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Inverse eigenvalue problem for mass–spring–inerter systems ☆

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ABSTRACT

This paper has solved the inverse eigenvalue problem for a "fixed–free" mass-chain system with inerters. It is well known that for a spring–mass system wherein the adjacent masses are linked through a spring, the natural frequency assignment can be achieved by choosing appropriate masses and spring stiffnesses if and only if the given positive eigenvalues are distinct. However, when we involve inerters, multiple eigenvalues in the assignment are allowed. In fact, for a set of arbitrarily given positive real numbers, we derive a necessary and sufficient condition on the multiplicities of these numbers, which are assigned as the natural frequencies of the concerned mass–spring–inerter system.

1. Introduction

Natural frequency, an inherent attribute of a mechanical vibration system, has attracted wide attention for its importance. In particular, purposefully allocating the natural frequencies to some pre-specified values provides an effective way to induce or evade resonance (see [1,2]). This naturally raises an inverse eigenvalue problem (IEP), that is, to construct a vibration system whose natural frequencies, or mathematically known as eigenvalues, are given beforehand [3–6].

A well-known result on this problem is due to [7] and [8], which is addressed for a mass–spring system. Observe that in such a basic system, the adjacent masses are linked merely by a spring. Therefore, the IEP turns out to be the construction of a Jacobi matrix with its eigenvalues being assigned to a set of specified positive numbers. Employing the tools for Jacobi matrices, [7] and [8] assert that the IEP is solvable if and only if the given positive eigenvalues are distinct. However, multiple eigenvalues often occur in vibration systems, especially in optimized structures [9–15]. As a matter of fact, the IEP admits multiple eigenvalues, if we introduce a mechanical element called the inerter. This new mechanical device can simulate masses by changing its inertance. It was theoretically first studied by [16], completing the force–current analogy between electrical and mechanical networks (see Section II-B of [16]). Through physical realization, inerters have been applied to many engineering fields such as vibration isolators, landing gears, train suspensions, vehicle suspensions, building vibration control, and so on [17–22]. In a mass–spring–inerter system, neighboring masses are linked by a parallel combination of a spring and an inerter. As a starting point, we restrict our interest in this paper to a "fixed–free" system. The term "fixed–free" means one end of the mass-chain system is attached to the ground while the other end is hanging free, as shown in Fig. 1.

The free vibration equation of such a mass–spring–inerter system is described by

$$(\boldsymbol{M}+\boldsymbol{B})\ddot{x}+\boldsymbol{K}\boldsymbol{x}=0,$$

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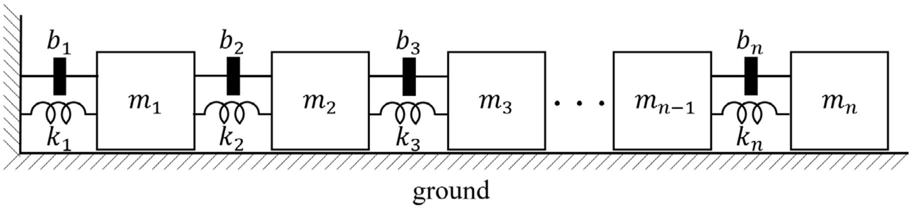

Fig. 1. Mass–spring-inerter system.

where

$$\boldsymbol{M} = \mathrm{diag}\{m_1, m_2, \ldots, m_n\}, \tag{1}$$

$$\boldsymbol{K} = \begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & -k_n & k_n \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} b_1+b_2 & -b_2 & & & \\ -b_2 & b_2+b_3 & -b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & b_{n-1}+b_n & -b_n \\ & & & -b_n & b_n \end{bmatrix}. \tag{2}$$

Here, real numbers $m_j > 0$, $k_j > 0$, $b_j \geq 0$ for $j = 1, \ldots, n$ stand for the masses, spring stiffnesses and inertances. Unlike the reconstruction of a mass–spring system, the IEP for the mass–spring–inerter system cannot be illuminated by the well-studied JIEP since the inertial matrix in (2) is a tridiagonal matrix. Recently, [23] found that inerters render the multiple eigenvalues possible for a mass–chain system. It showed that the multiplicity t_i of a natural frequency λ_i must fulfill $n \geq 2t_i - 1$. Beyond that, little is known for the multiple eigenvalue case.

The purpose of this paper is to give a theoretical answer to the IEP for a mass–spring-inerter system, where the eigenvalues are arbitrarily specified to n positive real numbers. We deduce a necessary and sufficient condition for this assignment on the multiplicities of the given numbers. Our construction further implies that m masses of the system can be arbitrarily fixed beforehand in the assignment, where m is the amount of the distinct assigned eigenvalues. More precisely, our construction is carried out by only adjusting $n - m$ masses, n spring stiffnesses and n inertances. It degenerates to the claim that, if the pre-specified eigenvalues are all distinct ($m = n$), the IEP can be worked out by recovering $\boldsymbol{K}$ and $\boldsymbol{B}$, whereas $\boldsymbol{M}$ is fixed arbitrarily. This claim is exactly the main result of [16], which demonstrates an advantage of using inerters in the mass-fixed situation. However, when $m < n$, we point out that not all the natural frequency assignments are realizable by merely adjusting spring stiffnesses and inertances. An example of five-degree-of-freedom system in this paper shows that there exist some restrictive relationships between masses and given eigenvalues. It is actually an inherent phenomenon of a mass–spring–inerter system.

The organization of this paper is as follows. In Section 2, we state the main result by deducing a necessary and sufficient condition of the IEP for the mass–spring–inerter system, while the proofs are included in Sections 3 and 4. Conclusions are drawn in Section 5.

Notation: In this paper, let $\mathbb{N}^+$ be the set of positive integers. We use the notation "[ ]" to express an interval of integers. For example, $[n] = \{1, 2, \ldots, n\}$ and $[m\ n] = \{m, m+1, \ldots, n\}$, $m, n \in \mathbb{N}^+$. Write $(f(\lambda), g(\lambda))$ as the monic greatest common divisor of two polynomials $f(\lambda)$ and $g(\lambda)$.

2. Main result

The natural frequencies of a mass–spring-inerter system are completely determined by the eigenvalues of matrix pencil $\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B})$, where $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$ are defined by (1) and (2), respectively. So, with a slight abuse of language, we will not distinguish the term "eigenvalues" from the "natural frequencies" in this article. We now raise our problem.

Problem 1. Arbitrarily given a set of real numbers $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, is it possible to recover matrices $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$ in (1) and (2) by choosing $m_j > 0$, $k_j > 0$ and $b_j \geq 0$ for $j = 1, \ldots, n$, so that the n eigenvalues of matrix pencil $\boldsymbol{K} - \mu(\boldsymbol{M} + \boldsymbol{B})$ are exactly μ_i, $i = 1, \ldots, n$?

Both [8] and [23] offered a positive answer to Problem 1 for the special case where the eigenvalues are all distinct. But the general situation should involve multiple eigenvalues, which is covered by the following theorem.

Theorem 2.1. *Let $\prod_{i=1}^{m}(\lambda - \lambda_i)^{t_i}$ be a polynomial with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m$ and $\sum_{i=1}^{m} t_i = n$. Then, there exist some matrices $\boldsymbol{K}, \boldsymbol{M}, \boldsymbol{B}$ in the forms of (1) and (2) such that*

$$\prod_{i=1}^{m}(\lambda - \lambda_i)^{t_i} \,\Big|\, \det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B})) \tag{3}$$

if and only if

$$t_i \leq i, \quad i = 1, \ldots, m. \tag{4}$$

Remark 2.1. Theorem 2.1 completely solves Problem 1 by providing the critical criterion (4). As indicated later (see Proposition 4.1 for details), we observe that:

When (4) holds, the recover of the mass matrix $\boldsymbol{M}$ allows a total of m masses being taken arbitrarily, where m is the number of distinct eigenvalues λ_i given beforehand.

Particularly, for $m = n$, each mass $m_i, 1 \le i \le n$ can be taken any fixed quantity in advance, as proved in [23]. However, when $m < n$, it is generally impossible to achieve the natural frequency assignment with all the masses arbitrarily given. Example 2.1 suggests a restrictive relation between the masses and eigenvalues.

Example 2.1. Let $n = 5$, $t_1 = t_2 = 1$, $t_3 = 3$ and $0 < \lambda_1 < \lambda_2 < \lambda_3$. If there exist some $m_j > 0$, $k_j > 0$, $b_j \ge 0$, $j = 1, 2, \dots, 5$ such that $\prod_{i=1}^{3}(\lambda - \lambda_i)^{t_i} \Big| \det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B}))$, then

$$\max_{2\leqslant j\leqslant 4} \frac{m_j}{m_{j+1}} > \frac{\lambda_1}{8\lambda_3}\left(1 - \left(\frac{\lambda_2}{\lambda_3}\right)^{\frac{1}{3}}\right). \tag{5}$$

Clearly, (4) holds, but the masses cannot be taken arbitrarily. The proof of (5) is included in Appendix A.

3. Proof of the necessity of Theorem 2.1

This section is devoted to proving the necessity of Theorem 2.1. We begin by expressing $\det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B}))$ in terms of a recursive sequence of polynomials. First, write

$$\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B}) = \begin{bmatrix} k_1+k_2-\lambda(m_1+b_1+b_2) & -k_2+\lambda b_2 & & & \\ -k_2+\lambda b_2 & k_2+k_3-\lambda(m_2+b_2+b_3) & -k_3+\lambda b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1}+\lambda b_{n-1} & k_{n-1}+k_n-\lambda(m_{n-1}+b_{n-1}+b_n) & -k_n+\lambda b_n \\ & & & -k_n+\lambda b_n & k_n-\lambda(m_n+b_n) \end{bmatrix}.$$

For $j = 1, \dots, n$, let $\boldsymbol{M}_j$, $\boldsymbol{K}_j$ and $\boldsymbol{B}_j$ be matrices with order j defined by $\boldsymbol{M}_j = \mathrm{diag}\{m_1, m_2, \dots, m_j\}$,

$$\boldsymbol{K}_j = \begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{j-1} & k_{j-1}+k_j & -k_j \\ & & & -k_j & k_j \end{bmatrix}, \quad \boldsymbol{B}_j = \begin{bmatrix} b_1+b_2 & -b_2 & & & \\ -b_2 & b_2+b_3 & -b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{j-1} & b_{j-1}+b_j & -b_j \\ & & & -b_j & b_j \end{bmatrix}.$$

Next, denote $f_j(\lambda)$ as the determinant of $\boldsymbol{K}_j - \lambda(\boldsymbol{M}_j + \boldsymbol{B}_j)$, $j = 1, \dots, n$. Let $g_1(\lambda) = 1$ and $g_j(\lambda)$ be the leading principal minor of $\boldsymbol{K}_j - \lambda(\boldsymbol{M}_j + \boldsymbol{B}_j)$ of order $j - 1$, where $j = 2, \dots, n$. So, to calculate $\det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B}))$, we only need to treat $f_n(\lambda)$.

Remark 3.1. For each $j = 1, \dots, n$, since $\det \boldsymbol{K}_j = \prod_{l=1}^{j} k_l > 0$ and $\det \boldsymbol{B}_j = \prod_{l=1}^{j} b_l \ge 0$, the Gershgorin's circle theorem indicates that both $\boldsymbol{K}_j$ and $\boldsymbol{M}_j + \boldsymbol{B}_j$ are positive definite matrices and so do their leading principal submatrices. Then, it follows that the roots of $f_j(\lambda)$ and $g_j(\lambda)$ are all real and positive (see [7, Theorem 1.4.3]).

To facilitate the subsequent analysis, we introduce the following definition.

Definition 3.1. Let $f(\lambda)$ and $g(\lambda)$ be two polynomials with degree s, where $s \in \mathbb{N}^+$. Suppose $f(\lambda)$ and $g(\lambda)$ both have s distinct real roots, which are denoted by $\alpha_1 < \cdots < \alpha_s$ and $\beta_1 < \cdots < \beta_s$, respectively. We say $g(\lambda) \prec f(\lambda)$, if their leading coefficients are of the same sign and $\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_s < \alpha_s$.

The proof of the theorem depends on a simple observation below.

Lemma 3.1. *The polynomials* $\{f_j(\lambda)\}_{j=1}^{n}$ *and* $\{g_j(\lambda)\}_{j=1}^{n}$ *satisfy*

$$\begin{cases} f_{j+1}(\lambda) = (-\lambda m_{j+1})g_{j+1}(\lambda) + (k_{j+1} - \lambda b_{j+1})f_j(\lambda), \\ g_{j+1}(\lambda) = f_j(\lambda) + (k_{j+1} - \lambda b_{j+1})g_j(\lambda), \end{cases} \quad j = 1, \dots, n-1, \tag{6}$$

with $g_1(\lambda) = 1$ *and* $f_1(\lambda) = k_1 - \lambda(m_1 + b_1)$.

Proof. By the definition of $\boldsymbol{K}_1 - \lambda(\boldsymbol{M}_1 + \boldsymbol{B}_1)$, it is trivial that $f_1(\lambda) = k_1 - \lambda(m_1 + b_1)$. For each $j = 1, \dots, n-1$, consider the leading principal minor of $\boldsymbol{K}_{j+1} - \lambda(\boldsymbol{M}_{j+1} + \boldsymbol{B}_{j+1})$ of order j. The multilinearity of this determinant with respect to the jth row shows $g_{j+1}(\lambda) = f_j(\lambda) + (k_{j+1} - \lambda b_{j+1})g_j(\lambda)$. Furthermore, the expansion of $\det(\boldsymbol{K}_{j+1} - \lambda(\boldsymbol{M}_{j+1} + \boldsymbol{B}_{j+1}))$ by cofactors of the $(j+1)$th row yields

$$\begin{aligned} f_{j+1}(\lambda) &= (k_{j+1} - \lambda(m_{j+1} + b_{j+1}))g_{j+1}(\lambda) - (k_{j+1} - \lambda b_{j+1})^2 g_j(\lambda) \\ &= (-\lambda m_{j+1})g_{j+1}(\lambda) + (k_{j+1} - \lambda b_{j+1})f_j(\lambda) + (k_{j+1} - \lambda b_{j+1})^2 g_j(\lambda) - (k_{j+1} - \lambda b_{j+1})^2 g_j(\lambda) \\ &= (-\lambda m_{j+1})g_{j+1}(\lambda) + (k_{j+1} - \lambda b_{j+1})f_j(\lambda), \end{aligned}$$

as desired. □

Lemma 3.2. *Let* $f(\lambda)$ *and* $g(\lambda)$ *be two polynomials such that* $g(\lambda) \prec f(\lambda)$. *Then for any* $a, b > 0$, $g(\lambda) \prec ag(\lambda) + bf(\lambda) \prec f(\lambda)$.

Proof. Since $g(\lambda) \prec f(\lambda)$, by Definition 3.1, $\deg(f(\lambda)) = \deg(g(\lambda)) = s$ for some $s \in \mathbb{N}^+$. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_s$ and $\beta_1 < \beta_2 < \cdots < \beta_s$ be the roots of $f(\lambda)$ and $g(\lambda)$, respectively. Clearly, $\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_s < \alpha_s$. Without loss of generality, assume the leading coefficients of $f(\lambda)$ and $g(\lambda)$ are both positive. Then, $\operatorname{sgn}(ag(\beta_i) + bf(\beta_i)) = (-1)^{s+1-i} = -(-1)^{s-i} = -\operatorname{sgn}(ag(\alpha_i) + bf(\alpha_i))$. This implies that for each $1 \le i \le s$, there is a root of $ag(\lambda) + bf(\lambda)$ in the interval (β_i, α_i). Observing that the degree of $ag(\lambda) + bf(\lambda)$ is s as well, the result follows immediately. □

We present an important property enjoyed by sequence $\{f_j(\lambda), g_j(\lambda)\}_{j=1}^n$.

Lemma 3.3. *Suppose for some $j \in [n-1]$,*

$$\frac{(-\lambda)g_j(\lambda)}{(f_j(\lambda), g_j(\lambda))} \prec \frac{f_j(\lambda)}{(f_j(\lambda), g_j(\lambda))}. \tag{7}$$

Then $\frac{(-\lambda)g_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$. Moreover,

(i) if $b_{j+1} \neq 0$ and $k_{j+1} - \lambda b_{j+1} | \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$, then $(f_{j+1}(\lambda), g_{j+1}(\lambda)) = (f_j(\lambda), g_j(\lambda))(\lambda - \frac{k_{j+1}}{b_{j+1}})$ and $\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{-f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$;

(ii) if $b_{j+1} \neq 0$ and $k_{j+1} - \lambda b_{j+1} \nmid \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$, then $(f_{j+1}(\lambda), g_{j+1}(\lambda)) = (f_j(\lambda), g_j(\lambda))$ and $\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{f_j(\lambda)(k_{j+1}-\lambda b_{j+1})}{(f_j(\lambda),g_j(\lambda))}$;

(iii) if $b_{j+1} = 0$, then $(f_{j+1}(\lambda), g_{j+1}(\lambda)) = (f_j(\lambda), g_j(\lambda))$ and $\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))} \prec \frac{g_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$ and $\frac{(-\lambda)f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$.

