Further Results on Inverse Eigenvalue Problem For Mass-spring-Inerter Systems★

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Abstract

This paper is concerned with the inverse eigenvalue problem (IEP) for a "fixed-fixed" mass-spring-inerter system. Unlike the "fixed-free" case, a "fixed-fixed" system has its both ends attached to the ground. This brings some essential differences. We find that the construction strategy proposed in [1] cannot be readily applied here. So we endeavor to develop a new strategy in this paper to solve the IEP and accordingly derive a necessary and sufficient condition for the "fixed-fixed" system.

Keywords: Inverse eigenvalue problem, Natural frequency assignment, Inerter, Mass-spring-inerter system, Multiple eigenvalues

1. Introduction

Natural frequency assignment is a basic problem in classical vibration theory. It is often mathematically associated with the reconstruction of specific structured matrices from designated spectral data [2]. A typical inverse eigenvalue problem (IEP) in this regard concerns a mass-spring system, in which each mass is connected to its neighbour by a spring. Accordingly, the matrices to be sought are the mass and stiffness matrices. This IEP was solved by translating it into an inverse problem for a Jacobi matrix.

A prominent property of a Jacobi matrix is that its eigenvalues are distinct and strictly interlace the eigenvalues of its truncated matrix. Conversely, [3, 4] proved that given a set of interlacing real numbers $0 \le \lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n$, there exists a unique Jacobi matrix such that $\{\lambda_i\}_{i=1}^{n}$ and $\{\mu_i\}_{i=1}^{n-1}$ are the eigenvalues of the Jacobi matrix and its truncated matrix, respectively. It was based on these results that researchers recovered a mass-spring system in the IEP [5, 6]. More precisely, they untangled the mass and stiffness matrices from a Jacobi matrix, which was constructed according to the prescribed eigenvalues. Variants of IEPs for Jacobi operators and modified mass-spring systems are also addressed in the literature [6–9]. Among these studies, it is noteworthy that for physical realization, the prescribed eigenvalues must be distinct due to the inherent property of Jacobi matrices mentioned above.

But the situation changes when involving inerters. The inerter is a new mechanical element first proposed to complete the force-current analogy between mechanical and electrical systems in [10]. Owing to the special merits in practice, inerters are applied to many engineering fields such as building vibration control, landing gears, vibration isolators, train suspensions, vehicle suspensions, and so on [11–16]. Recall that the masses should be appropriately chosen in the IEP of a mass-spring system. But when inerters are introduced in parallel to the springs, [17] found that as long as the given eigenvalues are positive and distinct, the IEP for the corresponding mass-chain system can be solved with all the masses arbitrarily taken. This demonstrates

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the superiority of using inerters since masses are frequently fixed in practice. Besides, [17] pointed out that a mass-spring-inerter system may have multiple eigenvalues. Such a phenomenon is mainly attributed to the involvement of inerters, which makes the IEP no longer a Jacobi inverse eigenvalue problem. So, tools for Jacobi matrices fail to work here. Recently, [1] solved the IEP for "fixed-free" mass-spring-inerter systems by deriving a necessary and sufficient condition, under which the natural frequencies can be designated arbitrarily positive. Specifically speaking, [1] shows that if $\lambda_1 < \cdots < \lambda_m$ are m positive real numbers, where λ_i has multiplicity t_i, then these numbers can be assigned as the natural frequencies of a "fixed-free" mass-spring-inerter system if and only if $t_i \leq i$, $i = 1, \ldots, m$. In addition, m masses of the system are allowed to be fixed beforehand in the assignment.

This paper concerns the IEP for a "fixed-fixed" mass-spring-inerter system, as shown in Figure 1. In contrast to the "fixed-free" case, the system has its both ends attached to the ground. The free vibration equation also has the form

$$(\boldsymbol{M} + \boldsymbol{B})\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = 0,$$

though the above three matrices turn out to be

$$\boldsymbol{M} = \mathrm{diag}\{m_1, m_2, \ldots, m_n\}, \tag{1}$$

$$\boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n + k_{n+1} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} b_1 + b_2 & -b_2 & & & \\ -b_2 & b_2 + b_3 & -b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & b_{n-1} + b_n & -b_n \\ & & & -b_n & b_n + b_{n+1} \end{bmatrix}. \tag{2}$$

Here, real numbers $m_j > 0$ for $j = 1, \ldots, n$ and $k_j > 0$, $b_j \geq 0$ for $j = 1, \ldots, n+1$ represent the masses, spring stiffnesses and inertances. We shall prove that the IEP for a "fixed-fixed" mass-spring-inerter system is solvable if and only if $t_i \leq i + 1, i = 1, \ldots, m$, where t_i is the multiplicity of λ_i and $0 < \lambda_1 < \cdots < \lambda_n$ are the prescribed eigenvalues.

As a matter of fact, if we replace one of the terminal ground by a new mass m_{n+1}, the system becomes "fixed-free" with $n+1$ degree of freedom. Therefore, it is quite natural to apply the construction strategy proposed in [1] to the corresponding "fixed-free" mass-spring-inerter system and thereby construct the desired matrices $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$, as illustrated in Example 1.1 below.

Example 1.1. Let λ_1 be a positive number. We shall construct $\boldsymbol{M} = \mathrm{diag}\{m_1, m_2\}$,

$$\boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} b_1 + b_2 & -b_2 \\ -b_2 & b_2 + b_3 \end{bmatrix}$$

such that $(\lambda - \lambda_1)^2 \Big| \det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B}))$. For this, we introduce one more mass m_3 and aim to recover $\boldsymbol{M}' = \mathrm{diag}\{m_1, m_2, m_3\}$,

$$\boldsymbol{K}' = \begin{bmatrix} k_1 + k_2 & -k_2 & \\ -k_2 & k_2 + k_3 & -k_3 \\ & -k_3 & k_3 \end{bmatrix}, \quad \boldsymbol{B}' = \begin{bmatrix} b_1 + b_2 & -b_2 & \\ -b_2 & b_2 + b_3 & -b_3 \\ & -b_3 & b_3 \end{bmatrix}.$$

We achieve this recover by applying the construction strategy in [1] and then obtain $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$ directly. The construction procedure is contained in Appendix A.

Unfortunately, this idea is not always viable. It fails in Example 1.2.

Example 1.2. Let $t_1 = 2, t_2 = t_3 = 3$ and $\lambda_1, \lambda_2, \lambda_3$ be three positive numbers satisfying

$$\lambda_3 - \lambda_2 - 1 > \frac{\lambda_3}{2} > \lambda_1 > 8(\lambda_2 - \lambda_1) > 16. \tag{3}$$

Clearly, $t_i \le i+1$, $i = 1,2,3$. Theorem 2.1 we provide later shows that there are some $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$ in the form of (1)–(2) such that $\prod_{i=1}^{3}(\lambda - \lambda_i)^{t_i} \Big| \det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B}))$. To apply the construction strategy proposed by [1], we also introduce a mass m_9 and try to recover $\boldsymbol{M'} = \operatorname{diag}\{m_1, m_2, \ldots, m_9\}$,

$$\boldsymbol{K'} = \begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_8 & k_8+k_9 & -k_9 \\ & & & -k_9 & k_9 \end{bmatrix}, \quad \boldsymbol{B'} = \begin{bmatrix} b_1+b_2 & -b_2 & & & \\ -b_2 & b_2+b_3 & -b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_8 & b_8+b_9 & -b_9 \\ & & & -b_9 & b_9 \end{bmatrix}.$$

However, we shall prove in Appendix A that this construction strategy is infeasible.

Actually, the multiplicity of each designated eigenvalue matters for the solvability of the IEP. Now, the necessary and sufficient condition becomes $t_i \le i+1$, whereas it is $t_i \le i$ for the "fixed-free" case. This results in the possible failure of the construction strategy proposed in [1]. So, we shall develop a new strategy to recover the "fixed-fixed" mass-spring-inerter system from a set of prescribed natural frequencies. In the new strategy, almost all our endeavors are focused on the one more multiplicity of $\lambda_i, i = 1, \ldots, m$. As before, the construction admits a total of m masses being taken arbitrarily, where m is the number of the given distinct eigenvalues.

This paper is built up as follows. In Section 2, we present the main result by deducing a necessary and sufficient condition of the IEP for "fixed-fixed" mass-spring-inerter systems. The necessity and sufficiency of the theorem are proved in Sections 3 and 4, respectively. Conclusions are drawn in Section 5.

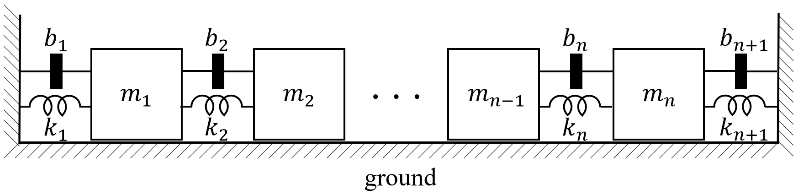

Figure 1: Mass-chain system with inerters.

Notation: We shall adopt the following notations hereinafter. Let "[ ]" represent an interval of integers. For instance, $[n] = \{1, 2, \ldots, n\}$ and $[m\ n] = \{m, m+1, \ldots, n\}$, $m, n \in \mathbb{N}^+$, where $\mathbb{N}^+$ is the set of positive integers. Moreover, denote by $(f(\lambda), g(\lambda))$ the greatest common divisor of two polynomials $f(\lambda)$ and $g(\lambda)$.

2. Main Result

Since the roots of the following characteristic equation

$$\det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B})) = 0$$

determine the natural frequencies of the mass-spring-inerter system, consistent with [1] and [17], we use the terms "eigenvalues" and the "natural frequencies" interchangeably. For the "fixed-fixed" case, we put forward the IEP as follows.

Problem 1. Arbitrarily given a set of real numbers $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, is it possible to recover matrices $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$ in (1) and (2) by choosing $m_j > 0$, $k_j > 0$ and $b_j \ge 0$ for $j = 1, \ldots, n$, so that the n eigenvalues of matrix pencil $\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B})$ are exactly λ_i, $i = 1, \ldots, n$?

The IEP has been completely solved by [1] for "fixed-free" systems. We naturally expect an analogous answer in this paper. It is indeed the case, except the previous construction strategy developed by [1] encounters an essential obstacle in the current situation. This obstacle is fatal, as illustrated in Example 1.2. So, we are forced to seek a new construction strategy.

Here is the main result of the paper.

Theorem 2.1. Let $\prod_{i=1}^{m}(\lambda-\lambda_i)^{t_i}$ be a polynomial with $0<\lambda_1<\lambda_2<\ldots<\lambda_m$ and $\sum_{i=1}^{m} t_i = n$. Then, there exist some matrices $\boldsymbol{K}, \boldsymbol{M}, \boldsymbol{B}$ defined by (1)–(2) such that

$$\prod_{i=1}^{m}(\lambda-\lambda_i)^{t_i} \Big| \det(\boldsymbol{K}-\lambda(\boldsymbol{M}+\boldsymbol{B})) \tag{4}$$

if and only if

$$t_i \le i+1, \quad i=1,\ldots,m. \tag{5}$$

Remark 2.1. Similar to [1], our new construction strategy preserves m degree of freedom in the recover of the mass matrix $\boldsymbol{M}$. More precisely, Proposition 4.1 in Section 4 interprets:

> The construction in Theorem 2.1 admits a total of m masses being taken arbitrarily, where m is the number of the given distinct eigenvalues λ_i.

In particular, if $m=n$ in Theorem 2.1, Remark 2.1 indicates that all the masses in the system can be prescribed arbitrarily.

Corollary 2.1. Let $m_i>0, i=1,2,\cdots,n$ and $0<\lambda_1<\lambda_2<\cdots<\lambda_n$ be some arbitrarily given real numbers. Then, $\lambda_i, i=1,2,\cdots,n$ can be assigned as the eigenvalues of the mass-spring-inerter system by appropriately adjusting $n+1$ springs and $n+1$ inerters.

3. Proof of the necessity of Theorem 2.1

Borrowing the idea from [1, Section 3], the argument in this section becomes easy. To be self-contained, some necessary definitions and lemmas of [1, Section 3] are reviewed below.

Let $f_1(\lambda)=k_1-\lambda(m_1+b_1)$ and $g_1(\lambda)=1$. Moreover, by introducing a new mass m_{n+1}, for $j=2,\ldots,n+1$, we can define

$$f_j(\lambda)=\det\begin{bmatrix} k_1+k_2-\lambda(m_1+b_1+b_2) & -k_2+\lambda b_2 & & & \\ -k_2+\lambda b_2 & k_2+k_3-\lambda(m_2+b_2+b_3) & -k_3+\lambda b_3 & & \\ & \ddots & \ddots & & \\ & -k_{j-1}+\lambda b_{j-1} & k_{j-1}+k_j-\lambda(m_{j-1}+b_{j-1}+b_j) & -k_j+\lambda b_j \\ & & -k_j+\lambda b_j & k_j-\lambda(m_j+b_j) \end{bmatrix},$$

$$g_j(\lambda)=\det\begin{bmatrix} k_1+k_2-\lambda(m_1+b_1+b_2) & -k_2+\lambda b_2 & & & \\ -k_2+\lambda b_2 & k_2+k_3-\lambda(m_2+b_2+b_3) & -k_3+\lambda b_3 & & \\ & \ddots & \ddots & & \\ & -k_{j-2}+\lambda b_{j-2} & k_{j-2}+k_j-\lambda(m_{j-2}+b_{j-2}+b_{j-1}) & -k_{j-1}+\lambda b_{j-1} \\ & & -k_{j-1}+\lambda b_{j-1} & k_{j-1}+k_j-\lambda(m_{j-1}+b_{j-1}+b_j) \end{bmatrix}.$$

Particularly, we remark that $g_{n+1}(\lambda)=\det(\boldsymbol{K}-\lambda(\boldsymbol{M}+\boldsymbol{B}))$. The following definition on "interlacing polynomials" plays a role throughout the paper.

Definition 3.1. [1] Let $f(\lambda)$ and $g(\lambda)$ be two polynomials with degree s, where $s\in\mathbb{N}^+$. Suppose $f(\lambda)$ and $g(\lambda)$ both have s distinct real roots, which are denoted by $\alpha_1<\cdots<\alpha_s$ and $\beta_1<\cdots<\beta_s$, respectively. We say $g(\lambda)\prec f(\lambda)$, if their leading coefficients are of the same sign and

$$\beta_1<\alpha_1<\beta_2<\alpha_2<\cdots<\beta_s<\alpha_s.$$

Since $\{f_j\}_{j=1}^{n+1}$ and $\{g_j\}_{j=1}^{n+1}$ are exactly the polynomials studied by [1], the following two important lemmas still work.

Lemma 3.1. [1] The polynomials $\{f_j(\lambda)\}_{j=1}^{n+1}$ and $\{g_j(\lambda)\}_{j=1}^{n+1}$ satisfy

$$\begin{cases} f_{j+1}(\lambda)=(-\lambda m_{j+1})g_{j+1}(\lambda)+(k_{j+1}-\lambda b_{j+1})f_j(\lambda), \\ g_{j+1}(\lambda)=f_j(\lambda)+(k_{j+1}-\lambda b_{j+1})g_j(\lambda), \end{cases} \quad j=1,\ldots,n, \tag{6}$$

with $g_1(\lambda)=1$ and $f_1(\lambda)=k_1-\lambda(m_1+b_1)$.

Lemma 3.2. [1] Suppose for some $j \in [n]$,

$$\frac{(-\lambda)g_j(\lambda)}{(f_j(\lambda), g_j(\lambda))} \prec \frac{f_j(\lambda)}{(f_j(\lambda), g_j(\lambda))}. \tag{7}$$

Then $\frac{(-\lambda)g_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$. Moreover,
(i) if $b_{j+1} \neq 0$ and $k_{j+1} - \lambda b_{j+1} | \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$, then $(f_{j+1}(\lambda), g_{j+1}(\lambda)) = (f_j(\lambda), g_j(\lambda))(\lambda - \frac{k_{j+1}}{b_{j+1}})$ and $\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{-f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$;
(ii) if $b_{j+1} \neq 0$ and $k_{j+1} - \lambda b_{j+1} \nmid \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$, then $(f_{j+1}(\lambda), g_{j+1}(\lambda)) = (f_j(\lambda), g_j(\lambda))$ and $\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{f_j(\lambda)(k_{j+1}-\lambda b_{j+1})}{(f_j(\lambda),g_j(\lambda))}$;
(iii) if $b_{j+1} = 0$, then $(f_{j+1}(\lambda), g_{j+1}(\lambda)) = (f_j(\lambda), g_j(\lambda))$ and $\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))} \prec \frac{g_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$ and $\frac{(-\lambda)f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$.

