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Further results on inverse eigenvalue problem for mass–spring–inerter systems ☆

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ABSTRACT

This paper is concerned with the inverse eigenvalue problem (IEP) for "fixed–fixed" mass–spring–inerter systems. Unlike the "fixed–free" case, a "fixed–fixed" system has its both ends attached to the ground. This brings some essential differences in the IEP. We find that the construction strategy proposed in Liu et al. (2022) cannot be readily applied here. So we endeavor to develop a new strategy in this paper to solve the problem and accordingly derive a necessary and sufficient condition for the "fixed–fixed" case.

1. Introduction

Natural frequency assignment is a basic problem in classical vibration theory. It is often mathematically associated with the reconstruction of specific structured matrices from designated spectral data [1]. A typical inverse eigenvalue problem (IEP) in this regard concerns a mass–spring system, in which each mass is connected to its neighbor by a spring. Accordingly, the matrices to be sought are the mass and stiffness matrices. This IEP was solved by translating it into an inverse problem for a Jacobi matrix.

A prominent property of a Jacobi matrix is that its eigenvalues are distinct and strictly interlace the eigenvalues of its truncated matrix. Conversely, [2,3] proved that given a set of interlacing real numbers $0 \le \lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n$, there exists a unique Jacobi matrix such that $\{\lambda_i\}_{i=1}^{n}$ and $\{\mu_i\}_{i=1}^{n-1}$ are the eigenvalues of the Jacobi matrix and its truncated matrix, respectively. It was based on these results that researchers recovered a mass–spring system in the IEP [4,5]. More precisely, they untangled the mass and stiffness matrices from a Jacobi matrix, which was constructed according to the prescribed eigenvalues. Variants of IEPs for Jacobi operators and modified mass–spring systems are also addressed in the literature [5–8]. Among these studies, it is noteworthy that for physical realization, the prescribed eigenvalues must be distinct due to the inherent property of Jacobi matrices mentioned above.

But the situation changes when involving inerters. The inerter is a new mechanical element first proposed to complete the force-current analogy between mechanical and electrical systems in [9]. Owing to the special merits in practice, inerters are applied to many engineering fields such as building vibration control, landing gears, vibration isolators, train suspensions, vehicle suspensions, and so on [10–15]. Recall that the masses should be appropriately chosen in the IEP of a mass–spring system. But when inerters are introduced in parallel to the springs, [16] found that as long as the given eigenvalues are positive and distinct, the IEP for

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the corresponding mass-chain system can be solved with all the masses arbitrarily taken. This demonstrates the superiority of using inerters since masses are frequently fixed in practice. Besides, [16] pointed out that a mass–spring–inerter system may have multiple eigenvalues. Such a phenomenon can be primarily ascribed to the involvement of inerters, which renders the IEP no longer a Jacobi inverse eigenvalue problem. So, tools for Jacobi matrices fail to work here. Recently, [17] solved the IEP for "fixed–free" mass–spring–inerter systems by deriving a necessary and sufficient condition, under which the natural frequencies can be designated arbitrarily positive. The term "fixed–free" means one end of the mass-chain system is attached to the ground while the other end is hanging free. Specifically speaking, [17] shows that if $\lambda_1 < \cdots < \lambda_m$ are m positive real numbers, where λ_i has multiplicity t_i, then these numbers can be assigned as the natural frequencies of a "fixed–free" mass–spring–inerter system if and only if $t_i \le i$, $i = 1, \ldots, m$. In addition, m masses of the system are allowed to be fixed arbitrarily beforehand in the assignment.

This paper concerns the IEP for another common mass-chain system, called "fixed–fixed" system [5]. In contrast to the "fixed–free" case, both ends of the "fixed–fixed" system are attached to the ground, as shown in Fig. 1. The free vibration equation of a "fixed–fixed" mass–spring–inerter system also has the form

$$(\boldsymbol{M} + \boldsymbol{B})\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = 0,$$

though the above three matrices turn out to be

$$\boldsymbol{M} = \mathrm{diag}\{m_1, m_2, \ldots, m_n\}, \tag{1}$$

$$\boldsymbol{K} = \begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & -k_n & k_n+k_{n+1} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} b_1+b_2 & -b_2 & & & \\ -b_2 & b_2+b_3 & -b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & b_{n-1}+b_n & -b_n \\ & & & -b_n & b_n+b_{n+1} \end{bmatrix}. \tag{2}$$

Here, real numbers $m_j > 0$ for $j = 1, \ldots, n$ and $k_j > 0$, $b_j \ge 0$ for $j = 1, \ldots, n+1$ represent the masses, spring stiffnesses and inertances. We shall prove that the IEP for a "fixed–fixed" mass–spring–inerter system is solvable if and only if $t_i \le i+1, i = 1, \ldots, m$, where t_i is the multiplicity of λ_i and $0 < \lambda_1 < \cdots < \lambda_n$ are the prescribed eigenvalues.

Note that if we replace one of the terminal ground by a new mass m_{n+1}, the system becomes "fixed–free" with $n+1$ degree of freedom. Therefore, it comes quite naturally to apply the construction strategy proposed in [17] to the corresponding "fixed–free" mass–spring–inerter system and thereby construct the desired matrices $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$, as illustrated in Example 1.1 below.

Example 1.1. Let λ_1 be a positive number. We shall construct

$$\boldsymbol{M} = \mathrm{diag}\{m_1, m_2\}, \quad \boldsymbol{K} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} b_1+b_2 & -b_2 \\ -b_2 & b_2+b_3 \end{bmatrix}$$

such that $(\lambda - \lambda_1)^2 \,\Big|\, \det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B}))$. For this, we introduce one more mass m_3. We can recover

$$\boldsymbol{M}' = \mathrm{diag}\{m_1, m_2, m_3\}, \quad \boldsymbol{K}' = \begin{bmatrix} k_1+k_2 & -k_2 & \\ -k_2 & k_2+k_3 & -k_3 \\ & -k_3 & k_3 \end{bmatrix}, \quad \boldsymbol{B}' = \begin{bmatrix} b_1+b_2 & -b_2 & \\ -b_2 & b_2+b_3 & -b_3 \\ & -b_3 & b_3 \end{bmatrix}$$

by applying the construction strategy in [17] and then obtain $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$ directly.

Unfortunately, this idea is not always viable. It fails in the following example.

Example 1.2. Let $t_1 = 2, t_2 = t_3 = 3$ and $\lambda_1, \lambda_2, \lambda_3$ be three positive numbers satisfying

$$\lambda_3 - \lambda_2 - 1 > \frac{\lambda_3}{2} > \lambda_1 > 8(\lambda_2 - \lambda_1) > 16.$$

But the eigenvalues in Example 1.2 meet the condition of Theorem 2.1 in Section 2, which means that there exist some $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$ in the form of (1)–(2) such that $\prod_{i=1}^{3}(\lambda-\lambda_i)^{t_i} \,\Big|\, \det(\boldsymbol{K}-\lambda(\boldsymbol{M}+\boldsymbol{B}))$. Not just Example 1.2, actually the construction strategy proposed in [17] for the "fixed–free" case usually fails in the "fixed–fixed" situation. The fundamental cause of this matter is attributed to the different configuration of matrix $\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B})$. The key idea employed in [17] and this paper is to strictly restrict the roots of polynomials $F_j(\lambda)$ and $G_j(\lambda)$ (see Section 3) in the vicinity of the designated eigenvalues $\lambda_1, \ldots, \lambda_m$, so that the constructed parameters m_i, k_i, and b_i are positive. However, the previous strategy of [17] can no longer guarantee the generated roots all in the required positions. So, we have to develop an effective approach, both in strategies and analysis, for "fixed–fixed" mass–spring–inerter systems. In fact, the newly devised construction strategy comprises three steps, one more than the "fixed–free" case. The complicated problem here necessitates a more intricate and sophisticated proof, and a significant portion of this paper is dedicated to expounding upon the added step. As in the case of [17], the construction admits a total of m masses being taken arbitrarily, where m is the number of the given distinct eigenvalues.

This paper is built up as follows. In Section 2, we present the main result by deducing a necessary and sufficient condition of the IEP for "fixed–fixed" mass–spring–inerter systems. The necessity and sufficiency of the theorem are proved in Sections 3 and 4, respectively. Conclusions are drawn in Section 5.

Notation: We shall adopt the following notations hereinafter. Let "[ ]" represent an interval of integers. For instance, $[n] = \{1, 2, \ldots, n\}$ and $[m\ n] = \{m, m+1, \ldots, n\}$, $m, n \in \mathbb{N}^+$, where $\mathbb{N}^+$ is the set of positive integers. Moreover, denote $f(\lambda)|g(\lambda)$ if polynomial $f(\lambda)$ is divisible by polynomial $g(\lambda)$. Denote by $(f(\lambda), g(\lambda))$ the greatest common divisor of two polynomials $f(\lambda)$ and $g(\lambda)$.

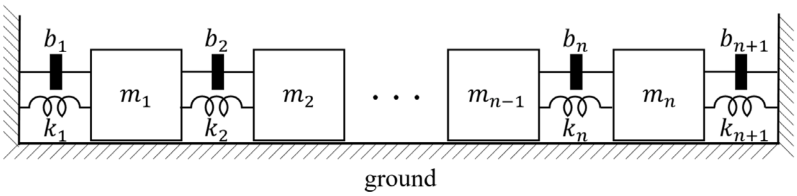

Fig. 1. Mass-chain system with inerters.

2. Main result

Since the roots of the following characteristic equation

$$\det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B})) = 0$$

determine the natural frequencies of the mass–spring–inerter system, consistent with [16,17], we use the terms "eigenvalues" and the "natural frequencies" interchangeably. For the "fixed–fixed" case, we put forward the IEP as follows.

Problem 1. Arbitrarily given a set of real numbers $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, is it possible to recover matrices $\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{B}$ in (1) and (2) by choosing $m_j > 0$, $k_j > 0$ and $b_j \geq 0$ for $j = 1, \ldots, n$, so that the n eigenvalues of matrix pencil $\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B})$ are exactly λ_i, $i = 1, \ldots, n$?

The IEP has been completely solved by [17] for "fixed–free" systems. But the construction strategy developed therein cannot work anymore, as explained in the Introduction. We are thus forced to seek a new construction strategy. Still, we expect an analogous answer of the IEP in this paper. It is indeed the case. Here is the main result of the paper.

Theorem 2.1. *Let $\prod_{i=1}^{m}(\lambda - \lambda_i)^{t_i}$ be a polynomial with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m$ and $\sum_{i=1}^{m} t_i = n$. Then, there exist some matrices $\boldsymbol{K}, \boldsymbol{M}, \boldsymbol{B}$ defined by (1)–(2) such that*

$$\prod_{i=1}^{m}(\lambda - \lambda_i)^{t_i} \,\Big|\, \det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B})) \tag{3}$$

if and only if

$$t_i \leq i + 1, \quad i = 1, \ldots, m. \tag{4}$$

Remark 2.1. The number of the unknown parameters is $3n + 2$, which is larger than n (the number of the eigenvalues). So the solution of the IEP is generally not unique. Usually, the unicity requires more constraints on the eigenpairs (eigenvalues and eigenvectors) [5,7]. But in this paper, the solvability issue is the major concern.

Remark 2.2. Similar to [17], our new construction strategy preserves m degree of freedom in the recover of the mass matrix $\boldsymbol{M}$. More precisely, Proposition 4.1 in Section 4 interprets:

> The construction in Theorem 2.1 admits a total of m masses being taken arbitrarily, where m is the number of the given distinct eigenvalues λ_i.

In particular, if $m = n$ in Theorem 2.1, Remark 2.2 indicates that all the masses in the system can be prescribed arbitrarily. The following corollary is an immediate result of Theorem 2.1 and Remark 2.2.

Corollary 2.1. *Let $m_i > 0, i = 1, 2, \ldots, n$ and $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ be some arbitrarily given real numbers. Then, $\lambda_i, i = 1, 2, \ldots, n$ can be assigned as the eigenvalues of a "fixed–fixed" mass–spring–inerter system by appropriately adjusting $n + 1$ springs and $n + 1$ inerters.*

3. Proof of the necessity of Theorem 2.1

Borrowing the idea from [17, Section 3], the argument in this section becomes easy. To be self-contained, some necessary definitions and lemmas of [17, Section 3] are reviewed below.

Let $f_1(\lambda) = k_1 - \lambda(m_1 + b_1)$ and $g_1(\lambda) = 1$. Moreover, by introducing a new mass m_{n+1}, for $j = 2, \ldots, n+1$, can define $f_j(\lambda) = \det \boldsymbol{F}_j$ and $g_j(\lambda) = \det \boldsymbol{G}_j$, where

$$\boldsymbol{F}_j = \begin{bmatrix} k_1+k_2-\lambda(m_1+b_1+b_2) & -k_2+\lambda b_2 & & & \\ -k_2+\lambda b_2 & k_2+k_3-\lambda(m_2+b_2+b_3) & -k_3+\lambda b_3 & & \\ & \ddots & \ddots & & \\ & & -k_{j-1}+\lambda b_{j-1} & k_{j-1}+k_j-\lambda(m_{j-1}+b_{j-1}+b_j) & -k_j+\lambda b_j \\ & & & -k_j+\lambda b_j & k_j-\lambda(m_j+b_j) \end{bmatrix},$$

$$\boldsymbol{G}_j = \begin{bmatrix} k_1+k_2-\lambda(m_1+b_1+b_2) & -k_2+\lambda b_2 & & & \\ -k_2+\lambda b_2 & k_2+k_3-\lambda(m_2+b_2+b_3) & -k_3+\lambda b_3 & & \\ & \ddots & \ddots & & \\ & & -k_{j-2}+\lambda b_{j-2} & k_{j-2}+k_j-\lambda(m_{j-2}+b_{j-2}+b_{j-1}) & -k_{j-1}+\lambda b_{j-1} \\ & & & -k_{j-1}+\lambda b_{j-1} & k_{j-1}+k_j-\lambda(m_{j-1}+b_{j-1}+b_j) \end{bmatrix}.$$

Particularly, we remark that $g_{n+1}(\lambda) = \det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B}))$. The following definition on "interlacing polynomials" plays a role throughout the paper.

Definition 3.1 (*[17]*). Let $f(\lambda)$ and $g(\lambda)$ be two polynomials with degree s, where $s \in \mathbb{N}^+$. Suppose $f(\lambda)$ and $g(\lambda)$ both have s distinct real roots, which are denoted by $\alpha_1 < \cdots < \alpha_s$ and $\beta_1 < \cdots < \beta_s$, respectively. We say $g(\lambda) \prec f(\lambda)$, if their leading coefficients are of the same sign and

$$\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_s < \alpha_s.$$

Since $\{f_j\}_{j=1}^{n+1}$ and $\{g_j\}_{j=1}^{n+1}$ are exactly the polynomials studied by [17], the following two important lemmas still work.

Lemma 3.1 (*[17]*). *The polynomials $\{f_j(\lambda)\}_{j=1}^{n+1}$ and $\{g_j(\lambda)\}_{j=1}^{n+1}$ satisfy*

$$\begin{cases} f_{j+1}(\lambda) = (-\lambda m_{j+1})g_{j+1}(\lambda) + (k_{j+1} - \lambda b_{j+1})f_j(\lambda), \\ g_{j+1}(\lambda) = f_j(\lambda) + (k_{j+1} - \lambda b_{j+1})g_j(\lambda), \end{cases} \quad j = 1, \ldots, n, \tag{5}$$

with $g_1(\lambda) = 1$ and $f_1(\lambda) = k_1 - \lambda(m_1 + b_1)$.