Proof. First, according to the definitions of $\{f_j(\lambda), g_j(\lambda)\}_{j=1}^n$, it is apparent that $\deg(f_j(\lambda)) = j$ and $\deg(g_j(\lambda)) = j-1$ for each $j \in [n]$. Let $\deg((f_j(\lambda), g_j(\lambda))) = j - s_j$ for some integer $s_j \in [j]$. Recalling (7), denote the roots of $\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$ and $\frac{(-\lambda)g_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$ by $\alpha_{j,1} < \cdots < \alpha_{j,s_j}$ and $0 < \beta_{j,1} < \cdots < \beta_{j,s_j-1}$, respectively. These roots fulfill

$$0 < \alpha_{j,1} < \beta_{j,1} < \cdots < \alpha_{j,s_j-1} < \beta_{j,s_j-1} < \alpha_{j,s_j}. \tag{8}$$

In addition, the second equation of (6) implies $(f_j(\lambda), g_j(\lambda)) | (f_j(\lambda), g_{j+1}(\lambda))$. Note that the leading coefficient of $(f_j(\lambda), g_j(\lambda))$ is positive, the definitions of $f_j(\lambda)$ and $g_j(\lambda)$ read

$$\begin{cases} f_j(\lambda) = c_j(-1)^{j-s_j}(f_j(\lambda), g_j(\lambda)) \prod_{l=1}^{s_j} (\alpha_{j,l} - \lambda), \\ g_j(\lambda) = d_j(-1)^{j-s_j}(f_j(\lambda), g_j(\lambda)) \prod_{l=1}^{s_j-1} (\beta_{j,l} - \lambda), \end{cases} \tag{9}$$

where c_j and d_j are the absolute values of the leading coefficients of $f_j(\lambda)$ and $g_j(\lambda)$, respectively. Now, we prove this lemma by discussing three cases.

(i) $b_{j+1} \neq 0$ and $k_{j+1} - \lambda b_{j+1} | \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$. For this case, there exists some $1 \le l_j \le s_j$ such that $\alpha_{j,l_j} = \frac{k_{j+1}}{b_{j+1}}$. We shall evaluate the sign of $\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))(k_{j+1}-\lambda b_{j+1})}$ at $\alpha_{j,i}, 1 \le i \le s_j$. In fact, according to the second equation of (6),

$$\left.\frac{g_{j+1}(\lambda)}{(f_j(\lambda), g_j(\lambda))(k_{j+1} - \lambda b_{j+1})}\right|_{\lambda=\alpha_{j,i}} = \left.\frac{f_j(\lambda)}{(f_j(\lambda), g_j(\lambda))(k_{j+1} - \lambda b_{j+1})}\right|_{\lambda=\alpha_{j,i}} + \left.\frac{g_j(\lambda)}{(f_j(\lambda), g_j(\lambda))}\right|_{\lambda=\alpha_{j,i}}, \qquad 1 \le i \le s_j.$$

Note that (8) and (9) yield that for $1 \le i \le s_j$,

$$\operatorname{sgn}\left(\left.\frac{f_j(\lambda)}{(f_j(\lambda), g_j(\lambda))(k_{j+1} - \lambda b_{j+1})}\right|_{\lambda=\alpha_{j,i}}\right) = (-1)^{j-s_j} \operatorname{sgn}\left(\prod_{1 \le l \le s_j, l \neq l_j} (\alpha_{j,l} - \alpha_{j,i})\right) = \begin{cases} 0, & \text{if} \quad i \neq l_j, \\ (-1)^{j-s_j+l_j-1}, & \text{if} \quad i = l_j, \end{cases}$$

and

$$\operatorname{sgn}\left(\left.\frac{g_j(\lambda)}{(f_j(\lambda), g_j(\lambda))}\right|_{\lambda=\alpha_{j,i}}\right) = (-1)^{j-s_j} \operatorname{sgn}\left(\prod_{l=1}^{s_j-1} (\beta_{j,l} - \alpha_{j,i})\right) = (-1)^{j-s_j+i-1}.$$

Therefore, $\operatorname{sgn}\left(\left.\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))(k_{j+1}-\lambda b_{j+1})}\right|_{\lambda=\alpha_{j,i}}\right) = (-1)^{j-s_j+i-1}$ for $1 \le i \le s_j$. This means that for each $i \in [s_j - 1]$, there exists exactly one root of $\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))(k_{j+1}-\lambda b_{j+1})}$ between $\alpha_{j,i}$ and $\alpha_{j,i+1}$, and hence $\frac{(-\lambda)g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))(k_{j+1}-\lambda b_{j+1})} \prec \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$.

Moreover, recall that $(f_j(\lambda), g_j(\lambda)) | (f_j(\lambda), g_{j+1}(\lambda))$, so (6) infers $(f_j(\lambda), g_{j+1}(\lambda)) = (f_j(\lambda), g_j(\lambda))\left(\lambda - \frac{k_{j+1}}{b_{j+1}}\right)$ and

$$\frac{(-\lambda)g_{j+1}(\lambda)}{(f_j(\lambda), g_{j+1}(\lambda))} = \frac{(-\lambda)g_{j+1}(\lambda)}{(f_j(\lambda), g_j(\lambda))\left(\lambda - \frac{k_{j+1}}{b_{j+1}}\right)} \prec \frac{-f_j(\lambda)}{(f_j(\lambda), g_j(\lambda))} = \frac{\left(\frac{k_{j+1}}{b_{j+1}} - \lambda\right) f_j(\lambda)}{(f_j(\lambda), g_{j+1}(\lambda))}.$$

Applying Lemma 3.2 to the first equation of (6), we thus deduce

$$\frac{(-\lambda)g_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))} \prec \frac{\left(\frac{k_{j+1}}{b_{j+1}}-\lambda\right)f_j(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))} = \frac{-f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}. \tag{10}$$

Now, $(f_{j+1}(\lambda),g_{j+1}(\lambda)) = (f_j(\lambda),g_{j+1}(\lambda))$ because of $\frac{(-\lambda)g_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}$, it follows from (10) that $\frac{(-\lambda)g_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{-f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$.

(ii) $b_{j+1}\neq 0$ and $(k_{j+1}-\lambda b_{j+1}) \nmid \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$. We first assume $\frac{k_{j+1}}{b_{j+1}} \in (\alpha_{j,l_j},\beta_{j,l_j})$ for some $1\le l_j\le s_j-1$ and evaluate the sign of $\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))}$ at points $\frac{k_{j+1}}{b_{j+1}}$ and $\alpha_{j,i}, 1\le i\le s_j$. Since Lemma 3.1 shows

$$\left.\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))}\right|_{\lambda=\alpha_{j,i}} = (k_{j+1}-\alpha_{j,i}b_{j+1})\left.\frac{g_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}\right|_{\lambda=\alpha_{j,i}}, \quad 1\le i\le s_j,$$

by (8) and (9),

$$\begin{aligned}\operatorname{sgn}\left(\left.\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))}\right|_{\lambda=\alpha_{j,i}}\right) &= (-1)^{j-s_j}\operatorname{sgn}\left((k_{j+1}-\alpha_{j,i}b_{j+1})\prod_{l=1}^{s_j-1}(\beta_{j,l}-\alpha_{j,i})\right)\\ &= \begin{cases}(-1)^{j-s_j+i-1}, & \text{if} \quad 1\le i\le l_j,\\ (-1)^{j-s_j+i}, & \text{if} \quad l_j+1\le i\le s_j.\end{cases}\end{aligned}$$

Similarly, by Lemma 3.1, (8) and (9) again,

$$\operatorname{sgn}\left(\left.\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))}\right|_{\lambda=\frac{k_{j+1}}{b_{j+1}}}\right) = \operatorname{sgn}\left(\left.\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}\right|_{\lambda=\frac{k_{j+1}}{b_{j+1}}} + \left.\frac{(k_{j+1}-\lambda b_{j+1})g_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}\right|_{\lambda=\frac{k_{j+1}}{b_{j+1}}}\right) = (-1)^{j-s_j+l_j}.$$

So, for each $i\in[s_j-1]$ with $i\neq l_j$, there is a root of $\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))}$ in $(\alpha_{j,i},\alpha_{j,i+1})$, and the rest two roots of $\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))}$ lie in $(\alpha_{j,l_j},\frac{k_{j+1}}{b_{j+1}})$ and $(\frac{k_{j+1}}{b_{j+1}},\alpha_{j,l_j+1})$, respectively. This infers $(f_j(\lambda),g_{j+1}(\lambda)) = (f_j(\lambda),g_j(\lambda))$ by noting that $(f_j(\lambda),g_j(\lambda))|(f_j(\lambda),g_{j+1}(\lambda))$. Hence, $\frac{(-\lambda)g_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))} \prec \frac{f_j(\lambda)(k_{j+1}-\lambda b_{j+1})}{(f_j(\lambda),g_{j+1}(\lambda))}$. Applying Lemma 3.2 to the first equation of (6), one has $\frac{(-\lambda)g_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))} \prec \frac{f_j(\lambda)(k_{j+1}-\lambda b_{j+1})}{(f_j(\lambda),g_{j+1}(\lambda))}$. Then, $(f_{j+1}(\lambda),g_{j+1}(\lambda)) = (f_j(\lambda),g_{j+1}(\lambda)) = (f_j(\lambda),g_j(\lambda))$. Consequently,

$$\frac{(-\lambda)g_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} = \frac{(-\lambda)g_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{f_j(\lambda)(k_{j+1}-\lambda b_{j+1})}{(f_j(\lambda),g_{j+1}(\lambda))} = \frac{f_j(\lambda)(k_{j+1}-\lambda b_{j+1})}{(f_j(\lambda),g_j(\lambda))}.$$

For the situations where $\frac{k_{j+1}}{b_{j+1}}\in(0,\alpha_{j,1})$, $\frac{k_{j+1}}{b_{j+1}}\in\bigcup_{l=1}^{s_j-1}[\beta_{j,l},\alpha_{j,l+1})$ and $\frac{k_{j+1}}{b_{j+1}}\in(\alpha_{j,s_j},+\infty)$, an analogous treatment can be employed.

(iii) $b_{j+1}=0$. We also first calculate the sign of $\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))}$ at the roots of $\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$ and $\frac{g_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$. As before,

$$\begin{cases}\operatorname{sgn}\left(\left.\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))}\right|_{\lambda=\alpha_{j,i}}\right) = (-1)^{j-s_j+i-1}, & \text{if} \quad 1\le i\le s_j,\\ \operatorname{sgn}\left(\left.\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))}\right|_{\lambda=\beta_{j,i}}\right) = (-1)^{j-s_j+i}, & \text{if} \quad 1\le i\le s_j-1.\end{cases} \tag{11}$$

Now, $\deg(g_{j+1}(\lambda)) = j$ and the leading coefficient of $(f_j(\lambda),g_j(\lambda))$ is positive, for any number $\theta_{j,1}>\alpha_{j,s_j}$ sufficiently large, it is evident that $(f_j(\lambda),g_{j+1}(\lambda))|_{\lambda=\theta_{j,1}}>0$ and $\operatorname{sgn}\left(\left.\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))}\right|_{\lambda=\theta_{j,1}}\right) = (-1)^{\deg(g_{j+1}(\lambda))} = (-1)^j$. So, each interval in $\{(\alpha_{j,s_j},+\infty),(\alpha_{j,i},\beta_{j,i}), i=1,\dots,s_j-1\}$ contains exactly one root of $\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_j(\lambda))}$, which concludes $(f_j(\lambda),g_{j+1}(\lambda)) = (f_j(\lambda),g_j(\lambda))$ and $\frac{f_j(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))} \prec \frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}$.

Let $\gamma_{j,1}<\cdots<\gamma_{j,s_j}$ be the roots of $\frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}$. From the first equation of (6) and (11), it follows that

$$\begin{cases}\operatorname{sgn}\left(\left.\frac{f_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}\right|_{\lambda=\alpha_{j,i}}\right) = \operatorname{sgn}\left(\left.\frac{(-\lambda)g_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}\right|_{\lambda=\alpha_{j,i}}\right) = (-1)^{j-s_j+i}, & \text{if} \quad 1\le i\le s_j,\\ \operatorname{sgn}\left(\left.\frac{f_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}\right|_{\lambda=\gamma_{j,i}}\right) = \operatorname{sgn}\left(\left.\frac{f_j(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}\right|_{\lambda=\gamma_{j,i}}\right) = (-1)^{j-s_j+i}, & \text{if} \quad 1\le i\le s_j,\\ \operatorname{sgn}\left(\left.\frac{f_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}\right|_{\lambda=0}\right) = \operatorname{sgn}\left(\left.\frac{f_j(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}\right|_{\lambda=0}\right) = (-1)^{j-s_j}.\end{cases} \tag{12}$$

Because $\deg(f_{j+1}(\lambda)) = j+1$ and the leading coefficient of $(f_j(\lambda),g_j(\lambda))$ is positive, we can choose a sufficiently large $\theta_{j,2}>\gamma_{j,s_j}>\alpha_{j,s_j}$ such that $(f_j(\lambda),g_{j+1}(\lambda))|_{\lambda=\theta_{j,2}}>0$ and $\operatorname{sgn}\left(\left.\frac{f_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}\right|_{\lambda=\theta_{j,2}}\right) = (-1)^{j+1}$. Recall that $\frac{f_j(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))} \prec \frac{g_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}$, so each interval in

$\{(0,\alpha_{j,1}),(\gamma_{j,s_j},+\infty),(\gamma_{j,i},\alpha_{j,i+1}), i=1,\ldots,s_j-1\}$ contains exactly one root of $\frac{f_{j+1}(\lambda)}{(f_j(\lambda),g_{j+1}(\lambda))}$. Because of this property, $(f_{j+1}(\lambda),g_{j+1}(\lambda)) = (f_j(\lambda),g_{j+1}(\lambda)) = (f_j(\lambda),g_j(\lambda))$, and hence $\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))} \prec \frac{g_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$, $\frac{(-\lambda)g_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$ and $\frac{(-\lambda)f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$. $\square$

It is ready to prove the necessity of Theorem 2.1. To this end, we introduce some notations. Let $f(\lambda)$ be a polynomial whose roots $\{z_i\}_{i=1}^p$ are all real and $z_1 < z_2 < \cdots < z_p$. Denote $\xi(f(\lambda),z_i)$ as the multiplicity of root z_i and for a real number α, define $\zeta(f(\lambda),\alpha) \triangleq \max\{i\in[p] : z_i<\alpha\}$.

The proof of the necessity of Theorem 2.1: First, in view of (3), we know that $\lambda_i, 1\le i\le m$ are the m distinct roots of $f_n(\lambda)$ with multiplicities t_i. To proceed the argument, note that Lemma 3.1 gives $\frac{(-\lambda)g_1(\lambda)}{(f_1(\lambda),g_1(\lambda))} \prec \frac{f_1(\lambda)}{(f_1(\lambda),g_1(\lambda))}$, then by using Lemma 3.3,

$$\frac{(-\lambda)g_j(\lambda)}{(f_j(\lambda),g_j(\lambda))} \prec \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))},\quad j=1,\ldots,n. \tag{13}$$

Particularly, it turns out that all the roots of $\frac{f_n(\lambda)}{(f_n(\lambda),g_n(\lambda))}$ are distinct. So $\xi\left(\frac{f_n(\lambda)}{(f_n(\lambda),g_n(\lambda))},\lambda_i\right)\le 1$ for all $1\le i\le m$ and then

$$\xi\left((f_n(\lambda),g_n(\lambda)),\lambda_i\right) = \xi\left(f_n(\lambda),\lambda_i\right) - \xi\left(\frac{f_n(\lambda)}{(f_n(\lambda),g_n(\lambda))},\lambda_i\right) \ge t_i - 1. \tag{14}$$

Now, by (13) and Lemma 3.3, for each $j=1,\ldots,n-1$,

$$(f_{j+1}(\lambda),g_{j+1}(\lambda)) = \begin{cases} (f_j(\lambda),g_j(\lambda))(\lambda-\frac{k_{j+1}}{b_{j+1}}), & \text{if } b_{j+1}\ne 0, (\lambda-\frac{k_{j+1}}{b_{j+1}}) \mid \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))},\\ (f_j(\lambda),g_j(\lambda)), & \text{otherwise,}\end{cases} \tag{15}$$

which yields that for all $1\le i\le m$,

$$\xi\left((f_{j+1}(\lambda),g_{j+1}(\lambda)),\lambda_i\right) = \begin{cases} \xi\left((f_j(\lambda),g_j(\lambda)),\lambda_i\right)+1, & \text{if } b_{j+1}\ne 0, k_{j+1}=\lambda_i b_{j+1}, (\lambda-\lambda_i) \mid \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))},\\ \xi\left((f_j(\lambda),g_j(\lambda)),\lambda_i\right), & \text{otherwise.}\end{cases} \tag{16}$$

On the other hand, given $j\in[n-1]$ and $i\in[m]$, we find that
(i) if $b_{j+1}\ne 0, k_{j+1}=\lambda_i b_{j+1}$ and $(\lambda-\lambda_i)\mid\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$, Lemma 3.3(i) indicates $\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{-f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$. Moreover, λ_i is a root of $\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$. It follows that $\zeta\left(\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))},\lambda_i\right) = \zeta\left(\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))},\lambda_i\right)+1$.