With the preliminaries ready, we now prove the necessity of Theorem 2.1. Let $f(\lambda)$ be a polynomial whose roots $\{z_i\}_{i=1}^p$ are all real and $z_1 < z_2 < \cdots < z_p$. Denote $\xi(f(\lambda), z_i)$ as the multiplicity of root z_i and $\zeta(f(\lambda), \alpha) \triangleq \max\{i \in [p] : z_i < \alpha\}$ for any $\alpha \in \mathbb{R}$.

The proof of the necessity of Theorem 2.1. Observe that $\det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B})) = g_{n+1}(\lambda)$, then (4) means $\lambda_i, 1 \leq i \leq m$ are m distinct roots of $g_{n+1}(\lambda)$ with multiplicities t_i. Since $\frac{(-\lambda)g_1(\lambda)}{(f_1(\lambda),g_1(\lambda))} \prec \frac{f_1(\lambda)}{(f_1(\lambda),g_1(\lambda))}$ by Lemma 3.1, repeatedly applying Lemma 3.2 shows that (7) holds for all $j \in [n+1]$. Thus, given any $1 \leq i \leq m$,

$$\zeta\left(\frac{g_{n+1}(\lambda)}{(f_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) \geq \zeta\left(\frac{f_{n+1}(\lambda)}{(f_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) - 1. \tag{8}$$

Moreover, it indicates that all the roots of $\frac{g_{n+1}(\lambda)}{(f_{n+1}(\lambda),g_{n+1}(\lambda))}$ are distinct. So, for each $1 \leq i \leq m$, $\xi\left(\frac{g_{n+1}(\lambda)}{(f_{n+1}(\lambda),g_{n+1}(\lambda))}, \lambda_i\right) \leq 1$. Consequently,

$$\xi\left((f_{n+1}(\lambda), g_{n+1}(\lambda)), \lambda_i\right) = \xi\left(g_{n+1}(\lambda), \lambda_i\right) - \xi\left(\frac{g_{n+1}(\lambda)}{(g_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) \geq t_i - 1. \tag{9}$$

Furthermore, since Lemma 3.2 holds, by almost the same argument as that for Eq.(19) in [1, Section 3], we conclude

$$\zeta\left(\frac{f_{n+1}(\lambda)}{(f_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) \geq \xi\left((f_{n+1}(\lambda), g_{n+1}(\lambda)), \lambda_i\right). \tag{10}$$

As a result, by (8), (9) and (10), for each $i = 1, \ldots, m$,

$$i - 1 = \zeta\left(g_{n+1}, \lambda_i\right) \geq \zeta\left(\frac{g_{n+1}(\lambda)}{(f_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) \geq \zeta\left(\frac{f_{n+1}(\lambda)}{(f_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) - 1 \geq \xi\left((f_{n+1}(\lambda), g_{n+1}(\lambda)), \lambda_i\right) - 1 \geq t_i - 2,$$

which completes the proof. □

4. Proof of sufficiency of Theorem 2.1

Now, we are in a position to treat the core part of the paper. We shall show

Proposition 4.1. Let $M_1, \ldots, M_m > 0$ be m real numbers and $\prod_{i=1}^m (\lambda - \lambda_i)^{t_i}$ be a polynomial with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m$ and $\sum_{i=1}^m t_i = n$. Suppose $t_i \leq i + 1$ for each $i = 1, \ldots, m$. Then, there exist some $m_j > 0$, $k_j > 0$, $b_j \geq 0$, $j = 1, \ldots, n$ and m distinct indices $i_h, h = 1, \ldots, m$ such that $m_{i_h} = M_h$ and

$$\prod_{i=1}^m (\lambda - \lambda_i)^{t_i} \Big| \det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B})).$$

With the above proposition, the sufficiency of Theorem 2.1 becomes trivial. So, this section is actually a proof of Proposition 4.1. Although the structure of this section is analogous to [1, Section 4], the construction strategy changes and accordingly we shall provide a deeper insight into this problem.

Our aim is to find some sequences of $\{m_j\}_{j=1}^n$, $\{k_j, b_j\}_{j=1}^{n+1}$ and m indices i_h such that $m_{i_h} = M_h$, $h = 1, \ldots, m$ and

$$\prod_{i=1}^{m} (\lambda - \lambda_i)^{t_i} | g_{n+1}(\lambda), \tag{11}$$

where $g_{n+1}(\lambda) = \det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B}))$. Before that, we recall some important definitions in [1, Section 4] on the multiple eigenvalues. Let $T = \max_{1 \leqslant i \leqslant m} t_i$. If $T > 1$, define

$$S_j \triangleq \{\lambda_i : t_i \geq j + 1\} \quad \text{and} \quad q_j \triangleq |S_j|, \quad j = 1, \ldots, T - 1. \tag{12}$$

We reorder the elements of S_j by $s_j(1) < \cdots < s_j(q_j)$ and remark that $\sum_{j=1}^{T-1} q_j = n - m$. Next, define a series of subsets R_j, $j = 1, 2, \cdots, T - 1$ by

$$\begin{cases} R_1 \triangleq [n - q_1 + 1 \quad n], \\ R_j \triangleq [j + \sum_{h=2}^{j-1} q_h \quad j - 1 + \sum_{h=2}^{j} q_h], \quad j = 2, \cdots, T - 1. \end{cases} \tag{13}$$

Alternatively, we write [1]

$$\begin{cases} R_1 = \{r_1(q_1) < r_1(q_1 - 1) < \cdots < r_1(1)\}, \\ R_j = \left\{r_j(1) < r_j(2) < \cdots < r_j(q_j)\right\}, \qquad j = 2, \cdots, T - 1. \end{cases} \tag{14}$$

Observe from Lemma 3.2 (also see [1, Eq. (15)]) that for each $j = 1, \ldots, n$,

$$(f_{j+1}(\lambda), g_{j+1}(\lambda)) = \begin{cases} (f_j(\lambda), g_j(\lambda))(\lambda - \frac{k_{j+1}}{b_{j+1}}), & \text{if } b_{j+1} \neq 0, (\lambda - \frac{k_{j+1}}{b_{j+1}}) | \frac{f_j(\lambda)}{(f_j(\lambda), g_j(\lambda))}, \\ (f_j(\lambda), g_j(\lambda)), & \text{otherwise.} \end{cases} \tag{15}$$

This indicates that every element in S_j, $j = 1, 2 \cdots, T - 1$ is equal to some $k_i/b_i, i \in [2 \ n]$. So it enlightens us to determine k_i/b_i with $i \in \bigcup_{j=1}^{T-1} R_j$ as follows:

$$\frac{k_i}{b_i} = s_j(q_j - l + 1), \qquad i = r_j(l). \tag{16}$$

Recall that the construction strategy proposed by [1] fails to recover $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$ in Example 1.2. Now, we apply rule (16) to determine $\frac{k_i}{b_i}$ for this mass-spring-inerter system.

Example 4.1. Take $n = 8, m = 3, t_1 = 2, t_2 = t_3 = 3$ and $0 < \lambda_1 < \lambda_2 < \lambda_3$ in Proposition 4.1. Sets S_j and R_j, $j = 1, 2$ are explicitly shown in Fig. 2. As a result, $k_2/b_2 = k_8/b_8 = \lambda_3$, $k_3/b_3 = k_7/b_7 = \lambda_2$, $k_6/b_6 = \lambda_1$.

As is clear from Example 4.1, a major difference between the construction strategies of [1] and this paper lies in how we connect $k_i/b_i, i \in \bigcup_{j=1}^{T-1} R_j$ to the entries of $S_j, j = 1, \ldots, T - 1$. In the next proof, we shall recover $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$ by the following three steps:

Step 1: Take a proper $m_n > 0$ and find some $b_{n+1}, k_{n+1} > 0$ according to Lemma 4.1 presented later.

Step 2: For each $i \in R_1$, we assign k_i/b_i a value taken from S_1 in the light of (16). Then, by virtue of Lemma 4.2 in this section, there exist some positive numbers m_i, k_i and b_i taken as the corresponding entries of $\boldsymbol{M}, \boldsymbol{K}$ and $\boldsymbol{B}$.

Step 3: We construct the rest $\{m_j, k_j, b_j\}_{j=1}^{n-q_1}$ in a way developed by [1]. It will be completed in view of Lemma 4.4.

[1] The definitions of $R_j, j = 1, \ldots, T - 1$ are different from those in [1].

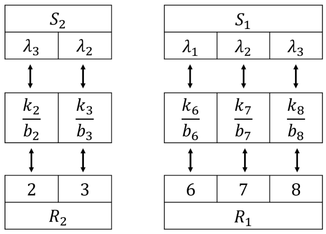

Figure 2: Association between S_i, $\frac{k_i}{b_i}$ and R_i of Example 4.1

Now, we begin our construction. Inspired by [1, Section 4], we first introduce some constants. For each $i = 1, \dots, m$, set $\rho_i = \varepsilon^{(n+1)^{m-i}}$, where

$$\varepsilon = \frac{\Delta^{2n^2+2n+6}}{n^2 2^{6(n+1)^4} \Lambda^{2(n+2)^2}}, \quad \Delta \triangleq \frac{1}{2} \min\{1, \lambda_1, \min_{1 \le i \le m-1} (\lambda_{i+1} - \lambda_i)\}, \quad \Lambda \triangleq 1 + \lambda_m. \tag{17}$$

Next, let

$$\tau = \rho_m^{\frac{1}{2}}, \quad \Omega = \frac{14\Lambda^2}{\Delta^2}, \quad M \triangleq \sum_{k=1}^{m} M_k. \tag{18}$$

$$C \triangleq \frac{2^{2n+1}(1 + \Lambda^n)}{C_1^{2n^2} \rho_1^{2n}}, \quad C_1 \triangleq \frac{\Delta}{2^{n+1}\Lambda}, \quad C_2(j) \triangleq \frac{2^{2(n+1)^2} \Lambda^{n+1}}{\Delta^n \varepsilon^{(n+1)^{m-j-1}}}, \quad j = 1, \dots, m-1. \tag{19}$$

Recall that $s_1(1) < s_1(2) < \cdots < s_1(q_1)$ are the entries of S_1 defined in (12), for each $i = 1, \dots, m-1$, let

$$C_{j,i} = \begin{cases} \frac{1}{\lambda_m - \lambda_i}, & \text{if} \quad j = 0, \\ \frac{s_1(q_1-j+1)-\lambda_i}{s_1(q_1-j+1)(\lambda_m-\lambda_i)}, & \text{if} \quad j = 1, \dots, q_1. \end{cases} \tag{20}$$

Remark 4.1. Clearly, $\varepsilon < 1$, $\Lambda/\Delta \ge 2$ in (17) and $C > 1$ in (19). Since $n \ge m$,

$$\begin{cases} m\Omega^{m+1} \frac{\Lambda^2}{\Delta} \tau < n25^{n+1} \frac{\Lambda^{2n+4}}{\Delta^{2n+3}} \frac{\Delta^{n^2+n+3}}{n 2^{3(n+1)^4} \Lambda^{(n+2)^2}} < 1, \\ m2^{m+1} \frac{\Lambda^2}{\Delta} \frac{\rho_i}{\rho_{i+1}^2} \le m2^{m+1} \frac{\Lambda^2}{\Delta} \varepsilon < 1, \quad i = 1, \dots, m-1 \text{ and } n \ge 2, \end{cases}$$

which immediately asserts

$$\begin{cases} m\Omega^{m+1}\tau < \frac{\Delta}{\Lambda^2}, \\ m2^{m+1} \frac{\Lambda^2}{\Delta} \rho_i < \rho_{i+1}^2, \quad i = 1, \dots, m-1 \text{ and } n \ge 2. \end{cases} \tag{21}$$

Moreover, [1, Lemma B.2.] indicates that $C_2(j), j = 1, \dots, m-1$ defined by (19) satisfy

$$\begin{cases} (1 + C_2(m-1))^n \rho_{m-1} < \frac{\Delta}{2}, \\ (1 + C_2(j))^n \rho_j < (1 + C_2(j+1))^n \rho_{j+1}, & \text{if} \quad j = 1, \dots, m-2, \quad m > 2, \\ \frac{n(1+C_2(j-1))^n}{\Delta} \rho_{j-1} < \frac{1}{4} \left(\frac{\Delta^n}{2^{(n+1)^2} \Lambda^{n+1}} \right) \rho_{j+1}, & \text{if} \quad j = 2, \dots, m-2, \quad m > 3. \end{cases} \tag{22}$$

Finally, we point out that for each $j \in [0\ q_1]$ and $i \in [1\ m-1]$,

$$\frac{2\Delta}{\Lambda^2} < |C_{j,i}| < \frac{1}{2\Delta}. \tag{23}$$

Lemma 4.1. Let $\nu > 0$, $L > 0$ and $G(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i)$, $\widetilde{F}(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i + \rho_i)$ be two monic polynomials.
(i) If $m > 1$, then there exist some $\widetilde{\lambda} > 0, \widetilde{b} > 0$, $\widetilde{\mu}, \widetilde{\nu}$ and monic polynomial $\widetilde{G}(\lambda)$ with distinct roots $\widetilde{\beta}_1 < \cdots < \widetilde{\beta}_{m-1}$, such that

$$\left|\frac{1}{\rho_m}\frac{\rho_i}{\widetilde{\beta}_i - \lambda_i} - C_{0,i}\right| < \tau, \quad i = 1, \ldots, m-1, \tag{24}$$

$$-\frac{\widetilde{\mu}}{\widetilde{\nu}}\frac{\widetilde{F}(\lambda_m)}{\lambda_m \widetilde{G}(\lambda_m)} = L, \tag{25}$$

$$\nu G(\lambda) = \widetilde{\mu}\widetilde{F}(\lambda) + \widetilde{b}(\widetilde{\lambda} - \lambda)\widetilde{\nu}\widetilde{G}(\lambda). \tag{26}$$

(ii) If $m = 1$, then there exist some $\widetilde{\lambda} > 0, \widetilde{b} > 0$, $\widetilde{\mu}, \widetilde{\nu}$ and $\widetilde{G}(\lambda) = 1$ such that (25) and (26) hold.

Now, we write $s_1(l) = \lambda_{z_l}, l = 1, \ldots, q_1$ and define $z_{q_1+1} \triangleq m$, $\beta_m \triangleq C_2(m) \triangleq +\infty$.

Lemma 4.2. Let $F(\lambda) = \prod_{i=1}^{m}(\lambda - \alpha_i)$, $G(\lambda) = \prod_{i=1}^{m-1}(\lambda - \beta_i)$ be two polynomials and μ, ν satisfy $-\frac{\mu}{\nu} > 0$. Suppose there exists an index $l \in [1\ q_1]$ with

$$\begin{cases} \lambda_i \in [\alpha_i, \beta_i), \quad \lambda_i - \alpha_i \le \rho_i, & \text{if} \quad i = 1, \ldots, z_{l+1}, \\ \lambda_i < \alpha_i < \beta_i < \lambda_i + (1 + C_2(i))^n \rho_i, & \text{if} \quad i = z_{l+1} + 1, \ldots, m \end{cases} \tag{27}$$

satisfying

$$\left|\frac{1}{\rho_m}\frac{\lambda_i - \alpha_i}{\beta_i - \lambda_i} - C_{q_1-l,i}\right| < \Omega^{q_1-l}\tau, \quad i = 1, \ldots, m-1 \tag{28}$$

and

$$\begin{cases} \alpha_m - \lambda_m < \frac{2\Lambda}{\lambda_1}\rho_m, & \text{if} \quad z_l \in [1\ m-1], \\ \alpha_m - \lambda_m = \rho_m, & \text{if} \quad z_l = m. \end{cases} \tag{29}$$

Then there exist some $\widetilde{m} > 0, \widetilde{b} > 0$, $\widetilde{\mu}, \widetilde{\nu}$ satisfying $-\frac{\widetilde{\mu}}{\widetilde{\nu}} > -\frac{\lambda_1}{\Lambda}\frac{\mu}{\nu}$, and monic polynomials $\widetilde{F}(\lambda)$, $\widetilde{G}(\lambda)$ with distinct roots $\widetilde{\alpha}_1 < \cdots < \widetilde{\alpha}_m$, $\widetilde{\beta}_1 < \cdots < \widetilde{\beta}_{m-1}$, respectively, such that

$$\begin{cases} \mu F(\lambda) = -\lambda \widetilde{m}\nu G(\lambda) - \widetilde{b}\widetilde{\mu}\widetilde{F}(\lambda), \\ (\lambda - \lambda_{z_l})\nu G(\lambda) = \widetilde{\mu}\widetilde{F}(\lambda) + \widetilde{b}(\lambda_{z_l} - \lambda)\widetilde{\nu}\widetilde{G}(\lambda). \end{cases} \tag{30}$$

In addition, the roots satisfy

$$\widetilde{\alpha}_{z_l} = \lambda_{z_l} \quad \text{and} \quad \widetilde{\alpha}_i \in (\alpha_i, \beta_i), \quad i = 1, \ldots, m, \tag{31}$$

$$\frac{1}{2} < \frac{\widetilde{\beta}_i - \lambda_i}{\beta_i - \lambda_i} < 2 \quad \text{and} \quad \left|\frac{1}{\rho_m}\frac{\lambda_i - \widetilde{\alpha}_i}{\widetilde{\beta}_i - \lambda_i} - C_{q_1-l+1,i}\right| < \Omega^{q_1-l+1}\tau, \quad i = 1, \ldots, m-1. \tag{32}$$

In particular, when $z_l < m$,

$$\frac{\alpha_m - \lambda_m}{\rho_m} < \frac{\widetilde{\alpha}_m - \lambda_m}{\rho_m} < \frac{\alpha_m - \lambda_m}{\rho_m} + \lambda_m \frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} C_{q_1-l,z_l} + \frac{3\Lambda^2}{\lambda_1}\Omega^{q_1-l}\tau. \tag{33}$$

The next lemma is based on Lemmas 4.1 and 4.2. By applying this lemma, we can construct a series of polynomials $\{F_j(\lambda)\}_{j=n-q_1}^{n}$ and $\{G_j(\lambda)\}_{j=n-q_1}^{n}$ with degrees $\deg F_j(\lambda) = m$ and $\deg G_j(\lambda) = m-1$. We shall see later that these functions are exactly $\{\frac{f_j(\lambda)}{(f_j(\lambda), g_j(\lambda))}, \frac{g_j(\lambda)}{(f_j(\lambda), g_j(\lambda))}\}_{j=n-q_1}^{n}$. Denote the roots of $F_j(\lambda)$ and $G_j(\lambda)$, $j \in [n - q_1\ n]$ by $\alpha_j(1) < \cdots < \alpha_j(m)$ and $\beta_j(1) < \cdots < \beta_j(m-1)$, respectively.

