(HW7)

(Ex 2.1)

We minimize with respect to u

$$\Rightarrow GD: \; u^{t+1} = u^t - \eta \nabla_u E_\theta^{FoE}(u^t, v)$$

$$\|u^t - v\|^2 = \langle u^t, u^t \rangle - 2\langle u^t, v\rangle + \langle v, v\rangle$$

$$\Rightarrow \nabla_u \|u^t - v\|^2 = 2(u^t - v)$$

$$\nabla_u \sum_{x\in\Omega} \sum_{i=1}^{N} \phi_i(k_i * u^t(x))$$

$$= \sum_{x\in\Omega} \sum_{i=1}^{N} \nabla_u (k_i * u^t(x))\, \phi_i'(k_i * u^t(x))$$

$$k_i * u^t(x) = \sum_{j=1}^{d} u^t(x-j)\, k_i(j)$$

$$\nabla_u (k_i * u^t(x)) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ k_i(d) \\ \vdots \\ k_i(1) \\ 0 \\ \vdots \end{pmatrix} \in \mathbb{R}^{\overset{\text{size of } u}{WH}}$$

$$\Rightarrow \nabla_u E_\theta^{FoE}(u^t, v) = \frac{1}{\sigma^2}(u^t - v) + \sum_{i=1}^{N} \sum_{x\in\Omega} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ k_i(d) \\ \vdots \\ k_i(1) \\ 0 \\ \vdots \end{pmatrix} \phi_i'(k_i * u^t(x))$$

$$= \frac{1}{\sigma^2}(u^t - v) + \sum_{i=1}^{N} \underbrace{\bar{k}_i}_{\pi\text{ rotation of } k_i} * \phi_i'(k_i * u^t)$$

since $\bar{k}_i = \begin{pmatrix} 0 \\ \vdots \\ k_i(d) \\ \vdots \\ k_i(1) \\ \vdots \end{pmatrix}$

Thus:

$$u^{t+1} = u^t - \eta \left[\frac{1}{\sigma^2}(u^t - v) + \sum_{i=1}^{N} \bar{k}_i * \phi_i'(k_i * u^t) \right]$$

(Ex 2.2)

$$\cancel{\max_\theta \mathcal{L}_\alpha(\theta) = \max_\theta \mathbb{E}[\log(p_\theta(u))]}$$

$$\cancel{= \max_\theta \int \log(p_\theta(u))\, p(u)\, du}$$

$$\min_\theta KL(p(u) \| p_\theta(u))$$

$$\Leftrightarrow \min_\theta \int \log\left(\frac{p(u)}{p_\theta(u)}\right) p(u)\, du$$

$$\Leftrightarrow \min_\theta \underbrace{\int \log(p(u))\, p(u)\, du}_{\text{ind of } \theta} - \int \log(p_\theta(u))\, p(u)\, du$$

$$\Leftrightarrow \min_\theta -\mathcal{L}_\alpha(\theta)$$

$$\Leftrightarrow \max_\theta \mathcal{L}_\alpha(\theta)$$