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Review Article

Minimax algebra and applications

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Abstract: We consider theories of linear and of polynomial algebra, over two scalar systems, often called *max-algebra* and *min-algebra*. Here, max-algebra is the system $M = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ where $x \oplus y = \max(x, y)$ and $x \otimes y = x + y$. Min-algebra is the dual system $M' = (\mathbb{R} \cup \{+\infty\}, \oplus', \otimes')$ with $x \oplus' y = \min(x, y)$ and $x \otimes' y = x + y$. Towards the end we also consider *minimax algebra*, the system $M'' = (\mathbb{R} \cup \{-\infty, +\infty\}, \oplus, \otimes, \oplus', \otimes')$. Application fields discussed include location problems, machine scheduling, cutting and packing problems, discrete-event systems and path-finding problems.

Keywords: Minimax algebra; combinatorial optimisation; discrete applied mathematics.

1. Introduction

Classical algebra, based on the familiar real-number system $(\mathbb{R}, +, \times)$, has two important branches: *linear algebra*, the theory of linear transformations and their matrix presentations; and *rational algebra*, the theory of polynomial and rational functions and forms.

Any system of scalars which enjoys sufficiently many of the axiomatic properties of $(\mathbb{R}, +, \times)$ can be used as a starting point for theories of linear algebra and rational algebra, which will mimic many of the rich properties of those based on $(\mathbb{R}, +, \times)$. What makes this fact useful, rather than merely interesting, is that some of the structures so devised are well-adapted to particular application fields, so bringing the mathematical study of those fields within the purview of familiar and well-understood algebraic processes. In the present paper we discuss in particular two scalar systems, often called *max-algebra* and *min-algebra*. Here, max-algebra is the system

$$M = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes) \tag{1.1}$$

where

$$x \oplus y = \max(x, y), \qquad x \otimes y = x + y.$$

Min-algebra is the system M', dual to M:

$$M' = (\mathbb{R} \cup \{+\infty\}, \oplus', \otimes') \tag{1.2}$$

where

$$x \oplus' y = \min(x, y), \qquad x \otimes' y = x + y.$$

Towards the end we shall also consider *minimax algebra*, the system M'' where

$$M'' = (\mathbb{R} \cup \{-\infty, +\infty\}, \oplus, \otimes, \oplus', \otimes'). \tag{1.3}$$

Application fields include location problems, machine scheduling, cutting and packing problems, discrete-event systems and path-finding problems.

The present paper is in no sense intended as a definitive survey paper, or comprehensive bibliography. It is a response to an invitation to give an appreciation of this subject to members of a community who may not be familiar with it, and to highlight some of the writer's own contributions.

Although questions of computational complexity will not be ignored, our object is to concentrate principally on the use of algebraic ideas. However we shall not attempt an axiomatic development of the kind undertaken in [4], but allow the applications and the theory to emerge hand-in-hand. First, we shall look at rational max-algebra, then at linear max-algebra and linear min-algebra, culminating in a discussion of the convergence of scalar and matrix series. A brief discussion of minimax algebra then concludes the paper.

2. A location problem

Figure 1 represents a large industrial site, with roads, buildings, and a number of production units $P_1, \ldots, P_7$. The production units are liable to breakdown, and it is decided to set up a central maintenance unit which can be called quickly to any of the production units. Land is available for building the maintenance unit anywhere along the West side of road AB.

Wherever the unit is built, at point X on AB, say, it will take a certain time, $t(X, P_j)$, to reach each of the production units P_j and we may take the worst of these,

$$\max_j t(X, P_j), \tag{2.1}$$

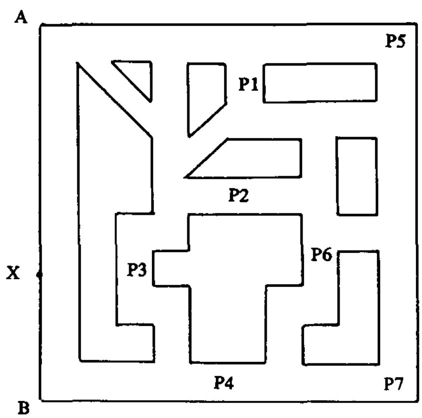

Fig. 1. A location problem.

as a measure of the *unsuitability* of point X. We may therefore define the following problem:

$$\text{Find } X \in \text{AB so as to minimise } \max_j t(X, P_j). \tag{2.2}$$

Assuming a constant average legal speed on the factory site, we may measure $t(X, P_j)$ conveniently by the distance $\Delta(X, P_j)$ from X to P_j by the shortest route and since travel from X to P_j must be either via A or via B:

$$\Delta(X, P_j) = \min(a_j + x, b_j + l - x) \tag{2.3}$$

where l is the length of AB; x is the distance from A of X on AB; and a_j, b_j are the distances to P_j from A, B respectively by the shortest route through the road network. Thus the minimand in problem (2.2) becomes

$$f(x) = \max_{j=1,\ldots,n} (\min(a_j + x, b_j + l - x)) \tag{2.4}$$

in which we now make the obvious generalisation to a system with n production units.

Problems of this nature, which arise in locational analysis, are known as (*local*) *absolute centre problems* [8].

