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IS IT POSSIBLE TO STABILIZE DISCRETE-TIME PARAMETERIZED UNCERTAIN SYSTEMS GROWING EXPONENTIALLY FAST?*

ZHAOBO LIU† AND CHANYING LI‡

Abstract. This paper derives a somewhat surprising but interesting result on the stabilizability of discrete-time parameterized uncertain systems. Contrary to intuition, it shows that the nonlinear growth rate of a discrete-time stabilizable system with linear parameterization is not necessarily small all the time. More specifically, to achieve stabilizability, the system function $f(x) = O(|x|^b)$ with $b < 4$ is only required for a very tiny fraction of x in $\mathbb{R}$, even if it grows exponentially fast for the other x. This x set is called a regular set, whose density determines the stabilizability of a nonlinear discrete-time control system. The densities of regular sets have also been computed for both stabilizable and unstabilizable systems with scalar parameters. As indicated herein, the density could be extremely sparse, while the corresponding system is stabilizable.

Key words. stochastic adaptive control, feedback limitations, stabilizability, nonlinear systems, discrete-time, least squares

AMS subject classifications. 93D15, 93D20, 93E24

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1. Introduction.

Linear systems [1], [2], [5], [7] and nonlinear systems with nonlinearities having linear growth rates [24], [26] are studied extensively in adaptive control theory. It is natural that we concentrate here on systems with output nonlinearities growing faster than linearities. Such investigations in the literature are mostly focused on control systems in continuous time [22], [10], [8]. Now comes the noteworthy part. The similarities of adaptive control between continuous- and discrete-time systems no longer exist. A large class of continuous-time nonlinear systems can be globally stabilized by applying nonlinear damping or back-stepping techniques, regardless of how fast its growth rate is [9], [11]. But its discrete-time counterpart is obviously lack of such good property. It was found early in [4] that fundamental difficulties arise for adaptive control of discrete-time nonlinear parameterized systems. Guo [4] proved that any feedback control law may fail to stabilize a discrete-time parameterized system if its nonlinearity is too high. Such a problem also troubles the control of discrete-time nonparametric nonlinear systems [27], [12], [30], semiparametric uncertain systems [6], [23], linear stochastic systems with unknown time-varying parameter processes [28], and continuous-time nonlinear systems with sampled-date observations for prescribed sampling rates [29].

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†Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, People's Republic of China (liuzhaobo15@mails.ucas.ac.cn).
‡Corresponding author. Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, People's Republic of China (cyli@amss.ac.cn).

All the phenomena suggest that a feedback has its limit in stabilizing a discrete-time uncertain system. The feedback limit was first characterized by an exponent $b = 4$ in [4], where a discrete-time nonlinear stochastic system with a scalar parameter was studied:

$$y_{t+1} = \theta y_t^b + u_t + w_{t+1}.$$

It showed the system is stabilizable if and only if $b < 4$. Later on, [25] confirmed the idea of [4] on feedback limitations by providing an "impossibility theorem" for the following multiparameter uncertain system:

$$y_{t+1} = \theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_n y_t^{b_n} + u_t + w_{t+1}. \tag{1}$$

A polynomial rule was proposed therein to describe the nonlinear growth rates that fail all discrete-time feedback control laws in stabilizing system (1). This rule was recently proved to be the necessary and sufficient condition of the stabilizability of system (1) in [17]. Note that the systems mentioned above are of linear parametrization. As to the nonlinear parametrization case, some initial research indicates as well that $b = 4$ is indeed an important exponent for the stabilizability of the underlying uncertain scalar-parameter systems [19]. Meanwhile, a parallel theory for the stabilizability of discrete-time systems in the deterministic framework has also been developed accordingly. Interested readers are referred to [13], [14], [15], [16], [18], [20], [21].

If we consider a random model

$$y_{t+1} = \theta^T f(y_t) + u_t + w_{t+1}, \quad \theta \in \mathbb{R}^n, \tag{2}$$

it is tempting to believe that function $f(x)$ for a stabilizable system should obey the polynomial rule characterized in [25] (this rule degenerates to $b = 4$ when the parameter is of one dimension), at least for most $x \in \mathbb{R}^n$. It may be a little frustrating that the polynomial rule forces the largest value of the exponents $b_1, b_2, \ldots, b_n$ around 1, whenever n (the number of the unknown parameters) is very large. That means, with sufficiently many parameters, the expected nonlinear growth rate of a stablizable system in form (2) would be close to linear. So, people might guess that discrete-time feedback control has very limited capability in dealing with nonlinear systems.

But, the truth is unexpected. For the scalar-parameter case, if we denote the set of x that $f(x) = O(|x|^b)$ with $b < 4$ by S_b^L, the results of this paper find that a stabilizable system could admit of S_b^L being a very tiny fraction of $\mathbb{R}$. How tiny? As long as the density of S_b^L in $\mathbb{R}$ does not equal to zero! Roughly speaking, for any small $\epsilon > 0$, a scalar-parameter system with

$$\liminf_{l \to +\infty} \frac{\ell\{x \in [-l, l] : f(x) \text{ grows slower than } |x|^b, b < 4\}}{\ell\{x \in [-l, l] : f(x) \text{ grows exponentially}\}} = \epsilon$$

is still stabilizable, where ℓ denotes the Lebesgue measure. The least-squares (LS) based self-tuning regulator, as shown later, is competent to perform the stabilizing task. This tells us that a nonlinear discrete-time parameterized system, which grows very fast for most of the time, still stands a chance to be stabilized by some discrete-time feedback controller. It is not a surprise that continuous-time controllers could fulfill such works, as they regulate systems at every moment. However, this is not obvious in discrete-time control. There is a certain amount of information loss during controller designs by using sampled data, especially for running a long time. In addition, our results also derive the density of S_4^L for unstabilizable systems.

The rest of the paper is organized as follows. Section 2 presents our main results on the stabilizabitlity of a basic class of nonlinear discrete-time parameterized systems. The proof of the stabilizablity theorem is contained in section 3, while section 4 treats the unstabilizability part. Concluding remarks are given in section 5.

2. Main results. We consider the following parameterized uncertain system:

$$y_{t+1} = \theta f(y_t) + u_t + w_{t+1}, \qquad t \geq 0, \tag{3}$$

where $\theta \in \mathbb{R}$ is an unknown parameter, and $y_t, u_t, w_t \in \mathbb{R}$ are the system output, input, and noise signals, respectively. Assume $f : \mathbb{R} \to \mathbb{R}$ is a known piecewise continuous function and the initial value y_0 is independent of θ and $\{w_t\}$. Moreover, we have the following:

A1. The noise $\{w_t\}$ is an independent and identically distributed random sequence with $w_1 \sim N(0, 1)$.

A2. Parameter $\theta \sim N(\theta_0, P_0)$ is independent of $\{w_t\}$.

We begin by studying the stabilizability of system (3), which is defined as follows.

Definition 2.1. *System* (3) *is said to be globally stabilizable if there exists a feedback control law*

$$u_t \in \mathscr{F}_t^y \triangleq \sigma\{y_i, 0 \leqslant i \leqslant t\}, \ t \geq 0, \tag{4}$$

such that for any initial $y_0 \in \mathbb{R}$,

$$\sup_{t\geq 1} \frac{1}{t} \sum_{i=1}^{t} y_i^2 < +\infty, \quad a.s.$$

2.1. How does the density of a regular set matter? As already noted by [4],

$$|f(x)| = O(|x|^b) + O(1) \quad \text{with} \quad b < 4, \ x \in \mathbb{R} \tag{5}$$

is a very important growth rate to guarantee the stabilizability of system (3). We might claim that at least the growth rate (5) should hold for x in the vast majority of $\mathbb{R}$, or unstabilizability would be inevitable. Surprisingly, this is not the case. Denote

$$S_b^L \triangleq \{x \in \mathbb{R} : |f(x)| < L|x|^b\}.$$

We call S_b^L a *regular set* of system (3) if $b < 4$. The paper in fact claims that the density of a regular set in the real number field $\mathbb{R}$ determines the stabilizability of system (3). This density could be extremely sparse, while system (3) is still stabilizable, even if $f(x)$ grows exponentially for x outside S_b^L. The fact is verified by the following.

Theorem 2.2. *Under assumptions* A1–A2, *system* (3) *is globally stabilizable if*

(i) *for some* $k_1, k_2 > 0$,

$$|f(x)| \leq k_1 e^{k_2|x|} \quad \forall x \in \mathbb{R}; \tag{6}$$

(ii) *there exist two numbers* $b < 4$ *and* $L > 0$ *such that*

$$\liminf_{l\to+\infty} \frac{\ell(S_b^L \cap [-l, l])}{l} > 0, \tag{7}$$

where ℓ *denotes the Lebesgue measure.*

Remark 1. Let $p_b \triangleq \liminf_{t\to+\infty} p_b^l$ with $p_b^l \triangleq \frac{\ell(S_b^L\cap[-l,l])}{2l}$; then p_b describes the density of the regular set $S_b^L, b<4$ in $\mathbb{R}$. Since $p_b>0$ can be taken as small as one likes, Theorem 2.2 produces an interesting finding that the growth rate (5) is only necessary for a set of x extremely sparse in $\mathbb{R}$.

If $f(x)$ grows no faster than a power function, then p_b^l, the density of the regular set S_b^L in interval $[-l,l]$, can converge to zero with a rate $\frac{1}{\log\log l}$ as $l\to+\infty$ for some properly small $b>0$.

THEOREM 2.3. *Under assumptions* A1–A2*, system* (3) *is globally stabilizable if*

(i) *for some* $a\ge 4$*,*

$$|f(x)| = O(|x|^a)+O(1), \quad \text{as } |x|\to+\infty;$$

(ii) *there exist two numbers* $b<(1+x_{\min})^2$ *and* $L>0$ *such that*

$$\liminf_{l\to+\infty}\frac{\ell(S_b^L\cap[-l,l])}{l}\cdot\log(\log l)>0,$$

where $x_{\min}$ *denotes the smallest solution of equation* $x^2-(a-2)x+1=0$*.*

2.2. What is the density of S_4^L for unstabilizable systems? Now, we turn to discuss the unstabilizability of system (3).

