(Ex 8.1)

For the image: $\tilde{u} = u + n$ of dimension M

$\hookrightarrow \mathcal{N}(0, I_{\text{work}})$

Writting it for the considered Patch:

$\tilde{P} = P + n \longrightarrow \mathcal{N}(0$ of dimension k^2

$\Rightarrow E(\tilde{P}) = \underbrace{E(P)}_{=\bar{P}} + \underbrace{E(n)}_{=0} = \bar{P}$

Thus: $C_{\tilde{P}} = E(\tilde{P}\tilde{P}^T) - \bar{P}\bar{P}^T$

$\hookrightarrow = E((P+n)(P^T+n^T))$

$= E(PP^T) + 2E(Pn^T) + E(nn^T)$

$P \perp\!\!\!\perp n \Rightarrow E(Pn^T) = E(P)E(n^T)$

$n \sim \mathcal{N}(0, \sigma^2 I) \Rightarrow \begin{cases} E(n^T) = 0 \\ E(nn^T) = \sigma^2 I \end{cases}$

Hence, $C_{\tilde{P}} = E(PP^T) + \sigma^2 I - \bar{P}\bar{P}^T$

$= [E(PP^T) - E(P)E(P)^T] + \sigma^2 I$

$= C_P + \sigma^2 I$

(Ex 8.3)

$C_{\tilde{P}}, C_{\hat{P}_1} \in S_N(\mathbb{R})$ (2 real symetric matrices)

$\Rightarrow \exists \tilde{M}, \hat{M} \in O_N(\mathbb{R})$;

$\begin{cases} C_{\tilde{P}} = \tilde{M}[\text{diag}(\lambda_k)]\tilde{M}^T \\ C_{\hat{P}_1} = \hat{M}[\text{diag}(\gamma_k)]\hat{M}^T \end{cases}$

Using (8.8), one can write:

$\hat{P}_1 - \bar{P} = (I - \sigma^2 C_{\tilde{P}}^{-1})(\tilde{P} - \bar{P})$

$= \tilde{M}\left[\text{diag}\left(1 - \frac{\sigma^2}{\lambda_k}\right)\right]\tilde{M}^T(\tilde{P} - \bar{P})$

$\Rightarrow \underbrace{\tilde{M}^T(\hat{P}_1 - \bar{P})}_{\downarrow} = \underbrace{\left[\text{diag}\left(\frac{\lambda_k - \sigma^2}{\lambda_k}\right)\right]}_{\downarrow}\tilde{M}^T(\tilde{P} - \bar{P})$

Similar to $D\tilde{u}_1$ from (4.10)

Similar to α_i from (4.10) $\alpha_i = \max\left(0, \frac{|\langle\tilde{u}, G_i\rangle|^2 - \sigma^2}{|\langle\tilde{u}, G_i\rangle|^2}\right)$

Similarly but using (8.12), one can write:

$\hat{P}_2 - \bar{P}^1 = \hat{M}\left[\text{diag}\left(1 + \frac{\sigma^2}{\gamma_k}\right)\right]^{-1}\hat{M}^T(\tilde{P} - \bar{P}^1)$

$\Rightarrow \underbrace{\hat{M}^T(\hat{P}_2 - \bar{P}^1)}_{\downarrow} = \underbrace{\left[\text{diag}\left(\frac{\gamma_k}{\gamma_k + \sigma^2}\right)\right]}_{\downarrow}\hat{M}^T(\tilde{P} - \bar{P}^1)$

Similar to $D\tilde{u}_2$ from (4.11)

Similar to α_i in (4.11) $\alpha_i = \frac{|\langle\tilde{u}_1, G_i\rangle|^2}{|\langle\tilde{u}_1, G_i\rangle|^2 + \sigma^2}$

Thus, indeed, one can interpret the two steps of the Bayesian method as an app of the Wiener empirical and oculer method.

The difference: In the Wiener method case, we use the same orthonormal basis (G_i) whereas here we use 2 different orthonormal basis $\tilde{M}^T$ and $\hat{M}^T$ (not necessarily the same).

(Ex 8.4)

$$MSE = \int_{P} \mathbb{P}(P) \int_{\tilde{P}} \mathbb{P}(\tilde{P}|P) \|P-\hat{P}\|^2 d\tilde{P}\, dP$$

$$= \int_{P}\int_{\tilde{P}} \mathbb{P}(P)\mathbb{P}(\tilde{P}|P) \|P-\hat{P}\|^2 d\tilde{P}\, dP$$

Using Bayes' rule: $\mathbb{P}(P)\mathbb{P}(\tilde{P}|P) = \mathbb{P}(\tilde{P},P)$
$= \mathbb{P}(P|\tilde{P})\mathbb{P}(\tilde{P})$,

one can write:

$$MSE = \int_{P}\int_{\tilde{P}} \underbrace{\mathbb{P}(\tilde{P})\mathbb{P}(P|\tilde{P}) \|P-\hat{P}\|^2} d\tilde{P}\, dP$$

Since this quantity ≥ 0, we use Fubini-Tonelli's thm to change order of integration as follows:

$$MSE = \int_{\tilde{P}}\int_{P} \mathbb{P}(\tilde{P})\mathbb{P}(P|\tilde{P}) \|P-\hat{P}\|^2 dP\, d\tilde{P}$$

$$= \int_{\tilde{P}} \mathbb{P}(\tilde{P}) \int_{P} \mathbb{P}(P|\tilde{P}) \|P-\hat{P}\|^2 dP\, d\tilde{P}$$

(Ex 8.5)

$$\frac{\partial}{\partial \hat{P}}\left[MMSE(\tilde{P})\right] = 0$$

$$\Rightarrow -2\int_{P} \mathbb{P}(P|\tilde{P})(P-\hat{P})\, dP = 0$$

$$\Rightarrow \int_{P} P\, \mathbb{P}(P|\tilde{P})\, dP = \hat{P}\int_{P} \mathbb{P}(P|\tilde{P})\, dP$$

$$\Rightarrow \underbrace{\mathbb{E}(P|\tilde{P})}_{\text{MMSE estimator of (8.15)}} = \hat{P}$$