3. Max-algebra

In Section 4 we shall show how functions like (2.4) may be reformulated using the system

$$M = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes) \tag{3.1}$$

where

$$x \oplus y = \max(x, y), \tag{3.2}$$

$$x \otimes y = x + y. \tag{3.3}$$

We call this system *max-algebra* and may readily verify that it has the following properties:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \tag{3.4}$$

$$x \oplus y = y \oplus x, \tag{3.5}$$

$$x \otimes (y \otimes z) = (x \otimes y) \otimes z, \tag{3.6}$$

$$x \otimes y = y \otimes x, \tag{3.7}$$

$$x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z, \tag{3.8}$$

$$x \otimes 0 = x, \tag{3.9}$$

$$x \oplus -\infty = x, \tag{3.10}$$

$$x \oplus x = x. \tag{3.11}$$

As can be seen, these axioms mimic the associativity, commutativity and distributivity properties of $(\mathbb{R}, +, \times)$, and present the elements 0 and $-\infty$ in the roles of 'unity element' and 'zero element' respectively. Axiom (3.11), the idempotent law of 'addition' in M, does however confer some special properties of this algebra.

Axiom systems of this nature are studied in detail in e.g. [12], where they are used as a starting-point for a theory of linear algebra. In order to carry out *rational* algebra, we must extend them in a natural way. First, an r-fold 'product' of an element x with itself will be written as a 'power':

$$x^{(r)} = x \otimes x \otimes \cdots \otimes x \quad (r \text{ times}). \tag{3.12}$$

The brackets around the exponent in (3.12) serve to distinguish it from an ordinary numerical power. Evidently $x^{(r)}$ is more familiarly denoted rx, so we may introduce zero and negative exponents with the definitions

$$x^{(0)} = 0, \tag{3.13}$$

$$x^{(-r)} = -rx \quad (r > 0). \tag{3.14}$$

An important, but easily proved, fact is

$$(x \oplus y)^{(r)} = x^{(r)} \oplus y^{(r)} \quad \text{when } r \geq 0. \tag{3.15}$$

It is also convenient to introduce 'quotients', using a double fraction-bar // to distinguish these from arithmetical quotients. Thus if U, V are expressions in max-algebra, we define

$$\frac{U}{V} = U \otimes V^{(-1)}. \tag{3.16}$$

By way of illustration, the expression

$$\frac{3 \oplus 2 \otimes x^{(2)}}{6 \otimes x \oplus 7 \otimes x^{(4)}}. \tag{3.17}$$

denotes, in a more usual notation:

$$\max(3, 2 + 2x) - \max(6 + x, 7 + 4x). \tag{3.18}$$

As a second illustration, we may render the operator 'min' as a rational operation in max-algebra. Clearly,

$$\min(x, y) = x + y - \max(x, y). \tag{3.19}$$

Hence

$$\min(x, y) = \frac{x \otimes y}{x \oplus y}. \tag{3.20}$$

Finally, to complete the notation, we need a 'sigma' and 'pi' notation to denote iterated 'sums' and 'products' of indexed expressions in max-algebra. Thus, for given terms $\xi_1, \ldots, \xi_n$,

$$\sum_{j=1}^{n} \oplus\, \xi_j \quad \text{denotes} \quad \xi_1 \oplus \cdots \oplus \xi_n, \tag{3.21}$$

$$\prod_{j=1}^{n} \otimes\, \xi_j \quad \text{denotes} \quad \xi_1 \otimes \cdots \otimes \xi_n. \tag{3.22}$$

4. An illustrative calculation

We now show how this formalism may be used for finding maxima and minima of functions of the general form:

$$\max_{j=1,\dots,n} (\min(a_j x + b_j, p_j x + q_j)), \tag{4.1}$$

which includes, of course, the location problem considered in Section 2.

To be specific, assume that we require to find minima and maxima of

$$f(x) \equiv \max(\min(3x, 5-2x), \min(x, 6-x), \min(x-2, 14-3x)). \tag{4.2}$$

This function (see Figure 2) is the upper envelope of the three functions

$$\min(3x, 5-2x),\ \min(x, 6-x),\ \min(x-2, 14-3x). \tag{4.3}$$

It is evidently piecewise-linear and its local minima and maxima arise at the intersections of certain pairs of the functions

$$3x,\ 5-2x,\ x,\ 6-x,\ x-2,\ 14-3x. \tag{4.4}$$

As it happens, the third of the functions in (4.3) is *dominated* and could be dropped from the definition of $f(x)$ without changing $f(x)$ as a function of x. We shall see how the algebra copes with this.

