(HW 1)

Ex 1)

(1.) Let $x, y \in E_1 = \{x \in \mathbb{R}^m \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \dots, m\}$

Let $\lambda \in [0; 1]$.

$$x, y \in E \Rightarrow \forall\, 1 \leq i \leq m \begin{cases} \alpha_i \leq x_i \leq \beta_i \\ \alpha_i \leq y_i \leq \beta_i \end{cases}$$

$$\Longrightarrow \forall\, 1 \leq i \leq m \begin{cases} \lambda\alpha_i \leq \lambda x_i \leq \lambda\beta_i & (\lambda \geq 0) \\ (1-\lambda)\alpha_i \leq (1-\lambda) y_i \leq (1-\lambda)\beta_i & (1-\lambda \geq 0) \end{cases}$$

$$\Longrightarrow \forall\, 1 \leq i \leq m,\ \alpha_i \leq \lambda x_i + (1-\lambda) y_i \leq \beta_i$$

$$\Longrightarrow \lambda x + (1-\lambda) y \in E$$

Thus, $\boxed{E \text{ is convex}}$

(2.) Let us denote by $E_2 = \{x \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\}$

$$= \{x \in \mathbb{R}_+^{*2} \mid x_1 x_2 \geq 1\}$$

Let $x, y \in E_2$ and $\lambda \in [0; 1]$.

$$[\lambda x_1 + (1-\lambda) y_1][\lambda x_2 + (1-\lambda) y_2]$$

$$= \lambda^2 x_1 x_2 + \lambda(1-\lambda)[x_1 y_2 + x_2 y_1] + (1-\lambda)^2 y_2$$

$$\geq \lambda^2 + (1-\lambda)^2 + \lambda(1-\lambda)[x_1 y_2 + x_2 y_1]$$

[since $x, y \in E_2 \Rightarrow x_1 x_2 \geq 1$ and $y_1 y_2 \geq 1$

$$\begin{cases} x_1 x_2 \geq 1 \\ y_1 y_2 \geq 1 \end{cases} \Rightarrow x_1 x_2 y_1 y_2 \geq 1$$

$$\xrightarrow[y_1 > 0]{x_2 > 0} x_1 y_2 \geq \frac{1}{x_2 y_1}$$

$$\Longrightarrow x_1 y_2 + \frac{1}{x_2 y_1} \geq \sqrt{x_2 y_1}^2 + \frac{1}{\sqrt{x_2 y_1}}^2$$

$$\geq \left(\sqrt{x_2 y_1} - \frac{1}{\sqrt{x_2 y_1}}\right)^2 + 2$$

$$\geq 2$$

Hence;

$$[\lambda x_1 + (1-\lambda) y_1][\lambda x_2 + (1-\lambda) y_2] \geq \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)$$

$$\geq [\lambda + 1 - \lambda]^2 = 1$$

Thus, $\boxed{E_2 \text{ is convex}}$

(3.) Let us denote by ~~$E_3 = \{x \mid \text{dist}(x, S) \leq \text{dist}(x, T)\}$~~

$$E_3 = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \quad \forall y \in S\}$$

Let $x, y \in E_3$ and $\lambda \in [0; 1]$

~~Let~~ $z \in S$ and $x_\lambda = \lambda x + (1-\lambda) y$

$$\cancel{\|\lambda x + (1-\lambda) y - x_0\|_2 = \|\lambda(x - x_0) + (1-\lambda)(y - x_0)\|_2}$$

$$\cancel{\leq |\lambda|\, \|x - x_0\|_2 + |1-\lambda|\, \|y - x_0\|_2}$$

$$\cancel{= \lambda \|x - x_0\|_2 + (1-\lambda) \|y - x_0\|_2}$$

$$\cancel{\overset{x, y \in E_3}{\leq} \lambda \|}$$

$$\|z - x_\lambda\|_2^2 = \|z\|_2^2 - 2\langle x_\lambda, z\rangle_2 + \|x_\lambda\|_2^2$$

