Data Structures and Algorithms
(ES221)

Graph Algorithms

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Depth First Search (DFS)

- Formally, DFS is an uninformed search that progresses by
  - expanding the first child node of the search tree (or graph) that appears, and
  - thus going deeper and deeper
  - until a goal node is found, or until it hits a node that has no children.
- Then the search backtracks, returning to the most recent node it hasn't finished exploring.
- In a non-recursive implementation, all freshly expanded nodes are added to a stack for exploration.

DFS Pseudocode

procedure DFS(*G*, *s*)

    for each vertex *u* in *G*

        u.status = **notVisited/white**

        u.pi = *NULL*

        time = 0;

    for each vertex *u* in *G*

        if *u.status* = **notVisited/white**

                DFS_visit(*G*, *u*)

DFS Pseudocode

```
procedure DFS_visit(G, u)
    u.status = visited / grey
      time = time + 1;
      u.d = time;
    for each v adjacent to u do
        if v.status = notVisited/white
            v.pi = u;
            DFS_visit(G, v)
    u.status = processed/ black
```


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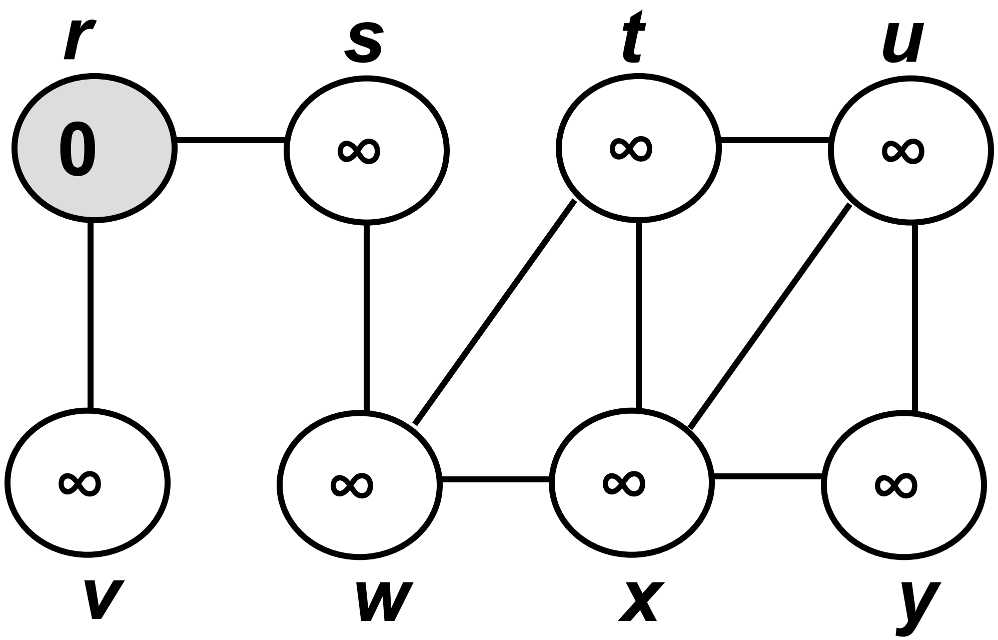
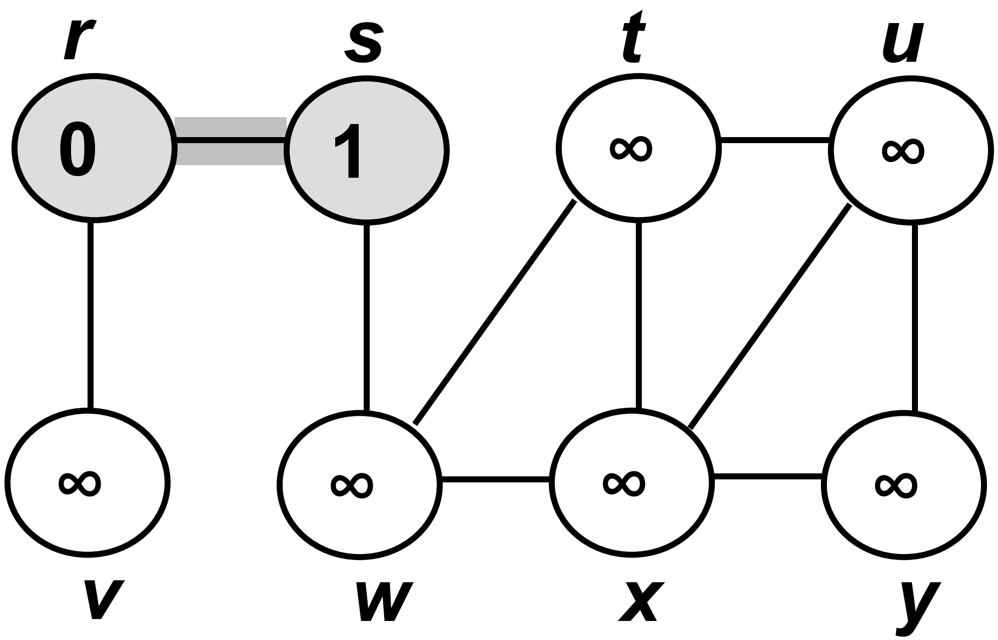
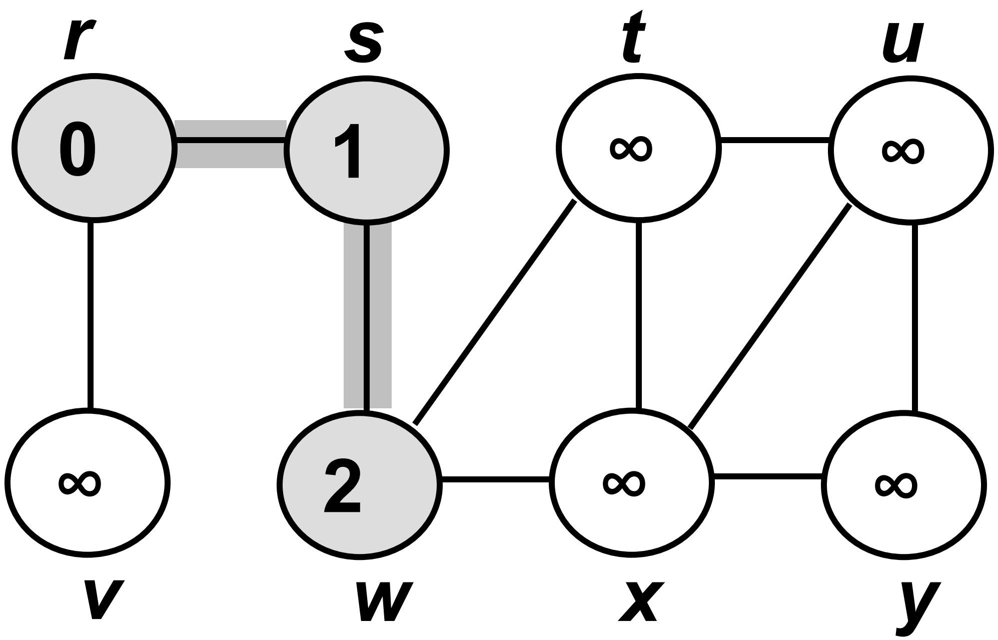
DFS Example