Lemma 4.3. Let $T > 1$. Given a polynomial $G_{n+1}(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i)$ and a number $\nu_{n+1} > 0$, there exist some monic polynomials $\{F_j(\lambda)\}_{j=n-q_1}^{n}$, $\{G_j(\lambda)\}_{j=n-q_1}^{n}$ and numbers λ^*, $\{b_j\}_{j=n-q_1+1}^{n+1}$, $\{m_j\}_{j=n-q_1+1}^{n}$, $\{(\mu_j, \nu_j)\}_{j=n-q_1}^{n}$ such that the following statements hold.
(i) λ^*, $\{b_j\}_{j=n-q_1+1}^{n+1}$, $\{m_j\}_{j=n-q_1+1}^{n}$ are positive and $-\frac{\mu_k}{\nu_k} > C(\frac{\Lambda}{\lambda_1})^k \frac{\Lambda M}{\Lambda-\lambda_1}$ for all $k \in [n - q_1\ n]$.
(ii) For each $k \in [n - q_1\ n]$, $\deg F_k(\lambda) = m$ and $\deg G_k(\lambda) = m - 1$. Furthermore, polynomials $F_n(\lambda)$ and $G_n(\lambda)$ satisfy

$$\nu_{n+1}G_{n+1}(\lambda) = \mu_n F_n(\lambda) + b_{n+1}(\lambda^* - \lambda)\nu_n G_n(\lambda). \tag{34}$$

For $k \in [n - q_1\ n - 1]$,

$$\begin{cases} \mu_{k+1}F_{k+1}(\lambda) = -\lambda m_{k+1}\nu_{k+1}G_{k+1}(\lambda) - b_{k+1}\mu_k F_k(\lambda), \\ (\lambda - s_1(k - n + q_1 + 1))\nu_{k+1}G_{k+1}(\lambda) = \mu_k F_k(\lambda) + b_{k+1}(s_1(k - n + q_1 + 1) - \lambda)\nu_k G_k(\lambda). \end{cases} \tag{35}$$

(iii) The roots of $F_{n-q_1}(\lambda)$ and $G_{n-q_1}(\lambda)$ satisfy $\alpha_{n-q_1}(m) \in [\lambda_m - \rho_m, \Lambda)$ and for $i = 1, \ldots, m - 1$,

$$\lambda_i + C_1^{q_1}\rho_i - \rho_i \leq \alpha_{n-q_1}(i) + C_1^{q_1}\rho_i < \beta_{n-q_1}(i) < \lambda_i + 2^{q_1+1}\Lambda\frac{\rho_i}{\rho_m}. \tag{36}$$

Proof. For $k = n$, let $F_n(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i + \rho_i)$. By applying Lemma 4.1 with $\nu = \nu_{n+1}$ and

$$L > C\left(\frac{\Lambda}{\lambda_1}\right)^n \frac{\Lambda M}{\Lambda - \lambda_1}, \tag{37}$$

we can obtain some $\lambda^* > 0, b_{n+1} > 0$, μ_n, ν_n, and a monic polynomial $G_n(\lambda)$ with $\deg G_n(\lambda) = m - 1$ satisfying (34) and

$$-\frac{\mu_n}{\nu_n}\frac{F_n(\lambda_m)}{\lambda_m G_n(\lambda_m)} = L. \tag{38}$$

Thus, by (96), $-\frac{\mu_n}{\nu_n} > C(\frac{\Lambda}{\lambda_1})^n \frac{\Lambda M}{\Lambda-\lambda_1}$. Moreover,

$$\left|\frac{1}{\rho_m}\frac{\rho_i}{\beta_n(i) - \lambda_i} - C_{0,1}\right| < \tau, \quad i = 1, \ldots, m - 1. \tag{39}$$

It follows from (17), (19) and (20) that, for $i = 1, \ldots, m - 1$,

$$\beta_n(i) - \lambda_i < \frac{\rho_i}{\rho_m}\left(\frac{1}{\lambda_m - \lambda_i} - \tau\right)^{-1} < \frac{2\Lambda\rho_i}{\rho_m}. \tag{40}$$

For $k = n - 1$, by (39) and $F_n(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i + \rho_i)$, it is easy to verify that $l = q_1$ satisfies (27)–(29). By applying Lemma 4.2 to $l = q_1$, $F(\lambda) = F_n(\lambda)$ and $G(\lambda) = G_n(\lambda)$, there exist some $b_n > 0, m_n > 0$, μ_{n-1}, ν_{n-1} with

$$-\frac{\mu_{n-1}}{\nu_{n-1}} > -\frac{\lambda_1}{\Lambda}\frac{\mu_n}{\nu_n} > -C(\frac{\Lambda}{\lambda_1})^{n-1}\frac{\Lambda M}{\Lambda - \lambda_1} \tag{41}$$

and two monic polynomials $F_{n-1}(\lambda)$, $G_{n-1}(\lambda)$ with $\deg F_{n-1}(\lambda) = m$ and $\deg G_{n-1}(\lambda) = m - 1$ such that (35) holds.

Next, we shall complete the construction inductively. Assume that for all $k \geqslant h \in (n - q_1\ n - 1]$, the conditions of Lemma 4.2 hold with $l = k - n + q_1 + 1$, $F(\lambda) = F_{k+1}(\lambda)$ and $G(\lambda) = G_{k+1}(\lambda)$. Then we can find the desired $b_{k+1}, m_{k+1}, \mu_k, \nu_k$ and polynomials $F_k(\lambda), G_k(\lambda)$ by applying Lemma 4.2, so that statements (i) and (ii) of Lemma 4.3 are valid.

We proceed to check the above assertion for $h - 1$. By (32) and (40), we deduce that for $i = 1, \ldots, m - 1$,

$$0 < \beta_h(i) - \lambda_i < 2(\beta_{h+1}(i) - \lambda_i) < \cdots < 2^{n-h}\frac{2\Lambda\rho_i}{\rho_m} < 2^{q_1+1}\Lambda\frac{\rho_i}{\rho_m} < (1 + C_2(i))^n\rho_i, \tag{42}$$

$$\left|\frac{1}{\rho_m}\frac{\lambda_i-\alpha_h(i)}{\beta_h(i)-\lambda_i}-C_{n-h,i}\right|<\Omega^{n-h}\tau. \tag{43}$$

Thus, (28) holds for $F_h(\lambda)$ and $G_h(\lambda)$ with $l=h-n+q_1$. Moreover, observe from (21) and (23) that, for $l\in[1\ q_1]$,

$$|C_{q_1-l,i}|>\frac{2\Delta}{\Lambda^2}>\Omega^{q_1-l}\tau,\quad i=1,\dots,m-1. \tag{44}$$

Therefore, (43) and (44) indicate $\frac{1}{\rho_m}\frac{\lambda_i-\alpha_h(i)}{\beta_h(i)-\lambda_i}$ and $C_{n-h,i}$ have the same sign for $i\neq z_{h-n+q_1+1}$. That is, by (20),

$$(\lambda_i-\alpha_h(i))(s_1(h-n+q_1+1)-\lambda_i)>0,\quad i\in[1\ m-1]\setminus\{z_{h-n+q_1+1}\}. \tag{45}$$

So, by (31) and (45),

$$\begin{cases}\alpha_h(z_{h-n+q_1+1})=\lambda_{z_{h-n+q_1+1}},\\ \lambda_i-\rho_i\le\alpha_{h+1}(i)<\alpha_h(i)<\lambda_i,\quad i\in[1\ z_{h-n+q_1+1}-1].\end{cases} \tag{46}$$

For $i\in[z_{h-n+q_1+1}+1\ m]$, by (45), $\lambda_i<\alpha_h(i)$. Moreover, by (23), $|C_{n-h,i}|<\frac{1}{2\Delta}<\frac{1}{\rho_m}$. Then (43) shows $|\alpha_h(i)-\lambda_i|<\beta_h(i)-\lambda_i$, $i\in[1\ m-1]$. Therefore, we deduce

$$\lambda_i<\alpha_h(i)<\beta_h(i),\quad i\in[z_{h-n+q_1+1}+1\ m]. \tag{47}$$

So, by (42), (46) and (47), we conclude (27) holds for $F_h(\lambda)$ and $G_h(\lambda)$ with $l=h-n+q_1$.

Now we show (29) with $l=h-n+q_1$. Note that $z_{h-n+q_1}<z_{q_1}\leqslant m$.

(a) If $z_{h-n+q_1+1}=m$, then (31) infers $\alpha_h(m)=\lambda_m$. Thus $0=\alpha_h(m)-\lambda_m<\frac{2\Lambda}{\lambda_1}\rho_m$.

(b) If $z_{h-n+q_1+1}<m$, by using (33) recursively for $l=h-n+q_1+1,\dots,q_1$, we deduce

$$\begin{aligned}0\le\frac{\alpha_h(m)-\lambda_m}{\rho_m}&<\frac{\alpha_n(m)-\lambda_m}{\rho_m}+\lambda_m\sum_{k=2}^{n-h}\left(\frac{1}{s_1(q_1-k+1)}-\frac{1}{s_1(q_1-k+2)}\right)+\frac{\lambda_m}{s_1(q_1)}+m\frac{3\Lambda^2}{\lambda_1}\Omega^m\tau\\&<\frac{\lambda_m}{\lambda_1}+m\frac{3\Lambda^2}{\lambda_1}\Omega^m\tau<\frac{2\Lambda}{\lambda_1},\end{aligned} \tag{48}$$

Therefore, (27)–(29) hold for $F_h(\lambda)$ and $G_h(\lambda)$ with $l=h-n+q_1$. Now, by Lemma 4.2, there exist some numbers $b_h>0,m_h>0$, μ_{h-1},ν_{h-1} with

$$-\frac{\mu_{h-1}}{\nu_{h-1}}>-\frac{\lambda_1}{\Lambda}\frac{\mu_h}{\nu_h}>-C(\frac{\Lambda}{\lambda_1})^{h-1}\frac{\Lambda M}{\Lambda-\lambda_1} \tag{49}$$

and two polynomials $F_{h-1}(\lambda),G_{h-1}(\lambda)$ such that (35) holds. The induction is completed for all $h=n-1,\dots,n-q_1$. Particularly, we remark that (42)–(49) hold for $h=n-q_1$.

The remainder is to verify (iii). Since (48) holds for $h=n-q_1$, $\alpha_{n-q_1}(m)\in(\lambda_m-\rho_m,\Lambda)$. We next show (36). Note that (42) is true for $h=n-q_1$, then

$$\beta_{n-q_1}(i)-\lambda_i<2^{q_1+1}\Lambda\frac{\rho_i}{\rho_m},\quad i=1,\dots,m-1. \tag{50}$$

On the other hand, (32) and (39) infer that for $i=1,\dots,m-1$,

$$\beta_{n-q_1}(i)-\lambda_i>\cdots>\frac{1}{2^{q_1}}(\beta_n(i)-\lambda_i)>\frac{1}{2^{q_1}}\left(\frac{1}{\lambda_m-\lambda_i}+\tau\right)^{-1}\frac{\rho_i}{\rho_m}. \tag{51}$$

Then, by (21), (23), (28) and (51), it follows that for $i=1,\dots,m-1$,

$$\begin{aligned}\beta_{n-q_1}(i)-\alpha_{n-q_1}(i)&>(\beta_{n-q_1}(i)-\lambda_i)+\rho_m(C_{q_1,i}-\Omega^{q_1}\tau)(\beta_{n-q_1}(i)-\lambda_i)\\&>\frac{1}{2^{q_1}}(1-(|C_{q_1,i}|+\Omega^{q_1})\tau\rho_m)\left(\frac{1}{\lambda_m-\lambda_i}+\tau\right)^{-1}\frac{\rho_i}{\rho_m}>\frac{1}{2^{q_1}}\left(\frac{1}{2\Delta}+\frac{1}{2\Delta}\right)^{-1}\frac{\rho_i}{2\rho_m}>C_1^{q_1}\rho_i.\end{aligned} \tag{52}$$

Recall that $\alpha_{n-q_1}(i) > \alpha_n(i) = \lambda_i - \rho_i$, then

$$\lambda_i + C_1^{q_1}\rho_i - \rho_i \le \alpha_{n-q_1}(i) + C_1^{q_1}\rho_i. \tag{53}$$

We conclude (36) by combining (50), (52) and (53). □

The lemma below sharpens [1, Lemma 4.1]. Its proof is contained in Appendix C.

Lemma 4.4. Let $F_{n-q_1}(\lambda) = \prod_{i=1}^{m}(\lambda - \alpha_i)$, $G_{n-q_1}(\lambda) = \prod_{i=1}^{m-1}(\lambda - \beta_i)$, μ_{n-q_1}, ν_{n-q_1} be constructed in Lemma 4.3. Then, there exist some monic polynomials $\{F_j(\lambda)\}_{j=1}^{n-q_1-1}$, $\{G_j(\lambda)\}_{j=1}^{n-q_1-1}$ and some sequences of numbers $\{(\lambda_j^*, b_j, m_j)\}_{j=2}^{n-q_1}$, $\{(\mu_j, \nu_j)\}_{j=1}^{n-q_1-1}$ such that for each $j \in [1\ n-q_1-1]$, the following two properties hold:
(i) $\lambda_{j+1}^* > 0, b_{j+1} > 0, m_{j+1} > 0$ and $-\frac{\mu_j}{\nu_j} > C(\frac{\Lambda}{\lambda_1})^j \frac{\Lambda M}{\Lambda - \lambda_1}$;
(ii) if $j+1 \in [2\ n-q_1] \setminus \bigcup_{l=2}^{T-1} R_l$, then

$$\begin{cases} \mu_{j+1}F_{j+1}(\lambda) = -\lambda m_{j+1}\nu_{j+1}G_{j+1}(\lambda) + b_{j+1}(\lambda_{j+1}^* - \lambda)\mu_j F_j(\lambda), \\ \nu_{j+1}G_{j+1}(\lambda) = \mu_j F_j(\lambda) + b_{j+1}(\lambda_{j+1}^* - \lambda)\nu_j G_j(\lambda), \end{cases} \tag{54}$$

otherwise, for $j+1 \in R_l$, $l \in [2\ T-1]$,

$$\begin{cases} \mu_{j+1}F_{j+1}(\lambda) = -\lambda m_{j+1}\nu_{j+1}G_{j+1}(\lambda) - b_{j+1}\mu_k F_j(\lambda), \\ \nu_{j+1}(\lambda - \lambda_{j+1}^*)G_{j+1}(\lambda) = \mu_j F_j(\lambda) + b_{j+1}\nu_j(\lambda_{j+1}^* - \lambda)G_j(\lambda), \\ \lambda_{j+1}^* = s_l(q_l - i + 1) \quad \text{where} \quad j+1 = r_l(i), \end{cases} \tag{55}$$

where $S_l = \{s_l(1) < s_l(2) < \cdots < s_l(q_l)\}$ and $R_l = \{r_l(1) < r_l(2) < \cdots < r_l(q_l)\}$ are defined in (12) and (14).

