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Least Squares Model Averaging for Distributed Data

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Abstract

Divide and conquer algorithm is a common strategy applied in big data. Model averaging has the natural divide-and-conquer feature, but its theory has not been developed in big data scenarios. The goal of this paper is to fill this gap. We propose two divide-and-conquer-type model averaging estimators for linear models with distributed data. Under some regularity conditions, we show that the weights from Mallows model averaging criterion converge in L_2 to the theoretically optimal weights minimizing the risk of the model averaging estimator. We also give the bounds of the in-sample and out-of-sample mean squared errors and prove the asymptotic optimality for the proposed model averaging estimators. Our conclusions hold even when the dimensions and the number of candidate models are divergent. Simulation results and a real airline data analysis illustrate that the proposed model averaging methods perform better than the commonly used model selection and model averaging methods in distributed data cases. Our approaches contribute to model averaging theory in distributed data and parallel computations, and can be applied in big data analysis to save time and reduce the computational burden.

Keywords: consistency, distributed data, divide and conquer algorithm, Mallows' criterion, model averaging, optimality.

1. Introduction

Modern science and technology make data collection easier and easier, and thus more and more big data have been obtained and stored. Usually, such data are with complicated, structured, varied, and various characteristics in economy, finance, biology, medicine, industry, agriculture, transportation, and other fields. See, for example, Misra et al. (2019),

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who provided plenty of real data examples that reflect the overall outlook of big data era. In this world of explosively large data, estimation faces big computational and statistical challenges, especially in scalability and storage bottlenecks of hardware and software issues, and invalidated exogeneous assumptions brought by incidental endogeneity in big data, seeing Fan et al. (2014) for a review. In big data applications, one often prefers to suggest a specific methodology for the problems he/she faces but without theoretical analysis. For example, Sienkiewicz et al. (2017) solved the computational problems on a single, multicore server to describe spiking activity in non-linear dynamic systems with the software MapReduce and Hoodop, but no theoretical property is discussed. Hence the effective distributed estimation procedures with theoretical supports are urgently needed to deal with the computational challenges arisen from large sample size and large number of parameters in massive data analysis. In this regard, some distributed statistical computing methods have been proposed. See, for example, Varian (2014) and Wang et al. (2016).

The large-scale datasets may not fit the memory of a single computer and thus are distributedly stored in multiple machines or servers. So statistical methods should be adjusted and modified to accommodate distributed data. The divide and conquer trick is a practicable and common approach to handle the massive data computation with memory constraints. It divides data into several groups and then aggregates all group estimators by a simple average to lessen the computational burden (Zhang et al., 2013b; Chen and Xie, 2014; Zhang et al., 2015; Xu et al., 2019). A number of problems have been studied for the divide and conquer method, including variable selection (Chen and Xie, 2014), statistical optimization (Zhang et al., 2013b), logistic regression (Xi et al., 2009), estimation equation (Lin and Xi, 2011), kernel ridge regression (Zhang et al., 2015; Xu et al., 2019), quantile regression (Chen et al., 2019, 2020), linear support vector machine (Wang et al., 2019), and distributed principal component analysis (PCA) (Balcan et al., 2012; Garber et al., 2017). Some distributed statistical methods based on likelihood framework are also proposed, and the theoretical upper bound of the information loss for the distributed algorithm is obtained (c.f., Battey et al., 2018). For data distributed over the nodes, Safarinejadian et al. (2010) proposed a distributed expectation maximization (DEM) algorithm with two important advantages of scalability and fault tolerance for density estimation and clustering in sensor networks, which can also be seen as a divide and conquer method. The DEM algorithm is scalable and robust under the Gaussian mixture model assumption, where the addition of more nodes does not affect the performance of the DEM algorithm and it can still produce the right results even if failures of some nodes occur. The diffusion speed and convergence of the DEM algorithm have also been studied in Safarinejadian et al. (2010).

However, numerous papers on the divide and conquer algorithm are not involved with model selection uncertainty. Model averaging is a feasible method to avoid such an uncertainty. There are four main reasons prompting us to choose model averaging instead of model selection. First, choosing a single model may not take full information provided by the training data, especially when it is hard to get a best model. For example, there may be more than one candidate model with similar quantitative scores under some model selection criteria. On the other hand, different candidate models capture different data characteristics. In this dilemma, combining all of those models will not lose the information from each candidate model and thus may be a better choice. Simple averaging of different machine learning models to get a more accurate prediction has been a popular method in

some big data applications. Model averaging can result in a smaller risk and get a more accurate prediction generally. In fact, model averaging often performs at least as well as the best algorithm in the candidate models. As commented by Schomaker and Heumann (2020), model averaging can improve the predictions and should be regarded as attractive complements for the machine learning and forecasting. Second, model averaging can be more stable. Based on different statistic analysis goals, model averaging can stabilize estimation and forecast by assigning different weights to candidate models, and is regarded as a smoothed extension of model selection. Third, model averaging can avoid selecting the worst candidate model. Last but not the least, model selection criteria based on likelihood, such as AIC (Akaike, 1974; Matsuda et al., 2021), BIC (Schwarz, 1978), and minimum description length (MDL, Maggioni and Murphy, 2019), can be invalid for some singular candidate models including artificial neural networks, normal mixtures, binomial mixtures, reduced rank regressions, Bayesian networks, and hidden Markov models, as the likelihood functions of these singular statistical models and learning machines cannot be approximated by any normal distribution (Watanabe, 2010, 2013). For so many singular models, model averaging, a valid solution, can be used to get more robust estimates and generalized machine learning methods. For all these reasons, compared with model selection, model averaging estimators often get higher prediction precision and better robustness, and thus have received extensive attention in recent years.

In the frequentist viewpoint, a key problem with the model averaging is the choice of weights assigned to different models. A variety of model averaging criteria have been suggested. See, for example, smoothed information criteria including smoothed AIC, smoothed BIC (Buckland et al., 1997), and smoothed FIC (Hjort and Claeskens, 2003; Claeskens and Carroll, 2007; Zhang and Liang, 2011; Zhang et al., 2012; Xu et al., 2014); adaptive method (Yang, 2001; Yuan and Yang, 2005); and asymptotically optimal methods, such as Mallows model averaging (MMA) method (Hansen, 2007; Wan et al., 2010), OPT method (Liang et al., 2011), jackknife model averaging (JMA) method (Hansen and Racine, 2012; Zhang et al., 2013a; Zhang and Zou, 2020), and leave-subject-out cross-validation method for time series data (Gao et al., 2016; Liao et al., 2019).

In this paper, we will focus on Hansen's MMA, which is the first model averaging method with optimality. Hansen (2007) proved that the Mallows criterion is asymptotically optimal in the sense of achieving the lowest possible squared error for the nested candidate models and discrete weight set. Further, Wan et al. (2010) provided an alternative proof for the non-nested candidate models and continuous model weights. Liu and Okui (2013) proposed a modified Mallows model averaging for heteroscedasticity data. Gao et al. (2019) suggested an adjusted MMA criterion for threshold auto-regressive model. Zhu et al. (2019) developed a Mallows-type model averaging estimator for the varying-coefficient partially linear model. A corrected Mallows model averaging method for small sample sizes can be found in Liao and Zou (2020).

In recent years, the property of model weight has attracted much attention. For model averaging, there are few articles on the uniqueness of the optimal weight choice except Hansen (2014) in which a unique empirical weight vector is obtained if the candidate models are appropriately restricted. Hansen (2014) investigated the asymptotic risks of nested least-squares averaging estimators with minimum mean squared error criterion in a local asymptotic framework and gave an explicit form of optimal weights based on asymptotic

risk in some common situations. Hansen (2014) also suggested a practical rule that model averaging estimators should be based on models where the regressors have been grouped. This rule will lead to a better implementation of averaging. Charkhi et al. (2016) noticed the uniqueness of weights of model averaging based on likelihood frameworks and recommended a suitable class of models which are so-called singleton models where each model includes only one candidate variable. This singleton model trick can result in a drastic reduction in the computational cost of model averaging and can be applied in big data area. Another interesting problem with the model averaging is the consistency of weights. There are a few articles on this topic (c.f., Chen et al., 2018; Liao et al., 2019; Liao and Zou, 2020). Chen et al. (2018) proposed a semi-parametric penalized model averaging method for marginal regressions of time series and derived the consistency and oracle property with the assumption that the weights are sparse and some other regularity conditions. Each candidate model in Chen et al. (2018) can be regarded as a projection from response variable to marginal regressions, and the weights assigned to different models are without any constraints, as in Li et al. (2015), who proposed a forecasting method by combining all marginal regressions in applications. Liao et al. (2019) derived the convergence rate of the weights based on leave-subject-out cross-validation model averaging method for VAR model. Liao and Zou (2020) proved the consistency of MMA weights. Some articles also focus on the other statistical limiting properties of Mallows model averaging. For example, Liu (2015) derived the limiting distributions of the weights based on Mallows criterion and nested least squares averaging estimators under the local misspecification framework.

For distributed and massive data, except simple averaging and Fang's et al (2018) approximating calculations, no model averaging theory is developed. The purpose of this paper is to fill this gap. We will propose efficient computational strategies and theory for model averaging on distributed data and divergent dimensional regressions. The contributions of this article are threefold. First, we prove that the weight vector selected by Mallows model averaging criterion for least squares estimators in linear regression models is L_2 convergent to the theoretical optimal weight. Our results of convergence type are different from those in Hansen (2014), Liu (2015), Chen et al. (2018), Liao et al. (2019), and Zhang et al. (2020). Second, we propose two types of model averaging estimators for distributed or parallel data. From our theoretical analysis, we find that the two tricks of grouping regressors and singleton models can be used to reduce the computation cost. Before model averaging, using model selection can throw away some clearly unreasonable models and will relieve of the computational burden. Based on some suitable candidate models, we may be able to get a better model averaging estimator. The grouping regressors models and singleton models can be used as some alternative tricks to build the candidate model set. Such tricks have been used by, say, Hansen (2014) and Charkhi et al. (2016) in the literature on model averaging. In fact, the idea of grouping regressors has been investigated previously in statistical literature, including Efron and Morris (1973), Berger and Dey (1983), Dey and Berger (1983), George (1986a), George (1986b), and Mougeot et al. (2013). Grouping strategies have been shown to improve the prediction performance and interpretability of the candidate models (Lounici et al., 2011). In model averaging, each learner based on grouping some similar regressors will be more useful and all these learners can comprise a candidate model set which will lead to a drastic reduction in the computational cost. Both singleton models and averaging across singletons are also two popular methods in

data analysis. For example, Hjort and Claeskens (2003) observed that the averaging across singletons method does quite well in achieving the smallest risk and leads to low standard deviation and short confidence intervals. In summary, both group strategies and singleton models can reduce the computational burden, make the optimal weights be identical and unique, and lead to a parsimonious model averaging estimator, and thus are useful tools in big data area. Third, it is inspired that we can use the weight calculated from a simple random sample without replacement with large size drawn from the extremely massive data as the weight estimation for the whole collected data.

The remainder of this paper is organized as follows. In Section 2, we build a general Mallows model averaging framework for distributed data. Then, we investigate the theoretical properties of the proposed weights and model averaging estimators in Section 3. Section 4 covers simulations. In Section 5, we apply our model averaging methods for distributed data to the real airline data. Theoretical proofs are included in Appendices.

2. Model Averaging Based on Distributed Data

2.1 Model averaging for subject

Let $\{(y_i, x_i) : i = 1, 2, \ldots, N\}$ be an i.i.d. sample from the following data-generating process,

$$y_i = \mu_i + e_i = \sum_{j=1}^{\infty} \theta_j x_{ij} + e_i, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $y_i \in \mathbb{R}$, $x_i = (x_{i1}, x_{i2}, \ldots)^T$ is countably infinite, and e_i is an error term. We assume that $\{e_i\}_{i\geq 1}$ are mutually independent with $\mathbf{E}(e_i|x_i) = 0$ and $\mathbf{E}\left(e_i^2|x_i\right) = \sigma^2$, and $\mathbf{E}\mu_i^2 < \infty$. The model set-up follows Hansen (2007). We assume a sequence of linear approximating models, where the sth model uses the first p_s regressors of x_i, $s = 1, \ldots, S$. That is, the sth candidate model is

$$y_i = \sum_{j=1}^{p_s} \theta_j x_{ij} + e_i, \quad i = 1, 2, \ldots, N. \tag{2}$$

The approximating error of the sth candidate model is $b_{i(s)} = \sum_{j=p_s+1}^{\infty} \theta_j x_{ij}$. Let $\beta_{(s)} = (\theta_1, \theta_2, \ldots, \theta_{p_s})^T$, $s = 1, \ldots, S$.

Since N is extremely large, we apply the divide and conquer trick to treat the collected data. Without loss of generality, we let $N = Kn$, where both K and n are positive integers. Then we divide the collected data set $\{(y_i, x_i), i = 1, \ldots, N\}$ evenly and uniformly at random among a total of K subjects. At each subject, denote the resultant data as $\left\{Y_k, X_{(k)}\right\}, k = 1, 2, \ldots, K$, where $Y_k = (y_{k,1}, y_{k,2}, \cdots, y_{k,n})^T$ and $X_{(k)} = (x_{k,1}, x_{k,2}, \cdots, x_{k,n})^T$ with $(y_{k,j}, x_{k,j})$, $j = 1, 2, \cdots, n$, being a random sample from the $\{(y_i, x_i), i = 1, \ldots, N\}$. Denote $\boldsymbol{\mu}_k = (\mu_{k,1}, \mu_{k,2}, \cdots, \mu_{k,n})^T$ and the error term for the kth subject as $e_{(k)} = (e_{k,1}, e_{k,2}, \cdots, e_{k,n})^T$ accordingly. At subject k, we consider model averaging procedure.