DEFINITION 2.4. *System* (3) *is unstabilizable if for any feedback control law* $\{u_t\}$ *defined by* (4)*, there exists an initial* y_0 *such that for some set* D *with* $P(D)>0$*,*

$$\limsup_{t\to+\infty}\frac{1}{t}\sum_{i=1}^{t} y_i^2 = +\infty \quad \text{on } D.$$

It is conceivable that the unstabilizability of system (3) depends on the sparsity of set $S_4^L=\{x:|f(x)|<L|x|^4\}$ in $\mathbb{R}$. Indeed, when the density of set S_4^L in any given interval with length l tends to zero rapidly as $l\to+\infty$, system (3) becomes unstabilizable. The required convergence rate is specified below.

THEOREM 2.5. *Under assumptions* A1–A2*, system* (3) *is unstabilizable if there exist two numbers* $\delta, L>0$ *such that as* $l\to+\infty$*,*

$$\sup_{x\in\mathbb{R}}\frac{\ell(S_4^L\cap[x-l,x+l])}{l} = O\left(\frac{1}{(\log(\log l))^{1+\delta}}\right).$$

Remark 2. Note that in Theorem 2.3, $b<(1+x_{\min})^2=4$ when $a=4$. Then, in view of Theorem 2.5, if $|f(x)|=O(|x|^4+1)$, we in fact derive a law of iterated logarithm $\frac{1}{\log\log l}$ that almost describes the "critical convergence rate" of p_b^l to guarantee the stabilizability of system (3).

Theorem 2.5 can be sharpened. Assume $h:[0,+\infty)\to[0,+\infty)$ is a nonnegative monotone increasing piecewise continuous function and satisfies $h(x)=O(x^4)+O(1)$. Let $g(x)\triangleq|x|^{-\frac14}h^{-1}(|x|)$, where h^{-1} denotes the inverse function of h. Theorem 2.5 is a direct consequence of the following theorem by taking $h(x)=Lx^4$ and $g(x)\equiv L^{-\frac14}$.

THEOREM 2.6. *Under assumptions* A1–A2*, system* (3) *is unstabilizable if there is a* $\delta>0$ *such that*

$$\sup_{x\in\mathbb{R}}\frac{\ell(S_h\cap[x-l,x+l])}{l} = O\left(\frac{1}{(\log(\log l))^{1+\delta}}\right),$$

where $S_h \triangleq \{x : |f(x)| < h(|x|)\}$ *with* h *satisfying*

$$\sum_{t=1}^{+\infty} \sup_{x\in[e^{2^t},+\infty)} x^{-\frac{1}{16t^2}} g(x) < +\infty. \tag{8}$$

3. Proof of Theorem 2.2. In order to prove the stabilizability of system (3), we construct a feedback control law based on the LS algorithm. The standard LS estimate θ_t for parameter θ can be recursively defined by

$$\begin{cases} \theta_{t+1} = \theta_t + a_t P_t \phi_t (y_{t+1} - u_t - \phi_t^T \theta_t), \\ P_{t+1} = P_t - a_t P_t \phi_t \phi_t^T P_t, \quad P_0 > 0, \\ \phi_t \triangleq f(y_t), \quad t \geq 0, \end{cases} \tag{9}$$

where $a_t \triangleq (1 + \phi_t^T P_t \phi_t)^{-1}$ and (θ_0, P_0) is the deterministic initial value of the algorithm. According to the "certainty equivalence principle," it is a natural way to design the stabilizing control by

$$u_t = -\theta_t f(y_t), \quad t \geq 0. \tag{10}$$

Now, for the closed-loop system (3), (9), and (10), one has

$$\begin{aligned} \tilde{\theta}_t &= \frac{1}{r_{t-1}} \left\{ \tilde{\theta}_0 - \sum_{i=0}^{t-1} \phi_i w_{i+1} \right\}, \\ y_{t+1} &= \tilde{\theta}_t f(y_t) + w_{t+1}, \end{aligned} \tag{11}$$

where $\tilde{\theta}_t \triangleq \theta - \theta_t$, $r_{-1} \triangleq P_0^{-1}$, $r_t \triangleq P_{t+1}^{-1} = P_0^{-1} + \sum_{i=0}^t \phi_i^2$, $t \geq 0$. Since the LS algorithm (9) is exactly the standard Kalman filter for $\theta \sim N(\theta_0, P_0)$, it yields that $\theta_t = E[\theta|\mathscr{F}_t^y]$ and $P_t = E[(\tilde{\theta}_t)^2|\mathscr{F}_t^y]$. Hence, y_{t+1} possesses a conditional Gaussian distribution given $\mathscr{F}_t^y$. For any $t \geq 0$, the conditional mean and variance are

$$m_t \triangleq E[y_{t+1}|\mathscr{F}_t^y] = u_t + \theta_t \phi_t = 0, \quad \text{a.s.} \tag{12}$$

$$\sigma_t^2 \triangleq Var(y_{t+1}|\mathscr{F}_t^y) = 1 + \phi_t P_t \phi_t = \frac{\phi_t^2}{r_{t-1}} + 1 = \frac{r_t}{r_{t-1}}, \quad \text{a.s.} \tag{13}$$

The proof of Theorem 2.2 is prefaced with several technique lemmas. The first presents a very simple fact, which is repeatedly used in the subsequent computations.

LEMMA 3.1. *If* $\{c_t\}_{t\geq 1}$ *satisfies* $\liminf_{t\to+\infty} \frac{c_t}{\log t} > 0$*, then*

$$\sum_{t=1}^{+\infty} \int_{|x|\geqslant c_t} e^{-\frac{x^2}{2}}\, dx < +\infty.$$

Proof. Since $\liminf_{t\to+\infty} \frac{c_t}{\log t} > 0$ implies that there is a $c > 0$ such that for any sufficiently large $t > 0$, $c_t > c \log t$, it suffices to prove $\sum_{t=1}^{+\infty} \int_{|x|\geqslant c\log t} e^{-\frac{x^2}{2}}\, dx < +\infty$. Note that for $t \geq \max\{e^{\frac{4}{c^2}}, 2\}$,

$$\begin{aligned} \int_{|x|\geqslant c\log t} e^{-\frac{x^2}{2}}\, dx &= \sum_{i=t}^{+\infty} \int_{c\log i}^{c\log(i+1)} e^{-\frac{x^2}{2}}\, dx < \sum_{i=t}^{+\infty} \int_{c\log i}^{c\log(i+1)} e^{-\frac{c^2\log^2 i}{2}}\, dx \\ &= \sum_{i=t}^{+\infty} c \log\left(\frac{i+1}{i}\right) i^{-\frac{c^2\log i}{2}} < \sum_{i=t}^{+\infty} \frac{c}{i^{1+\frac{c^2\log i}{2}}} \\ &\leq \sum_{i=t}^{+\infty} \frac{c}{i^3} < \int_{t-1}^{+\infty} \frac{c}{x^3}\, dx = \frac{c}{2(t-1)^2}, \end{aligned}$$

which leads to Lemma 3.1 immediately. ∎

LEMMA 3.2. *For any* $n \in \mathbb{Z}^+$, *let* $\{A_m^n\}_{m\geq 1}$ *be a sequence of events that* $A_m^n \triangleq \{y_{mn}, y_{mn+1}, \ldots, y_{mn+n-1} \in S_b^L\}$. *If* (7) *holds, then* $\sum_{m=1}^{+\infty} I_{A_m^n} = +\infty$ *almost surely.*

Proof. At first, the piecewise continuity of f infers that S_b^L contains a nonempty interval. Taking a point ρ from this interval, by (7), there exists a $c_1 > 0$ such that $\inf_{l>0} \frac{S_b^L \cap [\rho-l,\rho+l]}{l} > c_1$. Note that y_{i+1} is conditional Gaussian with the conditional mean $m_i = 0$ and variance σ_i^2 by (12) and (13); it yields that

$$\begin{aligned}
P\left(y_{i+1} \in S_b^L | \mathscr{F}_i^y\right) &= \frac{1}{\sqrt{2\pi}} \int_{|x\sigma_i| \in S_b^L} e^{-\frac{x^2}{2}}\, dx \\
&\geq \frac{1}{\sqrt{2\pi}} \int_{|x\sigma_i| \in S_b^L, |x-\rho\sigma_i^{-1}| \leqslant 1} e^{-\frac{x^2}{2}}\, dx \\
&\geq \ell(x : |x\sigma_i| \in S_b^L, |x - \rho\sigma_i^{-1}| \leqslant 1) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(1+|\rho|)^2}{2}} \\
&= \frac{\ell(S_b^L \cap [\rho - \sigma_i, \rho + \sigma_i])}{\sigma_i} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(1+|\rho|)^2}{2}} > \frac{c_1}{\sqrt{2\pi}} e^{-\frac{(1+|\rho|)^2}{2}}.
\end{aligned}$$

As a result, for all $m \geq 1$,

$$\begin{aligned}
P(A_m^n | \mathscr{F}_{mn-1}^y) &= E\left\{ \prod_{i=mn}^{mn+n-1} I_{A_i^1} \middle| \mathscr{F}_{mn-1}^y \right\} \\
&= E\left\{ E[I_{A_{mn+n-1}^1} | \mathscr{F}_{mn+n-2}^y] \cdot \prod_{i=mn}^{mn+n-2} I_{A_i^1} \middle| \mathscr{F}_{mn-1}^y \right\} \\
&= E\left\{ P(y_{mn+n-1} \in S_b^L | \mathscr{F}_{mn+n-2}^y) \cdot \prod_{i=mn}^{mn+n-2} I_{A_i^1} \middle| \mathscr{F}_{mn-1}^y \right\} \\
&\geq \frac{c_1}{\sqrt{2\pi}} e^{-\frac{(1+|\rho|)^2}{2}} \cdot E\left\{ \prod_{i=mn}^{mn+n-2} I_{A_i^1} \middle| \mathscr{F}_{mn-1}^y \right\} \geq \cdots \geq \left(\frac{c_1}{\sqrt{2\pi}} e^{-\frac{(1+|\rho|)^2}{2}} \right)^n.
\end{aligned}$$

Consequently, we conclude that $\sum_{m=1}^{+\infty} P(A_m^n | \mathscr{F}_{mn-1}^y) = +\infty$. The lemma then follows from the Borel–Cantelli–Levy theorem. ∎