Proof of Proposition 4.1. First, consider the case where $T > 1$. By taking $\nu_{n+1} = 1$ in Lemma 4.3, we obtain some $\{F_j(\lambda)\}_{j=n-q_1}^{n}$, $\{G_j(\lambda)\}_{j=n-q_1}^{n}$, and λ^*, $\{b_j\}_{j=n-q_1+1}^{n+1}$, $\{m_j\}_{j=n-q_1+1}^{n}$, $\{(\mu_j, \nu_j)\}_{j=n-q_1}^{n}$ such that statements (i) and (ii) of Lemma 4.3 hold. Next, we apply Lemma 4.4 to construct a sequence of $\{(\lambda_k^*, b_k, m_k)\}_{k=2}^{n-q_1}$, $\{(\mu_k, \nu_k)\}_{k=1}^{n-q_1-1}$, $\{F_k(\lambda)\}_{k=1}^{n-q_1-1}$, $\{G_k(\lambda)\}_{k=1}^{n-q_1-1}$ so that (54) holds for $j+1 \in [2\ n-q_1] \setminus \bigcup_{l=2}^{T-1} R_l$, and (55) holds for $j+1 \in R_l$, $l \in [2\ T-1]$. Noticing that $-\frac{\mu_1}{\nu_1} > M > M_1$, set

$$\begin{cases} k_j = \lambda_j^* b_j, \quad j = 2, \ldots, n, \quad k_1 = -\alpha_1(1)\frac{\mu_1}{\nu_1}, \\ m_1 = M_1, \quad b_1 = -\frac{\mu_1}{\nu_1} - M_1. \end{cases} \tag{56}$$

The remainder is to verify $\{k_j, b_j\}_{j=1}^{n+1}$ and $\{m_j\}_{j=1}^{n}$ meet the requirements.

Define a sequence of polynomials $\{D_j(\lambda)\}_{j=1}^{n}$ as follows:

$$\begin{cases} D_1(\lambda) = 1, \\ D_{k+1}(\lambda) = D_k(\lambda), & k+1 \in [2\ n] \setminus \bigcup_{l=1}^{T-1} R_l, \\ D_{k+1}(\lambda) = (\lambda - \lambda_{k+1}^*)D_k(\lambda), & k+1 \in \bigcup_{l=2}^{T-1} R_l, \\ D_{k+1}(\lambda) = (\lambda - s_1(k-n+q_1+1))D_k(\lambda), & k+1 \in R_1, \end{cases} \tag{57}$$

and let

$$\begin{cases} f_j(\lambda) = \frac{\mu_j}{\nu_1} D_j(\lambda)F_j(\lambda), \quad j = 1, \ldots, n, \\ g_j(\lambda) = \frac{\nu_j}{\nu_1} D_j(\lambda)G_j(\lambda), \quad j = 1, \ldots, n+1. \end{cases}$$

Clearly, $g_{n+1}(\lambda) = \frac{\nu_{n+1}}{\nu_1} D_{n+1}(\lambda)G_{n+1}(\lambda) = \frac{1}{\nu_1}\prod_{j=1}^{m}(\lambda - \lambda_j)^{t_j}$. The rest of the proof is to check whether $\{f_i(\lambda)\}_{i=1}^{n}$ and $\{g_i(\lambda)\}_{i=1}^{n+1}$ satisfy the recursive formulas (6). It is essentially the same as that of [1].

For the case where $T = 1$, we remark that Lemma 4.3 and Lemma 4.4 hold if we set $q_1 = 0$ and $R_1 = \bigcup_{l=2}^{T-1} R_l = \emptyset$. So, (57) is still valid and the above analysis works. □

5. Concluding remarks

This paper has investigated the inverse eigenvalue problem for "fixed-fixed" mass-spring-inerter systems. Since the construction cannot follow directly from that for "fixed-fixed" version, we provide a modified construction method with wider applicability and conclude an analogous necessary and sufficient condition. Moreover, we preserve the degree of freedom in the recover of the mass matrix. In particular, if all the given eigenvalues are distinct, the assignment can be realized by adjusting springs and inerters only.

A. Proof of Examples 1.1–1.2

The proof of Example 1.1. Arbitrarily take two numbers $m_1, m_3 > 0$. Let $f_3(\lambda) = \det(\boldsymbol{K}' - \lambda(\boldsymbol{M}' + \boldsymbol{B}'))$ and $g_3(\lambda) = \det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B}))$. According to [1, Section 3], the two polynomials can be rewritten by

$$f_3(\lambda) = \mu_3(\lambda - \lambda_1)(\lambda - \alpha_3(1))(\lambda - \alpha_3(2)), \quad g_3(\lambda) = \nu_3(\lambda - \lambda_1)^2,$$

where $\mu_3\nu_3 < 0$. Set $\alpha_3(2) = \lambda_1 + 1$ and $\alpha_3(1) = \lambda_1 - \rho_1$, where

$$\rho_1 = \varepsilon^4, \quad \varepsilon = \frac{\Delta^{14}}{3 \times 2^{192}\Lambda^{16}}, \quad \Delta \triangleq \frac{1}{2}, \quad \Lambda \triangleq \lambda_1 + 2.$$

It is obvious that $\alpha_3(1) < \lambda_1 < \alpha_3(2)$, $\Delta = \frac{1}{2}\min\{1, \alpha_3(2) - \alpha_3(1)\}$ and $\Lambda = \alpha_3(2) + 1$. Choose a pair μ_3, ν_3 such that $-\frac{\mu_3}{\nu_3}$ is sufficiently large. By applying [1, Lemma B.3. (ii)] to $\frac{f_3(\lambda)}{(f_3(\lambda), g_3(\lambda))}$ and $\frac{g_3(\lambda)}{(f_3(\lambda), g_3(\lambda))}$, we get $k_3, b_3 > 0$ and

$$f_2(\lambda) = \mu_2(\lambda - \lambda_1)(\lambda - \alpha_2(1)), \quad g_2(\lambda) = \nu_2(\lambda - \lambda_1),$$

where $\alpha_3(1) < \alpha_2(1) < \lambda_1$ and $-\frac{\mu_2}{\nu_2}$ is appropriately large. Next, applying [1, Lemma B.4. (ii)] to $\frac{f_2(\lambda)}{(f_2(\lambda), g_2(\lambda))}$ and $\frac{g_2(\lambda)}{(f_2(\lambda), g_2(\lambda))}$, there exist some $m_2, k_2, b_2 > 0$ and

$$f_1(\lambda) = \mu_1(\lambda - \lambda_1), \quad g_1(\lambda) = \nu_1 \quad \text{with} \quad -\frac{\mu_1}{\nu_1} > m_1.$$

Finally, for the given m_1, set $k_1 = -\lambda_1\frac{\mu_1}{\nu_1}$ and $b_1 = -\frac{\mu_1}{\nu_1} - m_1$. Recall the proof of [1, Proposition 4.1], we have constructed $\boldsymbol{M}', \boldsymbol{K}', \boldsymbol{B}'$ and thus obtained the required $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$.

□

The proof of Example 1.2. We prove this example by using reduction to absurdity. Let $\beta_9(i) = \lambda_i$ and $\alpha_9(i) = \lambda_i - \rho_i$ for $i = 1, 2, 3$, where

$$\rho_i = \varepsilon^{10^{4-i}}, \quad \varepsilon = \frac{\Delta^{91}}{9 \times 2^{3000}\Lambda^{100}}, \quad \Delta \triangleq \frac{1}{2}, \quad \Lambda \triangleq 1 + \alpha_9(4).$$

Moreover, set $\alpha_9(4) > \lambda_3 + 1$. Observe from (3) that $\Delta = \frac{1}{2}\min\{1, \min_{1\le i\le 3}(\alpha_9(i+1) - \alpha_9(i)\}$.

Now, let $F_9(\lambda) = \prod_{i=1}^{4}(\lambda - \alpha_9(i))$ and $G_9(\lambda) = \prod_{i=1}^{3}(\lambda - \beta_9(i))$. Suppose we can apply the construction strategy developed in [1] to construct $\boldsymbol{M}', \boldsymbol{K}', \boldsymbol{B}'$ such that $F_9(\lambda)$ and $G_9(\lambda)$ are the monic polynomials of $\det(\boldsymbol{K}' - \lambda(\boldsymbol{M}' + \boldsymbol{B}'))$ and $\det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B}))$, respectively. Accordingly, we obtain a series of $\{F_i(\lambda), G_i(\lambda)\}_{i=1}^{8}$ by repeatedly using [1, Lemmas B.3. and Lemmas B.4.]. In particular,

$$\begin{cases} F_8(\lambda) = \prod_{i=1}^{3}(\lambda - \alpha_8(i)), & G_8(\lambda) = (\lambda - \beta_8(1))(\lambda - \beta_8(2)), \\ F_7(\lambda) = (\lambda - \alpha_7(1))(\lambda - \alpha_7(2)), & G_7(\lambda) = \lambda - \beta_7(1), \\ F_6(\lambda) = (\lambda - \alpha_6(1))(\lambda - \lambda_2), & G_6(\lambda) = \lambda - \beta_6(1), \\ F_5(\lambda) = (\lambda - \alpha_5(1))(\lambda - \lambda_3), & G_5(\lambda) = \lambda - \beta_5(1). \end{cases} \tag{58}$$

Here, we denote $\alpha_i(1) < \cdots < \alpha_i(s_i)$ and $\beta_i(1) < \cdots < \beta_i(s_i - 1)$ as the roots of $F_i(\lambda)$ and $G_i(\lambda)$, respectively. According to [1], the following relations hold

$$\begin{cases} \alpha_7(1) < \alpha_6(1) < \alpha_5(1) < \lambda_1 < \beta_8(1) < \beta_7(1) < \beta_6(1) < \lambda_1 + \frac{\Delta}{2}, \\ \lambda_2 - \frac{\Delta}{2} < \alpha_7(2) < \lambda_2 < \beta_8(2) < \lambda_2 + \frac{\Delta}{2} < \alpha_8(3) < \lambda_3. \end{cases} \tag{59}$$

Define

$$\Delta_1 \triangleq \lambda_1 - \alpha_6(1), \quad \Delta_2 \triangleq \beta_6(1) - \lambda_1, \quad \Delta_3 \triangleq \beta_8(1) - \alpha_7(1), \quad \Delta_4 \triangleq \beta_7(1) - \beta_8(1). \tag{60}$$

It is evident from (59) that $\frac{\Delta_1}{\Delta_2} < \frac{\Delta_3}{\Delta_4}$. Note that $G_7(\lambda)$ and $F_7(\lambda)$ in (58) are constructed by [1, Lemma B.3.]. Then, there exists a number $\lambda^* > \alpha_8(3)$ defined in [1, Eq. (65)] such that

$$\begin{aligned} \Delta_4 &= -G_7(\beta_8(1)) = -\frac{F_7(\beta_8(1))}{(1 - \frac{F_7(\lambda^*)}{G_8(\lambda^*)})(\beta_8(1) - \lambda^*)} = -\frac{(\beta_8(1) - \alpha_7(1))(\beta_8(1) - \alpha_7(2))}{(1 - \frac{F_7(\lambda^*)}{G_8(\lambda^*)})(\beta_8(1) - \lambda^*)} \\ &= \Delta_3 \frac{\beta_8(1) - \alpha_7(2)}{\beta_8(1) - \lambda^*} \frac{G_8(\lambda^*)}{F_7(\lambda^*) - G_8(\lambda^*)} = \Delta_3 \frac{(\alpha_7(2) - \beta_8(1))(\lambda^* - \beta_8(2))}{F_7(\lambda^*) - G_8(\lambda^*)}. \end{aligned} \tag{61}$$

By the first equation of [1, Eq.(59)], [1, Eq.(63)] and (59), one has

$$\max\{\beta_8(1) - \alpha_7(1), \beta_8(2) - \alpha_7(2)\} < \max\{\beta_8(1) - \alpha_8(1), \beta_8(2) - \alpha_8(2)\} < \frac{\Delta}{2},$$

which, together with (59) and $\frac{\Delta}{4} < 16 < \lambda_2 < \alpha_8(3) < \lambda^*$, yields

$$\begin{aligned} F_7(\lambda^*) - G_8(\lambda^*) &= (\lambda^* - \alpha_7(1))(\lambda^* - \alpha_7(2)) - (\lambda^* - \beta_8(1))(\lambda^* - \beta_8(2)) \\ &= (\lambda^* - \beta_8(1) + \beta_8(1) - \alpha_7(1))(\lambda^* - \beta_8(2) + \beta_8(2) - \alpha_7(2)) - (\lambda^* - \beta_8(1))(\lambda^* - \beta_8(2)) \\ &< \frac{\Delta}{2}(2\lambda^* - \beta_8(1) - \beta_8(2)) + \frac{\Delta^2}{4} < 2\Delta\lambda^*. \end{aligned} \tag{62}$$

Note from (59) that

$$\alpha_7(2) - \beta_8(1) > (\lambda_2 - \frac{\Delta}{2}) - (\lambda_1 + \frac{\Delta}{2}) = \lambda_2 - \lambda_1 - \Delta > \Delta. \tag{63}$$

By (3), (59) and $\lambda^* > \alpha_8(3) > \alpha_3 = \lambda_3 - \rho_3 > \lambda_3 - 1$,

$$\lambda^* - \beta_8(2) = \frac{\lambda^*}{2} + \frac{\lambda^*}{2} - \beta_8(2) > \frac{\lambda^*}{2} + \frac{\lambda_3 - 1}{2} - \beta_8(2) > \frac{\lambda^*}{2} + \lambda_2 + \frac{1}{2} - \beta_8(2) > \frac{\lambda^*}{2}. \tag{64}$$

As a result, by (58), (61), (62), (63) and (64), the following inequality holds

$$\frac{\Delta_1}{\Delta_2} < \frac{\Delta_3}{\Delta_4} = \frac{F_7(\lambda^*) - G_8(\lambda^*)}{(\alpha_7(2) - \beta_8(1))(\lambda^* - \beta_8(2))} < \frac{2\Delta\lambda^*}{\Delta\frac{\lambda^*}{2}} = 4. \tag{65}$$

Since $F_5(\lambda)$ and $G_5(\lambda)$ are constructed by [1, Lemma B.4.], applying Vieta Theorem to the first equation of [1, Eq.(83)], we conclude

$$\begin{aligned} \alpha_5(1) &= \frac{(\lambda_3 - \beta_6(1))\alpha_6(1)\lambda_2}{\lambda_3(\alpha_6(1) + \lambda_2 - \beta_6(1)) - \alpha_6(1)\lambda_2} = \frac{(\lambda_3 - \lambda_1 - \Delta_2)(\lambda_1 - \Delta_1)\lambda_2}{\lambda_3(\lambda_1 - \Delta_1 + \lambda_2 - \lambda_1 - \Delta_2) - (\lambda_1 - \Delta_1)\lambda_2} \\ &= \frac{\lambda_2(\lambda_1 - \Delta_1)(\lambda_3 - \lambda_1 - \Delta_2)}{\lambda_3(\lambda_2 - \Delta_1 - \Delta_2) - \lambda_2(\lambda_1 - \Delta_1)}. \end{aligned}$$

So, by (3), (60) and (65),

$$\begin{aligned} \alpha_5(1) - \lambda_1 &= \frac{\lambda_2(\lambda_1 - \Delta_1)(\lambda_3 - \Delta_2) - \lambda_1\lambda_3(\lambda_2 - \Delta_1 - \Delta_2)}{\lambda_3(\lambda_2 - \Delta_1 - \Delta_2) - \lambda_2(\lambda_1 - \Delta_1)} = \frac{\Delta_1\Delta_2\lambda_2 + \Delta_2\lambda_1(\lambda_3 - \lambda_2) - \Delta_1\lambda_3(\lambda_2 - \lambda_1)}{\lambda_3(\lambda_2 - \Delta_1 - \Delta_2) - \lambda_2(\lambda_1 - \Delta_1)} \\ &> \frac{\Delta_1\Delta_2\lambda_2}{\lambda_3(\lambda_2 - \Delta_1 - \Delta_2) - \lambda_2(\lambda_1 - \Delta_1)} > 0, \end{aligned}$$

which completes the proof. □

B. Proofs of Lemmas 4.1–4.2

The proof of Lemma 4.1. (i) Let $m > 1$. Since $\lim_{\lambda\to+\infty} \frac{G(\lambda)}{\widetilde{F}(\lambda)} = 1$ and $\frac{G(\lambda_m)}{\widetilde{F}(\lambda_m)} = 0$, there is a sufficiently large number $\widetilde{\lambda} > \lambda_m$, such that

$$\frac{G(\widetilde{\lambda})}{\widetilde{F}(\widetilde{\lambda})} \in \left(1 - \frac{m}{\Lambda}\rho_{m-1}, 1\right). \tag{66}$$