The estimator of $\beta_{(s)}$ in the sth candidate model under the kth subject is given by

$$\hat{\beta}_{k,s} = (X_{k,s}^T X_{k,s})^{-1} X_{k,s}^T Y_k,$$

where $X_{k,s}$ is an $n \times p_s$ matrix with full column rank, including the first p_s columns of $X_{(k)}$ related to the sth candidate model, $s = 1, \ldots, S$. For simplicity, we denote $X_k = X_{k,S}$. Then the model averaging estimator for $\boldsymbol{\mu}_k$ has the form

$$\hat{\boldsymbol{\mu}}_k(W_k) = \sum_{s=1}^{S} w_{k,s} X_{k,s} \hat{\beta}_{k,s}$$

with $W_k = (w_{k,1}, \ldots, w_{k,S})^T \in Q$ and

$$Q \triangleq \left\{ w = (w_1, \ldots, w_S)^T : \sum_{s=1}^{S} w_s = 1, w_s \geq 0, s = 1, 2, \ldots, S \right\}. \tag{3}$$

A key problem with the estimator $\hat{\boldsymbol{\mu}}_k(W_k)$ is the choice of weights. To choose a proper W_k, we minimize the following Mallows criterion

$$C_{k,n}(W_k) = \frac{1}{n} \left(Y_k - \hat{\boldsymbol{\mu}}_k(W_k)\right)^T \left(Y_k - \hat{\boldsymbol{\mu}}_k(W_k)\right) + \frac{2}{n} \sigma^2 tr\left[P_k(W_k)\right] \tag{4}$$

in Q to get

$$\hat{W}_k = (\hat{w}_{k,1}, \hat{w}_{k,2}, \cdots, \hat{w}_{k,S})^T = \underset{w \in Q}{\operatorname{argmin}} C_{k,n}(w), \tag{5}$$

where

$$P_k(W_k) \triangleq \sum_{s=1}^{S} w_{k,s} X_{k,s} \left(X_{k,s}^T X_{k,s}\right)^{-1} X_{k,s}^T \triangleq \sum_{s=1}^{S} w_{k,s} P_{k,s},$$

and $tr\left[P_k(W_k)\right] = \sum_{s=1}^{S} w_{k,s} p_s$. When σ^2 is unknown, (4) needs to be computed with a sample estimate. There are several ways to estimate σ^2. We use the following estimator

$$\hat{\sigma}_k^2 = \frac{\left(Y_k - X_{k,S}\hat{\beta}_{k,S}\right)^T \left(Y_k - X_{k,S}\hat{\beta}_{k,S}\right)}{n - p_S},$$

which is based on the largest candidate model (Hansen, 2007; Wan et al., 2010) for the kth subject. The resultant Mallows model averaging estimator for $\boldsymbol{\mu}_k$ is given by

$$\hat{\boldsymbol{\mu}}_k(\hat{W}_k) = \sum_{s=1}^{S} \hat{w}_{k,s} X_{k,s} \hat{\beta}_{k,s}.$$

2.2 Model averaging for distributed data

Let Π_s be a selection matrix for the sth candidate model, so that $X_{k,s} = X_k \Pi_s^T$ and $\Pi_s \Pi_s^T = I_{p_s}$, where I_{p_s} is an identity matrix of order p_s. The model averaging estimator of $\beta_{(S)}$ at subject k is

$$\hat{\beta}_k(\hat{W}_k) = \sum_{s=1}^{S} \hat{w}_{k,s} \Pi_s^T \hat{\beta}_{k,s} \tag{6}$$

for $k = 1, \ldots, K$. In the following, we construct two types of model averaging estimators.

(1^0) Simple aggregation of model averaging estimators

We aggregate the K local estimators together by simple averaging to obtain the simple aggregated model averaging estimator of $\beta_{(S)}$, that is,

$$\overline{\beta} = \frac{1}{K}\sum_{k=1}^{K}\hat{\beta}_k(\hat{W}_k). \tag{7}$$

Accordingly, the simple aggregated model averaging estimator of $\boldsymbol{\mu}_k$ is given by

$$\overline{\boldsymbol{\mu}}_k = X_k\overline{\beta}.$$

(2^0) Doubly simple aggregation of model averaging estimators

The other aggregated model averaging procedure is as follows. First, we aggregate the least squares estimators $\hat{\beta}_{k,s}$ and the weights $\hat{w}_{k,s}$ respectively, that is,

$$\tilde{\beta}_s = \frac{1}{K}\sum_{k=1}^{K}\hat{\beta}_{k,s}, \quad s = 1,\ldots,S, \tag{8}$$

and

$$\overline{w}_s = \frac{1}{K}\sum_{k=1}^{K}\hat{w}_{k,s}, \quad s = 1,\ldots,S. \tag{9}$$

Second, we aggregate $\tilde{\beta}_s$ and $\overline{w}_s$ of each candidate model to obtain the doubly simple aggregated model averaging estimator of $\beta_{(S)}$, that is

$$\overline{\tilde{\beta}} = \sum_{s=1}^{S}\overline{w}_s\Pi_s^T\tilde{\beta}_s. \tag{10}$$

The doubly simple aggregated model averaging estimator of $\boldsymbol{\mu}_k$ is

$$\overline{\tilde{\boldsymbol{\mu}}}_k = X_k\overline{\tilde{\beta}}.$$

3. Theoretical Results

We first introduce some notations. We use ℓ_2 to denote the usual Euclidean norm $\|\theta\| = \sqrt{\sum_{j=1}^{d}\theta_j^2}$ with $\theta = (\theta_1,\theta_2,\cdots,\theta_d)^T$. The ℓ_2-operator norm of a matrix $A \in \mathbb{R}^{d_1\times d_2}$ is its maximum singular value, defined by

$$\|A\|_2 \triangleq \sup_{v\in\mathbb{R}^{d_2},\|v\|\leq 1}\|Av\|.$$

We denote the minimum eigenvalue of a matrix A by $\lambda_{\min}(A)$. Let λ_1, λ_2, $\cdots$, and λ_d be the real eigenvalues of a matrix $A \in \mathbb{R}^{d\times d}$. Then its spectral radius $\rho_r(A)$ is defined as

$$\rho_r(A) \triangleq \max_{1\leq i\leq d}|\lambda_i|.$$

A convex function F is λ-strongly convex on a set $U \subseteq \mathbb{R}^d$ if for arbitrary $u \in U$ and $v \in U$, we have

$$F(u) \geq F(v) + \langle \nabla F(v), u - v \rangle + \frac{\lambda}{2}\|u - v\|_2^2,$$

where ∇F is the derivative of the function F. In addition, if F is not differentiable, we may replace ∇F by any subgradient of F.

Consider the quadratic loss function

$$\begin{aligned} L_N(w) &= \frac{1}{N}\left\|\hat{\boldsymbol{\mu}}\left(w\right) - \boldsymbol{\mu}\right\|^2 \\ &= \frac{1}{nK}\sum_{k=1}^{K}\|\hat{\boldsymbol{\mu}}_k(w) - \boldsymbol{\mu}_k\|^2, \end{aligned}$$

where $w = (w_1, w_2, \cdots, w_S)^T \in Q$, and $\hat{\boldsymbol{\mu}}\left(w\right) = \left(\hat{\boldsymbol{\mu}}_1^T\left(w\right), \hat{\boldsymbol{\mu}}_2^T\left(w\right), \cdots, \hat{\boldsymbol{\mu}}_K^T\left(w\right)\right)^T$ and $\boldsymbol{\mu} = \left(\boldsymbol{\mu}_1^T, \boldsymbol{\mu}_2^T, \cdots, \boldsymbol{\mu}_K^T\right)^T$ are two $N \times 1$ vectors. The population risk R_N is given by

$$\begin{aligned} R_N(w) &= \mathbf{E}L_N(w) \\ &= \frac{1}{n}\mathbf{E}\left\|\hat{\boldsymbol{\mu}}_1\left(w\right) - \boldsymbol{\mu}_1\right\|^2, \end{aligned}$$

where the second equality is by the assumption that the data are independent and identically distributed. For the weight $w \in Q$, we can rewrite $R_N(w)$ as

$$R_0\left(w_1, w_2, \cdots, w_{S-1}\right) = \frac{1}{n}\mathbf{E}\left\|\sum_{s=1}^{S-1} w_s X_{1,s}\hat{\beta}_{1,s} + \left(1 - \sum_{s=1}^{S-1} w_s\right) X_{1,S}\hat{\beta}_{1,S} - \boldsymbol{\mu}_1\right\|^2 \tag{11}$$

with the constraint of $\left(w_1, w_2, \cdots, w_{S-1}\right)^T \in Q_0$ and

$$Q_0 \triangleq \left\{ \left(w_1, w_2, \cdots, w_{S-1}\right)^T \,\middle|\, w_s \geq 0, s = 1, 2, \cdots, S-1; 0 \leq \sum_{s=1}^{S-1} w_s \leq 1 \right\}.$$

Denote $w_0 = \left(w_1, w_2, \cdots, w_{S-1}\right)^T$. For the model averaging framework, we need to determine the weights assigned to candidate models. So, our goal is to estimate the parameter vector minimizing the risk $R_0\left(w_0\right)$, namely the quantity

$$w_0^* \triangleq \underset{w_0 \in Q_0}{\operatorname{argmin}} R_0\left(w_0\right),$$

which is equivalent to estimate

$$w^* \triangleq \underset{w \in Q}{\operatorname{argmin}} R_N(w).$$

By first calculating the weight W_k at the kth subject by (5), and then averaging the weights by (9) to get the averaged weight $\overline{w}$, where $\overline{w} = \left(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_S\right)^T$, we can show the consistency of the weight $\overline{w}_0 = \left(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_{S-1}\right)^T$ to w_0^*. We will establish the theoretical properties of $\overline{w}_0$ in Subsection 3.1 and the proposed estimators (7) and (10) in Subsection 3.2.

3.1 Convergence of the weight estimator

In the following, we assume the candidate models $s = 1, 2, \cdots, S$ are nested, and then $0 < p_1 < p_2 < \cdots < p_S$. Without loss of generality, we assume that $Ex_{k,i,j} = 0$, and the covariance of $x_{k,i,j}$ and x_{k,i,j_1} is σ_{j,j_1} with $j \neq j_1$. Denote $\Sigma_s = \{\sigma_{j,j_1}\}_{1 \leq j,j_1 \leq s}$ and let the pseudo-true value of $\beta_{(s)}$ be

$$\begin{aligned}
\beta_{\star,s} &\triangleq \arg \min_{\beta_{k,s} \in \mathbb{R}^{p_s}} \frac{1}{n} \mathbf{E} \left\| X_{k,s}\beta_{k,s} - \boldsymbol{\mu}_k \right\|^2 \\
&= \arg_{\beta_{k,s} \in \mathbb{R}^{p_s}} \left(\frac{1}{n} \mathbf{E} \left\{ X_{k,s}^T \left(X_{k,s}\beta_{k,s} - \boldsymbol{\mu}_k \right) \right\} = 0 \right) \\
&= \mathbf{E} \left(X_{k,s}^T X_{k,s} \right)^{-1} \mathbf{E} \left(X_{k,s}^T \boldsymbol{\mu}_k \right) \\
&= (\theta_1, \theta_2, \ldots, \theta_{p_s})^T + \Sigma_s^{-1} \left(\sum_{j=p_s+1}^{\infty} \theta_j \sigma_{1,j}, \sum_{j=p_s+1}^{\infty} \theta_j \sigma_{2,j}, \ldots, \sum_{j=p_s+1}^{\infty} \theta_j \sigma_{p_s,j} \right)^T \\
&\triangleq \beta_{(s)} + \Sigma_s^{-1} \gamma_s.
\end{aligned}$$

Further, define

$$R_N^*(w) = \frac{1}{N} \sum_{k=1}^{K} \mathbf{E} \left\| X_k \beta_\star(w) - \boldsymbol{\mu}_k \right\|^2,$$

where

$$\beta_\star(w) \triangleq \sum_{s=1}^{S} w_s \Pi_s^T \beta_{\star,s}.$$

Accordingly,

$$R_0^*(w_0) = \frac{1}{N} \sum_{k=1}^{K} \mathbf{E} \left\| X_k \left\{ \sum_{s=1}^{S-1} w_s \Pi_s^T \beta_{\star,s} + \left(1 - \sum_{s=1}^{S-1} w_s \right) \Pi_S^T \beta_{\star,S} \right\} - \boldsymbol{\mu}_k \right\|^2.$$

We now define the error of the pseudo-true model as

$$\begin{aligned}
\delta_{k,s} &= \boldsymbol{\mu}_k - X_{k,s}\beta_{\star,s} \\
&= \left(\boldsymbol{\mu}_k - X_{k,s}\beta_{(s)} \right) - X_{k,s}\Sigma_s^{-1}\gamma_s \\
&\triangleq \boldsymbol{b}_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s,
\end{aligned}$$

and

$$\Sigma_{\infty|s} \triangleq \mathbf{E} \left(\delta_{k,s}\delta_{k,s}^T | X_{k,s} \right), \quad \Sigma'_{\infty|s} \triangleq \frac{1}{n} \mathbf{E} (\delta_{k,s}\delta_{k,s}^T \delta_{k,s}\delta_{k,s}^T | X_{k,s}).$$

To derive the consistency of our weight estimator, we need the following regularity conditions.

Condition 1 $w_0^* \in int Q_0$.

Condition 2 $\max_{1 \leq s \leq S} \mathbf{E} \left(|x_{(i)}^T \Pi_s^T \beta_{(s)}|^{\eta+2} + |x_{(i)}^T \Pi_s^T \beta_{\star,s}|^{\eta+2} \right) < C_b < \infty$ for some $\eta \geq 2$, where $x_{(i)} = (x_{i,1}, x_{i,2}, \ldots, x_{i,P_S})$, and $\mathbf{E} e_{k,i}^4 \leq \omega < \infty$ for $k = 1, 2, \ldots, K$ and $i = 1, 2, \ldots, n$.

Condition 3 There is a constant $\sigma_n^2 > 0$ such that $\lambda_{\max}\left(\Sigma_{\infty|s}\right) + \lambda_{\max}\left(\Sigma'_{\infty|s}\right) \leq \sigma_n^2$ for $s = 1, \ldots, p_s$.

Condition 4 $\frac{S^2 p_S \sigma_n^2}{n\overline{\lambda}_S^2} = o(1)$, where $\overline{\lambda}_S = \lambda_{\min}\left[\nabla^2 R_0^*\left(w_0^*\right)\right]$.

Remark 1 *Condition 1 is common in optimization theory to ensure the solution can be calculated by some gradient descent algorithms or iterative algorithms. Since $R_0(w_0)$ is twice differentiable with respect to w_0, and Condition 1 requires that $R_0(w_0)$ have a local minimum at the interior point w_0^* of Q_0, which means that $R_N(w)$ has a local minimum at the interior point w^* of the simplex Q, then we have*

$$\lambda_n \triangleq \lambda_{\min}[\nabla^2 R_0\left(w_0^*\right)] > 0. \tag{12}$$

This condition may hold when all the candidate models are useful or competitive. Condition 1 is a valuable alternative to Definition 2 of Watanabe (2010) in Bayesian learning theory, by which Bayesian learning theory can be investigated directly.

Condition 2 places some bounds on the moments of error term $e_{k,i}$, candidate models and pseudo-true candidate models. When $\sigma_{j_1,j_2} = 0$ for $j_1 \neq j_2$, Condition 2 is easily satisfied with the assumption $\mathbf{E}\mu_i^2 < \infty$.

