AMS 570 Professor Wei Zhu

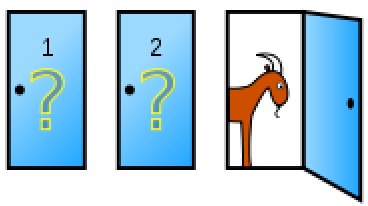
Lecture 1

Textbook for AMS 570 Mathematical Statistics:
Statistical Inference, 2nd edition;
Casella and Berger;
Brooks/Cole, Cengage Learning 2002

A good introductory book for probability:
A First Course in Probability, 10th edition;
Sheldon Ross;
Pearson 2019

1. Review of Probability, the Monty Hall Problem

(http://en.wikipedia.org/wiki/Monty_Hall_problem)



The **Monty Hall problem** is a probability puzzle loosely based on the American television game show *Let's Make a Deal* and named after the show's original host, Monty Hall.

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1 [but the door is not opened], and the host, who knows what's behind the doors, opens another door (*always a door you did not choose, and behind which there is no car), say No. 3, which has a goat. He then says to you, "Do you want to switch (*i.e. pick door No. 2), or to stay (*i.e., stay with door No. 1 you picked initially)?" Is it to your advantage to switch your choice?

The answer will be clear by computing and comparing the following two probabilities: (1) what is your winning chance if your strategy is to stay? (2) what is your winning chance if your strategy is to switch?

Solutions: (*many possible ways – but not all of them are correct even if your answers are the right numbers. Here we present one approach using conditional probability.)

$$P(Win\_By\_Stay)$$

$$= P(WBST \mid First\_Door\_Chosen\_Has\_\Pr ize)\cdot P(F.D.C\_Has\_\Pr ize)$$

$$+P(WBST \mid First\_Door\_Chosen\_Has\_No\_\Pr ize)\cdot P(F.D.C\_Has\_No\_\Pr ize)$$

$$=1\cdot\frac{1}{3}+0\cdot\frac{2}{3}=\frac{1}{3}$$

$$P(Win\_By\_Switch)$$

$$= P(WBSW \mid First\_Door\_Chosen\_Has\_\Pr ize)\cdot P(F.D.C\_Has\_\Pr ize)$$

$$+P(WBSW \mid First\_Door\_Chosen\_Has\_No\_\Pr ize)\cdot P(F.D.C\_Has\_No\_\Pr ize)$$

$$=0\cdot\frac{1}{3}+1\cdot\frac{2}{3}=\frac{2}{3}$$

2. Review of Probability (continued)

Exercise:

A write to B and does not receive an answer. Assuming that one letter in n is lost in the mail, find the chance that B received the letter. It is to be assumed that B would have answered the letter if she had received it.

Answer:

We set event A and event B to be:
A: person A got no answer;
B: person B received the letter.

In the following, we apply
(1) definition of conditional probability,
(2) Bayes' rule, and
(3) the law of total probability respectively to obtain the answer.

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$
$$= \frac{P(A|B)P(B)}{P(A)}$$
$$= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^C)P(B^C)}$$
$$= \frac{\frac{1}{n} \times (1 - \frac{1}{n})}{\frac{1}{n} \times \left(1 - \frac{1}{n}\right) + 1 \times \frac{1}{n}} = \frac{n-1}{2n-1}$$

3. Review of Probability (continued): Probability distributions.

(1) Binomial distribution

Eg. 1. Suppose each child's birth will result in either a boy or a girl with equal probability. For a randomly selected family with 2 children, what is the chance that the chosen family has 1) 2 boys? 2) 2 girls? 3) a boy and a girl?
Solution: 25%; 25%; 50%
P(B and B)= P(B∩B)= P(B)·P(B)=0.5·0.5=0.25
P(G and G)= P(G∩G)= P(G)·P(G)=0.5·0.5=0.25
P(B and G)= $P(B_1 \cap G_2 \text{ or } B_2 \cap G_1) = P((B_1 \cap G_2) \cup (B_2 \cap G_1))$

Binomial Experiment:

1) It consists of n trials
2) Each trial results in 1 of 2 possible outcomes, "S" or "F"
3) The probability of getting a certain outcome, say "S", remains the same, from trial to trial, say P("S")=p
4) These trials are independent, that is the outcomes from the previous trials will not affect the outcomes of the up-coming trials

Eg. 1 (continued) n=2, let "S"=B, P(B)=0.5

Let X denotes the total # of "S" among the n trials in a binomial experiment, then X~B(n, p), that is, Binomial Distribution with parameters n and p.
Its probability density function (pdf) is

$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n$$

Here $\binom{n}{x} = \frac{n(n-1)\ldots(n-x+1)}{x(x-1)\ldots 3\cdot 2\cdot 1}$***; (please note there are exactly x terms in the numerator, and in the denominator); for example,*** $\binom{5}{2} = \frac{5\cdot 4}{2\cdot 1} = 10$***; also note:*** $\binom{n}{x} = \binom{n}{n-x}$