LEMMA 3.3. *Denote* $S_b(l_1, l_2) \triangleq \{x : |f(x)| < l_1 + l_2|x|^b\}$ *for* $b > 1$, $l_1 \geq 0$, $l_2 > 0$. *Let* $\{B_m\}$ *and* $\{C_m\}$ *be two sequences of the events defined by*

$$\begin{aligned}
B_{m+1} &\triangleq \left\{ y_{m+1} \in S_b(l_1, l_2), \sigma_{m+1}^2 \geqslant r_m^q, \sigma_m^2 \leqslant r_{m-1}^{\frac{1+q}{b-1-q} - \varepsilon} \right\}, \\
C_{m+1} &\triangleq \left\{ y_{m+1} \in S_b(l_1, l_2), \sigma_{m+1}^2 \geqslant \lambda, \sigma_m^2 \leqslant r_{m-1}^{\frac{1}{b-1} - \varepsilon} \right\},
\end{aligned}$$

where $q \in (0, b-1)$, $\varepsilon \in (0,1)$ *and* $\lambda > 1$. *If* $\liminf_{t\to+\infty} \frac{r_t}{t} > 0$ *a.s., then*

$$\sum_{m=1}^{+\infty} I_{B_{m+1}} < +\infty \quad \text{and} \quad \sum_{m=1}^{+\infty} I_{C_{m+1}} < +\infty, \quad \text{a.s.}$$

Proof. Let $A \triangleq \frac{\varepsilon(b-1-q)^2}{b-\varepsilon(b-1-q)} > 0$ and

$$\begin{aligned}
Q_m &\triangleq (r_m^A - r_m^{A-q})^{\frac{1}{2b}} (1 - l_1(r_m^{1+q} - r_m)^{-\frac{1}{2}})^{\frac{1}{b}} l_2^{-\frac{1}{b}} \\
&= l_2^{-\frac{1}{b}} r_m^{\frac{A}{2b}} (1 - r_m^{-q})^{\frac{1}{2b}} (1 - l_1(r_m^{1+q} - r_m)^{-\frac{1}{2}})^{\frac{1}{b}}.
\end{aligned}$$

Since $|f(y_{m+1})| < l_1 + l_2|y_{m+1}|^b$ on set $\{y_{m+1} \in S_b(l_1, l_2)\}$ and $(r_m^{1+q} - r_m)^{\frac{1}{2}} > l_1$ for all sufficiently large m, one has

$$
\begin{aligned}
P(B_{m+1}|\mathscr{F}_m^y) &\leq P\left(|y_{m+1}| \geqslant ((r_m^{1+q} - r_m)^{\frac{1}{2}} - l_1)^{\frac{1}{b}} l_2^{-\frac{1}{b}}, \sigma_m^2 \leqslant r_{m-1}^{\frac{1+q}{b-1-q}-\varepsilon} \middle| \mathscr{F}_m^y\right)\\
&= I_{\left\{\sigma_m^2 \leqslant r_m^{\frac{1+q-\varepsilon(b-1-q)}{b-\varepsilon(b-1-q)}}\right\}} \cdot E\left\{ I_{\left\{|y_{m+1}| \geqslant ((r_m^{1+q}-r_m)^{\frac{1}{2}}-l_1)^{\frac{1}{b}} l_2^{-\frac{1}{b}}\right\}} \middle| \mathscr{F}_m^y \right\}\\
&= I_{\left\{\sigma_m^2 \leqslant r_m^{\frac{1+q-\varepsilon(b-1-q)}{b-\varepsilon(b-1-q)}}\right\}} \cdot \frac{1}{\sqrt{2\pi}} \int_{|x\cdot\sigma_m| \geqslant ((r_m^{1+q}-r_m)^{\frac{1}{2}}-l_1)^{\frac{1}{b}} l_2^{-\frac{1}{b}}} e^{-\frac{x^2}{2}}\, dx\\
&\leq \frac{1}{\sqrt{2\pi}} \int_{\left|x \cdot r_m^{\frac{1}{2}\frac{1+q-\varepsilon(b-1-q)}{b-\varepsilon(b-1-q)}}\right| \geqslant ((r_m^{1+q}-r_m)^{\frac{1}{2}}-l_1)^{\frac{1}{b}} l_2^{-\frac{1}{b}}} e^{-\frac{x^2}{2}}\, dx\\
&= \frac{1}{\sqrt{2\pi}} \int_{|x| \geqslant Q_m} e^{-\frac{x^2}{2}}\, dx. \qquad (14)
\end{aligned}
$$

By $\liminf_{m\to+\infty} \frac{r_m}{m} > 0$, $\liminf_{m\to+\infty} \frac{Q_m}{\log m} > 0$. Then, Lemma 3.1 shows that

$$
\sum_{m=1}^{+\infty} \frac{1}{\sqrt{2\pi}} \int_{|x| \geqslant Q_m} e^{-\frac{x^2}{2}}\, dx < +\infty.
$$

Consequently, by (14), $\sum_{m=1}^{+\infty} P(B_{m+1}|\mathscr{F}_m^y) < +\infty$, which leads to $\sum_{m=1}^{+\infty} I_{B_{m+1}} < +\infty$, in view of the Borel–Cantelli–Levy theorem.

Let $B \triangleq \frac{\varepsilon(b-1)^2}{b-\varepsilon(b-1)} > 0$ and $Q_m^{(1)} \triangleq ((\lambda-1)r_m^B)^{\frac{1}{2b}}(1 - l_1((\lambda-1)r_m)^{-\frac{1}{2}})^{\frac{1}{b}} l_2^{-\frac{1}{b}}$. The next claim is treated in a similar manner by noting that for all sufficiently large m,

$$
\begin{aligned}
P(C_{m+1}|\mathscr{F}_m^y) &\leq P\left(|y_{m+1}| \geqslant (((\lambda-1)r_m)^{\frac{1}{2}} - l_1)^{\frac{1}{b}} l_2^{-\frac{1}{b}}, \sigma_m^2 \leqslant r_{m-1}^{\frac{1}{b-1}-\varepsilon} \middle| \mathscr{F}_m^y\right)\\
&= I_{\left\{\sigma_m^2 \leqslant r_m^{\frac{1-\varepsilon(b-1)}{b-\varepsilon(b-1)}}\right\}} \cdot \frac{1}{\sqrt{2\pi}} \int_{|x\cdot\sigma_m| \geqslant (((\lambda-1)r_m)^{\frac{1}{2}}-l_1)^{\frac{1}{b}} l_2^{-\frac{1}{b}}} e^{-\frac{x^2}{2}}\, dx\\
&\leq \frac{1}{\sqrt{2\pi}} \int_{\left|x \cdot r_m^{\frac{1}{2}\frac{1-\varepsilon(b-1)}{b-\varepsilon(b-1)}}\right| \geqslant (((\lambda-1)r_m)^{\frac{1}{2}}-l_1)^{\frac{1}{b}} l_2^{-\frac{1}{b}}} e^{-\frac{x^2}{2}}\, dx\\
&= \frac{1}{\sqrt{2\pi}} \int_{|x| \geqslant Q_m^{(1)}} e^{-\frac{x^2}{2}}\, dx. \qquad (15)
\end{aligned}
$$

Then, by $\liminf_{m\to+\infty} \frac{Q_m^{(1)}}{\log m} > 0$ and Lemma 3.1,

$$
\sum_{m=1}^{+\infty} \frac{1}{\sqrt{2\pi}} \int_{|x| \geqslant Q_m^{(1)}} e^{-\frac{x^2}{2}}\, dx < +\infty,
$$

which, together with (15), implies $\sum_{m=1}^{+\infty} P(C_{m+1}|\mathscr{F}_m^y) < +\infty$. Finally, with probability 1, $\sum_{m=1}^{+\infty} I_{C_{m+1}} < +\infty$ according to the Borel–Cantelli–Levy theorem again. ☐

LEMMA 3.4. *If* $\ell(\{x : |f(x)| > 0\}) > 0$, *then* $\liminf_{t\to+\infty} \frac{r_t}{t} > 0$ *almost surely.*

Proof. For any $c > 0$, denote $T_c \triangleq \{x \in \mathbb{R} : |f(x)| > c\}$. Since $\{x : |f(x)| > 0\} = \bigcup_{n=1}^{+\infty} T_{\frac{1}{n}}$, there exists an integer $n \geq 1$ such that $\ell(T_{\frac{1}{n}}) > 0$. Moreover, $\cup_{i=1}^{+\infty}(T_{\frac{1}{n}} \cap$

$[-i,i]) = T_{\frac{1}{n}} \cap \left(\cup_{i=1}^{+\infty}[-i,i]\right) = T_{\frac{1}{n}}$, which shows that there is an $i \geq 1$ satisfying $\ell(T_{\frac{1}{n}} \cap [-i,i]) > 0$. For the two integers n and i defined above, denote

$$d \triangleq \ell\left(T_{\frac{1}{n}} \cap [-i,i]\right) \quad \text{and} \quad E_{m+1} \triangleq \left\{y_{m+1} \in T_{\frac{1}{n}}, \sigma_m^2 < 2\right\}, \quad m \geq 1.$$

We estimate the conditional probability of E_{m+1} for each $m \geq 1$ by

$$\begin{aligned} P(E_{m+1}|\mathscr{F}_m^y) &= I_{\{\sigma_m^2<2\}} \cdot \frac{1}{\sqrt{2\pi}} \int_{|x\cdot\sigma_m| \in T_{\frac{1}{n}}} e^{-\frac{x^2}{2}}\,dx \\ &\geq I_{\{\sigma_m^2<2\}} \cdot \frac{1}{\sqrt{2\pi}} \int_{|x\cdot\sigma_m| \in T_{\frac{1}{n}} \cap [-i,i]} e^{-\frac{x^2}{2}}\,dx \geq I_{\{\sigma_m^2<2\}} \cdot \frac{1}{\sqrt{2\pi}} \frac{d}{\sigma_m} e^{-\frac{i^2}{2\sigma_m^2}} \\ &\geq I_{\{\sigma_m^2<2\}} \cdot \frac{1}{\sqrt{2\pi}} \frac{d}{\sigma_m} e^{-\frac{i^2}{2}} \geq c_2 I_{\{\sigma_m^2<2\}}, \end{aligned} \tag{16}$$

where $c_2 \triangleq \frac{1}{\sqrt{2\pi}}\frac{d}{\sqrt{2}}e^{-\frac{i^2}{2}} \in (0,1)$. Next, for each $m \geq 1$, denote

$$F_{m+1} \triangleq \left\{\sigma_m^2 \geqslant 2\right\} \cup \left\{y_{m+1} \in T_{\frac{1}{n}}\right\} = \left\{\sigma_m^2 \geqslant 2\right\} \cup E_{m+1},$$

which, together with (16), leads to

$$P(F_{m+1}|\mathscr{F}_m^y) = E\left\{I_{\{\sigma_m^2 \geqslant 2\}} + I_{E_{m+1}} \middle| \mathscr{F}_m^y\right\} \geq I_{\{\sigma_m^2 \geqslant 2\}} + c_2 \cdot I_{\{\sigma_m^2 < 2\}} \geq c_2. \tag{17}$$