Lemma 3.2 (*[17]*). *Suppose for some $j \in [n]$,*

$$\frac{(-\lambda)g_j(\lambda)}{(f_j(\lambda), g_j(\lambda))} \prec \frac{f_j(\lambda)}{(f_j(\lambda), g_j(\lambda))}. \tag{6}$$

Then $\frac{(-\lambda)g_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$. Moreover,
(i) if $b_{j+1} \neq 0$ and $k_{j+1} - \lambda b_{j+1} | \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$, then $(f_{j+1}(\lambda), g_{j+1}(\lambda)) = (f_j(\lambda), g_j(\lambda))(\lambda - \frac{k_{j+1}}{b_{j+1}})$ and $\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{-f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$;
(ii) if $b_{j+1} \neq 0$ and $k_{j+1} - \lambda b_{j+1} \nmid \frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))}$, then $(f_{j+1}(\lambda), g_{j+1}(\lambda)) = (f_j(\lambda), g_j(\lambda))$ and $\frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))} \prec \frac{f_j(\lambda)(k_{j+1}-\lambda b_{j+1})}{(f_j(\lambda),g_j(\lambda))}$;
(iii) if $b_{j+1} = 0$, then $(f_{j+1}(\lambda), g_{j+1}(\lambda)) = (f_j(\lambda), g_j(\lambda))$ and $\frac{f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))} \prec \frac{g_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$ and $\frac{(-\lambda)f_j(\lambda)}{(f_j(\lambda),g_j(\lambda))} \prec \frac{f_{j+1}(\lambda)}{(f_{j+1}(\lambda),g_{j+1}(\lambda))}$.

With the preliminaries ready, we now prove the necessity of Theorem 2.1. Let $f(\lambda)$ be a polynomial whose roots $\{z_i\}_{i=1}^p$ are all real and $z_1 < z_2 < \cdots < z_p$. Denote $\xi(f(\lambda), z_i)$ as the multiplicity of root z_i and $\zeta(f(\lambda), \alpha) = \max\{i \in [p] : z_i < \alpha\}$ for any $\alpha \in \mathbb{R}$.

The proof of the necessity of Theorem 2.1. Observe that $\det(\boldsymbol{K} - \lambda(\boldsymbol{M} + \boldsymbol{B})) = g_{n+1}(\lambda)$, then (3) means $\lambda_i, 1 \le i \le m$ are m distinct roots of $g_{n+1}(\lambda)$ with multiplicities t_i. Since $\frac{(-\lambda)g_1(\lambda)}{(f_1(\lambda),g_1(\lambda))} \prec \frac{f_1(\lambda)}{(f_1(\lambda),g_1(\lambda))}$ by Lemma 3.1, repeatedly applying Lemma 3.2 shows that (6) holds for all $j \in [n+1]$. Thus, given any $1 \le i \le m$,

$$\zeta\left(\frac{g_{n+1}(\lambda)}{(f_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) \ge \zeta\left(\frac{f_{n+1}(\lambda)}{(f_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) - 1. \tag{7}$$

Moreover, it indicates that all the roots of $\frac{g_{n+1}(\lambda)}{(f_{n+1}(\lambda),g_{n+1}(\lambda))}$ are distinct. So, for each $1 \le i \le m$, $\xi\left(\frac{g_{n+1}(\lambda)}{(f_{n+1}(\lambda),g_{n+1}(\lambda))}, \lambda_i\right) \le 1$. Consequently,

$$\xi\left((f_{n+1}(\lambda), g_{n+1}(\lambda)), \lambda_i\right) = \xi\left(g_{n+1}(\lambda), \lambda_i\right) - \xi\left(\frac{g_{n+1}(\lambda)}{(g_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) \ge t_i - 1. \tag{8}$$

Furthermore, since Lemma 3.2 holds, by almost the same argument as that for Eq.(19) in [17, Section 3], we conclude

$$\zeta\left(\frac{f_{n+1}(\lambda)}{(f_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) \ge \xi\left((f_{n+1}(\lambda), g_{n+1}(\lambda)), \lambda_i\right). \tag{9}$$

As a result, by (7), (8) and (9), for each $i = 1, \ldots, m$,

$$i - 1 = \zeta\left(g_{n+1}, \lambda_i\right) \ge \zeta\left(\frac{g_{n+1}(\lambda)}{(f_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) \ge \zeta\left(\frac{f_{n+1}(\lambda)}{(f_{n+1}(\lambda), g_{n+1}(\lambda))}, \lambda_i\right) - 1 \ge \xi\left((f_{n+1}(\lambda), g_{n+1}(\lambda)), \lambda_i\right) - 1 \ge t_i - 2,$$

which completes the proof. □

4. Proof of sufficiency of Theorem 2.1

Now, we are in a position to treat the core part of the paper. We shall show

Step 1	Construct b_{n+1} and k_{n+1} by Lemma 4.1
Step 2	Construct m_i, b_i and k_i, $i = n - q_1 + 1, \cdots, n$ by Lemma 4.3
Step 3	Construct m_i, b_i and k_i, $i = 1, \cdots, n - q_1$ by Lemma 4.4

Fig. 2. Construction strategy.

Proposition 4.1. *Let $M_1, \ldots, M_m > 0$ be m real numbers and $\prod_{i=1}^{m}(\lambda-\lambda_i)^{t_i}$ be a polynomial with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m$ and $\sum_{i=1}^{m} t_i = n$. Suppose $t_i \leq i+1$ for each $i = 1, \ldots, m$. Then, there exist some $m_j > 0$, $k_j > 0$, $b_j \geq 0$, $j = 1, \ldots, n$ and m distinct indices $i_h, h = 1, \ldots, m$ such that $m_{i_h} = M_h$ and*

$$\prod_{i=1}^{m}(\lambda-\lambda_i)^{t_i}\Big|\det(\boldsymbol{K}-\lambda(\boldsymbol{M}+\boldsymbol{B})).$$

With the above proposition, the sufficiency of Theorem 2.1 becomes trivial. So, this section is actually a proof of Proposition 4.1.

4.1. Construction strategy

Our aim is to find some sequences of $\{m_j\}_{j=1}^{n}$, $\{k_j, b_j\}_{j=1}^{n+1}$ and m indices i_h such that $m_{i_h} = M_h$, $h = 1, \ldots, m$ and

$$\prod_{i=1}^{m}(\lambda-\lambda_i)^{t_i}|g_{n+1}(\lambda), \quad g_{n+1}(\lambda) = \det(\boldsymbol{K}-\lambda(\boldsymbol{M}+\boldsymbol{B})). \tag{10}$$

The construction can be divided into three steps contained in Fig. 2, where Lemmas 4.1, 4.3 and 4.4 appear later in Section 4.2.

We recall some important definitions in [17, Section 4]. Let $T = \max_{1\leqslant i\leqslant m} t_i$. If $T > 1$, define the sets of multiple eigenvalues by

$$S_j = \{\lambda_i : t_i \geq j+1\} \quad \text{and} \quad q_j = |S_j|, \quad j = 1, \ldots, T-1. \tag{11}$$

We reorder the elements of S_j as $s_j(1) < \cdots < s_j(q_j)$ and remark that $\sum_{j=1}^{T-1} q_j = n - m$. Now, by Lemma 3.1, $(f_1(\lambda), g_1(\lambda)) = 1$. Moreover, from Lemma 3.2 (also see [17, Eq. (15)]), for each $j = 1, \ldots, n$,

$$(f_{j+1}(\lambda), g_{j+1}(\lambda)) = \begin{cases} (f_j(\lambda), g_j(\lambda))(\lambda - \frac{k_{j+1}}{b_{j+1}}), & \text{if } b_{j+1} \neq 0, (\lambda - \frac{k_{j+1}}{b_{j+1}})| \frac{f_j(\lambda)}{(f_j(\lambda), g_j(\lambda))}, \\ (f_j(\lambda), g_j(\lambda)), & \text{otherwise.} \end{cases} \tag{12}$$

This indicates that every element in S_j, $j = 1, 2\cdots, T-1$ is equal to some $k_i/b_i, i \in [2\ n]$.

We point out that how we match $s_j(l)$ and k_i/b_i is vital to the validity of Lemmas 4.3 and 4.4 in the subsequent construction. Unfortunately, the matching rule in [17] is not feasible for the "fixed–fixed" case and we have to find a new one. To this end, define a series of subsets R_j, $j = 1, 2, \ldots, T-1$ by

$$\begin{cases} R_1 = [n - q_1 + 1 \quad n], \\ R_j = [j + \sum_{h=2}^{j-1} q_h \quad j - 1 + \sum_{h=2}^{j} q_h], \quad j = 2, \ldots, T-1. \end{cases} \tag{13}$$

Alternatively, we write

$$\begin{cases} R_1 = \left\{r_1(q_1) < r_1(q_1 - 1) < \cdots < r_1(1)\right\}, \\ R_j = \left\{r_j(1) < r_j(2) < \cdots < r_j(q_j)\right\}, \qquad j = 2, \ldots, T-1. \end{cases} \tag{14}$$

Determine k_i/b_i with $i \in \bigcup_{j=1}^{T-1} R_j$ as the following rule:

$$\frac{k_i}{b_i} = s_j(q_j - l + 1), \qquad i = r_j(l). \tag{15}$$

We summarize the similarities and differences of the construction strategies for the "fixed–free" and "fixed–fixed" cases by Fig. 3. The example below shows the differences of the two matching rules.

Example 4.1. Take $n = 8, m = 3, t_1 = 2, t_2 = t_3 = 3$ and $0 < \lambda_1 < \lambda_2 < \lambda_3$ in Proposition 4.1. In the current matching rule, $k_2/b_2 = k_8/b_8 = \lambda_3$, $k_3/b_3 = k_7/b_7 = \lambda_2$, $k_6/b_6 = \lambda_1$, whereas the previous rule in [17] yields $k_2/b_2 = k_6/b_6 = \lambda_3$, $k_3/b_3 = k_7/b_7 = \lambda_2$, $k_4/b_4 = \lambda_1$. Sets S_j and R_j, $j = 1, 2$ are explicitly shown in Fig. 4.

4.2. Proof of Proposition 4.1

Now, we begin our construction. Inspired by [17, Section 4], we first introduce some constants. As the proof of the "fixed–fixed" case is much more involved, we have to select these constants more carefully and sharpen all the inequalities in [17]. For each

		"fixed-free" case	"fixed-fixed" case
Similarities	Sets of multiple eigenvalues	$S_j=\{\lambda_i: t_i\ge j+1\}=\{s_j(1)<\cdots<s_j(q_j)\}$ and $q_j=\lvert S_j\rvert$, $j=1,\cdots,T-1$	
Differences	Aim of strategy	Construct $\boldsymbol{F}_n$ satisfying $\prod_{i=1}^{m}(\lambda-\lambda_i)^{t_i}\lvert f_n(\lambda)$	Construct $\boldsymbol{G}_{n+1}$ satisfying $\prod_{i=1}^{m}(\lambda-\lambda_i)^{t_i}\lvert g_{n+1}(\lambda)$
	Number of construction steps	two	three
	Definitions of R_j	$R_j=[j+1+\sum_{h=1}^{j-1}q_h \quad j+\sum_{h=1}^{j}q_h]$, $j=1,\cdots,T-1$	$R_1=[n-q_1+1 \quad n]$, $R_j=[j+\sum_{h=2}^{j-1}q_h \quad j-1+\sum_{h=2}^{j}q_h]$, $j=2,\cdots,T-1$
	Definitions of $r_j(l)$	$r_j(l)=j+l+\sum_{h=1}^{j-1}q_h$, $l=1,\cdots,q_j$, $j=1,\cdots,T-1$	$r_1(l)=n-l+1$, $l=1,\cdots,q_1$, $r_j(l)=j-1+l+\sum_{h=2}^{j-1}q_h$, $l=1,\cdots,q_j$, $j=2,\cdots,T-1$
	Matching rule of k_i/b_i	$\frac{k_i}{b_i}=s_j(q_j-l+1)$, $i=r_j(l)$, where $r_j(l)$ are different in the two cases	

Fig. 3. Similarities and differences of the construction strategies for the "fixed–free" case [17] and "fixed–fixed" case.

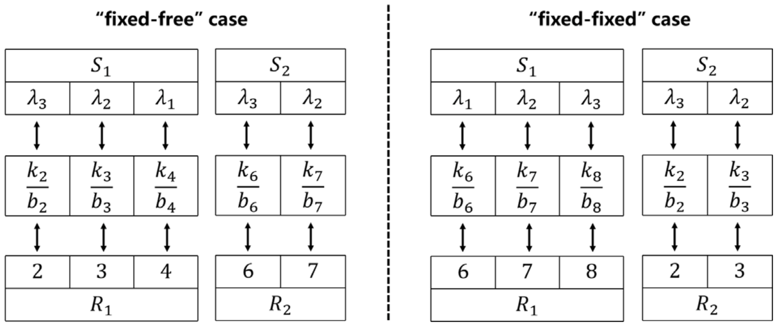

Fig. 4. Associations between S_i, $\frac{k_i}{b_i}$ and R_i of Example 4.1.

$i=1,\dots,m$, set $\rho_i=\varepsilon^{(n+1)^{m-i}}$, where

$$\varepsilon=\frac{\Delta^{2n^2+2n+6}}{n^2 2^{6(n+1)^4}\Lambda^{2(n+2)^2}},\quad \Delta=\frac{1}{2}\min\{1,\lambda_1,\min_{1\le i\le m-1}(\lambda_{i+1}-\lambda_i)\},\quad \Lambda=1+\lambda_m. \tag{16}$$

Next, let

$$\tau=\rho_m^{\frac{1}{2}},\quad \Omega=\frac{14\Lambda^2}{\Delta^2},\quad M=\sum_{k=1}^{m}M_k. \tag{17}$$

$$C=\frac{2^{2n+1}(1+\Lambda^n)}{C_1^{2n^2}\rho_1^{2n}},\quad C_1=\frac{\Delta}{2^{n+1}\Lambda},\quad C_2(j)=\frac{2^{2(n+1)^2}\Lambda^{n+1}}{\Delta^n\varepsilon^{(n+1)^{m-j-1}}},\quad j=1,\dots,m-1. \tag{18}$$

Recall that $s_1(1)<s_1(2)<\cdots<s_1(q_1)$ are the entries of S_1 defined in (11), for each $i=1,\dots,m-1$, let

$$C_{j,i}=\begin{cases}\frac{1}{\lambda_m-\lambda_i}, & \text{if } j=0,\\ \frac{s_1(q_1-j+1)-\lambda_i}{s_1(q_1-j+1)(\lambda_m-\lambda_i)}, & \text{if } j=1,\dots,q_1.\end{cases} \tag{19}$$

Remark 4.1. Clearly, $\varepsilon<1$, $\Lambda/\Delta\ge 2$ in (16) and $C>1$ in (18). Since $n\ge m$,

$$\begin{cases} m\Omega^{m+1}\frac{\Lambda^2}{\Delta}\tau<n25^{n+1}\frac{\Lambda^{2n+4}}{\Delta^{2n+3}}\frac{\Delta^{n^2+n+3}}{n2^{3(n+1)^4}\Lambda^{(n+2)^2}}<1,\\ m2^{m+1}\frac{\Lambda^2}{\Delta}\frac{\rho_i}{\rho_{i+1}^2}\le m2^{m+1}\frac{\Lambda^2}{\Delta}\varepsilon<1,\quad i=1,\dots,m-1 \text{ and } n\ge 2,\end{cases}$$

which immediately asserts

$$\begin{cases} m\Omega^{m+1}\tau<\frac{\Delta}{\Lambda^2},\\ m2^{m+1}\frac{\Lambda^2}{\Delta}\rho_i<\rho_{i+1}^2,\quad i=1,\dots,m-1 \text{ and } n\ge 2.\end{cases} \tag{20}$$

