.

- *straight-line* homotopy. $t \in [0,1]$, $H : (x,t) \mapsto (1-t)f(x) + tg(x)$.

- equivalence relation: homotopy is an equivalence relation on set of cts maps $X \to Y$.

properties.

- $\{x\} \to Y$ where $\{x\}$ is a single point, then homotopy defines paths between $f(x)$'s and $g(x)$'s.

- composition of maps. $g \simeq h$ implies $gf \simeq hf$ and $kg \simeq kh$.

homotopy equivalent spaces. $X \simeq Y$ if exists $f : X \to Y$, $g : Y \to X$ s.t. $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$.

- homotopy equivalence space is also an equivalence relation.

- homomorphism is a special case of homotopy. e.g. letters **X** and **Y** homotopic but not homomorphic.

***contractible* space.** if X is homotopy equivalence to singleton space $X = \{x\}$.

- X contractible iff $\mathrm{id}_X \simeq c_x$ for where c_x is constant map maps to some $x \in X$.

homotopy *retract*. $A \subseteq X$, $i : A \to X$. say map $r : X \to A$ s.t. $ri = \mathrm{id}_A$ and $ir \simeq \mathrm{id}_X$ a *homotopy retract*.

- e.g. $\mathbb{R}^n \setminus \{0\} \simeq S^1$, and Möbius strip $M \simeq S^1 \simeq S^1 \times I$.

homotopic *relative*. $A \subseteq X$, $f, g : X \to Y$ are homotopic relative to A if $f \mid_A = g \mid_A$ and there is homotopy $H : f \simeq g$ s.t. $H(x,t) = f(x) = g(x) \ \forall \ x \in A$ and $t \in I$.

7.2 Simplicial approximation theorem.

motivation. cts map $f : |K| \to |L|$ where K and L are simplicial complexes. we want a *simplicial map* g that is homotopic to f. i.e. use a simplicial map g to approximate cts map f

Theorem. K and L are two simplicial complexes, K finite, $f : |K| \to |L|$ cts map. then $\exists$ subdivision K' of K and simplicial map $g : K' \to L$ s.t. $|g| \simeq f$.

lemmas and definitions used.

- $\mathrm{st}_K(x)$ is open in $|K|$ for any $x \in |K|$.

- construct simplicial map. prop. $g : |K| \to |L|$ be a cts map. suppose each vertex v of K, there is a vertex $g(v)$ of L s.t. $f(\mathrm{st}_K(v)) \subset \mathrm{st}_L(g(v))$, then g is a simplicial map $V(K) \to V(L)$ and $|g| \simeq f$.

- for subcomplex. A, B be subcomplexes of K, L respectively, s.t. $f(|A|) \subset |B|$. then g maps A to B, and still $|g| \simeq f$ on $|A|$ to $|B|$.

- construct subdivision.

  - *standard metric* on $|K|$:

  $$d\left(\sum_i \lambda_i v_i, \sum_i \lambda'_i v_i\right) = \sum_i |\lambda_i - \lambda'_i|.$$

  - *coarseness* of subdivision K':

  $$\sup\{d(x,y) : x \text{ and } y \text{ belong to the star of the same vertex of } K'\}.$$

- Lebesgue covering thm. see before.

- result (duplicate). K finite, $f : |K| \to |L|$ a cts map. then $\exists \ \delta > 0$ s.t. there is a simplicial map g ($\forall \ K'$ with coarseness $< \delta$) s.t. $|g| \simeq f$.

8   Fundamental Groups

motivation.   assigning a group to each topological space.

8.1   definitions.

composite path. $u.v$ runs along u, then along v.

loop.   based on $b \in X$, the loop is a path $l : I \to X$ s.t. $l(0) = l(1) = b$. we then vary loops via homotopy.

fundamental group of (X, b).   denoted $\pi_1(X, b)$, is the homotopy class relative to ∂I of loops based at b. l and l' are loops with $[l]$ and $[l']$ homotopy classes relative to ∂I. their composition $[l].[l']$ in the group is defined to be $[l.l']$.

- the group is well-defined, associative, has inverse (these proofs are not trivial).

- $\mathbb{R}^2$ associates trivial group, but S^1 is not trivial as $\pi_1(S^1, 1) \cong \mathbb{Z}$.

- fundamental group is the simplest homotopy group.

changing base point.   if b and b' lie in the same path-component of X, then $\pi_1(X, b) \cong \pi_1(X, b')$, with group homomorphism $w_\# : \pi_1(X, b) \to \pi_1(X, b'); [l] \mapsto [w^{-1}.l.w]$ where $w^{-1}.l.w$ is a loop based at b'.

- this obtains a well-defined conjugacy class in $\pi_1(X, b)$ from any homotopy class of loops in X (change base means act on $\pi_1(X, b)$ by conjugacy).

relation between topological homotopy and group hom / iso.

- **topological cts map induces homomorphism.** $f : (X, x) \to (Y, y)$ induces a homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, y)$.

- **topological homotopy induces group isomorphism.** if X and Y are path-cnt s.t. $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$. in particular, contractable space induces a trivial group.

simply-connected.   if the space is path-connected and has trivial fundamental group. but simply-cnt may not imply contractible (e.g. S^2 or $\mathbb{R}^2 \setminus \{0\}$).

8.2   Simplicial version.

edge path.   a finite sequence $(a_0, ..., a_n)$ of vertices of K, where each $\{a_{i-1}, a_i\}$ spans a simplex of K. *length* is n. *edge loop* has $a_n = a_0$. also we have composition path.

elementary contraction.   reduce redundancy $(0,1,2$ elements$)$ finite times from α yields β, then forms equivalence class $\alpha \sim \beta$ (result with same initial and end points).

edge loop group.   the equivalence classes of edge loops in K based at b forms a group denoted $E(K, b)$.

w.r.t. fundamental group. $E(K, b) \cong \pi_1(|K|, b)$. significance:

- can compute $\pi_1(|K|, b)$ from $E(K, b)$.

- $E(K, b)$ does not depend on choice of triangulation.

n-skeleton. n-skeleton of K, $\mathrm{skel}^n(K)$ is the subcomplex consisting simplices with dim $\leq n$.

- then $\pi_1(|K|, b) \cong \pi_1(|\mathrm{skel}^2(K)|, b)$.

fundamental group of the circle. $\pi_1(S^1) \cong \mathbb{Z}$. prove this by triangulation and elementary contraction.

fundamental theorem of algebra. any non-constant polynomial with complex coefficients has at least one root in $\mathbb{C}$.

8.3 Fundamental group of graphs.

fundamental group of a connected graph is a free group. claim there is isomorphism $\phi : F(E(\Gamma) - E(T)) \to E(\Gamma', b)$, where Γ is connected graph, and T is a tree, E denotes edges.