$$= \lambda[\|z\|_2^2 - 2\langle x, z\rangle_2]$$

$$+ (1-\lambda)[\|z\|_2^2 - 2\langle y, z\rangle_2] + \|x_\lambda\|_2^2$$

$$= \lambda[\|z - x\|_2^2 - \|x\|_2^2]$$

$$+ (1-\lambda)[\|z - y\|_2^2 - \|y\|_2^2] + \|x_\lambda\|_2^2$$

$$\overset{x, y \in E_3}{\geq} \lambda[\|x_0 - x\|_2^2 - \|x\|_2^2]$$

$$+ (1-\lambda)[\|x_0 - y\|_2^2 - \|y\|_2^2] + \|x_\lambda\|_2^2$$

$$\geq \lambda[\|x_0\|_2^2 - 2\langle x, x_0\rangle]$$

$$+ (1-\lambda)[\|x_0\|_2^2 - 2\langle y, x_0\rangle] + \|x_\lambda\|_2^2$$

$$\geq \|x_0\|_2^2 - 2\langle x_\lambda, x_0\rangle + \|x_\lambda\|_2^2$$

$$= \|x_0 - x_\lambda\|_2^2$$

Thus, $\|z - x_\lambda\| \geq \|x_0 - x_\lambda\|$
$\forall z \in S$

$\Longrightarrow x_\lambda = \lambda x + (1-\lambda)y \in E_3$

$\Rightarrow$ $\boxed{E_3 \text{ is convex}}$

(4.) Let us denote by

$E_3(s) = \{x \mid \|x - s\|_2 \leq \|x - t\|_2 \ \forall t \in T\}, \forall s \in S$

$= \{x \mid \|x-s\|_2 \leq \text{dist}(x,T)\}$, $\forall s \in S$

we just take the infimum over T

According to (3.), $\forall s \in S$, $E_3(s)$ is cvx

$\Longrightarrow \bigcap_{s \in S} E_3(s)$ is cvx as well

but $\bigcap_{s \in S} E_3(s) = \{x \mid \|x-s\|_2 \leq \text{dist}(x,T), \forall s \in S\}$

$\overset{\inf_{s\in S}}{=} \{x \mid \text{dist}(x,S) \leq \text{dist}(x,T)\}$

$= E_4$

Thus, $\boxed{E_4 \text{ is convex}}$

(5.) Let us denote by $E_5 = \{x \mid x + S_2 \subseteq S_1\}$

Let $x, y \in E_5$ and $\lambda \in [0,1]$

$\Rightarrow \forall s \in S_2, \begin{cases} x + s \in S_1 \\ y + s \in S_1 \end{cases}$

$\xrightarrow{S_1 \text{ convex}} \lambda(x+s) + (1-\lambda)(y+s) \in S_1$

$\Rightarrow [\lambda x + (1-\lambda)y] + s \in S_1$

$\Rightarrow \lambda x + (1-\lambda)y \in E_5$

Hence, $\boxed{E_5 \text{ is convex}}$

(Ex 2)

(1.) $f: x \mapsto x_1 x_2$ is twice differentiable on $\mathbb{R}^2_{++}$ and we have $\forall x \in \mathbb{R}^2_{++}$:

$\nabla f(x) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$, $Hess(f)(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Let λ_1 and λ_2 be the eigenvalues of $Hess(f)(x)$

$\begin{cases} \lambda_1 + \lambda_2 = tr(Hess(f)(x)) = 0 \\ \lambda_1 \lambda_2 = \det(Hess(f)(x)) = -1 < 0 \end{cases}$

$\Rightarrow$ λ_1 and λ_2 have opposite signs and they are both non-zero

(we see easily $\lambda_1 = 1$ and $\lambda_2 = -1$ which are sol° to $\lambda^2 - 1$)

So, $\boxed{f \text{ is neither cvx nor concave}}$

- $f(2, \frac{1}{2}) = 1 \leq 1 = f(1,1)$.