Condition 3 gives an upper bound for the maximum eigenvalues of $\Sigma_{\infty|s}$ and $\Sigma'_{\infty|s}$ that depends on n. Noting that

$$\Sigma_{\infty|s} = \left\{\mathbf{E}\left[\left(\mu_{k,i} - x_{k,i}\Pi_s^T\beta_{\star,s}\right)\cdot\left(\mu_{k,j} - x_{k,j}\Pi_s^T\beta_{\star,s}\right)|x_{k,i}, x_{k,j}\right]\right\}_{1\leq i,j\leq n},$$

it is not difficult to show that

$$\begin{aligned}\lambda_{\max}(\Sigma_{\infty|s}) \quad \leq \quad & \max_{1\leq j\leq n}\left(\mathbf{E}\left[\left\|\mu_{k,j} - x_{k,j}\Pi_s^T\beta_{\star,s}\right\|^2|x_{k,j}\right]\right. \\ & \left. + \sum_{i\neq j}\mathbf{E}\left[\left(\mu_{k,i} - x_{k,i}\Pi_s^T\beta_{\star,s}\right)\cdot\left(\mu_{k,j} - x_{k,j}\Pi_s^T\beta_{\star,s}\right)|x_{k,i}, x_{k,j}\right]\right).\end{aligned}$$

Let us consider a special scenario where $\mu_{k,i} - x_{k,i}\Pi_s^T\beta_{\star,s}$ are mutually independent random variables conditionally given $X_{k,s}$. Then it follows that

$$\begin{aligned}& \mathbf{E}\left[\left(\mu_{k,i} - x_{k,i}\Pi_s^T\beta_{\star,s}\right)\cdot\left(\mu_{k,j} - x_{k,j}\Pi_s^T\beta_{\star,s}\right)|x_{k,i}, x_{k,j}\right] \\ = \quad & \mathbf{E}\left[\left(\mu_{k,i} - x_{k,i}\Pi_s^T\beta_{\star,s}\right)|x_{k,i}\right]\mathbf{E}\left[\left(\mu_{k,j} - x_{k,j}\Pi_s^T\beta_{\star,s}\right)|x_{k,j}\right].\end{aligned}$$

Clearly, as $n \to \infty$, $\mathbf{E}\left[\mu_{k,i} - x_{k,i}\Pi_s^T\beta_{\star,s}|x_{k,i}\right]$ plays a decisive role in the size of $\lambda_{\max}(\Sigma_{\infty|s})$. Similarly, let $q_{il} = \mathbf{E}\left[(\mu_{k,j} - x_{k,j}\Pi_s^T\beta_{\star,s})^l|x_{k,j}\right]$, $l = 1, 2, 3, 4$, then

$$\lambda_{\max}(\Sigma'_{\infty|s}) \quad \leq \quad \frac{1}{n}\max_{1\leq j\leq n}\left(q_{j4} + \sum_{i\neq j}q_{i2}q_{j2} + \sum_{i\neq j}(q_{i3}q_{j1} + q_{j3}q_{i1}) + \sum_{h\neq i,j}q_{h2}q_{i1}q_{j1}\right).$$

As $n^2 - n$ of the n^2 terms on the right-hand side of the above inequality contain q_{i1}, $i = 1, \ldots, n$, $\lambda_{\max}(\Sigma'_{\infty|s})$ is also dominated by $\mathbf{E}\left[\mu_{k,i} - x_{k,i}\Pi_s^T\beta_{\star,s}|x_{k,i}\right]$, $i = 1, \ldots, n$. Observing that $\mathbf{E}\left[\mu_{k,i}|x_{k,i}\right]$ is the optimal estimator of $\mu_{k,i}$ in L_2 sense, $\mathbf{E}\left[\mu_{k,1} - x_{k,i}\Pi_s^T\beta_{\star,s}|x_{k,i}\right]$

represents the gap between the optimal L_2 estimator and the linear minimum variance estimator based on the sth candidate model. Specially, when $x_{k,i,j}$ is Gaussian, it follows that $\mathbf{E}\left[\mu_{k,1} - x_{k,1}\Pi_s^T\beta_{\star,s}|x_{k,1}\right] = 0$ *and*

$$\Sigma_{\infty|s} = \left\{\sum_{j_1,j_2=p_s+1}^{\infty}\theta_{j_1}\theta_{j_2}\sigma_{j_1,j_2} - \gamma_s^T\Sigma_s^{-1}\gamma_s\right\}I_n = \sigma_{\infty|s}^2 I_n,$$

hence $\Sigma_{\infty|s}^{'} = \frac{n+2}{n}\sigma_{\infty|s}^4 I_n$. *Thus,* $\sigma_n^2 = \max_{1\le s\le S}(\sigma_{\infty|s}^2 + 3\sigma_{\infty|s}^4)$ *satisfies Condition 3.*

Condition 4 allows $\overline{\lambda}_S$ *to tend to zero at a rate slower than* $\sqrt{S^2 p_S \sigma_n^2 n^{-1}}$ *with the dimension of regressor vector and/or the number of candidate models being divergent when n tends to* ∞. *Further, with the assumption that the data are independent and identically distributed, after some calculations, it can be seen that,*

$$\begin{aligned}\nabla^2 R_0^*(w_0^*) &= \nabla^2 R_0^*(w_0)\\ &= 2\mathbf{E}\left[\left\{\left(\Pi_{s_1}^T\beta_{\star,s_1} - \Pi_S^T\beta_{\star,S}\right)^T x_1^T x_1\left(\Pi_{s_2}^T\beta_{\star,s_2} - \Pi_S^T\beta_{\star,S}\right)\right\}_{1\le s_1,s_2\le S-1}\right]\\ &= 2\left\{\left(\Pi_{s_1}^T\beta_{\star,s_1} - \Pi_S^T\beta_{\star,S}\right)^T\Sigma_S\left(\Pi_{s_2}^T\beta_{\star,s_2} - \Pi_S^T\beta_{\star,S}\right)\right\}_{1\le s_1,s_2\le S-1},\end{aligned}$$

which is similar to the A6 of Chen et al. (2018). Like Chen et al. (2018), if we do not take account of the constraint $\sum_{s=1}^S w_s = 1$, *then*

$$\nabla^2 R_N^*(w^*) = 2\mathbf{E}\left[\left\{\left(\Pi_{s_1}^T\beta_{\star,s_1}\right)^T x_1^T x_1\left(\Pi_{s_2}^T\beta_{\star,s_2}\right)\right\}_{1\le s_1,s_2\le S}\right].$$

In this case, Condition 4 only requires that

$$\frac{S\sqrt{p_S}\sigma_n}{\sqrt{n}\lambda_{\min}\left[\nabla^2 R_N^*(w^*)\right]} = o(1),$$

which is weaker than Condition A6 of Chen et al. (2018) when $S^2 p_S = o(n)$.

Now, denoting $\hat{W}_{k,0} = (\hat{w}_{k,1}, \hat{w}_{k,2}, \cdots, \hat{w}_{k,S-1})^T$ and then $\overline{w}_0 = \frac{1}{K}\sum_{k=1}^K \hat{W}_{k,0}$, we have the following theoretical results.

Theorem 1 *Under Conditions 1-4, we have*

$$\mathbf{E}\left\|\overline{w}_0 - w_0^*\right\|^2 = O\left(\frac{Sp_S(S+\sigma_n^2)}{Kn\overline{\lambda}_S^2}\right) + O\left(\frac{S^3 p_S^2(S+\sigma_n^2)}{n^2\overline{\lambda}_S^4}\right),$$

and so

$$\mathbf{E}\left\|\overline{w} - w^*\right\|^2 = O\left(\frac{S^2 p_S(S+\sigma_n^2)}{Kn\overline{\lambda}_S^2}\right) + O\left(\frac{S^4 p_S^2(S+\sigma_n^2)}{n^2\overline{\lambda}_S^4}\right).$$

Proof See Appendix B. ■

Corollary 1 *Under Conditions 1,2 and 4, if the covariates $x_{i,j}$ are Gaussian, and S and $p_s, s = 1, 2, \cdots, S$ are fixed, then*

$$\mathbf{E}\left\|\overline{w} - w^*\right\|^2 = O\left(\frac{1}{Kn}\right) + O\left(\frac{1}{n^2}\right).$$

3.2 Mean squared errors of model averaging estimators for regression coefficients

In this subsection, we first show some limiting results about $\min_{w\in Q} R_N(w)$ and the proposed two model averaging estimators of $\beta_{(S)}$, and then provide the upper bounds of the mean squared errors of Mallows model averaging estimators.

Condition 5 $\frac{p_S\sigma_n^2}{n\xi_{\star,N}} = o(1)$, where $\xi_{\star,N} = \inf_{w\in Q} R_N^*(w)$.

Remark 2 *Condition 5 requires that the rate of $n\xi_{\star,N}$ tending to ∞ should be faster than that of $p_S\sigma_n^2$, which is similar to Condition (C4) of Zhang et al. (2020). If $x_{k,i,j}$ is Gaussian, then $\sigma_n^2 = \max_{1\le s\le S}(\sigma_{\infty|s}^2 + 3\sigma_{\infty|s}^4)$, and in this case, this condition is easily satisfied.*

Theorem 2 *Under Conditions 1-5, we have*

$$\sup_{w\in Q}\left|\frac{R_N(w)}{R_N^*(w)} - 1\right| = o(1), \tag{13}$$

and

$$\frac{R_N(w^*)}{\xi_{\star,N}} = 1 + o(1). \tag{14}$$

Proof See Appendix B. ■

Remark 3 *From Theorem 2, $\xi_{\star,N}$ can be seen as the limit of $R_N(w^*)$, the optimal risk of Mallows model averaging estimator. Condition 5 and (14) show that the rate of $NR_N(w^*)$ tending to ∞ should be faster than that of $Kp_S\sigma_n^2$. This property is also consistent with the requirement that the true model should not be in the candidate model set, which is a condition commonly arisen in optimal model averaging. When the true model is an infinite dimensional model, $NR_N(w^*)/Kp_S\sigma_n^2 \to \infty$ is an alternative to Assumption 2 of Zhang (2021).*

In the following, we derive the differences between (7), (10) and $\beta_\star(w^*)$, respectively. Define

$$m_S = \max_{s=1,\ldots,S}\left\|\mathbf{E}\left[\Sigma_S^{1/2}\left\{\hat{\beta}_{k,s} - \beta_{\star,s}\right\}\right]\right\|.$$

Condition 6 $tr\left(\mathbf{E}\left[\left(X_{k,s}^T X_{k,s}\right)^{-1}\right]\Sigma_s\right) = O\left(\frac{p_s}{n}\right)$, $1 \le s \le S$.

Remark 4 *This condition places restriction on the upper bound of $tr(\mathbf{E}[(X_{k,s}^T X_{k,s})^{-1}]\Sigma_s)$. The upper bound nearly matches the risk for Gaussian design. The sufficient conditions for Condition 6 are given in Theorem 3 of Mourtada (2022). From Mourtada (2022), it can be seen that our Condition 6 is mild. When $x_{k,i,j}$ is Gaussian, it is easy to verify that*

$$\mathbf{E}\left[\left(X_{k,s}^T X_{k,s}\right)^{-1}\right]\Sigma_s = (n - p_s - 1)^{-1} I_{p_s},$$

and so

$$tr\left(\mathbf{E}\left[\left(X_{k,s}^T X_{k,s}\right)^{-1}\right]\Sigma_s\right) = \frac{p_s}{n - p_s - 1}.$$

Theorem 3 *Under Conditions 1-4 and 6, we have*

$$\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}\right\|^2=O\left(\frac{S^3p_S(S+\sigma_n^2)}{\overline{\lambda}_S^2n}\right), \tag{15}$$

and

$$\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\overline{\tilde{\beta}}-\beta_\star\left(w^*\right)\right\}\right\|^2=O\left(m_S^2\right)+O\left(\frac{S^3p_S(S+\sigma_n^2)}{Kn\overline{\lambda}_S^2}\right)+O\left(\frac{S^5p_S^2(S+\sigma_n^2)}{n^2\overline{\lambda}_S^4}\right). \tag{16}$$

Proof See Appendix B. ■

Remark 5 *When $x_{k,j,i}$ is Gaussian, the ordinary least squares estimator $\hat{\beta}_{k,s}$ is an unbiased estimator of pseudo-true parameter $\beta_{\star,s}$, i.e., $m_S=0$. So Theorem 3 means that when $\overline{\lambda}_S$ has a uniform lower bound away from zero, if $K=O(1)$, then* (7) *and* (10) *have the same convergence rates to $\beta_\star\left(w^*\right)$; if K tends to ∞, then* (10) *has a faster convergence rate to $\beta_\star\left(w^*\right)$ than* (7)*.*

3.3 Mean squared errors of model averaging estimators for conditional mean

We now consider the mean squared errors of model averaging estimators for estimating conditional mean.

(1^0) Out-of-sample mean squared errors

Let (y_v,x_v) be an independent copy of (y_i,x_i), where $x_v=(x_{v1},x_{v2},\ldots)$ is countably infinite, $x_{v,s}=\left(x_{v1},x_{v2},\ldots,x_{vp_s}\right)^T$, $\mu_v=\sum_{j=1}^\infty\theta_jx_{vj}$. The simple aggregated model averaging estimator of μ_v is

$$\overline{\mu}_v=x_{v,S}^T\overline{\beta}.$$

The doubly simple aggregated model averaging estimator for μ_v is

$$\overline{\tilde{\mu}}_v=x_{v,S}^T\overline{\tilde{\beta}}.$$

Define the out-of-sample mean squared errors for $\overline{\mu}_v$ and $\overline{\tilde{\mu}}_v$ as $\mathbf{E}\left(\overline{\mu}_v-\mu_v\right)^2$ and $\mathbf{E}\left(\overline{\tilde{\mu}}_v-\mu_v\right)^2$, respectively, for which we give bounds in the following theorem.

