1. The gunner on a small assault boat fires three missiles at an attacking plane. Each has a 20% chance of being on target. If two or more of the shells find their mark, the plane will crash. At the same time, the pilot of the plane fires 4 air-to-surface rockets, each of which has a 0.1 chance of destroying the boat. If one or more of the rockets hit the boat, the boat will sink. Would you rather be on the plane or the boat? (That is, please calculate and compare the probability that the plane will crash and the probability that the boat will be destroyed.)

Solution: Let X denotes the # of shells find their mark. Then $X \sim B\,(3, 0.2)$.
Let Y denotes the # of rockets that will hit the boat. Then $Y \sim B\,(4, 0.1)$.

$$P(\text{Plane crashes}) = P(X \geq 2) = P(X = 2 \text{ or X} = 3) = P(X = 2) + \text{P}\,(X = 3)$$
$$= \binom{3}{2}(0.2)^2(1 - 0.2) + \binom{3}{3}(0.2)^3 = 0.104$$
$$P(\text{Boat destroyed}) = P(Y \geq 1)$$
$$= 1 - P(Y = 0)$$
$$= 1 - \binom{4}{0}(1 - 0.1)^4$$
$$= 0.3439$$

Therefore, on the plane will be better.

2. A miner is trapped in a mine with 3 doors.

• The 1st door leads to a tunnel that will take him to safety after 3 hours.
• The 2nd door leads to a tunnel that returns him to the mine after 5 hours.
• The 3rd door leads to a tunnel that returns him to the mine after 7 hours.
At all times, he is equally likely to choose any one of the doors.
Question: What is the expected time for the miner to reach safety?

Solution:

Let X represent the time (hours) to freedom and Y the initial door chosen, then we have (*using the law of total expectation: https://en.wikipedia.org/wiki/Law_of_total_expectation)

$$E(X) = E(X|Y = 1)P(Y = 1) + E(X|Y = 2)P(Y = 2) + E(X|Y = 3)P(Y = 3)$$

$$E(X) = 3 * \frac{1}{3} + [E(X) + 5] * \frac{1}{3} + [E(X) + 7] * \frac{1}{3}$$

$$E(X) = 15\ (hours)$$

Therefore, we **expect** the miner to reach freedom in 15 hours.

3. Let $X_1,\ldots,X_n \overset{i.i.d.}{\sim} N(\mu,\sigma^2)$, be a random sample from the normal population where μ is assumed known. Please derive:

(a) The maximum likelihood estimator for σ^2.
(b) The method of moment estimators for σ^2.
(c) Are the above estimator(s) for σ^2 unbiased?
(d) Are these two estimators identical?
(e) Which of these two estimators is better?

Solution:

(a) The likelihood function is

$$L = f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f\,(x_i; \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$
$$= (2\pi)^{-n/2}[\sigma^2]^{-n/2} exp\left[-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right]$$

The log likelihood function is

$$l = lnL = constant - \frac{n}{2} ln(\sigma^2) - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$$

Solving

$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^4} = 0$$

We obtain the MLE for σ^2:

$$\widehat{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{n}$$

(b) We now derive the MOME estimator for σ^2. Since the first population moment is $E[X] = \mu$ that does not involve σ^2, we only need to use the second population and sample moment. Setting them equal we have:

$$E(X^2) = \mu^2 + \sigma^2 = \frac{\sum_{i=1}^{n} X_i^2}{n}$$

One step further we have the MOME estimator of σ^2:

$$\widetilde{\sigma^2} = \frac{\sum_{i=1}^{n} X_i^2}{n} - \mu^2$$

(c) Since

$$E\left[\widehat{\sigma^2}\right] = \frac{E\{\sum_{i=1}^{n}(X_i-\mu)^2\}}{n} = \frac{\sum_{i=1}^{n} E(X_i-\mu)^2}{n} = \sigma^2$$

It is straight-forward to verify that the MLE

$$\widehat{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{n}$$

is an unbiased estimator for σ^2.

Since

$$E\left[\widetilde{\sigma^2}\right] = \frac{\sum_{i=1}^{n} E[X_i^2]}{n} - \mu^2 = \frac{n(\sigma^2+\mu^2)}{n} - \mu^2 = \sigma^2$$

We have shown that the MOME

$$\widetilde{\sigma^2} = \frac{\sum_{i=1}^{n} X_i^2}{n} - \mu^2$$

is an unbiased estimator for σ^2.