$\nabla f(1,1)^T ((2,\frac{1}{2}) - (1,1)) = (1 \ \ 1) \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} = \frac{1}{2} > 0$

Since also we know $\mathbb{R}^2_{++}$ is cvx, so

$\boxed{f \text{ is not quasiconvex}}$

- As we mentionned $\mathbb{R}^2_{++}$ is cvx.

Let $x, y \in \mathbb{R}^2_{++}$ s.t $f(y) \geq f(x)$

i.e $y_1 y_2 \geq x_1 x_2$

$\nabla f(x)^T . (y - x) = (x_2 \ \ x_1) \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix}$

$= x_2 y_1 - x_1 x_2 + x_1 y_2 - x_1 x_2$

$= x_1 y_2 + x_2 y_1 - 2 x_1 x_2$

However, since $\begin{cases} x_1 x_2 > 0 \\ y_1 y_2 \geq x_1 x_2 \end{cases} \Rightarrow y_1 y_2 x_1 x_2 \geq (x_1 x_2)^2$

$\Rightarrow x_2 y_1 \geq \frac{(x_1 x_2)^2}{x_1 y_2}$

$\Rightarrow \nabla f(x)^T . (y - x) \geq x_1 y_2 + \frac{(x_1 x_2)^2}{x_1 y_2} - 2 x_1 x_2$

$= \sqrt{x_1 y_2}^2 + \left(\frac{x_1 x_2}{\sqrt{x_1 y_2}}\right)^2 - 2 \frac{x_1 x_2}{\sqrt{x_1 y_2}} \sqrt{x_1 y_2}$

$= \left(\sqrt{x_1 y_2} - \frac{x_1 x_2}{\sqrt{x_1 y_2}}\right)^2 \geq 0$

(Ex 2)

(1.) "Follow-up"

Hence, $\boxed{f \text{ is quasiconcave}}$

(2.) $f : x \mapsto \frac{1}{x_1 x_2}$ is twice differentiable on $\mathbb{R}^2_{++}$ and we have $\forall x \in \mathbb{R}^2_{++}$:

$$\nabla f(x) = \begin{pmatrix} \frac{-1}{x_1^2 x_2} \\ \frac{-1}{x_1 x_2^2} \end{pmatrix} = \frac{-1}{(x_1 x_2)^2} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$Hess(f)(x) = \begin{pmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{pmatrix}$$

$\to$ trace $= \frac{2}{x_1^3 x_2} + \frac{2}{x_1 x_2^3} \succ 0$

$\to$ Det $= \frac{4}{x_1^4 x_2^4} - \frac{1}{x_1^4 x_2^4} = \frac{3}{x_1^4 x_2^4} \succ 0$

$\Rightarrow Hess(f)(x) \succ 0$

$\Rightarrow \boxed{f \text{ is cvx and not concave}}$

$\Rightarrow \boxed{f \text{ is quasi convex}}$

~~Let $x, y \in \mathbb{R}^2_{++}$ s.t. $f(y) \geq f(x)$~~

~~i.e $\frac{1}{\ }$~~

$f(2, \frac{1}{2}) = 1 \geq 1 = f(1,1)$

$\nabla f(1,1)^T ((2, \frac{1}{2}) - (1,1)) = -(1 \ \ 1) \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} = \frac{-1}{2} < 0$

and Since $\mathbb{R}^2_{++}$ is cvx $\Rightarrow \boxed{f \text{ is not quasi concave}}$