Otherwise, by virtue of Lemma 3.3, at least one of the following cases will happen:
(ii) if $b_{j+1}\ne 0, k_{j+1}\ne\lambda_i b_{j+1}$ and $(k_{j+1}-\lambda b_{j+1})\mid\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$, then $\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{-f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$;
(iii) if $b_{j+1}\ne 0$ and $k_{j+1}-\lambda b_{j+1}\nmid\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$, then $\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{f_j(\lambda)(k_{j+1}-\lambda b_{j+1})}{(f_j(\lambda),g_j(\lambda))}$;
(iv) if $b_{j+1}=0$, then $\frac{(-\lambda)f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$.

The three cases (ii)–(iv) lead to $\zeta\left(\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))},\lambda_i\right) \ge \zeta\left(\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))},\lambda_i\right)$. So,

$$\begin{cases} \zeta\left(\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))},\lambda_i\right) = \zeta\left(\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))},\lambda_i\right)+1, & \text{if case (i) occurs,}\\ \zeta\left(\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))},\lambda_i\right) \ge \zeta\left(\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))},\lambda_i\right), & \text{otherwise.}\end{cases} \tag{17}$$

Clearly, $\zeta\left(\frac{f_1(\lambda)}{(f_1(\lambda),g_1(\lambda))},\lambda_i\right) \ge \xi\left((f_1(\lambda),g_1(\lambda)),\lambda_i\right)=0$. By (16) and (17), it can be derived inductively that

$$\zeta\left(\frac{f_n(\lambda)}{(f_n(\lambda),g_n(\lambda))},\lambda_i\right) \ge \xi\left((f_n(\lambda),g_n(\lambda)),\lambda_i\right). \tag{18}$$

Together with (14), the above inequality shows that for each $1\le i\le m$,

$$i-1 = \zeta\left(f_n(\lambda),\lambda_i\right) \ge \zeta\left(\frac{f_n(\lambda)}{(f_n(\lambda),g_n(\lambda))},\lambda_i\right) \ge \xi\left((f_n(\lambda),g_n(\lambda)),\lambda_i\right) \ge t_i-1,$$

which completes the proof.

4. Proof of sufficiency of Theorem 2.1

The sufficiency of Theorem 2.1 is a direct consequence of the following proposition.

Proposition 4.1. *Let $M_1,\ldots,M_m>0$ be m real numbers and $\prod_{i=1}^m(\lambda-\lambda_i)^{t_i}$ be a polynomial with $0<\lambda_1<\lambda_2<\cdots<\lambda_m$ and $\sum_{i=1}^m t_i = n$. Suppose $t_i\le i$ for each $i=1,\ldots,m$. Then, there exist some $m_j>0$, $k_j>0$, $b_j\ge 0$, $j=1,\ldots,n$ and m distinct indices $i_h, h=1,\ldots,m$ such that $m_{i_h}=M_h$ and $\prod_{i=1}^m(\lambda-\lambda_i)^{t_i}\Big|\det(\boldsymbol{K}-\lambda(\boldsymbol{M}+\boldsymbol{B}))$.*

	All eigenvalues							
	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
S_1		λ_2	λ_3	λ_4		λ_6		
S_2			λ_3			λ_6		
S_3						λ_6		

Fig. 2. S_j of Example 4.1.

λ_6	λ_4	λ_3	λ_2		λ_6	λ_3		λ_6
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$		$\updownarrow$	$\updownarrow$		$\updownarrow$
k_2/b_2	k_3/b_3	k_4/b_4	k_5/b_5		k_7/b_7	k_8/b_8		k_{10}/b_{10}

Fig. 3. Association between λ_j and k_i/b_i.

We now begin the construction of the required mass-chain system for Proposition 4.1. That is, we find a sequence of $\{(m_j,k_j,b_j)\}_{j=1}^n$ and m indices i_h such that $m_{i_h}=M_h$, $h=1,\dots,m$ and

$$\prod_{i=1}^{m}(\lambda-\lambda_i)^{t_i}\,|\,f_n(\lambda), \tag{19}$$

where $f_n(\lambda)=\det(\boldsymbol{K}-\lambda(\boldsymbol{M}+\boldsymbol{B}))$. We consider the case where $T=\max_{1\leqslant i\leqslant m}t_i>1$, when taking account of [23, Theorem 4].[1] To this end, denote $q_j\triangleq|S_j|$, where

$$S_j\triangleq\{\lambda_i\,:\,t_i\geq j+1\},\quad j=1,\dots,T-1. \tag{20}$$

Evidently, $\sum_{j=1}^{T-1}q_j=n-m$. We reorder the elements of S_j by $s_j(1)<\cdots<s_j(q_j)$.

An important observation of Section 3 is that to ensure (19), every element in S_j, $j=1,2\cdots,T-1$ is equal to some $k_i/b_i, i\in[2\ n]$ as suggested by (15). This could help us to design a rule to determine which $k_i/b_i\in S_j$, $j=1,2\cdots,T-1$. It is the key idea of our proof. Now, set forth the rule as follows. Define a series of subsets R_j, $j=1,2,\dots,T-1$ by

$$R_j\triangleq[j+1+\sum_{h=1}^{j-1}q_h\quad j+\sum_{h=1}^{j}q_h]. \tag{21}$$

Or equivalently, we write

$$R_j=\left\{r_j(1)<r_j(2)<\cdots<r_j(q_j)\right\} \tag{22}$$

to express the entries of R_j. Then, let

$$\frac{k_i}{b_i}=s_j(q_j-l+1),\qquad i=r_j(l). \tag{23}$$

We offer an example to elaborate it.

Example 4.1. Take $n=15, m=8, t_1=t_5=t_7=t_8=1, t_2=t_4=2, t_3=3, t_6=4$ and $0<\lambda_1<\lambda_2<\cdots<\lambda_8$ in Proposition 4.1. Then, we get sets $S_j, j=1,2,3$ as shown in Fig. 2. By (21) and (23), $R_1=\{2,3,4,5\}, R_2=\{7,8\}, R_3=\{10\}$ and $k_2/b_2=k_7/b_7=k_{10}/b_{10}=\lambda_6$, $k_3/b_3=\lambda_4$, $k_4/b_4=k_8/b_8=\lambda_3$, $k_5/b_5=\lambda_2$ shown in Fig. 3.

Now, let $\{(m_j,k_j,b_j)\}_{j=1}^n$ be the required numbers in Proposition 4.1 and $\{(f_i,g_i)\}_{i=1}^n$ the corresponding polynomials defined in Section 3. Then, by Lemma 3.1, if $i\in[2\ n]\setminus\bigcup_{j=1}^{T-1}R_j$,

$$\begin{cases}\frac{f_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}=-\lambda m_i\frac{g_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}+(k_i-\lambda b_i)\frac{f_{i-1}(\lambda)}{(f_{i-1}(\lambda),g_{i-1}(\lambda))},\\ \frac{g_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}=\frac{f_{i-1}(\lambda)}{(f_{i-1}(\lambda),g_{i-1}(\lambda))}+(k_i-\lambda b_i)\frac{g_{i-1}(\lambda)}{(f_{i-1}(\lambda),g_{i-1}(\lambda))}.\end{cases} \tag{24}$$

Otherwise, for $i\in\bigcup_{j=1}^{T-1}R_j$, Lemma 3.3 implies that (23) must hold with $b_i\neq0$ and $(\lambda-\frac{k_i}{b_i})|\frac{f_{i-1}(\lambda)}{(f_{i-1}(\lambda),g_{i-1}(\lambda))}$. Consequently, (15) infers

$$\begin{cases}\frac{f_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}=-\lambda m_i\frac{g_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}-b_i\frac{f_{i-1}(\lambda)}{(f_{i-1}(\lambda),g_{i-1}(\lambda))},\\ (\lambda-\frac{k_i}{b_i})\frac{g_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}=\frac{f_{i-1}(\lambda)}{(f_{i-1}(\lambda),g_{i-1}(\lambda))}+(k_i-\lambda b_i)\frac{g_{i-1}(\lambda)}{(f_{i-1}(\lambda),g_{i-1}(\lambda))}.\end{cases} \tag{25}$$

[1] When $T=1$, we do not define S_j and R_j. In this case, we view $\bigcup_{j=1}^{T-1}R_j$ as $\emptyset$ in the later proof and show Proposition 4.1 with $m=n$.

This observation motivates us to construct $\{(m_j,k_j,b_j)\}_{j=1}^n$ based on the following strategies.

(i) Let $i\in[2\ n]\setminus\bigcup_{j=1}^{T-1}R_j$. Since the cardinality of the set $[2\ n]\setminus\bigcup_{j=1}^{T-1}R_j$ is $(n-1)-(n-m)=m-1$, we rewrite the elements of $\{1\}\cup[2\ n]\setminus\bigcup_{j=1}^{T-1}R_j$ by $1=i_1<\cdots<i_m$. Let $m_{i_l}=M_l$ for $l=1,\dots,m$. Clearly, $i=i_l$ for some $l\in[m]$, and hence $m_i=M_l$. If $\frac{f_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}$ and $\frac{g_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}$ are given, then we can seek for b_i and k_i according to (24). So $\frac{f_{i-1}(\lambda)}{(f_{i-1}(\lambda),g_{i-1}(\lambda))}$ and $\frac{g_{i-1}(\lambda)}{(f_{i-1}(\lambda),g_{i-1}(\lambda))}$ can be deduced accordingly. This task is mainly fulfilled through Lemma B.3 in the appendix.

(ii) Let $i\in\bigcup_{j=1}^{T-1}R_j$. We assign k_i/b_i a value taken from $\bigcup_{j=1}^{T-1}S_j$ in the light of (23). Once $\frac{f_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}$ and $\frac{g_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}$ are given, we can find some m_i, b_i and k_i satisfying (25), and then construct $\frac{f_{i-1}(\lambda)}{(f_{i-1}(\lambda),g_{i-1}(\lambda))}$ and $\frac{g_{i-1}(\lambda)}{(f_{i-1}(\lambda),g_{i-1}(\lambda))}$. We shall work it out by Lemma B.4 in the appendix.

Using the above strategies repeatedly, the construction of $\{\frac{f_i(\lambda)}{(f_i(\lambda),g_i(\lambda))},\frac{g_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}\}_{i=1}^n$ and $\{(m_j,k_j,b_j)\}_{j=1}^n$ can be achieved by a recursive method from n to 1. As a matter of fact, we shall present a lemma below to construct a sequence of polynomials $\{\mu_iF_i(\lambda),\nu_iG_i(\lambda)\}_{i=1}^n$ which correspond to $\{\frac{f_i(\lambda)}{(f_i(\lambda),g_i(\lambda))},\frac{g_i(\lambda)}{(f_i(\lambda),g_i(\lambda))}\}_{i=1}^n$. As for $(f_i(\lambda),g_i(\lambda))$, $i=1,2,\dots,n$, the construction is very simple and will be fulfilled directly later.

Lemma 4.1. *Let $F_n(\lambda)=\prod_{i=1}^m(\lambda-\lambda_i)$ and $G_n(\lambda)=\prod_{i=1}^{m-1}(\lambda-\lambda_i-\rho_i)$ be two polynomials with $\rho_i>0, i=1,\dots,m-1$ and μ_n,ν_n be two numbers satisfying*

$$-\frac{\mu_n}{\nu_n}>C\left(\frac{\Lambda}{\lambda_1}\right)^n\frac{\Lambda M}{\Lambda-\lambda_1}. \tag{26}$$

Then, by appropriately choosing $\rho_i>0, i=1,\dots,m-1$, there exist some monic polynomials $\{F_j(\lambda)\}_{j=1}^{n-1}$, $\{G_j(\lambda)\}_{j=1}^{n-1}$ with distinct roots and some sequences of numbers $\{(\lambda_j^*,b_j,m_j)\}_{j=2}^n$, $\{(\mu_j,\nu_j)\}_{j=1}^{n-1}$ such that for each $j\in[1\ n-1]$, $\deg(F_j(\lambda))=\deg(G_j(\lambda))+1$ and the following properties hold:

(i) $\lambda_{j+1}^*,b_{j+1},m_{j+1}>0$ and $-\frac{\mu_j}{\nu_j}>C(\frac{\Lambda}{\lambda_1})^j\frac{\Lambda M}{\Lambda-\lambda_1}$;

(ii) if $j+1\in[2\ n]\setminus\bigcup_{l=1}^{T-1}R_l$, then

$$\begin{cases}\mu_{j+1}F_{j+1}(\lambda)=-m_{j+1}\nu_{j+1}\lambda G_{j+1}(\lambda)+b_{j+1}\mu_j(\lambda_{j+1}^*-\lambda)F_j(\lambda),\\ \nu_{j+1}G_{j+1}(\lambda)=\mu_jF_j(\lambda)+b_{j+1}\nu_j(\lambda_{j+1}^*-\lambda)G_j(\lambda),\end{cases} \tag{27}$$

otherwise, for $j+1\in R_l$, $l\in[1\ T-1]$,

$$\begin{cases}\mu_{j+1}F_{j+1}(\lambda)=-m_{j+1}\nu_{j+1}\lambda G_{j+1}(\lambda)-b_{j+1}\mu_jF_j(\lambda),\\ \nu_{j+1}(\lambda-\lambda_{j+1}^*)G_{j+1}(\lambda)=\mu_jF_j(\lambda)+b_{j+1}\nu_j(\lambda_{j+1}^*-\lambda)G_j(\lambda),\\ \lambda_{j+1}^*=s_l(q_l-i+1)\quad\text{where}\quad j+1=r_l(i),\end{cases} \tag{28}$$

where $S_l=\{s_l(1)<\cdots<s_l(q_l)\}$ and $R_l=\{r_l(1)<\cdots<r_l(q_l)\}$ are defined in (20) and (22), respectively.

(iii) $\deg(F_1(\lambda))=1$ and $\deg(G_1(\lambda))=0$.

Proof. See Appendix B. □

Proof of Proposition 4.1. Let $\{\mu_n,\nu_n,F_n(\lambda),G_n(\lambda)\}$ be those defined in Lemma 4.1 for some appropriate $\rho_i>0, i=1,\dots,m-1$. Then we can construct a series of numbers $\{(\lambda_j^*,b_j,m_j)\}_{j=2}^n$, $\{(\mu_j,\nu_j)\}_{j=1}^{n-1}$ and some monic polynomials $\{F_j(\lambda)\}_{j=1}^{n-1}$, $\{G_j(\lambda)\}_{j=1}^{n-1}$ according to the lemma. Note that $-\frac{\mu_1}{\nu_1}>M>M_1$. Set

$$\begin{cases}m_1=M_1,\quad b_1=-\frac{\mu_1}{\nu_1}-M_1,\quad k_1=-\alpha_1(1)\frac{\mu_1}{\nu_1},\\ k_j=\lambda_j^*b_j,\quad j=2,\dots,n,\end{cases} \tag{29}$$

where $F_1(\lambda)=\lambda-\alpha_1(1)$. We shall see that $\{(k_j,b_j,m_j)\}_{j=1}^n$ meet the requirements of Proposition 4.1.

Define a sequence of polynomials $\{D_j(\lambda)\}_{j=1}^n$ as follows:

$$\begin{cases}D_1(\lambda)=1,\\ D_{k+1}(\lambda)=D_k(\lambda), & \text{if}\quad k+1\in[2\ n]\setminus\bigcup_{l=1}^{T-1}R_l,\\ D_{k+1}(\lambda)=(\lambda-\lambda_{k+1}^*)D_k(\lambda), & \text{if}\quad k+1\in\bigcup_{l=1}^{T-1}R_l,\end{cases}$$

and let

$$f_j(\lambda)=\frac{\mu_j}{\nu_1}D_j(\lambda)F_j(\lambda)\quad\text{and}\quad g_j(\lambda)=\frac{\nu_j}{\nu_1}D_j(\lambda)G_j(\lambda),\quad j=1,\dots,n. \tag{30}$$

Clearly, $f_n(\lambda)=\frac{\mu_n}{\nu_1}D_n(\lambda)F_n(\lambda)=\frac{\mu_n}{\nu_1}\prod_{j=1}^m(\lambda-\lambda_j)^{t_j}$. The rest of the proof is to check whether $\{f_j(\lambda)\}_{j=1}^n,\{g_j(\lambda)\}_{j=1}^n$ satisfy the recursive formula in Lemma 3.1.