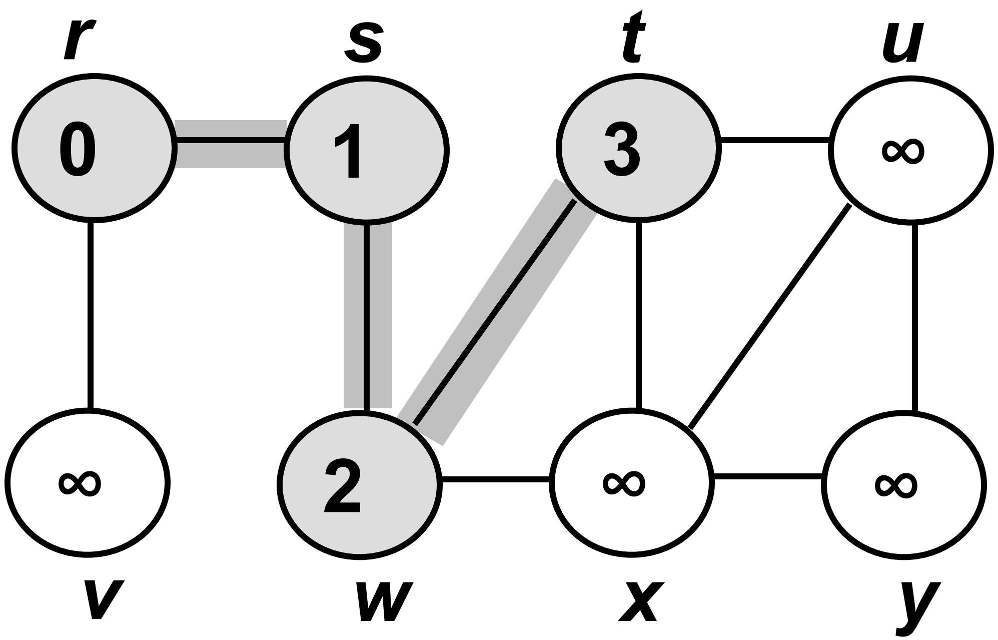
DFS Example



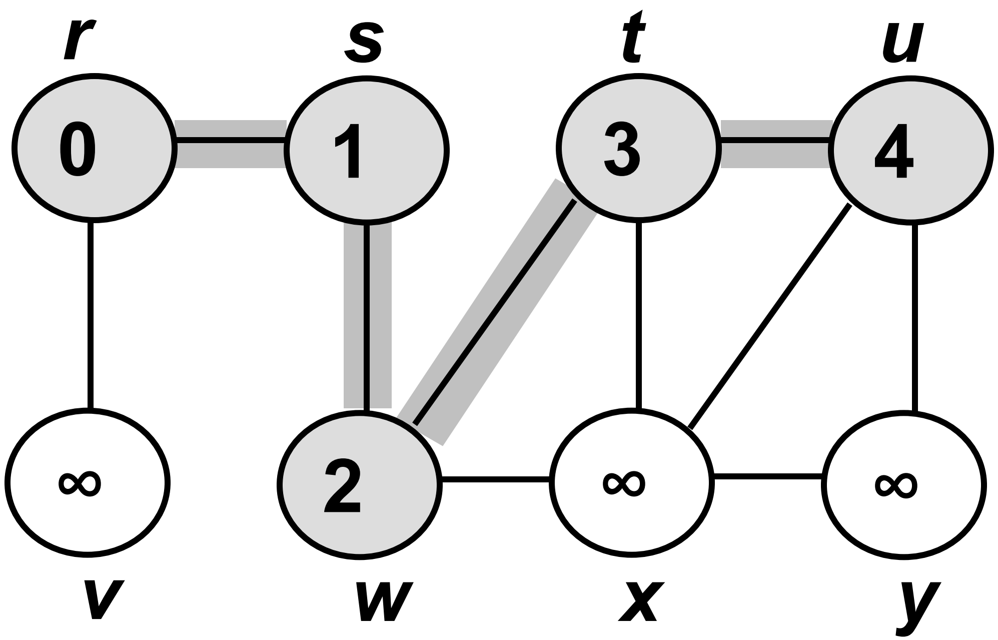
DFS Example



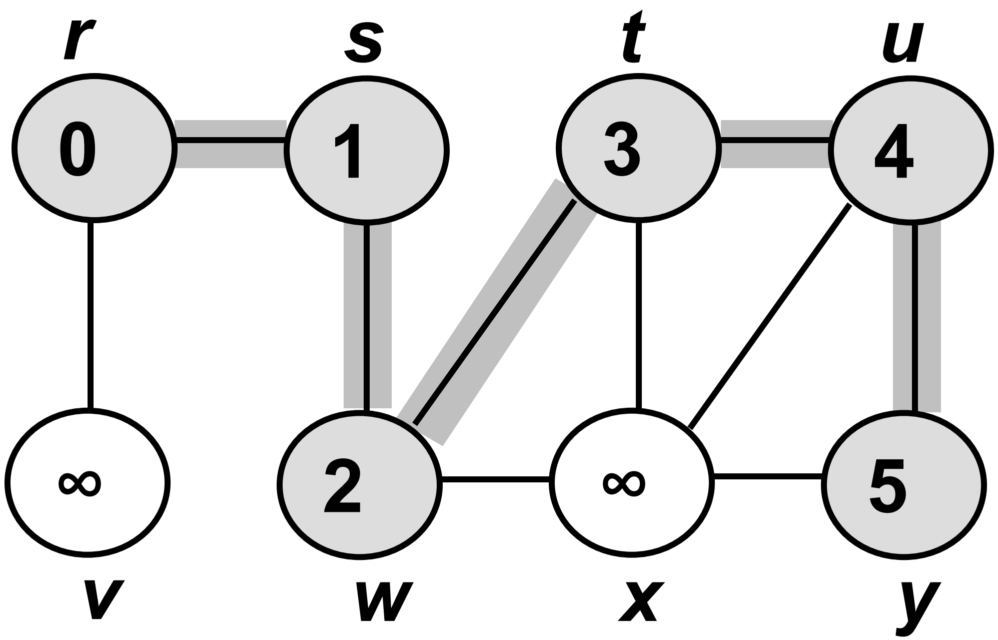
DFS Example



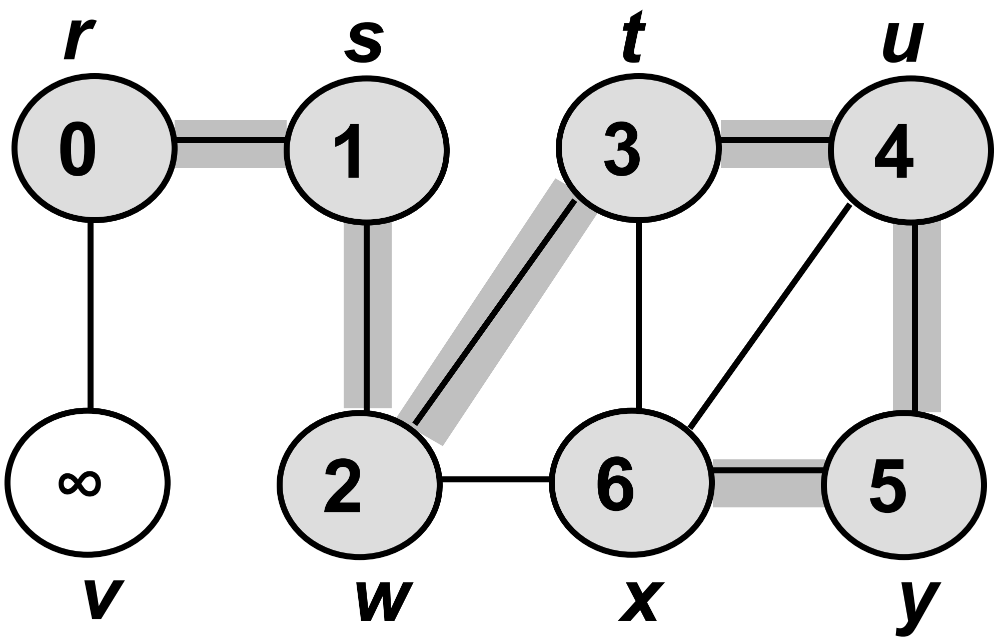
DFS Example



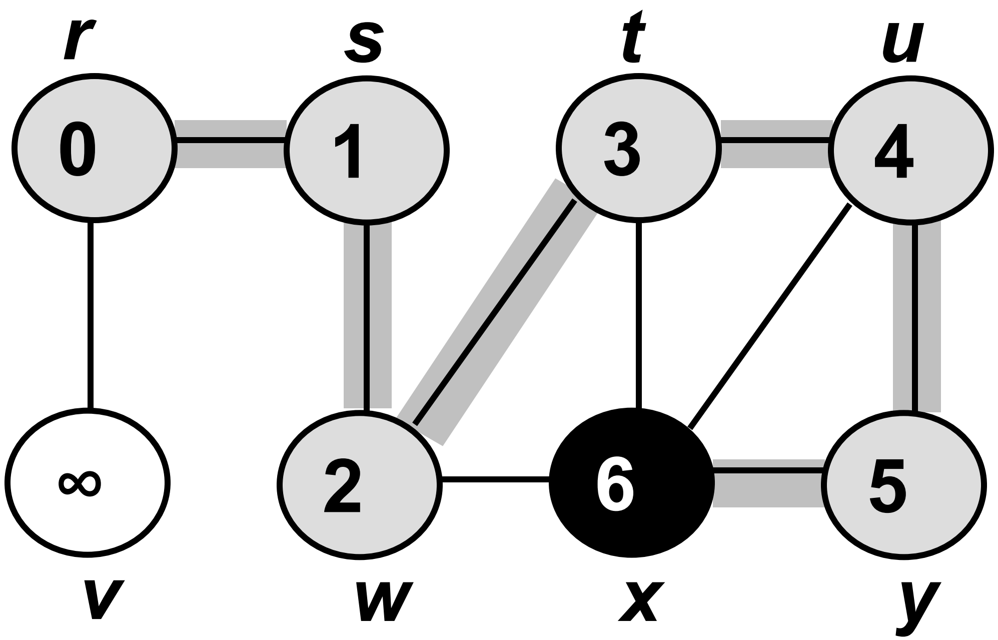
DFS Example



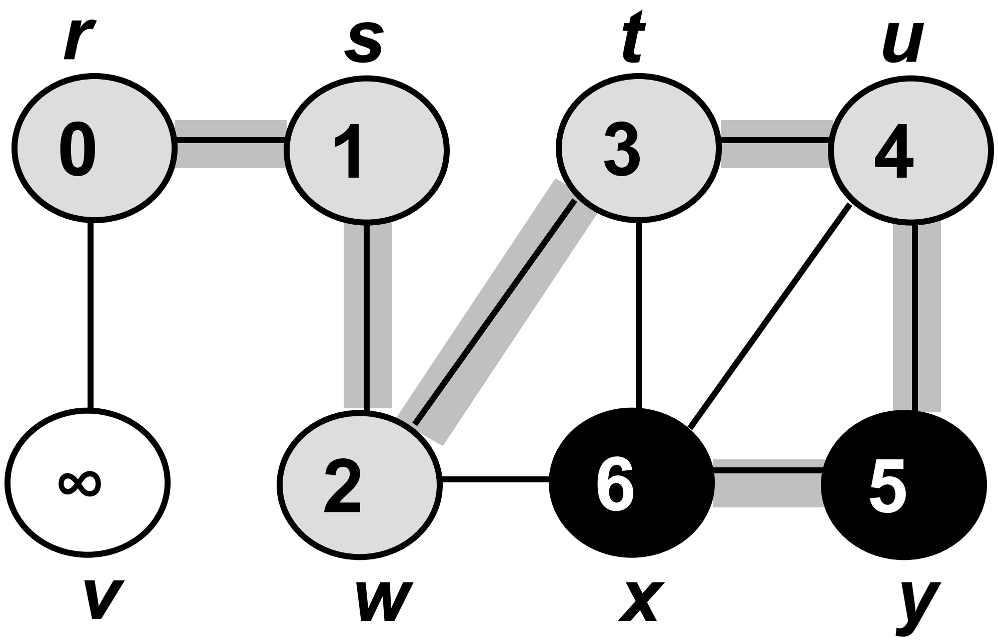
DFS Example



DFS Example



DFS Example



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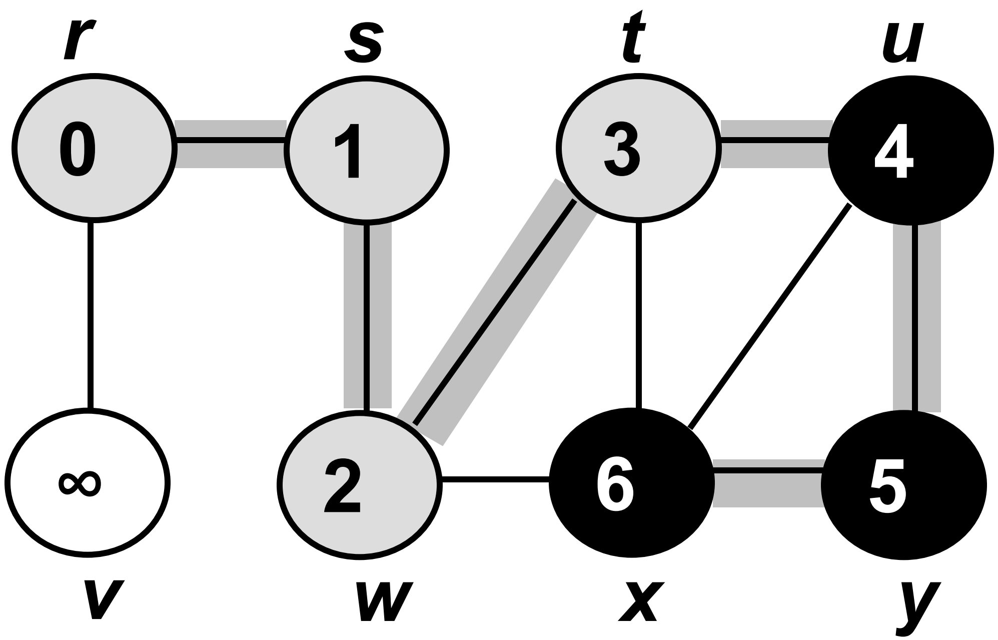

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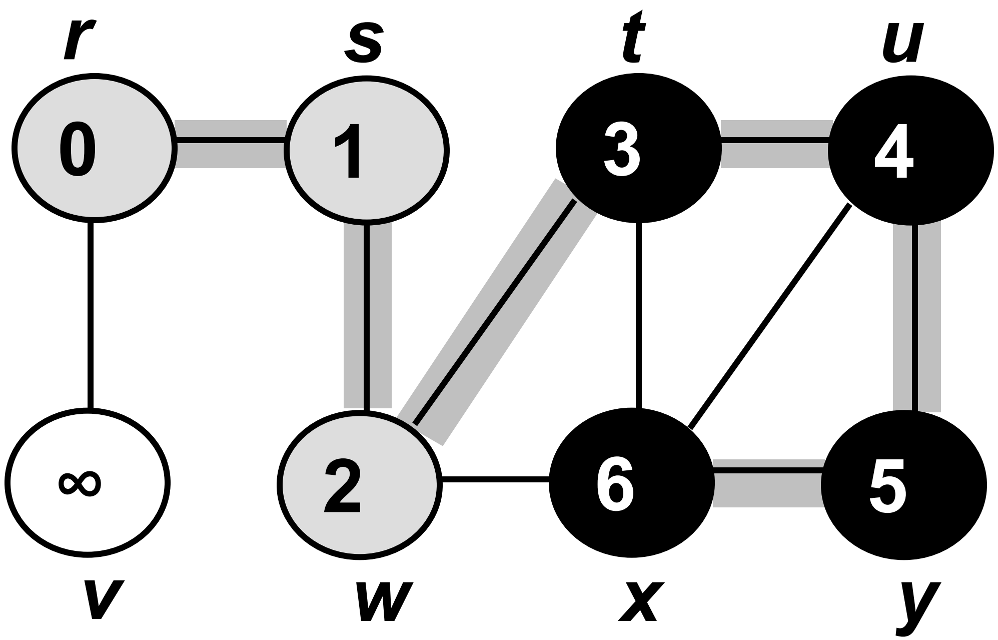
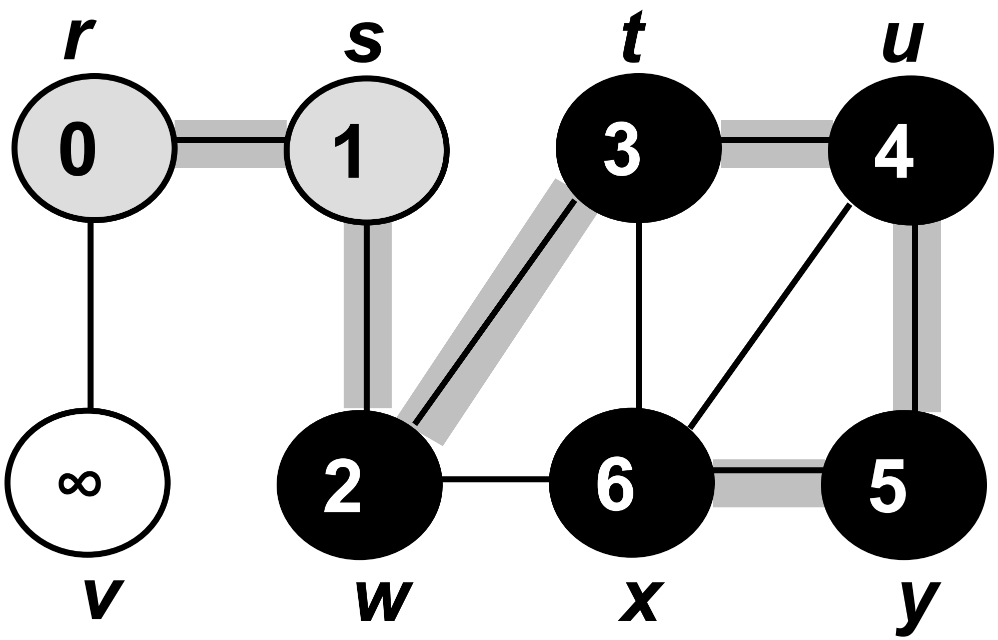
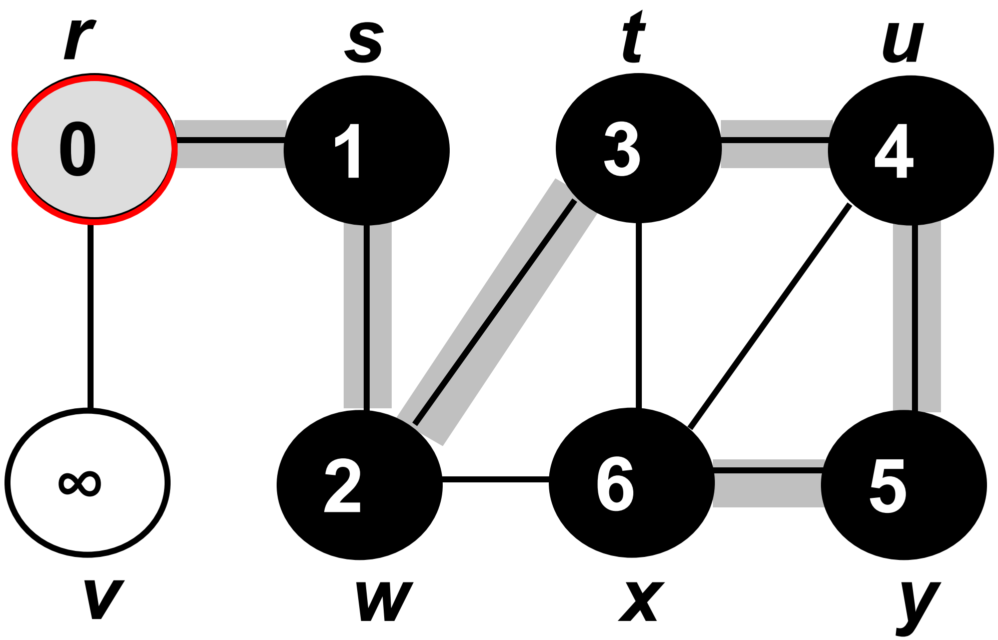


DFS Example



DFS Example



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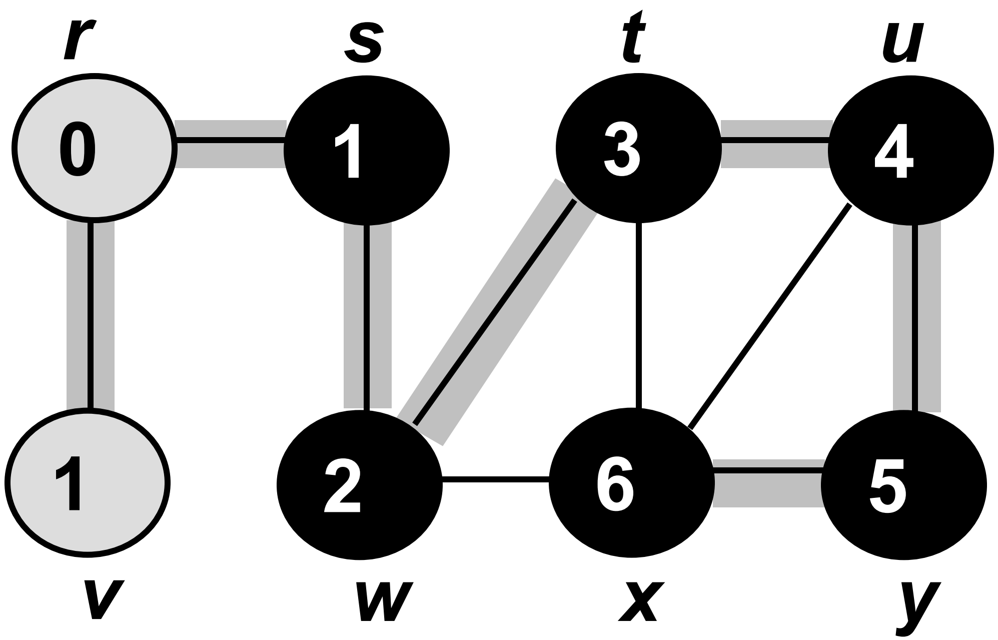

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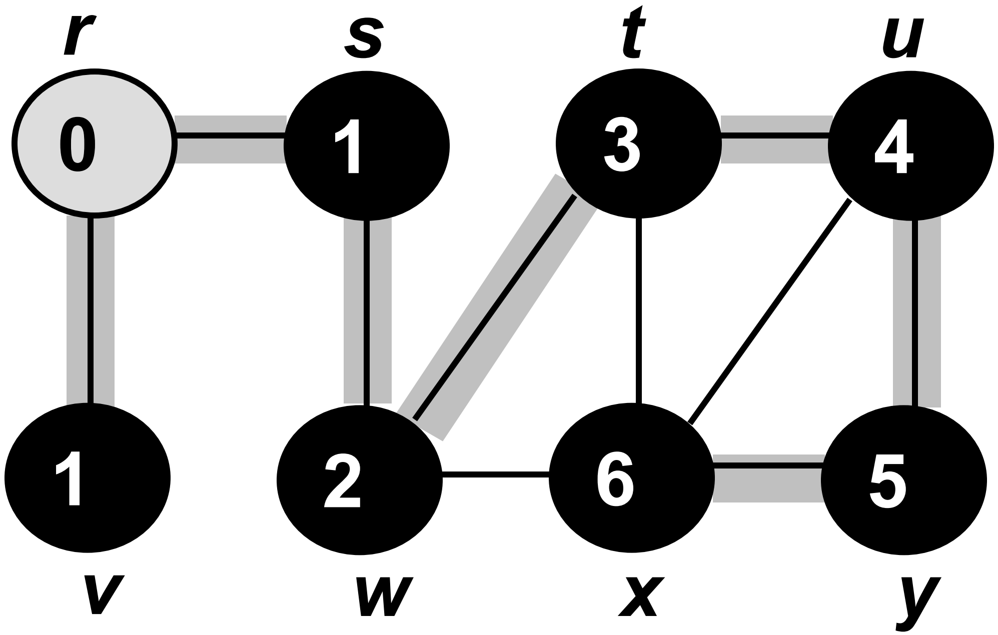