(3.) $f : x \mapsto \frac{x_1}{x_2}$ is twice differentiable on $\mathbb{R}^2_{++}$ and we have $\forall x \in \mathbb{R}^2_{++}$:

$$\nabla f(x) = \begin{pmatrix} \frac{1}{x_2} \\ \frac{-x_1}{x_2^2} \end{pmatrix} = \frac{1}{x_2^2} \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$$

$$Hess(f)(x) = \begin{pmatrix} 0 & \frac{-1}{x_2^2} \\ \frac{-1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$$

trace $= \frac{2x_1}{x_2^3} > 0$

det $= \frac{-1}{x_2^4} < 0$

$\Rightarrow \lambda_1$ and λ_2 are non-zero and of opposite signs

$\Rightarrow \boxed{f \text{ is neither cvx nor concave}}$

Let $x, y \in \mathbb{R}^2_{++}$.

$$\nabla f(x)^T . (y - x) = \frac{1}{x_2^2} \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}^T \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix}$$

$$= \frac{1}{x_2^2} (x_2 y_1 - \cancel{x_1 x_2} - x_1 y_2 + \cancel{x_1 x_2})$$

$$= \frac{y_2}{x_2} \left(\frac{y_1}{y_2} - \frac{x_1}{x_2} \right)$$

$$= \frac{y_2}{x_2} (f(y) - f(x))$$

Since $\frac{y_2}{x_2} > 0$ and $\mathbb{R}^2_{++}$ is cvx

. if $f(y) \leq f(x) \Rightarrow \nabla f(x)^T (y - x) \leq 0$

. if $f(y) \geq f(x) \Rightarrow \nabla f(x)^T (y - x) \geq 0$

Thus, $\boxed{f \text{ is quasilinear}}$

(4.) $f : x \mapsto x_1^\alpha x_2^{1-\alpha}$ is twice differentiable on $\mathbb{R}^2_{++}$ and we have $\forall x \in \mathbb{R}^2_{++}$:

$$\nabla f(x) = \begin{pmatrix} \alpha x_1^{\alpha-1} x_2^{1-\alpha} \\ (1-\alpha) x_1^\alpha x_2^{-\alpha} \end{pmatrix}$$

$$Hess(f)(x) = \begin{pmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & -\alpha(1-\alpha) x_1^\alpha x_2^{-1-\alpha} \end{pmatrix}$$

$\to$ trace $= \underbrace{\alpha}_{\geq 0} \underbrace{(\alpha - 1)}_{\leq 0} \underbrace{[x_1^{\alpha-2} x_2^{1-\alpha} + x_1^\alpha x_2^{-1-\alpha}]}_{\geq 0} \leq 0$

$\to$ det $= [\alpha(1-\alpha)]^2 x_1^{2\alpha-2} x_2^{-2\alpha} - [\alpha(1-\alpha)]^2 x_1^{2\alpha-2} x_2^{-2\alpha}$

$= 0 \leq 0$

$\Rightarrow \lambda_1 = 0$ and $\lambda_2 =$ trace ≤ 0

$\lambda_2 = 0$ iff $\underline{\alpha = 0 \text{ or } 1}$

$\boxed{\text{In this case, } f \text{ is convex and concave}}$

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Hence $\boxed{f \text{ is quasilinear if } \alpha = 0 \text{ or } 1}$

<u>if $\alpha \neq 0$ and 1</u> $\Rightarrow \lambda_2 < 0$

$\Rightarrow \boxed{f \text{ is concave and not convex}}$

$\Rightarrow \boxed{f \text{ is quasiconcave}}$

- $f(1,1) = 1 \geq 1 = f(m, m^{\frac{\alpha}{\alpha-1}})$

$\nabla f(1,1)^T \cdot ((m, m^{\frac{\alpha}{\alpha-1}}) - (1,1))$

$= (\alpha \quad 1-\alpha) \begin{pmatrix} m-1 \\ m^{\frac{\alpha}{\alpha-1}} - 1 \end{pmatrix}$