(d) The MLE for σ^2 can be rewritten as:

$$\widehat{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{n} = \frac{\sum_{i=1}^{n}(X_i^2-2\mu X_i+\mu^2)}{n} = \frac{\sum_{i=1}^{n} X_i^2}{n} - 2\mu\bar{X} + \mu^2$$

Comparing to the MOME, we found them NOT identical, although both are unbiased estimators for $\sigma^2\ unless\ \mu = 0$.

(e)

Let

$$W = \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2}$$

We know that $W \sim \chi_n^2$ (why? Please derive it on your own) and therefore,
$Var(W) = 2n$
Thus

$$Var\left(\widehat{\sigma^2}\right) = Var\left(\frac{\sigma^2 W}{n}\right) = \frac{\sigma^4}{n^2}2n = \frac{2\sigma^4}{n}$$

Using the moment generating function, we can show that the variance of the MOME is:

$$Var\left[\widetilde{\sigma^2}\right] = \frac{\sum_{i=1}^{n} Var[X_i^2]}{n^2} = \frac{Var[X^2]}{n} = \frac{2\sigma^4}{n} + \frac{4\mu^2\sigma^2}{n}$$

Therefore, one can see that:

$$Var\left[\widetilde{\sigma^2}\right] = \frac{2\sigma^4}{n} + \frac{4\mu^2\sigma^2}{n} \geq Var\left(\widehat{\sigma^2}\right) = \frac{2\sigma^4}{n}$$

The MLE has a smaller variance except for when $\mu = 0$.

(* In fact, based on theorems from mathematical statistics, one can show that the MLE in this case, has the smallest possible variance among all unbiased estimators of σ^2)

Below is a review of how to derive the moment generating functions. It also serves to help you review the concept of mathematical expectations.

Example 1. Please derive the moment generating function of $X \sim Binomial(n, p)$.

<u>Solution:</u>

$$\mathrm{M_X(t) = E(e^{tX})} = \sum_{\mathrm{x=0}}^{\mathrm{n}} \mathrm{e}^{\mathrm{t}x}\, \mathrm{P(X} = x)$$

$$= \sum_{x=0}^{\mathrm{n}} \mathrm{e}^{\mathrm{t}x} \binom{\mathrm{n}}{x} \mathrm{p}^{x}(1-\mathrm{p})^{\mathrm{n}-x}$$

$$= \sum_{x=0}^{\mathrm{n}} \binom{\mathrm{n}}{x} (\mathrm{e^t p})^{x}(1-\mathrm{p})^{\mathrm{n}-x}$$

$$= [\mathrm{pe^t} + (1-\mathrm{p})]^{\mathrm{n}}$$

Binomial Theorem (https://en.wikipedia.org/wiki/Binomial_theorem).

$$(\mathrm{a+b})^{\mathrm{n}} = \sum_{\mathrm{x=0}}^{\mathrm{n}} \binom{\mathrm{n}}{\mathrm{x}} \mathrm{a^x b^{n-x}}$$

Example 2. Please derive the moment generating function of $X \sim N(\mu, \sigma^2)$.
Solution:

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} * \frac{1}{\sqrt{2\pi}\sigma} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left(\frac{1}{2\sigma^2}[-(x-\mu)^2 + 2tx\sigma^2]\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2\sigma^2}[x^2 - 2x\mu - 2tx\sigma^2 + \mu^2]\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2\sigma^2}[x^2 - 2x(\mu + t\sigma^2) + \mu^2]\right) dx$$

$$\xRightarrow{\text{Complete the Square}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2\sigma^2}[x^2 - 2x(\mu + t\sigma^2) + \mu^2 + (\mu + t\sigma^2)^2 - (\mu + t\sigma^2)^2]\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2\sigma^2}\left[\left(x - (\mu + t\sigma^2)\right)^2 - (\mu + t\sigma^2)^2 + \mu^2\right]\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2\sigma^2}\left[\left(x - (\mu + t\sigma^2)\right)^2 - \mu^2 - 2\mu\sigma^2 t - t^2\sigma^4 + \mu^2\right]\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2\sigma^2}[x - (\mu + t\sigma^2)]^2\right) exp\left(-\frac{1}{2\sigma^2}[-2\mu\sigma^2 t - t^2\sigma^4]\right) dx$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2\sigma^2}[x - (\mu + t\sigma^2)]^2\right) dx;$$

$$Note: \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2\sigma^2}[x - (\mu + t\sigma^2)]^2\right) dx = 1$$

Therefore, we have:

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

That is why for the standard normal distribution (also called a unit normal distribution), $Z \sim N(0,1)$, its moment generating function (mgf) is:

$$M_Z(t) = E(e^{tZ}) = e^{t^2/2}$$