From the discussion in Section 3 we may write (4.2) in max-algebra notation as

$$f(x) \equiv \frac{x^{(3)} \otimes 5 \otimes x^{(-2)}}{x^{(3)} \oplus 5 \otimes x^{(-2)}} \oplus \frac{x \otimes 6 \otimes x^{(-1)}}{x \oplus 6 \otimes x^{(-1)}} \oplus \frac{2^{(-1)} \otimes x \otimes 14 \otimes x^{(-3)}}{2^{(-1)} \otimes x \oplus 14 \otimes x^{(-3)}}. \tag{4.5}$$

This expression may be 'rationalised' by exact analogy with school algebra to give

$$f(x) \equiv \frac{6 \otimes x^{(10)} \oplus 22 \otimes x^{(6)} \oplus 27 \otimes x^{(3)} \oplus 27 \otimes x}{(x^{(5)} \oplus 5) \otimes (x^{(2)} \oplus 6) \otimes (x^{(4)} \oplus 16)}. \tag{4.6}$$

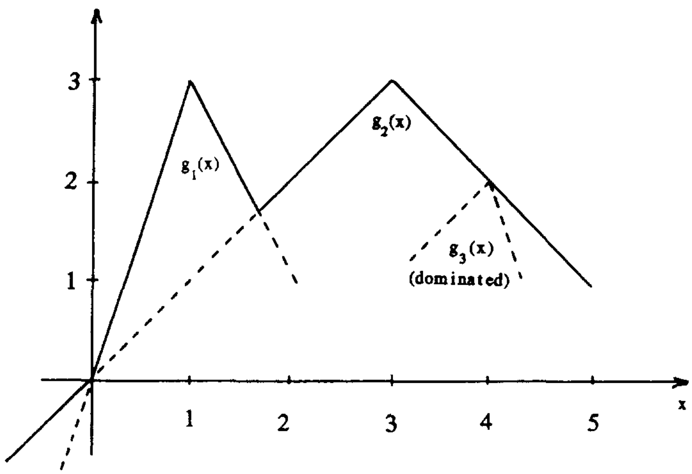

Fig. 2. The piecewise-linear function $f(x) = \max(g_1(x), g_2(x), g_3(x))$.

The numerator in (4.6) is a 'polynomial' of the form $\sum\oplus_{j=1}^{N} a_j \otimes x^{(k_j)}$.

Theorem 1. *The function* $\sum \oplus_{j=1}^{N} a_j \otimes x^{(k_j)}$ *always possesses a unique 'resolution into linear factors':* $a_N \otimes \prod\otimes_{r=1}^{s} \alpha_r^{(e_r)}$ *where each* α_r *is either* x *or a 'linear factor'* $x \oplus \beta_r$.

The values β_r are called the *corners* of the function. The proof of Theorem 1, together with straightforward procedures for computing the corners β_r and exponents e_r from the coefficients a_j, is given in [5]. Applying these and (3.15),

$$f(x) \equiv \frac{6 \otimes x \otimes (x \oplus 0)^{(2)} \otimes (x \oplus (5/3))^{(3)} \otimes (x \oplus 4)^{(4)}}{(x \oplus 1)^{(5)} \otimes (x \oplus 3)^{(2)} \otimes (x \oplus 4)^{(4)}}. \tag{4.7}$$

After cancelling $(x \oplus 4)^{(4)}$,

$$f(x) \equiv 6 \otimes x \otimes (x \oplus 0)^{(2)} \otimes (x \oplus 1)^{(-5)} \otimes (x \oplus (5/3))^{(3)} \otimes (x \oplus 3)^{(-2)}. \tag{4.8}$$

As may be seen by reference to Figure 2, the expression (4.8) fully specifies the shape of $f(x)$. The exponent of the term in x (i.e. unity) gives the slope of the function at $x = -\infty$; the exponent of each term $(x \oplus \beta_r)$ (i.e. e_r) gives the *change in slope at* $x = \beta_r$. So from (4.8), the slope of $f(x)$ is 1 at $x = -\infty$; increases to 3 at $x = 0$; decreases to -2 at $x = 1$; increases to 1 at $x = \frac{5}{3}$; decreases to -1 at $x = 3$.

Since local minima and maxima occur only where the slope changes sign appropriately, we can read the x-values of the local minima and maxima almost trivially from (4.8). Cancellation of $(x \oplus 4)^{(4)}$ removed the point $x = 4$ from consideration, corresponding to the irrelevance of the third of the functions in (4.3).

5. Planar computational geometry

The problem of cutting two-dimensional shapes from sheets of material is one which occurs widely in many different industries, and the associated calculations require us to find numerical representations of shapes and of standard geometrical transformations such as translations and rotations.

Frequently, given shapes are replaced for computational purposes by polygonal approximations, and the piecewise-linear character which this gives to the mathematics lends itself well to the use of rational max-algebra. For example, any shape consisting of a simple, closed polygonal curve together with its interior may be characterised as the set of points (x, y) satisfying the inequality

$$W(x, y) \leqslant 0 \tag{5.1}$$

where W is a suitable max-algebra 'rational function' of two variables. We shall illustrate this for the non-convex quadrilateral ABCD of Figure 3.

