

# (HN 8)

(Ex 3.1) (i)  $\forall u, \mathbb{E}_\sigma(v|u) = u$   
 (ii)  $v_x \perp v_y | u \quad \forall x \neq y$

$$\begin{aligned} (1) & \mathbb{E}_{\sigma, u} \{ \|F^\lambda(v) - u\|^2 \} \\ &= \mathbb{E}_{\sigma, u} \{ \|\lambda(F(v) - u) + (1-\lambda)(v - u)\|^2 \} \\ &= \lambda^2 \mathbb{E}_{\sigma, u} \{ \|F(v) - u\|^2 \} + (1-\lambda)^2 \mathbb{E}_{\sigma, u} \{ \|v - u\|^2 \} \\ &\quad + 2\lambda(1-\lambda) \mathbb{E}_{\sigma, u} \{ \langle F(v) - u, v - u \rangle \} \end{aligned}$$

(2.) Inspired by the proof of proposition 3.4, one can write:

$$\begin{aligned} & \mathbb{E}_{\sigma, u} \{ \langle F(v) - u, v - u \rangle \} \\ &= \mathbb{E}_u \mathbb{E}_\sigma \{ \langle F(v) - u, v - u \rangle | u \} \\ &= \mathbb{E}_u \mathbb{E}_\sigma \left\{ \sum_{j \in J} \langle F(v)_j - u_j, v_j - u_j \rangle | u \right\} \\ &\quad \hookrightarrow \text{b/c it's a partition} \end{aligned}$$

We focus on one set of pixels  $j \in J$ :

$$\begin{aligned} & \mathbb{E}_\sigma \{ \langle F(v)_j - u_j, v_j - u_j \rangle | u \} \\ &= \mathbb{E}_{\sigma_j, v_j^c} \{ \langle F(v)_j - u_j, v_j - u_j \rangle | u \} \\ &= \mathbb{E}_{\sigma_j^c} \left\{ \mathbb{E}_{\sigma_j} \{ \langle F(v)_j - u_j, v_j - u_j \rangle | v_j^c, u \} | u \right\} \\ &\quad \perp v_j \text{ since } F \text{ is } j\text{-invariant} \\ &= \mathbb{E}_{\sigma_j^c} \{ \langle F(v)_j - u_j, \mathbb{E}_{\sigma_j}(v_j | v_j^c, u) - u_j \rangle | u \} \\ &= \mathbb{E}_{\sigma_j^c}(v_j | u) \text{ b/c } v_j \perp v_j^c | u \quad (ii) \\ &= u_j \text{ b/c (i)} \end{aligned}$$

$$= 0$$

Thus,  $\mathbb{E}_{\sigma, u} \{ \langle F(v) - u, v - u \rangle \} = 0$

(3.) Let  $f: \lambda \mapsto \mathbb{E}_{\sigma, u} \{ \|F^\lambda(v) - u\|^2 \}$   
 $f$  is convex and quadratic  
 $(f''(\lambda) = 2 \mathbb{E}_{\sigma, u} \{ \|F(v) - u\|^2, \|v - u\|^2 \} \geq 0)$   
 $\Rightarrow f$  has a global minimum obtained via  $f'(\lambda) = 0$

$$\Rightarrow 2\lambda \mathbb{E}_{\sigma, u} \{ \|F(v) - u\|^2 \} - 2(1-\lambda) \mathbb{E}_{\sigma, u} \{ \|v - u\|^2 \} = 0$$

$$\Rightarrow \lambda \left( \mathbb{E}_{\sigma, u} \{ \|F(v) - u\|^2 \} + \mathbb{E}_{\sigma, u} \{ \|v - u\|^2 \} \right) = \mathbb{E}_{\sigma, u} \{ \|v - u\|^2 \}$$

$$\Rightarrow \lambda^* = \frac{\mathbb{E}_{\sigma, u} \{ \|v - u\|^2 \}}{\mathbb{E}_{\sigma, u} \{ \|F(v) - u\|^2 \} + \mathbb{E}_{\sigma, u} \{ \|v - u\|^2 \}}$$

$$\text{However, } \mathbb{E}_{\sigma, u} \{ \|v - u\|^2 \} = \mathbb{E}_u \left\{ \mathbb{E}_\sigma \{ \|v - \mathbb{E}_\sigma(v|u)\|^2 | u \} \right\} = \mathbb{E}_u \{ V(v|u) \}$$

$$\Rightarrow \lambda^* = \frac{\mathbb{E}_u \{ V(v|u) \}}{\mathbb{E}_{\sigma, u} \{ \|F(v) - u\|^2 \} + \mathbb{E}_u \{ V(v|u) \}}$$

(4.) Using proposition 3.4, one can write:

$$\lambda^* = \frac{\mathbb{E}_u \{ V(v|u) \}}{\mathbb{E}_\sigma \{ \|F(v) - u\|^2 \}}$$

$$= \frac{\mathbb{E}_u \{ d \sigma^2 \}}{R_{N2S}(F)}$$

$$\Rightarrow \lambda^* = \frac{d \sigma^2}{R_{N2S}(F)}$$

(Ex 3.2)

$$\mathbb{E}_v \{ \|\hat{X}(v) - \mu\|^2 \mid \mu \}$$

$$= \mathbb{E}_v \{ \|\hat{X}(v) - \mathbb{E}_v(\hat{X}(v) \mid \mu) + \mathbb{E}_v(\hat{X}(v) \mid \mu) - \mu\|^2 \mid \mu \}$$

$$= \mathbb{E}_v \{ \|\hat{X}(v) - \mathbb{E}_v(\hat{X}(v) \mid \mu)\|^2 \mid \mu \}$$

$$+ \underbrace{\mathbb{E}_v \{ \|\mathbb{E}_v(\hat{X}(v) \mid \mu) - \mu\|^2 \mid \mu \}}_{\text{doesn't depend on } v}$$

$$- 2 \mathbb{E}_v \{ \langle \hat{X}(v) - \mathbb{E}_v(\hat{X}(v) \mid \mu), \mathbb{E}_v(\hat{X}(v) \mid \mu) - \mu \rangle \mid \mu \}$$

$$= \mathbb{E}_v \{ \|\hat{X}(v) - \mathbb{E}_v(\hat{X}(v) \mid \mu)\|^2 \mid \mu \}$$

$$+ \|\mathbb{E}_v(\hat{X}(v) \mid \mu) - \mu\|^2$$

$$+ 2 \langle \underbrace{\mathbb{E}_v \{ \hat{X}(v) - \mathbb{E}_v(\hat{X}(v) \mid \mu) \}}_{=0}, \mathbb{E}_v(\hat{X}(v) \mid \mu) - \mu \rangle$$

$$= \mathbb{E}_v(\hat{X}(v) \mid \mu) - \mathbb{E}_v(\hat{X}(v) \mid \mu) \\ = 0$$

Hence :

$$\mathbb{E}_v(\|\hat{X}(v) - \mu\|^2 \mid \mu)$$

$$= \|\mathbb{E}_v \{ \hat{X}(v) \mid \mu \} - \mu\|^2$$

$$+ \mathbb{E}_v \{ \|\hat{X}(v) - \mathbb{E}_v(\hat{X}(v) \mid \mu)\|^2 \mid \mu \}$$