Now, set $x_m \triangleq I_{F_m} - E[I_{F_m}|\mathscr{F}_{m-1}^y]$, and it is clear that $\sup_{m\geq 1} E\left\{x_m^2|\mathscr{F}_{m-1}^y\right\} < +\infty$. Since $\{x_m, \mathscr{F}_m^y\}_{m\geq 1}$ forms a martingale difference sequence, by applying the strong law of large numbers for the martingale differences, one has $\sum_{m=1}^t x_m = o(t)$ almost surely. Therefore, by (17), for all sufficiently large t,

$$\frac{\sum_{m=1}^t I_{F_m}}{t} = o(1) + \frac{\sum_{m=1}^t P(F_m|\mathscr{F}_{m-1}^y)}{t} > \frac{c_2}{2},$$

and hence $\sum_{m=1}^t I_{F_m} > \frac{c_2}{2}t$ almost surely. As a consequence,

$$\begin{aligned} \max\left\{\sum_{m=1}^t I_{\{\sigma_m^2 \geqslant 2\}}, \sum_{m=1}^t I_{\left\{y_m \in T_{\frac{1}{n}}\right\}}\right\} &\geq \frac{1}{2}\left(\sum_{m=1}^t I_{\{\sigma_m^2 \geqslant 2\}} + \sum_{m=1}^t I_{\left\{y_m \in T_{\frac{1}{n}}\right\}}\right) \\ &= \frac{1}{2}\sum_{m=1}^t I_{F_m} > \frac{c_2}{4}t, \quad \text{a.s.} \end{aligned}$$

We complete the remainder of the proof by considering the following two cases:

Case 1. $\sum_{m=1}^t I_{\{\sigma_m^2 \geqslant 2\}} > \frac{c_2}{4}t$. This means the events $\{\sigma_m^2 \geqslant 2\}$, $1 \leq m \leq t$, occur at least $\lceil \frac{c_2}{4}t \rceil$ times, and hence $r_t = r_0 \cdot \prod_{m=1}^t \sigma_m^2 \geq 2^{\frac{c_2}{4}t} P_0^{-1}$.

Case 2. $\sum_{m=1}^t I_{\{y_m \in T_{\frac{1}{n}}\}} > \frac{c_2}{4}t$. Then, $\{y_m \in T_{\frac{1}{n}}\}$, $1 \leq m \leq t$, occur at least $\lceil \frac{c_2}{4}t \rceil$ times. Since $\{y_m \in T_{\frac{1}{n}}\} = \{f^2(y_m) > \frac{1}{n^2}\}$, one has $r_t = P_0^{-1} + \sum_{m=0}^t f^2(y_m) > \frac{c_2}{4n^2}t$.

Finally, by combining Case 1 and Case 2, it shows that $r_t \geq \min\{2^{\frac{c_2}{4}t}P_0^{-1}, \frac{c_2}{4n^2}t\}$ for all sufficiently large t almost surely, which proves the lemma. ∎

LEMMA 3.5. *Let* (6) *hold for some* $k_1, k_2 > 0$. *If there is a set* D *with* $P(D) > 0$ *such that* $\sup_t \sigma_t = +\infty$ *on* D *and* $\liminf_{t\to+\infty} \frac{r_t}{t} > 0$ *almost surely, then*

$$\lim_{t\to+\infty} \sigma_t = +\infty, \quad a.s. \quad on\ D.$$

Proof. Given a number $z > 1$, define $D_{m+1} \triangleq \left\{\sigma_m^2 \leqslant z, \sigma_{m+1}^2 \geqslant z\right\}$ and $Q_m^{(2)} \triangleq \frac{1}{2k_2\sqrt{z}} \log(\frac{(z-1)r_m}{k_1^2})$, $m \geq 0$. Therefore,

$$\begin{aligned} P(D_{m+1}|\mathscr{F}_m^y) &= P\left(f^2(y_{m+1}) \geqslant (z-1)r_m, \sigma_m^2 \leqslant z \middle| \mathscr{F}_m^y\right) \\ &\leq P\left(|y_{m+1}| \geqslant \frac{1}{k_2}\log\left(\frac{\sqrt{(z-1)r_m}}{k_1}\right), \sigma_m^2 \leqslant z \middle| \mathscr{F}_m^y\right) \\ &= I_{\{\sigma_m^2 \leqslant z\}} \cdot \frac{1}{\sqrt{2\pi}} \int_{|x\cdot\sigma_m| \geqslant \frac{1}{2k_2}\log\left(\frac{(z-1)r_m}{k_1^2}\right)} e^{-\frac{x^2}{2}}\, dx \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{|x\cdot\sqrt{z}| \geqslant \frac{1}{2k_2}\log\left(\frac{(z-1)r_m}{k_1^2}\right)} e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}} \int_{|x| \geqslant Q_m^{(2)}} e^{-\frac{x^2}{2}}\, dx. \end{aligned} \tag{18}$$

Since $\liminf_{m\to+\infty} \frac{r_m}{m} > 0$ shows $\liminf_{m\to+\infty} \frac{Q_m^{(2)}}{\log m} > 0$, by Lemma 3.1 and (18),

$$\sum_{m=1}^{+\infty} P(D_{m+1}|\mathscr{F}_m^y) \leq \sum_{m=1}^{+\infty} \frac{1}{\sqrt{2\pi}} \int_{|x| \geqslant Q_m^{(2)}} e^{-\frac{x^2}{2}}\, dx < +\infty, \quad \text{a.s.}$$

Taking account of the Borel–Cantelli–Levy theorem, events $\{D_m\}$ occur only finite times almost surely. Therefore, if $\sigma_m^2 \leqslant z$ for some sufficiently large m on a set $D' \subset D$ with $P(D') > 0$, then $\sigma_{m+1}^2 \leqslant z$, and hence $\sigma_t^2 \leqslant z$ for all $t \geq m+2$ with probability $P(D') > 0$. That is, $\sup_t \sigma_t < +\infty$ almost surely on D', which contradicts the assumption that $\sup_t \sigma_t = +\infty$ a.s. on D. Hence, for any $z > 1$, with probability $P(D)$, there is a random $m > 0$ such that $\sigma_t^2 > z$ for all $t \geq m$. This is exactly $\lim_{t\to+\infty} \sigma_t = +\infty$ a.s. on D by taking z over all the natural numbers. ∎

LEMMA 3.6. *For some $a_0 \geq 0$ and $\varepsilon_i \in (0, \frac{1}{i+1})$, define a sequence $\{a_i\}$ by*

$$a_{i+1} \triangleq \frac{1+a_i}{b-1-a_i} - \varepsilon_i, \quad i \geq 0. \tag{19}$$

(i) *If $b \in (1,4)$ and $a_0 = 0$, then there exists a positive integer k and a sequence $\{\varepsilon_i\}_{i=0}^{k-1}$ such that $a_i \in (0, b-1)$ for $1 \leq i \leq k-1$ and $a_k > b-1$.*

(ii) *If $b \geq 4$ and $x_1 < a_0 < b-1$, where x_1 is the maximal real solution of equation $x^2 - (b-2)x + 1 = 0$, then there exists a positive integer k and a sequence $\{\varepsilon_i\}_{i=0}^{k-1}$ such that $a_i \in (x_1, b-1)$ for $1 \leq i \leq k-1$ and $a_k > b-1$.*

(iii) *If $b \geq 4$ and $a_0 = 0$, then there exists a sequence $\{\varepsilon_i\}_{i\geq 0}$, such that $\lim_{i\to+\infty} a_i = x_2$ and $a_i < a_{i+1} < x_2$ for all $i \geq 0$, where x_2 is the minimum real solution of equation $x^2 - (b-2)x + 1 = 0$.*

Proof. (i) Since $b \in (1,4)$, it is clear that $a_0^2 - (b-2)a_0 + 1 > 0$, and hence

$$\frac{1+a_0}{b-1-a_0} > a_0. \tag{20}$$

Note that $a_0 = 0 < b-1$, there exists some $\varepsilon_0' \in (0,1)$ such that

$$a_1' \triangleq \frac{1+a_0}{b-1-a_0} - \varepsilon_0' > a_0 = 0. \tag{21}$$

We now prove assertion (i) by reduction to absurdity. Suppose it is not true; then $a_1' \in (0, b-1]$. Obviously, (19) means any

$$\varepsilon_0 \in \left(\varepsilon_0', \min\left\{\frac{1+a_0}{b-1-a_0} - a_0, 1\right\}\right) \tag{22}$$

achieves $a_1 \in (0, b-1)$. If for some $k \geq 1$, there is a sequence $\{\varepsilon_i\}_{i=0}^{k-1}$ with $\varepsilon_i \in (0, \frac{1}{i+1})$ such that $a_i \in (0, b-1)$ for all $i \in [1,k]$, by the same arguments for (20) and (21), one has $a_{k+1}' \triangleq \frac{1+a_k}{b-1-a_k} - \varepsilon_k' > a_k > 0$ for some $0 < \varepsilon_k' < \frac{1}{k+1}$. Take some $\varepsilon_k \in (0, \frac{1}{k+1})$ in a similar way as that of (22), then the corresponding $a_{k+1} \in (0, b-1)$ in view of the hypothesis that (i) does not hold. This means, by induction, there exists a sequence $\{\varepsilon_i\}_{i=0}^{+\infty}$ with $\varepsilon_i \in (0, \frac{1}{i+1})$ such that $0 < a_i < a_{i+1} < b-1$ for all $i \geq 1$. Hence, $\lim_{i\to+\infty} a_i$ exists and $\lim_{i\to+\infty} \varepsilon_i = 0$.

Denote $a \triangleq \lim_{i\to+\infty} a_i$, then

$$a = \lim_{i\to+\infty} a_{i+1} = \lim_{i\to+\infty} \frac{1+a_i}{b-1-a_i} - \lim_{i\to+\infty} \varepsilon_i = \frac{1+a}{b-1-a}. \tag{23}$$

Therefore, $a \in (0, b-1)$ and it serves as a solution of equation $x^2 - (b-2)x + 1 = 0$, which is impossible due to $b < 4$.

