(Ex 4.1.)

$\bullet \mathbb{E}[X] = \sum_{n=0}^{+\infty} n \mathbb{P}[X=n]$

$= \sum_{n=0}^{+\infty} n \frac{\lambda^n e^{-\lambda}}{n!}$

$= e^{-\lambda} \sum_{n=1}^{+\infty} n \frac{\lambda^n}{n!}$

$= e^{-\lambda} \sum_{n=1}^{+\infty} \frac{\lambda^n}{(n-1)!}$

$= e^{-\lambda} \sum_{n=0}^{+\infty} \frac{\lambda^{n+1}}{n!}$

$= \lambda e^{-\lambda} \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$

$\Rightarrow \boxed{\mathbb{E}[X] = \lambda}$

$\bullet \mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\mathbb{E}[X^2] = \sum_{n=0}^{+\infty} n^2 \mathbb{P}(X=n]$

$= \sum_{n=1}^{+\infty} [n(n-1)+n] \frac{\lambda^n e^{-\lambda}}{n!}$

$= \sum_{n=1}^{+\infty} n(n-1) \frac{\lambda^n e^{-\lambda}}{n!} + \underbrace{\sum_{n=1}^{+\infty} n \frac{\lambda^n e^{-\lambda}}{n!}}_{= \mathbb{E}[X] = \lambda}$

$= \sum_{n=2}^{+\infty} n(n-1) \frac{\lambda^n e^{-\lambda}}{n!} + \lambda$

$= e^{-\lambda} \sum_{n=2}^{+\infty} \frac{\lambda^n}{(n-2)!} + \lambda$

$= e^{-\lambda} \sum_{n=0}^{+\infty} \frac{\lambda^{n+2}}{n!} + \lambda$

$= \lambda^2 e^{-\lambda} \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} + \lambda$

$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda = \lambda^2 + \lambda$

$\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda \Rightarrow \boxed{\mathbb{V}[X] = \lambda}$

(Ex 4.2.)

Let's prove by induction:

$(\mathcal{P}_m): \left(\begin{array}{l} \forall m \in \mathbb{N}^*, \text{ if } (X_i)_{1 \le i \le m} \text{ are indep} \\ \text{rand var s.t. } X_i \sim \text{Poisson}(\lambda_i) \\ \Rightarrow \sum_{i=1}^{m} X_i \sim \text{Poisson}\left(\sum_{i=1}^{m} \lambda_i\right) \end{array} \right)$

$\underline{m=1}$: $\sum_{i=1}^{1} X_i = X_1 \sim \text{Poisson}(\lambda_1) \equiv \text{Poisson}\left(\sum_{i=1}^{1} \lambda_i\right)$

Trivial

Let $m \in \mathbb{N}^*$ s.t. $(\mathcal{P}_m)$ holds:

Let $(X_i)_{1 \le i \le m+1}$ be indep rand var s.t.

$X_i \sim \text{Poisson}(\lambda_i)$.

By $(\mathcal{P}_m)$, we have $\sum_{i=1}^{m} X_i \sim \text{Poisson}\left(\sum_{i=1}^{m} \lambda_i\right)$

Let $k \in \mathbb{N}$

$\mathbb{P}\left(\sum_{i=1}^{m+1} X_i = k\right) = \sum_{j=0}^{k} \mathbb{P}\left(\sum_{i=1}^{m+1} X_i = k, X_{m+1} = j\right)$

$= \sum_{j=0}^{k} \mathbb{P}\left(\sum_{i=1}^{m} X_i = k-j, X_{m+1} = j\right)$

$(X_i)_{1 \le i \le m+1}$ indp $\Rightarrow \sum_{i=1}^{m} X_i \perp\!\!\!\perp X_{m+1}$

$= \sum_{j=0}^{k} \mathbb{P}\left(\sum_{i=1}^{m} X_i = k-j\right) \mathbb{P}(X_{m+1} = j)$