Theorem 4 *Under Conditions 1-4 and 6, we obtain*

$$\frac{\mathbf{E}\left(\overline{\mu}_v-\mu_v\right)^2}{\xi_{\star,N}}=1+O\left(\frac{p_S\sigma_n^2}{n\xi_{\star,N}}\right)+O\left(\frac{S^3p_S(S+\sigma_n^2)}{Kn\overline{\lambda}_S^2\xi_{\star,N}}\right)+O\left(\frac{S^5p_S^2(S+\sigma_n^2)}{n^2\overline{\lambda}_S^4\xi_{\star,N}}\right), \tag{17}$$

and

$$\frac{\mathbf{E}\left(\overline{\tilde{\mu}}_v-\mu_v\right)^2}{\xi_{\star,N}}=1+O\left(\frac{m_s^2}{\xi_{\star,N}}\right)+O\left(\frac{S^3p_S(S+\sigma_n^2)}{Kn\overline{\lambda}_S^2\xi_{\star,N}}\right)+O\left(\frac{S^5p_S^2(S+\sigma_n^2)}{n^2\overline{\lambda}_S^4\xi_{\star,N}}\right). \tag{18}$$

Proof See Appendix B. ■

Remark 6 *Theorems 4 suggests the following points:*

1. *Noting that*
$$m_S^2 \leq \max_{1\leq s\leq S} \mathbf{E}\left\|\Sigma_S^{1/2}\left\{\hat{\beta}_{k,s}-\beta_{\star,s}\right\}\right\|^2 = O\left(\frac{p_S\sigma_n^2}{n}\right),$$
the doubly simple aggregation may be a better choice than simple aggregation since it has a smaller out-of-sample mean squared errors bound. Specifically, when the least squares is close to the unbiased estimator of the pseudo-true value (for example, when $x_{k,i,j}$ is Gaussian, all $m_s^2 = 0$), the advantages of doubly simple aggregation will be more prominent.

2. *When the total number of observations $N = nK$ is fixed, the optimal choice of the number of subjects K with the two methods satisfy*
$$K^* \asymp \begin{cases} \left(\frac{\sigma_n^2\overline{\lambda}_S^2}{(S+\sigma_n^2)S^3}+\sqrt{\frac{S^2p_S}{N\overline{\lambda}_S^2}}\right)^{-1}, & \text{for simple aggregation,} \\ \left(\frac{m_S^2n^2\overline{\lambda}_S^2}{(S+\sigma_n^2)S^3p_S}+\sqrt{\frac{S^2p_S}{N\overline{\lambda}_S^2}}\right)^{-1}, & \text{for doubly simple aggregation.} \end{cases}$$
Here we use symbol $a_n \asymp b_n$, which means both $a_n = O(b_n)$ and $b_n = O(a_n)$. The optimality of K implies that choosing any $K = O(K^*)$ cannot reduce the upper bound of out-of-sample mean squared errors (instead, n will increase and so more storage space and computational resources are needed at each subject), while choosing any K with K/K^* tending to ∞ will increase the upper bound of out-of-sample mean squared errors. If $x_{k,i,j}$ is Gaussian, it follows that the optimal choice of K with the doubly simple aggregation method satisfies $K^* \asymp \left(\frac{N\overline{\lambda}_S^2}{S^2p_S}\right)^{1/2}$. In such a setting, the boundedness of σ_n^2 can be obtained, so that the optimal K^* with the simple aggregation method satisfies
$$K^* \asymp \min\left\{\frac{S^4}{\overline{\lambda}_S^2}, \left(\frac{N\overline{\lambda}_S^2}{S^2p_S}\right)^{1/2}\right\},$$
as $n \to \infty$.

3. *In practical prediction, it is difficult to determine the value of $\overline{\lambda}_S$. To facilitate the selection of K, we can assume that $\overline{\lambda}_S \asymp 1$ holds, so the optimal $K^* \asymp \left(\frac{N}{S^2p_S}\right)^{1/2}$ and $K^* \asymp \min\left\{S^4, \left(\frac{N}{S^2p_S}\right)^{1/2}\right\}$ for the proposed methods, respectively. Assumption $\overline{\lambda}_S \asymp 1$ is not restricted, and it is consistent with Condition A6 of Chen et al. (2018). Essentially, $\overline{\lambda}_S \asymp 1$ represents the eigenvalues of a positive definite information matrix based on S pseudo-true models to be away from 0.*

(2^0) In-sample mean squared errors

The in-sample mean squared errors for simple aggregated model averaging estimator and doubly simple aggregated model averaging estimator are defined as

$$\overline{MSE} = \frac{1}{N}\sum_{k=1}^{K}\mathbf{E}\left\|\overline{\boldsymbol{\mu}}_k - \boldsymbol{\mu}_k\right\|^2,$$

and

$$\widetilde{MSE} = \frac{1}{N}\sum_{k=1}^{K}\mathbf{E}\left\|\tilde{\overline{\boldsymbol{\mu}}}_k - \boldsymbol{\mu}_k\right\|^2,$$

respectively.

Theorem 5 *Under Conditions 1-4 and 6, we obtain*

$$\frac{\overline{MSE}}{\xi_{\star,N}} = 1 + O\left(\xi_{\star,N}^{-1}\cdot\left\{\frac{S^3 p_S\left(S+\sigma_n^2\right)}{n\overline{\lambda}_S^2} + \frac{S^{\frac{\eta+4}{\eta+2}}}{K}\left(\frac{S^2 p_S\left(S+\sigma_n^2\right)}{n\overline{\lambda}_S^2}\right)^{\frac{\eta}{\eta+2}}\right\}\right), \tag{19}$$

and

$$\frac{\widetilde{MSE}}{\xi_{\star,N}} = 1 + O\left(\xi_{\star,N}^{-1}\cdot\left\{\overline{m}_S^2 + S^{\frac{\eta+4}{\eta+2}}\left(\frac{S^2 p_S\left(S+\sigma_n^2\right)}{Kn\overline{\lambda}_S^2} + \frac{S^4 p_S^2\left(S+\sigma_n^2\right)}{n^2\overline{\lambda}_S^4}\right)^{\frac{\eta}{\eta+2}}\right\}\right), \tag{20}$$

where

$$\overline{m}_S = \max_{1\le s\le S}\left\|\mathbf{E}\left[x_{k,1}\Pi_s^T\left\{\hat{\beta}_{k,s} - \beta_{\star,s}\right\}\right]\right\|. \tag{21}$$

Proof See Appendix B. ■

Remark 7 *Theorems 5 suggests the following points:*

1. *By Lemma 5, we see that $\overline{m}_s^2 = O(p_S\sigma_n^2/n)$, so by some simple calculations and Condition 4, for any K with*

$$\left(\frac{n\overline{\lambda}_S^2}{Sp_S(S+\sigma_n^2)}\right)^{\frac{2}{\eta}}/K = o(1), \tag{22}$$

   the doubly simple aggregation is a more appropriate choice when we focus on the in-sample mean squared errors. When η is large, (22) is easy to satisfy. If all $x_{k,i,j}$ are Gaussian, then η can be chosen as ∞.

2. *Similar to Remark 6, for the total number of observations $N = nK$, the optimal selection of K is given by*

$$K^* \asymp \begin{cases} \left(\frac{N\overline{\lambda}_S^2}{Sp_S(S+\sigma_n^2)}\right)^{\frac{2}{\eta+4}}, & \text{for simple aggregation,} \\ \left(\frac{N\overline{\lambda}_S^2}{S^2p_S}\right)^{1/2} + \frac{N\overline{\lambda}_S^2}{p_S\sqrt{S^{5+4/\eta}(S+\sigma_n^2)}}\overline{m}_S^{\frac{\eta+2}{\eta}}, & \text{for doubly simple aggregation.} \end{cases}$$

When $x_{k,i,j}$ is Gaussian, $\overline{m}_S = 0$, and η can be sufficiently large. So it follows that the optimal values of K^* for simple aggregation and doubly simple aggregation satisfy

$$K^* \asymp 1 \quad \textit{and} \quad K^* \asymp \left(\frac{N\overline{\lambda}_S^2}{S^2 p_S} \right)^{1/2},$$

respectively. This shows that for simple aggregation, as N tends to infinity, the optimal K is always less than some constant, and for doubly simple aggregation with in-sample estimation, K^* tends to infinity with the same rate as that of the out-of-sample estimation.

3.4 Asymptotic optimality

This subsection focuses on the optimality of the proposed methods in the asymptotic sense. In the distributed data framework, the definition of asymptotic optimality of model averaging estimator differs a little from the traditional definition. Since the least squares $\hat{\beta}_{k,s}$ at each subject uses only n observations, the loss

$$R_N(w^*) = \frac{1}{n}\mathbf{E} \left\| \sum_{s=1}^{S} w_s^* X_{1,s} \hat{\beta}_{1,s} - \boldsymbol{\mu}_1 \right\|^2$$

cannot represent the L_2 loss of the optimal model averaging estimator using the full N observations. To address this issue, we note that as long as $n \to \infty$, the least squares estimator based on either n observations or N observations converges to the pseudo-true value, so $R_N^*(w^*)$ defined in Subsection 3.1 can be used to represent the minimum risk of model averaging estimator in the distributed data case. Thus, we define that a model averaging estimator has asymptotic optimality if its mean squared error (MSE) satisfies $MSE/R_N^*(w^*) \to 1$.

Theorem 6 below reveals that under the framework of distributed data, our proposed model averaging methods are asymptotically optimal for both out-of-sample and in-sample estimations.

Theorem 6 *Under Conditions 1-6, and $\frac{S^3 p_S(S+\sigma_n^2)}{\xi_{\star,N}\overline{\lambda}_S^2 n} = o(1)$, we have*

(i) for out-of-sample mean squared errors,

$$\frac{\mathbf{E}\left(\overline{\mu}_v - \mu_v\right)^2}{R_N^*(w^*)} = 1 + o(1) \quad \textit{and} \quad \frac{\mathbf{E}\left(\widetilde{\overline{\mu}}_v - \mu_v\right)^2}{R_N^*(w^*)} = 1 + o(1);$$

(ii) for in-sample mean squared errors,

$$\frac{\overline{MSE}}{R_N^*(w^*)} = 1 + o(1) \quad \textit{and} \quad \frac{\widetilde{\overline{MSE}}}{R_N^*(w^*)} = 1 + o(1). \tag{23}$$

Specifically, when $N \to \infty$ and $K = 1$, (23) degenerates to a typical form

$$\frac{R_N(\hat{w})}{\inf_{w \in Q} R_N(w)} = 1 + o(1). \tag{24}$$

Proof This theorem is a direct corollary of Theorems 2, 4 and 5. ■

Remark 8 *Theorem 6 shows that our simple aggregated model averaging estimators and doubly simple aggregated model averaging estimators achieve the optimality in out-of-sample and in-sample mean squared errors.*

Unlike the existing literature on the asymptotic optimality based on the loss function, (24) *indicates that the Mallows' model averaging method has an asymptotic optimality in the sense of minimizing the risk. This is an interesting finding, which also shows that* $R_N(\hat{w}) \leq C \inf_{w \in Q} R_N(w)$*, where* C *is a constant that depends only on* $\theta_j, j = 1, 2, \ldots, \infty$*, and so implies the adaptation of Mallows' model averaging in the sense that the risk of Mallows' model averaging estimator has a minimax convergence rate.*

3.5 Minimaxity of model averaging estimators

We now turn to deriving the minimax optimal convergence rate of proposed estimators. For simple and doubly simple aggregation model averaging estimators, denote

$$\overline{Mse}\,(W_1, W_2, \ldots, W_K) = \frac{1}{N} \sum_{k=1}^{K} \mathbf{E} \left\| X_k \frac{1}{K} \sum_{k=1}^{K} \hat{\beta}_k(W_k) - \boldsymbol{\mu}_k \right\|^2,$$

$$\widetilde{Mse}(W_1, W_2, \ldots, W_K) = \frac{1}{N} \sum_{k=1}^{K} \mathbf{E} \left\| X_k \sum_{s=1}^{S} \left(\frac{1}{K} \sum_{k=1}^{K} w_{k,s} \right) \Pi_s^T \left(\frac{1}{K} \sum_{k=1}^{K} \hat{\beta}_{k,s} \right) - \boldsymbol{\mu}_k \right\|^2,$$

and then $\overline{MSE} = \overline{Mse}(\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_K)$ and $\widetilde{MSE} = \widetilde{Mse}(\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_K)$.

Notice that the analysis technique for model averaging in this paper to bound the prediction risk requires assuming some conditions about the relationship between the true parameter $\theta = (\theta_1, \theta_2, \ldots)^T$ and the candidate models (e.g., Condition 1), and if the candidate models vary as n increases, then the true parameter θ is also subject to an n-dependent constraint, so we cannot find a compact parameter set independent of n such that the minimax property of the proposed model averaging estimators over this set holds. For this reason, throughout the rest of the discussion in this section, we assume that the candidate models are fixed, i.e., that $p_1, \ldots, p_S$ and S are fixed integers.

Now, for any $q \geq 4$, we consider the true parameter $\theta = (\theta_1, \theta_2, \ldots)^T$ on the Banach space

$$\ell_q = \left\{ \theta : \sum_{j=1}^{\infty} |\theta_j|^q < \infty \right\} \tag{25}$$

with the norm

$$\|\theta\| = \left(\sum_{j=1}^{\infty} |\theta_j|^q \right)^{\frac{1}{q}}.$$

We construct

$$\Theta = \Theta(\epsilon_1, \epsilon_2, \varepsilon_3) = S_1 \cap S_2 \cap S_3 \tag{26}$$

with

$$S_1 = \left\{ \theta \in \ell_q : w_s^{**} \in [0, 1-\varepsilon_1], s = 1, \ldots, S-1, (w_1^{**}, \ldots, w_{S-1}^{**})^T = \underset{w \in Q_0}{\operatorname{argmin}} R_N^\star(w) \right\} \quad (27)$$

and

$$S_2 = \left\{ \theta \in \ell_q : \lambda_{\min} \left[\nabla^2 R_0^* \left(w_0^* \right) \right] \geq \epsilon_2, \inf_{w \in Q} R_N^*(w) \geq \epsilon_2 \right\}, \quad S_3 = \{ \theta \in \ell_q : \|\theta\| \leq \epsilon_3 \}, \quad (28)$$

where $\epsilon_1 \in (0,1)$, $\epsilon_2, \epsilon_3 > 0$ are constants.