Firstly, (29) and (30) imply $f_1(\lambda)=k_1-\lambda(m_1+b_1)$. Moreover, in view of the proof of Lemma 4.1, $G_1(\lambda)=1$, which indicates $g_1(\lambda)=1$. Next, by Lemma 4.1 again, if $j+1\in[2\ n]\setminus\bigcup_{l=1}^{T-1}R_l$,

$$\begin{cases}f_{j+1}(\lambda)=-m_{j+1}\lambda g_{j+1}(\lambda)+(k_{j+1}-\lambda b_{j+1})f_j(\lambda),\\ g_{j+1}(\lambda)=f_j(\lambda)+(k_{j+1}-\lambda b_{j+1})g_j(\lambda),\end{cases}$$

and if $j+1 \in \bigcup_{l=1}^{T-1} R_l$, we compute

$$\begin{aligned} f_{j+1}(\lambda) &= \frac{1}{\nu_1}\mu_{j+1}D_{j+1}(\lambda)F_{j+1}(\lambda) = \frac{1}{\nu_1}D_{j+1}(\lambda)(-m_{j+1}\nu_{j+1}\lambda G_{j+1}(\lambda) - b_{j+1}\mu_j F_j(\lambda)) \\ &= -m_{j+1}\lambda g_{j+1}(\lambda) - \frac{(\lambda-\lambda_{j+1}^*)D_j(\lambda)}{\nu_1}b_{j+1}\mu_j F_j(\lambda) = -m_{j+1}\lambda g_{j+1}(\lambda) + (k_{j+1}-\lambda b_{j+1})f_j(\lambda), \\ g_{j+1}(\lambda) &= \frac{\nu_{j+1}}{\nu_1}D_{j+1}(\lambda)G_{j+1}(\lambda) = \frac{D_j(\lambda)}{\nu_1}\nu_{j+1}(\lambda-\lambda_{j+1}^*)G_{j+1}(\lambda) \\ &= \frac{D_j(\lambda)}{\nu_1}(\mu_j F_j(\lambda) + b_{j+1}\nu_j(\lambda_{j+1}^* - \lambda)G_j(\lambda)) = f_j(\lambda) + (k_{j+1}-\lambda b_{j+1})g_j(\lambda). \end{aligned}$$

The assertion thus follows. □

5. Discussions and concluding remarks

The emergence of inerters in engineering brings some new phenomena in the study of inverse problems. Particularly, it enables a mass-chain system to possess multiple eigenvalues. As we have seen, given any positive eigenvalues (including multiplicities), one can construct a mass–spring–inerter system by following the construction strategy in Section 4. So, we theoretically solved Problem 1 proposed in Section 2. It is worth pointing out that, in order to prove the existence of the solution, the authors choose the parameters particularly small in Section 4, which in fact is not necessary except for some extreme cases. For practical purpose, one may employ Algorithm 1 of [24] to compute. Let $\hat{\boldsymbol{K}} = \mathrm{diag}\{k_1,\dots,k_n\}$, $\hat{\boldsymbol{B}} = \mathrm{diag}\{b_1,\dots,b_n\}$ and $\hat{\boldsymbol{M}} = \sum_{i=1}^n m_i(\sum_{j=1}^i \boldsymbol{e}_i)(\sum_{j=1}^i \boldsymbol{e}_i)^T$, where $\boldsymbol{e}_i$ is the standard ith unit column vector. Then, equation $\det(\boldsymbol{K}-\lambda(\boldsymbol{M}+\boldsymbol{B}))=0$ is transformed into $\det(\boldsymbol{I}_n - \hat{\boldsymbol{K}}^{-1}(\hat{\boldsymbol{M}}+\hat{\boldsymbol{B}})\lambda)=0$. The specified natural frequencies turn out to be the reciprocal of the eigenvalues of $k_1^{-1}\boldsymbol{A}_1+\cdots+k_n^{-1}\boldsymbol{A}_n$, where $\boldsymbol{A}_j = \sum_{i=1}^n \boldsymbol{e}_i\boldsymbol{a}_i^\tau$ and $\boldsymbol{a}_i$ is the ith row of $\hat{\boldsymbol{M}}+\hat{\boldsymbol{B}}$. Algorithm 1 of [24] can work here by choosing suitable $\boldsymbol{A}_j$ and initial eigenvector matrix.

In conclusion, this paper has solved the IEP for "fixed–free" mass–spring–inerter systems. Another common situation is when both ends of the system are fixed to the ground. For such a "fixed–fixed" system, the construction cannot follow readily from the method developed in Section 4. To address this issue, a more detailed analysis is required and it would be our next work.

CRediT authorship contribution statement

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proof of Example 2.1

For each $j=1,\dots,5$, let $F_j(\lambda)$ and $G_j(\lambda)$ be two monic polynomials such that $\mu_j F_j(\lambda) = \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$ and $\nu_j G_j(\lambda) = \frac{g_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$, where μ_j and ν_j are the leading coefficients of $f_j(\lambda)$ and $g_j(\lambda)$, respectively. Apparently, $\mu_j\nu_j < 0$ and

$$(F_j(\lambda), G_j(\lambda)) = 1, \quad j=1,\dots,5. \tag{31}$$

Moreover, denote $\alpha_j(1) < \cdots < \alpha_j(s_j)$ and $\beta_j(1) < \cdots < \beta_j(s_j-1)$ as the roots of $F_j(\lambda)$ and $G_j(\lambda)$, respectively. Define

$$J \triangleq \{j : k_j = \lambda_3 b_j, j=2,3,4,5\} \quad \text{and} \quad H \triangleq \{j : F_{j-1}(\lambda_3)=0, j=2,3,4,5\}.$$

We first assert that $J\cap H$ cannot contain any adjacent natural numbers. Otherwise, suppose there is a number $i\in\{1,2,3\}$ such that $i+1, i+2 \in J\cap H$. So, $k_{i+1}/b_{i+1} = \lambda_3$ and by Lemma 3.3,

$$\mu_{i+1}F_{i+1}(\lambda) = -m_{i+1}\nu_{i+1}\lambda G_{i+1}(\lambda) - b_{i+1}\mu_i F_i(\lambda). \tag{32}$$

Since the definition of H indicates $F_i(\lambda_3) = F_{i+1}(\lambda_3) = 0$, then by (32), $m_{i+1}\nu_{i+1}\lambda_3 G_{i+1}(\lambda_3) = -b_{i+1}\mu_i F_i(\lambda_3) - \mu_{i+1}F_{i+1}(\lambda_3) = 0$. Hence, $(\lambda-\lambda_3)|(F_{i+1}(\lambda), G_{i+1}(\lambda))$, which contradicts to (31) that $(F_{i+1}(\lambda), G_{i+1}(\lambda)) = 1$.

We in fact have derived $|J\cap H| \le 2$, which together with Lemma 3.2 implies $F(\lambda_3)=0$ and $5\notin H$. On the other hand, (14) and (16) in Section 3 infer $|J\cap H|\ge 2$, and hence $|J\cap H|=2$. Observe that $J\cap H \ne \{2,3\}$ or $\{3,4\}$, it then follows that $J\cap H = \{2,4\}$. So, $\alpha_1(1) = \lambda_3$ and by Lemma 3.3(i), $s_2 = 1$ and $\alpha_2(1) < \lambda_3$. For the roots of $F_j(\lambda), j=3,4,5$, they can be discussed similarly by using Lemma 3.3 repeatedly. Indeed, $3\notin J\cap H$ and $4\in H$ lead to $s_3=2$ and $F_3(\lambda_3)=0$, respectively. So,

$$\begin{cases} 0 < \alpha_3(1) < \alpha_2(1) < \alpha_3(2) = \lambda_3, & \text{if} \quad b_3 = 0, \\ \alpha_3(1) < \min\{\alpha_2(1), k_3/b_3\} < \lambda_3, \quad \text{and} \quad \alpha_3(2) = \lambda_3 < k_3/b_3, & \text{if} \quad b_3 > 0. \end{cases}$$

In addition, $4 \in J \cap H$ indicates $s_4 = 2$ and

$$\alpha_4(1) < \alpha_3(1) < \alpha_4(2) < \alpha_3(2) = \lambda_3. \tag{33}$$

At last, since $5 \notin J \cap H$, $s_5 = 3$. By $|J \cap H| = 2$ and (16), $(f_5(\lambda), g_5(\lambda)) = (\lambda - \lambda_3)^2$, which immediately gives $F_5(\lambda) = \prod_{i=1}^{3}(\lambda - \lambda_i)$. So, $\alpha_5(1) = \lambda_1 < \alpha_5(2) = \lambda_2 < \alpha_5(3) = \lambda_3$. If $b_5 > 0$, then (33) infers that $\lambda_3 < k_5/b_5$. Therefore,

$$\begin{cases} 0 < \alpha_5(1) < \alpha_4(1) < \alpha_5(2) < \alpha_4(2) < \alpha_5(3) = \lambda_3, & \text{if} \quad b_5 = 0, \\ \alpha_5(1) < \alpha_4(1) < \alpha_5(2) < \alpha_4(2) < \alpha_5(3) = \lambda_3 < k_5/b_5, & \text{if} \quad b_5 > 0. \end{cases}$$

We thus summarize

$$\begin{cases} s_5 = 3, \quad s_4 = s_3 = 2, \quad s_2 = s_1 = 1, \\ \alpha_2(s_2), \alpha_4(s_4) < \lambda_3 \quad \text{and} \quad \alpha_1(s_1) = \alpha_3(s_3) = \alpha_5(s_5) = \lambda_3, \\ \alpha_{j+1}(i) < \alpha_j(i) < \alpha_{j+1}(i+1), \quad \text{if} \quad j \in [1\ 4], \quad i \in [1\ s_{j+1} - 1], \end{cases} \tag{34}$$

and

$$\begin{cases} k_j/b_j = \lambda_3 \quad \text{and} \quad b_j > 0, \quad \text{if} \quad j = 2, 4, \\ k_j/b_j > \lambda_3 \quad \text{and} \quad b_j > 0, \quad \text{if} \quad j = 3, 5. \end{cases} \tag{35}$$

This means $J = J \cap H = \{2, 4\}$ and as a consequence, by Lemma 3.3,

$$\begin{cases} \mu_{j+1} F_{j+1}(\lambda) = -\lambda m_{j+1} \nu_{j+1} G_{j+1}(\lambda) - b_{j+1} \mu_j F_j(\lambda), \\ \left(\lambda - \dfrac{k_{j+1}}{b_{j+1}}\right) \nu_{j+1} G_{j+1}(\lambda) = \mu_j F_j(\lambda) + (k_{j+1} - \lambda b_{j+1}) \nu_j G_j(\lambda), \end{cases} \quad \text{if} \quad j = 1, 3, \tag{36}$$

and

$$\begin{cases} \mu_{j+1} F_{j+1}(\lambda) = -\lambda m_{j+1} \nu_{j+1} G_{j+1}(\lambda) + (k_{j+1} - \lambda b_{j+1}) \mu_j F_j(\lambda), \\ \nu_{j+1} G_{j+1}(\lambda) = \mu_j F_j(\lambda) + (k_{j+1} - \lambda b_{j+1}) \nu_j G_j(\lambda), \end{cases} \quad \text{if} \quad j = 2, 4. \tag{37}$$

With the above properties, we can also present the relationship between the roots of $F_j(\lambda)$ and $G_{j+1}(\lambda)$ for $j = 2, 4$. In fact, when $j = 2, 4$, $s_{j+1} = s_j + 1$ and by (13), (34) and (35), we deduce $\mu_j F_j(\lambda) \prec -\nu_j(\lambda b_{j+1} - k_{j+1}) G_j(\lambda)$. Then, (13), (34), (37) and Lemma 3.2 imply

$$\alpha_j(1) < \beta_{j+1}(1) < \alpha_j(2) < \cdots < \alpha_j(s_j) < \beta_{j+1}(s_{j+1} - 1) < \alpha_{j+1}(s_{j+1}) = \lambda_3, \ j = 2, 4. \tag{38}$$

Now, we prove (5) by using reduction to absurdity. Suppose

$$\max_{2 \leqslant j \leqslant 4} \frac{m_j}{m_{j+1}} \leq \sigma \triangleq \frac{\lambda_1}{8\lambda_3} \left(1 - \left(\frac{\lambda_2}{\lambda_3}\right)^{\frac{1}{3}}\right). \tag{39}$$

First, it is evident that for each $j \in [1\ 4]$, (34), (36) and (37) yield

$$m_{j+1} = -\frac{\mu_{j+1}}{\nu_{j+1}} \frac{F_{j+1}(\alpha_j(1))}{\alpha_j(1) G_{j+1}(\alpha_j(1))} = -\frac{\mu_{j+1}}{\nu_{j+1}} \frac{\prod_{i=1}^{s_{j+1}}(\alpha_j(1) - \alpha_{j+1}(i))}{\alpha_j(1) \prod_{i=1}^{s_{j+1}-1}(\alpha_j(1) - \beta_{j+1}(i))} \geq -\frac{\mu_{j+1}}{\nu_{j+1}} \frac{\alpha_j(1) - \alpha_{j+1}(1)}{\alpha_j(1)}. \tag{40}$$

In particular, when $j = 1, 3$, (13) further implies

$$m_{j+1} = -\frac{\mu_{j+1}}{\nu_{j+1}} \frac{F_{j+1}(\alpha_j(s_j))}{\alpha_j(s_j) G_{j+1}(\alpha_j(s_j))} = -\frac{\mu_{j+1}}{\nu_{j+1}} \frac{\prod_{i=1}^{s_{j+1}}(\lambda_3 - \alpha_{j+1}(i))}{\lambda_3 \prod_{i=1}^{s_{j+1}-1}(\lambda_3 - \beta_{j+1}(i))} < -\frac{\mu_{j+1}}{\nu_{j+1}}. \tag{41}$$

Note that by comparing the leading coefficients of the polynomials in (36)–(37), we assert that for each $j \in [1\ 4]$ with $b_{j+1} > 0$,

$$-\frac{\mu_j}{\nu_j} = -\frac{\mu_{j+1} + m_{j+1}\nu_{j+1}}{\nu_{j+1} - \mu_j}. \tag{42}$$

Finally, let us complete the proof by considering the following four cases.
Case 1: $b_3 b_5 > 0$. First, we estimate m_j/m_{j+1} for $j = 2, 4$. Let $\varsigma_j = k_{j+1}/b_{j+1}$, then $\varsigma_j > \alpha_{j+1}(s_{j+1}) = \lambda_3$ by (34)–(35). Therefore, noting that $\nu_{j+1}\mu_j > 0$, (13) and (42) lead to

$$\begin{aligned} -\frac{\mu_j}{\nu_j} &= -\frac{\mu_{j+1} + m_{j+1}\nu_{j+1}}{\nu_{j+1} - \mu_j} > -\frac{\mu_{j+1} + m_{j+1}\nu_{j+1}}{\nu_{j+1}} = -\frac{\mu_{j+1}}{\nu_{j+1}} \left(1 - \frac{F_{j+1}(\varsigma_j)}{\varsigma_j G_{j+1}(\varsigma_j)}\right) \\ &= -\frac{\mu_{j+1}}{\nu_{j+1}} \frac{\varsigma_j \prod_{i=1}^{s_{j+1}-1}(\varsigma_j - \beta_{j+1}(i)) - \prod_{i=1}^{s_{j+1}}(\varsigma_j - \alpha_{j+1}(i))}{\varsigma_j \prod_{i=1}^{s_{j+1}-1}(\varsigma_j - \beta_{j+1}(i))} \\ &> -\frac{\mu_{j+1}}{\nu_{j+1}} \frac{\alpha_{j+1}(1) \prod_{i=1}^{s_{j+1}-1}(\varsigma_j - \beta_{j+1}(i))}{\varsigma_j \prod_{i=1}^{s_{j+1}-1}(\varsigma_j - \beta_{j+1}(i))} = -\frac{\mu_{j+1}}{\nu_{j+1}} \frac{\alpha_{j+1}(1)}{\varsigma_j} \geq -\frac{\mu_{j+1}}{\nu_{j+1}} \frac{\lambda_1}{\varsigma_j}, \quad j = 2, 4. \end{aligned} \tag{43}$$