***** For a discrete random variable, its pdf is also called its probability mass function (pmf) . For the above binomial pdf, we have:**

$$\sum_{i=0}^{n} P(X = x) = 1$$

Eg. 1 (continued) n=2, p=0.5, S=B, birth=trial
Answer: Let X denotes the total of boys form the 2 births.
Then X~B(n=2, p=0.5)

1) P(2 boys)=P(X=2)= $\binom{2}{2} 0.5^2 (1-0.5)^{2-2}$ =.25

2) P(2 girls)=P(X=0)= $\binom{2}{0}0.5^0(1-0.5)^{2-0}$ =.25

3) P(1 boys and a girl)=P(X=1)= $\binom{2}{1}0.5^1(1-0.5)^{2-1}$ =.5

4) What is the probability of having at least 1 boy?

   P(at least 1 boy)=P($X \geq 1$)=P(X=1)+P(X=2)
   =.5+.25=.75

Eg 2. An exam consists of 10 multiple choice questions. Each question has 4 possible choices. Only 1 is correct. Jeff did not study for the exam. So he just guesses at the right answer for each question (pure guess, not an educated guess). What is his chance of passing the exam? That is, to make at least 6 correct answers.

Answer: Yes, this is a binomial experiment with n=10, p=0.25, "S"=choose the right answer for each question.

Let X be the total # of "S"

P(pass)=P($X \geq 6$)=P(X=6 or X=7 or X=8 or X=9 or X=10)

= P(X=6)+ P(X=7)+ P(X=8)+ P(X=9)+ P(X=10)

$= \binom{10}{6}0.25^6(1-0.5)^4 + \ldots$

Eg 3 . A person tried by a 3-judge panel is declared guilty if at least 2 judges cast votes of guilty. Suppose that when the defendant is, in fact, guilty, each judge will independently vote guilty with probability 0.7, whereas when the defendant is, in fact, innocent, this probability drops to 0.2. If 70 percent of defendants are guilty, compute the probability that

a. the jury would render a correct decision;
b. an innocent man would be found innocent.

Solution:

*(*This problem reviews the binomial distribution and the conditional probability.)*

Define event as follows

G – the defendant is in fact guilty

VG – the defendant is voted guilty by the 3-judge panel

I – the defendant is in fact innocent

VI – the defendant is voted innocent by the 3-judge panel *(a)* *Let X be the number of votes of guilty cast.*

(a1). If the defendant is in fact guilty, X follows the binomial distribution B(n=3, p=0.7). In this case

$P(VG\ |G\ ) = P(X=3)+P(X=2)= \binom{3}{3}0.7^3 + \binom{3}{2}0.7^2 0.3 = 0.784$

(a2). If the defendant is in fact innocent, X follows the binomial distribution B(n=3, p=0.2). In this case

$P(VG\ |\ I) = P(X=3)+P(X=2)= \binom{3}{3}0.2^3 + \binom{3}{2}0.2^2 0.8 = 0.104$

So $P(VI\ |\ I) = 1 - 0.104 = 0.896$

Thus, P(Correct decision) = P({A guilty defendant is found guilty} U {An innocent defendant is found innocent})

= P(A guilty defendant is found guilty) +

P(An innocent defendant is found innocent)

$= P(VG \cap G) + P(VI \cap I)$

$= P(VG\ |\ G)\ P(G) + P(VI\ |\ I)\ P(I)$

$= 0.784 \times 0.7 + 0.896 * 0.3$

$= 0.8176$

(b). As shown above

$P(VI\ |\ I) = 1 - 0.104 = 0.896$

(2) <u>Bernoulli Distribution</u>

- X~Bernoulli(p) . It can take on two possible values, say success (S) or failure (F) with probability p and (1-) respectively.
- That is:

$$P(X = 'S') = p\ ;\ P(X = 'F') = 1 - p$$

- Let X = number of "S", then $X = \begin{cases} 0, 1-p \\ 1, p \end{cases}$

  The pdf of X can be written as:
  $f(x) = P(X = x) = p^x(1-p)^{1-x}\,; x = 0,1$

Relation between Bernoulli RV and Binomial RV.

(1) $X \sim \text{Bernoulli}(p)$ => it is indeed a special case of Binomial random variable when n = 1 (*only one trial), that is: $B(n = 1, p)$

(2) Let $X_i \sim \text{Bernoulli}(p)$, $i = 1, \cdots, n$. Furthermore, X_i's are all independent. Let $X = \sum_{i=1}^{n} X_i$. Then, $X \sim B(n, p)$ (***Exercise, prove this!)

Note: *** This links directly to the Binomial Experiment with X_i denotes the number of 'S' for the i^{th} trial.

Jacob Bernoulli (1654-1705)
http://en.wikipedia.org/wiki/Jacob_Bernoulli

Homework:
Please read Chapter 1, review the concepts of probability, random variable, pdf, cdf, conditional probability and independence, etc.

1.25, 1.32, 1.37, 1.44, 1.46, 1.51, 1.53