Triangle ABD, being the intersection of the 3 half-planes

$$-x - y \leqslant 0, \quad x \leqslant 0, \quad -1 + y \leqslant 0, \tag{5.2}$$

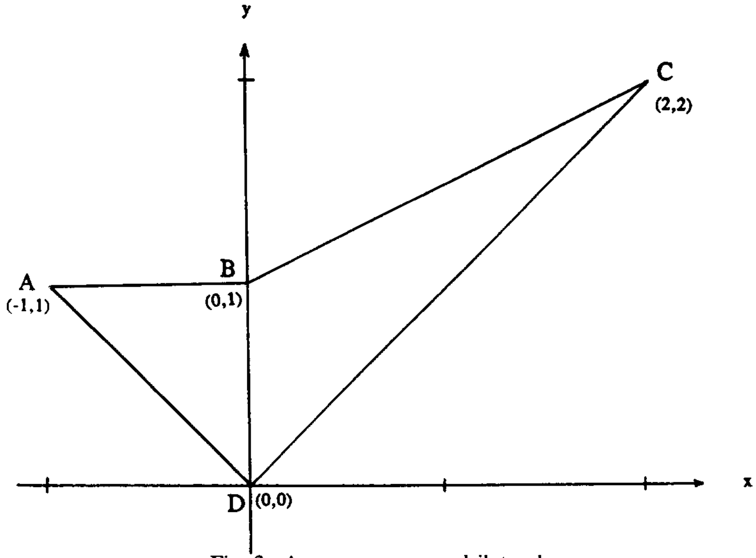

Fig. 3. A non-convex quadrilateral.

has the description

$$\{(x, y)\colon \max(-x-y, x, -1+y) \leqslant 0\}, \tag{5.3}$$

or in the notation of max-algebra,

$$\{(x, y)\colon x^{(-1)} \otimes y^{(-1)} \oplus x \oplus 1^{(-1)} \otimes y \leqslant 0\} \tag{5.4}$$

which may be 'rationalised' to $U(x, y) \leqslant 0$ where

$$U(x, y) = \frac{(1 \oplus 1 \otimes x^{(2)} \otimes y \oplus x \otimes y^{(2)})}{1 \otimes x \otimes y}. \tag{5.5}$$

Similarly triangle BCD is given by $V(x, y) \leqslant 0$, where

$$V(x, y) = \frac{(2 \otimes y \oplus 2 \otimes x^{(2)} \oplus y^{(3)})}{2 \otimes x \otimes y}. \tag{5.6}$$

Since quadrilateral ABCD is the union of these two triangles, it is given by $W(x, y) \leqslant 0$, where

$$W(x, y) = \min(U(x, y), V(x, y)). \tag{5.7}$$

And, by (3.20),

$$W(x, y) = U(x, y) \otimes V(x, y) // (U(x, y) \oplus V(x, y)) \tag{5.8}$$

which, using (5.5), (5.6) and elementary algebraic manipulation, gives

$$\begin{aligned} W(x, y) = {} & [(3 \otimes y \oplus 1 \otimes y^{(3)}) \oplus (2 \otimes y^{(3)} \oplus y^{(5)}) \otimes x \\ & \oplus (3 \oplus 3 \otimes y^{(2)} \oplus 1 \otimes y^{(4)}) \otimes x^{(2)} \oplus 2 \otimes y^{(2)} \otimes x^{(3)} \oplus 3 \otimes y \otimes x^{(4)}] \\ & //[(3 \otimes y \oplus 3 \otimes y^{(2)} \oplus 1 \otimes y^{(4)}) \otimes x \\ & \oplus 2 \otimes y^{(3)} \otimes x^{(2)} \oplus (3 \otimes y \oplus 3 \otimes y^{(2)}) \otimes x^{(3)}]. \end{aligned} \tag{5.9}$$

From a characterisation of a shape in the form (5.1) we may derive algebraically a similar characterisation of the image of the shape under a given geometrical transformation. For example, if τ is the *translation* which takes each point (p, q) to $(p+\alpha, q+\beta)$, and S is the shape characterised by (5.1) then (x, y) belongs to $\tau(S)$ if and only if $(x-\alpha, y-\beta)$ belongs to S, i.e. if and only if

$$W\left(\frac{x}{\alpha}, \frac{y}{\beta}\right) \leqslant 0. \tag{5.10}$$

Obviously (5.10) gives us a 'rational' inequality characterising $\tau(S)$. In a similar fashion we may, for example, develop a 'rational' condition on the parameters α, β such that $\tau(S)$ shall not intersect some other, 'rationally' described, shape T. In this way, many calculations concerned with moving a set of geometrical shapes so as to arrange them, without overlapping, into a suitable design for cutting or packing, may be handled by simple algebraic routines.

6. Schedule algebra

We turn now to applications of *linear* max-algebra. One common feature of industrial processes is that machines do not act independently, and a typical machine cannot begin a fresh cycle of activity until certain other machines have all completed their current cycles.

A natural way of describing such a system is to label the machines, e.g. $1, \ldots, n$, and to describe the interferences by recurrence relations such as

$$x_3(r+1) = \max(x_1(r) + t_1(r), x_2(r) + t_2(r)). \tag{6.1}$$

This expresses the fact that machine 3 must wait to begin its $(r+1)$-st cycle until machines 1 and 2 have both finished their r-th cycle, the symbol $x_i(r)$ denoting the starting time of the r-th cycle of machine i, and the symbol $t_i(r)$ denoting the corresponding activity duration. This mode of analysis gives rise to formidable-looking systems of recurrence relations:

$$x_i(r+1) = \max(x_1(r) + a_{i1}(r), \ldots, x_n(r) + a_{in}(r)) \quad (i = 1, \ldots, n), \tag{6.2}$$

where, for notational uniformity, terms $a_{ij}(r)$ and $x_j(r)$ are made to occur for all $i = 1, \ldots, n$ and all $j = 1, \ldots, n$ by introducing where necessary quantities $a_{ij}(r)$ equal to $-\infty$ for each combination (i, j) which has no physical significance; the operator *max* will then 'ignore' these terms.