$= \sum_{j=0}^{k} \frac{(\lambda_1 + \dots + \lambda_m)^{k-j} e^{-(\lambda_1 + \dots + \lambda_m)}}{(k-j)!} \frac{\lambda_{m+1}^j e^{-\lambda_{m+1}}}{j!}$

$= \frac{e^{-(\lambda_1 + \dots + \lambda_{m+1})}}{k!} \sum_{j=0}^{k} C_k^j (\lambda_1 + \dots + \lambda_m)^{k-j} \lambda_{m+1}^j$

$= \frac{(\lambda_1 + \dots + \lambda_{m+1})^k}{k!} e^{-(\lambda_1 + \dots + \lambda_{m+1})}$

$\Longrightarrow \sum_{i=1}^{m+1} X_i \sim \text{Poisson}\left(\sum_{i=1}^{m+1} \lambda_i\right) \Rightarrow (\mathcal{P}_{m+1})$ holds

Thus, by induction,

$\boxed{\forall m \in \mathbb{N}^*, \ (\mathcal{P}_m) \text{ holds}}$

Scanné avec CamScanner

(Ex 4.3.) Indeed, the denoising procedure with the std var stabilizing transformation (VST) procedure follows 3 steps:

(1.) Apply VST to approximate homoscedasticity:

Given the Poisson model with the Gaussian approximation, we can write the noisy image $\tilde{u}(i)$ as: $\tilde{u}(i) \simeq u(i) + \sqrt{u(i)}\, n(i)$

$\forall i$ a pixel.

Then applying the VST: $f : x \longmapsto 2c\sqrt{x}$ for a given cst $c \neq 0$ (usually $c \simeq 1$), one can write the 1^{st} order Taylor approximation as follows: $2c\sqrt{\tilde{u}(i)} \simeq 2c\sqrt{u(i)} + \frac{c}{\sqrt{u(i)}}\sqrt{u(i)}\, n(i)$

$$\Longrightarrow \boxed{2c\sqrt{\tilde{u}(i)} \simeq 2c\sqrt{u(i)} + \underline{c\, n(i)}}$$

$\longrightarrow$ no longer signal dependent

(2.) Now, to denoise the transformed data $2c\sqrt{u(i)}$, we write it as follows:

$$\boxed{2c\sqrt{u(i)} \simeq 2c\sqrt{\tilde{u}(i)} - c\, n(i)}$$

(3.) Now, we apply the inverse VST: $f^{-1} : x \longmapsto \frac{x^2}{4c^2}$, we get:

$$u(i) \simeq \frac{\left[2c\sqrt{\tilde{u}(i)} - c\, n(i)\right]^2}{4c^2}$$

$$\Longrightarrow \boxed{u(i) \simeq \left[\sqrt{\tilde{u}(i)} - \tfrac{1}{2} n(i)\right]^2}$$

(Ex 4.5.)

In the orthonormal basis $\mathcal{B} = (G_i)_{1 \leq i \leq M}$ of $\mathbb{R}^M$, one can write:

$$u = \left((\langle u, G_i\rangle)_{1 \leq i \leq M}\right)^T = (u_1, \dots, u_M)^T$$

$$N = \left((\langle N, G_i\rangle)_{1 \leq i \leq M}\right)^T = (n_1, \dots, n_M)^T$$

$$\tilde{u} = (u_1 + n_1, \dots, u_M + n_M)^T$$

For any given diagonal operator D we have: $D\tilde{u} = (a(1)(u_1 + n_1), \dots, a(M)(u_M + n_M))^T$

$$\Rightarrow \|u - D\tilde{u}\|^2 = \sum_{i=1}^{M} \left[u_i - a(i)(u_i + n_i)\right]^2$$

$$= \sum_{i=1}^{M} \left[(1 - a(i))u_i - a(i) n_i\right]^2$$

$$= \sum_{i=1}^{M} (1 - a(i))^2 u_i^2 + a(i)^2 n_i^2 + 2(1 - a(i))a(i) u_i n_i$$

$$\Rightarrow \mathbb{E}(\|u - D\tilde{u}\|^2) = \sum_{i=1}^{M} (1 - a(i))^2 u_i^2 + a(i)^2 \sigma^2$$