Moreover, [17, Lemma B.2.] indicates that $C_2(j), j = 1, \dots, m-1$ defined by (18) satisfy

$$\begin{cases} (1 + C_2(m-1))^n \rho_{m-1} < \frac{\Delta}{2}, \\ (1 + C_2(j))^n \rho_j < (1 + C_2(j+1))^n \rho_{j+1}, & \text{if} \quad j = 1, \dots, m-2, \quad m > 2, \\ \frac{n(1+C_2(j-1))^n}{\Delta} \rho_{j-1} < \frac{1}{4} \left(\frac{\Delta^n}{2^{(n+1)^2} \Lambda^{n+1}} \right) \rho_{j+1}, & \text{if} \quad j = 2, \dots, m-2, \quad m > 3. \end{cases} \tag{21}$$

Finally, we point out that for each $j \in [0\ q_1]$ and $i \in [1\ m-1]$,

$$\frac{2\Delta}{\Lambda^2} < |C_{j,i}| < \frac{1}{2\Delta}. \tag{22}$$

Lemma 4.1. *Let $\nu > 0$, $L > 0$ and $G(\lambda) = \prod_{i=1}^m (\lambda - \lambda_i)$, $\widetilde{F}(\lambda) = \prod_{i=1}^m (\lambda - \lambda_i + \rho_i)$ be two monic polynomials.*
(i) If $m > 1$, then there exist some $\widetilde{\lambda} > 0, \widetilde{b} > 0$, $\widetilde{\mu}, \widetilde{\nu}$ and monic polynomial $\widetilde{G}(\lambda)$ with distinct roots $\widetilde{\beta}_1 < \cdots < \widetilde{\beta}_{m-1}$, such that

$$\left| \frac{1}{\rho_m} \frac{\rho_i}{\widetilde{\beta}_i - \lambda_i} - C_{0,i} \right| < \tau, \quad i = 1, \dots, m-1, \tag{23}$$

$$-\frac{\widetilde{\mu}}{\widetilde{\nu}} \frac{\widetilde{F}(\lambda_m)}{\lambda_m \widetilde{G}(\lambda_m)} = L, \tag{24}$$

$$\nu G(\lambda) = \widetilde{\mu} \widetilde{F}(\lambda) + \widetilde{b}(\widetilde{\lambda} - \lambda) \widetilde{\nu} \widetilde{G}(\lambda). \tag{25}$$

(ii) If $m = 1$, then there exist some $\widetilde{\lambda} > 0, \widetilde{b} > 0$, $\widetilde{\mu}, \widetilde{\nu}$ and $\widetilde{G}(\lambda) = 1$ such that (24) and (25) *hold.*

Now, we write $s_1(l) = \lambda_{z_l}, l = 1, \dots, q_1$ and define $z_{q_1+1} = m$, $\beta_m = C_2(m) = +\infty$.

Lemma 4.2. *Let $F(\lambda) = \prod_{i=1}^m (\lambda - \alpha_i)$, $G(\lambda) = \prod_{i=1}^{m-1} (\lambda - \beta_i)$ be two polynomials and μ, ν satisfy $-\frac{\mu}{\nu} > 0$. Suppose there exists an index $l \in [1\ q_1]$ with*

$$\begin{cases} \lambda_i \in [\alpha_i, \beta_i), \quad \lambda_i - \alpha_i \le \rho_i, & \textit{if} \quad i = 1, \dots, z_{l+1}, \\ \lambda_i < \alpha_i < \beta_i < \lambda_i + (1 + C_2(i))^n \rho_i, & \textit{if} \quad i = z_{l+1} + 1, \dots, m \end{cases} \tag{26}$$

satisfying

$$\left| \frac{1}{\rho_m} \frac{\lambda_i - \alpha_i}{\beta_i - \lambda_i} - C_{q_1 - l, i} \right| < \Omega^{q_1 - l} \tau, \quad i = 1, \dots, m-1 \tag{27}$$

and

$$\begin{cases} \alpha_m - \lambda_m < \frac{2\Lambda}{\lambda_1} \rho_m, & \textit{if} \quad z_l \in [1\ m-1], \\ \alpha_m - \lambda_m = \rho_m, & \textit{if} \quad z_l = m. \end{cases} \tag{28}$$

Then there exist some $\widetilde{m} > 0, \widetilde{b} > 0$, $\widetilde{\mu}, \widetilde{\nu}$ satisfying $-\frac{\widetilde{\mu}}{\widetilde{\nu}} > -\frac{\lambda_1}{\Lambda} \frac{\mu}{\nu}$, and monic polynomials $\widetilde{F}(\lambda)$, $\widetilde{G}(\lambda)$ with distinct roots $\widetilde{\alpha}_1 < \cdots < \widetilde{\alpha}_m$, $\widetilde{\beta}_1 < \cdots < \widetilde{\beta}_{m-1}$, respectively, such that

$$\begin{cases} \mu F(\lambda) = -\lambda \widetilde{m} \nu G(\lambda) - \widetilde{b} \widetilde{\mu} \widetilde{F}(\lambda), \\ (\lambda - \lambda_{z_l}) \nu G(\lambda) = \widetilde{\mu} \widetilde{F}(\lambda) + \widetilde{b}(\lambda_{z_l} - \lambda) \widetilde{\nu} \widetilde{G}(\lambda). \end{cases} \tag{29}$$

In addition, the roots satisfy

$$\widetilde{\alpha}_{z_l} = \lambda_{z_l} \quad \textit{and} \quad \widetilde{\alpha}_i \in (\alpha_i, \beta_i), \quad i = 1, \dots, m, \tag{30}$$

$$\frac{1}{2} < \frac{\widetilde{\beta}_i - \lambda_i}{\beta_i - \lambda_i} < 2 \quad \textit{and} \quad \left| \frac{1}{\rho_m} \frac{\lambda_i - \widetilde{\alpha}_i}{\widetilde{\beta}_i - \lambda_i} - C_{q_1 - l + 1, i} \right| < \Omega^{q_1 - l + 1} \tau, \quad i = 1, \dots, m-1. \tag{31}$$

In particular, when $z_l < m$,

$$\frac{\alpha_m - \lambda_m}{\rho_m} < \frac{\widetilde{\alpha}_m - \lambda_m}{\rho_m} < \frac{\alpha_m - \lambda_m}{\rho_m} + \lambda_m \frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} C_{q_1 - l, z_l} + \frac{3\Lambda^2}{\lambda_1} \Omega^{q_1 - l} \tau. \tag{32}$$

The next lemma is based on Lemmas 4.1 and 4.2. By applying this lemma, we can construct a series of polynomials $\{F_j(\lambda)\}_{j=n-q_1}^n$ and $\{G_j(\lambda)\}_{j=n-q_1}^n$ with degrees $\deg F_j(\lambda) = m$ and $\deg G_j(\lambda) = m-1$. We shall see later that these functions are exactly $\{\frac{f_j(\lambda)}{(f_j(\lambda), g_j(\lambda))}, \frac{g_j(\lambda)}{(f_j(\lambda), g_j(\lambda))}\}_{j=n-q_1}^n$. Denote the roots of $F_j(\lambda)$ and $G_j(\lambda)$, $j \in [n-q_1\ n]$ by $\alpha_j(1) < \cdots < \alpha_j(m)$ and $\beta_j(1) < \cdots < \beta_j(m-1)$, respectively.

Lemma 4.3. *Let $T > 1$. Given a polynomial $G_{n+1}(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i)$ and a number $\nu_{n+1} > 0$, there exist some monic polynomials $\{F_j(\lambda)\}_{j=n-q_1}^{n}$, $\{G_j(\lambda)\}_{j=n-q_1}^{n}$ and numbers λ^*, $\{b_j\}_{j=n-q_1+1}^{n+1}$, $\{m_j\}_{j=n-q_1+1}^{n}$, $\{(\mu_j,\nu_j)\}_{j=n-q_1}^{n}$ such that the following statements hold.*

(i) λ^*, $\{b_j\}_{j=n-q_1+1}^{n+1}$, $\{m_j\}_{j=n-q_1+1}^{n}$ are positive and $-\frac{\mu_k}{\nu_k} > C(\frac{\Lambda}{\lambda_1})^k \frac{\Lambda M}{\Lambda-\lambda_1}$ for all $k \in [n-q_1\ n]$.

(ii) For each $k \in [n-q_1\ n]$, $\deg F_k(\lambda) = m$ and $\deg G_k(\lambda) = m-1$. Furthermore, polynomials $F_n(\lambda)$ and $G_n(\lambda)$ satisfy

$$\nu_{n+1}G_{n+1}(\lambda) = \mu_n F_n(\lambda) + b_{n+1}(\lambda^* - \lambda)\nu_n G_n(\lambda). \tag{33}$$

For $k \in [n-q_1\ n-1]$,

$$\begin{cases} \mu_{k+1}F_{k+1}(\lambda) = -\lambda m_{k+1}\nu_{k+1}G_{k+1}(\lambda) - b_{k+1}\mu_k F_k(\lambda), \\ (\lambda - s_1(k-n+q_1+1))\nu_{k+1}G_{k+1}(\lambda) = \mu_k F_k(\lambda) + b_{k+1}(s_1(k-n+q_1+1)-\lambda)\nu_k G_k(\lambda). \end{cases} \tag{34}$$

(iii) The roots of $F_{n-q_1}(\lambda)$ and $G_{n-q_1}(\lambda)$ satisfy $\alpha_{n-q_1}(m) \in [\lambda_m - \rho_m, \Lambda)$ and for $i = 1,\dots,m-1$,

$$\lambda_i + C_1^{q_1}\rho_i - \rho_i \le \alpha_{n-q_1}(i) + C_1^{q_1}\rho_i < \beta_{n-q_1}(i) < \lambda_i + 2^{q_1+1}\Lambda\frac{\rho_i}{\rho_m}. \tag{35}$$

Proof. For $k = n$, let $F_n(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i + \rho_i)$. By applying Lemma 4.1 with $\nu = \nu_{n+1}$ and

$$L > C\left(\frac{\Lambda}{\lambda_1}\right)^n \frac{\Lambda M}{\Lambda - \lambda_1}, \tag{36}$$

we can obtain some $\lambda^* > 0, b_{n+1} > 0$, μ_n, ν_n, and a monic polynomial $G_n(\lambda)$ with $\deg G_n(\lambda) = m-1$ satisfying (33) and

$$-\frac{\mu_n}{\nu_n}\frac{F_n(\lambda_m)}{\lambda_m G_n(\lambda_m)} = L. \tag{37}$$

Thus, by (87), $-\frac{\mu_n}{\nu_n} > C(\frac{\Lambda}{\lambda_1})^n\frac{\Lambda M}{\Lambda-\lambda_1}$. Moreover,

$$\left|\frac{1}{\rho_m}\frac{\rho_i}{\beta_n(i)-\lambda_i} - C_{0,1}\right| < \tau, \quad i = 1,\dots,m-1. \tag{38}$$

It follows from (16), (18) and (19) that, for $i = 1,\dots,m-1$,

$$\beta_n(i) - \lambda_i < \frac{\rho_i}{\rho_m}\left(\frac{1}{\lambda_m - \lambda_i} - \tau\right)^{-1} < \frac{2\Lambda\rho_i}{\rho_m}. \tag{39}$$

For $k = n-1$, by (38) and $F_n(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i + \rho_i)$, it is easy to verify that $l = q_1$ satisfies (26)–(28). By applying Lemma 4.2 to $l = q_1$, $F(\lambda) = F_n(\lambda)$ and $G(\lambda) = G_n(\lambda)$, there exist some $b_n > 0, m_n > 0$, μ_{n-1}, ν_{n-1} with

$$-\frac{\mu_{n-1}}{\nu_{n-1}} > -\frac{\lambda_1}{\Lambda}\frac{\mu_n}{\nu_n} > -C(\frac{\Lambda}{\lambda_1})^{n-1}\frac{\Lambda M}{\Lambda-\lambda_1} \tag{40}$$

and two monic polynomials $F_{n-1}(\lambda)$, $G_{n-1}(\lambda)$ with $\deg F_{n-1}(\lambda) = m$ and $\deg G_{n-1}(\lambda) = m-1$ such that (34) holds.

Next, we shall complete the construction inductively. Assume that for all $k \geqslant h \in (n-q_1\ n-1]$, the conditions of Lemma 4.2 hold with $l = k-n+q_1+1$, $F(\lambda) = F_{k+1}(\lambda)$ and $G(\lambda) = G_{k+1}(\lambda)$. Then we can find the desired b_{k+1}, m_{k+1}, μ_k, ν_k and polynomials $F_k(\lambda), G_k(\lambda)$ by applying Lemma 4.2, so that statements (i) and (ii) of Lemma 4.3 are valid.

We proceed to check the above assertion for $h-1$. By (31) and (39), we deduce that for $i = 1,\dots,m-1$,

$$0 < \beta_h(i) - \lambda_i < 2(\beta_{h+1}(i) - \lambda_i) < \cdots < 2^{n-h}\frac{2\Lambda\rho_i}{\rho_m} < 2^{q_1+1}\Lambda\frac{\rho_i}{\rho_m} < \left(1 + C_2(i)\right)^n \rho_i, \tag{41}$$

$$\left|\frac{1}{\rho_m}\frac{\lambda_i - \alpha_h(i)}{\beta_h(i) - \lambda_i} - C_{n-h,i}\right| < \Omega^{n-h}\tau. \tag{42}$$

Thus, (27) holds for $F_h(\lambda)$ and $G_h(\lambda)$ with $l = h-n+q_1$. Moreover, observe from (20) and (22) that, for $l \in [1\ q_1]$,

$$|C_{q_1-l,i}| > \frac{2\Delta}{\Lambda^2} > \Omega^{q_1-l}\tau, \quad i = 1,\dots,m-1. \tag{43}$$

Therefore, (42) and (43) indicate $\frac{1}{\rho_m}\frac{\lambda_i-\alpha_h(i)}{\beta_h(i)-\lambda_i}$ and $C_{n-h,i}$ have the same sign for $i \ne z_{h-n+q_1+1}$. That is, by (19),

$$(\lambda_i - \alpha_h(i))(s_1(h-n+q_1+1) - \lambda_i) > 0, \quad i \in [1\ m-1] \setminus \{z_{h-n+q_1+1}\}. \tag{44}$$

So, by (30) and (44),

$$\begin{cases} \alpha_h(z_{h-n+q_1+1}) = \lambda_{z_{h-n+q_1+1}}, \\ \lambda_i - \rho_i \le \alpha_{h+1}(i) < \alpha_h(i) < \lambda_i, \quad i \in [1\ z_{h-n+q_1+1} - 1]. \end{cases} \tag{45}$$

For $i \in [z_{h-n+q_1+1}+1\ m]$, by (44), $\lambda_i < \alpha_h(i)$. Moreover, by (22), $|C_{n-h,i}| < \frac{1}{2\Delta} < \frac{1}{\rho_m}$. Then (42) shows $|\alpha_h(i) - \lambda_i| < \beta_h(i) - \lambda_i$, $i \in [1\ m-1]$. Therefore, we deduce

$$\lambda_i < \alpha_h(i) < \beta_h(i), \quad i \in [z_{h-n+q_1+1}+1\ m]. \tag{46}$$

So, by (41), (45) and (46), we conclude (26) holds for $F_h(\lambda)$ and $G_h(\lambda)$ with $l = h - n + q_1$.

Now we show (28) with $l = h - n + q_1$. Note that $z_{h-n+q_1} < z_{q_1} \leqslant m$.

(a) If $z_{h-n+q_1+1} = m$, then (30) infers $\alpha_h(m) = \lambda_m$. Thus $0 = \alpha_h(m) - \lambda_m < \frac{2\Lambda}{\lambda_1}\rho_m$.