Theorem 7 *Assume Conditions 3 and 6 hold, and* $\sigma_n^2 = o(n)$. *If* $\sup_{j \geq 1} \mathbf{E}|x_{k,i,j}|^q < \infty$, *and* $\mathbf{E}e_{k,i}^4 \leq \omega < \infty$ *for* $k = 1, 2, \ldots, K$ *and* $i = 1, 2, \ldots, n$, *then for any* $\epsilon_1 \in (0,1)$, $\epsilon_2, \epsilon_3 > 0$,

$$\sup_{\theta \in \Theta} \overline{MSE} = \left\{ 1 + O\left(\frac{\sigma_n^2}{n} + \frac{1}{K} \left(\frac{\sigma_n^2}{n} \right)^{\frac{q-2}{q}} \right) \right\} \inf_{w \in Q} \sup_{\theta \in \Theta} R_N^\star(w) \quad (29)$$

and

$$\sup_{\theta \in \Theta} \widetilde{MSE} = \left\{ 1 + O\left(\frac{\sigma_n^2}{n} + \left(\frac{\sigma_n^2}{N} + \frac{\sigma_n^2}{n^2} \right)^{\frac{q-2}{q}} \right) \right\} \inf_{w \in Q} \sup_{\theta \in \Theta} R_N^\star(w), \quad (30)$$

where Θ *is defined by* (26). *Moreover,*

$$\inf_{W_1 \in Q, W_2 \in Q, \ldots, W_K \in Q} \sup_{\theta \in \Theta} \overline{Mse}\left(W_1, W_2, \ldots, W_K\right) = \left\{ 1 + O\left(\frac{\sigma_n^2}{n} \right) \right\} \inf_{w \in Q} \sup_{\theta \in \Theta} R_N^\star(w) \quad (31)$$

and

$$\inf_{W_1 \in Q, W_2 \in Q, \ldots, W_K \in Q} \sup_{\theta \in \Theta} \widetilde{Mse}\left(W_1, W_2, \ldots, W_K\right) = \left\{ 1 + O\left(\frac{\sigma_n^2}{n} \right) \right\} \inf_{w \in Q} \sup_{\theta \in \Theta} R_N^\star(w). \quad (32)$$

Proof See Appendix B. ■

Remark 9 *Theorem 7 implies that the simple and doubly simple aggregation model averaging estimators proposed in this paper are both minimax for the parameter set* Θ.

4. Simulation

In this section, we conduct simulation experiments to compare the finite sample performance of our distributed model averaging methods and some commonly used model selection and model averaging methods. In detail, we compare three simple aggregated model selection estimators: (i) AIC model selection (AIC), (ii) BIC model selection (BIC), (iii) Mallows' model selection (Mallows C_p); three simple aggregated model averaging estimators: (iv) simple aggregated smoothed AIC estimator (SAIC), (v) simple aggregated smoothed BIC estimator (SBIC), (vi) simple aggregated Mallows' model averaging estimator (MMA); and three doubly simple aggregated model averaging estimators: (vii) doubly simple aggregated smoothed AIC estimator (dSAIC), (viii) doubly simple aggregated smoothed BIC estimator (dSBIC), (ix) doubly simple aggregated Mallows' model averaging estimator (dMMA). Thus, we compare totally nine estimators.

4.1 Simulation setup

We report the simulation studies of the infinite order regression first. The data generating process is exactly the same as that in Hansen (2007):

$$y_i = \sum_{j=1}^{\infty} \theta_j x_{j,i} + e_i,$$

where $x_{1,i} = 1$, and $x_{j,i}(j = 1, 2, \ldots)$ and error e_i are independent and identically distributed as $N(0, 1)$. We set $\theta_j = c\sqrt{2\alpha}j^{-\alpha-1/2}$, and consider the parameter α varied at $0.5, 1.0$ and 1.5. As in Hansen (2007), the parameter c is selected such that $R^2 = c^2/(1 + c^2)$ changes from 0.1 to 0.9.

4.2 Results on in-sample risk

In this subsection, we compare the in-sample risks of the above nine distributed estimators. For the distributed data, we set the sample size for each subject to be varied at $n = 50, 150, 400, 1000, 5000$ and 10000. The number of subjects is set as $K = 1, 2, 3, 5$ and 10. Let p_S equal to $\left[4n^{1/2}\right]_+$ ($[\cdot]_+$ means round to get an integer, and so $p_S = 28, 49, 80, 126, 283$ and 400 for the above six sample sizes), and the number of candidate models S be $\lceil n^{1/3} \rceil + 1$($\lceil \cdot \rceil$ means round up to get an integer, and so $S = 5, 7, 9, 11, 19$ and 22 for the six sample sizes). All the candidate models are nested and the dimension for the sth candidate model is $1 + d \times (s - 1)$, where $d = \lceil (p_S - 1)/(S - 1) \rceil$ and $s = 1, 2, \ldots, S - 1$, while the dimension for the Sth candidate model is p_S.

To evaluate different estimators, similar to Hansen (2007), we normalize the risk based on average across 5000 simulation draws by dividing by the risk of the best-fitting simple aggregated estimator $\tilde{\beta}_s$ (i.e., (8)). For the simulated in-sample risk, we define it as

$$\frac{1}{D} \sum_{r=1}^{D} \sum_{k=1}^{K} \left(\hat{\boldsymbol{\mu}}_{k,(j)}^{(r)} - \boldsymbol{\mu}_k^{(r)} \right)^2,$$

where r means the rth simulation replication, $D = 5000$, and j means the jth method considered in our simulation.

For $j =$i, ii, and iii, $\hat{\boldsymbol{\mu}}_{k,(j)}^{(r)}$ is determined by AIC, BIC, and Mallows' model selection methods, respectively, i.e.,

$$\hat{\boldsymbol{\mu}}_{k,(j)}^{(r)} = X_k \left\{ \frac{1}{K} \sum_{k=1}^{K} \hat{\beta}_k(W_k) \right\},$$

where $W_k = (w_{k,1}, w_{k,2}, \ldots, w_{k,S}) \in \{0, 1\}^S$ with $\sum_{s=1}^{S} w_{k,s} = 1$. The AIC for the sth model at the kth subject is given by

$$\mathrm{AIC}_k^{(s)} = n_k \log\left(\hat{\sigma}_{k,(s)}^2 \right) + 2p_s$$

with

$$\hat{\sigma}_{k,(s)}^2 = \left\| Y_k - X_{k,s} \hat{\beta}_{k,s} \right\|^2 / n_k,$$

and the model selected by AIC is

$$\hat{W}_k^{AIC} = \arg \min_{\substack{W_k \in \{0,1\}^S \\ \sum_{s=1}^S w_{k,s}=1}} \sum_{s=1}^S w_{k,s} \mathrm{AIC}_k^{(s)}.$$

Similarly, the BIC for the sth model at the kth subject is

$$\mathrm{BIC}_k^{(s)} = n_k \log\left(\hat{\sigma}_{k,(s)}^2\right) + \log(n_k) p_s$$

and the model selected by BIC is

$$\hat{W}_k^{BIC} = \arg \min_{\substack{W_k \in \{0,1\}^S \\ \sum_{s=1}^S w_{k,s}=1}} \sum_{s=1}^S w_{k,s} \mathrm{BIC}_k^{(s)}.$$

Furthermore, the Mallows' C_p of the sth model at the kth subject is

$$\textsc{Mallows}_k^{(s)} = \left\| Y_k - X_{k,s}\hat{\beta}_{k,s} \right\|^2 + 2\tilde{\sigma}^2 p_s,$$

where

$$\tilde{\sigma}^2 = (n - p_S)^{-1} \left\| Y_k - X_{k,S}\hat{\beta}_{k,S} \right\|^2.$$

The model selected by Mallows' C_p is

$$\hat{W}_k^{Mallows} = \arg \min_{\substack{W_k \in \{0,1\}^S \\ \sum_{s=1}^S w_{k,s}=1}} \sum_{s=1}^S w_{k,s} \textsc{Mallows}_k^{(s)}.$$

For $j =$iv, v, and vi, $\hat{\boldsymbol{\mu}}_{k,(j)}^{(r)}$ is determined by three simple aggregated model averaging estimators SAIC, SBIC, and MMA, respectively, i.e.,

$$\hat{\boldsymbol{\mu}}_{k,(j)}^{(r)} = X_k \left\{ \frac{1}{K} \sum_{k=1}^K \hat{\beta}_k(W_k) \right\},$$

where $W_k \in Q$ is calculated by

$$\left(\frac{\exp(-\mathrm{AIC}_k^{(1)}/2)}{\sum_{s=1}^S \exp(-\mathrm{AIC}_k^{(s)}/2)}, \ldots, \frac{\exp(-\mathrm{AIC}_k^{(S)}/2)}{\sum_{s=1}^S \exp(-\mathrm{AIC}_k^{(s)}/2)} \right),$$

$$\left(\frac{\exp(-\mathrm{BIC}_k^{(1)}/2)}{\sum_{s=1}^S \exp(-\mathrm{BIC}_k^{(s)}/2)}, \ldots, \frac{\exp(-\mathrm{BIC}_k^{(S)}/2)}{\sum_{s=1}^S \exp(-\mathrm{BIC}_k^{(s)}/2)} \right),$$

and (5) with σ^2 being replaced by $\tilde{\sigma}^2$, respectively.

As for j =vii, viii, and ix, $\hat{\boldsymbol{\mu}}_{k,(j)}^{(r)}$ is generated by three doubly simple aggregated model averaging estimators dSAIC, dSBIC, and dMMA, respectively, i.e.,

$$\hat{\boldsymbol{\mu}}_{k,(j)}^{(r)} = X_k \left\{ \sum_{s=1}^{S} w_s \Pi_s^T \tilde{\beta}_s \right\},$$

where w_s is calculated by

$$\frac{1}{K} \sum_{k=1}^{K} \frac{\exp(-\mathrm{AIC}_k^{(s)}/2)}{\sum_{s=1}^{S} \exp(-\mathrm{AIC}_k^{(s)}/2)},$$

$$\frac{1}{K} \sum_{k=1}^{K} \frac{\exp(-\mathrm{BIC}_k^{(s)}/2)}{\sum_{s=1}^{S} \exp(-\mathrm{BIC}_k^{(s)}/2)},$$

and (9), respectively.

The simulation results for $K = 1, 2$ and 3 are similar. When $K = 1$, the three doubly simple aggregated model averaging estimators are equal to the three simple aggregated model averaging estimators respectively, and all the model selection and averaging estimators perform closely to those in Hansen (2007) where all data are at the same subject. The performance of $K = 10$ is similar to that of $K = 5$. So to save space, we present only the results on $K = 2$ and 5, which are summarized in Figures 1-6. The risks of estimators under other settings are available from the authors upon request.

We reveal some interesting commonalities from Figures 1-6 as follows:

1) Model averaging methods are frequently better than their model selection counterparts, e.g., dMMA and MMA get smaller risks than Mallows, and dSAIC (dSBIC) and SAIC (SBIC) behave better than AIC (BIC), especially when $n = 50$ and 150. In model selection methods, Mallows performs the best in most of cases. The difference between AIC and Mallows gets small when n increases for all of figures. It is also observed that the difference of all methods becomes small as α varies from 0.5 to 1.5.

2) For model averaging methods, doubly simple aggregated model averaging estimators achieve lower risks than their corresponding simple aggregated model averaging estimators in most of cases. MMA often performs the best among the simple aggregated model averaging estimators. The difference between SAIC and MMA decreases with n tending to 10000.

3) It can be seen that our dMMA gets the smallest risks in most of cases, especially in the case of $n = 50$, followed by MMA and dSAIC. An exception is when $\alpha = 0.5$, $K = 5$ and $n = 50$. In this situation, dSAIC is better than MMA and dMMA. MMA and dMMA always perform the best when the sample sizes are 5000 and 10000. With n tending to 10000, the difference between dMMA and MMA gets smaller and smaller, which is consistent with Theorem 5 that shows dMMA and MMA may not have big difference in the sense of in-sample risk. In addition, the behavior of dMMA and dSAIC becomes similar as n increases to 10000.

4) Observing the effect of n, we can see that when n is small ($n = 50$ and 150), MMA and dMMA perform the best in most of cases. When n becomes large ($n = 400, 1000$

and 5000), Mallows type methods and AIC type methods behave similarly. When $n = 10000$, the risks of all approaches are close.

5) BIC type methods (e.g., BIC, SBIC, and dSBIC) always fluctuate a lot as R^2 goes from 0.1 to 0.9, and often behave well when $n = 50$ and $R^2 = 0.1$. The risks of AIC type methods (e.g., AIC, SAIC, and dSAIC) and Mallows type methods (e.g., Mallows, MMA, and dMMA) regularly decrease slowly as R^2 increases, and Mallows type methods frequently have the smallest risks. These indicate that the Mallows type methods are the most favored methods in most of cases.

6) As for the number of subjects K, comparing figures with $K = 2$ and $K = 5$, for small n, like $n = 50$, the improvements of dMMA over MMA when $K = 2$ are larger than those when $K = 5$; for big n, like $n > 50$, the improvements of dMMA over MMA become smaller and smaller as K increases from 2 to 5. This means that in the cases of smaller n and smaller K, dMMA has greater advantages. For bigger n and bigger K, MMA is more applicable since MMA requires less computation and has similar performance to dMMA. This phenomenon is consistent with the results by Theorem 5 in the case of $\overline{m}_S = 0$ and $\eta = \infty$.

In summary, for in-sample estimation, our simulation results show that doubly simple aggregated model averaging methods are better than their simple aggregated counterparts and model selection methods. Further, dMMA performs the best in most of cases, followed by MMA or dSAIC.

4.3 Results on out-of-sample risk

For the simulated out-of-sample risk, we define it as

$$\frac{1}{D}\sum_{r=1}^{D}\sum_{i=1}^{n}\left(\hat{\mu}_{i,o,(j)}^{(r)} - \mu_{i,o}^{(r)}\right)^2,$$

where the definitions of D, r, and j are the same as before. We compare the normalized out-of-sample risks of the above nine distributed estimators. The sample size for every subject is set as $n = 50, 150, 400$ and 1000. The number of subjects is set as $K = 1, 2, 3, 5$ and 10. We let p_S equal to $\left[9n^{1/3}\right]_+$, and S be $\lceil (p-1)/5 \rceil + 1$. All the candidate models are nested and the dimension for the sth candidate model is $5 \times (s-1) + 1$, $s = 1, 2, \ldots, M-1$, while the dimension for the Sth candidate model is p_S.

To save space, we still present only the results on $K = 2$ and 5, which are summarized in Figures 7-12. Other results are available from the authors. Some common phenomena, which are a frequent occurrence in Figures 7-12, are listed below:

1) It is clear that model averaging methods are better than their model selection counterparts in the sense of minimizing the out-of-sample risks. For example, SAIC is better than AIC, SBIC is better than BIC, and MMA is better than Mallows' C_p, especially for the cases where $n = 50$ and 150.

2) Comparing all model averaging methods, doubly simple aggregated model averaging estimators outdo their corresponding simple aggregated model averaging estimators

in the most of scenarios, particularly when $n = 50$ and 150. For example, dSAIC is superior to SAIC, and dMMA is superior to MMA. BIC type methods are still not robust for different R^2. With $K = 5$, AIC and Mallows C_p type methods behave closely to each other when n increases from 150 to 400 and then 1000.