$= \underbrace{\alpha(m-1)}_{\substack{\to +\infty \\ \alpha > 0}} + (1-\alpha)(\underbrace{m^{\frac{\alpha}{\alpha-1}}}_{\substack{\to 0 \\ b/c\ \frac{\alpha}{\alpha-1} < 0}} - 1) \xrightarrow[m \to +\infty]{} +\infty$

So $\exists m ;\ \nabla f(1,1)^T ((m, m^{\frac{\alpha}{\alpha-1}}) - (1,1)) > 0$

$\Rightarrow \boxed{f \text{ is not quasiconvex}}$

(Ex 3)

(1.) $\langle \nabla f(X), H \rangle = \lim_{t \to 0} \frac{f(X+tH) - f(X)}{t}$

$= \lim_{t \to 0} tr\left(\frac{(X+tH)^{-1} - X^{-1}}{t} \right)$

$= \cancel{\lim_{t \to 0} tr \frac{X^{-1}}{t} (I + tHX^{-1})}$

$= \lim_{t \to 0} tr\left(\frac{X^{-1}}{t} [(I + tHX^{-1})^{-1} - I] \right)$

$= \lim_{t \to 0} tr\left(\frac{X^{-1}}{t} \sum_{m=1}^{+\infty} (-tHX^{-1})^m \right)$

$= \lim_{t \to 0} tr\left(-X^{-1}HX^{-1} + \sum_{m=2}^{+\infty} (-HX^{-1})^m t^{m-1} \right)$

$= tr(-X^{-1}HX^{-1})$

$= tr(-X^{-2}H) = \langle -X^{-2}, H \rangle$

$\Rightarrow \forall X, H \in S_{++}^m,\ \langle \nabla f(X), H \rangle = \langle -X^{-2}, H \rangle$

$\Rightarrow \forall X \in S_{++}^m,\ \nabla f(X) = -X^{-2}$

Let $X, H \in S_{++}^m$

$Hess(f)(X)H = \lim_{t \to 0} \frac{-1}{t} \left[(X+tH)^{-2} - X^{-2} \right]$

$= \lim_{t \to 0} \frac{-1}{t} \left[X^{-1}(I + tHX^{-1})^{-1}(I + tX^{-1}H)^{-1}X^{-1} - X^{-2} \right]$

$= \lim_{t \to 0} -\frac{X^{-1}}{t} \left[(I + tX^{-1}H)^{-1} \sum_{m=0}^{+\infty} (-tHX^{-1})^m - I \right] X^{-1}$

$= \lim_{t \to 0} -X^{-1} \left[\frac{(I + tX^{-1}H)^{-1} - I}{t} + (I + tX^{-1}H)^{-1} \sum_{m=1}^{+\infty} (-1)^m (HX^{-1})^m t^{m-1} \right] X^{-1}$

$= \cancel{-X^{-1}(-HX^{-1})X^{-1}}$

$= -X^{-1}(-X^{-1}H)X^{-1}$

$= X^{-1}X^{-1}HX^{-1}$

$\Rightarrow \forall H, X \in S_{++}^m$

$H \cdot Hess(f)(X) \cdot H = HX^{-1}X^{-1}HX^{-1}$

$=$

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(Ex 3)

(2.) Let us consider

$g(X,y,z) = 2y^Tz - z^TXz$; $y,z \in \mathbb{R}^n$, $X \in S^n_{++}$

$\sup_{z\in\mathbb{R}^n} g(X,y,z)$ is an unconstrained optimization pb having a quadratic fuction with $Hess_z(g)(X,y,z) = -2X \in S^n_{--}$

$\Rightarrow$ It has a ! supremum obtained for z verifying $\nabla_z g(X,y,z) = 0$

$$2y - 2Xz = 0$$

$$\Rightarrow z = X^{-1}y$$

Thus $\sup_{z\in\mathbb{R}^n} g(X,y,z) = 2y^TX^{-1}y - y^TX^{-1}XX^{-1}y$
$= y^TX^{-1}y = f(X,y)$