(b) If $z_{h-n+q_1+1} < m$, by using (32) recursively for $l = h - n + q_1 + 1, \dots, q_1$, we deduce

$$\begin{aligned} 0 \le \frac{\alpha_h(m) - \lambda_m}{\rho_m} &< \frac{\alpha_n(m) - \lambda_m}{\rho_m} + \lambda_m \sum_{k=2}^{n-h} \left(\frac{1}{s_1(q_1 - k + 1)} - \frac{1}{s_1(q_1 - k + 2)} \right) + \frac{\lambda_m}{s_1(q_1)} + m\frac{3\Lambda^2}{\lambda_1}\Omega^m \tau \\ &< \frac{\lambda_m}{\lambda_1} + m\frac{3\Lambda^2}{\lambda_1}\Omega^m \tau < \frac{2\Lambda}{\lambda_1}, \end{aligned} \tag{47}$$

Therefore, (26)–(28) hold for $F_h(\lambda)$ and $G_h(\lambda)$ with $l = h - n + q_1$. Now, by Lemma 4.2, there exist some numbers $b_h > 0, m_h > 0$, μ_{h-1}, ν_{h-1} with

$$-\frac{\mu_{h-1}}{\nu_{h-1}} > -\frac{\lambda_1}{\Lambda}\frac{\mu_h}{\nu_h} > -C(\frac{\Lambda}{\lambda_1})^{h-1}\frac{\Lambda M}{\Lambda - \lambda_1} \tag{48}$$

and two polynomials $F_{h-1}(\lambda), G_{h-1}(\lambda)$ such that (34) holds. The induction is completed for all $h = n-1, \dots, n-q_1$. Particularly, we remark that (41)–(48) hold for $h = n - q_1$.

The remainder is to verify (iii). Since (47) holds for $h = n - q_1$, $\alpha_{n-q_1}(m) \in (\lambda_m - \rho_m, \Lambda)$. We next show (35). Note that (41) is true for $h = n - q_1$, then

$$\beta_{n-q_1}(i) - \lambda_i < 2^{q_1+1}\Lambda\frac{\rho_i}{\rho_m}, \quad i = 1, \dots, m-1. \tag{49}$$

On the other hand, (31) and (38) infer that for $i = 1, \dots, m-1$,

$$\beta_{n-q_1}(i) - \lambda_i > \cdots > \frac{1}{2^{q_1}}(\beta_n(i) - \lambda_i) > \frac{1}{2^{q_1}}\left(\frac{1}{\lambda_m - \lambda_i} + \tau\right)^{-1}\frac{\rho_i}{\rho_m}. \tag{50}$$

Then, by (20), (22), (27) and (50), it follows that for $i = 1, \dots, m-1$,

$$\begin{aligned} \beta_{n-q_1}(i) - \alpha_{n-q_1}(i) &> (\beta_{n-q_1}(i) - \lambda_i) + \rho_m(C_{q_1,i} - \Omega^{q_1}\tau)(\beta_{n-q_1}(i) - \lambda_i) \\ &> \frac{1}{2^{q_1}}(1 - (|C_{q_1,i}| + \Omega^{q_1})\tau\rho_m)\left(\frac{1}{\lambda_m - \lambda_i} + \tau\right)^{-1}\frac{\rho_i}{\rho_m} > \frac{1}{2^{q_1}}\left(\frac{1}{2\Delta} + \frac{1}{2\Delta}\right)^{-1}\frac{\rho_i}{2\rho_m} > C_1^{q_1}\rho_i. \end{aligned} \tag{51}$$

Recall that $\alpha_{n-q_1}(i) > \alpha_n(i) = \lambda_i - \rho_i$, then

$$\lambda_i + C_1^{q_1}\rho_i - \rho_i \le \alpha_{n-q_1}(i) + C_1^{q_1}\rho_i. \tag{52}$$

We conclude (35) by combining (49), (51) and (52). □

The lemma below sharpens [17, Lemma 4.1]. Its proof is contained in Appendix B.

Lemma 4.4. *Let $F_{n-q_1}(\lambda) = \prod_{i=1}^{m}(\lambda - \alpha_i)$, $G_{n-q_1}(\lambda) = \prod_{i=1}^{m-1}(\lambda - \beta_i)$, μ_{n-q_1}, ν_{n-q_1} be constructed in Lemma 4.3. Then, there exist some monic polynomials $\{F_j(\lambda)\}_{j=1}^{n-q_1-1}$, $\{G_j(\lambda)\}_{j=1}^{n-q_1-1}$ and some sequences of numbers $\{(\lambda_j^*, b_j, m_j)\}_{j=2}^{n-q_1}$, $\{(\mu_j, \nu_j)\}_{j=1}^{n-q_1-1}$ such that for each $j \in [1\ n - q_1 - 1]$, the following two properties hold:*
(i) $\lambda_{j+1}^* > 0, b_{j+1} > 0, m_{j+1} > 0$ and $-\frac{\mu_j}{\nu_j} > C(\frac{\Lambda}{\lambda_1})^j \frac{\Lambda M}{\Lambda - \lambda_1}$;
(ii) if $j + 1 \in [2\ n - q_1] \setminus \bigcup_{l=2}^{T-1} R_l$, then

$$\begin{cases} \mu_{j+1}F_{j+1}(\lambda) = -\lambda m_{j+1}\nu_{j+1}G_{j+1}(\lambda) + b_{j+1}(\lambda_{j+1}^* - \lambda)\mu_j F_j(\lambda), \\ \nu_{j+1}G_{j+1}(\lambda) = \mu_j F_j(\lambda) + b_{j+1}(\lambda_{j+1}^* - \lambda)\nu_j G_j(\lambda), \end{cases} \tag{53}$$

otherwise, for $j + 1 \in R_l$, $l \in [2\ T - 1]$,

$$\begin{cases} \mu_{j+1}F_{j+1}(\lambda) = -\lambda m_{j+1}\nu_{j+1}G_{j+1}(\lambda) - b_{j+1}\mu_k F_j(\lambda), \\ \nu_{j+1}(\lambda - \lambda_{j+1}^*)G_{j+1}(\lambda) = \mu_j F_j(\lambda) + b_{j+1}\nu_j(\lambda_{j+1}^* - \lambda)G_j(\lambda), \\ \lambda_{j+1}^* = s_l(q_l - i + 1) \quad \text{where} \quad j + 1 = r_l(i), \end{cases} \tag{54}$$

where $S_l = \{s_l(1) < s_l(2) < \cdots < s_l(q_l)\}$ and $R_l = \{r_l(1) < r_l(2) < \cdots < r_l(q_l)\}$ are defined in (11) and (14).

Proof of Proposition 4.1. First, consider the case where $T > 1$. By taking $\nu_{n+1} = 1$ in Lemma 4.3, we obtain some $\{F_j(\lambda)\}_{j=n-q_1}^{n}$, $\{G_j(\lambda)\}_{j=n-q_1}^{n}$, and λ^*, $\{b_j\}_{j=n-q_1+1}^{n+1}$, $\{m_j\}_{j=n-q_1+1}^{n}$, $\{(\mu_j, \nu_j)\}_{j=n-q_1}^{n}$ such that statements (i) and (ii) of Lemma 4.3 hold. Next, we apply Lemma 4.4 to construct a sequence of $\{(\lambda_k^*, b_k, m_k)\}_{k=2}^{n-q_1}$, $\{(\mu_k, \nu_k)\}_{k=1}^{n-q_1-1}$, $\{F_k(\lambda)\}_{k=1}^{n-q_1-1}$, $\{G_k(\lambda)\}_{k=1}^{n-q_1-1}$ so that (53) holds for $j + 1 \in [2\ n - q_1] \setminus \bigcup_{l=2}^{T-1} R_l$, and (54) holds for $j + 1 \in R_l$, $l \in [2\ T - 1]$. Noticing that $-\frac{\mu_1}{\nu_1} > M > M_1$, set

$$\begin{cases} k_j = \lambda_j^* b_j, \quad j = 2, \dots, n, \quad k_1 = -\alpha_1(1)\frac{\mu_1}{\nu_1}, \\ m_1 = M_1, \quad b_1 = -\frac{\mu_1}{\nu_1} - M_1. \end{cases} \tag{55}$$

The remainder is to verify $\{k_j, b_j\}_{j=1}^{n+1}$ and $\{m_j\}_{j=1}^{n}$ meet the requirements.

Define a sequence of polynomials $\{D_j(\lambda)\}_{j=1}^{n}$ as follows:

$$\begin{cases} D_1(\lambda) = 1, & \\ D_{k+1}(\lambda) = D_k(\lambda), & k+1 \in [2\ n] \setminus \bigcup_{l=1}^{T-1} R_l, \\ D_{k+1}(\lambda) = (\lambda - \lambda_{k+1}^{*}) D_k(\lambda), & k+1 \in \bigcup_{l=2}^{T-1} R_l, \\ D_{k+1}(\lambda) = (\lambda - s_1(k - n + q_1 + 1)) D_k(\lambda), & k+1 \in R_1, \end{cases} \tag{56}$$

and let

$$\begin{cases} f_j(\lambda) = \frac{\mu_j}{\nu_1} D_j(\lambda) F_j(\lambda), & j = 1, \ldots, n, \\ g_j(\lambda) = \frac{\nu_j}{\nu_1} D_j(\lambda) G_j(\lambda), & j = 1, \ldots, n+1. \end{cases}$$

Clearly, $g_{n+1}(\lambda) = \frac{\nu_{n+1}}{\nu_1} D_{n+1}(\lambda) G_{n+1}(\lambda) = \frac{1}{\nu_1} \prod_{j=1}^{m} (\lambda - \lambda_j)^{t_j}$. The rest of the proof is to check whether $\{f_i(\lambda)\}_{i=1}^{n}$ and $\{g_i(\lambda)\}_{i=1}^{n+1}$ satisfy the recursive formulas (5). It is essentially the same as that of [17].

For the case where $T = 1$, we remark that Lemmas 4.3 and 4.4 hold if we set $q_1 = 0$ and $R_1 = \bigcup_{l=2}^{T-1} R_l = \emptyset$. So, (56) is still valid and the above analysis works. □

5. Concluding remarks

This paper has investigated the inverse eigenvalue problem for the "fixed–fixed" mass–spring–inerter systems. Since the construction cannot follow directly from that for "fixed–free" version, we provide a new construction strategy and derive a necessary and sufficient condition of the problem. Moreover, we preserve the degree of freedom in the recover of the mass matrix. In particular, if all the given eigenvalues are distinct, the assignment can be realized by adjusting springs and inerters only. It is noteworthy that the underlying models here and in [17] are basic. Nonetheless, their IEPs are not tivial owing to the involvement of multiple eigenvalues. As to the IEPs for complicated structures [18–20], the research calls for more analytical tools.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Proofs of Lemmas 4.1–4.2

The proof of Lemma 4.1. (i) Let $m > 1$. Since $\lim_{\lambda\to+\infty} \frac{G(\lambda)}{\widetilde{F}(\lambda)} = 1$ and $\frac{G(\lambda_m)}{\widetilde{F}(\lambda_m)} = 0$, there is a sufficiently large number $\widetilde{\lambda} > \lambda_m$, such that

$$\frac{G(\widetilde{\lambda})}{\widetilde{F}(\widetilde{\lambda})} \in \left(1 - \frac{m}{\Lambda}\rho_{m-1}, 1\right). \tag{57}$$

Here, $0 < 1 - \frac{m}{\Lambda}\rho_{m-1} < 1$. We define

$$\begin{cases} \widetilde{\mu} = \nu \frac{G(\widetilde{\lambda})}{\widetilde{F}(\widetilde{\lambda})}, \\ \widetilde{G}(\lambda) = \frac{\nu G(\lambda) - \widetilde{\mu}\widetilde{F}(\lambda)}{(\nu - \widetilde{\mu})(\lambda - \widetilde{\lambda})}, \\ \widetilde{b} = \frac{\lambda_m \widetilde{G}(\lambda_m)(\widetilde{F}(\widetilde{\lambda}) - G(\widetilde{\lambda}))}{\widetilde{F}(\lambda_m) G(\widetilde{\lambda})} L, \\ \widetilde{\nu} = \frac{\widetilde{\mu} - \nu}{\widetilde{b}}, \end{cases} \tag{58}$$

Observe that $\widetilde{\mu} \in (0, \nu)$ by (57), then $\widetilde{G}(\lambda)$ is well-defined. Consequently, (25) follows and

$$-\frac{\widetilde{\mu}}{\widetilde{\nu}} \frac{\widetilde{F}(\lambda_m)}{\lambda_m \widetilde{G}(\lambda_m)} = \frac{G(\widetilde{\lambda})}{\widetilde{F}(\widetilde{\lambda}) - G(\widetilde{\lambda})} \frac{\widetilde{F}(\lambda_m)}{\lambda_m \widetilde{G}(\lambda_m)} \widetilde{b} = L.$$

It remains to show (23). First, by applying [17, Lemma B.1.], we know

$$\widetilde{\beta}_i \in (\lambda_i, \lambda_{i+1} - \rho_{i+1}), \quad i \in [1\ m-1]. \tag{59}$$

Fix $i \in [1\ m-1]$. Then, (59) yields

$$\frac{\Lambda}{\Lambda+\rho_{i-1}} \geq \prod_{j=1}^{i-1} \frac{\Lambda}{\Lambda+\rho_j} > \prod_{j=1}^{i-1} \frac{\widetilde{\beta}_i-\lambda_j}{\widetilde{\beta}_i-\lambda_j+\rho_j} > \prod_{j=1}^{i-1} \frac{\lambda_{j+1}-\lambda_j}{\lambda_{j+1}-\lambda_j+\rho_j} > \left(\frac{2\Delta}{2\Delta+\rho_{i-1}}\right)^i > 1-\frac{i\rho_{i-1}}{\Delta+\rho_{i-1}}, \tag{60}$$

where the last inequality follows from the *Bernoulli inequality*. In addition, by (59) again,

$$\prod_{j=i+2}^{m} \frac{\widetilde{\beta}_i-\lambda_j}{\widetilde{\beta}_i-\lambda_j+\rho_j} > \prod_{j=i+2}^{m} \frac{\lambda_j-\lambda_i}{\lambda_j-\lambda_i-\rho_j} > 1. \tag{61}$$

Now, we estimate $\left(1+\frac{\rho_i}{\widetilde{\beta}_i-\lambda_i}\right)\left(1-\frac{\rho_{i+1}}{\lambda_{i+1}-\widetilde{\beta}_i}\right)$. Substituting $\lambda=\widetilde{\beta}_i$ into (25), we compute

$$\prod_{j=1}^{m}(\widetilde{\beta}_i-\lambda_j+\rho_j) = \widetilde{F}(\widetilde{\beta}_i) = \frac{1}{\widetilde{\mu}}(\nu G(\widetilde{\beta}_i)-\widetilde{b}\widetilde{\nu}(\widetilde{\lambda}-\widetilde{\beta}_i)\widetilde{G}(\widetilde{\beta}_i)) = \frac{\widetilde{F}(\widetilde{\lambda})}{G(\widetilde{\lambda})}\prod_{j=1}^{m}(\widetilde{\beta}_i-\lambda_j). \tag{62}$$

Note that (57), (60) and (62) infer

$$\left(1+\frac{\rho_i}{\widetilde{\beta}_i-\lambda_i}\right)\left(1-\frac{\rho_{i+1}}{\lambda_{i+1}-\widetilde{\beta}_i}\right) = \frac{\widetilde{F}(\widetilde{\lambda})}{\widetilde{G}(\widetilde{\lambda})}\prod_{j\neq i,i+1}^{m} \frac{\widetilde{\beta}_i-\lambda_j}{\widetilde{\beta}_i-\lambda_j+\rho_j} \tag{63}$$

$$> \left(1-\frac{i\rho_{i-1}}{\Delta+\rho_{i-1}}\right)\prod_{j=i+2}^{m} \frac{\widetilde{\beta}_i-\lambda_j}{\widetilde{\beta}_i-\lambda_j+\rho_j}. \tag{64}$$