If we now change the notation to that of max-algebra, expression (6.2) becomes:

$$x_i(r+1) = (a_{i1}(r) \otimes x_1(r)) \oplus \cdots \oplus (a_{in}(r) \otimes x_n(r)) \quad (i = 1, \ldots, n) \tag{6.3}$$

which is an inner product. Introduce the obvious vector-matrix notation

$$A(r) = [a_{ij}(r)], \qquad x(r) = [x_i(r)], \tag{6.4}$$

and (6.3) becomes

$$x(r+1) = A(r) \otimes x(r). \tag{6.5}$$

Expression (6.5) is a very intuitive 'change-of-state' equation. By iteration we have

$$x(r+1)=A(r)\otimes A(r-1)\otimes\cdots\otimes A(1)\otimes x(1), \tag{6.6}$$

showing how the state $x(r)$ of the system evolves with time, from a given initial state $x(1)$.

For simplicity of exposition assume for the moment that the quantities $a_{ij}(r)$ are independent of r. Define $A^{(r)}=A\otimes A\cdots\otimes A$, r times (associativity holds!). The 'orbit' of the system is then

$$x(1),\ A\otimes x(1),\ A^{(2)}\otimes x(1),\ \ldots \tag{6.7}$$

and it is clear that the sequence of matrices

$$A,\ A^{(2)},\ A^{(3)},\ \ldots \tag{6.8}$$

will determine the long-term behavior of the system: does it oscillate? Does it, in some suitable sense, achieve a stable state? Detailed consideration of these questions is given in [3, 4].

7. Max-algebra eigenvalue problem

An operational question, which occurs in relation to the problem of controlling systems of the type just considered, is this: "how must the system be set in motion to ensure that it moves forward in regular steps; i.e. so that for some constant λ, the interval between the beginnings of consecutive cycles on every machine is λ? And what are the possible values of λ?" Reference to the notation (3.1) to (3.3) shows that we are concerned with the problem

$$x(r+1)=\lambda\otimes x(r). \tag{7.1}$$

But $x(r+1)=A\otimes x(r)$, so we must solve

$$A\otimes x(r)=\lambda\otimes x(r). \tag{7.2}$$

Clearly we have arrived at an eigenvector–eigenvalue problem. In [4] it is shown that the only possible value for the eigenvalue of the matrix $[a_{ij}]$ is the greatest of all 'circuit-averages'

$$a_{ii},\ \frac{a_{ij}+a_{ji}}{2},\ \frac{a_{ij}+a_{jk}+a_{ki}}{3},\ \ldots. \tag{7.3}$$

Calculation of λ may be formulated as a linear-programming problem or, more efficiently, viewed as a special case of an optimal cycle problem for doubly-weighted graphs, for which algorithms of low-order polynomial complexity are known [10].

In [4], a detailed analysis of this eigenvector–eigenvalue problem is given. It is shown that all eigenvectors are generated by certain columns of the matrix Γ given by

$$\Gamma=B\oplus B^{(2)}\oplus\cdots \tag{7.4}$$

where $B = \lambda^{(-1)} \otimes A$ is obtained from A by arithmetical subtraction of λ from all elements. The matrix Γ generalises the *transitive closure* matrix of Boolean algebra and the matrix series (7.4) has been widely studied [12]. Γ may be calculated in low-order polynomial time by any of a number of algorithms of combinatorial optimisation, including the well-known Floyd–Warshall algorithm. An elegant feature of the theory is that many of these algorithms, when articulated in the notation of max-algebra, are direct analogues of Gauss–Seidel, Jordan and other well-known iterative procedures from classical linear algebra [2].

8. The shortest path problem

Given a *graph* consisting of *n nodes* $P_1, \ldots, P_n$ certain ordered pairs (P_i, P_j) of which are connected by *arcs* bearing *weights* β_{ij}, we wish to find, for each ordered pair of nodes (P_i, P_j), the quantity γ'_{ij} which represents the least total weight of the arcs traversed in any path from P_i to P_j (where $\gamma'_{ij} = +\infty$ if no such path exists; $=0$ if $i = j$).

Figure 4 depicts such a graph. There are several paths from P_1 to P_3. The path (P_1, P_2, P_3) has total arc-weight equal to 6, but path (P_1, P_4, P_2, P_3) has total arc-weight equal to 5 and this is in fact minimal. So $\gamma'_{13} = 5$ for this graph. And e.g. $\gamma'_{34} = +\infty$ because there is no (directed) path from P_3 to P_4.

In this way we may define the *shortest-distances matrix* $\Gamma' = [\gamma'_{ij}]$. For the given graph,

$$\Gamma' = \begin{bmatrix} 0 & 3 & 5 & 2 \\ +\infty & 0 & 2 & +\infty \\ +\infty & 3 & 0 & +\infty \\ +\infty & 1 & 3 & 0 \end{bmatrix}. \tag{8.1}$$

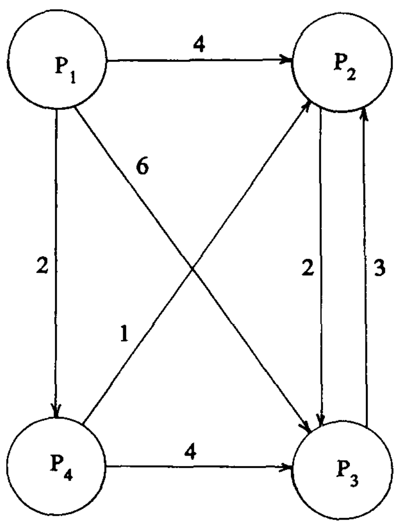

Fig. 4. A weighted graph.