So, if $\varsigma_j \le 2\lambda_3$, the above inequality reduces to

$$-\frac{\mu_j}{\nu_j} > -\frac{\mu_{j+1}}{\nu_{j+1}}\frac{\lambda_1}{2\lambda_3}, \quad j = 2, 4. \tag{44}$$

We next treat the case $\varsigma_2 > 2\lambda_3$. Since (13) and (34) imply $\beta_3(1) < \alpha_3(2)$ and $\lambda_1 = \alpha_5(1) < \alpha_3(1)$, then $(\varsigma_2 - \lambda_1)G_3(\varsigma_2) = (\varsigma_2 - \lambda_1)(\varsigma_2 - \beta_3(1)) > (\varsigma_2 - \alpha_3(1))(\varsigma_2 - \alpha_3(2)) = F_3(\varsigma_2)$, which is equivalent to $1 - \frac{F_3(\varsigma_2)}{\varsigma_2 G_3(\varsigma_2)} > \frac{\lambda_1}{\varsigma_2}$. Further, by (37) again, $\mu_2/\nu_3 = G_3(\varsigma_2)/F_2(\varsigma_2)$, so (38) shows

$$-\frac{\mu_2}{\nu_2} = -\frac{\mu_3 + m_3\nu_3}{\nu_3 - \mu_2} = -\frac{\mu_3}{\nu_3}\frac{1 - \frac{F_3(\varsigma_2)}{\varsigma_2 G_3(\varsigma_2)}}{1 - \frac{G_3(\varsigma_2)}{F_2(\varsigma_2)}} > -\frac{\mu_3}{\nu_3}\frac{\lambda_1}{\varsigma_2}\frac{F_2(\varsigma_2)}{F_2(\varsigma_2) - G_3(\varsigma_2)} \tag{45}$$

$$= -\frac{\mu_3}{\nu_3}\frac{\lambda_1}{\beta_3(1) - \alpha_2(1)}\frac{\varsigma_2 - \alpha_2(1)}{\varsigma_2} > -\frac{\mu_3}{\nu_3}\frac{\lambda_1}{\lambda_3}\left(1 - \frac{\lambda_3}{2\lambda_3}\right) > -\frac{\mu_3}{\nu_3}\frac{\lambda_1}{8\lambda_3}. \tag{46}$$

For $\varsigma_4 > 2\lambda_3$, similarly to (45), we compute by (38) that

$$\begin{aligned}
-\frac{\mu_4}{\nu_4} &> -\frac{\mu_5}{\nu_5}\frac{\lambda_1}{\varsigma_4}\frac{F_4(\varsigma_4)}{F_4(\varsigma_4) - G_5(\varsigma_4)} = -\frac{\mu_5}{\nu_5}\frac{\lambda_1}{\varsigma_4}\frac{(\varsigma_4 - \alpha_4(1))(\varsigma_4 - \alpha_4(2))}{(\varsigma_4 - \alpha_4(1))(\varsigma_4 - \alpha_4(2)) - (\varsigma_4 - \beta_5(1))(\varsigma_4 - \beta_5(2))} \\
&= -\frac{\mu_5}{\nu_5}\frac{\lambda_1}{\varsigma_4}\frac{(\varsigma_4 - \alpha_4(1))(\varsigma_4 - \alpha_4(2))}{(\beta_5(1) + \beta_5(2) - \alpha_4(1) - \alpha_4(2))\varsigma_4 + \alpha_4(1)\alpha_4(2) - \beta_5(1)\beta_5(2)} \\
&> -\frac{\mu_5}{\nu_5}\frac{\lambda_1}{\beta_5(1) + \beta_5(2) - \alpha_4(1) - \alpha_4(2)}\frac{\varsigma_4 - \alpha_4(1)}{\varsigma_4}\frac{\varsigma_4 - \alpha_4(2)}{\varsigma_4} > -\frac{\mu_5}{\nu_5}\frac{\lambda_1}{2\lambda_3}\left(1 - \frac{\lambda_3}{2\lambda_3}\right)^2 = -\frac{\mu_5}{\nu_5}\frac{\lambda_1}{8\lambda_3}.
\end{aligned} \tag{47}$$

As a result, by (44), (46) and (47), the following inequality always holds:

$$-\frac{\mu_j}{\nu_j} > -\frac{\mu_{j+1}}{\nu_{j+1}}\frac{\lambda_1}{8\lambda_3}, \quad j = 2, 4,$$

and thus (40) and (41) yield

$$m_j > -\frac{\mu_{j+1}}{\nu_{j+1}}\frac{\lambda_1}{8\lambda_3}\frac{\alpha_{j-1}(1) - \alpha_j(1)}{\alpha_{j-1}(1)} > m_{j+1}\frac{\lambda_1}{8\lambda_3}\frac{\alpha_{j-1}(1) - \alpha_j(1)}{\alpha_{j-1}(1)}, \quad j = 2, 4. \tag{48}$$

We proceed to the calculation of m_3/m_4. Note that $\lambda_1 < \beta_4(1) < \alpha_4(2) < \alpha_3(2) = \lambda_3$, then similarly to (43), we can prove $-\frac{\mu_3}{\nu_3} > -\frac{\mu_4}{\nu_4}\frac{\lambda_1}{\lambda_3}$, which together with (40) and (41) leads to

$$m_3 \ge -\frac{\mu_4}{\nu_4}\frac{\lambda_1}{\lambda_3}\frac{\alpha_2(1) - \alpha_3(1)}{\alpha_2(1)} > m_4\frac{\lambda_1}{\lambda_3}\frac{\alpha_2(1) - \alpha_3(1)}{\alpha_2(1)}. \tag{49}$$

Now, combining (48) and (49), (39) derives $\frac{\lambda_1}{8\lambda_3}\frac{\alpha_j(1) - \alpha_{j+1}(1)}{\alpha_j(1)} \le \sigma, j = 1, 2, 3$. That is, $\frac{\alpha_j(1)}{\alpha_{j+1}(1)} \le \frac{\lambda_1}{\lambda_1 - 8\lambda_3\sigma}, j = 1, 2, 3$. Consequently, $\lambda_3 = \alpha_1(1) \le \left(\frac{\lambda_1}{\lambda_1 - 8\lambda_3\sigma}\right)^3 \alpha_4(1) < \left(\frac{\lambda_1}{\lambda_1 - 8\lambda_3\sigma}\right)^3 \lambda_2$, which contradicts to the definition of σ in (39).

Case 2: $b_3 = b_5 = 0$. Then, (37) infers $m_3 = -\frac{\mu_3}{\nu_3}$ and $m_5 = -\frac{\mu_5}{\nu_5}$. In this case, $\nu_5 = \mu_4$ and then (37) becomes

$$\begin{cases} \mu_5(F_5(\lambda) - \lambda G_5(\lambda)) = k_5\nu_5 F_4(\lambda), \\ \mu_4(G_5(\lambda) - F_4(\lambda)) = k_5\nu_4 G_4(\lambda). \end{cases} \tag{50}$$

Since $\deg(F_5(\lambda) - \lambda G_5(\lambda)) = 2$ and $\deg(G_5(\lambda) - F_4(\lambda)) = 1$, comparing the leading coefficients of the polynomials in (50), we calculate

$$-\frac{\mu_4}{\nu_4}\left(\sum_{i=1}^{2}\beta_5(i) - \sum_{i=1}^{2}\alpha_4(i)\right) = -\frac{\mu_5}{\nu_5}\left(\sum_{i=1}^{3}\alpha_5(i) - \sum_{i=1}^{2}\beta_5(i)\right) = k_5.$$

Consequently, by (13) and (38),

$$-\frac{\mu_4}{\nu_4} = -\frac{\mu_5}{\nu_5}\frac{\sum_{i=1}^{3}\alpha_5(i) - \sum_{i=1}^{2}\beta_5(i)}{\sum_{i=1}^{2}\beta_5(i) - \sum_{i=1}^{2}\alpha_4(i)} > -\frac{\mu_5}{\nu_5}\frac{\alpha_5(1)}{2\lambda_3} > \frac{\lambda_1}{2\lambda_3}m_5, \tag{51}$$

and hence $m_3 = -\frac{\mu_3}{\nu_3} = -\frac{\mu_4 + m_4\nu_4}{\nu_4 - \mu_3} > -\frac{\mu_4}{\nu_4} - m_4 = \frac{\lambda_1}{2\lambda_3}m_5 - m_4$. Therefore, by (39), $\sigma^2 m_5 \ge m_3 > \frac{\lambda_1}{2\lambda_3}m_5 - m_4 \ge \frac{\lambda_1}{2\lambda_3}m_5 - \sigma m_5$, which infers $\sigma^2 + \sigma > \frac{\lambda_1}{2\lambda_3}$. This is impossible due to the definition of σ in (39).

Case 3: $b_3 = 0$ and $b_5 > 0$. Then, $m_3 = -\frac{\mu_3}{\nu_3}$ and a similar argument as (51) indicates $-\frac{\mu_2}{\nu_2} = -\frac{\mu_3}{\nu_3}\frac{\sum_{i=1}^{2}\alpha_3(i) - \beta_3(1)}{\beta_3(1) - \alpha_2(1)}$. Then by (13) and (38), $m_2 = -\frac{\mu_2}{\nu_2}\frac{\lambda_3 - \alpha_2(1)}{\lambda_3} = -\frac{\mu_3}{\nu_3}\frac{\lambda_3 - \alpha_2(1)}{\lambda_3}\frac{\sum_{i=1}^{2}\alpha_3(i) - \beta_3(1)}{\beta_3(1) - \alpha_2(1)} > -\frac{\mu_3}{\nu_3}\frac{\sum_{i=1}^{2}\alpha_3(i) - \beta_3(1)}{\lambda_3} > m_3\frac{\lambda_1}{\lambda_3}$. It contradicts to (39) that $\frac{m_2}{m_3} \le \sigma < \frac{\lambda_1}{\lambda_3}$.

Case 4: $b_3 > 0$ and $b_5 = 0$. Then, (51) holds and according to (40), $m_4 > -\frac{\mu_4}{\nu_4}\frac{\alpha_3(1) - \alpha_4(1)}{\alpha_3(1)} > m_5\frac{\lambda_1}{2\lambda_3}\frac{\alpha_3(1) - \alpha_4(1)}{\alpha_3(1)}$. By (43)–(46) and (49) in **Case 1**, we can demonstrate $m_j > m_{j+1}\frac{\lambda_1}{8\lambda_3}\frac{\alpha_{j-1}(1) - \alpha_j(1)}{\alpha_{j-1}(1)}, j = 2, 3$. The rest of the proof keeps the same as that in **Case 1**. □

Appendix B. Proof of Lemma 4.1

The proof of Lemma 4.1 is based on several technical lemmas.

Lemma B.1. *Let $F(\lambda)=\mu\prod_{i=1}^{p}(\lambda-\alpha_i)$ and $G(\lambda)=\nu\prod_{i=1}^{p}(\lambda-\beta_i)$ be two polynomials of degree p such that $G(\lambda)\prec F(\lambda)$, where $|\mu|>|\nu|$, $\alpha_1<\cdots<\alpha_p$ and $\beta_1<\cdots<\beta_p$. Then, $F(\lambda)-G(\lambda)$ has p real roots $\gamma_1<\cdots<\gamma_p$ satisfying $\gamma_i\in(\alpha_i,\beta_{i+1})$ for $i=1,\dots,p-1$ and $\gamma_p>\alpha_p$. Moreover, the following two statements hold:*
(i) when $p>1$, for any $\eta\in(0,\min_{1\leqslant i\leqslant p-1}(\alpha_{i+1}-\alpha_i))$, if

$$\frac{|\nu|}{|\mu|}<\frac{\min\{1,\left(\min_{1\leqslant i\leqslant p-1}(\alpha_{i+1}-\alpha_i)-\eta\right)^p\}\min\{1,\eta^p\}}{2+2\max_{1\leqslant j\leqslant p}\left|\prod_{i=1}^{p}(\alpha_j-\beta_i)\right|} \tag{52}$$

then

$$\max_{1\leqslant i\leqslant p}(\gamma_i-\alpha_i)<\eta; \tag{53}$$

(ii) when $p=1$, for any $\eta\in(0,\frac{1}{2})$, (53) holds provided that

$$|\nu|<\frac{|\mu|}{2}\min\left\{\frac{\eta}{\alpha_1-\beta_1},1\right\}. \tag{54}$$

Proof. Note that $G(\lambda)\prec F(\lambda)$ indicates $\mu\nu>0$ and for each $j=1,\dots,p-1$,

$$(F(\alpha_j)-G(\alpha_j))(F(\beta_{j+1})-G(\beta_{j+1}))=-\mu\nu\prod_{i=1}^{p}(\alpha_j-\beta_i)(\beta_{j+1}-\alpha_i)<0.$$

This means $F(\lambda)-G(\lambda)$ has a root γ_j in (α_j,β_{j+1}). Further, since $\mu\nu>0$ and $|\mu|>|\nu|$, $\mu(F(\lambda)-G(\lambda))>0$ holds for all sufficiently large $\lambda>\alpha_p$. On the other hand, $\mu(F(\alpha_p)-G(\alpha_p))=-\mu\nu\prod_{i=1}^{p}(\alpha_p-\beta_i)<0$, so $F(\lambda)-G(\lambda)$ has a root γ_p in (α_p,∞). Clearly, $\gamma_1<\cdots<\gamma_p$.

Next, we show statement (i). Observe that $F(\lambda)-G(\lambda)=(\mu-\nu)\prod_{i=1}^{p}(\lambda-\gamma_i)$, then for each $j\in[1\ p]$,

$$(\mu-\nu)\prod_{i=1}^{p}(\alpha_j-\gamma_i)=F(\alpha_j)-G(\alpha_j)=-\nu\prod_{i=1}^{p}(\alpha_j-\beta_i). \tag{55}$$

Let $p>1$. If (53) fails, denote l as the smallest subscript $i\in[1\ p]$ such that $\gamma_i-\alpha_i\geq\eta$. Then,

$$0<\alpha_{i+1}-\alpha_i-\eta\leq\alpha_{i+1}-\gamma_i\leq\alpha_l-\gamma_i,\quad i=1,\dots,l-1.$$

Moreover, it is clear that $\gamma_i-\alpha_l\geq\gamma_l-\alpha_l>0$ for all $i\geq l$. Hence by (52) and (55),

$$\begin{aligned}&\left(\min_{1\leqslant i\leqslant p-1}(\alpha_{i+1}-\alpha_i)-\eta\right)^{l-1}(\gamma_l-\alpha_l)^{p-l+1}\leq\left|\prod_{i=1}^{p}(\alpha_l-\gamma_i)\right|\leq\left|\frac{\nu/\mu}{1-\nu/\mu}\right|\max_{1\leqslant j\leqslant p}\left|\prod_{i=1}^{p}(\alpha_j-\beta_i)\right|\\&<2\left|\frac{\nu}{\mu}\right|\max_{1\leqslant j\leqslant p}\left|\prod_{i=1}^{p}(\alpha_j-\beta_i)\right|<\min\left\{1,\left(\min_{1\leqslant i\leqslant p-1}(\alpha_{i+1}-\alpha_i)-\eta\right)^p\right\}\min\{1,\eta^p\}\leq\left(\min_{1\leqslant i\leqslant p-1}(\alpha_{i+1}-\alpha_i)-\eta\right)^{l-1}\eta^{p-l+1},\end{aligned}$$

which contradicts to $\gamma_l-\alpha_l\geq\eta$. So, statement (i) is true.

When $p=1$, (54) and (55) lead to $\gamma_1-\alpha_1\leq\left|\frac{\nu/\mu}{1-\nu/\mu}\right|(\alpha_1-\beta_1)<2\left|\frac{\nu}{\mu}\right|(\alpha_1-\beta_1)<\eta$. The statement (ii) is proved. □

To prove Lemma 4.1, a key point is to select $\rho_i=\varepsilon^{(n+1)^{m-i}}$, $i=1,\dots,m-1$, where

$$\varepsilon=\frac{\Delta^{n^2+n+1}}{n2^{3(n+1)^3}\Lambda^{(n+1)^2}},\quad\Delta\triangleq\frac{1}{2}\min\{1,\min_{1\leq i\leq m-1}(\lambda_{i+1}-\lambda_i)\},\quad\Lambda\triangleq1+\lambda_m. \tag{56}$$

Next, define

$$C_1\triangleq\frac{\Delta}{2^{n+1}\Lambda}\quad\text{and}\quad C_2(j)\triangleq\frac{2^{2(n+1)^2}\Lambda^{n+1}}{\Delta^n\varepsilon^{(n+1)^{m-j-1}}},\quad j=1,\dots,m-1, \tag{57}$$

as well as

$$C\triangleq\frac{2^{2n+1}(1+\Lambda^n)}{C_1^{2n^2}\rho_1^{2n}}\quad\text{and}\quad M\triangleq\sum_{h=1}^{m}M_h. \tag{58}$$

It is evident that $\varepsilon<1$, $\Lambda/\Delta\geq2$ in (56), $C_1<1$ in (57) and $C>1$ in (58).