Combining (20) and (64), we deduce that

$$\left(1+\frac{\rho_i}{\widetilde{\beta}_i-\lambda_i}\right)\left(1-\frac{\rho_{i+1}}{\lambda_{i+1}-\lambda_i}\right) > \left(1+\frac{\rho_i}{\widetilde{\beta}_i-\lambda_i}\right)\left(1-\frac{\rho_{i+1}}{\lambda_{i+1}-\widetilde{\beta}_i}\right) > 1-\frac{i\rho_{i-1}}{\Delta+\rho_{i-1}} > 1-\frac{m\rho_i}{\Delta}$$
$$> 1-\frac{\rho_{i+1}^2}{\Lambda^2} > 1-\frac{\rho_{i+1}^2}{(\lambda_{i+1}-\lambda_i)^2} = (1+\frac{\rho_{i+1}}{\lambda_{i+1}-\lambda_i})(1-\frac{\rho_{i+1}}{\lambda_{i+1}-\lambda_i}).$$

Consequently, by letting $e_i=\frac{\rho_i}{\rho_{i+1}}$, we immediately assert

$$\widetilde{\beta}_i-\lambda_i < e_i(\lambda_{i+1}-\lambda_i) < e_i(\lambda_j-\lambda_i), \quad j\in[i+1\ m].$$

Therefore,

$$\lambda_j-\widetilde{\beta}_i > (1-e_i)\left(\lambda_j-\lambda_i\right), \quad j\in[i+1\ m]. \tag{65}$$

As a result,

$$\prod_{j=i+2}^{m-1} \frac{\widetilde{\beta}_i-\lambda_j}{\widetilde{\beta}_i-\lambda_j+\rho_j} < \prod_{j=i+2}^{m-1} \frac{(\lambda_j-\lambda_i)(1-e_i)}{(\lambda_j-\lambda_i)(1-e_i)-\rho_j}. \tag{66}$$

Finally, in view of (57), (60), (63) and (66), it follows that

$$\left(1+\frac{\rho_i}{\widetilde{\beta}_i-\lambda_i}\right)\left(1-\frac{\rho_{i+1}}{\lambda_{i+1}-\widetilde{\beta}_i}\right) < \frac{1}{1-\frac{m}{\Lambda}\rho_{m-1}}\frac{\Lambda}{\Lambda+\rho_{i-1}}\frac{\widetilde{\beta}_i-\lambda_m}{\widetilde{\beta}_i-\lambda_m+\rho_m}\prod_{j=i+2}^{m-1}\frac{(\lambda_j-\lambda_i)(1-e_i)}{(\lambda_j-\lambda_i)(1-e_i)-\rho_j}. \tag{67}$$

On the other hand, (61) and (64) infer

$$\left(1+\frac{\rho_i}{\widetilde{\beta}_i-\lambda_i}\right)\left(1-\frac{\rho_{i+1}}{\lambda_{i+1}-\widetilde{\beta}_i}\right) > \left(1-\frac{i\rho_{i-1}}{\Delta+\rho_{i-1}}\right)\prod_{j=i+2}^{m}\frac{\lambda_j-\lambda_i}{\lambda_j-\lambda_i-\rho_j}. \tag{68}$$

Next, we prove (23) by reduction to absurdity. We first consider the case where $i<m-1$. Suppose (23) fails. Since (61) and (68) lead to

$$\frac{\rho_i}{\widetilde{\beta}_i-\lambda_i} > \left(1-\frac{i\rho_{i-1}}{\Delta+\rho_{i-1}}\right)\frac{\lambda_m-\lambda_i}{\lambda_m-\lambda_i-\rho_m}-1 = \frac{\lambda_m-\lambda_i}{\lambda_m-\lambda_i-\rho_m}-1-\frac{i\rho_{i-1}}{\Delta+\rho_{i-1}}\frac{\lambda_m-\lambda_i}{\lambda_m-\lambda_i-\rho_m}$$
$$> \frac{\rho_m}{\lambda_m-\lambda_i-\rho_m}-\frac{m\rho_{m-1}}{\Delta+\rho_{m-1}}\frac{\Delta}{\Delta-\rho_m} > \frac{\rho_m}{\lambda_m-\lambda_i-\rho_m}-\frac{2m\rho_{m-1}}{\Delta} > \frac{\rho_m}{\lambda_m-\lambda_i}-\tau\rho_m, \tag{69}$$

we obtain $\frac{1}{\rho_m}\frac{\rho_i}{\widetilde{\beta}_i-\lambda_i}-\frac{1}{\lambda_m-\lambda_i} > -\tau$. So,

$$\frac{1}{\rho_m}\frac{\rho_i}{\widetilde{\beta}_i-\lambda_i}-\frac{1}{\lambda_m-\lambda_i} > \tau. \tag{70}$$

Note that $e_i < \frac{1}{2}$, (65), (67) and (70) imply

$$\begin{aligned}
&\left(1-\frac{1}{1-e_i}\frac{\rho_m}{\lambda_m-\lambda_i}\right)\left(1+\rho_m\left(\frac{1}{\lambda_m-\lambda_i}+\tau\right)\right)\left(1-\frac{\rho_{i+1}}{\lambda_{i+1}-\widetilde{\beta}_i}\right)\\
&<\left(1-\frac{\rho_m}{\lambda_m-\widetilde{\beta}_i}\right)\left(1+\frac{\rho_i}{\widetilde{\beta}_i-\lambda_i}\right)\left(1-\frac{\rho_{i+1}}{\lambda_{i+1}-\widetilde{\beta}_i}\right)<\frac{1}{1-\frac{m}{\Lambda}\rho_{m-1}}\frac{\Lambda}{\Lambda+\rho_{i-1}}\prod_{j=i+2}^{m-1}\frac{(\lambda_j-\lambda_i)(1-e_i)}{(\lambda_j-\lambda_i)(1-e_i)-\rho_j}\\
&<\frac{1}{1-\frac{m}{\Lambda}\rho_{m-1}}\frac{\Lambda}{\Lambda+\rho_{i-1}}\frac{1}{1-\sum_{j=i+2}^{m-1}\frac{2\rho_j}{\lambda_j-\lambda_i}}<1+\frac{2m}{\Lambda}\rho_{m-1}+4\sum_{j=i+2}^{m-1}\frac{\rho_j}{\lambda_j-\lambda_i},
\end{aligned}\tag{71}$$

where the third inequality follows from the *Bernoulli inequality* and the last inequality is due to $\frac{m}{\Lambda}\rho_{m-1}+\sum_{j=i+2}^{m-1}\frac{2\rho_j}{\lambda_j-\lambda_i}<\frac{1}{2}$. Furthermore, due to (20), we obtain

$$m2^{m+1}\frac{\Lambda^2}{\Delta}e_i<\rho_{i+1}\le\rho_m<1,$$

which implies $\frac{\tau}{4}>\frac{\rho_m}{4}>\frac{e_i}{\Delta}$. Besides, $\rho_m=\varepsilon<\frac{\Delta^4}{16}$, it then follows that

$$\frac{\tau}{4}-\rho_m\frac{1}{\Delta}\left(\frac{1}{2\Delta}+\tau\right)>\frac{\tau}{4}-\rho_m\frac{1}{\Delta^2}=\rho_m^{\frac{1}{2}}\left(\frac{1}{4}-\rho_m^{\frac{1}{2}}\frac{1}{\Delta^2}\right)>0.$$

Therefore

$$\begin{aligned}
&\left(1-\frac{1}{1-e_i}\frac{\rho_m}{\lambda_m-\lambda_i}\right)\left(1+\rho_m\left(\frac{1}{\lambda_m-\lambda_i}+\tau\right)\right)\\
&=1+\rho_m\left(\tau-\frac{e_i}{1-e_i}\frac{1}{\lambda_m-\lambda_i}\right)-\rho_m^2\frac{1}{1-e_i}\frac{1}{\lambda_m-\lambda_i}\left(\frac{1}{\lambda_m-\lambda_i}+\tau\right)\\
&>1+\frac{\tau}{2}\rho_m+\rho_m\left(\frac{\tau}{2}-\frac{e_i}{\Delta}-\rho_m\frac{1}{\Delta}\left(\frac{1}{2\Delta}+\tau\right)\right)>1+\frac{\tau}{2}\rho_m.
\end{aligned}\tag{72}$$

So, by observing $\frac{4\rho_m^2}{\Lambda^2 2^{m+1}}\le\frac{\rho_m^2}{\Lambda^2}<\tau<\frac{1}{4}$, (71) and (72) lead to

$$\begin{aligned}
\lambda_{i+1}-\widetilde{\beta}_i&<\frac{\rho_{i+1}}{1-\left(1+\frac{\tau}{2}\rho_m\right)^{-1}\left(1+\frac{2m}{\Lambda}\rho_{m-1}+4\sum_{j=i+2}^{m-1}\frac{\rho_j}{\lambda_j-\lambda_i}\right)}<\frac{\rho_{i+1}}{1-\left(1+\frac{\tau}{2}\rho_m\right)^{-1}\left(1+\frac{4m}{\Delta}\rho_{m-1}\right)}\\
&=\frac{1+\frac{\tau}{2}\rho_m}{\frac{\tau}{2}\rho_m-\frac{4m}{\Delta}\rho_{m-1}}\rho_{i+1}<\frac{1+\frac{\tau}{2}\rho_m}{\frac{\tau}{2}\rho_m-\frac{4\rho_m^2}{\Lambda^2 2^{m+1}}}\rho_{i+1}<\frac{2}{\tau\rho_m(\frac{1}{2}-\tau)}\rho_{i+1}<\frac{8\rho_{i+1}}{\tau\rho_m},
\end{aligned}\tag{73}$$

where the third inequality follows from $m2^{m+1}\frac{\Lambda^2}{\Delta}\rho_{m-1}<\rho_m^2$. At last, combining (20), (70) and (73) gives

$$\begin{aligned}
\lambda_{i+1}-\lambda_i&=(\lambda_{i+1}-\widetilde{\beta}_i)+(\widetilde{\beta}_i-\lambda_i)<\frac{8\rho_{i+1}}{\tau\rho_m}+\left(\frac{1}{\lambda_m-\lambda_i}+\tau\right)^{-1}\frac{\rho_i}{\rho_m}\\
&<\frac{8\rho_{m-1}}{\tau\rho_m}+\Lambda\frac{\rho_{m-1}}{\rho_m}<(8+\Lambda)\frac{1}{\tau}\frac{\rho_{m-1}}{\rho_m}<(8+\Lambda)\tau<2\Delta,
\end{aligned}$$

which contradicts to (16).

So far, we have shown (23) except $i=m-1$. As to $i=m-1$, (67) turns out to be

$$\left(1+\frac{\rho_{m-1}}{\widetilde{\beta}_{m-1}-\lambda_{m-1}}\right)\left(1-\frac{\rho_m}{\lambda_m-\widetilde{\beta}_{m-1}}\right)<\frac{1}{1-\frac{m}{\Lambda}\rho_{m-1}}\frac{\Lambda}{\Lambda+\rho_{m-2}}.\tag{74}$$

If $\frac{1}{\rho_m}\frac{\rho_{m-1}}{\widetilde{\beta}_{m-1}-\lambda_{m-1}}-\frac{1}{\lambda_m-\lambda_{m-1}}>\tau$, by (65), (74) and $e_m<\rho_m<\frac{\Delta^2\tau}{4}<\frac{1}{2}$,

$$\begin{aligned}
\frac{1}{1-\frac{m}{\Lambda}\rho_{m-1}}\frac{\Lambda}{\Lambda+\rho_{m-2}}&>\left(1+\rho_m\left(\frac{1}{\lambda_m-\lambda_{m-1}}+\tau\right)\right)\left(1-\frac{\rho_m}{\lambda_m-\widetilde{\beta}_{m-1}}\right)\\
&>\left(1+\rho_m\left(\frac{1}{\lambda_m-\lambda_{m-1}}+\tau\right)\right)\left(1-\frac{\rho_m}{(1-e_m)(\lambda_m-\lambda_{m-1})}\right)\\
&>1+\tau\rho_m+\frac{\rho_m}{\lambda_m-\lambda_{m-1}}-\frac{\rho_m}{(1-e_m)(\lambda_m-\lambda_{m-1})}-\rho_m\left(\frac{1}{\lambda_m-\lambda_{m-1}}+\tau\right)\frac{\rho_m}{(1-e_m)(\lambda_m-\lambda_{m-1})}\\
&>1+\tau\rho_m-\frac{e_m}{1-e_m}\frac{\rho_m}{\lambda_m-\lambda_{m-1}}-\frac{\rho_m^2}{\Delta^2}>1+\tau\rho_m-\frac{\rho_m^2}{\Delta}-\frac{\rho_m^2}{\Delta^2}>1+\frac{\tau}{4}\rho_m.
\end{aligned}\tag{75}$$

However, because (20) infers $\rho_{m-1} < \frac{\Delta}{m2^{m+1}\Lambda^2}\rho_m^2 < \frac{\Lambda}{8m}\tau\rho_m$, we conclude

$$1+\frac{\tau}{4}\rho_m > \frac{1}{1-\frac{\tau}{8}\rho_m} > \frac{1}{1-\frac{m}{\Lambda}\rho_{m-1}} > \frac{1}{1-\frac{m}{\Lambda}\rho_{m-1}}\frac{\Lambda}{\Lambda+\rho_{m-2}},$$

which contradicts to (75).

Suppose $\frac{1}{\rho_m}\frac{\rho_{m-1}}{\widetilde{\beta}_{m-1}-\lambda_{m-1}} - \frac{1}{\lambda_m-\lambda_{m-1}} < -\tau$. Observe that (74) reduces to

$$\left(1+\frac{\rho_{m-1}}{\widetilde{\beta}_{m-1}-\lambda_{m-1}}\right)\left(1-\frac{\rho_m}{\lambda_m-\widetilde{\beta}_{m-1}}\right) > 1-\frac{(m-1)\rho_{m-2}}{\Delta+\rho_{m-2}}, \tag{76}$$

then

$$\begin{aligned} 1-\tau\rho_m &> \left(1+\frac{\rho_m}{\lambda_m-\lambda_{m-1}}\right)\left(1-\frac{\rho_m}{\lambda_m-\lambda_{m-1}}\right) > \left(1+\rho_m\left(\frac{1}{\lambda_m-\lambda_{m-1}}-\tau\right)\right)\left(1-\frac{\rho_m}{\lambda_m-\lambda_{m-1}}\right) \\ &> \left(1+\frac{\rho_{m-1}}{\widetilde{\beta}_{m-1}-\lambda_{m-1}}\right)\left(1-\frac{\rho_m}{\lambda_m-\lambda_{m-1}}\right) > \left(1+\frac{\rho_{m-1}}{\widetilde{\beta}_{m-1}-\lambda_{m-1}}\right)\left(1-\frac{\rho_m}{\lambda_m-\widetilde{\beta}_{m-1}}\right) \\ &> 1-\frac{(m-1)\rho_{m-2}}{\Delta+\rho_{m-2}}, \end{aligned} \tag{77}$$

which is impossible due to (20) that

$$\frac{m-1}{\Delta+\rho_{m-2}}\frac{\rho_{m-2}}{\rho_m} < \frac{m}{\Delta}\frac{1}{m2^{m+1}}\frac{\Delta}{\Lambda^2}\rho_m < \tau.$$

In summary, (23) holds.

