Dear all, this is a close book exam. Please turn in by 9:50am. No calculator is allowed. Please scan your exam in one PDF file and submit to the class Brightspace site. Academic dishonesty will result in severe consequence.

1. A person tried by a 3-judge panel is declared guilty if at least 2 judges cast votes of guilty. Suppose that when the defendant is, in fact, guilty, each judge will independently vote guilty with probability 0.8, whereas when the defendant is, in fact, innocent, this probability drops to 0.3. If 50 percent of defendants are guilty, compute the probability that:

   (a) a guilty man would be found guilty.
   (b) the jury would render a correct decision.

Solution:

Define the events as follows:

G – the defendant is in fact guilty

VG – the defendant is voted guilty by the 3-judge panel.

I – the defendant is in fact innocent.

VI – the defendant is voted innocent by the 3-judge panel.

(a) Let X be the number of votes of guilty cast.

If the defendant is in fact guilty, X follows the binomial distribution B(n=3, p=0.8). In this case

$P(VG\,|G\,) = P(X=3)+P(X=2) = \binom{3}{3}(0.8)^3 + \binom{3}{2}(0.8)^2(1-0.8) = 0.896$

(b). If the defendant is in fact innocent, X follows the binomial distribution B(n=3, p=0.3). In this case

$P(VG\,|\,I) = P(X=3)+P(X=2) = \binom{3}{3}(0.7)^3 + \binom{3}{2}(0.7)^2(1-0.7) = 0.784$

Thus, P(Correct decision) = P({A guilty defendant is found guilty} U {An innocent defendant is found innocent})

= P(A guilty defendant is found guilty) +

P(An innocent defendant is found innocent)

$=P(VG \cap G) + P(VI \cap I)$

= P(VG | G) P(G) + P(VI | I) P(I)

=0.896*0.5+0.784*0.5

=0.84

2. A miner is trapped in a totally dark underground tunnel system. He is standing at the intersection of four tunnels – tunnel 1 will lead to freedom in 2 days, tunnel 2 will lead to freedom in 4 days; however, tunnel 3 and tunnel 4 will return him to the exact same intersection in 1 day's journey! Assuming that the intersection is pitch black and the miner lost his headlight and is entirely disoriented so that each time he will choose one tunnel completely at random. **Question** – in how many days do we expect the miner to reach freedom?

Solution:

Let X represent the time (days) to freedom and Y the initial door chosen, then we have (*using the law of total expectation: https://en.wikipedia.org/wiki/Law_of_total_expectation)

$$E(X) = E(X|Y=1)P(Y=1) + E(X|Y=2)P(Y=2) + E(X|Y=3)P(Y=3) + E(X|Y=4)P(Y=4)$$

$$E(X) = 2 * \frac{1}{4} + 4 * \frac{1}{4} + [E(X)+1] * \frac{1}{4} + [E(X)+1] * \frac{1}{4}$$

$$E(X) = 4\ (days)$$

Therefore, we **expect** the miner to reach freedom in 4 days.

3. Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, be a random sample from the normal population where σ^2 is assumed known. Please derive:

(a) The maximum likelihood estimator for μ.
(b) The method of moment estimators for μ.
(c) Is the MLE for μ unbiased?
(d) Please derive the mean squared error (MSE) of the MLE for μ.

Solution:

(a) The likelihood function is

$$L = f(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f\,(x_i; \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$
$$= (2\pi)^{-n/2}[\sigma^2]^{-n/2} exp\left[-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right]$$

The log likelihood function is

$$l = lnL = constant - \frac{n}{2} ln(\sigma^2) - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$$

Solving

$$\frac{dl}{d\mu} = 2\frac{\sum_{i=1}^{n}(x_i-\mu)}{2\sigma^2} = 0$$

We obtain the MLE for μ:

$$\hat{\mu} = \bar{X}$$

(b) We now derive the MOME estimator for μ.

Since the first population moment is:

$$E[X] = \mu,$$

while the first sample moment is:

$$\frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}$$

Setting them equal we have:

$$\mu = \bar{X}$$

Therefore, the MOME estimator of μ:

$$\tilde{\mu} = \bar{X}$$

(c) Since

$$E[\bar{X}] = E\left[\frac{\sum_{i=1}^{n} X_i}{n}\right] = \frac{E[\sum_{i=1}^{n} X_i]}{n} = \frac{\sum_{i=1}^{n} E[X_i]}{n} = \frac{n\mu}{n} = \mu$$

It is straight-forward to verify that the MLE

$$\hat{\mu} = \bar{X}$$

is an unbiased estimator for μ.

(d) Because the random variables $X_i, i = 1,2, \dots, n$, are independent, we have

$$Var[\bar{X}] = Var\left[\frac{\sum_{i=1}^{n} X_i}{n}\right] = \frac{Var[\sum_{i=1}^{n} X_i]}{n^2} = \frac{\sum_{i=1}^{n} Var[X_i]}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Therefore, the MSE of $\hat{\mu} = \bar{X}$ is:

$$\mathrm{MSE}(\bar{X}) = \mathrm{Var}(\bar{X}) + (\mathrm{E}(\bar{X}) - \mu)^2 = \frac{\sigma^2}{n} + 0 = \frac{\sigma^2}{n}$$