Here, $0 < 1 - \frac{m}{\Lambda}\rho_{m-1} < 1$. We define

$$\begin{cases} \widetilde{\mu} = \nu \frac{G(\widetilde{\lambda})}{\widetilde{F}(\widetilde{\lambda})}, \\ \widetilde{G}(\lambda) = \frac{\nu G(\lambda) - \widetilde{\mu}\widetilde{F}(\lambda)}{(\nu-\widetilde{\mu})(\lambda-\widetilde{\lambda})}, \\ \widetilde{b} = \frac{\lambda_m \widetilde{G}(\lambda_m)(\widetilde{F}(\widetilde{\lambda})-G(\widetilde{\lambda}))}{\widetilde{F}(\lambda_m)G(\widetilde{\lambda})} L, \\ \widetilde{\nu} = \frac{\widetilde{\mu}-\nu}{\widetilde{b}}, \end{cases} \tag{67}$$

Observe that $\widetilde{\mu} \in (0, \nu)$ by (66), then $\widetilde{G}(\lambda)$ is well-defined. Consequently, (26) follows and

$$-\frac{\widetilde{\mu}}{\widetilde{\nu}} \frac{\widetilde{F}(\lambda_m)}{\lambda_m \widetilde{G}(\lambda_m)} = \frac{G(\widetilde{\lambda})}{\widetilde{F}(\widetilde{\lambda}) - G(\widetilde{\lambda})} \frac{\widetilde{F}(\lambda_m)}{\lambda_m \widetilde{G}(\lambda_m)} \widetilde{b} = L.$$

It remains to show (24). First, by applying [1, Lemma B.1.], we know

$$\widetilde{\beta}_i \in (\lambda_i, \lambda_{i+1} - \rho_{i+1}), \quad i \in [1\ m-1]. \tag{68}$$

Fix $i \in [1\ m-1]$. Then, (68) yields

$$\frac{\Lambda}{\Lambda + \rho_{i-1}} \geq \prod_{j=1}^{i-1} \frac{\Lambda}{\Lambda + \rho_j} > \prod_{j=1}^{i-1} \frac{\widetilde{\beta}_i - \lambda_j}{\widetilde{\beta}_i - \lambda_j + \rho_j} > \prod_{j=1}^{i-1} \frac{\lambda_{j+1} - \lambda_j}{\lambda_{j+1} - \lambda_j + \rho_j} > \left(\frac{2\Delta}{2\Delta + \rho_{i-1}}\right)^i > 1 - \frac{i\rho_{i-1}}{\Delta + \rho_{i-1}}, \tag{69}$$

where the last inequality follows from the Bernoulli inequality. In addition, by (68) again,

$$\prod_{j=i+2}^{m} \frac{\widetilde{\beta}_i - \lambda_j}{\widetilde{\beta}_i - \lambda_j + \rho_j} > \prod_{j=i+2}^{m} \frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_i - \rho_j} > 1. \tag{70}$$

Now, we estimate $\left(1 + \frac{\rho_i}{\widetilde{\beta}_i - \lambda_i}\right)\left(1 - \frac{\rho_{i+1}}{\lambda_{i+1} - \widetilde{\beta}_i}\right)$. Substituting $\lambda = \widetilde{\beta}_i$ into (26), we compute

$$\prod_{j=1}^{m}(\widetilde{\beta}_i - \lambda_j + \rho_j) = \widetilde{F}(\widetilde{\beta}_i) = \frac{1}{\widetilde{\mu}}(\nu G(\widetilde{\beta}_i) - \widetilde{b}\widetilde{\nu}(\widetilde{\lambda} - \widetilde{\beta}_i)\widetilde{G}(\widetilde{\beta}_i)) = \frac{\widetilde{F}(\widetilde{\lambda})}{G(\widetilde{\lambda})} \prod_{j=1}^{m}(\widetilde{\beta}_i - \lambda_j). \tag{71}$$

Note that (66), (69) and (71) infer

$$\begin{aligned} \left(1 + \frac{\rho_i}{\widetilde{\beta}_i - \lambda_i}\right)\left(1 - \frac{\rho_{i+1}}{\lambda_{i+1} - \widetilde{\beta}_i}\right) &= \frac{\widetilde{F}(\widetilde{\lambda})}{G(\widetilde{\lambda})} \prod_{j\neq i,i+1}^{m} \frac{\widetilde{\beta}_i - \lambda_j}{\widetilde{\beta}_i - \lambda_j + \rho_j} && (72) \\ &> \left(1 - \frac{i\rho_{i-1}}{\Delta + \rho_{i-1}}\right) \prod_{j=i+2}^{m} \frac{\widetilde{\beta}_i - \lambda_j}{\widetilde{\beta}_i - \lambda_j + \rho_j}. && (73) \end{aligned}$$

Combining (21) and (73), we deduce that

$$\begin{aligned} \left(1 + \frac{\rho_i}{\widetilde{\beta}_i - \lambda_i}\right)\left(1 - \frac{\rho_{i+1}}{\lambda_{i+1} - \lambda_i}\right) &> \left(1 + \frac{\rho_i}{\widetilde{\beta}_i - \lambda_i}\right)\left(1 - \frac{\rho_{i+1}}{\lambda_{i+1} - \widetilde{\beta}_i}\right) > 1 - \frac{i\rho_{i-1}}{\Delta + \rho_{i-1}} > 1 - \frac{m\rho_i}{\Delta} \\ &> 1 - \frac{\rho_{i+1}^2}{\Lambda^2} > 1 - \frac{\rho_{i+1}^2}{(\lambda_{i+1} - \lambda_i)^2} = (1 + \frac{\rho_{i+1}}{\lambda_{i+1} - \lambda_i})(1 - \frac{\rho_{i+1}}{\lambda_{i+1} - \lambda_i}). \end{aligned}$$

Consequently, by letting $e_i \triangleq \frac{\rho_i}{\rho_{i+1}}$, we immediately assert

$$\widetilde{\beta}_i - \lambda_i < e_i(\lambda_{i+1} - \lambda_i) < e_i(\lambda_j - \lambda_i), \quad j \in [i+1\ m].$$

Therefore,

$$\lambda_j - \widetilde{\beta}_i > (1 - e_i)\left(\lambda_j - \lambda_i\right), \quad j \in [i+1\ m]. \tag{74}$$

As a result,

$$\prod_{j=i+2}^{m-1} \frac{\widetilde{\beta}_i - \lambda_j}{\widetilde{\beta}_i - \lambda_j + \rho_j} < \prod_{j=i+2}^{m-1} \frac{(\lambda_j - \lambda_i)(1 - e_i)}{(\lambda_j - \lambda_i)(1 - e_i) - \rho_j}. \tag{75}$$

Finally, in view of (66), (69), (72) and (75), it follows that

$$\left(1 + \frac{\rho_i}{\widetilde{\beta}_i - \lambda_i}\right)\left(1 - \frac{\rho_{i+1}}{\lambda_{i+1} - \widetilde{\beta}_i}\right) < \frac{1}{1 - \frac{m}{\Lambda}\rho_{m-1}} \frac{\Lambda}{\Lambda + \rho_{i-1}} \frac{\widetilde{\beta}_i - \lambda_m}{\widetilde{\beta}_i - \lambda_m + \rho_m} \prod_{j=i+2}^{m-1} \frac{(\lambda_j - \lambda_i)(1 - e_i)}{(\lambda_j - \lambda_i)(1 - e_i) - \rho_j}. \tag{76}$$

On the other hand, (70) and (73) infer

$$\left(1 + \frac{\rho_i}{\widetilde{\beta}_i - \lambda_i}\right)\left(1 - \frac{\rho_{i+1}}{\lambda_{i+1} - \widetilde{\beta}_i}\right) > \left(1 - \frac{i\rho_{i-1}}{\Delta + \rho_{i-1}}\right) \prod_{j=i+2}^{m} \frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_i - \rho_j}. \tag{77}$$

Next, we prove (24) by reduction to absurdity. We first consider the case where $i < m - 1$. Suppose (24) fails. Since (70) and (77) lead to

$$\begin{aligned} \frac{\rho_i}{\widetilde{\beta}_i - \lambda_i} &> \left(1 - \frac{i\rho_{i-1}}{\Delta + \rho_{i-1}}\right) \frac{\lambda_m - \lambda_i}{\lambda_m - \lambda_i - \rho_m} - 1 = \frac{\lambda_m - \lambda_i}{\lambda_m - \lambda_i - \rho_m} - 1 - \frac{i\rho_{i-1}}{\Delta + \rho_{i-1}} \frac{\lambda_m - \lambda_i}{\lambda_m - \lambda_i - \rho_m} \\ &> \frac{\rho_m}{\lambda_m - \lambda_i - \rho_m} - \frac{m\rho_{m-1}}{\Delta + \rho_{m-1}} \frac{\Delta}{\Delta - \rho_m} > \frac{\rho_m}{\lambda_m - \lambda_i - \rho_m} - \frac{2m\rho_{m-1}}{\Delta} > \frac{\rho_m}{\lambda_m - \lambda_i} - \tau\rho_m, \end{aligned} \tag{78}$$

we obtain $\frac{1}{\rho_m} \frac{\rho_i}{\widetilde{\beta}_i - \lambda_i} - \frac{1}{\lambda_m - \lambda_i} > -\tau$. So,

$$\frac{1}{\rho_m} \frac{\rho_i}{\widetilde{\beta}_i - \lambda_i} - \frac{1}{\lambda_m - \lambda_i} > \tau. \tag{79}$$

Note that $e_i < \frac{1}{2}$, (74), (76) and (79) imply

$$\begin{aligned} &\left(1 - \frac{1}{1 - e_i} \frac{\rho_m}{\lambda_m - \lambda_i}\right)\left(1 + \rho_m\left(\frac{1}{\lambda_m - \lambda_i} + \tau\right)\right)\left(1 - \frac{\rho_{i+1}}{\lambda_{i+1} - \widetilde{\beta}_i}\right) \\ < \; &\left(1 - \frac{\rho_m}{\lambda_m - \widetilde{\beta}_i}\right)\left(1 + \frac{\rho_i}{\widetilde{\beta}_i - \lambda_i}\right)\left(1 - \frac{\rho_{i+1}}{\lambda_{i+1} - \widetilde{\beta}_i}\right) < \frac{1}{1 - \frac{m}{\Lambda}\rho_{m-1}} \frac{\Lambda}{\Lambda + \rho_{i-1}} \prod_{j=i+2}^{m-1} \frac{(\lambda_j - \lambda_i)(1 - e_i)}{(\lambda_j - \lambda_i)(1 - e_i) - \rho_j} \\ < \; &\frac{1}{1 - \frac{m}{\Lambda}\rho_{m-1}} \frac{\Lambda}{\Lambda + \rho_{i-1}} \frac{1}{1 - \sum_{j=i+2}^{m-1} \frac{2\rho_j}{\lambda_j - \lambda_i}} < 1 + \frac{2m}{\Lambda}\rho_{m-1} + 4\sum_{j=i+2}^{m-1} \frac{\rho_j}{\lambda_j - \lambda_i}, \end{aligned} \tag{80}$$

where the third inequality follows from the Bernoulli inequality and the last inequality is due to $\frac{m}{\Lambda}\rho_{m-1} + \sum_{j=i+2}^{m-1} \frac{2\rho_j}{\lambda_j - \lambda_i} < \frac{1}{2}$. Furthermore, due to (21), we obtain

$$m2^{m+1}\frac{\Lambda^2}{\Delta} e_i < \rho_{i+1} \leq \rho_m < 1,$$

which implies $\frac{\tau}{4} > \frac{\rho_m}{4} > \frac{e_i}{\Delta}$. Besides, $\rho_m = \varepsilon < \frac{\Delta^4}{16}$, it then follows that

$$\frac{\tau}{4} - \rho_m \frac{1}{\Delta}\left(\frac{1}{2\Delta} + \tau\right) > \frac{\tau}{4} - \rho_m \frac{1}{\Delta^2} = \rho_m^{\frac{1}{2}}\left(\frac{1}{4} - \rho_m^{\frac{1}{2}}\frac{1}{\Delta^2}\right) > 0.$$

Therefore

$$\begin{aligned} & \left(1 - \frac{1}{1-e_i}\frac{\rho_m}{\lambda_m - \lambda_i}\right)\left(1 + \rho_m\left(\frac{1}{\lambda_m - \lambda_i} + \tau\right)\right) \\ = \;& 1 + \rho_m\left(\tau - \frac{e_i}{1-e_i}\frac{1}{\lambda_m - \lambda_i}\right) - \rho_m^2 \frac{1}{1-e_i}\frac{1}{\lambda_m - \lambda_i}\left(\frac{1}{\lambda_m - \lambda_i} + \tau\right) \\ > \;& 1 + \frac{\tau}{2}\rho_m + \rho_m\left(\frac{\tau}{2} - \frac{e_i}{\Delta} - \rho_m\frac{1}{\Delta}\left(\frac{1}{2\Delta} + \tau\right)\right) > 1 + \frac{\tau}{2}\rho_m. \end{aligned} \tag{81}$$

So, by observing $\frac{4\rho_m^2}{\Lambda^2 2^{m+1}} \le \frac{\rho_m^2}{\Lambda^2} < \tau < \frac{1}{4}$, (80) and (81) lead to

$$\begin{aligned} \lambda_{i+1} - \widetilde{\beta}_i \;&<\; \frac{\rho_{i+1}}{1 - \left(1 + \frac{\tau}{2}\rho_m\right)^{-1}\left(1 + \frac{2m}{\Lambda}\rho_{m-1} + 4\sum_{j=i+2}^{m-1}\frac{\rho_j}{\lambda_j - \lambda_i}\right)} < \frac{\rho_{i+1}}{1 - \left(1 + \frac{\tau}{2}\rho_m\right)^{-1}\left(1 + \frac{4m}{\Delta}\rho_{m-1}\right)} \\ &=\; \frac{1 + \frac{\tau}{2}\rho_m}{\frac{\tau}{2}\rho_m - \frac{4m}{\Delta}\rho_{m-1}}\rho_{i+1} < \frac{1 + \frac{\tau}{2}\rho_m}{\frac{\tau}{2}\rho_m - \frac{4\rho_m^2}{\Lambda^2 2^{m+1}}}\rho_{i+1} < \frac{2}{\tau\rho_m(\frac{1}{2} - \tau)}\rho_{i+1} < \frac{8\rho_{i+1}}{\tau\rho_m}, \end{aligned} \tag{82}$$

where the third inequality follows from $m2^{m+1}\frac{\Lambda^2}{\Delta}\rho_{m-1} < \rho_m^2$. At last, combining (21), (79) and (82) gives

$$\begin{aligned} \lambda_{i+1} - \lambda_i \;&=\; (\lambda_{i+1} - \widetilde{\beta}_i) + (\widetilde{\beta}_i - \lambda_i) < \frac{8\rho_{i+1}}{\tau\rho_m} + \left(\frac{1}{\lambda_m - \lambda_i} + \tau\right)^{-1}\frac{\rho_i}{\rho_m} \\ &<\; \frac{8\rho_{m-1}}{\tau\rho_m} + \Lambda\frac{\rho_{m-1}}{\rho_m} < (8+\Lambda)\frac{1}{\tau}\frac{\rho_{m-1}}{\rho_m} < (8+\Lambda)\tau < 2\Delta, \end{aligned}$$

which contradicts to (17).