(ii) The proof is almost the same as that of (i), by noting that $x_1 < b-1$ and for any $a_i \in (x_1, b-1)$, $b \geq 4$ yields $\frac{1+a_i}{b-1-a_i} > a_i$. As a matter of fact, if the assertion fails, one can take a series of $\{\varepsilon_i\}_{i=0}^{+\infty}$ with $\varepsilon_i \in (0, \frac{1}{i+1})$ such that $x_1 < a_i < a_{i+1} < b-1$ for all $i \geq 0$. There thus arises a contradiction between $a \triangleq \lim_{i\to+\infty} a_i \geq a_0 > x_1$ and $a^2 - (b-2)a + 1 = 0$ by (23) with $b \geq 4$.

(iii) With $b \geq 4$, any $a_i < x_2$ implies that there is a $\varepsilon_i \in (0, \frac{1}{i+1})$ such that $a_{i+1} = \frac{1+a_i}{b-1-a_i} - \varepsilon_i > a_i$. Since $a_i < x_2$ also leads to $\frac{1+a_i}{b-1-a_i} < x_2$,

$$a_i < a_{i+1} = \frac{1+a_i}{b-1-a_i} - \varepsilon_i < x_2. \tag{24}$$

Noting that $a_0 = 0$, (24) yields that $a_0 < a_1 < x_2$ for some $\varepsilon_0 \in (0,1)$. By induction, there is a sequence $\{\varepsilon_i\}_{i\geq 0}$ satisfying $\lim_{i\to+\infty} \varepsilon_i = 0$ such that (24) holds for all $i \geq 0$. Thus, $\lim_{i\to+\infty} a_i$ exists. Letting $a \triangleq \lim_{i\to+\infty} a_i$ shows that $a = \frac{1+a}{b-1-a}$ by (23), and hence $a = x_2$. ∎

LEMMA 3.7. *Let system* (3) *satisfy* (7) *and* $\ell(\{x : |f(x)| > 0\}) > 0$; *then*

$$\sup_t \sigma_t < +\infty, \quad a.s.$$

Proof. We only need to consider the case $b > 1$. Since $\ell(\{x : |f(x)| > 0\}) > 0$, Lemma 3.4 shows $\liminf_{t\to+\infty} \frac{r_t}{t} > 0$ a.s. Assume $D \triangleq \{\sup_t \sigma_t = +\infty\}$ satisfies $P(D) > 0$; then by Lemma 3.5, $\lim_{t\to+\infty} \sigma_t = +\infty$ on D almost surely. That is, for any $\lambda > 1$, there is a random $N > 0$ such that

$$\sigma_t^2 > \lambda \quad \text{if} \quad t > N, \quad \text{a.s.} \quad \text{on } D. \tag{25}$$

Now, according to (i) of Lemma 3.6, for some integer $k \geq 1$, one can construct a finite sequence $\{a_i\}_{i=1}^k$ satisfying $0 = a_0 < \cdots < a_{k-1} < b-1$, $a_k > b-1$, and

$$a_{i+1} = \frac{1+a_i}{b-1-a_i} - \varepsilon_i, \tag{26}$$

where $\varepsilon_i \in (0,1)$ for $0 \leq i \leq k-1$. Fix this k. Applying Lemma 3.2 indicates that $\sum_{m=1}^{+\infty} I_{A_m^{k+1}} = +\infty$ a.s. on D, that is, $\{y_{m(k+1)}, y_{m(k+1)+1}, \ldots, y_{m(k+1)+k} \in S_b^L\}$ occurs infinitely many times for m on D. Let

$$T \triangleq \{t_j : y_{t_j(k+1)}, y_{t_j(k+1)+1}, \ldots, y_{t_j(k+1)+k} \in S_b^L\}. \tag{27}$$

Clearly, $|T| = \aleph_0$ on D almost surely. In view of (25) and (27), as long as $t_j \in T$ is sufficiently large,

$$y_{t_j(k+1)+k} \in S_b^L \quad \text{and} \quad \sigma^2_{t_j(k+1)+k} > \lambda \quad \text{a.s.} \quad \text{on } D. \tag{28}$$

Now, we use the induction method to prove that if $t_j \in T$ is sufficiently large,

$$\begin{cases} y_{t_j(k+1)+k-i} \in S_b^L \\ \sigma^2_{t_j(k+1)+k-i} > r^{a_i}_{t_j(k+1)+k-i-1} \end{cases} \quad \text{a.s.} \quad \text{on } D \tag{29}$$

holds for all $1 \leq i \leq k$, where a_i is defined by (26). In fact, for $i = 1$, Lemma 3.3 with $l_1 = 0$, $l_2 = L$, and $\varepsilon = \varepsilon_0$ yields $\sum_{m=1}^{+\infty} I_{C_{m+1}} < +\infty$ a.s., which indicates that the events $\{y_{m+1} \in S_b^L, \sigma^2_{m+1} \geqslant \lambda, \sigma^2_m \leqslant r_{m-1}^{\frac{1}{b-1}-\varepsilon_0}\}$ occur only finite times for m. This, together with (26) and (28), leads to

$$\sigma^2_{t_j(k+1)+k-1} > r_{t_j(k+1)+k-2}^{\frac{1}{b-1}-\varepsilon_0} = r^{a_1}_{t_j(k+1)+k-2}, \quad \text{a.s.} \quad \text{on } D,$$

when $t_j \in T$ is sufficiently large. Thus, (29) holds for $i = 1$ due to (27).

Assume that (29) is true for some $i = s \in [1, k)$, when $t_j \in T$ is sufficiently large. By Lemma 3.3 again with $l_1 = 0$, $l_2 = L$, $q = a_i$ and $\varepsilon = \varepsilon_i$, one has $\sum_{m=1}^{+\infty} I_{B_{m+1}} < +\infty$. Hence, events $\{y_{m+1} \in S_b^L, \sigma^2_{m+1} \geqslant r_m^{a_i}, \sigma^2_m \leqslant r_{m-1}^{\frac{1+a_i}{b-1-a_i}-\varepsilon_i}\}$ occur finite times for m. By the hypothesis, for the sufficiently large t_j,

$$\sigma^2_{t_j(k+1)+k-s-1} > r_{t_j(k+1)+k-s-2}^{\frac{1+a_s}{b-1-a_s}-\varepsilon_s} = r^{a_{s+1}}_{t_j(k+1)+k-s-2}, \quad \text{a.s.} \quad \text{on } D,$$

and hence (29) also holds for $i = s+1$ in view of (27). The assertion is thus proved.

Now, it follows immediately from (29) that for all sufficiently large $t_j \in T$,

$$y_{t_j(k+1)} \in S_b^L \quad \text{and} \quad \sigma^2_{t_j(k+1)} > r^{a_k}_{t_j(k+1)-1}, \quad \text{a.s.} \quad \text{on } D.$$

Denote $G_{m+1} \triangleq \{y_{m+1} \in S_b^L, \sigma^2_{m+1} > r_m^{a_k}\}$, then $\sum_{m=1}^{+\infty} I_{G_{m+1}} = +\infty$, a.s. on D, as $|T| = \aleph_0$ almost surely. This means $P(D) \leq P(\sum_{m=1}^{+\infty} I_{G_{m+1}} = +\infty)$. Lemma 3.7 thus becomes straightforward if we could prove

$$P\left(\sum_{m=1}^{+\infty} I_{G_{m+1}} = +\infty\right) = 0, \tag{30}$$

which contradicts the hypothesis that $P(D) > 0$.

To this end, for each sufficiently large m, compute

$$
\begin{aligned}
P(G_{m+1}|\mathscr{F}_m^y) &= P\left(y_{m+1} \in S_b^L, f^2(y_{m+1}) > r_m^{1+a_k} - r_m|\mathscr{F}_m^y\right)\\
&\leq P\left(y_{m+1} \in S_b^L, |y_{m+1}| > (r_m^{1+a_k} - r_m)^{\frac{1}{2b}} L^{-\frac{1}{b}}\middle|\mathscr{F}_m^y\right)\\
&\leq \frac{1}{\sqrt{2\pi}} \int_{|x\cdot\sigma_m|\geqslant (r_m^{1+a_k}-r_m)^{\frac{1}{2b}} L^{-\frac{1}{b}}} e^{-\frac{x^2}{2}}\, dx\\
&\leq \frac{1}{\sqrt{2\pi}} \int_{|x\cdot\sqrt{P_0 r_m}|\geqslant (r_m^{1+a_k}-r_m)^{\frac{1}{2b}} L^{-\frac{1}{b}}} e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}} \int_{|x|\geqslant Q_m^{(3)}} e^{-\frac{x^2}{2}}\, dx, \qquad (31)
\end{aligned}
$$

where $Q_m^{(3)} \triangleq P_0^{-\frac{1}{2}} L^{-\frac{1}{b}} (r_m^{a_k+1-b} - r_m^{1-b})^{\frac{1}{2b}}$. Since $a_k > b-1$ and $\liminf_{m\to+\infty} \frac{r_m}{m} > 0$, one has $\liminf_{m\to+\infty} \frac{Q_m^{(3)}}{\log m} > 0$. By virtue of Lemma 3.1,

$$
\sum_{m=1}^{+\infty} \frac{1}{\sqrt{2\pi}} \int_{|x|\geqslant Q_m^{(3)}} e^{-\frac{x^2}{2}}\, dx < +\infty.
$$

So, (31) yields that $\sum_{m=1}^{+\infty} P(G_{m+1}|\mathscr{F}_m^y) < +\infty$. This shows $\sum_{m=1}^{+\infty} I_{G_{m+1}} < +\infty$ a.s. by the Borel–Cantelli–Levy theorem and (30) follows as desired. ∎

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. First of all, if $\ell(\{x : |f(x)| > 0\}) = 0$, then for any $i \geq 0$, $E\left[I_{\{f(y_{i+1})\neq 0\}}|\mathscr{F}_i^y\right] = P(f(y_{i+1}) \neq 0|\mathscr{F}_i^y) = 0$, which yields

$$
E\left[\sum_{i=0}^{+\infty} I_{\{f(y_{i+1})\neq 0\}}\right] = \sum_{i=0}^{+\infty} E\{E[I_{\{f(y_{i+1})\neq 0\}}|\mathscr{F}_i^y]\} = 0.
$$

This infers that $\sum_{i=0}^{+\infty} I_{\{f(y_{i+1})\neq 0\}} = 0$, a.s. That is, with probability 1, $f(y_{i+1}) = 0$ for all $i \geq 0$. By (10), $u_{i+1} \equiv 0, i \geq 0$. Then, system (3) reduces to

$$
y_{i+2} = w_{i+2}, \quad i \geq 0 \quad \text{a.s.},
$$

and the stabilizability is verified by $\sum_{i=1}^t y_i^2 = y_1^2 + \sum_{i=2}^t w_i^2 = O(t)$, as $t \to +\infty$.