Lemma B.2. *Constants $C_2(j), j=1,\dots,m-1$ defined by (57) satisfy*

$$\begin{cases}(1+C_2(m-1))^n\rho_{m-1}<\frac{\Delta}{2},\\(1+C_2(j))^n\rho_j<(1+C_2(j+1))^n\rho_{j+1},& j=1,\dots,m-2,\quad m>2,\\\frac{n(1+C_2(j-1))^n}{\Delta}\rho_{j-1}<\frac{1}{4}\left(\frac{\Delta^n}{2^{(n+1)^2}\Lambda^{n+1}}\right)\rho_{j+1},& j=2,\dots,m-2,\quad m>3.\end{cases} \tag{59}$$

Proof. First, note that $C_2(m-1) > 1$ and $\Lambda/\Delta \geq 2$, so

$$(1+C_2(m-1))^n \rho_{m-1} < 2^n C_2^n(m-1)\rho_{m-1} = 2^n \left(\frac{2^{2n(n+1)^2} \Lambda^{n(n+1)}}{\Delta^{n^2} \varepsilon^n} \right) \varepsilon^{n+1} < \frac{2^{2(n+1)^3} \Lambda^{(n+1)^2}}{2\Delta^{n^2+n}} \varepsilon < \frac{\Delta}{2n} < \frac{\Delta}{2}.$$

Let $m > 2$. Since $2\varepsilon < 1$, for each $j \in [1\ m-2]$,

$$\begin{aligned}(1+C_2(j))^n \rho_j &< 2^n C_2^n(j)\rho_j = 2^n \left(\frac{2^{2n(n+1)^2} \Lambda^{n(n+1)}}{\Delta^{n^2} \varepsilon^{n(n+1)^{m-j-1}}} \right) \varepsilon^{(n+1)^{m-j}} = 2^n \left(\frac{2^{2n(n+1)^2} \Lambda^{n(n+1)}}{\Delta^{n^2}} \right) \left(\varepsilon^{(n+1)^{m-j-2}} \right)^{n+1} \\ &< \frac{2^{2n(n+1)^2} \Lambda^{n(n+1)}}{\Delta^{n^2}} \varepsilon^{(n+1)^{m-j-2}} = C_2^n(j+1)\rho_{j+1} < (1+C_2(j+1))^n \rho_{j+1}.\end{aligned}$$

When $m > 3$, for each $j \in [2\ m-2]$,

$$\begin{aligned}\frac{n(1+C_2(j-1))^n \rho_{j-1}}{\Delta} &< \frac{n2^n C_2^n(j-1)\rho_{j-1}}{\Delta} = n2^n \left(\frac{2^{2n(n+1)^2} \Lambda^{n(n+1)}}{\Delta^{n^2+1}} \right) \varepsilon^{(n+1)^{m-j}} \\ &= n2^n \left(\frac{2^{2n(n+1)^2} \Lambda^{n(n+1)}}{\Delta^{n^2+1}} \right) \varepsilon^{(n+1)^{m-j} - (n+1)^{m-j-1}} \rho_{j+1} < n2^n \left(\frac{2^{2n(n+1)^2} \Lambda^{n(n+1)} \varepsilon}{\Delta^{n^2+1}} \right) \rho_{j+1} \\ &< \frac{\Delta^n}{2^{(n+1)^3} \Lambda^{n+1}} \rho_{j+1} \leq \frac{1}{4} \left(\frac{\Delta^n}{2^{(n+1)^2} \Lambda^{n+1}} \right) \rho_{j+1}.\end{aligned}$$

Hence, (59) follows immediately. □

The next two lemmas concern the following two polynomials

$$F(\lambda) = \prod_{i=1}^{p} (\lambda - \alpha_i) \quad \text{and} \quad G(\lambda) = \prod_{i=1}^{p-1} (\lambda - \beta_i), \qquad p \in [2\ m], \tag{60}$$

whose roots $\{\alpha_i\}_{i=1}^p$ and $\{\beta_i\}_{i=1}^{p-1}$ satisfy

$$\alpha_p \in [\lambda_p, \bar{\lambda}) \qquad \text{for some number } \bar{\lambda} > \lambda_p \tag{61}$$

and for each $i = 1, \dots, p-1$,

$$\lambda_i + C_1^n \rho_i \leq \alpha_i + C_1^n \rho_i < \beta_i < \lambda_i + (1+C_2(i))^n \rho_i < \lambda_{i+1}. \tag{62}$$

Lemma B.3. *Let $F(\lambda)$ and $G(\lambda)$ be two polynomials satisfying (60)–(62) with $\bar{\lambda} = \Lambda$ in (61). For any given constants $\mu, \nu, \widetilde{m}$ satisfying $0 < \widetilde{m} < M$ and $-\frac{\mu}{\nu} > CM$, the following two statements hold.*
(i) If $p > 2$, then there exist two monic polynomials $\widetilde{F}(\lambda)$ and $\widetilde{G}(\lambda)$ with distinct roots $\widetilde{\alpha}_1 < \cdots < \widetilde{\alpha}_{p-1}$ and $\widetilde{\beta}_1 < \cdots < \widetilde{\beta}_{p-2}$, respectively, satisfying $\widetilde{\alpha}_{p-1} \in (\alpha_{p-1}, \lambda_p)$ and for $i \in [1\ p-2]$,

$$\widetilde{\alpha}_i \in (\alpha_i, \beta_i), \qquad C_1(\beta_i - \alpha_i) < \widetilde{\beta}_i - \widetilde{\alpha}_i < C_2(i)(\beta_i - \alpha_i). \tag{63}$$

In addition, for some numbers $\widetilde{\lambda}, \widetilde{b} > 0$ and $\widetilde{\mu}, \widetilde{\nu}$ with $-\frac{\widetilde{\mu}}{\widetilde{\nu}} > -\frac{\mu}{\nu} - \widetilde{m}$, $\widetilde{F}(\lambda)$ and $\widetilde{G}(\lambda)$ fulfill

$$\begin{cases} \mu F(\lambda) = -\widetilde{m}\nu\lambda G(\lambda) + \widetilde{b}\widetilde{\mu}(\widetilde{\lambda} - \lambda)\widetilde{F}(\lambda), \\ \nu G(\lambda) = \widetilde{\mu}\widetilde{F}(\lambda) + \widetilde{b}\widetilde{\nu}(\widetilde{\lambda} - \lambda)\widetilde{G}(\lambda). \end{cases} \tag{64}$$

(ii) If $p = 2$, then there are some numbers $\widetilde{\lambda}, \widetilde{b} > 0$, $\widetilde{\alpha}_1 \in (\alpha_1, \lambda_2)$ and $\widetilde{\mu}, \widetilde{\nu}$ with $-\frac{\widetilde{\mu}}{\widetilde{\nu}} > -\frac{\mu}{\nu} - \widetilde{m}$ such that polynomials $\widetilde{F}(\lambda) = \lambda - \widetilde{\alpha}_1$ and $\widetilde{G}(\lambda) = 1$ satisfy equation (64).

Proof. Let $p \geq 2$. We first take a number $\widetilde{\lambda}$ satisfying

$$\widetilde{\lambda} > \alpha_p \quad \text{and} \quad -\frac{\mu}{\nu} \frac{F(\widetilde{\lambda})}{\widetilde{\lambda} G(\widetilde{\lambda})} = \widetilde{m}. \tag{65}$$

Such $\widetilde{\lambda}$ indeed exists because of (60)–(62), which yield $-\frac{\mu}{\nu} \frac{F(\alpha_p)}{\alpha_p G(\alpha_p)} = 0$ and $\lim_{\lambda \to +\infty} -\frac{\mu}{\nu} \frac{F(\lambda)}{\lambda G(\lambda)} = -\frac{\mu}{\nu} > CM > \widetilde{m}$. Then, let

$$\widetilde{F}(\lambda) = \frac{\mu F(\lambda) + \widetilde{m}\nu\lambda G(\lambda)}{(\mu + \widetilde{m}\nu)(\lambda - \widetilde{\lambda})}, \quad \widetilde{b} = -\frac{\mu + \widetilde{m}\nu}{\nu} \frac{\widetilde{F}(\widetilde{\lambda})}{G(\widetilde{\lambda})}, \quad \widetilde{\mu} = \nu \frac{G(\widetilde{\lambda})}{\widetilde{F}(\widetilde{\lambda})}, \quad \widetilde{G}(\lambda) = \frac{\nu G(\lambda) - \widetilde{\mu}\widetilde{F}(\lambda)}{(\nu - \widetilde{\mu})(\lambda - \widetilde{\lambda})}, \quad \widetilde{\nu} = \frac{\widetilde{\mu} - \nu}{\widetilde{b}}. \tag{66}$$

We shall show that all the above defined numbers and polynomials are well-defined and fulfill our requirements.

Observe that $\mu\nu < 0$ and by (61)–(62),

$$\alpha_p \geq \lambda_p > \lambda_{p-1} + (1+C_2(p-1))^n \rho_{p-1} > \beta_{p-1} > \alpha_{p-1} > \cdots > \beta_1 > \alpha_1 > 0, \tag{67}$$

then $\lambda G(\lambda) \prec F(\lambda)$. As a result, by (65),

$$-\frac{\mu+\widetilde{m}\nu}{\nu} = -\frac{\mu}{\nu}\left(1-\frac{F(\widetilde{\lambda})}{\widetilde{\lambda}G(\widetilde{\lambda})}\right) > -\frac{\mu}{\nu}\left(1-\frac{\prod_{i=1}^{p}(\widetilde{\lambda}-\alpha_i)}{(\widetilde{\lambda}-\alpha_1)\prod_{i=1}^{p-1}(\widetilde{\lambda}-\alpha_{i+1})}\right) = 0. \tag{68}$$

Since (65) indicates $(\lambda-\widetilde{\lambda})|(\mu F(\lambda)+\lambda\widetilde{m}\nu G(\lambda))$, (68) means $\widetilde{F}(\lambda)$ is a well-defined monic polynomial of degree $p-1$. Recall that $\lambda G(\lambda) \prec F(\lambda)$ and $|\mu| > \widetilde{m}|\nu|$, applying Lemma B.1 to polynomials $\mu F(\lambda)$ and $-\widetilde{m}\nu\lambda G(\lambda)$ shows

$$\widetilde{\alpha}_i \in (\alpha_i, \beta_i), \quad i = 1,\dots,p-1. \tag{69}$$

Therefore,

$$\lambda\widetilde{F}(\lambda) \prec F(\lambda) \quad \text{and} \quad \widetilde{F}(\lambda) \prec G(\lambda), \tag{70}$$

which give $\widetilde{b} = -\frac{\mu+\widetilde{m}\nu}{\nu}\left(\frac{\widetilde{F}(\widetilde{\lambda})}{G(\widetilde{\lambda})}\right) > 0$,

$$\widetilde{\mu}\nu = \nu^2\left(\frac{G(\widetilde{\lambda})}{\widetilde{F}(\widetilde{\lambda})}\right) > 0 \quad \text{and} \quad |\widetilde{\mu}| = |\nu|\left(\frac{G(\widetilde{\lambda})}{\widetilde{F}(\widetilde{\lambda})}\right) < |\nu|. \tag{71}$$

Consequently, by (66) and (68),

$$-\frac{\widetilde{\mu}}{\widetilde{\nu}} = -\frac{\mu+\widetilde{m}\nu}{\nu-\widetilde{\mu}} = -\frac{\mu+\widetilde{m}\nu}{\nu}\frac{1}{1-\widetilde{\mu}/\nu} > -\frac{\mu}{\nu}-\widetilde{m}. \tag{72}$$

So far, we have verified that $\widetilde{\lambda},\widetilde{b},\widetilde{\mu}$ and $\widetilde{\nu}$ satisfy the condition of Lemma B.3. For these numbers, the first equality of (64) follows directly from (66). Considering (69), if the second inequality of (63) holds when $p > 2$, then $\widetilde{F}(\lambda)$ in (66) will be exactly the desired polynomial for both statements (i) and (ii).

Next, we check $\widetilde{G}(\lambda)$ in (66). Evidently, the definition of $\widetilde{\mu}$ infers $(\lambda-\widetilde{\lambda})\Big|(\nu G(\lambda)-\widetilde{\mu}\widetilde{F}(\lambda))$, and (71) implies that $\nu-\widetilde{\mu}$ is nonzero. So $\widetilde{G}(\lambda)$ is a well-defined monic polynomial. We discuss this part by considering two cases.
(i) $p = 2$. In this case, $\widetilde{G}(\lambda) = 1$ fulfills the second equality of (64) by (66).
(ii) $p > 2$. Taking account to (70) and (71), we can apply Lemma B.1 to polynomials $\nu G(\lambda)$ and $\widetilde{\mu}\widetilde{F}(\lambda)$ to derive

$$\widetilde{\beta}_i \in (\beta_i, \widetilde{\alpha}_{i+1}), \quad i = 1,\dots,p-2. \tag{73}$$

So the roots of $\widetilde{G}(\lambda)$ are distinct. Further, we can verify the second equality of (64) from (66) again.

Now, it remains to show the second inequality of (63) for $i = 1,\dots,p{-}2$ when $p > 2$. Observe that (69) and (73) show $\lambda\widetilde{G}(\lambda) \prec \widetilde{F}(\lambda)$. Fix an index $i \in [1\ p-2]$, we compute

$$\prod_{j=1}^{p-1}(\widetilde{\beta}_i-\widetilde{\alpha}_j) = \widetilde{F}(\widetilde{\beta}_i) = \frac{1}{\widetilde{\mu}}(\nu G(\widetilde{\beta}_i)-\widetilde{b}(\widetilde{\lambda}-\widetilde{\beta}_i)\widetilde{\nu}\widetilde{G}(\widetilde{\beta}_i)) = \frac{\widetilde{F}(\widetilde{\lambda})}{G(\widetilde{\lambda})}\prod_{j=1}^{p-1}(\widetilde{\beta}_i-\beta_j). \tag{74}$$

Note that $\frac{\widetilde{F}(\widetilde{\lambda})}{G(\widetilde{\lambda})} > 1$ by (70) and $\prod_{j>i+1}\frac{\widetilde{\beta}_i-\beta_j}{\widetilde{\beta}_i-\widetilde{\alpha}_j} \geq 1$ by (69) and (73), then (74) immediately leads to

$$\frac{\widetilde{\beta}_i-\widetilde{\alpha}_i}{\widetilde{\beta}_i-\beta_i} > \frac{\widetilde{\beta}_i-\beta_{i+1}}{\widetilde{\beta}_i-\widetilde{\alpha}_{i+1}}\prod_{j<i}\frac{\widetilde{\beta}_i-\beta_j}{\widetilde{\beta}_i-\widetilde{\alpha}_j}. \tag{75}$$

We are going to employ Lemma B.1 to estimate term $\frac{\widetilde{\beta}_i-\beta_{i+1}}{\widetilde{\beta}_i-\widetilde{\alpha}_{i+1}}$ in (75). For this, denote $\eta = \frac{1}{2}\min_{j\in[1\ p-1]}(\beta_j-\alpha_j)$. Recall that $C_1,\rho_1 < 1$, then by (62), for each $j \in [1\ p-1]$, $C_1^n\rho_1 \leq \min\{1, C_1^n\rho_j\} < \min\{1, \beta_j-\alpha_j\}$, which means $C_1^n\rho_1/2 < \min\{1,\eta\}$. As a result, by (58),

$$\begin{aligned}\left|\frac{\widetilde{m}\nu}{\mu}\right| &< \frac{\widetilde{m}}{CM} < \left(\frac{C_1^n\rho_1}{2}\right)^{2n}\frac{1}{2+2\Lambda^n} < \frac{\min\{1,\eta^{2p}\}}{2+2\Lambda^p}\\ &< \frac{\min\left\{1,\left(\frac{1}{2}\min_{1\leqslant l\leqslant p-1}\left(\alpha_{l+1}-\alpha_l\right)\right)^p\right\}\min\{1,\eta^p\}}{2+2\Lambda^p} < \frac{\min\left\{1,\left(\min_{1\leqslant l\leqslant p-1}(\alpha_{l+1}-\alpha_l)-\eta\right)^p\right\}\min\{1,\eta^p\}}{2+2\max_{1\leqslant j\leqslant p}\left|\alpha_j\prod_{l=1}^{p-1}(\alpha_j-\beta_l)\right|}.\end{aligned}$$

