(HW 8)

(Ex 3.1)

(i) $\forall \mu,\ \mathbb{E}_{\upsilon}(\upsilon|\mu) = \mu$

(ii) $\upsilon_x \perp\!\!\!\perp \upsilon_y \,|\, \mu \quad \forall x \neq y$

(1.) $\mathbb{E}_{\upsilon,\mu}\{\|\mathcal{F}^{\lambda}(\upsilon)-\mu\|^2\}$

$$= \mathbb{E}_{\upsilon,\mu}\{\|\lambda(\mathcal{F}(\upsilon)-\mu)+(1-\lambda)(\upsilon-\mu)\|^2\}$$

$$= \lambda^2 \mathbb{E}_{\upsilon,\mu}\{\|\mathcal{F}(\upsilon)-\mu\|^2\} + (1-\lambda)^2 \mathbb{E}_{\upsilon,\mu}\{\|\upsilon-\mu\|^2\}$$

$$+ 2\lambda(1-\lambda)\mathbb{E}_{\upsilon,\mu}\{\langle \mathcal{F}(\upsilon)-\mu, \upsilon-\mu\rangle\}$$

(2.) Inspired by the proof of proposition 3.4, one can write:

$$\mathbb{E}_{\upsilon,\mu}\{\langle \mathcal{F}(\upsilon)-\mu, \upsilon-\mu\rangle\}$$

$$= \mathbb{E}_{\mu}\mathbb{E}_{\upsilon}\{\langle \mathcal{F}(\upsilon)-\mu, \upsilon-\mu\rangle|\mu\}$$

$$= \mathbb{E}_{\mu}\mathbb{E}_{\upsilon}\{\sum_{J\in\mathcal{J}}\langle \mathcal{F}(\upsilon)_J-\mu_J, \upsilon_J-\mu_J\rangle|\mu\}$$

$\hookrightarrow$ b/c it's a partition

We focus on one set of pixels $J \in \mathcal{J}$:

$$\mathbb{E}_{\upsilon}\{\langle \mathcal{F}(\upsilon)_J-\mu_J, \upsilon_J-\mu_J\rangle|\mu\}$$

$$= \mathbb{E}_{\upsilon_J,\upsilon_{J^c}}\{\langle \mathcal{F}(\upsilon)_J-\mu_J, \upsilon_J-\mu_J\rangle|\mu\}$$

$$= \mathbb{E}_{\upsilon_{J^c}}\{\mathbb{E}_{\upsilon_J}\{\langle \underbrace{\mathcal{F}(\upsilon)_J-\mu_J}_{\perp\!\!\!\perp\, \upsilon_J \text{ since } \mathcal{F} \text{ is } \mathcal{J}\text{-invariant}}, \upsilon_J-\mu_J\rangle|\upsilon_{J^c},\mu\}|\mu\}$$

$$= \mathbb{E}_{\upsilon_J^c}\{\langle \mathcal{F}(\upsilon)_J-\mu_J, \underbrace{\mathbb{E}_{\upsilon_J}(\upsilon_J|\upsilon_{J^c},\mu)}-\mu_J\rangle|\mu\}$$

$= \mathbb{E}_{\upsilon_J}(\upsilon_J|\mu)$ b/c $\upsilon_J \perp\!\!\!\perp \upsilon_{J^c}|\mu$ (ii)

$= \mu_J$ b/c (i)

$= 0$

Thus, $\mathbb{E}_{\upsilon,\mu}\{\langle \mathcal{F}(\upsilon)-\mu, \upsilon-\mu\rangle\} = 0$

(3.) Let $f : \lambda \mapsto \mathbb{E}_{\upsilon,\mu}\{\|\mathcal{F}^{\lambda}(\upsilon)-\mu\|^2\}$

f is convex and quadratic

$(f''(\lambda) = 2\,\mathbb{E}_{\upsilon,\mu}\{\|\mathcal{F}(\upsilon)-\mu\|^2 + \|\upsilon-\mu\|^2\} \geq 0)$