(ii) Let $m=1$. Similarly to (i), there is a sufficiently large number $\widetilde{\lambda} > \lambda_1$ such that (57) holds. Then, it is easy to verify that

$$\begin{cases} \widetilde{\mu} = \nu\frac{\widetilde{\lambda}-\lambda_1}{\widetilde{\lambda}-\lambda_1+\rho_1}, \\ \widetilde{G}(\lambda) = 1, \\ \widetilde{b} = \frac{L\lambda_1}{\widetilde{\lambda}-\lambda_1}, \\ \widetilde{\nu} = \frac{\widetilde{\mu}-\nu}{\widetilde{b}} \end{cases}$$

are exactly the quantities we want. $\square$

Before proceeding to the proof of Lemma 4.2, we need a lemma below.

Lemma A.1. *Let $F(\lambda) = \mu\prod_{i=1}^p(\lambda-\alpha_i)$ and $G(\lambda) = \nu\prod_{i=1}^p(\lambda-\beta_i)$ be two polynomials of degree p, where $|\mu| > |\nu|$, $\alpha_1 < \cdots < \alpha_p$ and $\beta_1 < \cdots < \beta_p$. Then, $F(\lambda)-G(\lambda)$ has p real roots $\gamma_1 \leqslant \cdots \leqslant \gamma_p$ and the following two statements hold: (i) when $p>1$, for any $\eta \in (0, \frac{1}{2}\min_{1\leqslant i\leqslant p-1}(\alpha_{i+1}-\alpha_i))$, if*

$$\frac{|\nu|}{|\mu|} < \frac{\min\{1,\eta^p\}}{2+2\max_{j\in[1\ p]}\prod_{i=1}^p|\alpha_j-\beta_i|}, \tag{78}$$

then

$$\max_{1\leqslant i\leqslant p}|\gamma_i-\alpha_i| < \eta; \tag{79}$$

(ii) when $p=1$, for any $\eta\in(0,\frac{1}{2})$, (79) holds provided that

$$|\nu| < \frac{|\mu|}{2}\min\left\{\frac{\eta}{|\alpha_1-\beta_1|}, 1\right\}. \tag{80}$$

Proof. First, since $|\nu|<|\mu|$, $F(\lambda)-G(\lambda)$ has p real roots $\gamma_1\leqslant\cdots\leqslant\gamma_p$.

(i) Let $p>1$. Observe that $F(\lambda)-G(\lambda) = (\mu-\nu)\prod_{i=1}^p(\lambda-\gamma_i)$, then for each $j\in[1\ p]$,

$$(\mu-\nu)\prod_{i=1}^p(\alpha_j-\gamma_i) = F(\alpha_j)-G(\alpha_j) = -\nu\prod_{i=1}^p(\alpha_j-\beta_i). \tag{81}$$

We shall prove for each $j\in[1\ p]$, $\min_{1\leqslant i\leqslant p}|\alpha_j-\gamma_i| < \eta$ holds. Otherwise, assume $\min_{1\leqslant i\leqslant p}|\alpha_j-\gamma_i| \geq \eta$ for some $j\in[1\ p]$. Thus, by (78) and (81),

$$\eta^p \leq \prod_{i=1}^p\left|\alpha_j-\gamma_i\right| \leq \left|\frac{\nu/\mu}{1-\nu/\mu}\right|\max_{j\in[1\ p]}\prod_{i=1}^p\left|\alpha_j-\beta_i\right| < 2\left|\frac{\nu}{\mu}\right|\max_{j\in[1\ p]}\prod_{i=1}^p\left|\alpha_j-\beta_i\right| < \min\{1,\eta^p\} \leq \eta^p,$$

which is impossible. Hence, $\min_{1\leqslant i\leqslant p}|\alpha_j-\gamma_i|<\eta$ holds for $j=1,2,\dots,p$. On the other hand, observe that $\eta\in(0,\frac{1}{2}\min_{1\leqslant i\leqslant p-1}(\alpha_{i+1}-\alpha_i))$ implies

$$\begin{cases}\alpha_1+\eta<\alpha_2-\frac{1}{2}(\alpha_2-\alpha_1),\\ (\alpha_j-\eta,\alpha_j+\eta)\subset(\alpha_{j-1}+\frac{1}{2}(\alpha_j-\alpha_{j-1}),\alpha_{j+1}-\frac{1}{2}(\alpha_{j+1}-\alpha_j)), \quad j\in[2\ p-1],\\ \alpha_p-\eta>\alpha_{p-1}+\frac{1}{2}(\alpha_p-\alpha_{p-1}),\end{cases}$$

by $\gamma_1\leqslant\cdots\leqslant\gamma_p$, it is evident that $|\alpha_i-\gamma_i|<\eta$, $i=1,2,\dots,p$.

(ii) When $p=1$, (80) and (81) lead to

$$|\gamma_1-\alpha_1|\leq\left|\frac{\nu/\mu}{1-\nu/\mu}\right||\alpha_1-\beta_1|<2\left|\frac{\nu}{\mu}\right||\alpha_1-\beta_1|<\eta,$$

as desired. □

The proof of Lemma 4.2. We first claim $\lambda G(\lambda)\prec F(\lambda)$. As a matter of fact, by (20) and (22), it is evident that

$$|C_{q_1-l,j}|-\Omega^{q_1-l}\tau>\frac{\Delta}{\Lambda^2},\quad j\in[1\ m-1].$$

So, by (22)–(27), for each $j\in[1\ m-1]$,

$$\begin{aligned}\beta_j-\lambda_j &< \left(|C_{q_1-l,j}|-\Omega^{q_1-l}\tau\right)^{-1}\frac{|\lambda_j-\alpha_j|}{\rho_m}\leq\left(|C_{q_1-l,j}|-\Omega^{q_1-l}\tau\right)^{-1}\frac{\rho_j}{\rho_m}\\ &< \frac{\Lambda^2}{\Delta}\frac{\rho_j}{\rho_m}<\frac{\Lambda^2}{\Delta}\frac{1}{m2^{m+1}}\frac{\Delta}{\Lambda^2}\rho_m=\frac{\rho_m}{m2^{m+1}}.\end{aligned}\tag{82}$$

It then follows from (16) and (26) that when $j\in[1\ z_{l+1}-1]$,

$$\alpha_j<\beta_j=(\beta_j-\lambda_j)+(\lambda_j-\lambda_{j+1})+\lambda_{j+1}<\frac{\rho_m}{m2^{m+1}}-2\Delta+\lambda_{j+1}<\alpha_{j+1}<\beta_{j+1}.\tag{83}$$

Moreover, if $z_{l+1}<m$, in view of (21), (22), (26) and (82), we deduce

$$\beta_{z_{l+1}}<\lambda_{z_{l+1}}+\frac{\rho_m}{m2^{m+1}}<\lambda_{z_{l+1}+1}<\alpha_{z_{l+1}+1}<\beta_{z_{l+1}+1}<\lambda_{z_{l+1}+2}<\cdots<\beta_{m-1}<\alpha_m.\tag{84}$$

We thus conclude $\lambda G(\lambda)\prec F(\lambda)$ by (83) and (84).

We begin our construction. Let

$$\begin{cases}\widetilde{m}=-\frac{\mu}{\nu}\frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})},\\ \widetilde{F}(\lambda)=\frac{\mu F(\lambda)+\lambda\widetilde{m}\nu G(\lambda)}{\mu+\widetilde{m}\nu}.\end{cases}\tag{85}$$

It is easy to verify $\widetilde{m}>0$ in view of (26) and $\lambda G(\lambda)\prec F(\lambda)$. Then, by (20), (22) and (27), for $z_l\in[1\ m-1]$,

$$\left|\frac{\widetilde{m}\nu}{\mu}\right|=\frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}<\frac{\lambda_{z_l}-\alpha_{z_l}}{\beta_{z_l}-\lambda_{z_l}}\frac{\Lambda^{m-1}}{\Delta^{m-1}}<\rho_m(C_{q_1-l,z_l}+\Omega^{q_1-l}\tau)\frac{\Lambda^{m-1}}{\Delta^{m-1}}<\rho_m(\frac{1}{2\Delta}+\frac{\Delta}{\Lambda^2})\frac{\Lambda^{m-1}}{\Delta^{m-1}}<\frac{\Delta\tau}{\Lambda}<1.\tag{86}$$

When $z_l=m$, due to (26), (28) and $\lambda G(\lambda)\prec F(\lambda)$,

$$\left|\frac{\widetilde{m}\nu}{\mu}\right|=\frac{F(\lambda_m)}{\lambda_m G(\lambda_m)}=\frac{\prod_{j=1}^{m}(\lambda_m-\alpha_j)}{\lambda_m\prod_{j=1}^{m-1}(\lambda_m-\beta_j)}<\frac{\lambda_m-\alpha_m}{\lambda_m-\beta_{m-1}}<\frac{2\Lambda\rho_m}{\Delta\lambda_1}<\frac{\Delta\tau}{\Lambda}<1.\tag{87}$$

Hence $\widetilde{F}(\lambda)$ is well-defined and $\deg(\widetilde{F}(\lambda))=\deg(F(\lambda))=m$. Now, by $\lambda G(\lambda)\prec F(\lambda)$ again, applying [17, Lemma B.1.] to $\mu F(\lambda)$ and $-\lambda\widetilde{m}\nu G(\lambda)$ shows

$$\widetilde{\alpha}_j\in(\alpha_j,\beta_j),\quad j=1,\dots,m,\tag{88}$$

which indicates $F(\lambda)\prec\widetilde{F}(\lambda)$. Observe that $(\lambda-\lambda_{z_l})|\mu F(\lambda)+\lambda\widetilde{m}\nu G(\lambda)$ and $\alpha_{z_l}<\lambda_{z_l}<\beta_{z_l}$, we assert $\widetilde{\alpha}_{z_l}=\lambda_{z_l}$ at once. So (30) is proved.

Next, let

$$\begin{cases}\widetilde{\mu}=\epsilon\nu,\\ \widetilde{b}=-\frac{\mu+\widetilde{m}\nu}{\widetilde{\mu}},\\ \widetilde{\nu}=\frac{\widetilde{\mu}-\nu}{\widetilde{b}},\\ \widetilde{G}(\lambda)=\frac{\nu(\lambda-\lambda_{z_l})G(\lambda)-\widetilde{\mu}\widetilde{F}(\lambda)}{\widetilde{b}\widetilde{\nu}(\lambda-\lambda_{z_l})},\end{cases}\tag{89}$$

where

$$\epsilon=\begin{cases}\upsilon_1, & \text{if}\quad m>2,\\ \upsilon_2, & \text{if}\quad m=2,\end{cases}$$

and

$$\begin{aligned}
v_1 &= \frac{\min\{1, \eta_1^{m-1}\}}{4 + 4\max_{1\leqslant j\leqslant m-1}\prod_{h=1}^{m-1}\left|\beta_j - \widetilde{\alpha}_h\right|},\\
v_2 &= \frac{1}{4}\min\left\{\frac{\eta_2}{|\beta_1 - \widetilde{\alpha}_1|}, 1\right\},\\
\eta_1 &= \min\left\{\frac{1}{\Delta}\rho_m\tau \min_{1\leqslant j\leqslant m-1}(\beta_j - \lambda_j)^2, \frac{1}{4}\min_{1\leqslant l\leqslant m-2}(\beta_{l+1} - \beta_l)\right\},\\
\eta_2 &= \frac{1}{\Delta}\rho_m\tau(\beta_1 - \lambda_1)^2.
\end{aligned}$$

We shall see all the quantities in (89) are well-defined. By (86), (87) and $-\frac{\mu}{\nu} > 0$,

$$\widetilde{b} = -\frac{\mu + \widetilde{m}\nu}{\widetilde{\mu}} = -\frac{\mu}{\epsilon\nu}\left(1 + \frac{\widetilde{m}\nu}{\mu}\right) > 0. \tag{90}$$

This means $\widetilde{\mu}, \widetilde{b}, \widetilde{\nu}$ are well-defined. Note that $\epsilon < 1$, then $\widetilde{\nu}$ is nonzero. Moreover, by (85), $(\lambda - \lambda_{z_l})|\widetilde{F}(\lambda)$. Therefore, $\widetilde{G}(\lambda)$ is a well-defined monic polynomial of degree $m-1$. We immediately conclude (29) by (85) and (89). Besides, by (26) and (85)–(87), we deduce

$$-\frac{\widetilde{\mu}}{\widetilde{\nu}} = -\frac{\mu + \widetilde{m}\nu}{\nu - \widetilde{\mu}} > -\frac{\mu}{\nu}\left(1 + \frac{\widetilde{m}\nu}{\mu}\right) = -\frac{\mu}{\nu}\left(1 - \frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}\right) > -\frac{\mu}{\nu}\left(1 - \frac{\Delta\tau}{\Lambda}\right) > -\frac{\lambda_1}{\Lambda}\frac{\mu}{\nu} > 0. \tag{91}$$

We are now devoted to prove $\frac{\widetilde{\beta}_i - \lambda_i}{\beta_i - \lambda_i} < 2, i \in [1\ m-1]$. If $m > 2$,

$$\left|\frac{\widetilde{\mu}}{\nu}\right| = \epsilon = v_1 < \frac{\min\{1, \eta_1^{m-1}\}}{2 + 2\max_{1\leqslant j\leqslant m-1}\left|\prod_{h=1}^{m-1}(\beta_j - \widetilde{\alpha}_h)\right|}.$$

By applying Lemma A.1 to $\nu G(\lambda)$ and $\widetilde{\mu}\frac{\widetilde{F}(\lambda)}{\lambda - \lambda_{z_l}}$, we deduce

$$\max_{1\leqslant i\leqslant m-1}|\widetilde{\beta}_i - \beta_i| < \eta_1 \leq \frac{1}{\Delta}\rho_m\tau \min_{1\leqslant i\leqslant m-1}(\beta_i - \lambda_i)^2. \tag{92}$$

If $m = 2$, then $\left|\frac{\widetilde{\mu}}{\nu}\right| = v_2 < \frac{1}{2}\min\left\{\frac{\eta_2}{|\beta_1 - \widetilde{\alpha}_1|}, 1\right\}$. Hence Lemma A.1 leads to

$$|\widetilde{\beta}_1 - \beta_1| < \eta_2 = \frac{1}{\Delta}\rho_m\tau(\beta_1 - \lambda_1)^2. \tag{93}$$

In view of (92) and (93), we conclude

$$\max_{1\leqslant i\leqslant m-1}|\widetilde{\beta}_i - \beta_i| < \frac{\rho_m\tau}{\Delta}\min_{1\leqslant i\leqslant m-1}(\beta_i - \lambda_i)^2, \tag{94}$$

which, together with (82), yields

$$\left|\frac{\widetilde{\beta}_i - \lambda_i}{\beta_i - \lambda_i} - 1\right| = \left|\frac{\widetilde{\beta}_i - \beta_i}{\beta_i - \lambda_i}\right| < \frac{\rho_m}{\Delta}\tau(\beta_i - \lambda_i) < \frac{\rho_m^2\tau}{m2^{m+1}\Delta} < \frac{1}{2}, \quad i = 1, \ldots, m-1.$$

Therefore, we obtain

$$\frac{1}{2} < \frac{\widetilde{\beta}_i - \lambda_i}{\beta_i - \lambda_i} < 2, \quad i = 1, \ldots, m-1. \tag{95}$$