Therefore, it is sufficient to consider the case where $\ell(\{x : |f(x)| > 0\}) > 0$. Taking account of Lemma 3.7,

$$
\sup_t \sigma_t < +\infty, \quad \text{a.s.} \qquad (32)
$$

Moreover, recall from [3, Lemma 3.1] that $\sum_{i=0}^t \alpha_i = O(\log r_t)$ a.s., where $\alpha_i \triangleq \frac{(\phi_i\tilde{\theta}_i)^2}{1+\phi_i P_i \phi_i}$, (11), and (32) lead to

$$
\sum_{i=0}^{t} (y_{i+1} - w_{i+1})^2 = \sum_{i=0}^{t} (\phi_i\tilde{\theta}_i)^2 = \sum_{i=0}^{t} \alpha_i \frac{r_i}{r_{i-1}} = O(\log r_t). \qquad (33)
$$

Observe that

$$
\begin{aligned}
\log r_t &\leq \log\left(P_0^{-1} + (t+1)\max_{0\leqslant i\leqslant t} f^2(y_i)\right) \leq \log\left(P_0^{-1} + (t+1)k_1^2 e^{2k_2 \max_{0\leqslant i\leqslant t}|y_i|}\right)\\
&= O(1) + O(\log t) + O\left(\max_{0\leqslant i\leqslant t}|y_i|\right) = O(1) + O(\log t) + O\left(\left(\sum_{i=0}^t y_i^2\right)^{\frac{1}{2}}\right)
\end{aligned}
$$

and $\sum_{i=0}^{t} 2[(y_{i+1}-w_{i+1})^2+w_{i+1}^2] \geq \sum_{i=0}^{t} y_{i+1}^2$, (33) becomes

$$\begin{aligned}\frac{1}{2}\sum_{i=0}^{t} y_i^2 \leq \sum_{i=0}^{t}(y_{i+1}-w_{i+1})^2 + \sum_{i=0}^{t} w_{i+1}^2 + \frac{1}{2}y_0^2 = O(\log r_t) + \sum_{i=0}^{t} w_{i+1}^2 \\ \leq O(1) + O(\log t) + O\left(\left(\sum_{i=0}^{t} y_i^2\right)^{\frac{1}{2}}\right) + \sum_{i=0}^{t} w_{i+1}^2, \quad \text{a.s.}\end{aligned}$$

Moreover, since $\sum_{i=0}^{t} w_{i+1}^2 = O(t)$, as $t \to +\infty$,

$$\frac{1}{2t}\sum_{i=0}^{t} y_i^2 = O(1) + O\left(\frac{\log t}{t}\right) + o\left(\frac{1}{t}\sum_{i=0}^{t} y_i^2\right)^{\frac{1}{2}},$$

which is exactly $\frac{1}{t}\sum_{i=0}^{t} y_i^2 = O(1)$ almost surely. ∎

4. Proof of unstabilizability.

Because Theorem 2.5 is an immediate corollary of Theorem 2.6, we only provide the proof of Theorem 2.6 here.

Proof of Theorem 2.6. Denote

$$\begin{cases} H_0 \triangleq \left\{\omega : r_0 > e^2, y_0 \notin S_h\right\}, \\ H_t \triangleq \left\{\omega : r_t > r_{t-1}^{2+\frac{1}{t}}, y_t \notin S_h\right\}, \qquad t \geq 1, \end{cases}$$

and $H \triangleq \bigcap_{t=0}^{+\infty} H_t$. The following argument is mainly devoted to verifying $P(H) > 0$.

Now, for any $t \geq 1$,

$$\begin{aligned} P(H_{t+1}^c|\mathscr{F}_t^y) &= P\left(\left\{r_{t+1} \leqslant r_t^{2+\frac{1}{t+1}}, y_{t+1} \notin S_h\right\} \cup \{y_{t+1} \in S_h\}\Big|\mathscr{F}_t^y\right) \\ &= P\left(f^2(y_{t+1}) \leqslant r_t^{2+\frac{1}{t+1}} - r_t, y_{t+1} \notin S_h\Big|\mathscr{F}_t^y\right) + P(y_{t+1} \in S_h|\mathscr{F}_t^y). \end{aligned} \tag{34}$$

To compute this probability, observe that by the assumption of the theorem, there is a $k_3 > 0$ such that for any $x \in \mathbb{R}$,

$$\ell(S_h \cap [x-l, x+l]) \leq \frac{k_3 l}{(\log(\log l))^{1+\delta}}, \quad l \geq 3. \tag{35}$$

Denote $J_t \triangleq \frac{2}{\sqrt{2\pi}}\sigma_t^{-1} r_t^{\frac{1}{4}\left(1+\frac{1}{2(t+1)}\right)} g(r_t^{1+\frac{1}{2(t+1)}})$; then in view of (12) and (13),

$$\begin{aligned} &P\left(f^2(y_{t+1}) \leqslant r_t^{2+\frac{1}{t+1}} - r_t, y_{t+1} \notin S_h|\mathscr{F}_t^y\right) \\ &\leq P\left(h^2(|y_{t+1}|) \leqslant r_t^{2+\frac{1}{t+1}}, y_{t+1} \notin S_h|\mathscr{F}_t^y\right) \\ &\leq \frac{1}{\sqrt{2\pi}}\int_{|m_t+x\cdot\sigma_t| \leqslant r_t^{\frac{1}{4}\left(1+\frac{1}{2(t+1)}\right)} g\left(r_t^{1+\frac{1}{2(t+1)}}\right)} e^{-\frac{x^2}{2}}\, dx \leq J_t. \end{aligned} \tag{36}$$

Furthermore, by (35) and letting $K_t \triangleq \frac{1}{\sqrt{2\pi}} \int_{|x|>3\log(t+e)} e^{-\frac{x^2}{2}}\,dx$, it follows that

$$
\begin{aligned}
P(y_{t+1} \in S_h|\mathscr{F}_t^y) - K_t &= \frac{1}{\sqrt{2\pi}} \int_{m_t + x\cdot\sigma_t \in S_h, |x|\leqslant 3\log(t+e)} e^{-\frac{x^2}{2}}\,dx \\
&\le \frac{\ell(S_h \cap [m_t - 3\sigma_t \log(t+e), m_t + 3\sigma_t\log(t+e)])}{\sqrt{2\pi}\sigma_t} \\
&\le \frac{3k_3 \log(t+e)}{\sqrt{2\pi}\Big(\log\log\Big(3\sigma_t\log(t+e)\Big)\Big)^{1+\delta}} \triangleq L_t. \qquad (37)
\end{aligned}
$$

Since $r_t \geqslant e^{2^{t+1}}, \sigma_t^2 > e^{2^t}$ on $\bigcap_{i=0}^t H_i$, it derives

$$
\bigcap_{i=0}^{t} H_i \subseteq \left\{ r_t > r_{t-1}^{2+\frac{1}{t}}, r_t \geqslant e^{2^{t+1}}, \sigma_t^2 > e^{2^t} \right\}, \qquad (38)
$$

and hence

$$
\prod_{i=1}^{t} I_{H_i} = I_{\left\{r_t > r_{t-1}^{2+\frac{1}{t}}, r_t \geqslant e^{2^{t+1}}\right\}} \cdot \prod_{i=1}^{t} I_{H_i} = I_{\left\{\sigma_t^2 > e^{2^t}\right\}} \cdot \prod_{i=1}^{t} I_{H_i}. \qquad (39)
$$

Substituting (36) and (37) into (34), one has $P(H_{t+1}^c|\mathscr{F}_t^y) \le J_t + K_t + L_t$ and by (39),

$$
\begin{aligned}
P\left(\bigcap_{i=0}^{t+1} H_i\right) &= E\left\{E[I_{H_{t+1}}|\mathscr{F}_t^y]\prod_{i=0}^{t} I_{H_i}\right\} \ge E\left\{(1 - J_t - K_t - L_t)\prod_{i=0}^{t} I_{H_i}\right\} \\
&= E\Bigg\{\prod_{i=0}^{t} I_{H_i} - J_t \cdot I_{\left\{r_t > r_{t-1}^{2+\frac{1}{t}}, r_t \geqslant e^{2^{t+1}}\right\}} \prod_{i=0}^{t} I_{H_i} - K_t \prod_{i=0}^{t} I_{H_i} \\
&\qquad - L_t \cdot I_{\left\{\sigma_t^2 > e^{2^t}\right\}} \prod_{i=0}^{t} I_{H_i}\Bigg\}. \qquad (40)
\end{aligned}
$$

At the same time,

$$
\begin{aligned}
J_t \cdot I_{\left\{r_t > r_{t-1}^{2+\frac{1}{t}}, r_t \geqslant e^{2^{t+1}}\right\}} &= \frac{2}{\sqrt{2\pi}} \frac{r_t^{\frac{1}{4}\left(\frac{2t+3}{2(t+1)}\right)} g\left(r_t^{1+\frac{1}{2(t+1)}}\right)}{\sigma_t} \cdot I_{\left\{\sigma_t^2 > r_t^{\frac{1+t}{1+2t}}, r_t \geqslant e^{2^{t+1}}\right\}} \\
&\le \frac{2}{\sqrt{2\pi}} \frac{r_t^{\frac{1}{4}\left(\frac{2t+3}{2(t+1)}\right)} g\left(r_t^{1+\frac{1}{2(t+1)}}\right)}{\sqrt{r_t^{\frac{1+t}{1+2t}}}} \cdot I_{\left\{\sigma_t^2 > r_t^{\frac{1+t}{1+2t}}, r_t \geqslant e^{2^{t+1}}\right\}} \\
&= \frac{2}{\sqrt{2\pi}} \frac{g\left(r_t^{1+\frac{1}{2(t+1)}}\right)}{r_t^{\frac{1}{8(1+t)(1+2t)}}} \cdot I_{\left\{r_t \geqslant e^{2^{t+1}}, \sigma_t^2 > r_t^{\frac{1+t}{1+2t}}\right\}} \\
&\le \frac{2}{\sqrt{2\pi}} \frac{g\left(r_t^{1+\frac{1}{2(t+1)}}\right)}{r_t^{\left(1+\frac{1}{2(t+1)}\right)\frac{1}{16(t+1)^2}}} \cdot I_{\left\{r_t > r_{t-1}^{2+\frac{1}{t}}, r_t \geqslant e^{2^{t+1}}\right\}} \\
&\le \frac{2}{\sqrt{2\pi}} \sup_{x \in [e^{2^{t+1}}, +\infty)} x^{-\frac{1}{16(t+1)^2}} g(x) \cdot I_{\left\{r_t > r_{t-1}^{2+\frac{1}{t}}, r_t \geqslant e^{2^{t+1}}\right\}}. \qquad (41)
\end{aligned}
$$