So, by applying Lemma B.1 to polynomials $\mu F(\lambda)$ and $-\widetilde{m}\nu\lambda G(\lambda)$, we conclude

$$\max_{1\leqslant j\leqslant p-1}(\widetilde{\alpha}_j-\alpha_j) < \eta = \frac{1}{2}\min_{j\in[1\ p-1]}(\beta_j-\alpha_j). \tag{76}$$

Then, $\beta_{i+1}-\widetilde{\alpha}_{i+1} \geq \frac{\beta_{i+1}-\alpha_{i+1}}{2}$, which together with (62) indicates

$$\frac{\widetilde{\beta}_i-\beta_{i+1}}{\widetilde{\beta}_i-\widetilde{\alpha}_{i+1}} = 1+\frac{\beta_{i+1}-\widetilde{\alpha}_{i+1}}{\widetilde{\alpha}_{i+1}-\widetilde{\beta}_i} \geq 1+\frac{\beta_{i+1}-\alpha_{i+1}}{2(\beta_{i+1}-\widetilde{\beta}_i)} \geq 1+\frac{C_1^n\rho_{i+1}}{2(\lambda_{i+2}-\widetilde{\beta}_i)} \geq 1+\frac{\Delta^n}{2^{(n+1)^2}\Lambda^{n+1}}\rho_{i+1}. \tag{77}$$

Next, we deal with $\prod_{j<i} \frac{\widetilde{\beta}_i-\beta_j}{\widetilde{\beta}_i-\widetilde{\alpha}_j}$ in (75) for $i \geq 2$. As a matter of fact, by (62), for any $j < i$,

$$\frac{\widetilde{\beta}_i - \beta_j}{\widetilde{\beta}_i - \widetilde{\alpha}_j} > \frac{\widetilde{\beta}_i - \beta_j}{\widetilde{\beta}_i - \alpha_j} = 1 - \frac{\beta_j - \alpha_j}{\widetilde{\beta}_i - \alpha_j} \geq 1 - \frac{\beta_j - \lambda_j}{\widetilde{\beta}_i - \lambda_j} > 1 - \frac{\beta_j - \lambda_j}{\lambda_i - \lambda_j} \geq 1 - \frac{(1+C_2(j))^n \rho_j}{\Delta}. \tag{78}$$

Now, if $i \geq 2$, substituting (77) and (78) into (75) yields

$$\begin{aligned}\frac{\widetilde{\beta}_i - \widetilde{\alpha}_i}{\widetilde{\beta}_i - \beta_i} &> \left(1 + \frac{\Delta^n}{2^{(n+1)^2} \Lambda^{n+1}} \rho_{i+1}\right) \prod_{j<i} \left(1 - \frac{(1+C_2(j))^n \rho_j}{\Delta}\right) \geq \left(1 + \frac{\Delta^n}{2^{(n+1)^2} \Lambda^{n+1}} \rho_{i+1}\right) \left(1 - \frac{(1+C_2(i-1))^n \rho_{i-1}}{\Delta}\right)^{i-1} \\ &> \left(1 + \frac{\Delta^n}{2^{(n+1)^2} \Lambda^{n+1}} \rho_{i+1}\right) \left(1 - \frac{(1+C_2(i-1))^n \rho_{i-1}}{\Delta}\right)^{n},\end{aligned} \tag{79}$$

where the second and third inequalities follow from (59) in Lemma B.2. Since the *Bernoulli inequality* gives

$$\left(1 - \frac{(1+C_2(i-1))^n \rho_{i-1}}{\Delta}\right)^n > 1 - \frac{n(1+C_2(i-1))^n \rho_{i-1}}{\Delta},$$

due to the third inequality in (59), (79) reduces to

$$\begin{aligned}\frac{\widetilde{\beta}_i - \widetilde{\alpha}_i}{\widetilde{\beta}_i - \beta_i} &> \left(1 + \frac{\Delta^n}{2^{(n+1)^2} \Lambda^{n+1}} \rho_{i+1}\right) \left(1 - \frac{n(1+C_2(i-1))^n \rho_{i-1}}{\Delta}\right) \\ &> \left(1 + \frac{\Delta^n}{2^{(n+1)^2} \Lambda^{n+1}} \rho_{i+1}\right) \left(1 - \frac{1}{4}\left(\frac{\Delta^n}{2^{(n+1)^2} \Lambda^{n+1}} \rho_{i+1}\right)\right) > 1 + \frac{\Delta^n}{2^{(n+1)^2+1} \Lambda^{n+1}} \rho_{i+1},\end{aligned} \tag{80}$$

where the last inequality follows from $\frac{\Delta^n}{2^{(n+1)^2} \Lambda^{n+1}} \rho_{i+1} < 1$. For $i = 1$, it is trivial from (75) and (77) that $\frac{\widetilde{\beta}_i-\widetilde{\alpha}_i}{\widetilde{\beta}_i-\beta_i} > 1 + \frac{\Delta^n}{2^{(n+1)^2} \Lambda^{n+1}} \rho_{i+1}$. So, both the two cases $i \geq 2$ and $i = 1$ lead to (80). Hence,

$$\frac{\widetilde{\beta}_i - \widetilde{\alpha}_i}{\beta_i - \alpha_i} < \frac{\widetilde{\beta}_i - \widetilde{\alpha}_i}{\beta_i - \widetilde{\alpha}_i} = 1 + \frac{1}{\frac{\widetilde{\beta}_i-\widetilde{\alpha}_i}{\widetilde{\beta}_i-\beta_i} - 1} \leq 1 + \frac{2^{(n+1)^2+1} \Lambda^{n+1}}{\Delta^n \rho_{i+1}} < C_2(i). \tag{81}$$

On the other hand, in view of (76),

$$\frac{\widetilde{\beta}_i - \widetilde{\alpha}_i}{\beta_i - \alpha_i} > \frac{\beta_i - \widetilde{\alpha}_i}{\beta_i - \alpha_i} \geq \frac{1}{2} \geq C_1. \tag{82}$$

Therefore, (63) is a direct result of (81) and (82). □

Lemma B.4. *Let $\mu, \nu, \widetilde{\lambda}$ be real numbers with $\lambda_p < \widetilde{\lambda} < \Lambda$ and $\frac{\mu}{\nu} < 0$, the following two statements hold.*
(i) If $p > 1$, for polynomials $F(\lambda)$ and $G(\lambda)$ satisfying (60)–(62) with $\bar{\lambda} = \widetilde{\lambda}$ in (61), there exist two monic polynomials $\widetilde{F}(\lambda)$ and $\widetilde{G}(\lambda)$ with distinct roots $\widetilde{\alpha}_1 < \cdots < \widetilde{\alpha}_p$ and $\widetilde{\beta}_1 < \cdots < \widetilde{\beta}_{p-1}$, respectively, such that $\widetilde{\alpha}_p = \widetilde{\lambda}$ and (63) holds for all $i \in [1\ p-1]$. In addition, for some numbers $\widetilde{m}, \widetilde{b} > 0$ and $\widetilde{\mu}, \widetilde{\nu}$ with $-\frac{\widetilde{\mu}}{\widetilde{\nu}} > -\frac{\lambda_1}{\Lambda}\frac{\mu}{\nu}$, $\widetilde{F}(\lambda)$ and $\widetilde{G}(\lambda)$ fulfill

$$\begin{cases} \mu F(\lambda) = -\widetilde{m}\nu\lambda G(\lambda) - \widetilde{b}\widetilde{\mu}\widetilde{F}(\lambda), \\ \nu(\lambda - \widetilde{\lambda})G(\lambda) = \widetilde{\mu}\widetilde{F}(\lambda) + \widetilde{b}\widetilde{\nu}(\widetilde{\lambda} - \lambda)\widetilde{G}(\lambda). \end{cases} \tag{83}$$

(ii) For $F(\lambda) = \lambda - \alpha_1$ with $\alpha_1 < \widetilde{\lambda}$ and $G(\lambda) = 1$, there are some numbers $\widetilde{m}, \widetilde{b} > 0$ and $\widetilde{\mu}, \widetilde{\nu}$ with $-\frac{\widetilde{\mu}}{\widetilde{\nu}} > -\frac{\lambda_1}{\Lambda}\frac{\mu}{\nu}$ such that polynomials $\widetilde{F}(\lambda) = \lambda - \widetilde{\lambda}$ and $\widetilde{G}(\lambda) = 1$ satisfy equation (83).

Proof. (i) Let $p \geq 2$ and set

$$\widetilde{m} = -\frac{\mu}{\nu}\frac{F(\widetilde{\lambda})}{\widetilde{\lambda}G(\widetilde{\lambda})} \quad \text{and} \quad \widetilde{F}(\lambda) = \frac{\mu F(\lambda) + \widetilde{m}\nu\lambda G(\lambda)}{\mu + \widetilde{m}\nu}. \tag{84}$$

By (60) and (62), it is clear that $\widetilde{m} > 0$. Observe that

$$\left|\frac{\widetilde{m}\nu}{\mu}\right| = \frac{\prod_{j=1}^{p}(\widetilde{\lambda} - \alpha_j)}{\widetilde{\lambda}\prod_{j=1}^{p-1}(\widetilde{\lambda} - \beta_j)} < \frac{\prod_{j=1}^{p}(\widetilde{\lambda} - \alpha_j)}{(\widetilde{\lambda} - \alpha_1)\prod_{j=1}^{p-1}(\widetilde{\lambda} - \alpha_{j+1})} = 1,$$

then $\widetilde{F}(\lambda)$ is a well-defined monic polynomial of $\deg(\widetilde{F}(\lambda)) = \deg(F(\lambda)) = p$. Now, (62) indicates $\lambda G(\lambda) \prec F(\lambda)$. By applying Lemma B.1 to polynomials $\mu F(\lambda)$ and $-\widetilde{m}\nu\lambda G(\lambda)$, it follows that the first $p-1$ roots of $\widetilde{F}(\lambda)$ satisfy

$$\widetilde{\alpha}_j \in (\alpha_j, \beta_j), \quad j = 1, \ldots, p-1. \tag{85}$$

Moreover, the definition of $\widetilde{m}$ in (84) shows $(\lambda - \widetilde{\lambda})|(\mu F(\lambda) + \widetilde{m}\nu\lambda G(\lambda))$, which yields $\widetilde{\alpha}_p = \widetilde{\lambda}$. Hence, $F(\lambda) \prec \widetilde{F}(\lambda)$.

Next, let

$$\widetilde{\mu} = \tau\nu, \quad \widetilde{b} = -\frac{\mu + \widetilde{m}\nu}{\widetilde{\mu}}, \quad \widetilde{\nu} = \frac{\widetilde{\mu} - \nu}{\widetilde{b}}, \quad \widetilde{G}(\lambda) = \frac{\nu(\lambda - \widetilde{\lambda})G(\lambda) - \widetilde{\mu}\widetilde{F}(\lambda)}{\widetilde{b}\widetilde{\nu}(\widetilde{\lambda} - \lambda)}, \tag{86}$$

where

$$\tau = \begin{cases} v_1, & p > 2, \\ v_2, & p = 2, \end{cases} \tag{87}$$

and

$$v_1 = \frac{1}{2}\frac{\min\{1, \left(\min_{1\leqslant l\leqslant p-2}(\beta_{l+1}-\beta_l)-\eta_1\right)^{p-1}\}\min\{1,\eta_1^{p-1}\}}{2+2\max_{1\leqslant j\leqslant p-1}\left|\prod_{h=1}^{p-1}(\beta_j-\widetilde{\alpha}_h)\right|}, \quad v_2 = \frac{1}{4}\min\left\{\frac{\eta_2}{\beta_1-\widetilde{\alpha}_1}, 1\right\},$$
$$\eta_1 = \min\left\{\frac{\min_{1\leqslant l\leqslant p-2}(\beta_{l+1}-\beta_l)}{4}, \min_{1\leqslant l\leqslant p-1}(\beta_l-\alpha_l)\right\}, \quad \eta_2 = \min\left\{\beta_1-\alpha_1, \frac{1}{4}\right\}.$$

Therefore, $\widetilde{b} = -\frac{\mu+\widetilde{m}v}{\widetilde{\mu}} = -\frac{\mu}{\tau v}\left(1-\frac{F(\widetilde{\lambda})}{\widetilde{\lambda}G(\widetilde{\lambda})}\right) > 0$. Note that $\tau \in (0,1)$, then (86) gives $\widetilde{\mu}/v \in (0,1)$ and $\widetilde{v}$ is nonzero. So, in view of $(\lambda-\widetilde{\lambda})|\widetilde{F}(\lambda)$, $\widetilde{G}(\lambda)$ is a well-defined monic polynomial of degree $p-1$.

As a result, plugging (84) and (86) into (83) immediately shows the validity of (83). Observe that by $\lambda G(\lambda) \prec F(\lambda)$,

$$-\frac{\widetilde{\mu}}{\widetilde{v}} = -\frac{\mu+\widetilde{m}v}{v-\widetilde{\mu}} > -\frac{\mu}{v}\left(1-\frac{F(\widetilde{\lambda})}{\widetilde{\lambda}G(\widetilde{\lambda})}\right) > -\frac{\mu}{v}\frac{\widetilde{\lambda}G(\widetilde{\lambda})-(\widetilde{\lambda}-\alpha_1)G(\widetilde{\lambda})}{\widetilde{\lambda}G(\widetilde{\lambda})} > -\frac{\lambda_1}{\Lambda}\frac{\mu}{v} > 0.$$

At last, we show the second inequality of (63) for $i = 1,\dots,p-1$. Fix an index $i \in [1\ \ p-1]$. By plugging $\lambda = \beta_i$ into the first equation of (83), we obtain $\mu F(\beta_i) = -\widetilde{b}\widetilde{\mu}\widetilde{F}(\beta_i)$, which is equivalent to

$$\prod_{j=1}^{p}(\beta_i-\widetilde{\alpha}_j) = \frac{1}{1-\frac{F(\widetilde{\lambda})}{\widetilde{\lambda}G(\widetilde{\lambda})}}\prod_{j=1}^{p}(\beta_i-\alpha_j). \tag{88}$$

Since (85) means that $\frac{\widetilde{F}(\lambda)}{(\lambda-\widetilde{\lambda})} \prec G(\lambda)$, by applying Lemma B.1 to polynomials $vG(\lambda)$ and $\widetilde{\mu}\frac{\widetilde{F}(\lambda)}{(\lambda-\widetilde{\lambda})}$, we deduce $\widetilde{\beta}_{p-1} > \beta_{p-1}$ and

$$\widetilde{\beta}_j \in (\beta_j, \widetilde{\alpha}_{j+1}), \quad j = 1,\dots,p-2. \tag{89}$$

Note that $\frac{\beta_i-\alpha_j}{\beta_i-\widetilde{\alpha}_j} > 1$ for all $j < i$ because of (85). As for $j \in [i+1\quad p-1]$, by (62) and (85),

$$\frac{\beta_i-\alpha_j}{\beta_i-\widetilde{\alpha}_j} = 1-\frac{\widetilde{\alpha}_j-\alpha_j}{\widetilde{\alpha}_j-\beta_i} \geq 1-\frac{\beta_j-\lambda_j}{\alpha_j-\beta_i} \geq 1-\frac{\beta_j-\lambda_j}{\lambda_{i+1}-\beta_i},$$

which together with (56), (59) and (62) yields

$$\frac{\beta_i-\alpha_j}{\beta_i-\widetilde{\alpha}_j} \geq 1-\frac{\beta_j-\lambda_j}{(\lambda_{i+1}-\lambda_i)-(\beta_i-\lambda_i)} \geq 1-\frac{(1+C_2(j))^n\rho_j}{2\Delta-(1+C_2(i))^n\rho_i} > \frac{1}{2}.$$

Furthermore, since $\alpha_p \in [\lambda_p, \widetilde{\lambda}) \subset [\lambda_p, \Lambda)$, (56), (59) and (62) lead to $\frac{\beta_i-\alpha_p}{\beta_i-\widetilde{\alpha}_p} \geq \frac{\lambda_{i+1}-\beta_i}{\Lambda} \geq \frac{\Delta-(1+C_2(i))^n\rho_i}{\Lambda} > \frac{\Delta}{2\Lambda}$. Consequently, it follows from (88) and (89) that

$$\frac{\widetilde{\beta}_i-\widetilde{\alpha}_i}{\beta_i-\alpha_i} > \frac{\beta_i-\widetilde{\alpha}_i}{\beta_i-\alpha_i} = \frac{1}{1-\frac{F(\widetilde{\lambda})}{\widetilde{\lambda}G(\widetilde{\lambda})}}\prod_{j\neq i}\frac{\beta_i-\alpha_j}{\beta_i-\widetilde{\alpha}_j} > \frac{1}{2^n}\frac{\Delta}{2\Lambda} = C_1.$$