So far, we have shown (24) except $i = m-1$. As to $i = m-1$, (76) turns out to be

$$\left(1 + \frac{\rho_{m-1}}{\widetilde{\beta}_{m-1} - \lambda_{m-1}}\right)\left(1 - \frac{\rho_m}{\lambda_m - \widetilde{\beta}_{m-1}}\right) < \frac{1}{1 - \frac{m}{\Lambda}\rho_{m-1}}\frac{\Lambda}{\Lambda + \rho_{m-2}}. \tag{83}$$

If $\frac{1}{\rho_m}\frac{\rho_{m-1}}{\widetilde{\beta}_{m-1} - \lambda_{m-1}} - \frac{1}{\lambda_m - \lambda_{m-1}} > \tau$, by (74), (83) and $e_m < \rho_m < \frac{\Delta^2\tau}{4} < \frac{1}{2}$,

$$\begin{aligned} \frac{1}{1 - \frac{m}{\Lambda}\rho_{m-1}}\frac{\Lambda}{\Lambda + \rho_{m-2}} \;&>\; \left(1 + \rho_m\left(\frac{1}{\lambda_m - \lambda_{m-1}} + \tau\right)\right)\left(1 - \frac{\rho_m}{\lambda_m - \widetilde{\beta}_{m-1}}\right) \\ &>\; \left(1 + \rho_m\left(\frac{1}{\lambda_m - \lambda_{m-1}} + \tau\right)\right)\left(1 - \frac{\rho_m}{(1-e_m)(\lambda_m - \lambda_{m-1})}\right) \\ &>\; 1 + \tau\rho_m + \frac{\rho_m}{\lambda_m - \lambda_{m-1}} - \frac{\rho_m}{(1-e_m)(\lambda_m - \lambda_{m-1})} - \rho_m\left(\frac{1}{\lambda_m - \lambda_{m-1}} + \tau\right)\frac{\rho_m}{(1-e_m)(\lambda_m - \lambda_{m-1})} \\ &>\; 1 + \tau\rho_m - \frac{e_m}{1-e_m}\frac{\rho_m}{\lambda_m - \lambda_{m-1}} - \frac{\rho_m^2}{\Delta^2} > 1 + \tau\rho_m - \frac{\rho_m^2}{\Delta} - \frac{\rho_m^2}{\Delta^2} > 1 + \frac{\tau}{4}\rho_m. \end{aligned} \tag{84}$$

However, because (21) infers $\rho_{m-1} < \frac{\Delta}{m2^{m+1}\Lambda^2}\rho_m^2 < \frac{\Lambda}{8m}\tau\rho_m$, we conclude

$$1 + \frac{\tau}{4}\rho_m > \frac{1}{1 - \frac{\tau}{8}\rho_m} > \frac{1}{1 - \frac{m}{\Lambda}\rho_{m-1}} > \frac{1}{1 - \frac{m}{\Lambda}\rho_{m-1}}\frac{\Lambda}{\Lambda + \rho_{m-2}},$$

which contradicts to (84).

Suppose $\frac{1}{\rho_m}\frac{\rho_{m-1}}{\widetilde{\beta}_{m-1}-\lambda_{m-1}} - \frac{1}{\lambda_m-\lambda_{m-1}} < -\tau$. Observe that (83) reduces to

$$\left(1+\frac{\rho_{m-1}}{\widetilde{\beta}_{m-1}-\lambda_{m-1}}\right)\left(1-\frac{\rho_m}{\lambda_m-\widetilde{\beta}_{m-1}}\right) > 1-\frac{(m-1)\rho_{m-2}}{\Delta+\rho_{m-2}}, \tag{85}$$

then

$$\begin{aligned}
1-\tau\rho_m &> \left(1+\frac{\rho_m}{\lambda_m-\lambda_{m-1}}\right)\left(1-\frac{\rho_m}{\lambda_m-\lambda_{m-1}}\right) > \left(1+\rho_m\left(\frac{1}{\lambda_m-\lambda_{m-1}}-\tau\right)\right)\left(1-\frac{\rho_m}{\lambda_m-\lambda_{m-1}}\right)\\
&> \left(1+\frac{\rho_{m-1}}{\widetilde{\beta}_{m-1}-\lambda_{m-1}}\right)\left(1-\frac{\rho_m}{\lambda_m-\lambda_{m-1}}\right) > \left(1+\frac{\rho_{m-1}}{\widetilde{\beta}_{m-1}-\lambda_{m-1}}\right)\left(1-\frac{\rho_m}{\lambda_m-\widetilde{\beta}_{m-1}}\right)\\
&> 1-\frac{(m-1)\rho_{m-2}}{\Delta+\rho_{m-2}},
\end{aligned} \tag{86}$$

which is impossible due to (21) that

$$\frac{m-1}{\Delta+\rho_{m-2}}\frac{\rho_{m-2}}{\rho_m} < \frac{m}{\Delta}\frac{1}{m2^{m+1}}\frac{\Delta}{\Lambda^2}\rho_m < \tau.$$

In summary, (24) holds.

(ii) Let $m=1$. Similarly to (i), there is a sufficiently large number $\widetilde{\lambda} > \lambda_1$ such that (66) holds. Then, it is easy to verify that

$$\begin{cases} \widetilde{\mu} = \nu\frac{\widetilde{\lambda}-\lambda_1}{\widetilde{\lambda}-\lambda_1+\rho_1}, \\ \widetilde{G}(\lambda) = 1, \\ \widetilde{b} = \frac{L\lambda_1}{\widetilde{\lambda}-\lambda_1}, \\ \widetilde{\nu} = \frac{\widetilde{\mu}-\nu}{\widetilde{b}} \end{cases}$$

are exactly the quantities we want. □

Before proceeding to the proof of Lemma 4.2, we need a lemma below.

Lemma B.1. Let $F(\lambda) = \mu\prod_{i=1}^{p}(\lambda-\alpha_i)$ and $G(\lambda) = \nu\prod_{i=1}^{p}(\lambda-\beta_i)$ be two polynomials of degree p, where $|\mu| > |\nu|$, $\alpha_1 < \cdots < \alpha_p$ and $\beta_1 < \cdots < \beta_p$. Then, $F(\lambda)-G(\lambda)$ has p real roots $\gamma_1 \leqslant \cdots \leqslant \gamma_p$ and the following two statements hold:

(i) when $p > 1$, for any $\eta \in (0, \frac{1}{2}\min_{1\leqslant i\leqslant p-1}(\alpha_{i+1}-\alpha_i))$, if

$$\frac{|\nu|}{|\mu|} < \frac{\min\{1,\eta^p\}}{2+2\max_{j\in[1\ p]}\prod_{i=1}^{p}|\alpha_j-\beta_i|}, \tag{87}$$

then

$$\max_{1\leqslant i\leqslant p}|\gamma_i-\alpha_i| < \eta; \tag{88}$$

(ii) when $p=1$, for any $\eta \in (0,\frac{1}{2})$, (88) holds provided that

$$|\nu| < \frac{|\mu|}{2}\min\left\{\frac{\eta}{|\alpha_1-\beta_1|}, 1\right\}. \tag{89}$$

Proof. First, since $|\nu| < |\mu|$, $F(\lambda)-G(\lambda)$ has p real roots $\gamma_1 \leqslant \cdots \leqslant \gamma_p$.

(i) Let $p > 1$. Observe that $F(\lambda)-G(\lambda) = (\mu-\nu)\prod_{i=1}^{p}(\lambda-\gamma_i)$, then for each $j \in [1\ p]$,

$$(\mu-\nu)\prod_{i=1}^{p}(\alpha_j-\gamma_i) = F(\alpha_j)-G(\alpha_j) = -\nu\prod_{i=1}^{p}(\alpha_j-\beta_i). \tag{90}$$

We shall prove for each $j \in [1\ p]$, $\min_{1\leqslant i\leqslant p}|\alpha_j-\gamma_i| < \eta$ holds. Otherwise, assume $\min_{1\leqslant i\leqslant p}|\alpha_j-\gamma_i| \geq \eta$ for some $j \in [1\ p]$. Thus, by (87) and (90),

$$\eta^p \leq \prod_{i=1}^{p}\left|\alpha_j-\gamma_i\right| \leq \left|\frac{\nu/\mu}{1-\nu/\mu}\right| \max_{j\in[1\ p]}\prod_{i=1}^{p}\left|\alpha_j-\beta_i\right| < 2\left|\frac{\nu}{\mu}\right| \max_{j\in[1\ p]}\prod_{i=1}^{p}\left|\alpha_j-\beta_i\right| < \min\{1,\eta^p\} \leq \eta^p,$$

which is impossible. Hence, $\min_{1\leqslant i\leqslant p}|\alpha_j-\gamma_i| < \eta$ holds for $j = 1,2,\ldots,p$. But observe that $\eta \in (0, \frac{1}{2}\min_{1\leqslant i\leqslant p-1}(\alpha_{i+1}-\alpha_i))$ implies

$$\begin{cases} \alpha_1+\eta < \alpha_2 - \frac{1}{2}(\alpha_2-\alpha_1), \\ (\alpha_j-\eta, \alpha_j+\eta) \subset (\alpha_{j-1}+\frac{1}{2}(\alpha_j-\alpha_{j-1}), \alpha_{j+1}-\frac{1}{2}(\alpha_{j+1}-\alpha_j)), \quad j \in [2\ p-1], \\ \alpha_p-\eta > \alpha_{p-1}+\frac{1}{2}(\alpha_p-\alpha_{p-1}), \end{cases}$$

by $\gamma_1 \leqslant \cdots \leqslant \gamma_p$, it is evident that $|\alpha_i-\gamma_i| < \eta$, $i = 1,2,\ldots,p$.

(ii) When $p = 1$, (89) and (90) lead to

$$|\gamma_1-\alpha_1| \leq \left|\frac{\nu/\mu}{1-\nu/\mu}\right| |\alpha_1-\beta_1| < 2\left|\frac{\nu}{\mu}\right| |\alpha_1-\beta_1| < \eta,$$

as desired. □

The proof of Lemma 4.2. We first claim $\lambda G(\lambda) \prec F(\lambda)$. As a matter of fact, by (21) and (23), it is evident that

$$|C_{q_1-l,j}| - \Omega^{q_1-l}\tau > \frac{\Delta}{\Lambda^2}, \quad j \in [1\ m-1].$$

So, by (23)–(28), for each $j \in [1\ m-1]$,

$$\begin{aligned} \beta_j - \lambda_j &< \left(|C_{q_1-l,j}| - \Omega^{q_1-l}\tau\right)^{-1} \frac{|\lambda_j-\alpha_j|}{\rho_m} \leq \left(|C_{q_1-l,j}| - \Omega^{q_1-l}\tau\right)^{-1} \frac{\rho_j}{\rho_m} \\ &< \frac{\Lambda^2}{\Delta}\frac{\rho_j}{\rho_m} < \frac{\Lambda^2}{\Delta}\frac{1}{m2^{m+1}}\frac{\Delta}{\Lambda^2}\rho_m = \frac{\rho_m}{m2^{m+1}}. \end{aligned} \tag{91}$$

It then follows from (17) and (27) that when $j \in [1\ z_{l+1}-1]$,

$$\alpha_j < \beta_j = (\beta_j-\lambda_j) + (\lambda_j-\lambda_{j+1}) + \lambda_{j+1} < \frac{\rho_m}{m2^{m+1}} - 2\Delta + \lambda_{j+1} < \alpha_{j+1} < \beta_{j+1}. \tag{92}$$

Moreover, if $z_{l+1} < m$, in view of (22), (23), (27) and (91), we deduce

$$\beta_{z_{l+1}} < \lambda_{z_{l+1}} + \frac{\rho_m}{m2^{m+1}} < \lambda_{z_{l+1}+1} < \alpha_{z_{l+1}+1} < \beta_{z_{l+1}+1} < \lambda_{z_{l+1}+2} < \cdots < \beta_{m-1} < \alpha_m. \tag{93}$$

We thus conclude $\lambda G(\lambda) \prec F(\lambda)$ by (92) and (93).

We begin our construction. Let

$$\begin{cases} \widetilde{m} = -\frac{\mu}{\nu}\frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}, \\ \widetilde{F}(\lambda) = \frac{\mu F(\lambda)+\lambda\widetilde{m}\nu G(\lambda)}{\mu+\widetilde{m}\nu}. \end{cases} \tag{94}$$

It is easy to verify $\widetilde{m} > 0$ in view of (27) and $\lambda G(\lambda) \prec F(\lambda)$. Then, by (21), (23) and (28), for $z_l \in [1\ m-1]$,

$$\left|\frac{\widetilde{m}\nu}{\mu}\right| = \frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})} < \frac{\lambda_{z_l}-\alpha_{z_l}}{\beta_{z_l}-\lambda_{z_l}}\frac{\Lambda^{m-1}}{\Delta^{m-1}} < \rho_m(C_{q_1-l,z_l}+\Omega^{q_1-l}\tau)\frac{\Lambda^{m-1}}{\Delta^{m-1}} < \rho_m(\frac{1}{2\Delta}+\frac{\Delta}{\Lambda^2})\frac{\Lambda^{m-1}}{\Delta^{m-1}} < \frac{\Delta\tau}{\Lambda} < 1. \tag{95}$$

When $z_l = m$, due to (27), (29) and $\lambda G(\lambda) \prec F(\lambda)$,

$$\left|\frac{\widetilde{m}\nu}{\mu}\right| = \frac{F(\lambda_m)}{\lambda_m G(\lambda_m)} = \frac{\prod_{j=1}^{m}(\lambda_m-\alpha_j)}{\lambda_m\prod_{j=1}^{m-1}(\lambda_m-\beta_j)} < \frac{\lambda_m-\alpha_m}{\lambda_m-\beta_{m-1}} < \frac{2\Lambda\rho_m}{\Delta\lambda_1} < \frac{\Delta\tau}{\Lambda} < 1. \tag{96}$$

Hence $\widetilde{F}(\lambda)$ is well-defined and $\deg(\widetilde{F}(\lambda)) = \deg(F(\lambda)) = m$. Now, by $\lambda G(\lambda) \prec F(\lambda)$ again, applying [1, Lemma B.1.] to $\mu F(\lambda)$ and $-\lambda\widetilde{m}\nu G(\lambda)$ shows

$$\widetilde{\alpha}_j \in (\alpha_j, \beta_j), \quad j = 1, \ldots, m, \tag{97}$$

which indicates $F(\lambda) \prec \widetilde{F}(\lambda)$. Observe that $(\lambda - \lambda_{z_l}) | \mu F(\lambda) + \lambda\widetilde{m}\nu G(\lambda)$ and $\alpha_{z_l} < \lambda_{z_l} < \beta_{z_l}$, we assert $\widetilde{\alpha}_{z_l} = \lambda_{z_l}$ at once. So (31) is proved.

Next, let

$$\begin{cases} \widetilde{\mu} = \epsilon\nu, \\ \widetilde{b} = -\frac{\mu + \widetilde{m}\nu}{\widetilde{\mu}}, \\ \widetilde{\nu} = \frac{\widetilde{\mu} - \nu}{\widetilde{b}}, \\ \widetilde{G}(\lambda) = \frac{\nu(\lambda - \lambda_{z_l})G(\lambda) - \widetilde{\mu}\widetilde{F}(\lambda)}{\widetilde{b}\widetilde{\nu}(\lambda - \lambda_{z_l})}, \end{cases} \tag{98}$$

where

$$\epsilon = \begin{cases} \nu_1, & \text{if} \quad m > 2, \\ \nu_2, & \text{if} \quad m = 2, \end{cases}$$

and

$$\begin{aligned} \nu_1 &= \frac{\min\{1, \eta_1^{m-1}\}}{4 + 4\max_{1 \leqslant j \leqslant m-1} \prod_{h=1}^{m-1} \left|\beta_j - \widetilde{\alpha}_h\right|}, \\ \nu_2 &= \frac{1}{4}\min\left\{\frac{\eta_2}{|\beta_1 - \widetilde{\alpha}_1|}, 1\right\}, \\ \eta_1 &= \min\left\{\frac{1}{\Delta}\rho_m\tau \min_{1 \leqslant j \leqslant m-1}(\beta_j - \lambda_j)^2, \frac{1}{4}\min_{1 \leqslant l \leqslant m-2}(\beta_{l+1} - \beta_l)\right\}, \\ \eta_2 &= \frac{1}{\Delta}\rho_m\tau(\beta_1 - \lambda_1)^2. \end{aligned}$$

We shall see all the quantities in (98) are well-defined. By (95), (96) and $-\frac{\mu}{\nu} > 0$,

$$\widetilde{b} = -\frac{\mu + \widetilde{m}\nu}{\widetilde{\mu}} = -\frac{\mu}{\epsilon\nu}\left(1 + \frac{\widetilde{m}\nu}{\mu}\right) > 0. \tag{99}$$

This means $\widetilde{\mu}, \widetilde{b}, \widetilde{\nu}$ are well-defined. Note that $\epsilon < 1$, then $\widetilde{\nu}$ is nonzero. Moreover, by (94), $(\lambda - \lambda_{z_l}) | \widetilde{F}(\lambda)$. Therefore, $\widetilde{G}(\lambda)$ is a well-defined monic polynomial of degree $m-1$. We immediately conclude (30) by (94) and (98). Besides, by (27) and (94)–(96), we deduce

$$-\frac{\widetilde{\mu}}{\widetilde{\nu}} = -\frac{\mu + \widetilde{m}\nu}{\nu - \widetilde{\mu}} > -\frac{\mu}{\nu}\left(1 + \frac{\widetilde{m}\nu}{\mu}\right) = -\frac{\mu}{\nu}\left(1 - \frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}\right) > -\frac{\mu}{\nu}\left(1 - \frac{\Delta\tau}{\Lambda}\right) > -\frac{\lambda_1}{\Lambda}\frac{\mu}{\nu} > 0. \tag{100}$$