So, it remains to show the second inequality of (31) and (32). Note that if $i = z_l$,

$$\left|\frac{1}{\rho_m}\frac{\lambda_i - \widetilde{\alpha}_i}{\widetilde{\beta}_i - \lambda_i} - C_{q_1-l+1,i}\right| < \Omega^{q_1-l+1}\tau \tag{96}$$

is trivial because the left hand side of the inequality becomes zero. Therefore, it suffices to prove (96) when $i \neq z_l$. For that, we first bring in some estimates on $\{\alpha_j, \beta_j, \widetilde{\alpha}_j\}_{j=1}^{m-1}$. Fix $i \in [1\ m-1] \setminus \{z_l\}$. Observe that (20) and (22)–(27) imply

$$\begin{cases}\beta_j - \alpha_j < \frac{\rho_j}{\rho_m}\left(C_{q_1-l,j} - \Omega^{q_1-l}\tau\right)^{-1} + \rho_j < \frac{2\Lambda^2\rho_j}{\Delta\rho_m} < \frac{\Delta^2\tau}{m\Lambda}, & \text{if} \quad 1 \leq j \leq z_{l+1},\\ \beta_j - \alpha_j \leq \beta_j - \lambda_j < (1 + C_2(m-1))^n\rho_{m-1} < \frac{\Delta^2\tau}{m\Lambda}, & \text{if} \quad z_{l+1} < j < m-1,\end{cases} \tag{97}$$

where the last inequality is due to $2^n(\frac{2^{2(n+1)^2}\Lambda^{n+1}}{\Delta^n})^n\varepsilon < \frac{\Delta^2\tau}{m\Lambda}$. Then it follows

$$\sum_{j=1}^{m-1}\frac{\beta_j - \alpha_j}{\Delta} < m\frac{\frac{\Delta^2\tau}{m\Lambda}}{\Delta} < \frac{\Delta\tau}{\Lambda} < \frac{1}{2}. \tag{98}$$

In view of (26) and (98), for $h \in [1\ m]$,[1]

$$\prod_{j=1,j\neq h}^{m-1} \frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j} = \prod_{j=1}^{h-1}\left(1+\frac{\beta_j-\alpha_j}{\lambda_h-\beta_j}\right)\prod_{j=h+1}^{m-1}\left(1-\frac{\beta_j-\alpha_j}{\beta_j-\lambda_h}\right) < \prod_{j=1}^{h-1}\left(1+\frac{\beta_j-\alpha_j}{\Delta}\right) < 1+\sum_{j=1}^{h-1}\frac{2(\beta_j-\alpha_j)}{\Delta}. \tag{99}$$

On the other hand, by the *Bernoulli inequality*, for $h \in [1\ m]$,

$$\prod_{j=1,j\neq h}^{m-1} \frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j} = \prod_{j=1}^{h-1}\left(1+\frac{\beta_j-\alpha_j}{\lambda_h-\beta_j}\right)\prod_{j=h+1}^{m-1}\left(1-\frac{\beta_j-\alpha_j}{\beta_j-\lambda_h}\right) > \prod_{j=h+1}^{m-1}\left(1-\frac{\beta_j-\alpha_j}{\Delta}\right) > 1-\sum_{j=h+1}^{m-1}\frac{\beta_j-\alpha_j}{\Delta}. \tag{100}$$

That is, by (98),

$$\left|\prod_{j=1,j\neq h}^{m-1} \frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j}-1\right| < \max\left\{\sum_{j=1}^{h-1}\frac{2(\beta_j-\alpha_j)}{\Delta}, \sum_{j=h+1}^{m-1}\frac{\beta_j-\alpha_j}{\Delta}\right\} < \frac{2\Delta\tau}{\Lambda}. \tag{101}$$

Particularly, taking $h = z_l, i, m$ in (101) successively gives

$$\max\left\{\left|\prod_{j=1,j\neq z_l}^{m-1} \frac{\lambda_{z_l}-\alpha_j}{\lambda_{z_l}-\beta_j}-1\right|, \left|\prod_{j=1,j\neq i}^{m-1} \frac{\lambda_i-\alpha_j}{\lambda_i-\beta_j}-1\right|, \left|\prod_{j=1}^{m-1} \frac{\lambda_m-\alpha_j}{\lambda_m-\beta_j}-1\right|\right\} < \frac{2\Delta\tau}{\Lambda} < 1. \tag{102}$$

Moreover, by (88), arguing as in (97)–(100), we obtain

$$\left|\prod_{j=1,j\neq i}^{m-1} \frac{\lambda_i-\widetilde{\alpha}_j}{\lambda_i-\beta_j}-1\right| < \max\left\{\sum_{j=1}^{i-1}\frac{2(\beta_j-\widetilde{\alpha}_j)}{\Delta}, \sum_{j=i+1}^{m-1}\frac{\beta_j-\widetilde{\alpha}_j}{\Delta}\right\} < \frac{2\Delta\tau}{\Lambda}. \tag{103}$$

Based on the above estimates, we can establish two important inequalities below. By (22), (27), (28) and (102), for $h \in [1\ m-1]$,

$$\begin{aligned}
&\left|\frac{F(\lambda_h)}{\rho_m G(\lambda_h)}-(\lambda_m-\lambda_h)C_{q_1-l,h}\right| = \left|(\alpha_m-\lambda_h)\frac{1}{\rho_m}\frac{\lambda_h-\alpha_h}{\beta_h-\lambda_h}\prod_{j=1,j\neq h}^{m-1}\frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j}-(\lambda_m-\lambda_h)C_{q_1-l,h}\right| \\
&\leq (\alpha_m-\lambda_h)\left|\prod_{j=1,j\neq h}^{m-1}\frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j}\right|\left|\frac{1}{\rho_m}\frac{\lambda_h-\alpha_h}{\beta_h-\lambda_h}-C_{q_1-l,h}\right|+2C_{q_1-l,h}(\alpha_m-\lambda_m)+(\lambda_m-\lambda_h)C_{q_1-l,h}\left|\prod_{j=1,j\neq h}^{m-1}\frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j}-1\right| \\
&\leq 2(\alpha_m-\lambda_h)\left|\frac{1}{\rho_m}\frac{\lambda_h-\alpha_h}{\beta_h-\lambda_h}-C_{q-l,h}\right|+2C_{q_1-l,h}(\alpha_m-\lambda_m)+(\lambda_m-\lambda_h)C_{q_1-l,h}\left|\prod_{j=1,j\neq h}^{m-1}\frac{\lambda_h-\alpha_j}{\lambda_h-\beta_j}-1\right| \\
&< 2\Lambda\Omega^{q_1-l}\tau+\frac{2\Lambda}{\Delta\lambda_1}\rho_m+\tau < 2\Lambda\Omega^{q_1-l}\tau+2\tau.
\end{aligned} \tag{104}$$

Furthermore, define

$$V_{z_l} = \left(1-\frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}\right)\prod_{j=1,j\neq i}^{m-1}\frac{\lambda_i-\widetilde{\alpha}_j}{\lambda_i-\beta_j}.$$

We remark that $V_{z_l} > \frac{1}{2}$. Indeed, (86), (87) and (103) infer

$$\left|V_{z_l}-1\right| \leq \frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}+\left(1-\frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}\right)\left|\prod_{j=1,j\neq i}^{m-1}\frac{\lambda_i-\widetilde{\alpha}_j}{\lambda_i-\beta_j}-1\right| < \frac{F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}+\left|\prod_{j=1,j\neq i}^{m-1}\frac{\lambda_i-\widetilde{\alpha}_j}{\lambda_i-\beta_j}-1\right| < \frac{3\Delta\tau}{\Lambda}. \tag{105}$$

We are now ready to prove (96). Substituting $\lambda = \lambda_i$ into (29), we obtain by (85) and (89) that

$$\frac{F(\lambda_i)}{G(\lambda_i)} = \lambda_i\left(-\frac{\widetilde{m}\nu}{\mu}\right)-\frac{\widetilde{b}\widetilde{\mu}}{\mu}\frac{\widetilde{F}(\lambda_i)}{G(\lambda_i)} = \frac{\lambda_i F(\lambda_{z_l})}{\lambda_{z_l}G(\lambda_{z_l})}+(\widetilde{\alpha}_m-\lambda_i)\frac{\lambda_i-\widetilde{\alpha}_i}{\beta_i-\lambda_i}V_{z_l}. \tag{106}$$

We discuss (106) by two cases.

(i) Let $z_l = m$. Note that $s_1(l) \leq s_1(q_1) \leq \lambda_m$ and $\lambda_{z_l} = s_1(l) = \lambda_m$ imply $l = q_1 = m$. Thus, $(\lambda_m-\lambda_i)C_{q_1-l,i} = 1$. In this case, (106) becomes

$$\frac{F(\lambda_i)}{G(\lambda_i)} = \rho_m\frac{\lambda_i}{\lambda_m}\prod_{j=1}^{m-1}\frac{\lambda_m-\alpha_j}{\lambda_m-\beta_j}+(\lambda_m-\lambda_i)\frac{\lambda_i-\widetilde{\alpha}_i}{\beta_i-\lambda_i}V_m,$$

[1] For $p,q \in \mathbb{N}^+$, we define $\prod_{i=p}^{q} a_i = 1$ and $\sum_{i=p}^{q} a_i = 0$ if $p > q$.

which, together with (87), (102), (104) with $h = i$ and $V_{z_l} > \frac{1}{2}$, infers

$$
\begin{aligned}
&\left|\frac{1}{\rho_m}\frac{\lambda_i - \widetilde{\alpha}_i}{\beta_i - \lambda_i} - \frac{1}{\lambda_m}\right| = \left|\frac{1}{(\lambda_m - \lambda_i)V_m}\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - \frac{\lambda_i}{\lambda_m}\prod_{j=1}^{m-1}\frac{\lambda_m - \alpha_j}{\lambda_m - \beta_j}\right) - \frac{1}{\lambda_m}\right| \\
&= \frac{1}{\lambda_m(\lambda_m - \lambda_i)V_m}\left|\lambda_m\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - 1\right) - \lambda_i\left(\prod_{j=1}^{m-1}\frac{\lambda_m - \alpha_j}{\lambda_m - \beta_j} - 1\right) - (\lambda_m - \lambda_i)\left(V_m - 1\right)\right| \\
&< \frac{2}{\Delta}\left(\left|\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - 1\right| + \left|\prod_{j=1}^{m-1}\frac{\lambda_m - \alpha_j}{\lambda_m - \beta_j} - 1\right| + \left|V_m - 1\right|\right) < \frac{2}{\Delta}\left(2\Lambda\tau + 2\tau + \frac{2\Delta\tau}{\Lambda} + \frac{3\Delta\tau}{\Lambda}\right) < \frac{13\Lambda^2}{\Delta^2}\tau.
\end{aligned} \tag{107}
$$

So, combining (95) and (107), we obtain for $i \in [1\ m-1]$,

$$
\begin{aligned}
\left|\frac{1}{\rho_m}\frac{\lambda_i - \widetilde{\alpha}_i}{\widetilde{\beta}_i - \lambda_i} - C_{q_1-l+1,i}\right| &< \frac{13\Lambda^2}{\Delta^2}\Omega^{q_1-l}\tau + \frac{|\lambda_i - \widetilde{\alpha}_i|}{\rho_m}\frac{|\widetilde{\beta}_i - \beta_i|}{|(\widetilde{\beta}_i - \lambda_i)(\beta_i - \lambda_i)|} < \frac{13\Lambda^2}{\Delta^2}\Omega^{q_1-l}\tau + \frac{\Lambda}{\rho_m}\frac{\rho_m\tau}{\Delta}\frac{|\beta_i - \lambda_i|}{|\widetilde{\beta}_i - \lambda_i|} \\
&< \frac{13\Lambda^2}{\Delta^2}\Omega^{q_1-l}\tau + \frac{2\Lambda}{\Delta}\tau < \Omega^{q_1-l+1}\tau.
\end{aligned} \tag{108}
$$

(ii) Let $z_l < m$. By (106), we deduce

$$
\begin{aligned}
&\left|\frac{1}{\rho_m}\frac{\lambda_i - \widetilde{\alpha}_i}{\beta_i - \lambda_i} - C_{q_1-l+1,i}\right| = \left|\frac{1}{(\widetilde{\alpha}_m - \lambda_i)V_{z_l}}\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - \frac{\lambda_i}{\lambda_{z_l}}\frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})}\right) - C_{q_1-l+1,i}\right| \\
&= \left|\frac{1}{V_{z_l}}\left(\frac{1}{\lambda_m - \lambda_i} + \frac{\lambda_m - \widetilde{\alpha}_m}{(\widetilde{\alpha}_m - \lambda_i)(\lambda_m - \lambda_i)}\right)\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - \frac{\lambda_i}{\lambda_{z_l}}\frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})}\right) - C_{q_1-l+1,i}\right| \\
&\le J + K,
\end{aligned} \tag{109}
$$

where

$$
\begin{aligned}
J &= \frac{1}{V_{z_l}(\lambda_m - \lambda_i)}\left|\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - \frac{\lambda_i}{\lambda_{z_l}}\frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})}\right) - V_{z_l}\left(\lambda_m - \lambda_i\right)C_{q_1-l+1,i}\right|, \\
K &= \left|\frac{\lambda_m - \widetilde{\alpha}_m}{V_{z_l}(\widetilde{\alpha}_m - \lambda_i)(\lambda_m - \lambda_i)}\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - \frac{\lambda_i}{\lambda_{z_l}}\frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})}\right)\right|.
\end{aligned}
$$

It suffices to estimate J and K in (109). To estimate J, note that $z_l \in [1\ m-1]$, then by the definition of $C_{j,i}$ with $i \in [1\ m-1]$,

$$
(\lambda_m - \lambda_i)C_{q_1-l,i} - \frac{\lambda_i}{\lambda_{z_l}}(\lambda_m - \lambda_{z_l})C_{q_1-l,z_l} = (\lambda_m - \lambda_i)C_{q_1-l+1,i}. \tag{110}
$$

Since $V_{z_l} > \frac{1}{2}$, it follows from (22), (103)–(105) and (110) that

$$
\begin{aligned}
J &= \frac{1}{V_{z_l}(\lambda_m - \lambda_i)}\left|\left(\frac{F(\lambda_i)}{\rho_m G(\lambda_i)} - (\lambda_m - \lambda_i)C_{q_1-l,i}\right) - \frac{\lambda_i}{\lambda_{z_l}}\left(\frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})} - (\lambda_m - \lambda_{z_l})C_{q_1-l,z_l}\right) - (\lambda_m - \lambda_i)C_{q_1-l+1,i}\left(V_{z_l} - 1\right)\right| \\
&< \frac{1}{\Delta}\left(2\Lambda\Omega^{q_1-l}\tau + 2\tau + \frac{\Lambda}{\Delta}\left(2\Lambda\Omega^{q_1-l}\tau + 2\tau\right) + \frac{3\Delta\tau}{\Lambda}\right) < \frac{11\Lambda^2}{\Delta^2}\Omega^{q_1-l}\tau.
\end{aligned} \tag{111}
$$

Next, we estimate K. By (85), plugging $\lambda = \widetilde{\alpha}_m$ into (29) gives

$$
\frac{F(\widetilde{\alpha}_m)}{\widetilde{\alpha}_m G(\widetilde{\alpha}_m)} = -\frac{\widetilde{m}\nu}{\mu} = \frac{F(\lambda_{z_l})}{\lambda_{z_l} G(\lambda_{z_l})}.
$$

We write it as

$$
\frac{\widetilde{\alpha}_m - \alpha_m}{\widetilde{\alpha}_m}\prod_{j=1}^{m-1}\frac{\widetilde{\alpha}_m - \alpha_j}{\widetilde{\alpha}_m - \beta_j} = \frac{\alpha_m - \lambda_{z_l}}{\lambda_{z_l}}\frac{\lambda_{z_l} - \alpha_{z_l}}{\beta_{z_l} - \lambda_{z_l}}\prod_{j=1, j\neq z_l}^{m-1}\frac{\lambda_{z_l} - \alpha_j}{\lambda_{z_l} - \beta_j}. \tag{112}
$$