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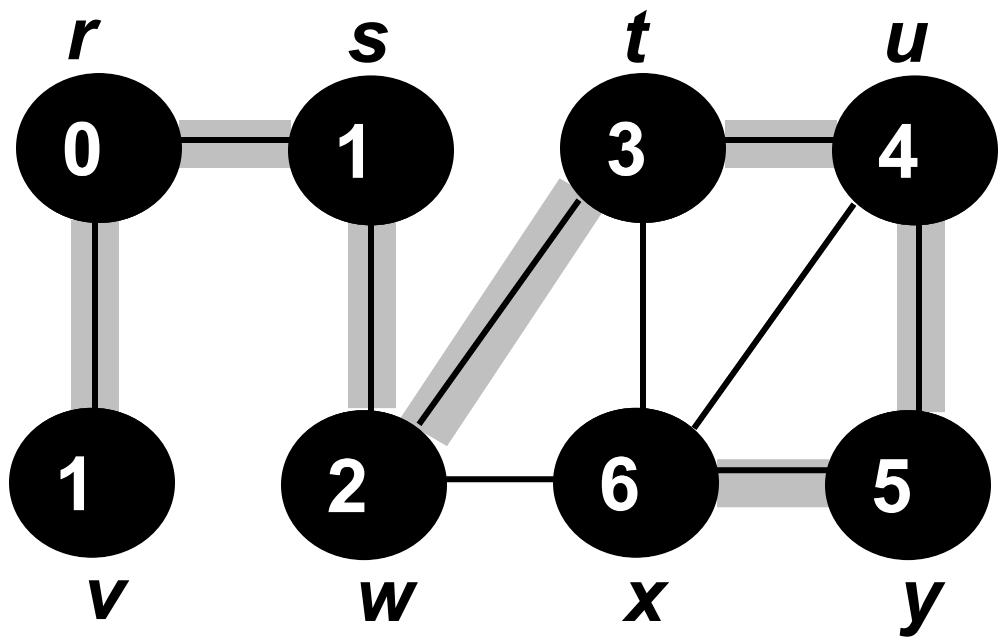
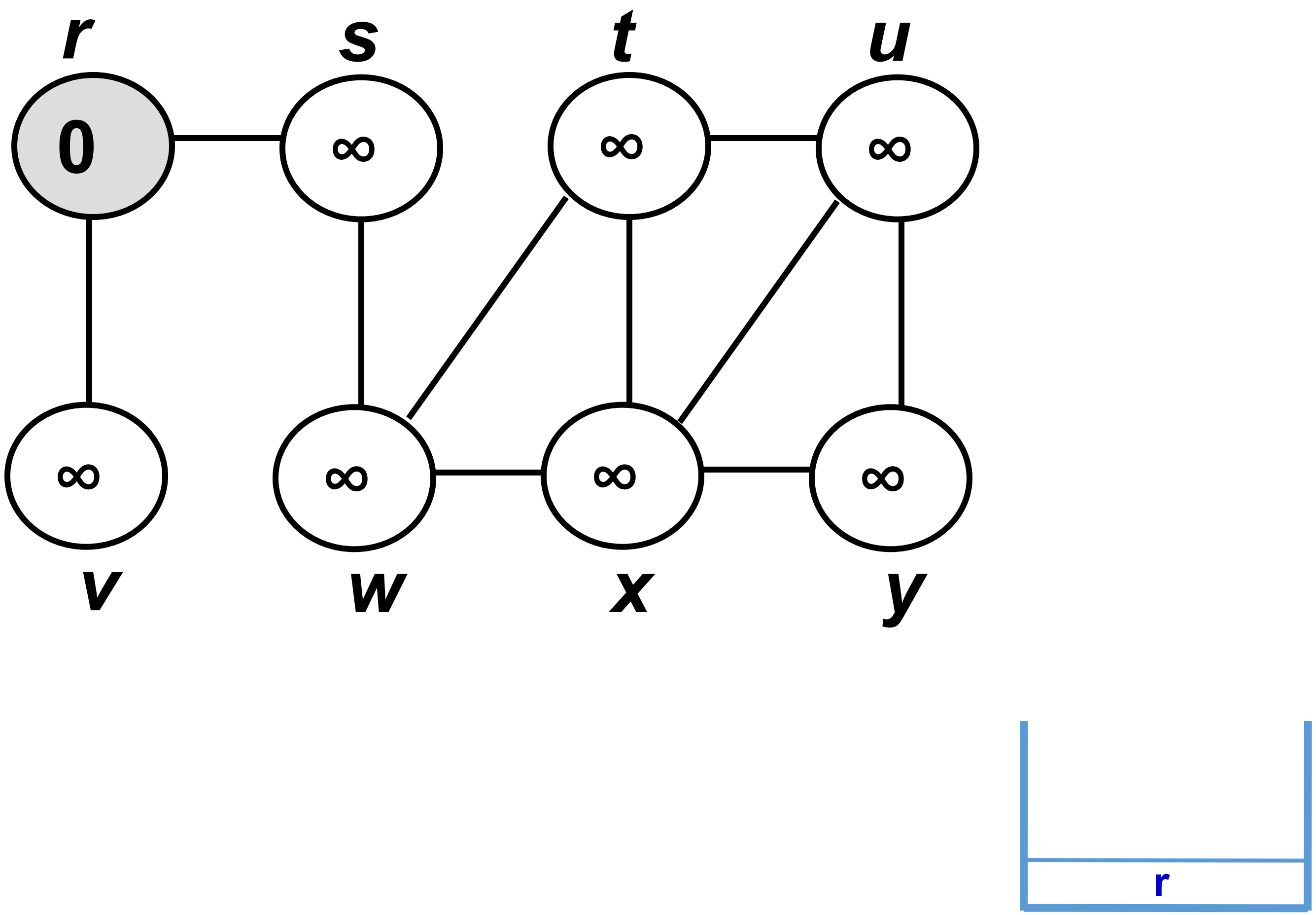


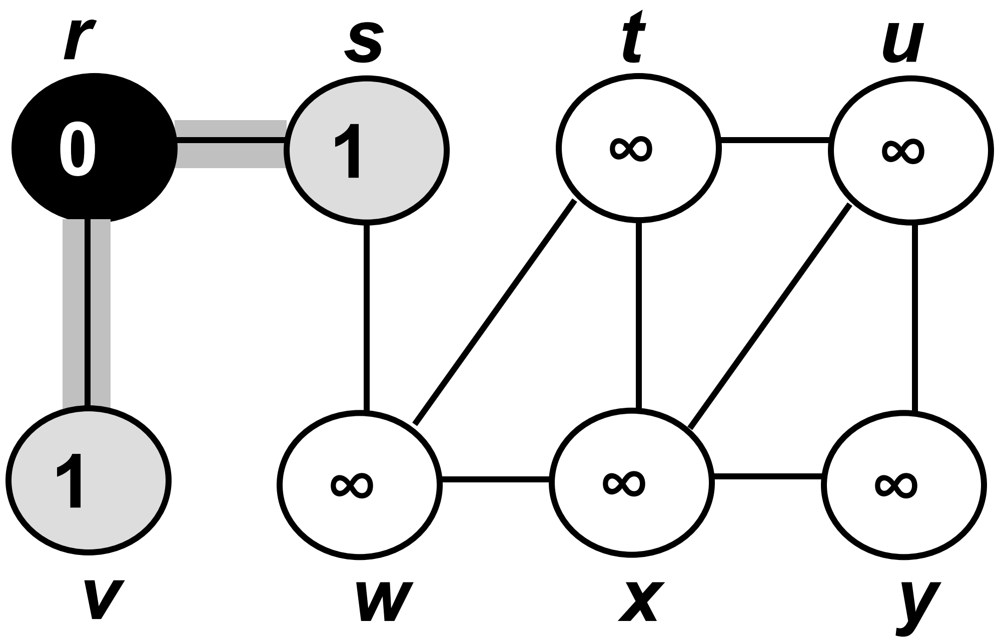
DFS: Non-recursive Implementation

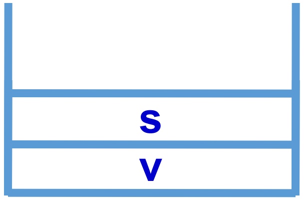
1. Initialize all nodes to the notVisited state (STATUS=1)
2. PUSH starting node A on stack and change its status to visited(STATUS=2)
3. Repeat 4-5 until stack is empty
    4. POP *N*. Process it and change its status to processed(STATUS=3)
    5. PUSH on stack all the neighbors of *N* that are still in notVisited state and change their status to visited
6. Exit

DFS Example



DFS Example





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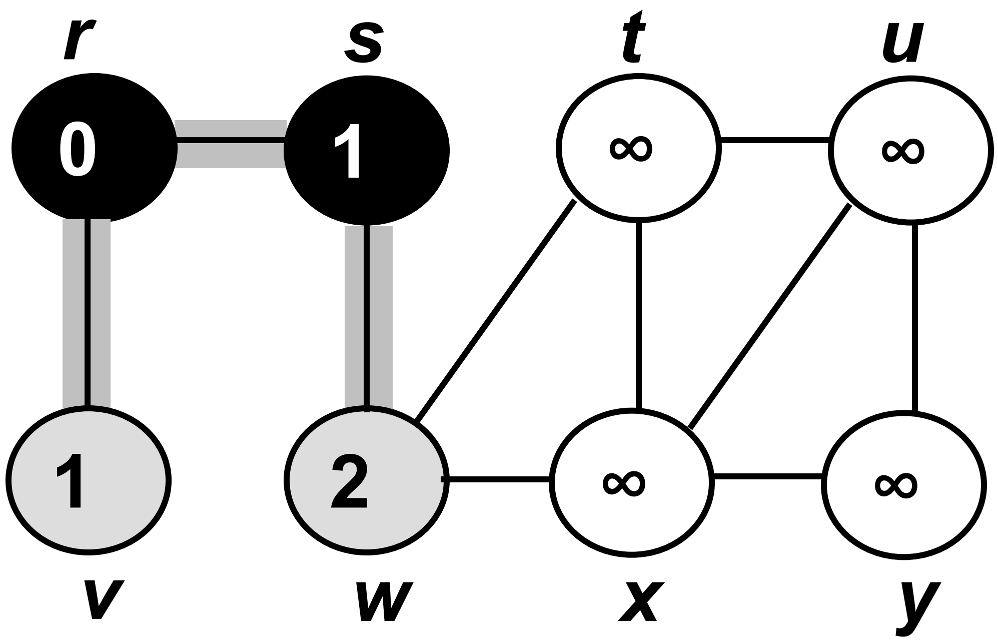

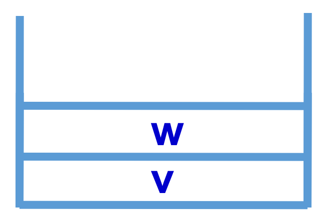

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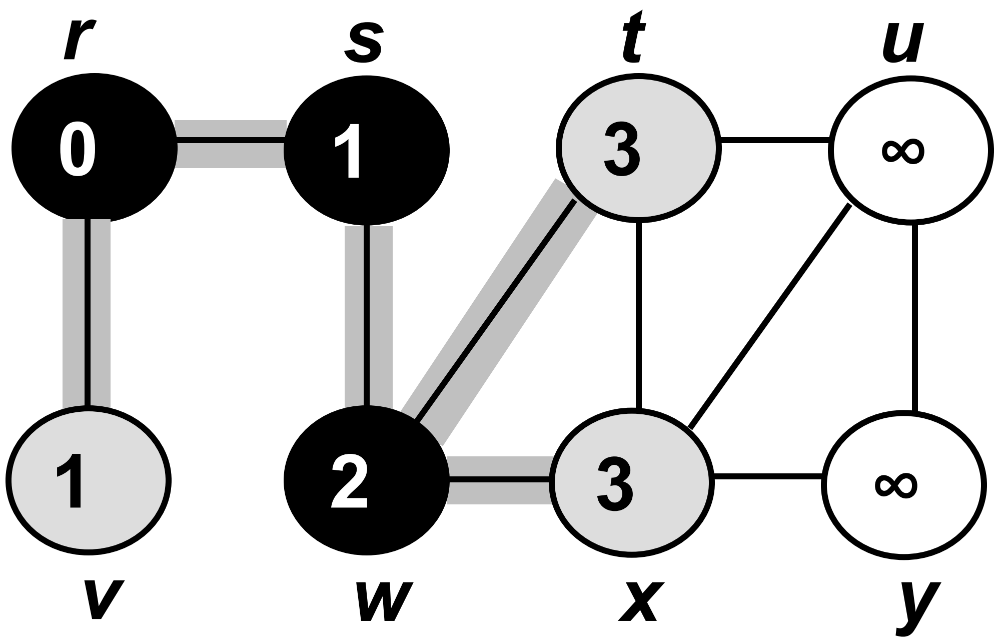
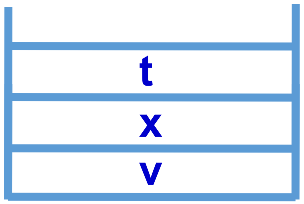
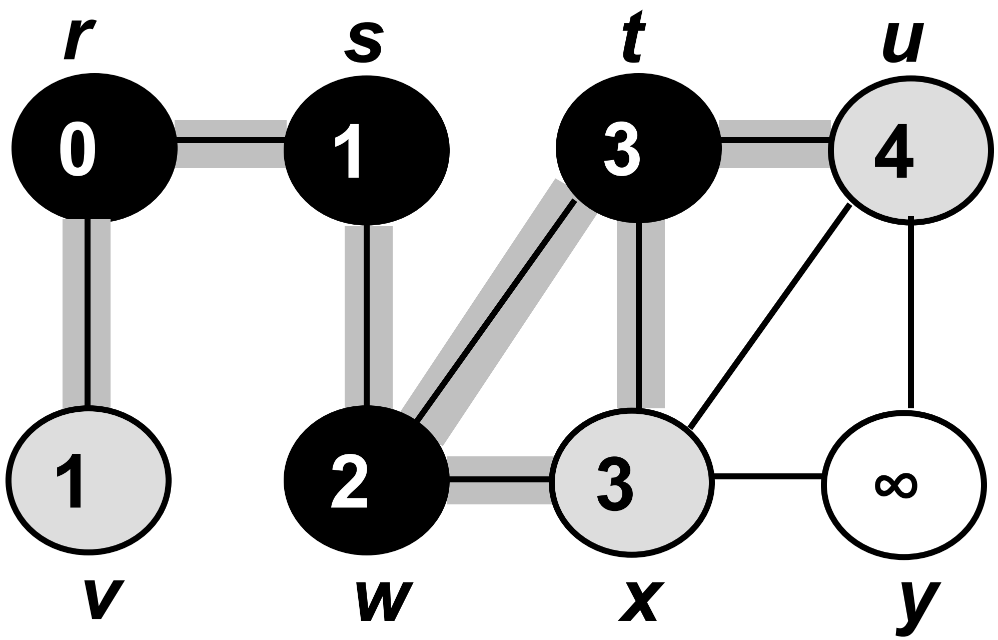
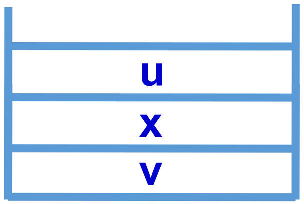




DFS Example





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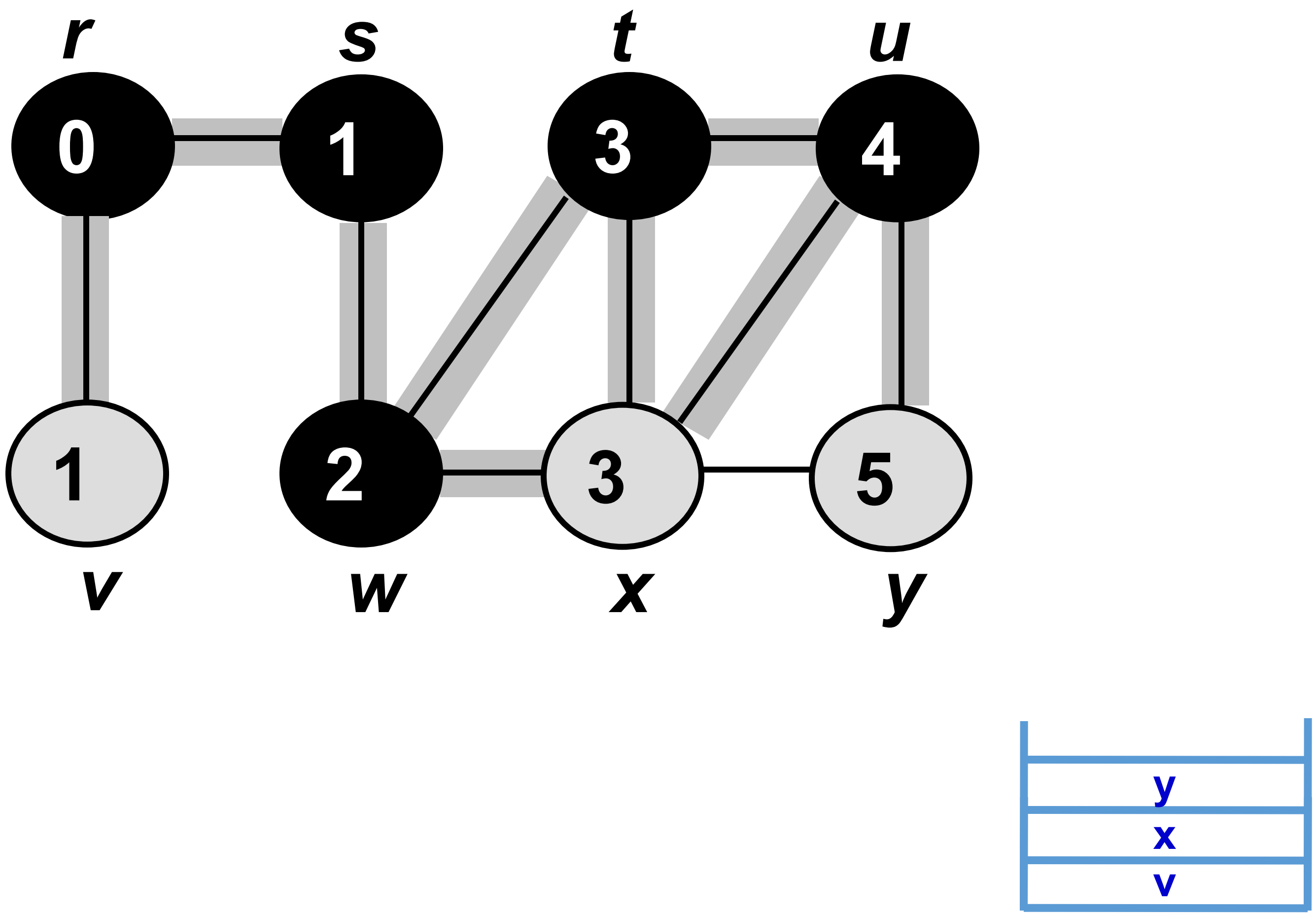