Denote

$$J_t^{(1)} \triangleq \frac{2}{\sqrt{2\pi}} \sup_{x\in[e^{2^{t+1}},+\infty)} x^{-\frac{1}{16(t+1)^2}} g(x), \tag{42}$$

$$L_t^{(1)} \triangleq \frac{3k_3 \log(t+e)}{\sqrt{2\pi}(\log\log(3e^{2^{t-1}}\log(t+e)))^{1+\delta}}, \tag{43}$$

then $\sum_{t=1}^{+\infty} J_t^{(1)} < +\infty$ because of (8), and

$$L_t \le L_t^{(1)} = O\left(\frac{\log t}{t^{1+\delta}}\right) \quad \text{on} \quad \left\{\sigma_t^2 > e^{2^t}\right\} \tag{44}$$

due to (37) that

$$L_t \cdot I_{\{\sigma_t^2 > e^{2^t}\}} \le \frac{3k_3 \log(t+e)}{\sqrt{2\pi}(\log\log(3e^{2^{t-1}}\log(t+e)))^{1+\delta}} \cdot I_{\{\sigma_t^2 > e^{2^t}\}}.$$

In addition, Lemma 3.1 yields $\sum_{t=1}^{+\infty} K_t < +\infty$, hence

$$\sum_{t=1}^{+\infty}(J_t^{(1)} + K_t + L_t^{(1)}) < +\infty. \tag{45}$$

Applying (41)–(44), (40) reduces to

$$\begin{aligned} P\left(\bigcap_{t=0}^{t+1} H_t\right) \ge E\Bigg\{ &\prod_{i=0}^{t} I_{H_i} - J_t^{(1)} \cdot I_{\left\{r_t > r_{t-1}^{2+\frac{1}{t}}, r_t \geqslant e^{2^{t+1}}\right\}} \prod_{i=0}^{t} I_{H_i} - K_t \prod_{i=0}^{t} I_{H_i} \\ &- L_t^{(1)} \cdot I_{\{\sigma_t^2 > e^{2^t}\}} \prod_{i=0}^{t} I_{H_i}\Bigg\} = \left(1 - J_t^{(1)} - K_t - L_t^{(1)}\right) P\left(\bigcap_{t=0}^{t} H_t\right). \end{aligned}$$

Since (45) implies that there is a $N_1 > 0$ such that for all $t \ge N_1$, $J_t^{(1)} + K_t + L_t^{(1)} < 1$, it follows that

$$\begin{aligned} P\left(\bigcap_{t=0}^{+\infty} H_t\right) &= \lim_{t=N_1}^{+\infty} P\left(\bigcap_{j=0}^{t+1} H_j\right) \ge \lim_{t=N_1}^{+\infty}\left(\prod_{i=N_1}^{t}\left(1 - J_i^{(1)} - K_i - L_i^{(1)}\right)\right) P\left(\bigcap_{j=0}^{N_1} H_j\right) \\ &= \prod_{i=N_1}^{+\infty}\left(1 - J_i^{(1)} - K_i - L_i^{(1)}\right) P\left(\bigcap_{j=0}^{N_1} H_j\right) > 0, \end{aligned}$$

which is exactly $P(H) > 0$.

On the other hand, by (38), $\sigma_t^2 > e^{2^t}$ for all $t \ge 1$, on H. Therefore, for any $C > 0$,

$$\begin{aligned} P\left(\left(\bigcap_{i=0}^{t} H_i\right) \cap \{|y_{t+1}| < C\} \middle| \mathscr{F}_t^y\right) &= I_{\bigcap_{i=0}^t H_i} \cdot P(|y_{t+1}| < C|\mathscr{F}_t^y) \\ &= I_{\bigcap_{i=0}^t H_i} \cdot \frac{1}{\sqrt{2\pi}} \int_{|m_t + x\cdot\sigma_t| < C} e^{-\frac{x^2}{2}}\, dx \le I_{\{\sigma_t^2 > e^{2^t}\}} \cdot \frac{1}{\sqrt{2\pi}} \frac{2C}{\sigma_t} \le \frac{1}{\sqrt{2\pi}} \frac{2C}{e^{2^{t-1}}}, \end{aligned}$$

and hence

$$\sum_{t=1}^{+\infty} P\left(\left(\bigcap_{i=0}^{t} H_i\right) \cap \{|y_{t+1}| < C\} \middle| \mathscr{F}_t^y\right) < +\infty \quad \text{on } H.$$

Invoking the Borel–Cantelli–Levy theorem, one has

$$\sum_{t=1}^{+\infty} I_{\bigcap_{i=0}^{t} H_i} \cdot I_{\{|y_{t+1}|<C\}} = \sum_{t=1}^{+\infty} I_{\{|y_{t+1}|<C\}\cap(\bigcap_{i=0}^{t} H_i)} < +\infty, \quad \text{a.s.} \quad \text{on } H. \tag{46}$$

Note that $I_{\bigcap_{i=0}^{t} H_i} = 1$ on H for every $t \geq 1$, $\sum_{t=1}^{+\infty} I_{\{|y_{t+1}|<C\}} < +\infty$ almost surely on H, in view of (46). This infers that $\liminf_{t\to+\infty} |y_t| \geq C$ on H, and consequently, $\lim_{t\to+\infty} |y_t| = +\infty$ on H by letting $C \to +\infty$. So, considering $P(H) > 0$,

$$\frac{1}{t}\sum_{i=1}^{t} y_i^2 \to \infty \quad \text{as} \quad t \to +\infty, \quad \text{on } H,$$

establishing the result. ∎

5. Concluding remarks. At the very beginning, the work was intended to seek a connection between the measure of S_b^L and the stabilizability of stochastic parameterized systems in discrete time. But the finding is interesting, as it turns out. It suggests that a discrete-time control law is also able to deal with high nonlinearity. This paper, of course, is just a starting point to provide some preliminary results for the scalar-parameter case. It calls for further investigations on this topic.

Appendix A. Proof of Theorem 2.3. This appendix addresses the proof of Theorem 2.3. Some lemmas are necessary.

LEMMA A.1. *Let $x_{\min} \leq x_{\max}$ denote the two solutions of equation $x^2-(a-2)x+1=0$ and $\ell(\{x : |f(x)| > 0\}) > 0$. Under* (i) *of Theorem* 2.3*, then*

(i) *$D_1 = D_2$ with $D_1 \triangleq \{\sup_t \sigma_t = +\infty\}$ and $D_2 \triangleq \{\liminf\limits_{t\to+\infty} \frac{\log r_t}{\log r_{t-1}} \geq 1 + x_{\min}\}$;*

(ii) *$P(D_3) = 0$ with $D_3 \triangleq \{\limsup\limits_{t\to+\infty} \frac{\log r_t}{\log r_{t-1}} > 1 + x_{\max}\}$.*

Proof. To prove (i), note that $\ell(\{x : |f(x)| > 0\}) > 0$ implies $\liminf_{t\to+\infty} \frac{r_t}{t} > 0$ almost surely by Lemma 3.4, so it is enough to show $D_1 \subseteq D_2$. As a matter of fact, in view of Lemma 3.5, $\lim_{t\to+\infty} \sigma_t = +\infty$ on D_1 almost surely. That is, for any $\lambda > 1$, there is a random $N > 0$ such that

$$\sigma_t^2 > \lambda \quad \text{if} \quad t > N, \quad \text{a.s.} \quad \text{on } D_1. \tag{47}$$

Furthermore, according to (iii) of Lemma 3.6, there is an infinite sequence $\{a_n\}_{n\geq 0}$ satisfying $0 = a_0 < \cdots < a_n < \cdots < x_{\min}$, $\lim_{n\to+\infty} a_n = x_{\min}$ and

$$a_{n+1} = \frac{1 + a_n}{a - 1 - a_n} - \varepsilon_n,$$

where $\varepsilon_n \in (0, 1)$ for all $n \geq 0$. We use the induction method to prove that for each $n \geq 1$, when m is sufficiently large,

$$\sigma_m^2 > r_{m-1}^{a_n}, \quad \text{a.s.} \quad \text{on } D_1. \tag{48}$$

Observe that $|f(x)| < l_1 + l_2|x|^a$ for some $l_1, l_2 > 0$, where $x \in \mathbb{R}$. When $n = 1$, Lemma 3.3 with $\varepsilon = \varepsilon_0$ indicates that the events $\{\sigma_{m+1}^2 \geqslant \lambda, \sigma_m^2 \leqslant r_{m-1}^{\frac{1}{a-1}-\varepsilon_0}\}$ occur finite times for m. Together with (47), it infers that for all sufficiently large m, $\sigma_m^2 > r_{m-1}^{\frac{1}{a-1}-\varepsilon_0} = r_{m-1}^{a_1}$ almost surely. Now, assume that (48) holds for some $n \geq 1$, whenever m is sufficiently large. We prove it for $n + 1$. Applying Lemma 3.3 again with $\varepsilon = \varepsilon_n$ and $q = a_n$, events $\{\sigma_{m+1}^2 \geqslant r_m^{a_n}, \sigma_m^2 \leqslant r_{m-1}^{\frac{1+a_n}{a-1-a_n}-\varepsilon_n}\}$ occur finite times for m as well. So, for all sufficiently large m,

$$\sigma_m^2 > r_{m-1}^{\frac{1+a_n}{a-1-a_n}-\varepsilon_n} = r_{m-1}^{a_{n+1}}, \quad \text{a.s.} \quad \text{on } D_1.$$

