2.1.1. a) Matrices represent linear transformations.
In the case of $A \cdot \vec{v} = \vec{w}$, A transforms $\vec{v}$ either directly by changing its position (translation) or indirectly by "distorting" the coordinate system around $\vec{v}$ (rotation and scaling).
With $A \cdot B = C$, each column vector of B gets transformed by A in the same way.

The inverse of a transform represented by the transformation matrix A is the transformation needed to return the object in question to its original state. This reversing transformation is represented by the inverse matrix of A.

b) Transformation matrices:

$$T = \begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \underset{(x\text{-axis})}{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\underline{R_1 \cdot R_2}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & \sin\beta & 0 \\ 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha+\beta) & \sin(\alpha+\beta) & 0 \\ 0 & -\sin(\alpha+\beta) & \cos(\alpha+\beta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & \sin\beta & 0 \\ 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\beta+\alpha) & \sin(\beta+\alpha) & 0 \\ 0 & -\sin(\beta+\alpha) & \cos(\beta+\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Addition is commutative $\Rightarrow R_1 \cdot R_2 = R_2 \cdot R_1 \Leftrightarrow$ commutative
(Same principle for the other combinations of rotations)

$\underline{T_1 \cdot T_2}$

$$\begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & p+s \\ 0 & 1 & 0 & q+t \\ 0 & 0 & 1 & r+u \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & s+p \\ 0 & 1 & 0 & t+q \\ 0 & 0 & 1 & u+r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Addition is commutative $\Rightarrow T_1 \cdot T_2$ is commutative

$\underline{S_1 \cdot S_2}$

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} ad & 0 & 0 & 0 \\ 0 & be & 0 & 0 \\ 0 & 0 & cf & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} d & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} da & 0 & 0 & 0 \\ 0 & eb & 0 & 0 \\ 0 & 0 & fc & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Multiplication is commutative $\Rightarrow S_1 \cdot S_2$ is commutative

$\underline{R \cdot T}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & p \\ 0 & \cos\alpha & \sin\alpha & q\cdot\cos\alpha + r\cdot\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha & r\cdot\cos\alpha - q\cdot\sin\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & p \\ 0 & \cos\alpha & \sin\alpha & q \\ 0 & -\sin\alpha & \cos\alpha & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Results are not the same, since the reference axis for the rotation is not translated with the object
$\Rightarrow$ not commutative
(Same applies to the other rotations)

$\underline{S \cdot T}$

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & ap \\ 0 & b & 0 & bq \\ 0 & 0 & c & cr \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ not commutative

$$\begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & p \\ 0 & b & 0 & q \\ 0 & 0 & c & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\underline{R \cdot S}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b\cdot\cos\alpha & c\cdot\sin\alpha & 0 \\ 0 & -b\cdot\sin\alpha & c\cdot\cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ not commutative

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b\cdot\cos\alpha & b\cdot\sin\alpha & 0 \\ 0 & -c\cdot\sin\alpha & c\cdot\cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Commuting: $R_1 \cdot R_2$, $T_1 \cdot T_2$, $S_1 \cdot S_2$

not commuting: $R \cdot T$, $S \cdot T$, $R \cdot S$

2.1.2. a)

$$\left(P^T \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right)^T$$

b) $P \cdot \mathbb{1}_{n \times 1}$

c) Length z_i: $z_i = \frac{(\underline{p}_i - \underline{e}) \circ \vec{d}}{|\vec{d}|^2} = (\underline{p}_i - \underline{e}) \circ \hat{d}$
(because $|\hat{d}| = 1$)

Step 1: $(\underline{p}_i - \underline{e})$

$$\begin{array}{ccc|cccc}
 & & & x_1 & x_2 & \cdots & x_n \\
 & & & y_1 & y_2 & \cdots & y_n \\
 & & & z_1 & z_2 & \cdots & z_n \\
\hline
1-x_e & 0 & 0 & x_1 - x_e & x_2 - x_e & \cdots & x_n - x_e \\
0 & 1-y_e & 0 & y_1 - y_e & y_2 - x_e & \cdots & y_n - y_e \\
0 & 0 & 1-z_e & z_1 - z_e & z_2 - x_e & \cdots & z_n - z_e
\end{array} \qquad \underline{e} = \begin{pmatrix} x_e \\ y_e \\ z_e \end{pmatrix}$$

Step 2: $(\underline{p}_i - \underline{e}) \circ \hat{d}$

$(\underline{p}_i - \underline{e}) \circ \hat{d} = \hat{d} \circ (\underline{p}_i - \underline{e})$ (dot product is commutative)

$$\begin{array}{ccc|ccc}
 & & & x_1 - x_e & \cdots & x_n - x_e \\
 & & & y_1 - y_e & \cdots & y_n - y_e \\
 & & & z_1 - z_e & \cdots & z_n - z_e \\
\hline
\multicolumn{3}{c|}{\underbrace{x_{\hat{d}} \quad y_{\hat{d}} \quad z_{\hat{d}}}_{\hat{d}^T}} & (\underline{p}_1 - \underline{e}) \circ \hat{d} & \cdots & (\underline{p}_n - \underline{e}) \circ \hat{d}
\end{array} \Rightarrow \begin{array}{ccc|ccc}
 & & & x_1 - x_e & \cdots & x_n - x_e \\
 & & & y_1 - y_e & \cdots & y_n - y_e \\
 & & & z_1 - z_e & \cdots & z_n - z_e \\
\hline
x_{\hat{d}} & y_{\hat{d}} & z_{\hat{d}} & z_1 & \cdots & z_n
\end{array}$$

$$x_{\hat{d}} \cdot (x_1 - x_e) + y_{\hat{d}} \cdot (y_1 - y_e) + z_{\hat{d}} \cdot (z_1 - z_e) = (\underline{p}_1 - \underline{e}) \circ \hat{d} = z_1$$

$$(z_1 \ z_2 \cdots z_n) = \hat{d}^T \cdot \left(\begin{pmatrix} 1-x_e & 0 & 0 \\ 0 & 1-y_e & 0 \\ 0 & 0 & 1-z_e \end{pmatrix} \cdot P \right)$$

$$|\vec{d}| = \sqrt{(-2)^2 + (-1)^2 + (-1)^2} = \sqrt{6}$$

$$\hat{d} = \left(-\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)^T$$

$$\left(-\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) \cdot \left(\begin{pmatrix} 1-x_e & 0 & 0 \\ 0 & 1-y_e & 0 \\ 0 & 0 & 1-z_e \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & 2 & -1 & 3 \\ -1 & 1 & 1 & -2 & -1 \\ 1 & -2 & -3 & 1 & 0 \end{pmatrix} \right)$$

$$= \left(-\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) \cdot \begin{pmatrix} -6 & -3 & -3 & -6 & -2 \\ -11 & -9 & -9 & -12 & -11 \\ -4 & -7 & -8 & -4 & -5 \end{pmatrix}$$

$$= \left(\frac{9}{2}\sqrt{6}, \frac{11}{3}\sqrt{6}, \frac{23}{6}\sqrt{6}, \frac{14}{3}\sqrt{6}, \frac{10}{3}\sqrt{6}\right)$$

$z_{near} = \frac{10}{3}\sqrt{6}$ $z_{far} = \frac{14}{3}\sqrt{6}$