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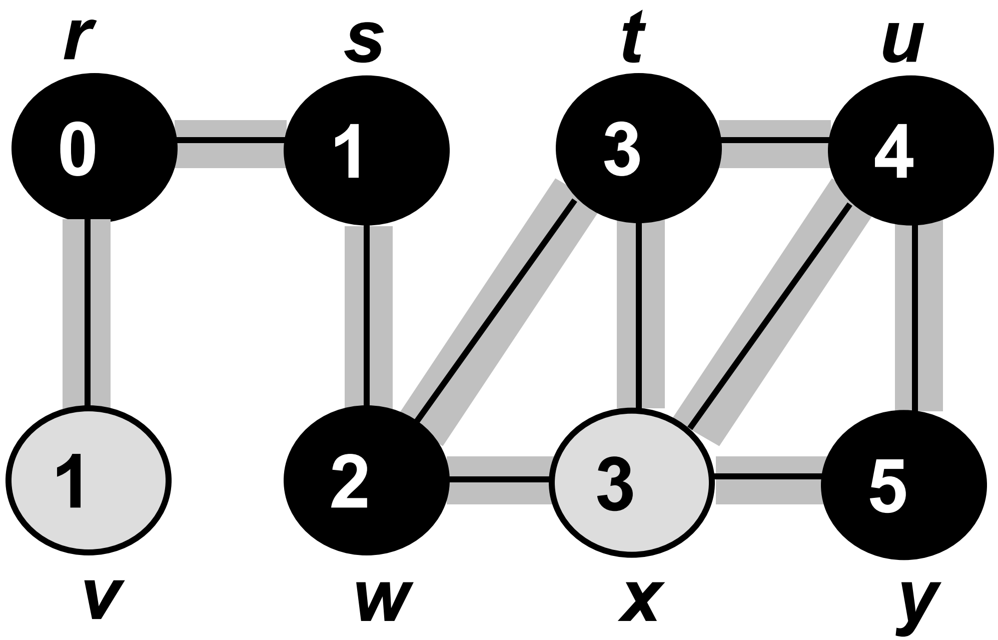

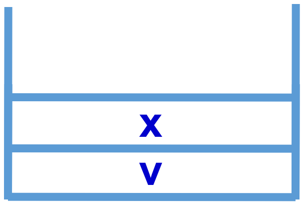

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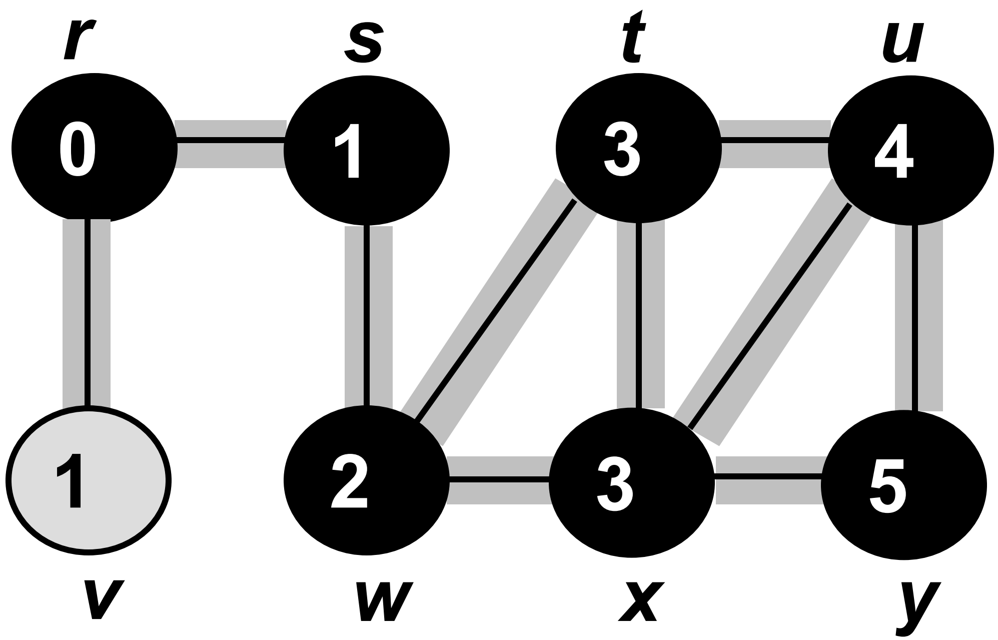

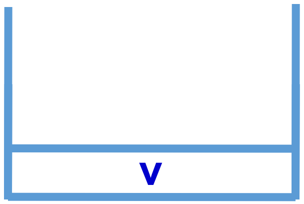

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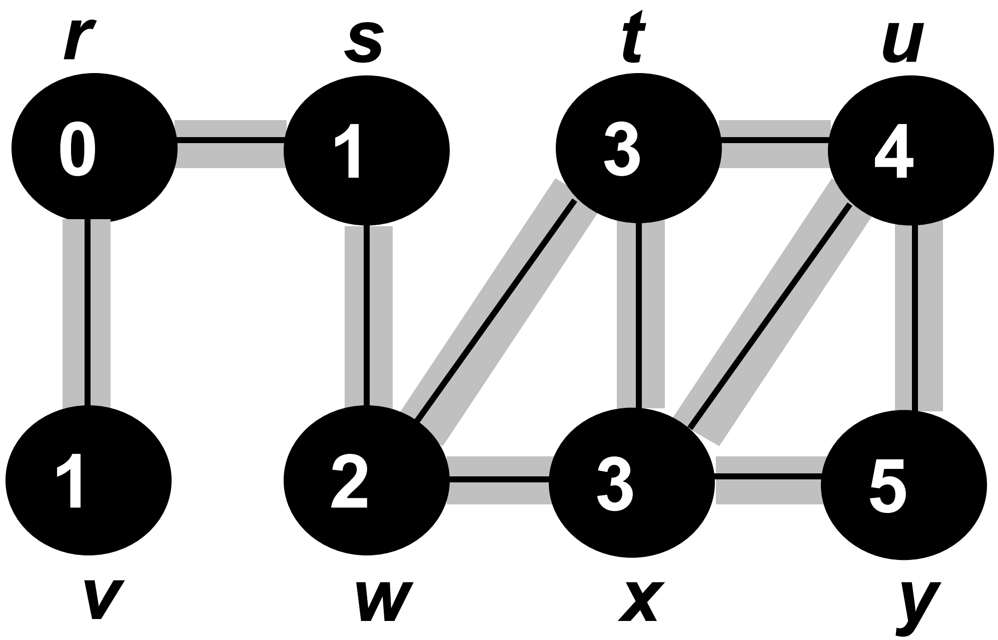


Dijkstra’ s algorithm

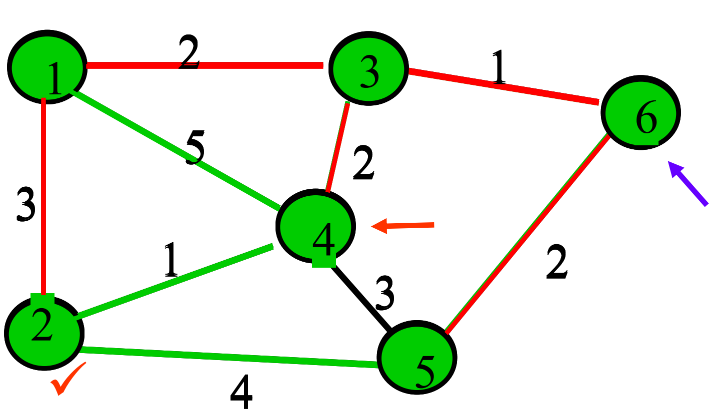
The **Dijkstra algorithm** is a **greedy algorithm** used to find the **shortest paths from a single source node** to all other nodes in a graph with **non-negative edge weights**

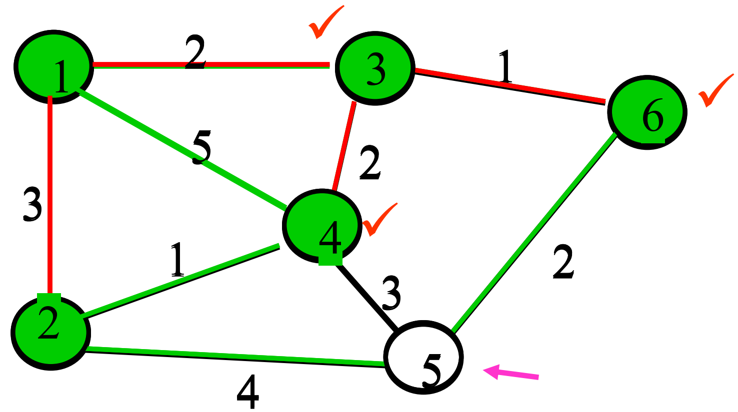
Dijkstra's algorithm

- N: set of nodes for which shortest path already found
- Initialization: (S*tart with source node s)*
  - $N = \{s\}$, $D_s = 0$, "s is distance zero from itself"
  - $D_j = C_{sj}$ for all $j \neq s$, distances of directly-connected neighbors
- Step A: (*Find next closest node i*)
  - Find $i \notin N$ such that
  - $D_i = \min Dj$ for $j \notin N$
  - Add i to N
  - If N contains all the nodes, stop
- Step B: (*update minimum costs)*
  - For each node $j \notin N$
  - $D_j = \min (D_j, D_i + C_{ij})$
  - Go to Step A

Minimum distance from s to j through node i in N

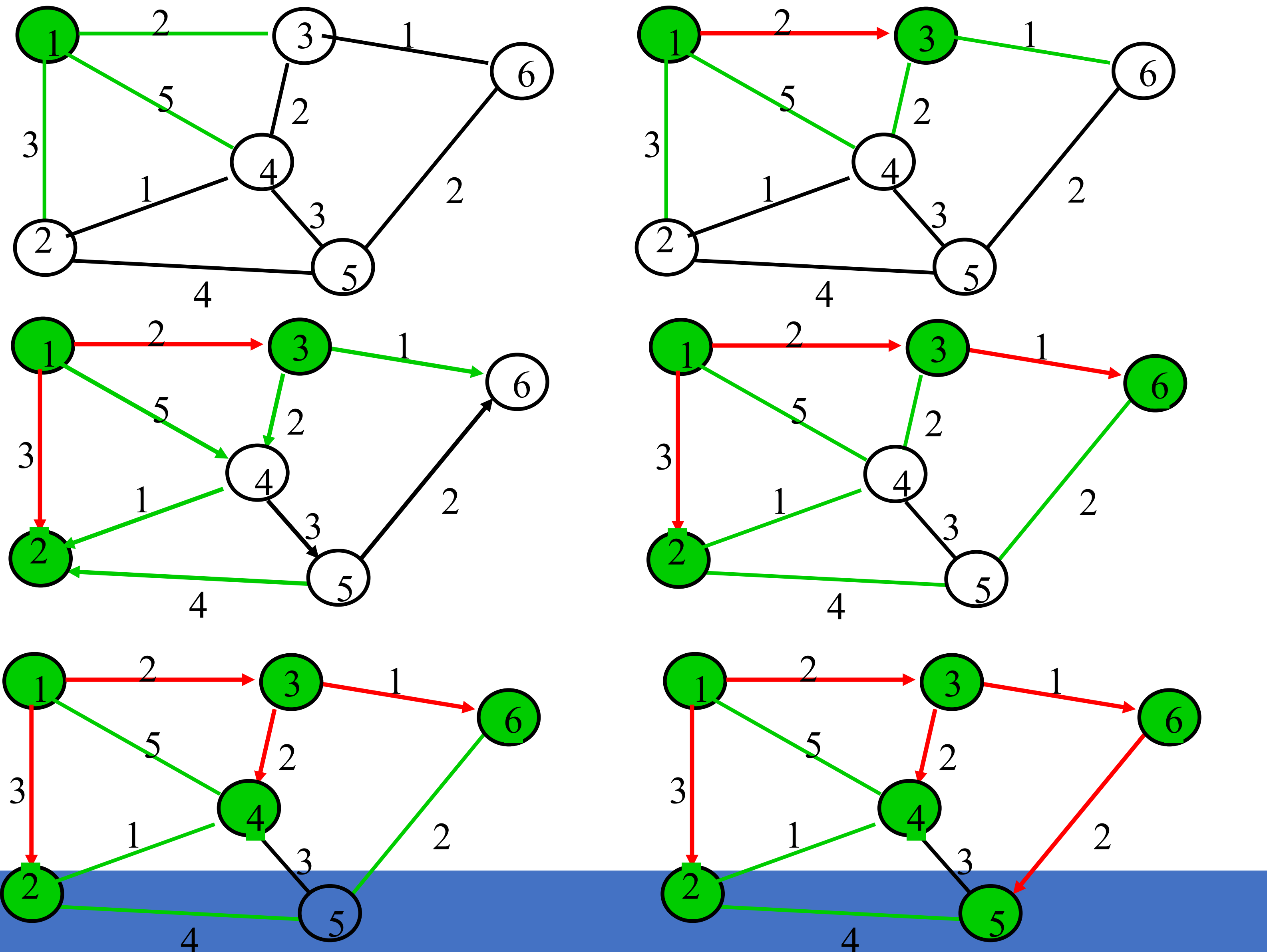
Execution of Dijkstra algorithm





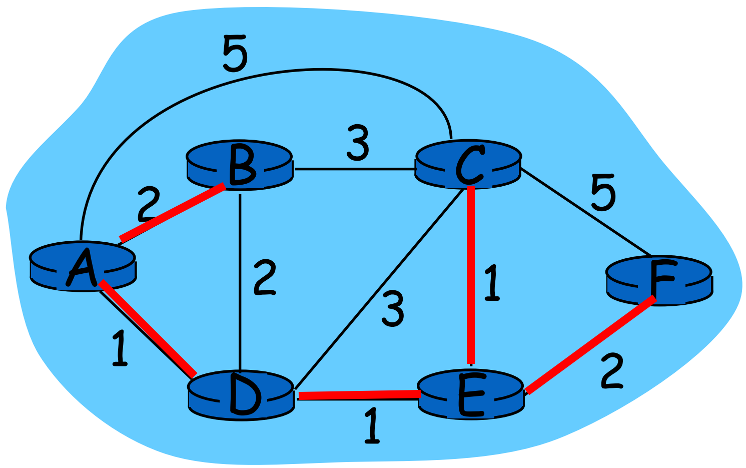
Iteration	N	D_2	D_3	D_4	D_5	D_6
Initial	{1}	3	2	5	$\propto$	$\propto$
1	{1,3}	3	**2**	4	$\propto$	3
2	{1,2,3}	**3**	**2**	4	7	3
3	{1,2,3,6}	**3**	**2**	4	5	**3**
4	{1,2,3,4,6}	**3**	**2**	**4**	5	**3**
5	{1,2,3,4,5,6}	**3**	**2**	**4**	**5**	**3**

Shortest Paths in Dijkstra's Algorithm



Dijkstra's algorithm: example

Step	start N	D(B),p(B)	D(C),p(C)	D(D),p(D)	D(E),p(E)	D(F),p(F)
0	A	2,A	5,A	1,A	infinity	infinity
1	AD	2,A	4,D		2,D	infinity
2	ADE	2,A	3,E			4,E
3	ADEB		3,E			4,E
4	ADEBC					4,E
5	ADEBCF					



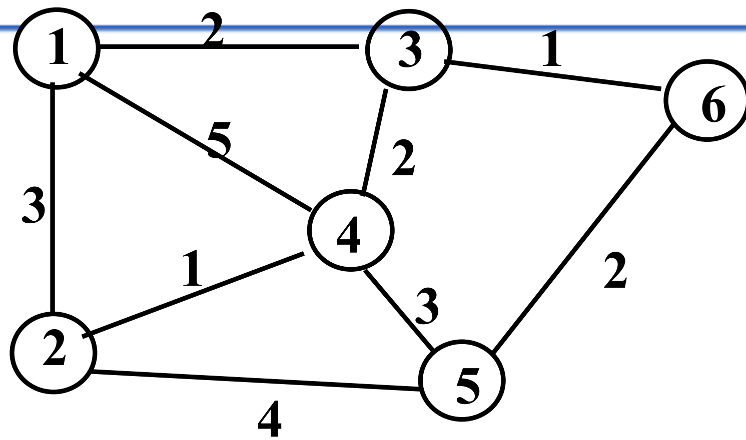
All pairs shortest path

- So far, we have discussed the problem of finding the shortest paths, starting from a specific node .
- How about finding the shorted path between every possible pair of nodes in a graph?
- Which algorithm can be used?
- Dijkstra’ s algorithm can be used
  - Call the Dijkstra’ s algorithm by setting each node as the source node, one by one
- Any Other solution?
  - Floyd -Warshal Algorithm can be used instead.