3) It is observed that dMMA often behaves the best in getting the smallest risks, followed by MMA and dSAIC. In particular, dMMA surpasses MMA more clearly when $K = 5$ than when $K = 2$ for $n = 50$. This phenomenon accords closely with Theorem 4. In addition, the difference between dMMA and MMA becomes small when n varies from 50 to $150, 400$, and 1000. This is expected because from Theorem 4, the difference between dMMA and MMA becomes smaller and smaller with n increasing.

4) Varying R from 0.1 to 0.9 causes significant variations for BIC type methods in a large number of simulation settings, except for the case of $n = 50$ and $K = 2$, where BIC type methods often behave well. BIC type methods are quite poor relative to the other methods when n increases from 150 to 400 and 1000, as shown in Figures 7-12. These indicate that the BIC type methods are not robust. AIC type methods and Mallows type methods gradually reduce the out-of sample risks as R tends to 0.9. Mallows type methods are the most stable methods in our simulations. Thus, dMMA and MMA are also efficient and stable in achieving minimum out-of-sample risks. On the other hand, dSAIC is frequently superior to dMMA in getting minimum risks when $K = 5$ and $n = 150$.

5) Comparing risks with the same n when $K = 2$ and $K = 5$, the difference between dMMA and dSAIC when $K = 2$ is smaller than that when $K = 5$, particularly for the case where $n = 50$. As for the effect of the number of subjects K, small K is perferred for dMMA and MMA with $n = 150, 400$ and 1000, but when $n = 50$, dMMA and MMA with big K have significant advantages over other methods.

In summary, for out-of-sample estimation, our methods dMMA and MMA perform the best in most of cases. Furthermore, Mallows and AIC type methods often perform equally well.

5. Real Data Analysis

In this section, we use our proposed distributed model averaging methods to analyze the airline on-time performance data from the 2009 ASA Data Expo (http://stat-computing.org/dataexpo/2009/the-data.html). The data set is publicly available and has been used for demonstration with big data in many papers. For instance, it was used as a case study to demonstrate a logistic model fitting with a massive dataset that exceeds the RAM of a single computer by Wang et al. (2016). This data set is collected from October 1987 to April 2008 for all commercial flights within the USA. It consists of 12 million flights with 29 variables. The big memory project (http://www.jstatsoft.org/index.php/jss/article /downloadSuppFile/v055i14/Airline.tar.bz2) presents a compressed version of the pre-processed data set, which is approximately 1.7 GB, and will take 12 GB when uncompressed.

The response variable of the regression is late time (in hours). We consider linear models, and the covariates include three continuous variables: departure delay time (DepDelay,

in hours), scheduled elapsed time (CRSElapsedTime, in hours), and distance from origin to destination (Distance, in 1000 miles); and five dummy variables: Weekend, departure hour (Dephour), origin (Origin), and destination (Dest). Since we consider a series of linear candidate models, we first rank the continuous variables by absolute marginal correlation coefficients to the response variable. The top three variables are DepDelay, CRSElapsed-Time, and Distance. We then consider two sets of models: (i) three models that range from the model with intercept and DepDelay to the model that includes all top three continuous variables, and (ii) three nested models that incorporate dummy variables such as Weekend, Dephour, and Oringe and Dest. Weekend and Dephour capture the impact of official and business activities, and Dephour also captures the effects of weather on flight delays, while Oringe and Dest capture the impact of different routes. Additionally, the regressor sets for the six nested candidate models are presented in Table 1.

Table 1: Regressor sets for the six models used in Airline Data.

Model	Regressor Set
1	Intercept + DepDelay
2	Intercept + DepDelay + CRSElapsedTime
3	Intercept + DepDelay + CRSElapsedTime + Distance
4	Intercept + DepDelay + CRSElapsedTime + Distance+ Weekend
5	Intercept + DepDelay + CRSElapsedTime + Distance+ Weekend + DepHour
6	Intercept + DepDelay + CRSElapsedTime + Distance+ Weekend + DepHour + Oringe + Dest

Due to computer memory limitation, we sort the data by the date of boarding time on schedule and divide the whole data with the sample size of 123,534,969 into 124 subjects, where the first 123 subjects each contain $N = 1,000,000$ records covering a week's flight data, and the last one contains 534,969 records. We use the ith subject data as training data to predict the late time at the $(i+1)$th subject data, $i = 1, 2, \ldots, 123$. For the ith subject, we apply simple random sampling scheme without replacement to the data and get K random samples, then we use our proposed distributed model averaging methods for data analysis. K is set to be $1, 2, 5, 10, 100, 200, 500$ and 1000. We compare the mean squared prediction errors (MSPEs) of the nine methods given in Section 4 for the $(i+1)$ subject data. We conduct 123 rounds. Since when $K = 1$, SAIC, SBIC, and MMA are the same as dSAIC, dSBIC and dMMA, respectively, we omit the results for the doubly simple aggregated model averaging methods in the case. To save space, we present only the results on the mean, median and optimal rate of 123 rounds MSPEs for each method in Table 2, and Diebold and Mariano test (Diebold and Mariano, 2002) results for the differences of MSPEs in Tables 3 and 4. The results on the other estimators such as those of weights and coefficients of candidate models are available from the authors upon request.

From Table 2, we observe that MMA and dMMA always achieve the lowest MSPEs, followed by AIC or SAIC. dMMA has a significant advantage when $K = 200, 500$ and 1000. Basically, the MSPEs of all methods decrease as K increases from 1 to 100 and increase as K increases from 100 to 1000. When $K = 100$, MMA obtains the smallest MSPEs, followed

Table 2: MSPEs of different methods for Airline Data.

K		AIC	BIC	Mallows	SAIC	SBIC	MMA	dSAIC	dSBIC	dMMA
1	Mean ($\times 10^{-2}$)	6.181	6.181	6.181	6.181	6.181	**6.174**			
	Median ($\times 10^{-2}$)	5.833	5.833	5.833	5.833	5.833	**5.811**			
	Optimal rate	0.051	0.047	0.051	0.088	0.088	**0.674**			
2	Mean ($\times 10^{-2}$)	6.179	6.179	6.179	6.179	6.179	**6.173**	6.179	6.179	**6.173**
	Median ($\times 10^{-2}$)	5.833	5.833	5.833	5.833	5.833	**5.811**	5.833	5.833	**5.811**
	Optimal rate	0.054	0.050	0.054	0.053	0.045	**0.285**	0.046	0.046	**0.366**
5	Mean ($\times 10^{-2}$)	6.177	6.177	6.177	6.177	6.177	**6.172**	6.177	6.177	**6.172**
	Median ($\times 10^{-2}$)	5.835	5.835	5.835	5.835	5.835	**5.803**	5.835	5.835	**5.803**
	Optimal rate	0.052	0.056	0.052	0.062	0.067	**0.252**	0.040	0.038	**0.382**
10	Mean ($\times 10^{-2}$)	6.174	6.174	6.174	6.174	6.174	**6.170**	6.175	6.175	**6.170**
	Median ($\times 10^{-2}$)	5.839	5.839	5.839	5.839	5.839	**5.807**	5.839	5.839	**5.806**
	Optimal rate	0.061	0.061	0.061	0.047	0.069	**0.260**	0.026	0.050	**0.366**
100	Mean ($\times 10^{-2}$)	6.162	6.164	6.162	6.162	6.164	**6.159**	6.162	6.163	6.160
	Median ($\times 10^{-2}$)	5.843	5.843	5.843	5.843	5.843	**5.840**	5.843	5.843	**5.839**
	Optimal rate	0.088	0.027	0.080	0.033	0.057	**0.244**	0.073	0.073	**0.325**
200	Mean ($\times 10^{-2}$)	6.172	6.176	6.172	6.172	6.175	**6.169**	**6.169**	6.174	**6.166**
	Median ($\times 10^{-2}$)	5.843	5.843	5.843	5.843	5.842	**5.838**	5.843	5.841	**5.838**
	Optimal rate	0.130	0.033	0.057	0.024	0.024	**0.203**	0.089	0.081	**0.358**
500	Mean ($\times 10^{-2}$)	6.214	6.221	6.214	6.215	6.221	**6.209**	6.211	6.223	**6.206**
	Median ($\times 10^{-2}$)	**5.760**	5.774	**5.760**	5.761	5.774	5.762	5.763	5.779	5.764
	Optimal rate	**0.228**	0.008	0.024	0.016	0.008	0.138	0.073	0.081	**0.423**
1000	Mean ($\times 10^{-2}$)	6.263	6.273	6.263	6.263	6.272	**6.255**	6.262	6.275	**6.253**
	Median ($\times 10^{-2}$)	5.785	5.799	5.779	5.783	5.798	**5.777**	5.780	5.800	**5.778**
	Optimal rate	0.260	0.008	0.024	0.024	0.016	**0.195**	0.016	0.073	**0.382**

by dMMA. In optimal rate, dMMA is superior to the rest methods in obtaining the highest optimal rates

Table 3: Diebold–Mariano test results for the differences between MMA and other methods.

K		$\frac{\text{AIC}}{\text{MMA}}$	$\frac{\text{BIC}}{\text{MMA}}$	$\frac{\text{Mallows}}{\text{MMA}}$	$\frac{\text{SAIC}}{\text{MMA}}$	$\frac{\text{SBIC}}{\text{MMA}}$	$\frac{\text{dSAIC}}{\text{MMA}}$	$\frac{\text{dSBIC}}{\text{MMA}}$	$\frac{\text{dMMA}}{\text{MMA}}$
1	DM	5.073	5.078	5.073	5.073	5.078	5.073	5.078	
	P-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
2	DM	4.306	4.321	4.306	4.306	4.318	4.461	4.473	1.263
	P-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.103
5	DM	3.939	3.882	3.939	3.938	3.888	4.179	4.122	1.974
	P-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.024
10	DM	3.783	3.835	3.783	3.773	3.798	4.165	4.192	2.343
	P-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.010
100	DM	3.948	6.028	3.959	4.101	5.932	1.864	3.348	0.543
	P-value	0.000	0.000	0.000	0.000	0.000	0.031	0.000	0.294
200	DM	6.438	11.997	6.667	7.054	11.695	8.530	11.685	-2.187
	P-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.986
500	DM	5.024	8.579	5.058	5.252	8.543	0.095	5.735	-1.762
	P-value	0.000	0.000	0.000	0.000	0.000	0.462	0.000	0.961
1000	DM	6.023	10.681	6.132	6.401	10.552	2.828	10.630	-3.277
	P-value	0.000	0.000	0.000	0.000	0.000	0.002	0.000	0.999

From Diebold and Mariano test results in Tables 3 and 4, MMA and dMMA are statistically significantly superior to other methods, and the difference between MMA and dMMA is not significant.

In conclusion, MMA and dMMA are effective methods to reduce risks in prediction for big data analysis.

6. Concluding Remarks

In this paper, we proposed two aggregated model averaging estimators for distributed data and proved that the weights based on Mallows model averaging criterion are L_2 convergent to the theoretically optimal weights. The bounds of mean squared errors and the asymptotic optimality for the proposed model averaging estimators are also established. These are the first theoretical results of applying model averaging method to big data analysis with divide and conquer trick. Simulations and real data analysis show that simple aggregation and

Table 4: Diebold–Mariano test results for the differences between dMMA and other methods.

K		AIC / dMMA	BIC / dMMA	Mallows / dMMA	SAIC / dMMA	SBIC / dMMA	dSAIC / dMMA	dSBIC / dMMA	MMA / dMMA
2	DM	4.134	4.149	4.134	4.134	4.146	4.294	4.306	-1.263
	P-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.897
5	DM	3.562	3.508	3.562	3.561	3.512	3.813	3.759	-1.974
	P-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.976
10	DM	2.992	3.085	2.992	3.001	3.052	3.466	3.517	-2.343
	P-value	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.990
100	DM	1.069	2.243	1.073	1.119	2.215	2.387	3.157	-0.543
	P-value	0.143	0.012	0.142	0.132	0.013	0.008	0.001	0.706
200	DM	4.835	10.356	4.958	5.152	9.975	5.979	10.969	2.187
	P-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.014
500	DM	3.216	4.933	3.229	3.278	4.926	3.829	5.699	1.762
	P-value	0.001	0.000	0.001	0.001	0.000	0.000	0.000	0.039
1000	DM	5.011	8.798	5.066	5.171	8.634	5.622	9.441	3.277
	P-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001

doubly simple aggregation methods for model averaging estimators are better than their model selection counterparts in situations where there are massive distributed or parallel data, and especially when K is large, dMMA has more advantages in getting the smallest mean squared errors. In practice, how to balance K and n is an unavoidable problem for big data computing. In our opinion, dMMA is more preferred in the cases of smaller n and bigger K where dMMA has more reduction in variances of both the coefficient estimators in candidate models and weight estimators in model averaging, and MMA is more applicable to the cases of bigger n and smaller K where MMA requires less computation and has similar performance to dMMA.

Our results hold in both the fixed and divergent dimensional models. For high-dimensional linear models, we can group the regressors in order and then build the nested group candidate models or single group candidate models to reduce the effect of dimension. For example, Ando and Li (2014) proposed cross-validation model averaging framework which groups variables by correlation first to reduce the dimension, and then combines the candidate models by model averaging.

Our theoretical results need a homoscedastic assumption on the error term. If the data at the subject are heteroscedastic or dependent, how to choose weights to aggregate each subject estimator is an important problem. In this regard, the Jackknife model averaging method in Hansen and Racine (2012), heteroscedasticity-robust C_p model averaging method in Liu and Okui (2013), and leave-subject-out cross-validation model averaging method in Gao et al. (2016) are useful and warrant our further research. Some other interesting researches can also be done in the next step. One is to extend the proposed aggregated model averaging methods to the case of big data streams (Xi et al., 2009; Wang et al., 2018). Investigating model averaging based on generalized linear model and other complex models for distributed data is another important topic.

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Appendices

To prove Theorems 1-5 in current paper, we first give some lemmas and their proofs in Appendix A, then provide the proofs of the theorems in Appendix B.