Observe that

$$
\frac{\widetilde{\alpha}_m - \alpha_m}{\widetilde{\alpha}_m}\prod_{j=1}^{m-1}\frac{\widetilde{\alpha}_m - \alpha_j}{\widetilde{\alpha}_m - \beta_j} > \frac{\widetilde{\alpha}_m - \alpha_m}{\widetilde{\alpha}_m}. \tag{113}
$$

On the other hand, by (20), (22), (27), (28) and (102),

$$
\begin{aligned}
\frac{\alpha_m - \lambda_{z_l}}{\lambda_{z_l}}\frac{\lambda_{z_l} - \alpha_{z_l}}{\beta_{z_l} - \lambda_{z_l}}\prod_{j=1, j\neq z_l}^{m-1}\frac{\lambda_{z_l} - \alpha_j}{\lambda_{z_l} - \beta_j} &= \left(\frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} + \frac{\alpha_m - \lambda_m}{\lambda_{z_l}}\right)\frac{\lambda_{z_l} - \alpha_{z_l}}{\beta_{z_l} - \lambda_{z_l}}\prod_{j=1, j\neq z_l}^{m-1}\frac{\lambda_{z_l} - \alpha_j}{\lambda_{z_l} - \beta_j} \\
&< \left(\frac{\lambda_m - \lambda_{z_l}}{\lambda_{z_l}} + \frac{2\Lambda\rho_m}{\lambda_1\lambda_{z_l}}\right)\rho_m(C_{q_1-l,z_l} + \Omega^{q_1-l}\tau)\left(1 + \frac{2\Delta\tau}{\Lambda}\right)
\end{aligned}
$$

$$
\begin{aligned}
&< \left(\frac{2\Lambda\rho_m}{\lambda_1\lambda_{z_l}}\left(\frac{1}{2\Delta}+\frac{\Delta}{\Lambda^2}\right)2+\left(\frac{\Lambda}{2\lambda_1\Delta}+\frac{\Lambda}{\lambda_1}\Omega^{q_1-l}\tau\right)\frac{2\Delta}{\Lambda}\tau+\frac{\Lambda}{\lambda_1}\Omega^{q_1-l}\tau\right)\rho_m \\
&\quad + \left(\frac{\lambda_m-\lambda_{z_l}}{\lambda_{z_l}}C_{q_1-l,z_l}\right)\rho_m \\
&< \left(\frac{\lambda_m-\lambda_{z_l}}{\lambda_{z_l}}C_{q_1-l,z_l}+\frac{2\Lambda}{\lambda_1}\Omega^{q_1-l}\tau\right)\rho_m. \qquad (114)
\end{aligned}
$$

Define $H=\frac{\lambda_m-\lambda_{z_l}}{\lambda_{z_l}}C_{q_1-l,z_l}+\frac{2\Lambda}{\lambda_1}\Omega^{q_1-l}\tau$, then by (16), (20) and (22),

$$
H<\frac{\Lambda}{2\Delta^2}+\frac{2}{\Lambda}.
$$

Hence, by (20), (28), (85) and (114),

$$
\begin{aligned}
\frac{\widetilde{\alpha}_m-\lambda_m}{\rho_m} &= \frac{\alpha_m-\lambda_m}{\rho_m}+\frac{\widetilde{\alpha}_m-\alpha_m}{\rho_m}<\frac{\alpha_m-\lambda_m}{\rho_m}+\frac{H}{1-H\rho_m}\alpha_m<\frac{\alpha_m-\lambda_m}{\rho_m}+\left(H+\frac{H^2\rho_m}{1-H\rho_m}\right)\left(\lambda_m+\frac{2\Lambda}{\lambda_1}\rho_m\right) \\
&< \frac{\alpha_m-\lambda_m}{\rho_m}+\lambda_m\frac{\lambda_m-\lambda_{z_l}}{\lambda_{z_l}}C_{q_1-l,z_l}+\frac{2\Lambda^2}{\lambda_1}\Omega^{q_1-l}\tau+H\frac{2\Lambda}{\lambda_1}\rho_m+2H^2\rho_m\left(\Lambda+\frac{2\Lambda}{\lambda_1}\rho_m\right) \\
&< \frac{\alpha_m-\lambda_m}{\rho_m}+\lambda_m\frac{\lambda_m-\lambda_{z_l}}{\lambda_{z_l}}C_{q_1-l,z_l}+\frac{3\Lambda^2}{\lambda_1}\Omega^{q_1-l}\tau. \qquad (115)
\end{aligned}
$$

Recall that $\widetilde{\alpha}_m>\lambda_m$ and $V_{z_l}>\frac{1}{2}$, by (28) and (115),

$$
\frac{\widetilde{\alpha}_m-\lambda_m}{V_{z_l}(\widetilde{\alpha}_m-\lambda_i)(\lambda_m-\lambda_i)}<\frac{\left(\frac{\alpha_m-\lambda_m}{\rho_m}+\frac{\widetilde{\alpha}_m-\alpha_m}{\rho_m}\right)\rho_m}{V_{z_l}(\lambda_m-\lambda_i)^2}<\frac{\left(\frac{2\Lambda}{\lambda_1}+\lambda_m\frac{\lambda_m-\lambda_{z_l}}{\lambda_{z_l}}C_{q_1-l,z_l}+\frac{3\Lambda^2}{\lambda_1}\Omega^{q_1-l}\tau\right)\rho_m}{2\Delta^2}. \qquad (116)
$$

Moreover, (26) and (104) imply

$$
\left|\frac{F(\lambda_i)}{\rho_m G(\lambda_i)}-\frac{\lambda_i}{\lambda_{z_l}}\frac{F(\lambda_{z_l})}{\rho_m G(\lambda_{z_l})}\right|<\left(1+\frac{\Lambda}{\Delta}\right)\left(\frac{\Lambda}{2\Delta}+\left(2\Lambda\Omega^{q_1-l}\tau+2\tau\right)\right). \qquad (117)
$$

So, $K<\tau$ by (16), (17), (116) and (117). This together with (109) and (111), infers

$$
\left|\frac{1}{\rho_m}\frac{\lambda_i-\widetilde{\alpha}_i}{\beta_i-\lambda_i}-C_{q_1-l+1,i}\right|<\frac{13\Lambda^2}{\Delta^2}\Omega^{q_1-l}\tau. \qquad (118)
$$

Analogously to (108), we obtain for $i\in[1\ m-1]$,

$$
\left|\frac{1}{\rho_m}\frac{\lambda_i-\widetilde{\alpha}_i}{\widetilde{\beta}_i-\lambda_i}-C_{q_1-l+1,i}\right|<\Omega^{q_1-l+1}\tau.
$$

So far, we have proved (30) and (31). We remark that (32) holds as well because of (88) and (115). The proof is completed. □

Appendix B. Proofs of Lemma 4.4

To prove Lemma 4.4, we need to modify [17, Lemmas B.3. and B.4.] to derive two lemmas. Both of the lemmas and their proofs will follow the notations used in [17, Appendix B].

Lemma B.1. *[17, Lemma B.3.] holds if [17, Eqs. (61)–(62)] are replaced respectively by*

$$
\alpha_p\in[\lambda_p-\rho_p,\bar{\lambda})\quad \textit{for some number}\quad \bar{\lambda}>\lambda_p, \qquad (119)
$$

$$
\lambda_i+C_1^n\rho_i-\rho_i\le\alpha_i+C_1^n\rho_i<\beta_i<\lambda_i+(1+C_2(i))^n\rho_i<\lambda_{i+1}-\rho_{i+1}. \qquad (120)
$$

Proof of Lemma B.1. The proof is the same as that of [17, Lemma B.3.] except [17, Eqs. (67) and (77)]. As a matter of fact, in view of (119) and (120), [17, Eq. (67)] becomes

$$
\alpha_p\ge\lambda_p-\rho_p>\lambda_{p-1}+(1+C_2(p-1))^n\rho_{p-1}>\beta_{p-1}>\alpha_{p-1}>\cdots>\beta_1>\alpha_1, \qquad (121)
$$

and hence $\lambda G(\lambda)\prec F(\lambda)$ still holds. By (20) and (120), for any $j<i$, [17, Eq. (77)] can be rewritten by

$$
\frac{\widetilde{\beta}_i-\beta_j}{\widetilde{\beta}_i-\widetilde{\alpha}_j}>\frac{\widetilde{\beta}_i-\beta_j}{\widetilde{\beta}_i-\alpha_j}=1-\frac{\beta_j-\alpha_j}{\widetilde{\beta}_i-\alpha_j}>1-\frac{\beta_j-\lambda_j+\rho_j}{\widetilde{\beta}_i-\lambda_j+\rho_j}>1-\frac{(1+C_2(j))^n\rho_j+\rho_j}{2\Delta}\ge1-\frac{(1+C_2(j))^n\rho_j}{\Delta}, \qquad (122)
$$

which does not change the result. □

Lemma B.2. *[17, Lemma B.4.] holds if [17, Eqs. (61)–(62)] are replaced by (119) and (120), respectively.*

Proof of Lemma B.2. Firstly, by (121), $\lambda G(\lambda) \prec F(\lambda)$. Then, it suffices to restate the following inequalities, which are used to prove [17, Eq. (62)]. According to the proof of [17, Lemma B.4.], $\widetilde{\alpha}_j \in (\alpha_j, \beta_j)$. As a result, for $j \in [i+1\ p-1]$, (120) gives

$$\frac{\beta_i - \alpha_j}{\beta_i - \widetilde{\alpha}_j} = 1 - \frac{\widetilde{\alpha}_j - \alpha_j}{\widetilde{\alpha}_j - \beta_i} \geq 1 - \frac{\beta_j - \alpha_j}{\alpha_j - \beta_i} \geq 1 - \frac{\beta_j - \lambda_j + \rho_j}{\lambda_{i+1} - \rho_{i+1} - \beta_i},$$

which together with (21) and (120) yield

$$\frac{\beta_i - \alpha_j}{\beta_i - \widetilde{\alpha}_j} \geq 1 - \frac{\beta_j - \lambda_j + \rho_j}{(\lambda_{i+1} - \lambda_i) - (\beta_i - \lambda_i) - \rho_{i+1}} \geq 1 - \frac{(1 + C_2(j))^n \rho_j + \rho_j}{2\Delta - (1 + C_2(i))^n \rho_i - \rho_{i+1}} > \frac{1}{2}.$$

Furthermore, $\alpha_p \in [\lambda_p - \rho_p, \widetilde{\lambda}) \subset [\lambda_p - \rho_p, \Lambda)$, (16), (21) and (120) lead to

$$\frac{\beta_i - \alpha_p}{\beta_i - \widetilde{\alpha}_p} \geq \frac{\lambda_{i+1} - \rho_{i+1} - \beta_i}{\Lambda} \geq \frac{2\Delta - (1 + C_2^n(i))\rho_i - \rho_{i+1}}{\Lambda} > \frac{\Delta}{2\Lambda}.$$

The remaining part keeps the same as that of [17, Lemma B.4.]. □

Proof of Lemma 4.4. Since the analysis is essentially the same as that of [17, Lemma 4.1], we still use the notations defined in the proof of [17, Lemma 4.1]. First of all, we remark that polynomials $F_{n-q_1}(\lambda)$, $G_{n-q_1}(\lambda)$ and numbers μ_{n-q_1}, ν_{n-q_1} constructed in Lemma 4.3 meet the conditions of Lemma B.1. In fact, by Lemma 4.3(iii), $\alpha_{n-q_1}(m) \in [\lambda_m - \rho_m, \Lambda)$. Moreover, since (18) implies $C_1^n < C_1^{q_1}$, (35) shows

$$\lambda_i + C_1^n \rho_i - \rho_i \leq \alpha_{n-q_1}(i) + C_1^n \rho_i < \beta_{n-q_1}(i), \quad i = 1, \dots, m-1. \tag{123}$$

Observe that by (16) and (21),

$$\lambda_i + (1 + C_2(i))^n \rho_i < \lambda_{i+1} - \rho_{i+1}, \quad i = 1, \dots, m-1, \tag{124}$$

which combined with (35) infers

$$\beta_{n-q_1}(i) < \lambda_i + (1 + C_2(i))^n \rho_i < \lambda_{i+1} - \rho_{i+1}, \quad i = 1, \dots, m-1. \tag{125}$$

Recall that Lemma 4.3 asserts $-\frac{\mu_{n-q_1}}{\nu_{n-q_1}} > C(\frac{\Lambda}{\lambda_1})^k \frac{\Lambda M}{\Lambda - \lambda_1}$. Consequently, we can construct polynomials $F_{n-q_1-1}(\lambda)$ and $G_{n-q_1-1}(\lambda)$ in light of Lemma B.1.

So, similarly to the proof of [17, Lemma 4.1], assume that for some $r \in [1+q_1\ n-2]$, we have constructed the desired $F_{n-j}(\lambda)$ and $G_{n-j}(\lambda)$, $j \in [1+q_1\ r]$ according to either Lemma B.1 or Lemma B.2.[2] Particularly, for all $j \in [q_1\ r-1]$,

$$\begin{aligned}\alpha_{n-q_1}(i) + C_1^j \rho_i &< \alpha_{n-j}(i) + C_1^j \rho_i < \beta_{n-j}(i) \\ &< \alpha_{n-q_1}(i) + (1 + C_2(i))^j (\beta_{n-q_1}(i) - \alpha_{n-q_1}(i)).\end{aligned} \tag{126}$$

The only difference from the proof of [17, Lemma 4.1] is that instead of verifying [17, Eq. (96)], we are devoted to check

$$\lambda_i + C_1^n \rho_i - \rho_i \leq \alpha_{n-r}(i) + C_1^n \rho_i < \beta_{n-r}(i) < \lambda_i + (1 + C_2(i))^n \rho_i < \lambda_{i+1} - \rho_{i+1}. \tag{127}$$

Note that by (20), (21), (35) and (126), Lemmas B.1–B.2 lead to

$$\begin{aligned}\beta_{n-r}(i) - \lambda_i &< \alpha_{n-q_1}(i) - \lambda_i + (1 + C_2(i))^r (\beta_{n-q_1}(i) - \lambda_i + \lambda_i - \alpha_{n-q_1}(i)) \\ &< 2^{m+1} \Lambda \frac{\rho_i}{\rho_m} + (1 + C_2(i))^r \left(2^{m+1} \Lambda \frac{\rho_i}{\rho_m} + \rho_i\right) \\ &< 2^{m+1} \Lambda \frac{\rho_i}{\rho_m} + (1 + C_2(i))^{r+1} \rho_i \\ &< (1 + C_2(i))^n \rho_i < \frac{\Delta}{2}.\end{aligned} \tag{128}$$

On the other hand, by Lemmas B.1–B.2 again, (35) and (126) infer

$$\beta_{n-r}(i) - \alpha_{n-r}(i) > C_1^{r-q_1} \left(\beta_{n-q_1}(i) - \alpha_{n-q_1}(i)\right) > C_1^r \rho_i.$$

Therefore,

$$\lambda_i + C_1^n \rho_i - \rho_i < \alpha_{n-r}(i) + C_1^n \rho_i < \alpha_{n-r}(i) + C_1^r \rho_i < \beta_{n-r}(i). \tag{129}$$

So (127) is true in view of (124), (128) and (129). Furthermore, observe that $\alpha_{n-r}(z_{n-r}) \in [\lambda_{z_{n-r}}, \Lambda) \subset [\lambda_{z_{n-r}} - \rho_{z_{n-r}}, \Lambda)$ due to the proof of [17, Lemma 4.1]. We thus omit the rest of the proof that is identical to that of [17, Lemma 4.1]. □

[2] In the proof, we do not mention μ_j, ν_j, $j = 1, \dots, n - q_1 - 1$, because the relevant analysis is the same as that of [17, Lemma 4.1].

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