Floyd-warshall algorithm

- Initially $d_{ij} = w_{ij}$
  - $w_{ij} = 0$ if $i = j$
  - $w_{ij} =$ weight of the directed edge if $i \neq j$, and $(i, j) \in E$
  - $w_{ij} = \infty$ if $i \neq j$, and $(i, j) \notin E$
- **Initially**
  - $p_{ij} = \text{Nil}$ if $i = j$ or $w_{ij} = \infty$
  - $p_{ij} = i$ if $i \neq j$ and $w_{ij} < \infty$

Floyd-warshall algorithm



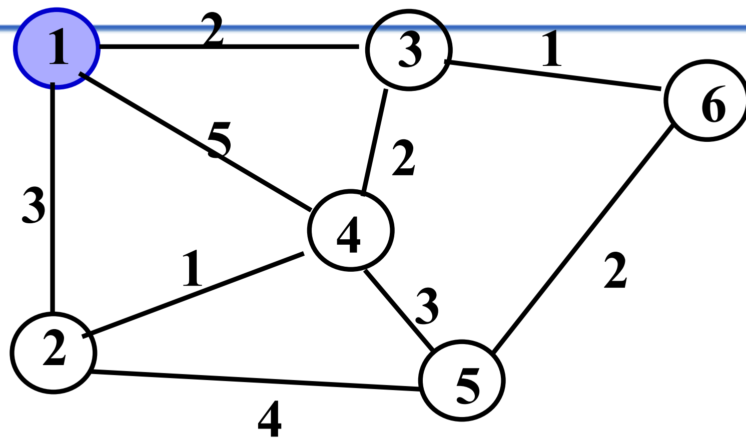
$D^{(0)}$

i / j	1	2	3	4	5	6
1	0	3	2	5	∞	∞
2	3	0	∞	1	4	∞
3	2	∞	0	2	∞	1
4	5	1	2	0	3	∞
5	∞	4	∞	3	0	2
6	∞	∞	1	∞	2	0

$P^{(0)}$

I / j	1	2	3	4	5	6
1	Nil	1	1	1	Nil	Nil
2	2	Nil	Nil	2	2	Nil
3	3	Nil	Nil	3	Nil	3
4	4	4	4	Nil	4	4
5	Nl	5	Nl	5	Nil	5
6	Nil	Nil	6	Nil	6	Nil

Floyd-warshall algorithm



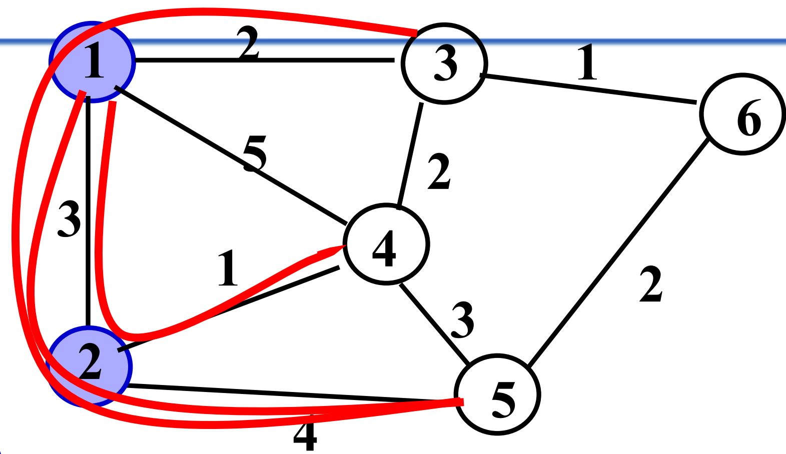
$D^{(1)}$

i / j	1	2	3	4	5	6
1	0	3	2	5	∞	∞
2	3	0	5	1	4	∞
3	2	5	0	2	∞	1
4	5	1	2	0	3	∞
5	∞	4	∞	3	0	2
6	∞	∞	1	∞	2	0

$P^{(1)}$

I / j	1	2	3	4	5	6
1	Nil	1	1	1	Nil	Nil
2	2	Nil	1	2	2	Nil
3	3	1	Nil	3	Nil	3
4	4	4	4	Nil	4	4
5	NI	5	NI	5	Nil	5
6	Nil	Nil	6	Nil	6	Nil

Floyd-warshall algorithm



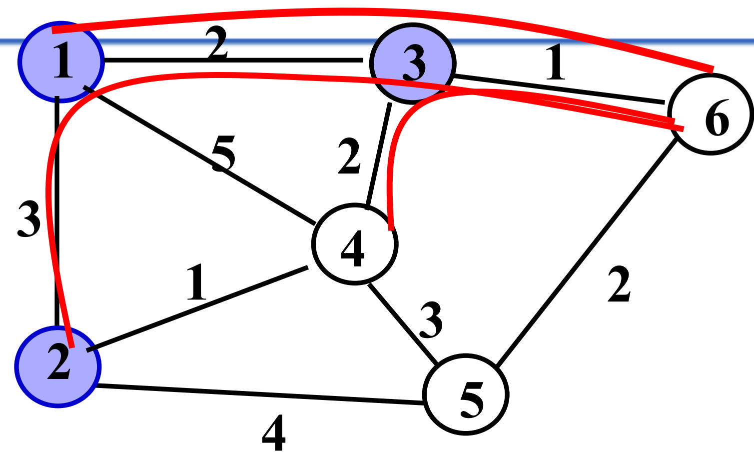
$D^{(2)}$

i / j	1	2	3	4	5	6
1	0	3	2	4	7	∞
2	3	0	5	1	4	∞
3	2	5	0	2	9	1
4	4	1	2	0	3	∞
5	7	4	9	3	0	2
6	∞	∞	1	∞	2	0

$P^{(2)}$

I / j	1	2	3	4	5	6
1	Nil	1	1	2	2	Nil
2	2	Nil	1	2	2	Nil
3	3	1	Nil	3	2	3
4	2	4	4	Nil	4	4
5	2	5	1	5	Nil	5
6	Nil	Nil	6	Nil	6	Nil

Floyd-warshall algorithm



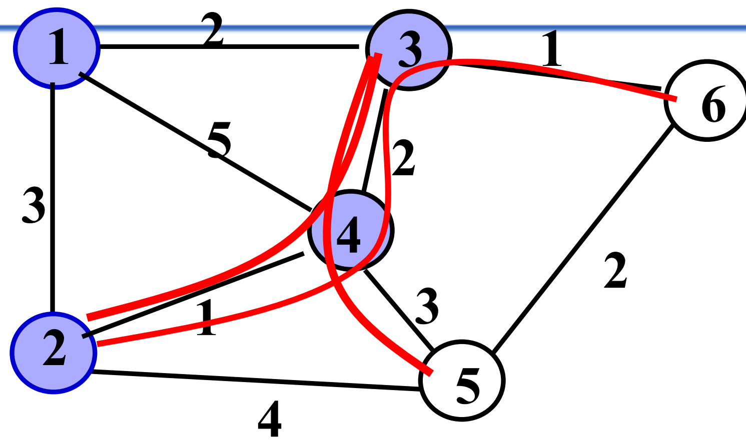
$D^{(3)}$

i / j	1	2	3	4	5	6
1	0	3	2	4	7	3
2	3	0	5	1	4	6
3	2	5	0	2	9	1
4	4	1	2	0	3	3
5	7	4	9	3	0	2
6	3	6	1	3	2	0

$P^{(3)}$

i / j	1	2	3	4	5	6
1	*Nil*	1	1	2	2	3
2	2	*Nil*	1	2	2	3
3	3	1	*Nil*	3	2	3
4	2	4	4	*Nil*	4	3
5	2	5	1	5	*Nil*	5
6	3	1	6	3	6	*Nil*

Floyd-warshall algorithm



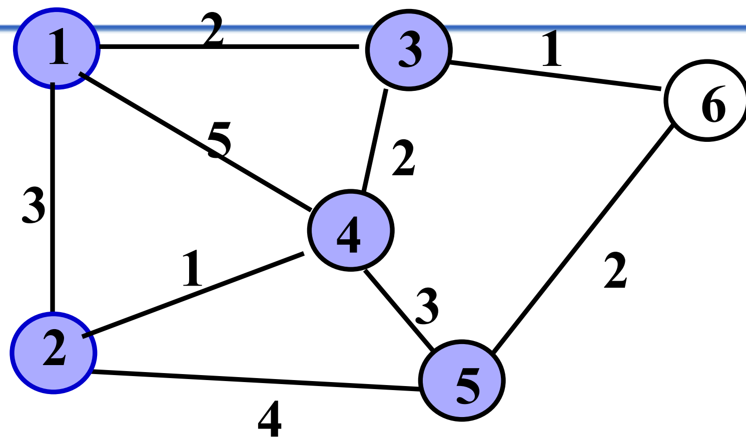
$D^{(4)}$

i / j	1	2	3	4	5	6
1	0	3	2	4	7	3
2	3	0	3	1	4	4
3	2	3	0	2	5	1
4	4	1	2	0	3	3
5	7	4	5	3	0	2
6	3	4	1	3	2	0

$P^{(4)}$

I / j	1	2	3	4	5	6
1	Nil	1	1	2	2	3
2	2	Nil	4	2	2	3
3	3	4	Nil	3	4	3
4	2	4	4	Nil	4	3
5	2	5	4	5	Nil	5
6	4	1	6	3	6	Nil

Floyd-warshall algorithm



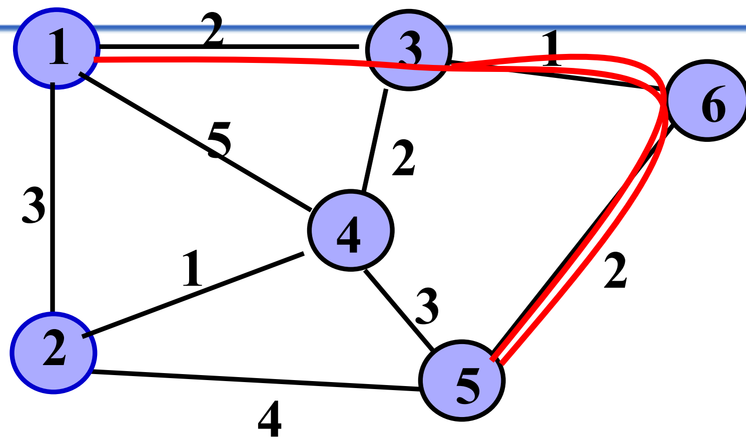
$D^{(4)}$

i / j	1	2	3	4	5	6
1	0	3	2	4	7	3
2	3	0	3	1	4	4
3	2	3	0	2	5	1
4	4	1	2	0	3	3
5	7	4	5	3	0	2
6	3	4	1	3	2	0

$P^{(4)}$

I / j	1	2	3	4	5	6
1	Nil	1	1	2	2	3
2	2	Nil	4	2	2	3
3	3	4	Nil	3	4	3
4	2	4	4	Nil	4	3
5	2	5	4	5	Nil	5
6	4	1	6	3	6	Nil

Floyd-warshall algorithm



$D^{(5)}$

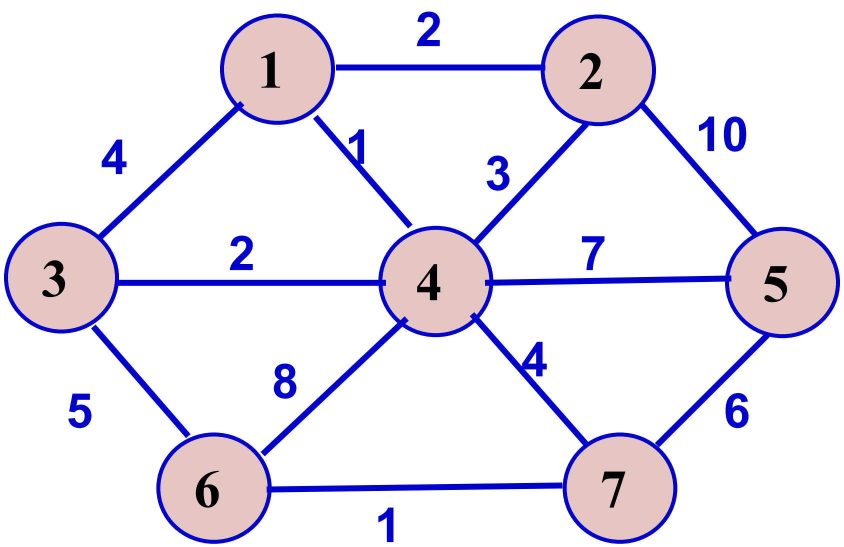
i / j	1	2	3	4	5	6
1	0	3	2	4	5	3
2	3	0	3	1	4	4
3	2	3	0	2	3	1
4	4	1	2	0	3	3
5	5	4	3	3	0	2
6	3	4	1	3	2	0

$P^{(5)}$

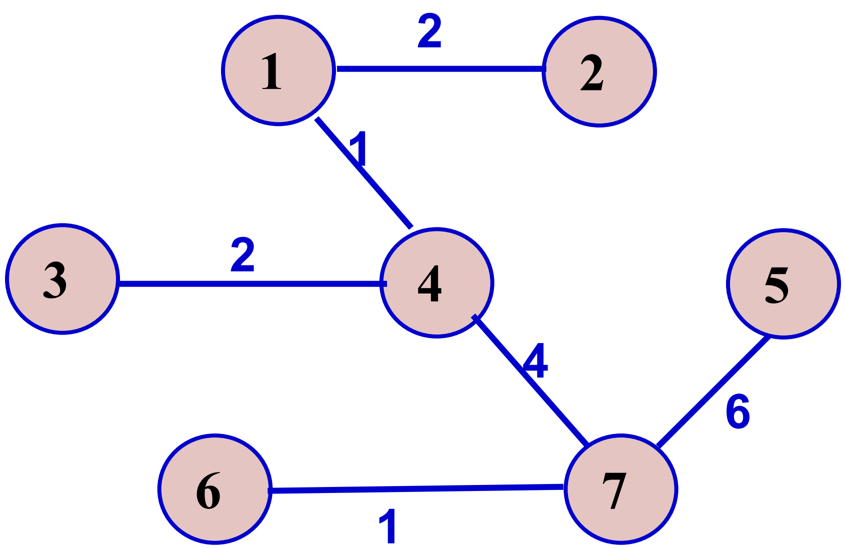
I / j	1	2	3	4	5	6
1	Nil	1	1	2	6	3
2	2	Nil	4	2	2	3
3	3	4	Nil	3	6	3
4	2	4	4	Nil	4	3
5	3	5	6	5	Nil	5
6	4	1	6	3	6	Nil

Minimum Spanning Tree (MST)

- *A* minimum spanning tree of an undirected graph G is a tree formed from graph edges that connects all the vertices of G at lowest total cost.
- A minimum spanning tree exists if and only if G is connected



A graph G



MST of G

Prim Algorithm

Connect all vertices with the **minimum total edge weight**, without forming any cycles.

Start with any vertex (this becomes the root of the MST).
Maintain two sets:

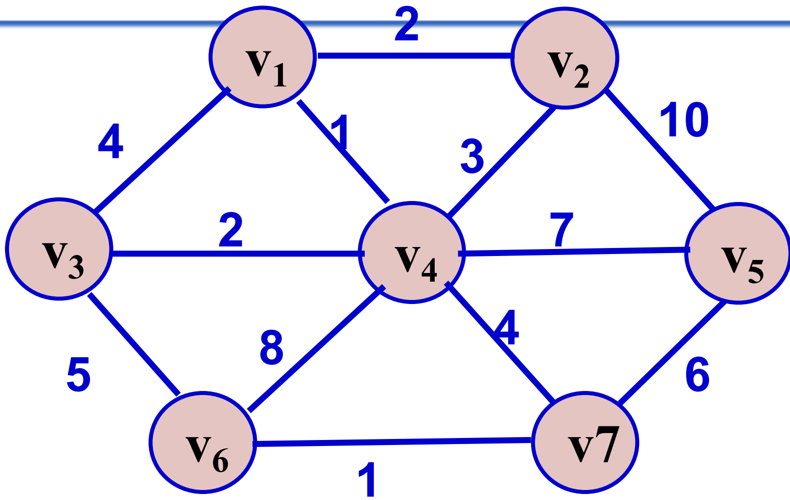
Tree vertices (T): already included in the MST.
Non-tree vertices: not yet in the MST.