But, $g(\cdot,\cdot,z)$ is a linear function

$\Rightarrow$ convex $\forall z \in \mathbb{R}^n$

Thus, the sup over $z \in \mathbb{R}^n$ is convex as well $\Rightarrow$

$$\boxed{\begin{array}{ll} f: S^m_{++}\times\mathbb{R}^n \to \mathbb{R} & \text{is convex} \\ (X,y) \mapsto y^TX^{-1}y & \end{array}}$$

(3.) Let us consider $g(Z,X) = tr(Z^TX)$, $Z \in \mathbb{R}^{m\times m}$, $X \in S^m_{++}$

$X \in S^m_{++} \Rightarrow \exists P \in O(m)$; $X = PDP^T$ where $D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_m \end{pmatrix}$

$g(Z,X) = tr(Z^TPDP^T) = tr(\underbrace{P^TZ^TP}_{=Y^T}D)$
$= tr(Y^TD)$
$= \sum_{i=1}^m y_{ii}\lambda_i$

Since $Z \sim Y = P^TZP \Rightarrow f(Z) = f(Y)$

For Z s.t $f(Z) = 1 \Rightarrow f(Y) = 1$

in particular $|y_{ii}| \leq 1$

$\Rightarrow g(Z,X) \leq \sum_{i=1}^m |\lambda_i| = f(X)$

However for $Z = diag(sgn(\lambda_i))_{1\leq i\leq m}$

we have $tr(Z^TX) = f(X)$

So $f(X) = \sup_{f(Z)=1} g(Z,X)$

Since $\forall Z \in \mathbb{R}^{m\times m}$, $g(Z,\cdot)$ is linear and thus cvx

$\Rightarrow \boxed{f \text{ is cvx}}$

(Ex 4)

(1.) . $\forall \lambda \geq 0$, if $x \in K_{m+}$

$\Rightarrow x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$

$\Rightarrow \lambda x_1 \geq \lambda x_2 \geq \cdots \geq \lambda x_n \geq 0$

$\Rightarrow \lambda x \in K_{m+}$

. $0 \in K_{m+}$

. Closed ?

Let $x^{(k)} \in K_{m+}$ $\forall k \in \mathbb{N}$, s.t

$x^{(k)} \longrightarrow x$

$x^{(k)} \in K_{m+} \Rightarrow x_1^{(k)} \geq \cdots \geq x_m^{(k)} \geq 0$

$\xrightarrow{\lim} x_1 \geq \cdots \geq x_m \geq 0$

$\Rightarrow x \in K_{m+}$

$\Rightarrow K_{m+}$ is closed

. $\mathring{K}_{m+} \neq \emptyset$?

$x_0 = (m, m-1, \ldots, 1) \in K_{m+}$

Let $z \in B(x_0, \frac{1}{2})$

$\Rightarrow \|x_0 - z\|_2^2 < \frac{1}{4}$

$$\Rightarrow \sum_{i=1}^{m} (m-i+1-z_i)^2 < \frac{1}{4}$$

$$\Rightarrow \forall\, 1\le i\le m,\quad |z_i-(m-i+1)| < \frac{1}{2}$$

$$\Rightarrow \forall\, 1\le i\le m,\ m-i+\frac{1}{2} < z_i < m-i+\frac{3}{2}$$

Let $i \in \{0,\dots,m-1\}$

$$z_i > m-i+\frac{1}{2} = m-(i+1)+\frac{3}{2} > z_{i+1}$$

$$\Rightarrow z \in K_{m+} \quad \forall z \in B(x_0, \tfrac{1}{2})$$

$$\Rightarrow B(x_0,\tfrac{1}{2}) \subset K_{m+} \Rightarrow \mathring{K}_{m+} \neq \emptyset$$

. K_{m+} is pointed?