$$= \sum_{i=1}^{M} f_i(a(i))$$

where $f_i : x \mapsto (1-x)^2 u_i^2 + x^2 \sigma^2$

$\forall x \in \mathbb{R}$, $f_i'(x) = 2(x-1)u_i^2 + 2x\sigma^2$

$f_i''(x) = 2u_i^2 + 2\sigma^2 > 0$

$\Longrightarrow \forall\, 1 \leq i \leq M$, f_i is strongly cvx

and since $\lim_{x \to \pm\infty} f_i(x) = +\infty$, then the minimum of f_i $\exists$ and is $!$.

Since the coefficients $a(i)$ are independent we can write

$$\min_{a(i) \in \mathbb{R}} \sum_{i=1}^{M} f_i(a(i)) = \sum_{i=1}^{M} \min_{a(i) \in \mathbb{R}} f_i(a(i))$$

Each $a(i)$ is characterized by: $f_i'(a(i)) = 0$

$$\Longrightarrow 2a(i)(u_i^2 + \sigma^2) - 2u_i^2 = 0$$

$$\Longrightarrow \boxed{\forall\, 1 \leq i \leq M, \quad a(i) = \frac{\langle u, G_i\rangle^2}{\langle u, G_i\rangle^2 + \sigma^2}}$$

For the corresponding operator D_{inf} one can write by the existence and uniqueness of the minimum:

$$\boxed{D_{inf} = \underset{D}{\operatorname{argmin}}\, \mathbb{E}\left[\|u - D\tilde{u}\|^2\right]}$$

Furthermore:

$$\mathbb{E}\left[\|u - D_{inf}\tilde{u}\|^2\right]$$

$$= \sum_{i=1}^{M} \left(1 - \frac{u_i^2}{u_i^2 + \sigma^2}\right)^2 u_i^2 + \frac{u_i^4}{(u_i^2 + \sigma^2)^2}\sigma^2$$

(Ex 4.5.) "Follow-up"

$$\mathbb{E}[\|\mu - D_{inf}\tilde{\mu}\|^2] = \sum_{i=1}^{M} \frac{\sigma^4\mu_i^2 + \mu_i^4\sigma^2}{(\mu_i^2+\sigma^2)^2}$$

$$= \sum_{i=1}^{M} \frac{\mu_i^2\sigma^2\cancel{(\sigma^2+\mu_i^2)}}{(\mu_i^2+\sigma^2)^{\cancel{2}}}$$

$$\Rightarrow \boxed{\mathbb{E}[\|\mu - D_{inf}\tilde{\mu}\|^2] = \sum_{i=1}^{M} \frac{\langle \mu, G_i\rangle^2\sigma^2}{\langle \mu, G_i\rangle^2+\sigma^2}}$$

(Ex 4.6.)

We use once more the same notation as in (Ex 4.5). Hence, as we have proved there, one can write:

$$\mathbb{E}(\|\mu - D_{inf}\tilde{\mu}\|^2) = \sum_{i=1}^{M} (1-a(i))^2\mu_i^2 + a(i)^2\sigma^2$$

Let $1 \le i \le M$.

- $\mu_i^2 \ge c\sigma^2$: $a(i)=1 \Rightarrow (1-a(i))^2\mu_i^2 + a(i)^2\sigma^2 = \sigma^2$

  Since $c>1 \Rightarrow \sigma^2 \le c\sigma^2 = \min(\mu_i^2, c\sigma^2)$

- $\mu_i^2 < c\sigma^2$: $a(i)=0 \Rightarrow (1-a(i))^2\mu_i^2 + a(i)^2\sigma^2 = \mu_i^2$

  Since $\mu_i^2 < c\sigma^2 \Rightarrow \min(\mu_i^2, c\sigma^2) = \mu_i^2 \ge \mu_i^2$