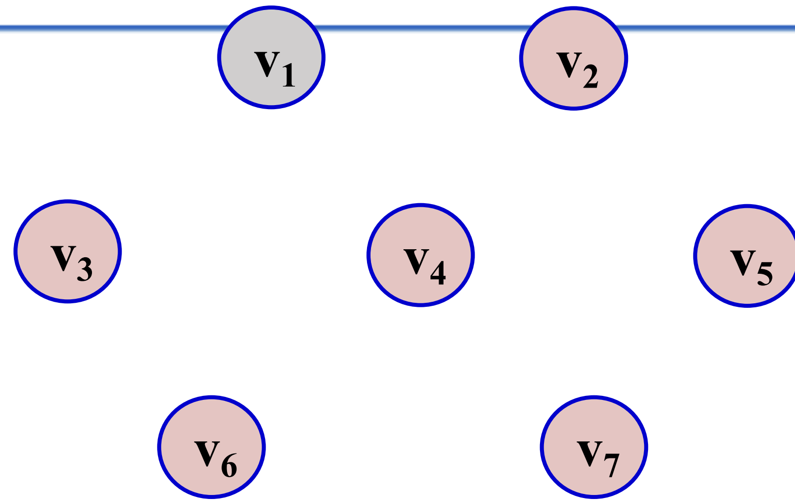
Prim Algorithm

- One way to compute a minimum spanning tree is to grow the tree in successive stages.
- In each stage,
  1. Pick a node as the root,
  2. add an edge, and
  3. the associated vertex, to the tree.
- At any point in the algorithm, we have
  - a set of vertices that have already been included in the tree;
  - the rest of the vertices are not in the tree yet
- At each stage, the algorithm finds
  - a new vertex to add to the tree by choosing the edge **(*u*, *v*)**
  - such that the cost of **(*u*, *v*)** is the smallest among all edges where ***u*** is in the tree and v is not.
- Same as Dijkstra's algo?

Prim Algorithm

v_1	v_4	v_2	v_3	v_7	v_6	v_5

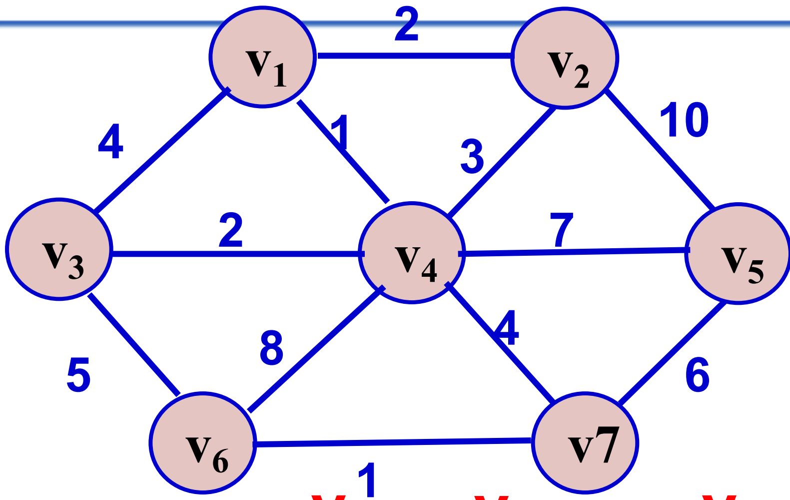


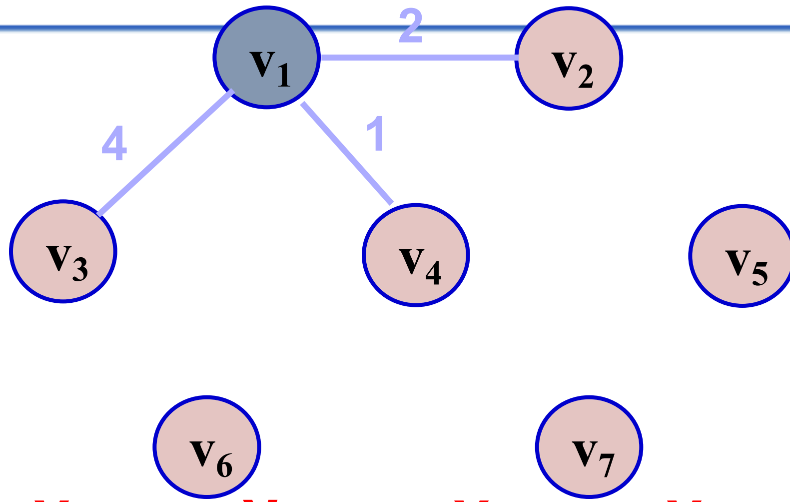


	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v	D_1 , p_1	D_2 , p_2	D_3 , p_3	D_4 , p_4	D_5 , p_5	D_6 , p_6	D_7 , p_7

Prim Algorithm

V_4	V_2	V_3	V_7	V_6	V_5

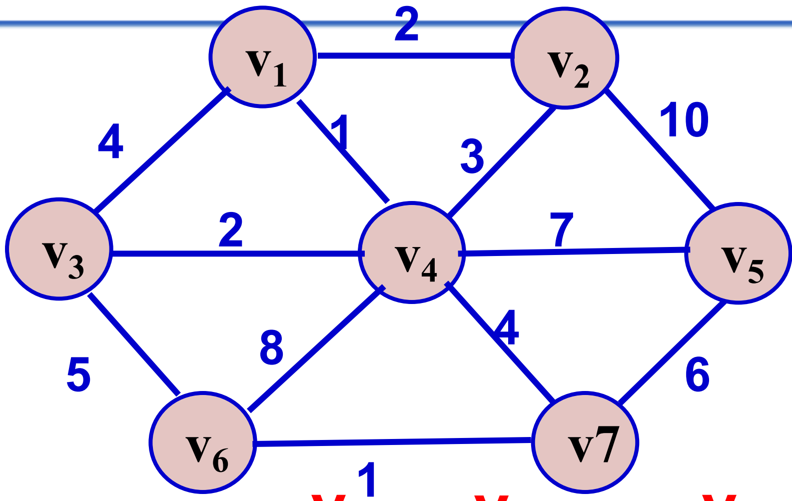


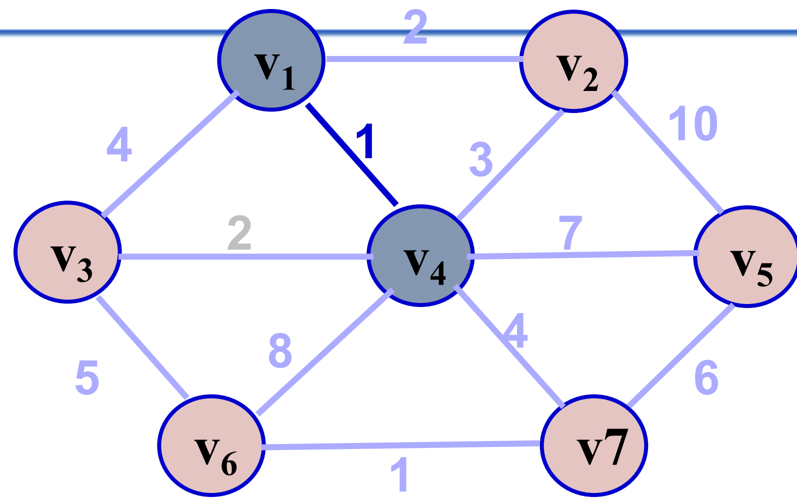


v	v_1 D_1 , p_1	v_2 D_2 , p_2	v_3 D_3 , p_3	v_4 D_4 , p_4	v_5 D_5 , p_5	v_6 D_6 , p_6	v_7 D_7 , p_7
1	0 , 0	2 , v_1	4 , v_1	1 , v_1	∞ , 0	∞ , 0	∞ , 0

Prim Algorithm

V_2	V_3	V_7	V_6	V_5

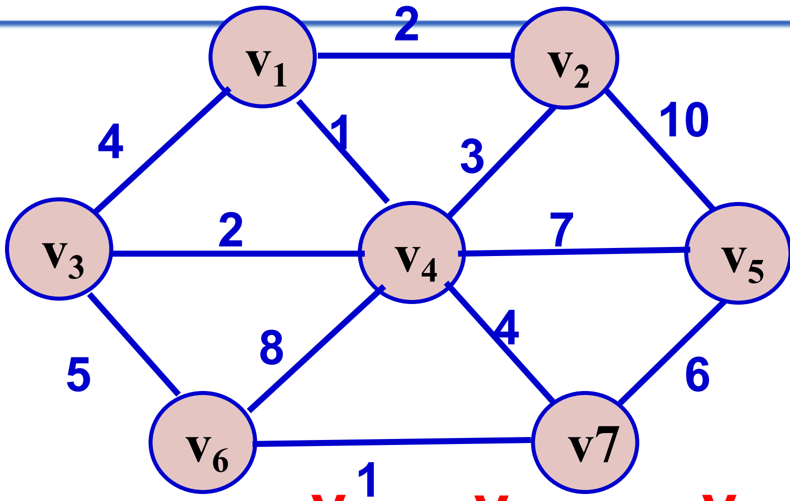


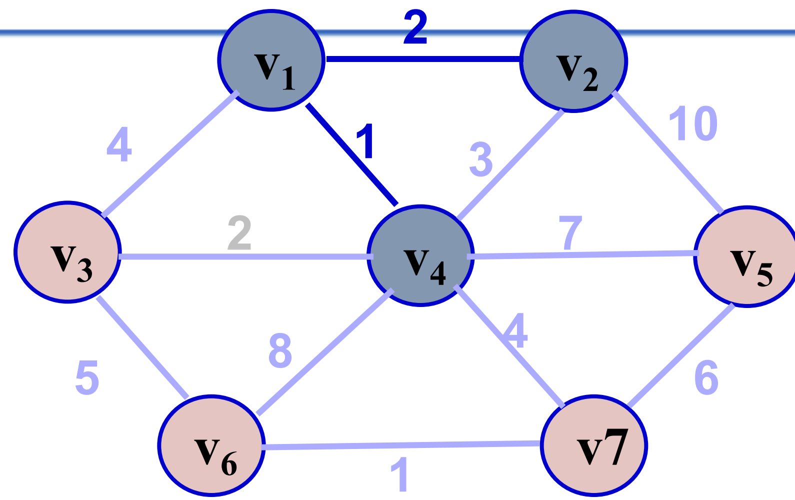


	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v	D_1 , p_1	D_2 , p_2	D_3 , p_3	D_4 , p_4	D_5 , p_5	D_6 , p_6	D_7 , p_7
1	0 , 0	2 , v_1	4 , v_1	1 , v_1	∞ , 0	∞ , 0	∞ , 0
1,4	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4

Prim Algorithm

V_3	V_7	V_6	V_5

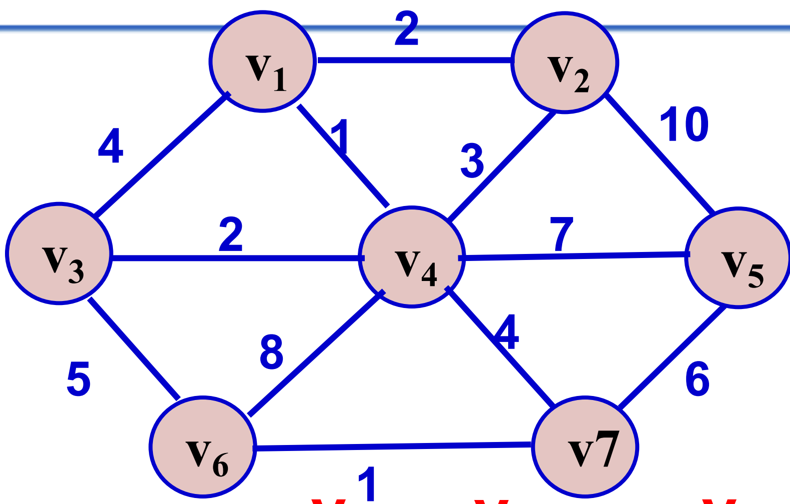


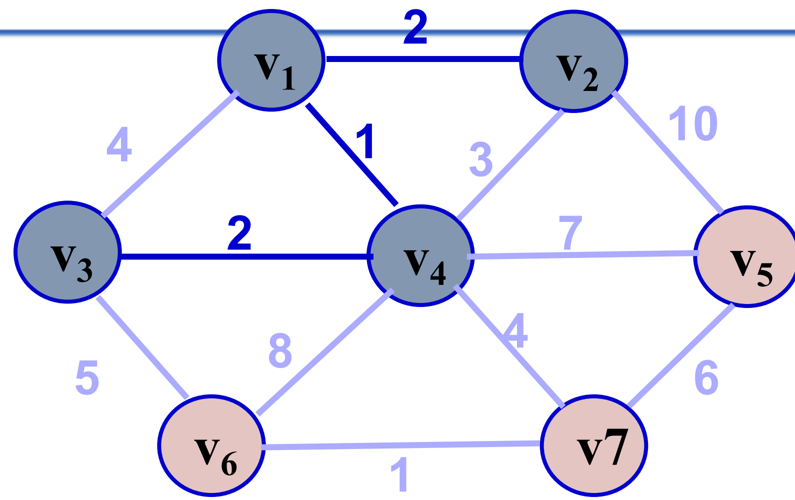


	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v	D_1 , p_1	D_2 , p_2	D_3 , p_3	D_4 , p_4	D_5 , p_5	D_6 , p_6	D_7 , p_7
1	0 , 0	2 , v_1	4 , v_1	1 , v_1	∞ , 0	∞ , 0	∞ , 0
1,4	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4
1,4,2	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4

Prim Algorithm

V_7	V_6	V_5

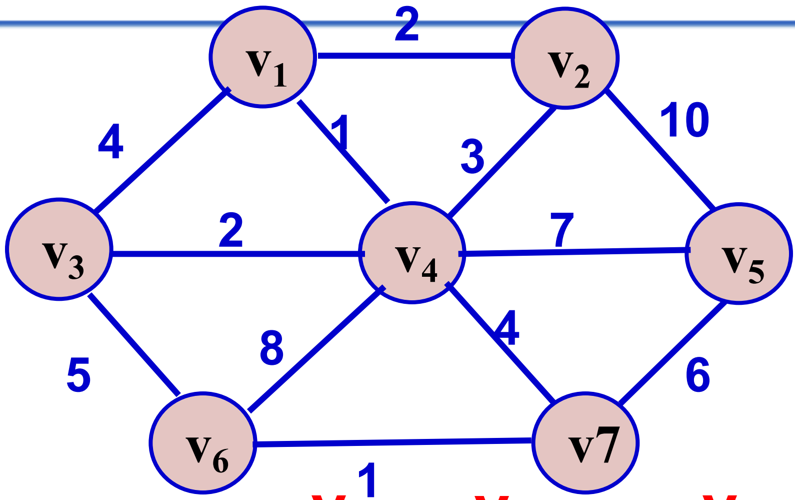


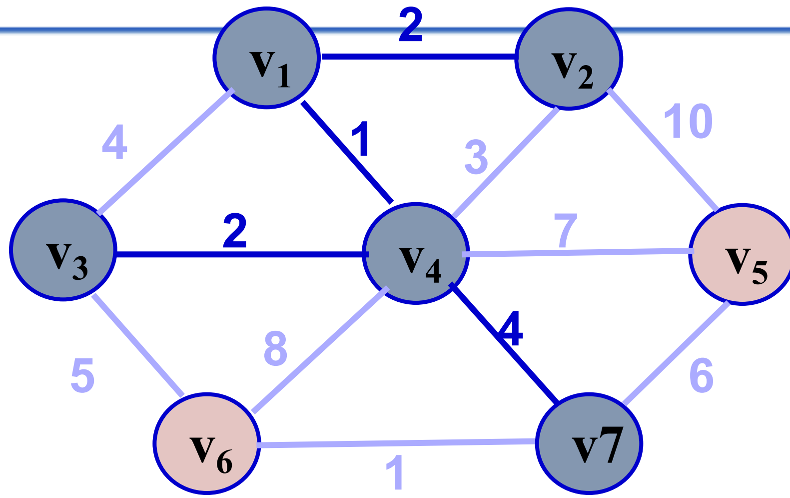


	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v	D_1 , p_1	D_2 , p_2	D_3 , p_3	D_4 , p_4	D_5 , p_5	D_6 , p_6	D_7 , p_7
1	0 , 0	2 , v_1	4 , v_1	1 , v_1	∞ , 0	∞ , 0	∞ , 0
1,4	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4
1,4,2	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4
1,4,2,3	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	5 , v_3	4 , v_4