We may also define the *direct-distance matrix* $E = [e_{ij}]$ where $e_{ij} = \beta_{ij}$ if there is an arc in the graph from P_i to P_j and $e_{ij} = +\infty$ otherwise. For the given graph,

$$E = \begin{bmatrix} +\infty & 4 & 6 & 2 \\ +\infty & +\infty & 2 & +\infty \\ +\infty & 3 & +\infty & +\infty \\ +\infty & 1 & 4 & +\infty \end{bmatrix}. \tag{8.2}$$

It is convenient to refer to the quantities in these matrices as *distances* although, in the more general situation considered in Section 9, some arc-weights β_{ij} may be negative.

So we may describe the problem in following terms: given E, compute Γ'. There are several well-known algorithms to achieve this [2] but in the present context it is instructive to make the following approach via *min-algebra*, defined in (1.2) as the system M'.

It is very easy to show that the following matrix I plays the role of the identity matrix in linear min-algebra:

$$I = \begin{bmatrix} 0 & & & (+\infty) \\ & 0 & & \\ & & \ddots & \\ (+\infty) & & & 0 \end{bmatrix}. \tag{8.3}$$

In relation to our shortest-path problem, we may regard the matrix I as telling us the distance from nodes P_i to nodes P_j if we make *no use of the arcs*. The direct-distances matrix E tells us the corresponding distances making use of paths which contain *exactly one arc*.

Next consider the product $E^{[2]} = E \otimes' E$ over M'. The (ij)-th element of this is

$$\min_{k=1,\ldots,n} (e_{ik} + e_{kj}) \tag{8.4}$$

and clearly this is either $+\infty$ or is the minimal distance from P_i to P_j via some P_k, i.e. using *exactly two arcs*.

Similarly, it is straightforward to show [2] that the (ij)-th element of the r-fold product

$$E^{[r]} = E \otimes' \cdots \otimes' E \tag{8.5}$$

is either $+\infty$ or gives the minimal distance from P_i to P_j using *exactly r arcs*.

Since the shortest route must make use of zero, one, two, ... or some number of arcs, γ'_{ij} must be the least of the (ij)-th elements in the matrices I, E, $E^{[2]}$, ... , i.e.

$$\Gamma' = I \oplus' E \oplus' E^{[2]} \oplus' \cdots = \sum\oplus' E^{[r]}. \tag{8.6}$$

The analogy to the matrix Γ defined in (7.4) is obvious, and motivates our use of the notation Γ'. Essentially the same algorithms and properties apply.

9. Discrete-event systems

Suppose now that the nodes P_j $(j = 1, \ldots, n)$ of our graph represent the *possible states of some finite multistage system.* The presence of a directed arc (P_i, P_j) indicates that a state-transition P_i to P_j is possible at any stage when the system is in state P_i, and β_{ij} is the associated cost of that transition (or $-\beta_{ij}$ is the profit if $\beta_{ij} < 0$).

Adapting the argument of Section 8, it is clear that the power $E^{[r]}$ of matrix E now gives the least possible total cost of each net transition from each P_i to each P_j in a run of exactly $r + 1$ stages. Let μ_r represent the *cost of stopping the system* at that point. Then the elements of the matrix $\mu_r \otimes' E^{[r]}$ give us a set of costs for getting from each P_i to each P_j by running the system through $r + 1$ stages and stopping.

If we are at liberty to stop the system after any chosen number of transitions, then the matrix

$$F = \mu_0 \otimes' I \oplus' \mu_1 \otimes' E \oplus' \mu_2 \otimes' E^{[2]} \oplus' \cdots = \sum_r \oplus' \mu_r \otimes' E^{[r]} \tag{9.1}$$

gives the cost of the cheapest 'ride' from any state to any other. What can be said about the convergence of such an infinite matrix power-series over M' (of which (8.6) is, of course, a special case)?

We shall say that the series (9.1) *converges finitely* if there is an integer q such that

$$\sum_{r=0}^{t} \oplus' \mu_r \otimes' E^{[r]} = \sum_{r=0}^{q} \oplus' \mu_r \otimes' E^{[r]} \quad \text{for any } t \geq q. \tag{9.2}$$

Intuitively, this means that no cheapest ride needs a run of more than $q + 1$ stages.