Thus, $\forall 1 \le i \le M$, $(1-a(i))^2\mu_i^2 + a(i)^2\sigma^2 \le \min(\mu_i^2, \sigma^2)$

$$\xrightarrow{\sum_{i=1}^{M}} \boxed{\mathbb{E}[\|\mu - D_{inf}\tilde{\mu}\|^2] \le \sum_{i=1}^{M} \min(\langle \mu, G_i\rangle^2, c\sigma^2)}$$

For the case $\mu_i^2 < c\sigma^2$ we always have the equality

$$(1-a(i))^2\mu_i^2 + a(i)^2\sigma^2 = \mu_i^2 = \min(\mu_i^2, c\sigma^2)$$

However, if $c=1$, we have as well the second equality ~~$(1-a(i))^2\mu_i^2$~~ in the case $\mu_i^2 \ge c\sigma^2 = \sigma^2$:

$$(1-a(i))^2\mu_i^2 + a(i)^2\sigma^2 = \sigma^2 = \min(\mu_i^2, \sigma^2)$$

Thus, the previous inequality becomes indeed an equality for $c=1$

(Ex 4.7.)

Let's denote by A and B the matrix of respectively DCT and IDCT.

Thus by (4.12) and (4.13), we have:

$$A_{kj} = 2\alpha_k \cos\left(\pi\left(j+\tfrac{1}{2}\right)\tfrac{k}{N}\right), \quad 0 \le k,j \le N-1$$

$$B_{jk} = \begin{cases} \beta_0 & \text{if } k=0 \\ 2\beta_k\cos\left(\pi\left(j+\tfrac{1}{2}\right)\tfrac{k}{N}\right) & \text{if } k \neq 0 \end{cases}, \quad 0 \le j \le N-1$$

$$= \begin{cases} 2\alpha_0 & \text{if } k=0 \\ 2\alpha_k\cos\left(\pi\left(j+\tfrac{1}{2}\right)\tfrac{k}{N}\right) & \text{if } k \neq 0 \end{cases}, \quad 0 \le j \le N-1$$

$$= A_{jk}$$

Thus $B = A^T$

Hence, if we prove that A is an isometry we have indeed B is an isometry also and it's its inverse i.e $B = A^{-1} = A^T$

To prove that, we show that the row vectors of A form an orthonormal basis of $\mathbb{R}^N$.

$$A = \begin{pmatrix} \varphi_0 \\ \vdots \\ \varphi_{N-1} \end{pmatrix} \quad \text{where } \varphi_l = (A_{l0}, \dots, A_{l,N-1})$$

Let $0 \le l,k \le N-1$

$$\langle \varphi_k, \varphi_l \rangle = \sum_{j=0}^{N-1} A_{kj}A_{lj}$$

$$= 4\alpha_k\alpha_l \sum_{j=0}^{N-1} \cos\left(\pi\left(j+\tfrac{1}{2}\right)\tfrac{k}{N}\right)\cos\left(\pi\left(j+\tfrac{1}{2}\right)\tfrac{l}{N}\right)$$

$$= 2\alpha_k\alpha_l \sum_{j=0}^{N-1} \cos\left(\tfrac{\pi}{N}\left(j+\tfrac{1}{2}\right)(k+l)\right) + \cos\left(\tfrac{\pi}{N}\left(j+\tfrac{1}{2}\right)(k-l)\right)$$

For this, we consider in general for m

$$\sum_{j=0}^{N-1} \cos\left(\tfrac{\pi}{N}\left(j+\tfrac{1}{2}\right)m\right) = Re\left(\sum_{j=0}^{N-1} e^{i\frac{\pi}{N}m\left(j+\frac{1}{2}\right)}\right)$$