We are now devoted to prove $\frac{\widetilde{\beta}_i - \lambda_i}{\beta_i - \lambda_i} < 2, i \in [1\ m-1]$. If $m > 2$,

$$\left|\frac{\widetilde{\mu}}{\nu}\right| = \epsilon = \nu_1 < \frac{\min\{1, \eta_1^{m-1}\}}{2 + 2\max_{1 \leqslant j \leqslant m-1}\left|\prod_{h=1}^{m-1}(\beta_j - \widetilde{\alpha}_h)\right|}.$$

By applying Lemma B.1 to $\nu G(\lambda)$ and $\widetilde{\mu}\frac{\widetilde{F}(\lambda)}{\lambda - \lambda_{z_l}}$, we deduce

$$\max_{1 \leqslant i \leqslant m-1} |\widetilde{\beta}_i - \beta_i| < \eta_1 \leq \frac{1}{\Delta}\rho_m\tau \min_{1 \leqslant i \leqslant m-1}(\beta_i - \lambda_i)^2. \tag{101}$$

If $m=2$, then $\left|\frac{\widetilde{\mu}}{\nu}\right| = \nu_2 < \frac{1}{2}\min\left\{\frac{\eta_2}{|\beta_1-\widetilde{\alpha}_1|}, 1\right\}$. Hence Lemma B.1 leads to

$$|\widetilde{\beta}_1 - \beta_1| < \eta_2 = \frac{1}{\Delta}\rho_m\tau(\beta_1-\lambda_1)^2. \tag{102}$$

In view of (101) and (102), we conclude

$$\max_{1\leqslant i\leqslant m-1} |\widetilde{\beta}_i - \beta_i| < \frac{\rho_m\tau}{\Delta}\min_{1\leqslant i\leqslant m-1}(\beta_i-\lambda_i)^2, \tag{103}$$

which, together with (91), yields

$$\left|\frac{\widetilde{\beta}_i-\lambda_i}{\beta_i-\lambda_i} - 1\right| = \left|\frac{\widetilde{\beta}_i-\beta_i}{\beta_i-\lambda_i}\right| < \frac{\rho_m}{\Delta}\tau(\beta_i-\lambda_i) < \frac{\rho_m^2\tau}{m2^{m+1}\Delta} < \frac{1}{2}, \quad i=1,\ldots,m-1.$$

Therefore, we obtain

$$\frac{1}{2} < \frac{\widetilde{\beta}_i-\lambda_i}{\beta_i-\lambda_i} < 2, \quad i=1,\ldots,m-1. \tag{104}$$

So, it remains to show the second inequality of (32) and (33). Note that if $i=z_l$,

$$\left|\frac{1}{\rho_m}\frac{\lambda_i-\widetilde{\alpha}_i}{\widetilde{\beta}_i-\lambda_i} - C_{q_1-l+1,i}\right| < \Omega^{q_1-l+1}\tau \tag{105}$$

is trivial because the left hand side of the inequality becomes zero. Therefore, it suffices to prove (105) when $i\neq z_l$. For that, we first bring in some estimates on $\{\alpha_j,\beta_j,\widetilde{\alpha}_j\}_{j=1}^{m-1}$. Fix $i\in[1\ m-1]\setminus\{z_l\}$. Observe that (21) and (23)–(28) imply

$$\begin{cases} \beta_j-\alpha_j < \frac{\rho_j}{\rho_m}\left(C_{q_1-l,j}-\Omega^{q_1-l}\tau\right)^{-1} + \rho_j < \frac{2\Lambda^2\rho_j}{\Delta\rho_m} < \frac{\Delta^2\tau}{m\Lambda}, & \text{if} \quad 1\le j\le z_{l+1}, \\ \beta_j-\alpha_j \le \beta_j-\lambda_j < (1+C_2(m-1))^n\rho_{m-1} < \frac{\Delta^2\tau}{m\Lambda}, & \text{if} \quad z_{l+1} < j < m-1, \end{cases} \tag{106}$$

where the last inequality is due to $2^n(\frac{2^{2(n+1)^2}\Lambda^{n+1}}{\Delta^n})^n\varepsilon < \frac{\Delta^2\tau}{m\Lambda}$. Then it follows

$$\sum_{j=1}^{m-1}\frac{\beta_j-\alpha_j}{\Delta} < m\frac{\frac{\Delta^2\tau}{m\Lambda}}{\Delta} < \frac{\Delta\tau}{\Lambda} < \frac{1}{2}. \tag{107}$$

In view of (27) and (107), for $h\in[1\ m]$, [2]

$$\prod_{j=1,j\neq h}^{m-1}\frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j} = \prod_{j=1}^{h-1}\left(1+\frac{\beta_j-\alpha_j}{\lambda_h-\beta_j}\right)\prod_{j=h+1}^{m-1}\left(1-\frac{\beta_j-\alpha_j}{\beta_j-\lambda_h}\right) < \prod_{j=1}^{h-1}\left(1+\frac{\beta_j-\alpha_j}{\Delta}\right) < 1+\sum_{j=1}^{h-1}\frac{2(\beta_j-\alpha_j)}{\Delta}. \tag{108}$$

On the other hand, by the Bernoulli inequality, for $h\in[1\ m]$,

$$\prod_{j=1,j\neq h}^{m-1}\frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j} = \prod_{j=1}^{h-1}\left(1+\frac{\beta_j-\alpha_j}{\lambda_h-\beta_j}\right)\prod_{j=h+1}^{m-1}\left(1-\frac{\beta_j-\alpha_j}{\beta_j-\lambda_h}\right) > \prod_{j=h+1}^{m-1}\left(1-\frac{\beta_j-\alpha_j}{\Delta}\right) > 1-\sum_{j=h+1}^{m-1}\frac{\beta_j-\alpha_j}{\Delta}. \tag{109}$$

That is, by (107),

$$\left|\prod_{j=1,j\neq h}^{m-1}\frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j} - 1\right| < \max\left\{\sum_{j=1}^{h-1}\frac{2(\beta_j-\alpha_j)}{\Delta}, \sum_{j=h+1}^{m-1}\frac{\beta_j-\alpha_j}{\Delta}\right\} < \frac{2\Delta\tau}{\Lambda}. \tag{110}$$

[2] For $p,q\in\mathbb{N}^+$, we define $\prod_{i=p}^q a_i \triangleq 1$ and $\sum_{i=p}^q a_i \triangleq 0$ if $p>q$.

Particularly, taking $h = z_l, i, m$ in (110) successively gives

$$\max\left\{\left|\prod_{j=1,j\neq z_l}^{m-1}\frac{\lambda_{z_l}-\alpha_j}{\lambda_{z_l}-\beta_j}-1\right|,\left|\prod_{j=1,j\neq i}^{m-1}\frac{\lambda_i-\alpha_j}{\lambda_i-\beta_j}-1\right|,\left|\prod_{j=1}^{m-1}\frac{\lambda_m-\alpha_j}{\lambda_m-\beta_j}-1\right|\right\}<\frac{2\Delta\tau}{\Lambda}<1. \tag{111}$$

Moreover, by (97), arguing as in (106)–(109), we obtain

$$\left|\prod_{j=1,j\neq i}^{m-1}\frac{\lambda_i-\widetilde{\alpha}_j}{\lambda_i-\beta_j}-1\right|<\max\left\{\sum_{j=1}^{i-1}\frac{2(\beta_j-\widetilde{\alpha}_j)}{\Delta},\sum_{j=i+1}^{m-1}\frac{\beta_j-\widetilde{\alpha}_j}{\Delta}\right\}<\frac{2\Delta\tau}{\Lambda}. \tag{112}$$

Based on the above estimates, we can establish two important inequalities below. By (23), (28), (29) and (111), for $h\in[1\ m-1]$,

$$\begin{aligned}
&\left|\frac{F(\lambda_h)}{\rho_m G(\lambda_h)}-(\lambda_m-\lambda_h)C_{q_1-l,h}\right| = \left|(\alpha_m-\lambda_h)\frac{1}{\rho_m}\frac{\lambda_h-\alpha_h}{\beta_h-\lambda_h}\prod_{j=1,j\neq h}^{m-1}\frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j}-(\lambda_m-\lambda_h)C_{q_1-l,h}\right|\\
\leq\ & (\alpha_m-\lambda_h)\left|\prod_{j=1,j\neq h}^{m-1}\frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j}\right|\left|\frac{1}{\rho_m}\frac{\lambda_h-\alpha_h}{\beta_h-\lambda_h}-C_{q_1-l,h}\right|+2C_{q_1-l,h}(\alpha_m-\lambda_m)+(\lambda_m-\lambda_h)C_{q_1-l,h}\left|\prod_{j=1,j\neq h}^{m-1}\frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j}-1\right|\\
\leq\ & 2(\alpha_m-\lambda_h)\left|\frac{1}{\rho_m}\frac{\lambda_h-\alpha_h}{\beta_h-\lambda_h}-C_{q-l,h}\right|+2C_{q_1-l,h}(\alpha_m-\lambda_m)+(\lambda_m-\lambda_h)C_{q_1-l,h}\left|\prod_{j=1,j\neq h}^{m-1}\frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j}-1\right|\\
<\ & 2\Lambda\Omega^{q_1-l}\tau+\frac{2\Lambda}{\Delta\lambda_1}\rho_m+\tau<2\Lambda\Omega^{q_1-l}\tau+2\tau.
\end{aligned} \tag{113}$$

Furthermore, define

$$V_{z_l}\triangleq\left(1-\frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}\right)\prod_{j=1,j\neq i}^{m-1}\frac{\lambda_i-\widetilde{\alpha}_j}{\lambda_i-\beta_j}.$$

We remark that $V_{z_l}>\frac{1}{2}$. Indeed, (95), (96) and (112) infer

$$\left|V_{z_l}-1\right|\leq\frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}+\left(1-\frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}\right)\left|\prod_{j=1,j\neq i}^{m-1}\frac{\lambda_i-\widetilde{\alpha}_j}{\lambda_i-\beta_j}-1\right|<\frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}+\left|\prod_{j=1,j\neq i}^{m-1}\frac{\lambda_i-\widetilde{\alpha}_j}{\lambda_i-\beta_j}-1\right|<\frac{3\Delta\tau}{\Lambda}. \tag{114}$$

We are now ready to prove (105). Substituting $\lambda=\lambda_i$ into (30), we obtain by (94) and (98) that

$$\frac{F(\lambda_i)}{G(\lambda_i)} = \lambda_i\left(-\frac{\widetilde{m}\nu}{\mu}\right)-\frac{\widetilde{b}\widetilde{\mu}}{\mu}\frac{\widetilde{F}(\lambda_i)}{G(\lambda_i)}=\frac{\lambda_i F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}+(\widetilde{\alpha}_m-\lambda_i)\frac{\lambda_i-\widetilde{\alpha}_i}{\beta_i-\lambda_i}V_{z_l}. \tag{115}$$

We discuss (115) by two cases.
(i) Let $z_l=m$. Note that $s_1(l)\leq s_1(q_1)\leq\lambda_m$ and $\lambda_{z_l}=s_1(l)=\lambda_m$ imply $l=q_1=m$. Thus, $(\lambda_m-\lambda_i)C_{q_1-l,i}=1$. In this case, (115) becomes

$$\frac{F(\lambda_i)}{G(\lambda_i)}=\rho_m\frac{\lambda_i}{\lambda_m}\prod_{j=1}^{m-1}\frac{\lambda_m-\alpha_j}{\lambda_m-\beta_j}+(\lambda_m-\lambda_i)\frac{\lambda_i-\widetilde{\alpha}_i}{\beta_i-\lambda_i}V_m,$$

which, together with (96), (111), (113) with $h = i$ and $V_{z_l} > \frac{1}{2}$, infers

$$\left|\frac{1}{\rho_m}\frac{\lambda_i - \widetilde{\alpha}_i}{\beta_i - \lambda_i} - \frac{1}{\lambda_m}\right| = \left|\frac{1}{(\lambda_m - \lambda_i)V_m}\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - \frac{\lambda_i}{\lambda_m}\prod_{j=1}^{m-1}\frac{\lambda_m - \alpha_j}{\lambda_m - \beta_j}\right) - \frac{1}{\lambda_m}\right|$$

$$\begin{aligned}
&= \frac{1}{\lambda_m(\lambda_m - \lambda_i)V_m}\left|\lambda_m\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - 1\right) - \lambda_i\left(\prod_{j=1}^{m-1}\frac{\lambda_m - \alpha_j}{\lambda_m - \beta_j} - 1\right) - (\lambda_m - \lambda_i)(V_m - 1)\right| \\
&< \frac{2}{\Delta}\left(\left|\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - 1\right| + \left|\prod_{j=1}^{m-1}\frac{\lambda_m - \alpha_j}{\lambda_m - \beta_j} - 1\right| + |V_m - 1|\right) < \frac{2}{\Delta}\left(2\Lambda\tau + 2\tau + \frac{2\Delta\tau}{\Lambda} + \frac{3\Delta\tau}{\Lambda}\right) < \frac{13\Lambda^2}{\Delta^2}\tau. \quad (116)
\end{aligned}$$

So, combining (104) and (116), we obtain for $i \in [1\ m-1]$,

$$\begin{aligned}
\left|\frac{1}{\rho_m}\frac{\lambda_i - \widetilde{\alpha}_i}{\widetilde{\beta}_i - \lambda_i} - C_{q_1-l+1,i}\right| &< \frac{13\Lambda^2}{\Delta^2}\Omega^{q_1-l}\tau + \frac{|\lambda_i - \widetilde{\alpha}_i|}{\rho_m}\frac{|\widetilde{\beta}_i - \beta_i|}{|(\widetilde{\beta}_i - \lambda_i)(\beta_i - \lambda_i)|} < \frac{13\Lambda^2}{\Delta^2}\Omega^{q_1-l}\tau + \frac{\Lambda}{\rho_m}\frac{\rho_m\tau}{\Delta}\frac{|\beta_i - \lambda_i|}{|\widetilde{\beta}_i - \lambda_i|} \\
&< \frac{13\Lambda^2}{\Delta^2}\Omega^{q_1-l}\tau + \frac{2\Lambda}{\Delta}\tau < \Omega^{q_1-l+1}\tau. \quad (117)
\end{aligned}$$

(ii) Let $z_l < m$. By (115), we deduce

$$\begin{aligned}
\left|\frac{1}{\rho_m}\frac{\lambda_i - \widetilde{\alpha}_i}{\beta_i - \lambda_i} - C_{q_1-l+1,i}\right| &= \left|\frac{1}{(\widetilde{\alpha}_m - \lambda_i)V_{z_l}}\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - \frac{\lambda_i}{\lambda_{z_l}}\frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})}\right) - C_{q_1-l+1,i}\right| \\
&= \left|\frac{1}{V_{z_l}}\left(\frac{1}{\lambda_m - \lambda_i} + \frac{\lambda_m - \widetilde{\alpha}_m}{(\widetilde{\alpha}_m - \lambda_i)(\lambda_m - \lambda_i)}\right)\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - \frac{\lambda_i}{\lambda_{z_l}}\frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})}\right) - C_{q_1-l+1,i}\right| \\
&\leq J + K, \quad (118)
\end{aligned}$$

where

$$\begin{aligned}
J &= \frac{1}{V_{z_l}(\lambda_m - \lambda_i)}\left|\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - \frac{\lambda_i}{\lambda_{z_l}}\frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})}\right) - V_{z_l}(\lambda_m - \lambda_i)C_{q_1-l+1,i}\right|, \\
K &= \left|\frac{\lambda_m - \widetilde{\alpha}_m}{V_{z_l}(\widetilde{\alpha}_m - \lambda_i)(\lambda_m - \lambda_i)}\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - \frac{\lambda_i}{\lambda_{z_l}}\frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})}\right)\right|.
\end{aligned}$$

It suffices to estimate J and K in (118). To estimate J, note that $z_l \in [1\ m-1]$, then by the definition of $C_{j,i}$ with $i \in [1\ m-1]$,

$$(\lambda_m - \lambda_i)C_{q_1-l,i} - \frac{\lambda_i}{\lambda_{z_l}}(\lambda_m - \lambda_{z_l})C_{q_1-l,z_l} = (\lambda_m - \lambda_i)C_{q_1-l+1,i}. \quad (119)$$