Appendix A. Lemmas and Proofs

For $w_0 = (w_1, \ldots, w_{S-1})^T \in \mathbb{R}^{S-1}$, denote

$$C_k(w_0) = C_{k,n}\left(\text{col}\left\{w_0, 1 - \sum_{s=1}^{S-1} w_s\right\}\right).$$

Choosing any small radius $\delta_\rho \leq \rho$, $\rho \in (0,1)$, we define the events

$$\mathcal{E}_1 \triangleq \left\{ \left\| \nabla^2 C_k(w_0^*) - \nabla^2 R_0(w_0^*) \right\|_2 \leq \rho\lambda_n \right\},$$

and

$$\mathcal{E}_2 \triangleq \left\{ \left\| \nabla C_k\left(w_0^*\right) \right\| \leq \frac{(1-\rho)\,\lambda_n \delta_\rho}{2} \right\}.$$

Lemma 1 *Under the events $\mathcal{E}_1$ and $\mathcal{E}_2$, for $k \in \{1, 2, \ldots, K\}$, we have*

$$\left\| \hat{W}_k - w_0^* \right\| \leq \frac{2\left\| \nabla C_k\left(w_0^*\right) \right\|}{(1-\rho)\,\lambda_n}, \tag{33}$$

and

$$\lambda_{\min}\left[\nabla^2 C_k(w_0) \right] \geqslant (1-\rho)\,\lambda_n, \tag{34}$$

where

$$w_0 \in U_{\delta_\rho} \triangleq \left\{ w_0 \in \mathbb{R}^{S-1} \middle| \, \|w_0 - w_0^*\| \leq \delta_\rho \right\} \subseteq Q.$$

Proof We first prove (34), which means that the function $C_k\left(w_0\right)$ is $(1-\rho)\,\lambda_n$-strongly convex over the feasible set U_{δ_ρ} under the conditions given in the lemma. In fact, for fixed $\tau \in U_{\delta_\rho}$, we have

$$\begin{aligned} \left\| \nabla^2 C_k\left(\tau\right) - \nabla^2 R_0\left(w_0^*\right) \right\|_2 &\leq \left\| \nabla^2 C_k(\tau) - \nabla^2 C_k(w_0^*) \right\|_2 + \left\| \nabla^2 C_k(w_0^*) - \nabla^2 R_0(w^*) \right\|_2 \\ &= 0 + \left\| \nabla^2 C_k(w_0^*) - \nabla^2 R_0(w_0^*) \right\|_2 \leq \rho\lambda_n. \end{aligned}$$

According to the properties of the spectral radius, it follows that

$$\begin{aligned} \left| \lambda_{\min}\left[\nabla^2 C_k(\tau) - \nabla^2 R_0(w_0^*) \right] \right| &\leq \rho_r\left[\nabla^2 C_k(\tau) - \nabla^2 R_0(w_0^*) \right] \leq \left\| \nabla^2 C_k(\tau) - \nabla^2 R_0(w_0^*) \right\|_2 \\ &\leq \rho\lambda_n. \end{aligned}$$

Hence

$$\begin{aligned} \lambda_{\min}\left[\nabla^2 C_k\left(w_0\right) \right] &\geq \lambda_{\min}\left[\nabla^2 C_k(\tau) - \nabla^2 R_0\left(w_0^*\right) \right] + \lambda_{\min}\left[\nabla^2 R_0\left(w_0^*\right) \right] \\ &\geq -\rho\lambda_n + \lambda_n = (1-\rho)\,\lambda_n, \end{aligned}$$

which implies that $C_k(w_0)$ is $(1-\rho)\lambda_n$-strongly convex on U_{δ_ρ}.

We now prove (33). We will follow the proof framework of Zhang et al. (2013b). Using the fact that $C_k\left(w_0\right)$ is strongly convex on the set U_{δ_ρ}, for any $w_0^{'} \in Q$, we obtain

$$\begin{aligned} C_k\left(w_0'\right) &\geqslant C_k\left(w_0^*\right) + \left\langle \nabla C_k\left(w_0^*\right), w_0' - w_0^* \right\rangle + \frac{(1-\rho)\lambda_n}{2}\left\| w_0' - w_0^* \right\|^2 \\ &\geqslant C_k(w_0^*) + \left\langle \nabla C_k\left(w_0^*\right), w_0' - w_0^* \right\rangle + \frac{(1-\rho)\lambda_n}{2}\min\left\{ \left\| w_0' - w_0^* \right\|^2, \delta_\rho^2 \right\}. \end{aligned}$$

Rewriting this inequality, it can be seen that

$$\begin{aligned}\min\left\{\left\|w_0'-w_0^*\right\|^2,\delta_\rho^2\right\} &\leq \frac{2}{(1-\rho)\lambda_n}\left[C_k\left(w_0'\right)-C_k\left(w_0^*\right)-\left\langle\nabla C_k\left(w_0^*\right),w_0'-w_0^*\right\rangle\right]\\ &\leq \frac{2}{(1-\rho)\lambda_n}\left[C_k(w_0')-C_k\left(w_0^*\right)+\|\nabla C_k(w_0^*)\|\left\|w_0'-w_0^*\right\|\right].\end{aligned} \tag{35}$$

Without loss of generality, let $w_0'=\kappa\hat{W}_{k,0}+(1-\kappa)w_0^*$ for $\kappa\in(0,1]$, then $\|w_0'-w_0^*\|>0$ and $\|w_0'-w_0^*\|^2=\kappa^2\left\|\hat{W}_{k,0}-w_0^*\right\|^2$. Dividing both sides of (35) by $\|w_0'-w_0^*\|$ leads to

$$\begin{aligned}&\min\left\{\kappa\left\|\hat{W}_{k,0}-w_0^*\right\|,\frac{\delta_\rho^2}{\kappa\left\|\hat{W}_{k,0}-w_0^*\right\|}\right\}\\ \leq\ &\frac{2\left[C_1\left(\kappa\hat{W}_{k,0}+(1-\kappa)\,w_0^*\right)-C_k(w_0^*)\right]}{\kappa\left\|\hat{W}_{k,0}-w_0^*\right\|(1-\rho)\,\lambda_n}+\frac{2\left\|\nabla C_1\left(w_0^*\right)\right\|}{(1-\rho)\,\lambda_n}.\end{aligned}$$

By the Jensen's inequality, we see that

$$C_1\left(\kappa\hat{W}_{k,0}+(1-\kappa)w_0^*\right)<C_1\left(w_0^*\right),$$

which gives the following inequality

$$\min\left\{\kappa\left\|\hat{W}_{k,0}-w_0^*\right\|,\frac{\delta_\rho^2}{\kappa\left\|\hat{W}_{k,0}-w_0^*\right\|}\right\}<\frac{2\left\|\nabla C_k\left(w_0^*\right)\right\|}{(1-\rho)\,\lambda_n}\leq\delta_\rho, \tag{36}$$

where the last inequality follows from the definition of $\mathcal{E}_2$ and the conditions in Lemma 1.

Since (36) holds for any $\kappa\in(0,1]$, if $\left\|\hat{W}_{k,0}-w_0^*\right\|>\delta_\rho$, we can set $\kappa=\frac{\delta_\rho}{\left\|\hat{W}_{k,0}-w_0^*\right\|}$, which yields a contradiction that $\min\{\delta_\rho,\delta_\rho\}<\delta_\rho$. Thus, we have

$$\left\|\hat{W}_{k,0}-w_0^*\right\|\leq\delta_\rho.$$

Therefore, from (35) and $w_0'=\kappa\hat{W}_{k,0}+(1-\kappa)w_0^*$ with $\kappa=1$, we obtain

$$\begin{aligned}\left\|\hat{W}_{k,0}-w_0^*\right\|^2 &\leq \frac{2}{(1-\rho)\,\lambda_n}\left[C_k\left(\hat{W}_{k,0}\right)-C_1\left(w_0^*\right)+\|\nabla C_k\left(w_0^*\right)\|\left\|\hat{W}_{k,0}-w_0^*\right\|\right]\\ &\leq \frac{2\left\|\nabla C_1\left(w_0^*\right)\right\|}{(1-\rho)\,\lambda_n}\left\|\hat{W}_{k,0}-w_0^*\right\|,\end{aligned}$$

which implies the inequality (33) immediately. ■

Lemma 2 *Assume Conditions 1-3 hold, then*

$$\mathbf{E}\left\|\triangledown C_k\left(w_0^*\right)\right\|^2 = O\left(\frac{\sigma_n^2 S p_S}{n}\right), \tag{37}$$

and

$$\mathbf{E}\left\|\triangledown^2 C_k\left(w_0^*\right) - \triangledown^2 R_0\left(w_0^*\right)\right\|_2^2 = O\left(\frac{S^2 p_S}{n}\right). \tag{38}$$

Proof By the definition of w_0^* and Condition 1, we see that $\triangledown R_0(w_0^*) = 0$, which together with (11) gives

$$\begin{aligned}
0 &= \left.\frac{\partial R_0(w_0)}{\partial w_0}\right|_{w_0=w_0^*} \\
&= \frac{2}{n}\mathbf{E}\begin{bmatrix} Y_k^T\left(P_{k,1}-P_{k,S}\right)\left\{P(w^*)Y_k-\boldsymbol{\mu}_k\right\} \\ \vdots \\ Y_k^T\left(P_{k,S-1}-P_{k,S}\right)\left\{P\left(w^*\right)Y_k-\boldsymbol{\mu}_k\right\}\end{bmatrix} \\
&= \frac{2}{n}\mathbf{E}\begin{bmatrix} \boldsymbol{\mu}_k^T\left(P_{k,1}-P_{k,S}\right)\left\{P(w^*)-I\right\}\boldsymbol{\mu}_k \\ \vdots \\ \boldsymbol{\mu}_k^T\left(P_{k,S-1}-P_{k,S}\right)\left\{P\left(w^*\right)-I\right\}\boldsymbol{\mu}_k\end{bmatrix} \\
&\quad +\frac{2}{n}\mathbf{E}\begin{bmatrix} e_{(k)}^T\left(P_{k,1}-P_{k,S}\right)P(w^*)e_{(k)} \\ \vdots \\ e_{(k)}^T\left(P_{k,S-1}-P_{k,S}\right)P(w^*)e_{(k)}\end{bmatrix}.
\end{aligned} \tag{39}$$

Moreover,

$$\begin{aligned}
&\triangledown C_k(w_0^*) \\
= &\frac{2}{n}\begin{bmatrix} Y_k^T\left(P_{k,1}-P_{k,S}\right)\left\{P(w^*)Y_k-\boldsymbol{\mu}_k\right\} \\ \vdots \\ Y_k^T\left(P_{k,S-1}-P_{k,S}\right)\left\{P\left(w^*\right)Y_k-\boldsymbol{\mu}_k\right\}\end{bmatrix} - \frac{2}{n}\begin{bmatrix} Y_k^T\left(P_{k,1}-P_{k,S}\right)e_{(k)} \\ \vdots \\ Y_k^T\left(P_{k,S-1}-P_{k,S}\right)e_{(k)}\end{bmatrix} \\
&+\frac{2\sigma^2}{n}\begin{bmatrix} tr(P_{k,1}-P_{k,S}) \\ \vdots \\ tr(P_{k,S-1}-P_{k,S})\end{bmatrix} \\
= &\frac{2}{n}\begin{bmatrix} \boldsymbol{\mu}_k^T\left(P_{k,1}-P_{k,S}\right)\left\{P(w^*)-I\right\}\boldsymbol{\mu}_k \\ \vdots \\ \boldsymbol{\mu}_k^T\left(P_{k,S-1}-P_{k,S}\right)\left\{P\left(w^*\right)-I\right\}\boldsymbol{\mu}_k\end{bmatrix} + \frac{2}{n}\begin{bmatrix} e_{(k)}^T\left(P_{k,1}-P_{k,S}\right)P(w^*)e_{(k)} \\ \vdots \\ e_{(k)}^T\left(P_{k,S-1}-P_{k,S}\right)P(w^*)e_{(k)}\end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{4}{n}\begin{bmatrix} e_{(k)}^T\left(P_{k,1}-P_{k,S}\right)\left\{P(w^*)-I\right\}\boldsymbol{\mu}_k \\ \vdots \\ e_{(k)}^T\left(P_{k,S-1}-P_{k,S}\right)\left\{P\left(w^*\right)-I\right\}\boldsymbol{\mu}_k \end{bmatrix}-\frac{2}{n}\begin{bmatrix} e_{(k)}^T\left(P_{k,1}-P_{k,S}\right)e_{(k)} \\ \vdots \\ e_{(k)}^T\left(P_{k,S-1}-P_{k,S}\right)e_{(k)} \end{bmatrix} \\
&\quad+\frac{2\sigma^2}{n}\begin{bmatrix} tr(P_{k,1}-P_{k,S}) \\ \vdots \\ tr(P_{k,S-1}-P_{k,S}) \end{bmatrix} \\
&\triangleq \frac{2}{n}\left(A+C_1+2B_1-D_2\right), \qquad (40)
\end{aligned}
$$

where

$$
A=\begin{bmatrix} \boldsymbol{\mu}_k^T\left(P_{k,1}-P_{k,S}\right)\left\{P(w^*)-I\right\}\boldsymbol{\mu}_k \\ \vdots \\ \boldsymbol{\mu}_k^T\left(P_{k,S-1}-P_{k,S}\right)\left\{P\left(w^*\right)-I\right\}\boldsymbol{\mu}_k \end{bmatrix},
$$

$$
B_1=\begin{bmatrix} e_{(k)}^T\left(P_{k,1}-P_{k,S}\right)\left\{P(w^*)-I\right\}\boldsymbol{\mu}_k \\ \vdots \\ e_{(k)}^T\left(P_{k,S-1}-P_{k,S}\right)\left\{P\left(w^*\right)-I\right\}\boldsymbol{\mu}_k \end{bmatrix},
$$

$$
C_1=\begin{bmatrix} e_{(k)}^T\left(P_{k,1}-P_{k,S}\right)P(w^*)e_{(k)} \\ \vdots \\ e_{(k)}^T\left(P_{k,S-1}-P_{k,S}\right)P(w^*)e_{(k)} \end{bmatrix},
$$

and

$$
D_2=\begin{bmatrix} e_{(k)}^T\left(P_{k,1}-P_{k,S}\right)e_{(k)} \\ \vdots \\ e_{(k)}^T\left(P_{k,S-1}-P_{k,S}\right)e_{(k)} \end{bmatrix}-\sigma^2\begin{bmatrix} tr(P_{k,1}-P_{k,S}) \\ \vdots \\ tr(P_{k,S-1}-P_{k,S}) \end{bmatrix}.
$$

Plugging (39) into (40) and by C_r-inequality, we have

$$
\begin{aligned}
\mathbf{E}\left\|\triangledown C_k\left(w_0^*\right)\right\|^2 &= \mathbf{E}\left\|\frac{2}{n}\left(A-\mathbf{E}A+2B_1+C_1-\mathbf{E}C_1-D_2\right)\right\|^2 \\
&\leq \frac{16}{n^2}\left\{\mathbf{E}\left\|A-EA\right\|^2+4\mathbf{E}\left\|B_1\right\|^2+\mathbf{E}\left\|C_1-EC_1\right\|^2+\mathbf{E}\left\|D_2\right\|^2\right\}.
\end{aligned}
$$