Now, we prove $\widetilde{\beta}_i-\widetilde{\alpha}_i \leq C_2(i)(\beta_i-\alpha_i)$ for each $i \in [1\ p-1]$. If $p > 2$,

$$\left|\frac{\widetilde{\mu}}{v}\right| = \tau = v_1 < \frac{\min\{1, \left(\min_{1\leqslant l\leqslant p-2}(\beta_{l+1}-\beta_l)-\eta_1\right)^{p-1}\}\min\{1,\eta_1^{p-1}\}}{2+2\max_{1\leqslant j\leqslant p-1}\left|\prod_{h=1}^{p-1}(\beta_j-\widetilde{\alpha}_h)\right|}.$$

By applying Lemma B.1(i) to polynomials $vG(\lambda)$ and $\widetilde{\mu}\frac{\widetilde{F}(\lambda)}{(\lambda-\widetilde{\lambda})}$, it infers

$$\max_{1\leqslant i\leqslant p-1}(\widetilde{\beta}_i-\beta_i) < \eta_1 \leq \min_{1\leqslant i\leqslant p-1}(\beta_i-\alpha_i).$$

When $p = 2$, $\left|\frac{\widetilde{\mu}}{v}\right| = \tau < \frac{1}{2}\min\left\{\frac{\eta_2}{\beta_1-\widetilde{\alpha}_1}, 1\right\}$. Applying Lemma B.1 (ii) to polynomials $vG(\lambda)$ and $\widetilde{\mu}\frac{\widetilde{F}(\lambda)}{(\lambda-\widetilde{\lambda})}$ shows $\widetilde{\beta}_1-\beta_1 < \eta_2 \leq \beta_1-\alpha_1$. So both cases $p > 2$ and $p = 2$ result in $\max_{1\leqslant i\leqslant p-1}(\widetilde{\beta}_i-\beta_i) < \min_{1\leqslant i\leqslant p-1}(\beta_i-\alpha_i)$, which implies that $\widetilde{\beta}_i-\alpha_i < 2(\beta_i-\alpha_i)$ for each $i \in [1\ p-1]$. Then,

$$\widetilde{\beta}_i-\widetilde{\alpha}_i < \widetilde{\beta}_i-\alpha_i < 2(\beta_i-\alpha_i) \leq C_2(i)(\beta_i-\alpha_i).$$

(ii) Let $\widetilde{m} = -\frac{\mu}{v}\frac{\widetilde{\lambda}-\alpha_1}{\widetilde{\lambda}}$, $\widetilde{\mu} = \frac{1}{2}v$, $\widetilde{b} = -\frac{\mu}{v}\frac{2\alpha_1}{\widetilde{\lambda}}$ and $\widetilde{v} = \frac{1}{4}\frac{v^2}{\mu}\frac{\widetilde{\lambda}}{\alpha_1}$. Clearly, $\widetilde{F}(\lambda) = \lambda-\widetilde{\lambda}$ and $\widetilde{G}(\lambda) = 1$ satisfy equation (83). □

Proof of Lemma 4.1. For $j = n-1,\dots,1$, we construct a series of numbers $\lambda^*_{j+1}, b_{j+1}, m_{j+1}, \mu_j, v_j$ and polynomials $F_j(\lambda)$, $G_j(\lambda)$ on the basis of μ_{j+1}, v_{j+1}, and $F_{j+1}(\lambda)$, $G_{j+1}(\lambda)$, according to the following strategies:

(a) if $j+1 \in [2\ n] \setminus \bigcup_{l=1}^{T-1} R_l$, we apply Lemma B.3 with $F(\lambda) = F_{j+1}(\lambda)$, $G(\lambda) = G_{j+1}(\lambda)$, $\widetilde{m} = m_{j+1}$, $\mu = \mu_{j+1}$, $\nu = \nu_{j+1}$ to obtain $\lambda^*_{j+1} = \widetilde{\lambda}, b_{j+1} = \widetilde{b}$, $\mu_j = \widetilde{\mu}, \nu_j = \widetilde{\nu}$ and $F_j(\lambda) = \widetilde{F}(\lambda)$, $G_j(\lambda) = \widetilde{G}(\lambda)$, where

$$m_{j+1} \triangleq \begin{cases} M_{l+1}, & \text{if} \quad j = r_l(q_l) \quad \text{for some} \quad l \in [1\ T-2], \\ M_{j+1-n+m}, & \text{if} \quad j > T-2+n-m; \end{cases} \tag{90}$$

(b) if $j+1 \in R_l$ for some $l \in [1\ T-1]$, we set $\lambda^*_{j+1} = s_l(q_l - i + 1)$ where $j+1 = r_l(i)$ and apply Lemma B.4 with $F(\lambda) = F_{j+1}(\lambda)$, $G(\lambda) = G_{j+1}(\lambda)$, $\widetilde{\lambda} = \lambda^*_{j+1}$, $\mu = \mu_{j+1}$, $\nu = \nu_{j+1}$ to obtain $m_{j+1} = \widetilde{m}, b_{j+1} = \widetilde{b}$, $\mu_j = \widetilde{\mu}, \nu_j = \widetilde{\nu}$, and $F_j(\lambda) = \widetilde{F}(\lambda)$, $G_j(\lambda) = \widetilde{G}(\lambda)$.

The construction of the desired quantities in the above two strategies is accomplished in the proof of Lemmas B.3–B.4. Now, we shall use the induction method to show that either strategy (a) or strategy (b) can be implemented for each $j = n-1, \ldots, 1$. First, let $j = n-1$. Observe that $T \le m$. It is easy to compute $\max \bigcup_{l=1}^{T-1} R_l = T - 1 + \sum_{l=1}^{T-1} R_l = T - 1 + n - m \le n - 1$. Hence, $n \notin \bigcup_{l=1}^{T-1} R_l$. Noting that the roots of $F_n(\lambda)$ and $G_n(\lambda)$ are $\{\lambda_i\}_{i=1}^n$ and $\{\lambda_i + \rho_i\}_{i=1}^{n-1}$, respectively, then (62) holds obviously. In addition, since (26) and (90) yield $-\frac{\mu_n}{\nu_n} > CM$ and $m_n = M_n \in (0, M)$, by applying Lemma B.3 (i) with $F(\lambda) = F_n(\lambda)$, $G(\lambda) = G_n(\lambda)$, $\widetilde{m} = m_n$, $\mu = \mu_n$, $\nu = \nu_n$, we can find some numbers $\lambda^*_n = \widetilde{\lambda}, b_n = \widetilde{b}, \mu_{n-1} = \widetilde{\mu}, \nu_{n-1} = \widetilde{\nu}$ with $\lambda^*_n, b_n > 0$ and $-\frac{\mu_{n-1}}{\nu_{n-1}} > -\frac{\mu_n}{\nu_n} - m_n > C\left(\frac{\Lambda}{\lambda_1}\right)^{n-1} \frac{\Lambda M}{\Lambda - \lambda_1}$, and two monic polynomials $F_{n-1}(\lambda) = \widetilde{F}(\lambda)$, $G_{n-1}(\lambda) = \widetilde{G}(\lambda)$ with distinct roots such that $\deg F_{n-1}(\lambda) = m-1$, $\deg G_{n-1}(\lambda) = m-2$ and (27) with $j+1 = n$ holds. So, strategy (a) is applicable and both (i) and (ii) with $j+1 = n$ are true for these $\lambda^*_n, b_n, \mu_{n-1}, \nu_{n-1}, F_{n-1}(\lambda), G_{n-1}(\lambda)$.

Now, assume that we have constructed the required $\{(\lambda^*_j, b_j, m_j)\}_{j=n-r+1}^n$, $\{(\mu_j, \nu_j)\}_{j=n-r}^{n-1}$ and $\{F_j(\lambda), G_j(\lambda)\}_{j=n-r}^{n-1}$ for some $r \in [1\ n-2]$ by following either strategy (a) or strategy (b), so that properties (i) and (ii) hold for $j = n-1, \ldots, n-r$. Considering Lemmas B.3 and B.4, we write $F_{n-j}(\lambda) = \prod_{i=1}^{z_{n-j}} (\lambda - \alpha_{n-j}(i))$ with $\alpha_{n-j}(1) < \cdots < \alpha_{n-j}(z_{n-j})$ and $G_{n-j}(\lambda) = \prod_{i=1}^{z_{n-j}-1} (\lambda - \beta_{n-j}(i))$ with $\beta_{n-j}(1) < \cdots < \beta_{n-j}(z_{n-j} - 1)$, $j = 0, 1, \ldots, r$, where $z_{n-j} = \deg F_{n-j}(\lambda)$. Here, for each $j = 0, \ldots, r-1$,

$$z_{n-j-1} = \begin{cases} z_{n-j} - 1, & \text{if} \quad n - j \notin \cup_{l=1}^{T-1} R_l, \\ z_{n-j}, & \text{otherwise}. \end{cases} \tag{91}$$

The roots of $\{F_{n-j}(\lambda), G_{n-j}(\lambda)\}_{j=1}^r$ fulfill

$$\begin{cases} \alpha_{n-j+1}(z_{n-j}) < \alpha_{n-j}(z_{n-j}) < \lambda_{z_{n-j}+1}, & \text{if} \quad z_{n-j+1} - 1 = z_{n-j}, \\ \alpha_{n-j}(z_{n-j}) = \lambda^*_{n-j+1} < \Lambda, & \text{if} \quad z_{n-j+1} = z_{n-j}. \end{cases} \tag{92}$$

Furthermore, if $z_{n-r} > 1$, for each $j \in [1\ r]$ and $i \in [1\ z_{n-j} - 1]$,

$$\begin{cases} \alpha_{n-j}(i) \in (\alpha_{n-j+1}(i), \beta_{n-j+1}(i)), \\ C_1(\beta_{n-j+1}(i) - \alpha_{n-j+1}(i)) \le \beta_{n-j}(i) - \alpha_{n-j}(i) \le C_2(i)(\beta_{n-j+1}(i) - \alpha_{n-j+1}(i)). \end{cases} \tag{93}$$

Recall that $\alpha_n(i) = \lambda_i$ and $\beta_n(i) = \lambda_i + \rho_i$, (93) implies that for $i \in [1\ z_{n-j} - 1]$,

$$\beta_{n-j}(i) - \alpha_{n-j}(i) > C_1^j(\beta_n(i) - \alpha_n(i)) = C_1^j \rho_i. \tag{94}$$

Moreover, by (93) again,

$$\begin{aligned} \beta_{n-j}(i) - \alpha_n(i) &= \beta_{n-j}(i) - \alpha_{n-j}(i) + \alpha_{n-j}(i) - \alpha_n(i) \\ &< C_2(i)(\beta_{n-j+1}(i) - \alpha_{n-j+1}(i)) + (\beta_{n-j+1}(i) - \alpha_n(i)) \le (1 + C_2(i))(\beta_{n-j+1}(i) - \alpha_n(i)), \end{aligned}$$

then a straightforward calculation leads to

$$\alpha_n(i) < \alpha_{n-j}(i) < \beta_{n-j}(i) < \alpha_n(i) + (1 + C_2(i))^j(\beta_n(i) - \alpha_n(i)). \tag{95}$$

Note that by (59) in Lemma B.2, $(1 + C_2(i))^j(\beta_n(i) - \alpha_n(i)) < (1 + C_2(i))^n \rho_i < \Delta$. So by virtue of (94) and (95), for $j \in [1\ r]$ and $i \in [1\ z_{n-j} - 1]$,

$$\lambda_i + C_1^n(i)\rho_i < \alpha_{n-j}(i) + C_1^n(i)\rho_i < \beta_{n-j}(i) < \lambda_i + (1 + C_2(i))^n \rho_i < \lambda_{i+1}. \tag{96}$$

Next, by (92) and (96), if $z_{n-r} = z_{n-r+1} - 1$,

$$\lambda_{z_{n-r}} < \alpha_{n-r+1}(z_{n-r}) < \alpha_{n-r}(z_{n-r}) < \lambda_{z_{n-r}+1} < \Lambda. \tag{97}$$

For $z_{n-r} = z_{n-r+1}$, we remark that $n - r + 1 \in R_l$ for some $l \in [1\ T-1]$. Observe from (91) that

$$z_{n-r} = T - 1 - (T - l - 1) = l. \tag{98}$$

Let $n - r + 1 = r_l(i)$. If we suppose $s_l(q_l - i + 1) = \lambda_h$ for some $h \in [1\ m]$, then $h \ge t_h \ge l + 1$ because of (20). As a consequence,

$$s_l(q_l - i + 1) \ge \lambda_{l+1}. \tag{99}$$

By (4), (28), (92) and (99), we deduce

$$\alpha_{n-r}(z_{n-r}) = \lambda^*_{n-r+1} = s_l(q_l - i + 1) \ge \lambda_{l+1} > \lambda_l = \lambda_{z_{n-r}}, \quad n - r + 1 = r_l(i).$$

So no matter applying strategy (a) or (b), it always infers

$$\alpha_{n-r}(z_{n-r}) \in [\lambda_{z_{n-r}}, \Lambda). \tag{100}$$

We now verify that at least one of the strategies (a) and (b) is valid for $j = n - r - 1$. It is discussed by two cases.

Case 1: $n - r \notin \cup_{l=1}^{T-1} R_l$. Because of (91), we estimate z_{n-r} directly by $z_{n-r} \geq m - \left(n - 2 - \sum_{h=1}^{T-1} q_h\right) = m - (n - 2 - (n - m)) = 2$. Now, $-\frac{\mu_{n-r}}{\nu_{n-r}} > C(\frac{\Lambda}{\lambda_1})^{n-r} \frac{\Lambda M}{\Lambda - \lambda_1}$, it thus gives $-\frac{\mu_{n-r}}{\nu_{n-r}} > CM$. Clearly, $m_{n-r} \in (0, M)$ by (90). So combining (96) and (100), it shows that the assumptions of Lemma B.3 are fulfilled. Therefore, Lemma B.3 is applicable and strategy (a) works. We thus conclude properties (i) and (ii) hold for $j = n - r - 1$ by Lemma B.3 and

$$-\frac{\mu_{n-r-1}}{\nu_{n-r-1}} > -\frac{\mu_{n-r}}{\nu_{n-r}} - m_{n-r} > C\left(\frac{\Lambda}{\lambda_1}\right)^{n-r} \frac{\Lambda M}{\Lambda - \lambda_1} - M > C\left(\frac{\Lambda}{\lambda_1}\right)^{n-r-1} \frac{\Lambda M}{\Lambda - \lambda_1}.$$

Case 2: $n-r \in R_l$ for some $l \in [1\ T-1]$. Let $n-r = r_l(i)$. If $n-r < \max R_l$, then $n-r+1 \in R_l$, and hence $F_{n-r}(\lambda)$ is constructed by strategy (b). In view of Lemma B.4, the maximal root of $F_{n-r}(\lambda)$ is equal to λ^*_{n-r+1}, which shows $s_l(q_l - i + 1) > s_l(q_l - i) = \lambda^*_{n-r+1} = \alpha_{n-r}(z_{n-r})$, since $n - r + 1 = r_l(i + 1)$. If $n - r = \max R_l$, then $F_{n-r}(\lambda)$ is constructed by strategy (a). As in (98), $z_{n-r} = l$ and hence by (97), $\alpha_{n-r}(z_{n-r}) < \lambda_{z_{n-r}+1} = \lambda_{l+1}$. So, (99) yields

$$s_l(q_l - i + 1) > \alpha_{n-r}(z_{n-r}). \tag{101}$$

We thus conclude (101) is always true in the both cases.

Now, by (100), $\alpha_{n-r}(z_{n-r}) \in [\lambda_{z_{n-r}}, s_l(q_l - i + 1))$. This combined with (96) and (100) verifies the assumptions of Lemma B.4 with $\lambda^* = s_l(q_l - i + 1)$. Hence, Lemma B.4 is applicable and strategy (b) can be implemented. So, properties (i) and (ii) hold for $j = n - r - 1$ due to Lemma B.4 and $-\frac{\mu_{n-r-1}}{\nu_{n-r-1}} > -\frac{\mu_{n-r}}{\nu_{n-r}} \frac{\lambda_1}{\Lambda} > C\left(\frac{\Lambda}{\lambda_1}\right)^{n-r-1} \frac{\Lambda M}{\Lambda - \lambda_1}$. The induction is completed.

Finally, recall that $\deg(\tilde{F}(\lambda)) = \deg(F(\lambda)) - 1$ in Lemmas B.3 and B.4, therefore applying strategies (a) and (b) repeatedly dwindles $\deg(F_1(\lambda))$ down to 1 and $\deg(G_1(\lambda))$ to 0. □

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