$\Rightarrow f$ has a global minimum obtained via $f'(\lambda) = 0$

$$\Rightarrow 2\lambda\,\mathbb{E}_{\upsilon,\mu}\{\|\mathcal{F}(\upsilon)-\mu\|^2\} - 2(1-\lambda)\,\mathbb{E}\|\upsilon-\mu\|^2 = 0$$

$$\Rightarrow \lambda\left(\mathbb{E}_{\upsilon,\mu}\{\|\mathcal{F}(\upsilon)-\mu\|^2\} + \mathbb{E}_{\upsilon,\mu}\{\|\upsilon-\mu\|^2\}\right) = \mathbb{E}_{\upsilon,\mu}\{\|\upsilon-\mu\|^2\}$$

$$\Rightarrow \lambda^* = \frac{\mathbb{E}_{\upsilon,\mu}\{\|\upsilon-\mu\|^2\}}{\mathbb{E}_{\upsilon,\mu}\{\|\mathcal{F}(\upsilon)-\mu\|^2\} + \mathbb{E}_{\upsilon,\mu}\{\|\upsilon-\mu\|^2\}}$$

However, $\mathbb{E}_{\upsilon,\mu}\{\|\upsilon-\mu\|^2\} = \mathbb{E}_{\mu}\{\mathbb{E}_{\upsilon}\{\|\upsilon-\mathbb{E}_{\upsilon}(\upsilon|\mu)\|^2|\mu\}\}$

$$= \mathbb{E}_{\mu}\{\mathbb{V}(\upsilon|\mu)\}$$

$$\Rightarrow \lambda^* = \frac{\mathbb{E}_{\mu}\{\mathbb{V}(\upsilon|\mu)\}}{\mathbb{E}_{\upsilon,\mu}\{\|\mathcal{F}(\upsilon)-\mu\|^2\} + \mathbb{E}_{\mu}(\mathbb{V}(\upsilon|\mu))}$$

(4.) Using proposition 3.4, one can write:

$$\lambda^* = \frac{\mathbb{E}_{\mu}\{\mathbb{V}(\upsilon|\mu)\}}{\mathbb{E}_{\upsilon}\{\|\mathcal{F}(\upsilon)-\upsilon\|^2\}}$$

$$= \frac{\mathbb{E}_{\mu}\{d\sigma^2\}}{R_{N2S}(\mathcal{F})}$$

$$\Rightarrow \lambda^* = \frac{d\sigma^2}{R_{N2S}(\mathcal{F})}$$

(Ex 3.2)

$$\mathbb{E}_v\{\|\hat{u}(v)-u\|^2 \mid u\}$$

$$= \mathbb{E}_v\{\|\hat{u}(v)-\mathbb{E}_v(\hat{u}(v)|u) + \mathbb{E}_v(\hat{u}(v)|u) - u\|^2 \mid u\}$$

$$= \mathbb{E}_v\{\|\hat{u}(v)-\mathbb{E}_v(\hat{u}(v)|u)\|^2 \mid u\}$$

$$+ \mathbb{E}_v\{\underbrace{\|\mathbb{E}_v(\hat{u}(v)|u) - u\|^2}_{\text{doesn't depend on } v} \mid u\}$$

$$- 2\mathbb{E}_v\{\langle \hat{u}(v)-\mathbb{E}_v(\hat{u}(v)|u), \mathbb{E}_v(\hat{u}(v)|u)-u\rangle \mid u\}$$

$$= \mathbb{E}_v\{\|\hat{u}(v)-\mathbb{E}_v(\hat{u}(v)|u)\|^2 \mid u\}$$

$$+ \|\mathbb{E}_v(\hat{u}(v)|u) - u\|^2$$

$$+ 2\langle \underbrace{\mathbb{E}_v\{\hat{u}(v)-\mathbb{E}_v(\hat{u}(v)|u)|u\}}_{\substack{= \mathbb{E}_v(\hat{u}(v)|u) - \mathbb{E}_v(\hat{u}(v)|u) \\ = 0}}, \mathbb{E}_v(\hat{u}(v)|u) - u\rangle$$

Hence :

$$\mathbb{E}_v(\|\hat{u}(v)-u\|^2 \mid u)$$

$$= \|\mathbb{E}_v\{\hat{u}(v)|u\} - u\|^2$$

$$+ \mathbb{E}_v\{\|\hat{u}(v)-\mathbb{E}_v(\hat{u}(v)|u)\|^2 \mid u\}$$