$m \in \mathbb{Z}^*$ and $|m| \le 2N-1$

$$\sum_{j=0}^{N-1} e^{i\frac{\pi}{2N}m} e^{i\frac{\pi}{N}mj}$$

$$= e^{i\frac{\pi}{2N}m} \frac{1-e^{im\pi}}{1-e^{i\frac{\pi}{N}m}}$$

$$= \begin{cases} 0 & \text{if } m \text{ is even} \\ e^{i\frac{\pi}{2N}m} \frac{2}{1-e^{i\frac{\pi}{N}m}} & \text{if } m \text{ is odd} \end{cases}$$

$$\hookrightarrow = 2e^{i\frac{\pi}{2N}m} \frac{1-e^{-i\frac{\pi}{N}m}}{|1-e^{i\frac{\pi}{N}m}|^2}$$

$$= \frac{2}{|-1|^2}\left(e^{i\frac{\pi}{2N}m} - e^{-i\frac{\pi}{2N}m}\right)$$

$$= \frac{4i}{|-1|^2} \sin\left(\frac{\pi}{2N}m\right) \in i\mathbb{R}$$

Thus, in both cases $Re\left(\sum_{j=0}^{N-1} e^{i\frac{\pi}{N}m(j+\frac{1}{2})}\right) = 0$

for all $m \in \mathbb{Z}^*$ s.t. $|m| < 2N$

Hence, if $0 \leq l+k \leq N-1$

we have indeed $k-l, l+k \in \mathbb{Z}^*$ and $|k-l| < 2N$ and $|l+k| < 2N$

$\Rightarrow \langle \varphi_k, \varphi_l \rangle = 0 = \delta_{kl}$

. If $l = k = 0$, ~~we have~~

$$\langle \varphi_k, \varphi_l \rangle = \frac{2}{4N} \sum_{j=0}^{N-1} (1+1) = \frac{4N}{4N} = 1 = \delta_{00} = \delta_{kl}$$

. If $l = k \neq 0 \Rightarrow l+k \in \mathbb{Z}^*$ and $|l+k| < 2N$

$$\Rightarrow \sum_{j=0}^{N-1} \cos\left(\frac{\pi}{N}(j+\tfrac{1}{2})(l+k)\right) = 0$$

$$\Rightarrow \langle \varphi_k, \varphi_l \rangle = \frac{2}{2N} \sum_{j=0}^{N-1} 1 = \frac{2N}{2N} = 1 = \delta_{kl}$$

$$\Rightarrow \forall\, 0 \leq k, l \leq N-1,\ \langle \varphi_k, \varphi_l \rangle = \delta_{kl}$$

So indeed A is an isometry and B is its inverse and is an isometry too.

(Ex 4.8.)

(PC): $\min\limits_{g(\alpha)=0} f(\alpha)$

where $\begin{cases} f(\alpha) = \sum_k \alpha_k^2 \sigma_k^2 \\ g(\alpha) = 1 - \sum_k \alpha_k \end{cases}$

Since g is differentiable, we have

$Jac(g)(\alpha) = (-1\ -1 \cdots -1)$

which is a non-zero linear form on $\mathbb{R}^{K^2}$

$\Rightarrow Jac(g)(\alpha)$ is surjective $\forall \alpha$

we also have $h \equiv 0$ (no inequality constraints)

Thus the surjectivity constraint qualification is fulfilled

$\xRightarrow{KKT}$ If α is a local optimal of (PC) then $\exists \lambda \in \mathbb{R}$, $\nabla f(\alpha) + \lambda \nabla g(\alpha) = 0$

$\Rightarrow \forall 1 \leq k \leq K^2,\ 2\alpha_k \sigma_k^2 - \lambda = 0$

$\Rightarrow \boxed{\forall k,\ 2\alpha_k \sigma_k^2 = \lambda}$

(Ex 4.9.)

For the k-th patch, an estimate of the variance with the modified noise remaining in the patch is

$$\sigma^2 \sum_j f_{P_k}(j)^2 = \sigma^2 \|f_{P_k}\|^2$$

where $f_{P_k} = (f_{P_k}(j))_j$

Thus, $\alpha_k = \dfrac{\sigma^{-2}\|f_{P_k}\|^{-2}}{\sum_j \sigma^{-2}\|f_{P_j}\|^{-2}} = \dfrac{\|f_{P_k}\|^{-2}}{\sum_j \|f_{P_j}\|^{-2}}$