Let $L := \{x_0 + td,\ t\in\mathbb{R}\}$ a line

where $d\in\mathbb{R}^m$ has at least one coefficient $d_i \neq 0$

and $x_0 \in \mathbb{R}^m$

$$(x_0)_i + td_i \xrightarrow[\substack{t\to-\infty \text{ if } d_i>0\\ t\to+\infty \text{ if } d_i<0}]{-\infty} \Rightarrow \exists t\in\mathbb{R};\ (x_0)_i+td_i=-1<0$$

$$\Longrightarrow L \not\subset K_{m+} \Rightarrow K_{m+} \text{ contains no line.}$$

Thus, $\boxed{K_{m+} \text{ is pointed}}$

(2.) $K_{m+}^* = \{y \mid y^T x \geq 0 \quad \forall x \in K_{m+}\}$

Let $y \in K^*_{m+}$

$x = (1\ 0\dots 0) \in K^*_{m+} \Rightarrow y_1 \geq 0$

$x = (1\ 1\ 0\dots 0) \in K^*_{m+} \Rightarrow y_2 \geq -y_1$

$\vdots$

$x = (\underbrace{1\dots 1}_{i \text{ times}}\ 0\dots 0) \in K^*_{m+} \Rightarrow y_i \geq -\sum_{j=1}^{i-1} y_j \quad \forall\, 2\le i\le m$

$\Rightarrow \{y \mid y_1\geq 0,\ y_i \geq -\sum_{j=1}^{i-1} y_j \ \forall\, 2\le i\le m\} \supset K^*_{m+}$

. If $y \in \{y \mid y_1 \geq 0,\ y_i \geq -\sum_{j=1}^{i-1} y_j\ \forall\, 2\le i\le m\}$:

Let $x \in K_{m+}$.

$$y^T x = \sum_{i=1}^{m} y_i x_i$$

$$\geq \sum_{i=1}^{m-1} y_i x_i + \left(\sum_{i=1}^{m-1} -y_i\right) x_m$$

$$= \sum_{i=1}^{m-1} y_i (x_i - x_m)$$

$$\geq \sum_{i=1}^{m-2} y_i (x_i - x_m) + \left(\sum_{i=1}^{m-2} -y_i\right)(x_{m-1} - x_m)$$

$$= \sum_{i=1}^{m-2} y_i (x_i - x_{m-1})$$

$\vdots$

$$\geq \underbrace{y_1}_{\geq 0}\ \underbrace{(x_1 - x_2)}_{\geq 0} \geq 0$$

b/c $x \in K_{m+}$

$$\Rightarrow y \in K^*_{m+}$$

Thus, $\boxed{K^*_{m+} = \{y \mid y_1 \geq 0,\ \forall\, 2\le i\le m,\ y_i \geq \sum_{j=1}^{i-1} -y_j\}}$

(Ex 5)

(1.) Let $y \in \mathbb{R}^m$

$$f^*(y) = \sup_{x\in\mathbb{R}^m} \{y^T x - \max_{1\le i\le n}(x_i)\}$$

if $\exists i;\ y_i > 1$

$\Rightarrow \forall k,\ x^{(k)} = (0\dots 0\ \overset{i^{th}}{k}\ 0\dots 0)$

$y^T x^{(k)} - \max_{1\le i\le m}((x^{(k)})_i) = (y_i - 1)k \xrightarrow[k\to+\infty]{+\infty}$

$\Rightarrow f^*(y) = +\infty$

if $\exists i;\ y_i < 0$

$\Rightarrow \forall k,\ x^{(k)} = (0\dots 0\ \overset{i^{th}}{-k}\ 0\dots 0)$

$y^T x^{(k)} - \max_{1\le i\le n}((x^{(k)})_i) = -ky_i \xrightarrow[k\to+\infty]{+\infty}$

$\Rightarrow f^*(y) = +\infty$

if $\forall i,\ 0 \le y_i \le 1$

$\Rightarrow \forall i,\ y_i x_i \le \max(x)\, y_i$