Further, we shall say that (9.1) *diverges* if there is (at least) one element-position (i, j) at which the sequence of elements decreases without bound in the sequence of matrices

$$\sum_{r=0}^{t} \otimes' \mu_r \otimes' E^{[r]} \quad (t = 0, 1, 2, \ldots). \tag{9.3}$$

Let $\lambda(E)$ be the min-algebra *eigenvalue* (supposed finite) of the matrix E, defined and computed in a manner dual to the max-algebra manner discussed in Section 7. Then the following result may be proved [6]:

Theorem 2. *There exists a number ρ such that*

If $\lambda(E) > \rho$ then (9.1) *converges finitely.*

If $\lambda(E) < \rho$ then (9.1) *diverges.*

As is usual in convergence theory in mathematics, the 'knife-edge' case $\lambda(E) = \rho$ may go either way. The constant ρ depends, of course, on the coefficients μ_r and is in fact given by

$$\rho = \lim\sup\left(\frac{-\mu_r}{r}\right). \tag{9.4}$$

For the shortest-path problem (6.5), all coefficients μ_r are zero, and hence $\sum\oplus_r' E^{[r]}$ converges finitely if $\lambda(E) > 0$ and diverges if $\lambda(E) < 0$. Since we know, dually to (7.3), that $\lambda(E)$ is given by least of the 'cycle-averages'

$$e_{ii}, \frac{e_{ij}+e_{ji}}{2}, \frac{e_{ij}+e_{jk}+e_{ki}}{3}, \ldots, \tag{9.5}$$

we reach the standard result [2] that if E contains any negative cycles then its shortest-distance matrix does not exist, whereas if it contains only positive cycles then Γ' is found by terminating (8.6) after a suitable finite number of terms.

A more detailed analysis shows that the number of terms need not exceed n, the number of nodes.

10. Characteristic max-algebra polynomial

In classical algebra, a link between matrices and polynomials comes through the characteristic polynomial of a matrix, which has the eigenvalues of the matrix as it roots. We show that a closely analogous result holds in max-algebra (and, dually, in min-algebra).

Let $[b_{ij}]$ be a given $n \times n$ matrix. In max-algebra, we cannot define a 'determinant' for $[b_{ij}]$ but we can define the *permanent* of $[b_{ij}]$ by

$$\mathrm{perm}[b_{ij}] = \sum_\sigma \oplus \left(\prod_{t=1}^{n} \otimes\, b_{t\sigma(t)}\right) \tag{10.1}$$

where the 'summation' is over all permutations σ in the symmetric group of order $n!$. Then we may define the characteristic maxpolynomial $\pi_A(x)$ of a given square matrix A by

$$\pi_A(x) = \mathrm{perm}(A, x), \tag{10.2}$$

where (A, x) is a matrix derived from A by replacing its diagonal elements a_{ii} by $a_{ii} \oplus x$ $(i = 1, \ldots, n)$.

Thus, if

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}, \tag{10.3}$$

then

$$\begin{aligned} \pi_A(x) &= \mathrm{perm}\begin{bmatrix} 2\oplus x & 1 & 4 \\ 1 & 0\oplus x & 1 \\ 2 & 2 & 1\oplus x \end{bmatrix} \qquad (10.4) \\ &= (2\oplus x)\otimes(0\oplus x)\otimes(1\oplus x)\oplus(2\oplus x)\otimes 1\otimes 2 \\ &\quad \oplus 1\otimes 1\otimes(1\oplus x)\oplus 1\otimes 1\otimes 2\oplus 4\otimes 1\otimes 2\oplus 4\otimes(0\oplus x)\otimes 2 \\ &= x^{(3)}\oplus 2\otimes x^{(2)}\oplus 6\otimes x\oplus 7, \qquad (10.5) \end{aligned}$$

by use of the associative, distributive, and commutative laws. Theorem 1 gives for maxpolynomial (10.5)

$$\pi_A(x) = (x \oplus 1) \otimes (x \oplus 3)^{(2)}. \tag{10.6}$$

(7.3) implies for A in (10.3),

$$\lambda(A) = \max\left(2,\ 0,\ 1,\ \frac{1+1}{2},\ \frac{4+2}{2},\ \frac{1+2}{2},\ \frac{1+1+2}{3},\ \frac{4+1+2}{3}\right) = 3. \tag{10.7}$$

Then (10.6) and (10.7) illustrate the following result, proved in [7]:

Theorem 3. *The eigenvalue of A is the greatest corner of $\pi(x)$.*

11. Minimax algebra

Earlier, we briefly mentioned the system *minimax algebra*:

$$M'' = (\mathbb{R} \cup \{-\infty, +\infty\}, \oplus, \otimes, \oplus', \otimes'). \tag{11.1}$$

From the meanings of $\oplus$, $\oplus'$ it is clear that

$$-\infty \oplus +\infty = +\infty \oplus -\infty = +\infty \quad \text{and} \quad -\infty \oplus' +\infty = +\infty \oplus' -\infty = -\infty, \tag{11.2}$$

but for the system to be well-defined we must also say how infinite elements are 'multiplied', namely

$$-\infty \otimes +\infty = +\infty \otimes -\infty = -\infty, \qquad -\infty \otimes' +\infty = +\infty \otimes' -\infty = +\infty. \tag{11.3}$$

For any $m \times n$ matrix $A = [a_{ij}]$ over M'' we may define the *conjugate* $(n \times m)$ matrix A^* by negation-and-transposition:

$$A^* = [-a_{ji}]. \tag{11.4}$$

The following properties are readily derived:

$$(A \oplus B)^* = A^* \oplus' B^*, \qquad (A \oplus' B)^* = A^* \oplus B^*, \tag{11.5}$$

$$(A \otimes B)^* = B^* \otimes' A^*, \qquad (A \otimes' B)^* = B^* \otimes A^*, \tag{11.6}$$

$$(A^{(r)})^* = (A^*)^{[r]}, \qquad (A^{[r]})^* = (A^*)^{(r)}. \tag{11.7}$$

From these, a rich duality theory may be derived [4] in which the following result is fundamental.