We thus in fact have verified (48) for all n, when m is sufficiently large. This means

$$\liminf_{t\to+\infty} \frac{\log r_t}{\log r_{t-1}} \geq 1 + a_n \quad \forall n \geq 1, \quad \text{a.s.} \quad \text{on } D_1,$$

and by letting $n \to +\infty$,

$$\liminf_{t\to+\infty} \frac{\log r_t}{\log r_{t-1}} \geq 1 + x_{\min}, \quad \text{a.s.} \quad \text{on } D_1.$$

Next, we show (ii). For each integer $l \geq 2$, denote $a_{0,l} \triangleq x_{\max} + l^{-1}(a-1-x_{\max}) \in (x_{\max}, a-1)$. According to (ii) of Lemma 3.6, there exists a finite sequence $\{a_{i,l}\}_{i=1}^{k_l}$ with some integer $k_l \geq 1$ depending on $a_{0,l}$ such that $x_{\max} < a_{0,l} < \cdots < a_{k_l-1,l} < a-1$, $a_{k_l,l} > a-1$, and

$$a_{i+1,l} = \frac{1 + a_{i,l}}{a - 1 - a_{i,l}} - \epsilon_{i,l},$$

where $\epsilon_{i,l} \in (0,1)$ for $0 \leq i \leq k_l - 1$. Similar to the induction argument of (48), we can prove that for all $l \geq 2$ and $0 \leq n \leq k_l$,

$$\{\sigma_m^2 > r_{m-1}^{a_{0,l}}, \text{ i.o.}\} \subset \{\sigma_m^2 > r_{m-1}^{a_{n,l}}, \text{ i.o.}\},$$

where i.o. is infinitely often. Suppose $P(D_3) > 0$. Then, there is a random number a_0 taking values from $\{a_{0,l}, l \in \mathbb{N}^+ \backslash \{1\}\}$ such that $\sigma_{m+1}^2 > r_m^{a_0}$ for infinitely many m on D_3. Hence

$$P(D_3) \leq \sum_{l=2}^{+\infty} P(\sigma_m^2 > r_{m-1}^{a_{0,l}}, \text{ i.o.}) \leq \sum_{l=2}^{+\infty} P(\sigma_m^2 > r_{m-1}^{a_{k_l,l}}, \text{ i.o.}),$$

and $P(D_3) = 0$ will hold if we could show that

$$P\left(\sigma_m^2 > r_{m-1}^{a_{k_l,l}}, \text{ i.o.}\right) = 0 \quad \forall l \geq 2. \tag{49}$$

Indeed, for any $a_{k_l,l}$ and all sufficiently large m, $(r_m^{1+a_{k_l,l}} - r_m)^{\frac{1}{2}} - l_1 > 0$ and

$$\begin{aligned} P\left(\sigma_{m+1}^2 > r_m^{a_{k_l,l}} \middle| \mathscr{F}_m^y\right) &= P\left(f^2(y_{m+1}) > r_m^{1+a_{k_l,l}} - r_m | \mathscr{F}_m^y\right) \\ &\leq P\left(|y_{m+1}| > l_2^{-\frac{1}{a}}((r_m^{1+a_{k_l,l}} - r_m)^{\frac{1}{2}} - l_1)^{\frac{1}{a}} \middle| \mathscr{F}_m^y\right) \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{|x\cdot\sigma_m| \geqslant l_2^{-\frac{1}{a}}((r_m^{1+a_{k_l,l}} - r_m)^{\frac{1}{2}} - l_1)^{\frac{1}{a}}} e^{-\frac{x^2}{2}}\, dx \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{|x\cdot\sqrt{P_0 r_m}| \geqslant l_2^{-\frac{1}{a}}((r_m^{1+a_{k_l,l}} - r_m)^{\frac{1}{2}} - l_1)^{\frac{1}{a}}} e^{-\frac{x^2}{2}}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{|x| \geqslant Q_m^{(4)}} e^{-\frac{x^2}{2}}\, dx, \end{aligned} \tag{50}$$

where $Q_m^{(4)} \triangleq P_0^{-\frac{1}{2}} l_2^{-\frac{1}{a}} (r_m^{a_{k_l,l}+1-a} - r_m^{1-a})^{\frac{1}{2a}} (1 - l_1(r_m^{1+a_{k_l,l}} - r_m)^{-\frac{1}{2}})^{\frac{1}{a}}$. Furthermore, since $\liminf_{m\to+\infty} \frac{r_m}{m} > 0$ and $a_{k_l,l} > a-1$, one has $\liminf_{m\to+\infty} \frac{Q_m^{(4)}}{\log m} > 0$. By Lemma 3.1, $\sum_{m=1}^{+\infty} \frac{1}{\sqrt{2\pi}} \int_{|x|\geqslant Q_m^{(4)}} e^{-\frac{x^2}{2}}\, dx < +\infty$, and hence (50) leads to

$$\sum_{m=1}^{+\infty} P\left(\sigma_{m+1}^2 > r_m^{a_{k_l,l}} | \mathscr{F}_m^y\right) < +\infty.$$

So, the Borel–Cantelli–Levy theorem yields (49) and $P(D_3) = 0$ follows. ∎

Proof of Theorem 2.3. Suppose $P(D_1) > 0$, where D_1 is defined in Lemma A.1. It suffices to consider Theorem 2.3 for $\ell(\{x : |f(x)| > 0\}) > 0$. Since $b < (1 + x_{\min})^2$ and $x_{\min} x_{\max} = 1$, there exist some $\delta_1, \delta_2 \in (0, x_{\min})$ such that

$$\frac{1 + x_{\min} - \delta_2}{\frac{x_{\max}+\delta_1}{1+x_{\max}+\delta_1}} > b. \tag{51}$$

By Lemma A.1, for all sufficiently large m,

$$\sigma_{m+1}^2 > r_m^{x_{\min}-\delta_2}, \quad \sigma_m^2 < r_{m-1}^{x_{\max}+\delta_1}, \quad \text{a.s.} \quad \text{on } D_1. \tag{52}$$

Now, by letting $Q_m^{(5)} \triangleq L^{-\frac{1}{b}} ((r_m^{1+x_{\min}-\delta_2} - r_m)^{\frac{1}{2}})^{\frac{1}{b}} \cdot r_m^{-\frac{1}{2}\frac{x_{\max}+\delta_1}{1+x_{\max}+\delta_1}}$,

$$\begin{aligned}
&P\left(y_{m+1} \in S_b^L, \sigma_{m+1}^2 > r_m^{x_{\min}-\delta_2}, \sigma_m^2 < r_{m-1}^{x_{\max}+\delta_1}\right) \\
&= P\left(y_{m+1} \in S_b^L, f^2(y_{m+1}) > r_m^{1+x_{\min}-\delta_2} - r_m, \sigma_m^2 < r_{m-1}^{x_{\max}+\delta_1}\right) \\
&\leq I_{\{\sigma_m^2 < r_{m-1}^{x_{\max}+\delta_1}\}} \cdot \frac{1}{\sqrt{2\pi}} \int_{|x\cdot\sigma_m| \geqslant L^{-\frac{1}{b}} ((r_m^{1+x_{\min}-\delta_2} - r_m)^{\frac{1}{2}})^{\frac{1}{b}}} e^{-\frac{x^2}{2}}\, dx \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{|x|\geqslant Q_m^{(5)}} e^{-\frac{x^2}{2}}\, dx.
\end{aligned}$$

Note that (51) and $\liminf_{m\to+\infty} \frac{r_m}{m} > 0$ imply $\liminf_{m\to+\infty} \frac{Q_m^{(5)}}{\log m} > 0$. By Lemma 3.1 and the Borel–Cantelli–Levy theorem, events $\{y_{m+1} \in S_b^L, \sigma_{m+1}^2 > r_m^{x_{\min}-\delta_2}, \sigma_m^2 < r_{m-1}^{x_{\max}+\delta_1}\}$ occur finite times for m. This fact together with (52) yileds that $y_{m+1} \notin S_b^L$ for all sufficiently large m on D_1 almost surely. Then,

$$\sum_{m=1}^{+\infty} P(y_{m+1} \in S_b^L | \mathscr{F}_m^y) < +\infty, \quad \text{a.s.} \quad \text{on } D_1. \tag{53}$$

On the other hand, Lemma A.1 implies

$$\sup_{m\geq 1} \frac{\log r_m}{\log r_{m-1}} < +\infty, \quad \text{a.s.} \quad \text{on } D_1,$$

thus there is a random number $M_1 > 1$ such that for all $m \geq 1$,

$$\log \sigma_m \leq \frac{1}{2}(\log P_0 + \log r_m) < M_1^m, \quad \text{a.s.} \quad \text{on } D_1.$$

Moreover, Lemma 3.5 yields that $\lim_{m\to+\infty} \sigma_m = +\infty$ on D_1. So, there is a positive constant M_2 such that for all sufficiently large m,

$$\frac{\ell(S_b^L \cap [-\sigma_m, \sigma_m])}{\sigma_m} \geq \frac{M_2}{\log(\log \sigma_m)} \geq \frac{M_2}{m \log M_1} \quad \text{on } D_1.$$

As a result,

$$\begin{aligned}
\sum_{m=1}^{+\infty} P(y_{m+1} \in S_b^L | \mathscr{F}_m^y) &= \frac{1}{\sqrt{2\pi}} \sum_{m=1}^{+\infty} \int_{|x\sigma_m| \in S_b^L} e^{-\frac{x^2}{2}}\, dx \\
&\geq \frac{1}{\sqrt{2\pi}} \sum_{m=1}^{+\infty} \int_{|x\sigma_m| \in S_b^L, |x| \leqslant 1} e^{-\frac{x^2}{2}}\, dx \\
&\geq \sum_{m=1}^{+\infty} \ell(x : |x\sigma_m| \in S_b^L, |x| \leqslant 1) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} \\
&= \sum_{m=1}^{+\infty} \frac{\ell(S_b^L \cap [-\sigma_m, \sigma_m])}{\sigma_m} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} \\
&\geq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} \cdot \frac{M_2}{\log M_1} \sum_{m=1}^{+\infty} \frac{1}{m} = +\infty \quad \text{on } D_1,
\end{aligned}$$

which contradicts (53). So, $P(D_1) = 0$, that is, $\sup_t \sigma_t < +\infty$ almost surely. The remainder of the proof is thus similar to that of Theorem 2.2. ∎

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