Prim Algorithm

V_6	V_5

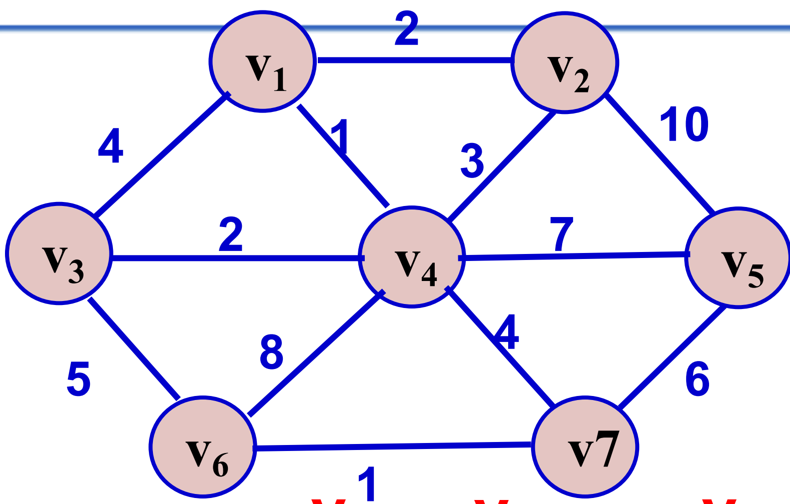


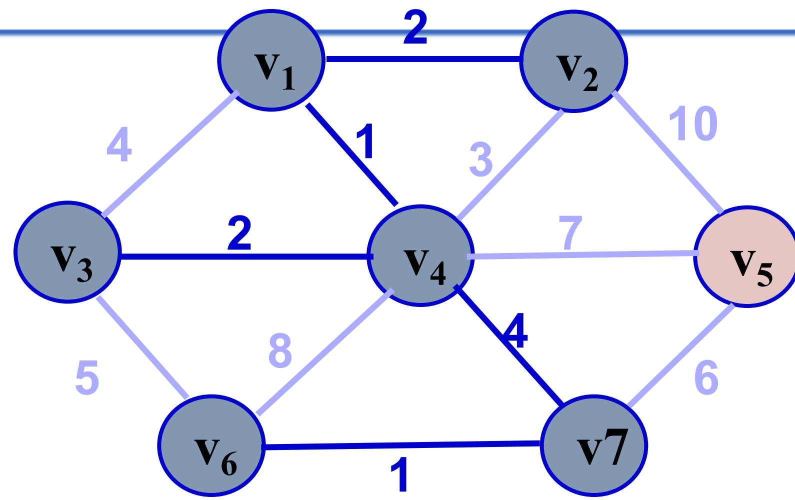


	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v	D_1 , p_1	D_2 , p_2	D_3 , p_3	D_4 , p_4	D_5 , p_5	D_6 , p_6	D_7 , p_7
1	0 , 0	2 , v_1	4 , v_1	1 , v_1	∞ , 0	∞ , 0	∞ , 0
1,4	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4
1,4,2	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4
1,4,2,3	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	5 , v_3	4 , v_4
1,4,2,3,7	0 , 0	2 , v_1	2 , v_4	1 , v_1	6 , v_7	1 , v_7	4 , v_4

Prim Algorithm

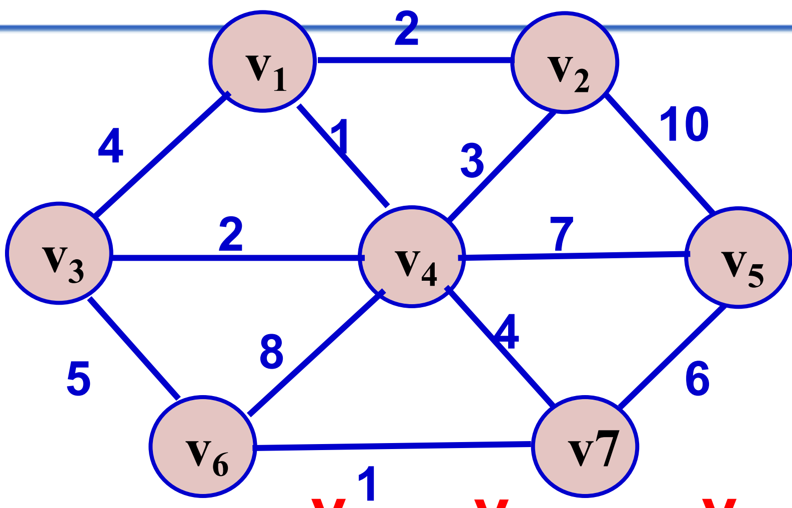
V_5

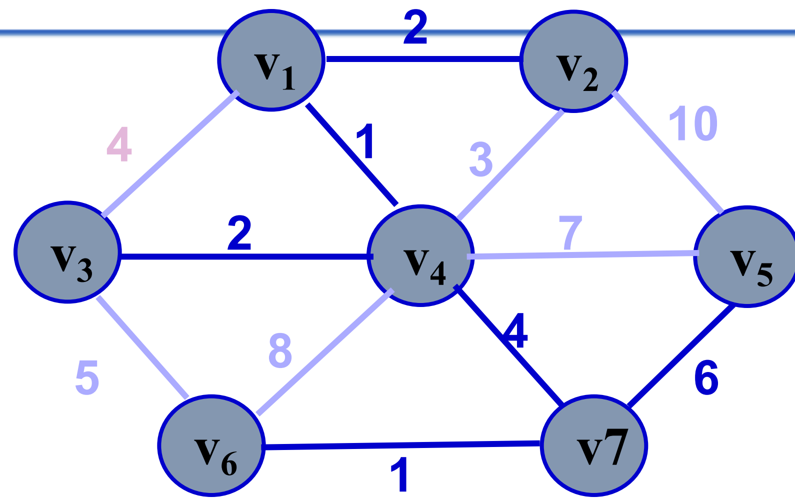




	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v	D_1 , p_1	D_2 , p_2	D_3 , p_3	D_4 , p_4	D_5 , p_5	D_6 , p_6	D_7 , p_7
1	0 , 0	2 , v_1	4 , v_1	1 , v_1	∞ , 0	∞ , 0	∞ , 0
1,4	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4
1,4,2	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4
1,4,2,3	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	5 , v_3	4 , v_4
1,4,2,3,7	0 , 0	2 , v_1	2 , v_4	1 , v_1	6 , v_7	1 , v_7	4 , v_4
1,4,2,3,7,6	0 , 0	2 , v_1	2 , v_4	1 , v_1	6 , v_7	1 , v_7	4 , v_4

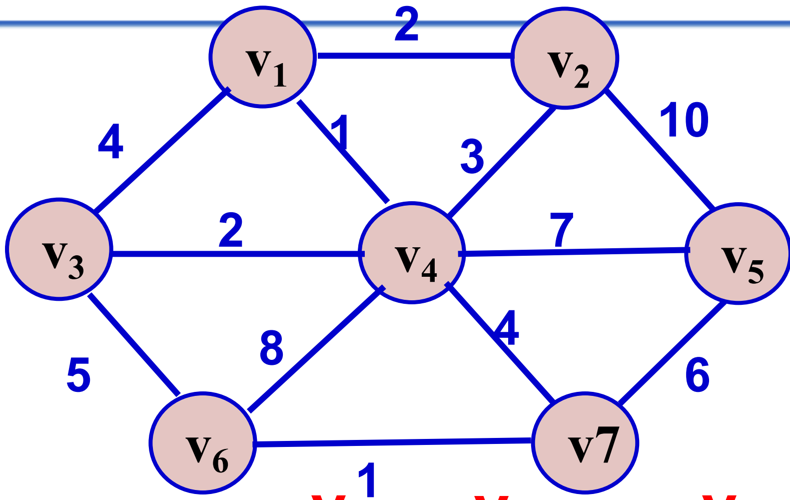
Prim Algorithm

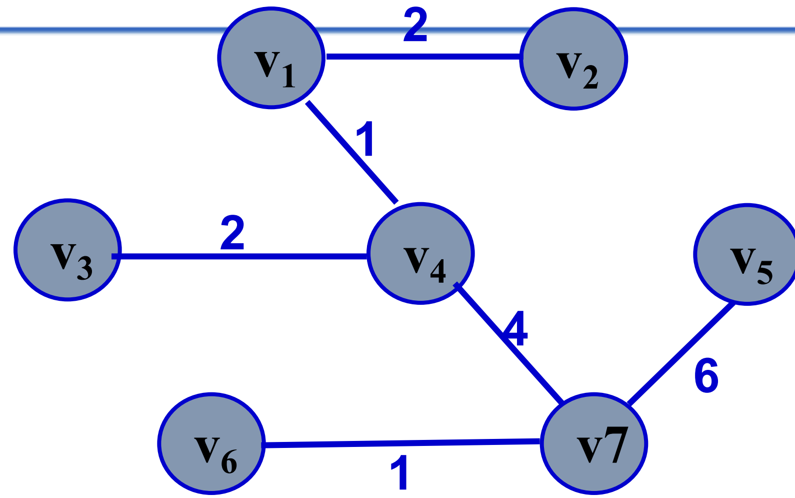




	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v	D_1 , p_1	D_2 , p_2	D_3 , p_3	D_4 , p_4	D_5 , p_5	D_6 , p_6	D_7 , p_7
1	0 , 0	2 , v_1	4 , v_1	1 , v_1	∞ , 0	∞ , 0	∞ , 0
1,4	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4
1,4,2	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4
1,4,2,3	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	5 , v_3	4 , v_4
1,4,2,3,7	0 , 0	2 , v_1	2 , v_4	1 , v_1	6 , v_7	1 , v_7	4 , v_4
1,4,2,3,7,6	0 , 0	2 , v_1	2 , v_4	1 , v_1	6 , v_7	1 , v_7	4 , v_4
1 to 7	0 , 0	2 , v_1	2 , v_4	1 , v_1	6 , v_7	1 , v_7	4 , v_4

Prim Algorithm





	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v	D_1 , p_1	D_2 , p_2	D_3 , p_3	D_4 , p_4	D_5 , p_5	D_6 , p_6	D_7 , p_7
1	0 , 0	2 , v_1	4 , v_1	1 , v_1	∞ , 0	∞ , 0	∞ , 0
1,4	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4
1,4,2	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	8 , v_4	4 , v_4
1,4,2,3	0 , 0	2 , v_1	2 , v_4	1 , v_1	7 , v_4	5 , v_3	4 , v_4
1,4,2,3,7	0 , 0	2 , v_1	2 , v_4	1 , v_1	6 , v_7	1 , v_7	4 , v_4
1,4,2,3,7,6	0 , 0	2 , v_1	2 , v_4	1 , v_1	6 , v_7	1 , v_7	4 , v_4
1 to 7	0 , 0	2 , v_1	2 , v_4	1 , v_1	6 , v_7	1 , v_7	4 , v_4

Kruskal Algorithm

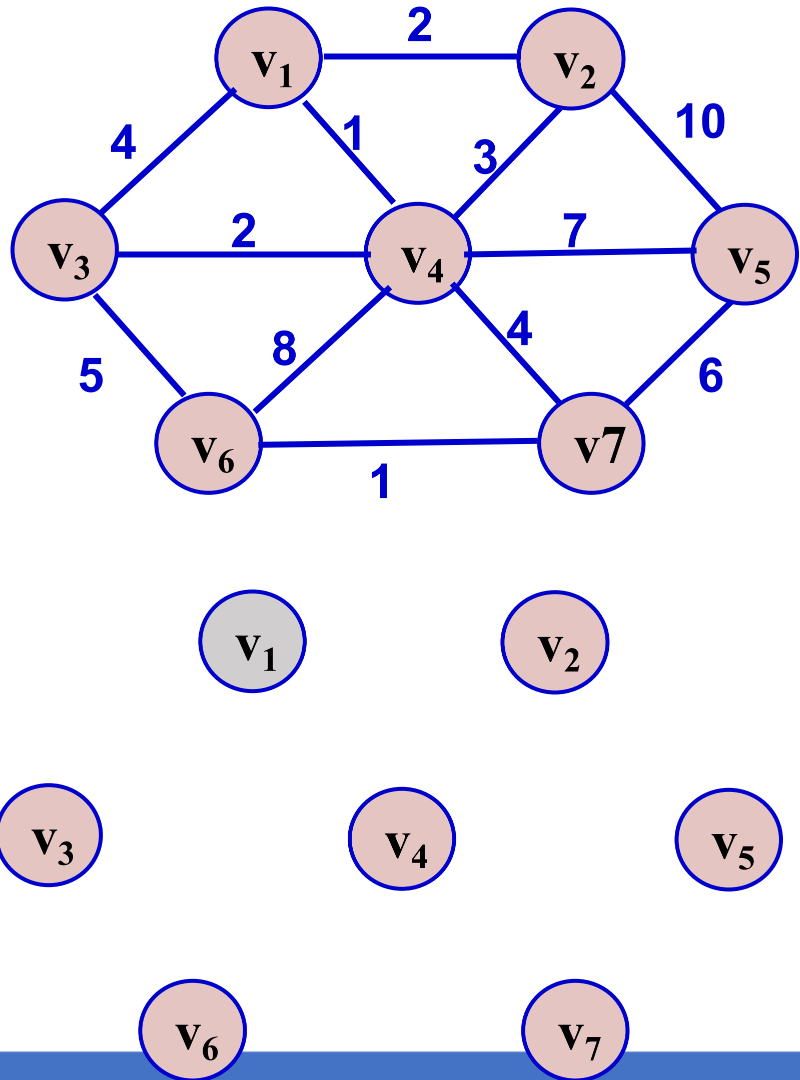
MST-Kruskal (G,w)

```
F = { }
For each vertex v ∈ V[G]
  MakeSet(v)
sort the edges(E) in increasing order by
weight w
for each edge (u,v) ∈ E
  if FindSet(u) ≠ FindSet(v)
      F = F U {(u,v)}
        UNION(u,v)
```

Kruskal Algorithm

- MakeSet(***v***): Create a new set whose only member is pointed to by ***v***. Note that for this operation ***v*** must already be in a set.
- FindSet(v): Returns a pointer to the set containing ***v***.
- UNION(**u, v**): Unites the dynamic sets that contain ***u*** and ***v*** into a new set that is union of these two sets.

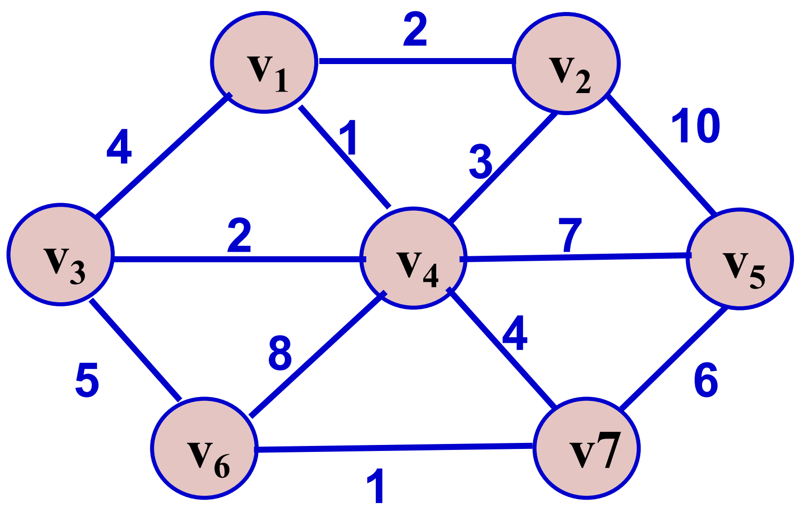
Kruskal’s Algorithm



SETS

$\{ v_1 \}$,
$\{ v_2 \}$,
$\{ v_3 \}$,
$\{ v_4 \}$,
$\{ v_5 \}$,
$\{ v_6 \}$,
$\{ v_7 \}$

Kruskal's Algorithm



SETS

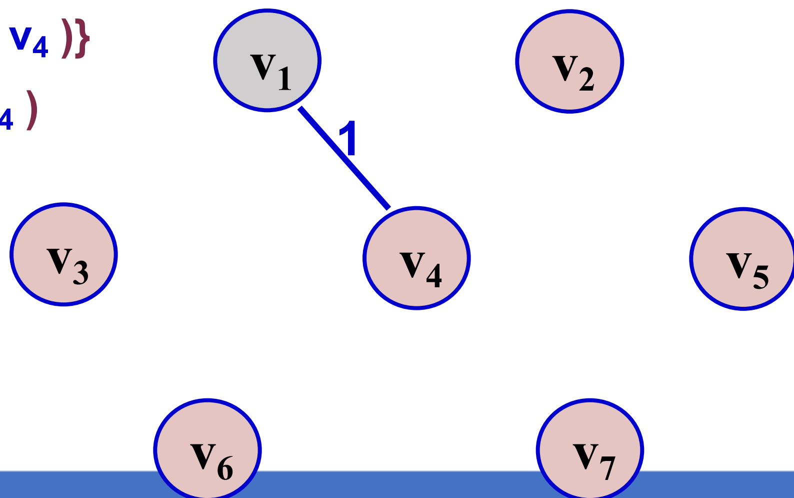
$\{ v_1 , v_4 \}$,
$\{ v_2 \}$,
$\{ v_3 \}$,
$\{ v_5 \}$,
$\{ v_6 \}$,
$\{ v_7 \}$