Theorem 4. *For any n-tuple x and m-tuple y,*

$$A \otimes x \leqslant y \ \textit{ if and only if } \ A^* \otimes' y \geqslant x. \tag{11.8}$$

The vector inequalities in (11.8) are read componentwise. Roughly speaking, (11.8) says that the multiplications $A^* \otimes'$ and $A \otimes$ act as each other's inverse in relation to inequalities; more precise expression of this requires the language of *residuation theory* as in [4].

Many useful theoretical results follow from this. For example, a criterion of solubility of 'linear equations' in max-algebra (or min-algebra) may be developed, leading to a theory of linear dependence, rank and dimensionality [4].

But Theorem 4 also has more practical consequences, as we shall see next.

12. Critical activities

Consider an industrial process of the kind discussed in Section 6 with, say, four machines. Let the matrix A be given by

$$\begin{bmatrix} 3 & 2 & -\infty & 4 \\ 3 & 4 & 5 & -\infty \\ -\infty & -\infty & 2 & 3 \\ -\infty & 3 & -\infty & 4 \end{bmatrix}. \tag{12.1}$$

A certain project requires the completion of five cycles on all machines. The first cycles are all completed at time zero and the customer then enquires what the earliest completion-time of the project will be. Carrying out matrix multiplications in max-algebra, we find

$$A^{(2)} = \begin{bmatrix} 6 & 7 & 7 & 8 \\ 7 & 8 & 9 & 8 \\ -\infty & 6 & 4 & 7 \\ 6 & 7 & 8 & 8 \end{bmatrix}. \tag{12.2}$$

'Squaring' again,

$$A^{(4)} = (A^{(2)})^{(2)} = \begin{bmatrix} 14 & 15 & 16 & 16 \\ 15 & 16 & 17 & 16 \\ 13 & 14 & 15 & 15 \\ 14 & 15 & 16 & 16 \end{bmatrix}. \tag{12.3}$$

Hence

$$t(5) = A^{(4)} \otimes t(1) = \begin{bmatrix} 14 & 15 & 16 & 16 \\ 15 & 16 & 17 & 16 \\ 13 & 14 & 15 & 15 \\ 14 & 15 & 16 & 16 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 16 \\ 17 \\ 15 \\ 16 \end{bmatrix}. \tag{12.4}$$

Thus the earliest possible completion-time for the whole project is 17. Suppose a delivery promise is made to the customer on this basis. It then turns out that the third cycles may be subject to delay for technical reasons. How much delay can be tolerated in the completion of each third cycle without prejudice to the delivery promise?

If in fact the third cycles are finished at time $T(3)$ then the earliest possible completion times for the project, since two more cycles must elapse, will be given by $A^{(2)} \otimes T(3)$ and this will be acceptable if and only if

$$A^{(2)} \otimes T(3) \leqslant [17, 17, 17, 17]^{\mathrm{T}}, \tag{12.5}$$

i.e. (using Theorem 4) if and only if

$$T(3) \leqslant (A^{(2)})^* \otimes' [17, 17, 17, 17]^{\mathrm{T}}$$
$$= \begin{bmatrix} -6 & -7 & +\infty & -6 \\ -7 & -8 & -6 & -7 \\ -7 & -9 & -4 & -8 \\ -8 & -8 & -7 & -8 \end{bmatrix} \otimes' \begin{bmatrix} 17 \\ 17 \\ 17 \\ 17 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 8 \\ 9 \end{bmatrix}. \tag{12.6}$$

So (12.6) gives the *latest tolerable* third-cycle completion times. On the other hand the *earliest possible* third-cycle completion times, if there were no delays, would be

$$t(3) = A^{(2)} \otimes [0, 0, 0, 0]^{\mathrm{T}}$$
$$= \begin{bmatrix} 6 & 7 & 7 & 8 \\ 7 & 8 & 9 & 8 \\ -\infty & 6 & 4 & 7 \\ 6 & 7 & 8 & 8 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 7 \\ 8 \end{bmatrix}. \tag{12.7}$$

Comparing (12.6), (12.7) we see that the tolerable delays in the completion of the third cycles on the four machines are 2, 0, 1, 1 respectively. In particular, no delay can be tolerated in the completion of the third cycle on machine 2, which would therefore be known in scheduling parlance as a *critical activity*.

13. Post script

An essay such as this is inevitably incomplete: the theory and applications of these and related algebraic structures have been vigorously studied for twenty-five years or so. The references which follow will give access to contributions by many researchers, notably in France, Germany, Austria and Czechoslovakia. These references have been selected by three criteria. First, there are those which have been expressly cited in the text; these have been chosen for expository convenience and do not always correspond to first publication of particular results. Second, there are those, notably the thorough and scholarly book of U. Zimmermann [12], which present much more exhaustive literature references, historical accounts and axiomatic developments, than could be attempted in the present article. Third, there are one or two more recent publications, which may be of interest.

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