Since $V_{z_l} > \frac{1}{2}$, it follows from (23), (112)–(114) and (119) that

$$\begin{aligned}
J &= \frac{1}{V_{z_l}(\lambda_m - \lambda_i)}\left|\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - (\lambda_m - \lambda_i)C_{q_1-l,i}\right) - \frac{\lambda_i}{\lambda_{z_l}}\left(\frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})} - (\lambda_m - \lambda_{z_l})C_{q_1-l,z_l}\right) - (\lambda_m - \lambda_i)C_{q_1-l+1,i}(V_{z_l} - 1)\right| \\
&< \frac{1}{\Delta}\left(2\Lambda\Omega^{q_1-l}\tau + 2\tau + \frac{\Lambda}{\Delta}\left(2\Lambda\Omega^{q_1-l}\tau + 2\tau\right) + \frac{3\Delta\tau}{\Lambda}\right) < \frac{11\Lambda^2}{\Delta^2}\Omega^{q_1-l}\tau. \quad (120)
\end{aligned}$$

Next, we estimate K. By (94), plugging $\lambda = \widetilde{\alpha}_m$ into (30) gives

$$\frac{F(\widetilde{\alpha}_m)}{\widetilde{\alpha}_m G(\widetilde{\alpha}_m)} = -\frac{\widetilde{m}\nu}{\mu} = \frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}.$$

We write it as

$$\frac{\widetilde{\alpha}_m - \alpha_m}{\widetilde{\alpha}_m} \prod_{j=1}^{m-1} \frac{\widetilde{\alpha}_m - \alpha_j}{\widetilde{\alpha}_m - \beta_j} = \frac{\alpha_m - \lambda_{z_l}}{\lambda_{z_l}} \frac{\lambda_{z_l} - \alpha_{z_l}}{\beta_{z_l} - \lambda_{z_l}} \prod_{j=1, j\neq z_l}^{m-1} \frac{\lambda_{z_l} - \alpha_j}{\lambda_{z_l} - \beta_j}. \tag{121}$$

Observe that

$$\frac{\widetilde{\alpha}_m - \alpha_m}{\widetilde{\alpha}_m} \prod_{j=1}^{m-1} \frac{\widetilde{\alpha}_m - \alpha_j}{\widetilde{\alpha}_m - \beta_j} > \frac{\widetilde{\alpha}_m - \alpha_m}{\widetilde{\alpha}_m}. \tag{122}$$

On the other hand, by (21), (23), (28), (29) and (111),

$$\begin{aligned}
\frac{\alpha_m - \lambda_{z_l}}{\lambda_{z_l}} \frac{\lambda_{z_l} - \alpha_{z_l}}{\beta_{z_l} - \lambda_{z_l}} \prod_{j=1, j\neq z_l}^{m-1} \frac{\lambda_{z_l} - \alpha_j}{\lambda_{z_l} - \beta_j} &= \left(\frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} + \frac{\alpha_m - \lambda_m}{\lambda_{z_l}}\right) \frac{\lambda_{z_l} - \alpha_{z_l}}{\beta_{z_l} - \lambda_{z_l}} \prod_{j=1, j\neq z_l}^{m-1} \frac{\lambda_{z_l} - \alpha_j}{\lambda_{z_l} - \beta_j} \\
&< \left(\frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} + \frac{2\Lambda\rho_m}{\lambda_1\lambda_{z_l}}\right)\rho_m(C_{q_1-l,z_l} + \Omega^{q_1-l}\tau)\left(1 + \frac{2\Delta\tau}{\Lambda}\right) \\
&< \left(\frac{2\Lambda\rho_m}{\lambda_1\lambda_{z_l}}\left(\frac{1}{2\Delta} + \frac{\Delta}{\Lambda^2}\right)2 + \left(\frac{\Lambda}{2\lambda_1\Delta} + \frac{\Lambda}{\lambda_1}\Omega^{q_1-l}\tau\right)\frac{2\Delta}{\Lambda}\tau + \frac{\Lambda}{\lambda_1}\Omega^{q_1-l}\tau\right)\rho_m \\
&\quad + \left(\frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} C_{q_1-l,z_l}\right)\rho_m \\
&< \left(\frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} C_{q_1-l,z_l} + \frac{2\Lambda}{\lambda_1}\Omega^{q_1-l}\tau\right)\rho_m.
\end{aligned} \tag{123}$$

Define $H \triangleq \frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} C_{q_1-l,z_l} + \frac{2\Lambda}{\lambda_1}\Omega^{q_1-l}\tau$, then by (17), (21) and (23),

$$H < \frac{\Lambda}{2\Delta^2} + \frac{2}{\Lambda}.$$

Hence, by (21), (29), (94) and (123),

$$\begin{aligned}
\frac{\widetilde{\alpha}_m - \lambda_m}{\rho_m} &= \frac{\alpha_m - \lambda_m}{\rho_m} + \frac{\widetilde{\alpha}_m - \alpha_m}{\rho_m} < \frac{\alpha_m - \lambda_m}{\rho_m} + \frac{H}{1 - H\rho_m}\alpha_m < \frac{\alpha_m - \lambda_m}{\rho_m} + \left(H + \frac{H^2\rho_m}{1 - H\rho_m}\right)\left(\lambda_m + \frac{2\Lambda}{\lambda_1}\rho_m\right) \\
&< \frac{\alpha_m - \lambda_m}{\rho_m} + \lambda_m \frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} C_{q_1-l,z_l} + \frac{2\Lambda^2}{\lambda_1}\Omega^{q_1-l}\tau + H\frac{2\Lambda}{\lambda_1}\rho_m + 2H^2\rho_m\left(\Lambda + \frac{2\Lambda}{\lambda_1}\rho_m\right) \\
&< \frac{\alpha_m - \lambda_m}{\rho_m} + \lambda_m \frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} C_{q_1-l,z_l} + \frac{3\Lambda^2}{\lambda_1}\Omega^{q_1-l}\tau.
\end{aligned} \tag{124}$$

Recall that $\widetilde{\alpha}_m > \lambda_m$ and $V_{z_l} > \frac{1}{2}$, by (29) and (124),

$$\frac{\widetilde{\alpha}_m - \lambda_m}{V_{z_l}(\widetilde{\alpha}_m - \lambda_i)(\lambda_m - \lambda_i)} < \frac{\left(\frac{\alpha_m - \lambda_m}{\rho_m} + \frac{\widetilde{\alpha}_m - \alpha_m}{\rho_m}\right)\rho_m}{V_{z_l}(\lambda_m - \lambda_i)^2} < \frac{\left(\frac{2\Lambda}{\lambda_1} + \lambda_m \frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} C_{q_1-l,z_l} + \frac{3\Lambda^2}{\lambda_1}\Omega^{q_1-l}\tau\right)\rho_m}{2\Delta^2}. \tag{125}$$

Moreover, (27) and (113) imply

$$\left|\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - \frac{\lambda_i}{\lambda_{z_l}} \frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})}\right| < \left(1 + \frac{\Lambda}{\Delta}\right)\left(\frac{\Lambda}{2\Delta} + \left(2\Lambda\Omega^{q_1-l}\tau + 2\tau\right)\right). \tag{126}$$

So, $K < \tau$ by (17), (18), (125) and (126). This together with (118) and (120), infers

$$\left|\frac{1}{\rho_m} \frac{\lambda_i - \widetilde{\alpha}_i}{\beta_i - \lambda_i} - C_{q_1-l+1,i}\right| < \frac{13\Lambda^2}{\Delta^2}\Omega^{q_1-l}\tau. \tag{127}$$

Analogously to (117), we obtain for $i \in [1\ m-1]$,

$$\left|\frac{1}{\rho_m}\frac{\lambda_i - \widetilde{\alpha}_i}{\widetilde{\beta}_i - \lambda_i} - C_{q_1-l+1,i}\right| < \Omega^{q_1-l+1}\tau.$$

So far, we have proved (31) and (32). We remark that (33) holds as well because of (97) and (124). The proof is completed.

□

C. Proofs of Lemmas 4.4

To prove Lemma 4.4, we need to modify [1, Lemmas B.3. and B.4.] to derive two lemmas. Both of the lemmas and their proofs will follow the notations used in [1, Appendix B].

Lemma C.1. [1, Lemma B.3.] holds if [1, Eqs. (61)–(62)] are replaced respectively by

$$\alpha_p \in [\lambda_p - \rho_p, \bar{\lambda}) \quad \text{for some number} \quad \bar{\lambda} > \lambda_p, \tag{128}$$

$$\lambda_i + C_1^n\rho_i - \rho_i \le \alpha_i + C_1^n\rho_i < \beta_i < \lambda_i + (1 + C_2(i))^n\rho_i < \lambda_{i+1} - \rho_{i+1}. \tag{129}$$

Proof of Lemma C.1. The proof is the same as that of [1, Lemma B.3.] except [1, Eqs. (67) and (77)]. As a matter of fact, in view of (128) and (129), [1, Eq. (67)] becomes

$$\alpha_p \ge \lambda_p - \rho_p > \lambda_{p-1} + (1 + C_2(p-1))^n\rho_{p-1} > \beta_{p-1} > \alpha_{p-1} > \cdots > \beta_1 > \alpha_1, \tag{130}$$

and hence $\lambda G(\lambda) \prec F(\lambda)$ still holds. By (21) and (129), for any $j < i$, [1, Eq. (77)] can be rewritten by

$$\frac{\widetilde{\beta}_i - \beta_j}{\widetilde{\beta}_i - \widetilde{\alpha}_j} > \frac{\widetilde{\beta}_i - \beta_j}{\widetilde{\beta}_i - \alpha_j} = 1 - \frac{\beta_j - \alpha_j}{\widetilde{\beta}_i - \alpha_j} > 1 - \frac{\beta_j - \lambda_j + \rho_j}{\widetilde{\beta}_i - \lambda_j + \rho_j} > 1 - \frac{(1 + C_2(j))^n\rho_j + \rho_j}{2\Delta} \ge 1 - \frac{(1 + C_2(j))^n\rho_j}{\Delta}, \tag{131}$$

which does not change the result. □

Lemma C.2. [1, Lemma B.4.] holds if [1, Eqs. (61)–(62)] are replaced by (128) and (129), respectively.

Proof of Lemma C.2. Firstly, by (130), $\lambda G(\lambda) \prec F(\lambda)$. Then, it suffices to restate the following inequalities, which are used to prove [1, Eq. (62)]. According to the proof of [1, Lemma B.4.], $\widetilde{\alpha}_j \in (\alpha_j, \beta_j)$. As a result, for $j \in [i+1\ p-1]$, (129) gives

$$\frac{\beta_i - \alpha_j}{\beta_i - \widetilde{\alpha}_j} = 1 - \frac{\widetilde{\alpha}_j - \alpha_j}{\widetilde{\alpha}_j - \beta_i} \ge 1 - \frac{\beta_j - \alpha_j}{\alpha_j - \beta_i} \ge 1 - \frac{\beta_j - \lambda_j + \rho_j}{\lambda_{i+1} - \rho_{i+1} - \beta_i},$$

which together with (22) and (129) yield

$$\frac{\beta_i - \alpha_j}{\beta_i - \widetilde{\alpha}_j} \ge 1 - \frac{\beta_j - \lambda_j + \rho_j}{(\lambda_{i+1} - \lambda_i) - (\beta_i - \lambda_i) - \rho_{i+1}} \ge 1 - \frac{(1 + C_2(j))^n\rho_j + \rho_j}{2\Delta - (1 + C_2(i))^n\rho_i - \rho_{i+1}} > \frac{1}{2}.$$

Furthermore, $\alpha_p \in [\lambda_p - \rho_p, \widetilde{\lambda}) \subset [\lambda_p - \rho_p, \Lambda)$, (17), (22) and (129) lead to

$$\frac{\beta_i - \alpha_p}{\beta_i - \widetilde{\alpha}_p} \ge \frac{\lambda_{i+1} - \rho_{i+1} - \beta_i}{\Lambda} \ge \frac{2\Delta - (1 + C_2^n(i))\rho_i - \rho_{i+1}}{\Lambda} > \frac{\Delta}{2\Lambda}.$$

The remaining part keeps the same as that of [1, Lemma B.4.]. □

Proof of Lemma 4.4. Since the analysis is essentially the same as that of [1, Lemma 4.1], we still use the notations defined in the proof of [1, Lemma 4.1]. First of all, we remark that polynomials $F_{n-q_1}(\lambda)$, $G_{n-q_1}(\lambda)$ and numbers μ_{n-q_1}, ν_{n-q_1} constructed in Lemma 4.3 meet the conditions of Lemma C.1. In fact, by Lemma 4.3 (iii), $\alpha_{n-q_1}(m) \in [\lambda_m - \rho_m, \Lambda)$. Moreover, since (19) implies $C_1^n < C_1^{q_1}$, (36) shows

$$\lambda_i + C_1^n\rho_i - \rho_i \le \alpha_{n-q_1}(i) + C_1^n\rho_i < \beta_{n-q_1}(i), \quad i = 1, \dots, m-1. \tag{132}$$

Observe that by (17) and (22),

$$\lambda_i + (1 + C_2(i))^n\rho_i < \lambda_{i+1} - \rho_{i+1}, \quad i = 1, \dots, m-1, \tag{133}$$

which combined with (36) infers

$$\beta_{n-q_1}(i) < \lambda_i + (1 + C_2(i))^n\rho_i < \lambda_{i+1} - \rho_{i+1}, \quad i = 1, \dots, m-1. \tag{134}$$

Recall that Lemma 4.3 asserts $-\frac{\mu_{n-q_1}}{\nu_{n-q_1}} > C(\frac{\Delta}{\lambda_1})^k \frac{\Lambda M}{\Lambda-\lambda_1}$. Consequently, we can construct polynomials $F_{n-q_1-1}(\lambda)$ and $G_{n-q_1-1}(\lambda)$ in light of Lemma C.1.

So, similarly to the proof of [1, Lemma 4.1], assume that for some $r \in [1 + q_1 \ n-2]$, we have constructed the desired $F_{n-j}(\lambda)$ and $G_{n-j}(\lambda)$, $j \in [1 + q_1 \ r]$ according to either Lemma C.1 or Lemma C.2 [3]. Particularly, for all $j \in [q_1 \ r-1]$,

$$\begin{aligned} \alpha_{n-q_1}(i) + C_1^j\rho_i &< \alpha_{n-j}(i) + C_1^j\rho_i < \beta_{n-j}(i) \\ &< \alpha_{n-q_1}(i) + (1 + C_2(i))^j(\beta_{n-q_1}(i) - \alpha_{n-q_1}(i)). \end{aligned} \tag{135}$$

The only difference from the proof of [1, Lemma 4.1] is that instead of verifying [1, Eq. (96)], we are devoted to check

$$\lambda_i + C_1^n\rho_i - \rho_i \le \alpha_{n-r}(i) + C_1^n\rho_i < \beta_{n-r}(i) < \lambda_i + (1 + C_2(i))^n\rho_i < \lambda_{i+1} - \rho_{i+1}. \tag{136}$$

Note that by (21), (22), (36) and (135), Lemmas C.1–C.2 lead to

$$\begin{aligned} \beta_{n-r}(i) - \lambda_i &< \alpha_{n-q_1}(i) - \lambda_i + (1 + C_2(i))^r(\beta_{n-q_1}(i) - \lambda_i + \lambda_i - \alpha_{n-q_1}(i)) \\ &< 2^{m+1}\Lambda\frac{\rho_i}{\rho_m} + (1 + C_2(i))^r\left(2^{m+1}\Lambda\frac{\rho_i}{\rho_m} + \rho_i\right) \\ &< 2^{m+1}\Lambda\frac{\rho_i}{\rho_m} + (1 + C_2(i))^{r+1}\rho_i \\ &< (1 + C_2(i))^n\rho_i < \frac{\Delta}{2}. \end{aligned} \tag{137}$$

On the other hand, by Lemmas C.1–C.2 again, (36) and (135) infer

$$\beta_{n-r}(i) - \alpha_{n-r}(i) > C_1^{r-q_1}\left(\beta_{n-q_1}(i) - \alpha_{n-q_1}(i)\right) > C_1^r\rho_i.$$

Therefore,

$$\lambda_i + C_1^n\rho_i - \rho_i < \alpha_{n-r}(i) + C_1^n\rho_i < \alpha_{n-r}(i) + C_1^r\rho_i < \beta_{n-r}(i). \tag{138}$$

So (136) is true in view of (133), (137) and (138). Furthermore, observe that $\alpha_{n-r}(z_{n-r}) \in [\lambda_{z_{n-r}}, \Lambda) \subset [\lambda_{z_{n-r}} - \rho_{z_{n-r}}, \Lambda)$ due to the proof of [1, Lemma 4.1]. We thus omit the rest of the proof that is identical to that of [1, Lemma 4.1]. □

[3] In the proof, we do not mention μ_j, ν_j, $j = 1, \dots, n - q_1 - 1$, because the relevant analysis is the same as that of [1, Lemma 4.1].

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