Min Edge = (v_1, v_4)

FindSet(v_1) $\neq$ FindSet(v_4)

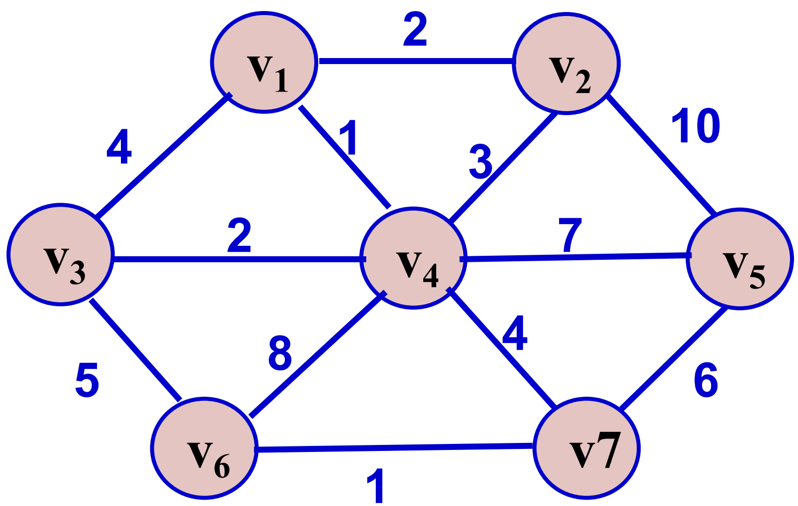
$F = F \cup \{(v_1 , v_4)\}$

UNION(v_1 , v_4)



$F = \{ (v_1 , v_4) \}$

Kruskal's Algorithm



SETS

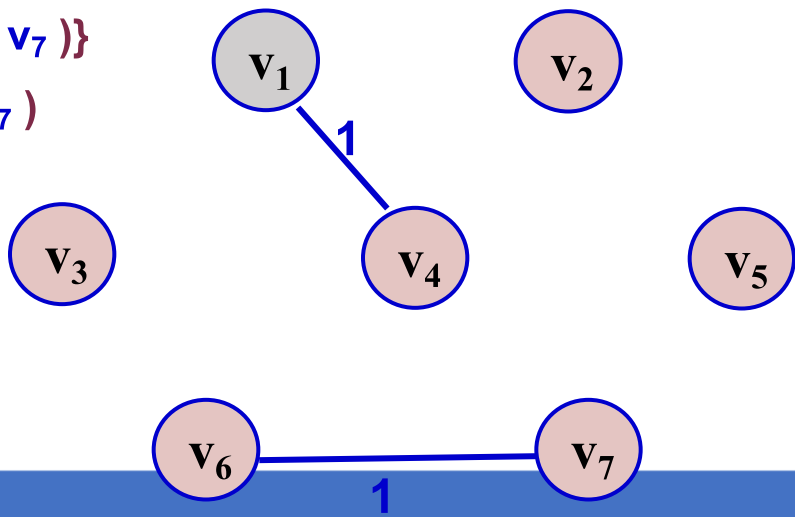
$\{ v_1 , v_4 \}$,
$\{ v_2 \}$,
$\{ v_3 \}$,
$\{ v_5 \}$,
$\{ v_6 , v_7 \}$

Min Edge = (v_6, v_7)

FindSet(v_6) $\neq$ FindSet(v_7)

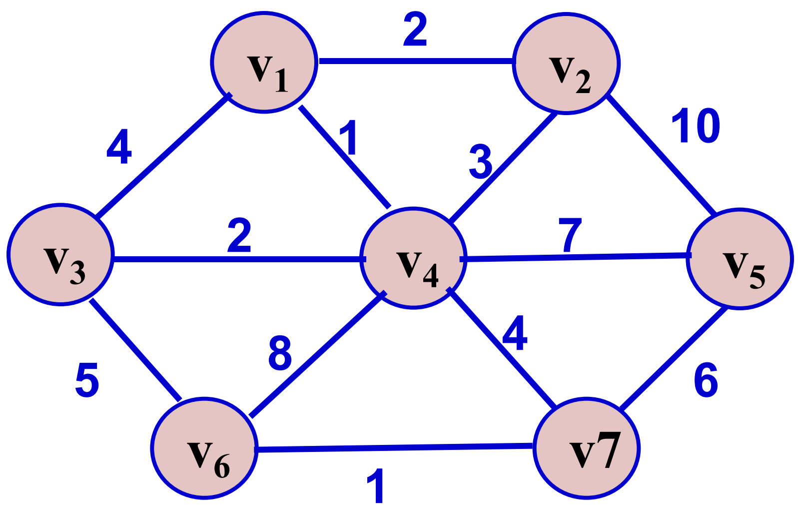
$F = F \cup \{(v_6 , v_7)\}$

UNION(v_6 , v_7)



$F = \{ (v_1 , v_4) ,$
$(v_6 , v_7) \}$

Kruskal’s Algorithm



SETS

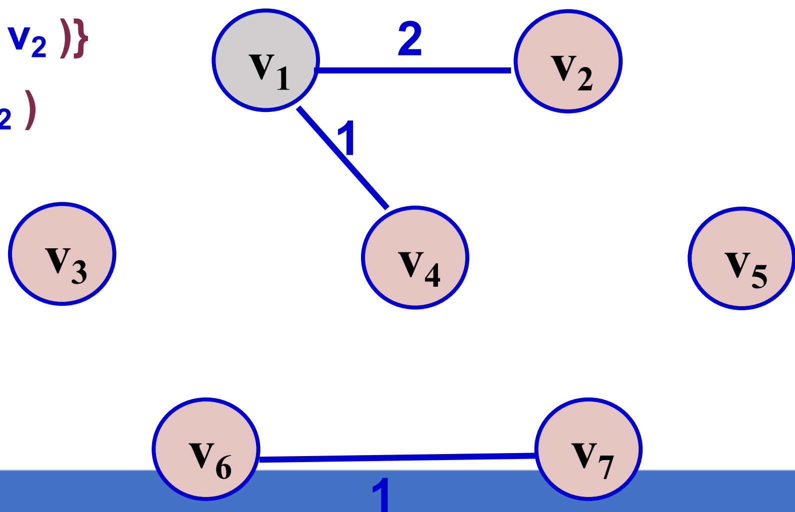
$\{v_1, v_2, v_4\}$,
$\{v_3\}$,
$\{v_5\}$,
$\{v_6, v_7\}$

Min Edge = (v_1, v_2)

FindSet(v_1) $\neq$ FindSet(v_2)

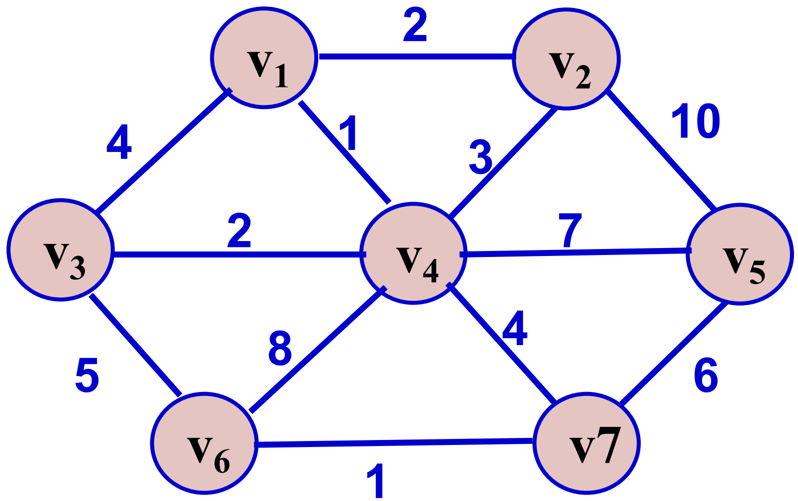
$F = F \cup \{(v_1, v_2)\}$

UNION(v_1 , v_2)



$F = \{ (v_1, v_4),$
$(v_6, v_7),$
$(v_1, v_2) \}$

Kruskal's Algorithm



SETS

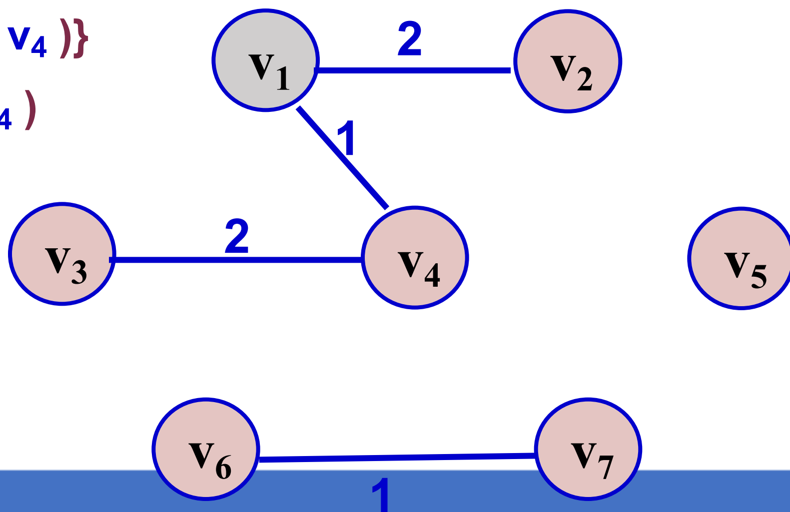
$\{v_1 , v_2 , v_3 , v_4\}$,
$\{ v_5 \}$,
$\{ v_6 , v_7 \}$

Min Edge = (v_3, v_4)

FindSet(v_3) $\neq$ FindSet(v_4)

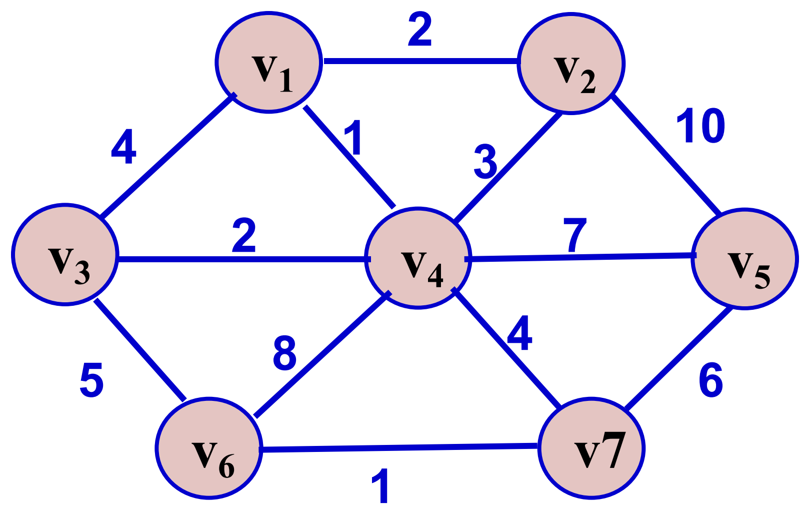
$F = F \cup \{(v_3 , v_4)\}$

UNION(v_3 , v_4)



$F = \{ (v_1 , v_4) ,$
$(v_6 , v_7) ,$
$(v_1 , v_2) ,$
$(v_3 , v_4) \}$

Kruskal's Algorithm



SETS

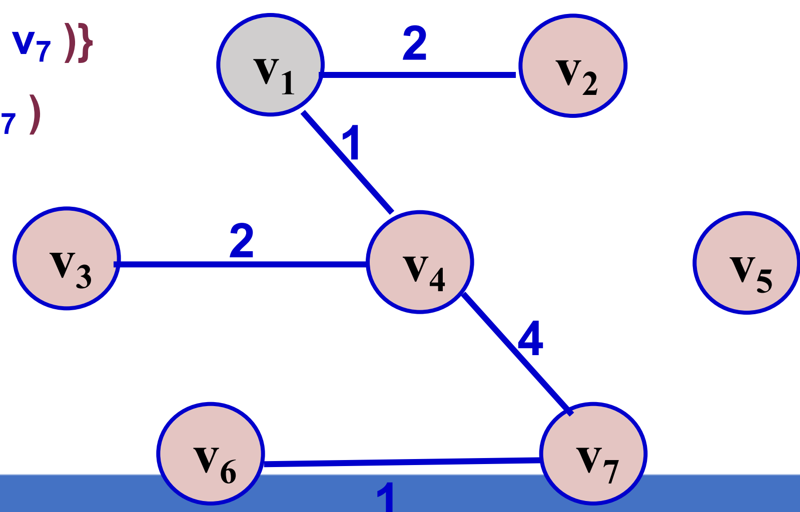
$\{v_1, v_2, v_3, v_4, v_6, v_7\}$,
$\{v_5\}$

Min Edge = (v_4, v_7)

FindSet(v_4) $\neq$ FindSet(v_7)

F = *F* U {(v_4 , v_7)}

UNION(v_4 , v_7)

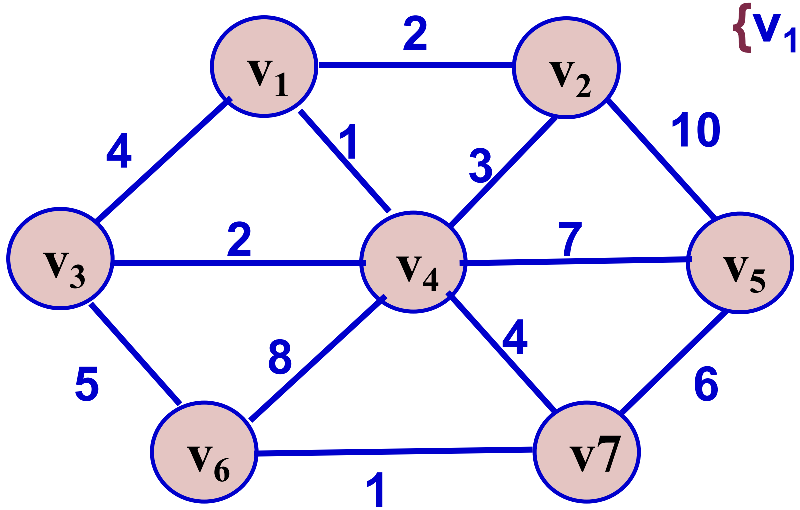


F ={ (v_1 , v_4) ,
(v_6 , v_7) ,
(v_1 , v_2) ,
(v_3 , v_4) ,
(v_4 , v_7) }

Kruskal’s Algorithm

FINAL SET

$\{v_1 , v_2 , v_3 , v_4 , v_5 , v_6 , v_7\}$,

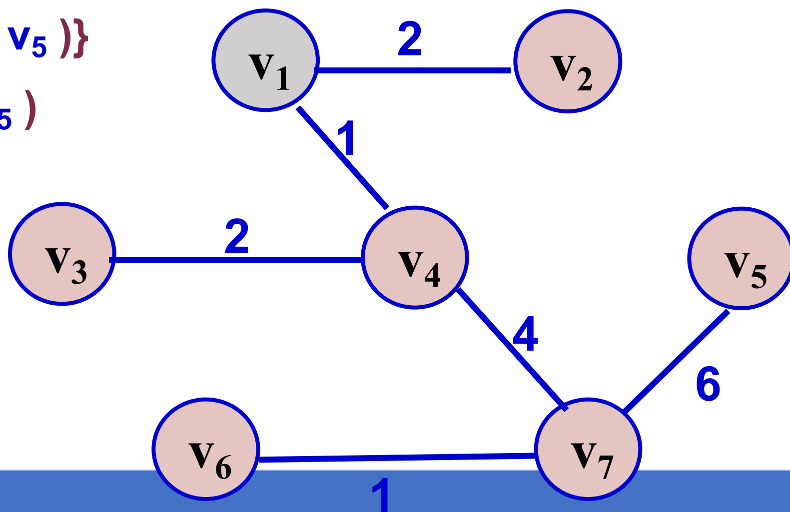


Min Edge = (v_7, v_5)

FindSet(v_7) $\neq$ FindSet(v_5)

$F = F \cup \{(v_7 , v_5)\}$

UNION(v_7 , v_5)



$F =\{ (v_1 , v_4) ,$
$(v_6 , v_7) ,$
$(v_1 , v_2) ,$
$(v_3 , v_4) ,$
$(v_4 , v_7) ,$
$(v_7 , v_5) \}$

Questions?

zahmaad.github.io