We first estimate $\mathbf{E}\left\|A-EA\right\|^2$. For $s\in\{1,2,\ldots,S\}$, since the transpose of each row in $X_{(k)}$ is independent of each other, and $\mathbf{E}\left|x_{(i)}^T\Pi_s^T\beta_{(s)}\right|^4\leq\mathbf{E}\left|x_{(i)}^T\Pi_s^T\beta_{(s)}\right|^{\eta+2}+1<C_b+1$, we obtain

$$
\mathbf{Var}[\beta_{(s)}^T\Pi_s x_{(i)}x_{(i)}^T\Pi_s^T\beta_{(s)}]\leq\mathbf{E}\left|x_{(i)}^T\Pi_s^T\beta_{(s)}\right|^4<C_b+1,
$$

and

$$
\mathbf{Var}\left[\beta_{(s)}^TX_{k,s}^TX_{k,s}\beta_{(s)}\right]=n\cdot\mathbf{Var}\left(\beta_{(s)}^T\Pi_s x_{(i)}x_{(i)}^T\Pi_s^T\beta_{(s)}\right)=O(n). \qquad (41)
$$

Notice that for any random vectors x and y with $\mathbf{E}\left(x^T y\right) = 0$,

$$\begin{aligned} & \mathbf{E}\left[\|x+y\|^2 - \mathbf{E}\|x+y\|^2\right]^2 \\ = \; & \mathbf{E}\left[\|x\|^4 + \left(\|y\|^2 + 2x^T y\right)^2 + 4\|x\|^2 x^T y + 2\|x\|^2\|y\|^2\right] - \left(\mathbf{E}\|x\|^2 + \mathbf{E}\|y\|^2\right)^2 \\ \leq \; & 3\mathbf{E}\|x\|^4 + \mathbf{Var}\left[\|y\|^2 + 2x^T y\right] + 4\mathbf{E}\|x\|^2\|y\|^2. \end{aligned} \tag{42}$$

Hence, by $\mathbf{E}[\delta_{k,s}^T P_{k,s} X_{k,s} \Sigma_s^{-1} \gamma_s] = 0$, and letting $x = P_{k,s}\delta_{k,s}$ and $y = P_{k,s}X_{k,s}\Sigma_s^{-1}\gamma_s$ in (42), it is seen that

$$\begin{aligned} \mathbf{Var}\left[\boldsymbol{b}_{k,s}^T P_{k,s} \boldsymbol{b}_{k,s}\right] \leq \; & 3\mathbf{E}\left[\delta_{k,s}^T P_{k,s}\delta_{k,s}\right]^2 + \mathbf{Var}\left[\left(2\boldsymbol{b}_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s\right)^T X_{k,s}\Sigma_s^{-1}\gamma_s\right] \\ & + 4\mathbf{E}\left[\delta_{k,s}^T P_{k,s}\delta_{k,s}\left(X_{k,s}\Sigma_s^{-1}\gamma_s\right)^T P_{k,s}X_{k,s}\Sigma_s^{-1}\gamma_s\right]. \end{aligned} \tag{43}$$

Since $(2\boldsymbol{b}_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s)^T X_{k,s}\Sigma_s^{-1}\gamma_s$ is a sum of n i.i.d. random variables, and Condition 2 implies

$$\begin{aligned} \mathbf{E}\left|x_{(i)}^T \Pi_s^T \Sigma_s^{-1}\gamma_s\right|^2 \leq \; & \frac{1}{2} + \frac{1}{2}\mathbf{E}\left|x_{(i)}^T \Pi_s^T\left(\beta_{(s)} - \beta_{\star,s}\right)\right|^4 \\ \leq \; & \frac{1}{2} + 4\mathbf{E}\left(\left|x_{(i)}^T \Pi_s^T \beta_{(s)}\right|^4 + \left|x_{(i)}^T \Pi_s^T \beta_{\star,s}\right|^4\right) < \frac{9}{2} + 4C_b, \end{aligned} \tag{44}$$

it follows that

$$\begin{aligned} & \mathbf{Var}\left[\left(2\boldsymbol{b}_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s\right)^T X_{k,s}\Sigma_s^{-1}\gamma_s\right] \\ \leq \; & \sum_{i=1}^n \mathbf{E}\left|(2\boldsymbol{b}_{k,i,s} - x_{(i)}^T \Pi_s^T \Sigma_s^{-1}\gamma_s)^T x_{(i)}^T \Pi_s^T \Sigma_s^{-1}\gamma_s\right|^2 \\ \leq \; & 4\sigma_n^2 n\mathbf{E}\left[\gamma_s^T \Sigma_s^{-1}\Pi_s x_{(i)}^T x_{(i)} \Pi_s^T \Sigma_s^{-1}\gamma_s\right] + 2n\mathbf{E}\left|x_{(i)}\Pi_s^T \Sigma_s^{-1}\gamma_s\right|^4 \\ \leq \; & O\left(\sigma_n^2 n\right) + O(n). \end{aligned} \tag{45}$$

Moreover, Conditions 2 and 3 lead to

$$\begin{aligned} & \mathbf{E}\left[\delta_{k,s}^T P_{k,s}\delta_{k,s}(X_{k,s}\Sigma_s^{-1}\gamma_s)^T P_{k,s}X_{k,s}\Sigma_s^{-1}\gamma_s\right] \\ = \; & \mathbf{E}\left[\delta_{k,s}^T P_{k,s}\delta_{k,s}\gamma_s^T \Sigma_s^{-1} X_{k,s}^T X_{k,s}\Sigma_s^{-1}\gamma_s\right] \\ \leq \; & \sigma_n^2 p_s \mathbf{E}\left[\gamma_s^T \Sigma_s^{-1} X_{k,s}^T X_{k,s}\Sigma_s^{-1}\gamma_s\right] \\ \leq \; & O(\sigma_n^2 p_s n), \end{aligned} \tag{46}$$

and

$$\begin{aligned} & \mathbf{E}\left[\delta_{k,s}^T P_{k,s}\delta_{k,s}\right]^2 \\ \leq \; & p_s \mathbf{E}\left[\lambda_{\max}\left(\mathbf{E}\left(\delta_{k,s}\delta_{k,s}^T \delta_{k,s}\delta_{k,s}^T \middle| X_{k,s}\right)\right)\right] \\ \leq \; & \sigma_n^2 n p_s. \end{aligned} \tag{47}$$

Combining (45)–(47), we obtain

$$\mathbf{E}\left[\boldsymbol{b}_{k,s}^T P_{k,s}\boldsymbol{b}_{k,s} - \mathbf{E}\left\{\boldsymbol{b}_{k,s}^T P_{k,s}\boldsymbol{b}_{k,s}\right\}\right]^2 = O\left(np_s\sigma_n^2\right). \tag{48}$$

Further, from Condition 2, we have

$$\begin{aligned}
& \mathbf{Var}\left[\boldsymbol{b}_{k,s}^T X_{k,s}\beta_{(s)}\right] \\
= & \sum_{i=1}^{n} \mathbf{Var}\left[b_{i(s)} x_{(i)}^T \Pi_s^T \beta_{(s)}\right] \\
\leq & 2n\mathbf{E}\left[\left(b_{i(s)} - x_{(i)}^T \Pi_s^T \Sigma_s^{-1}\gamma_s\right) x_{(i)}^T \Pi_s^T \beta_{(s)}\right]^2 \\
& +2n\mathbf{E}\left[\left(x_{(i)}^T \Pi_s^T \Sigma_s^{-1}\gamma_s\right) x_{(i)}^T \Pi_s^T \beta_{(s)}\right]^2 \\
\leq & O(n\sigma_n^2) + O\left(n\right). \qquad (49)
\end{aligned}$$

On the other hand, it is clear that $P_{k,s}$ is an idempotent matrix. According to the assumption that all candidate models are nested, we see that $P_{k,i}P_{k,j} = P_{k,j}P_{k,i} = P_{k,\min_{i,j}}$ holds. Thus,

$$\begin{aligned}
& \left(P_{k,s} - P_{k,S}\right)\left\{P_k\left(w^*\right) - I_n\right\} \\
= & \left(P_{k,s} - P_{k,S}\right)\left\{\sum_{j=1}^{S} w_j^* P_{k,j} - \left(\sum_{j=1}^{S} w_j^*\right) I_n\right\} \\
= & \sum_{j=1}^{S} w_j^* \left(P_{k,s} - P_{k,S}\right)\left(P_{k,j} - I_n\right) \\
= & \left(\sum_{j=1}^{S-1} w_j^*\right) P_{k,S} - \left(\sum_{j=1}^{s} w_j^*\right) P_{k,s} - \sum_{j=s+1}^{S-1} w_j^* P_{k,j}. \qquad (50)
\end{aligned}$$

With the help of (41), (49) and (50), one has

$$\begin{aligned}
& \mathbf{Var}\left[\boldsymbol{\mu}_k^T P_{k,s}\boldsymbol{\mu}_k\right] \\
= & \mathbf{E}\left[\beta_s^T X_{k,s}^T X_{k,s}\left(X_{k,s}^T X_{k,s}\right)^{-1} X_{k,s}^T X_{k,s}\beta_s - \mathbf{E}\left\{\beta_s^T X_{k,s}^T X_{k,s}\left(X_{k,s}^T X_{k,s}\right)^{-1} X_{k,s}^T X_{k,s}\beta_s\right\}\right. \\
& +\boldsymbol{b}_{k,s}^T X_{k,s}\left(X_{k,s}^T X_{k,s}\right)^{-1} X_{k,s}^T \boldsymbol{b}_{k,s} - \mathbf{E}\left\{\boldsymbol{b}_{k,s}^T X_{k,s}\left(X_{k,s}^T X_{k,s}\right)^{-1} X_{k,s}^T \boldsymbol{b}_{k,s}\right\} \\
& \left.+2\boldsymbol{b}_{k,s}^T X_{k,s}\left(X_{k,s}^T X_{k,s}\right)^{-1} X_{k,s}^T X_{k,s}\beta_s - 2\mathbf{E}[\boldsymbol{b}_{k,s}^T X_{k,s}\left(X_{k,s}^T X_{k,s}\right)^{-1} X_{k,s}^T X_{k,s}\beta_s]\right]^2 \\
\leq & 3\left(\mathbf{E}\left[\beta_{(s)}^T X_{k,s}^T X_{k,s}\beta_{(s)} - \mathbf{E}\left(\beta_{(s)}^T X_{k,s}^T X_{k,s}\beta_{(s)}\right)\right]^2\right. \\
& \left.+\mathbf{Var}\left[\boldsymbol{b}_{k,s}^T X_{k,s}\left(X_{k,s}^T X_{k,s}\right)^{-1} X_{k,s}^T \boldsymbol{b}_{k,s}\right] + 4\mathbf{Var}\left[\boldsymbol{b}_{k,s}^T X_{k,s}\beta_s\right]\right) \\
= & O\left(np_s\sigma_n^2\right),
\end{aligned}$$

and so

$$\begin{aligned}
& \mathbf{E}\left\|A-EA\right\|^2 \\
= \; & \mathbf{E}\left\{\sum_{s=1}^{S-1}\left[\left\{\left(\sum_{j=1}^{S-1}w_j^*\right)\boldsymbol{\mu}_k^T P_{k,S}\boldsymbol{\mu}_k-\left(\sum_{j=1}^{S-1}w_j^*\right)\mathbf{E}\left(\boldsymbol{\mu}_k^T P_{k,S}\boldsymbol{\mu}_k\right)\right\}\right.\right. \\
& -\left(\sum_{j=1}^{s}w_j^*\right)\left\{\boldsymbol{\mu}_k^T P_{k,s}\boldsymbol{\mu}_k-\mathbf{E}\left(\boldsymbol{\mu}_k^T P_{k,s}\boldsymbol{\mu}_k\right)\right\} \\
& \left.\left.-\sum_{j=s+1}^{S-1}w_j^*\left\{\boldsymbol{\mu}_k^T P_{k,j}\boldsymbol{\mu}_k-\mathbf{E}\left(\boldsymbol{\mu}_k^T P_{k,j}\boldsymbol{\mu}_k\right)\right\}\right]^2\right\} \\
\leq \; & \mathbf{E}\left\{3\sum_{s=1}^{S-1}\left[\left(\sum_{j=1}^{S-1}w_j^*\right)^2\left\{\boldsymbol{\mu}_k^T P_{k,S}\boldsymbol{\mu}_k-\mathbf{E}\left(\boldsymbol{\mu}_k^T P_{k,S}\boldsymbol{\mu}_k\right)\right\}^2\right.\right. \\
& +\left(\sum_{j=1}^{s}w_j^*\right)^2\left\{\boldsymbol{\mu}_k^T P_{k,s}\boldsymbol{\mu}_k-\mathbf{E}\left(\boldsymbol{\mu}_k^T P_{k,s}\boldsymbol{\mu}_k\right)\right\}^2 \\
& \left.\left.+\left(\sum_{j=s+1}^{S-1}w_j^*\left\{\boldsymbol{\mu}_k^T P_{k,j}\boldsymbol{\mu}_k-\mathbf{E}\left(\boldsymbol{\mu}_k^T P_{k,j}\boldsymbol{\mu}_k\right)\right\}\right)^2\right]\right\} \\
\leq \; & 3\sum_{s=1}^{S-1}\left(\mathbf{Var}\left[\boldsymbol{\mu}_k^T P_{k,S}\boldsymbol{\mu}_k\right]+\mathbf{Var}\left[\boldsymbol{\mu}_k^T P_{k,s}\boldsymbol{\mu}_k\right]+\max_{j=s+1,\ldots,S-1}\mathbf{Var}\left[\boldsymbol{\mu}_k^T P_{k,j}\boldsymbol{\mu}_k\right]\right) \\
= \; & O\left(nSp_S\sigma_n^2\right). \qquad (51)
\end{aligned}$$

Next, we will bound $\mathbf{E}\left\|B_1\right\|^2$. With (50), it is clear that

$$\begin{aligned}
\mathbf{E}\left\|B_1\right\|^2 &= \mathbf{E}\left[\sum_{s=1}^{S-1}\left\{e_{(k)}^T\left(P_{k,s}-P_{k,S}\right)P\left(w_0^*\right)\boldsymbol{\mu}_k\right\}^2\right] \\
&\leq S\max_{s=1,\ldots,S}\mathbf{E}\left[e_{(k)}^T P_{k,s}\boldsymbol{\mu}_k\right]^2 \\
&= O\left(nSp_S\right). \qquad (52)
\end{aligned}$$

For $\mathbf{E}\left\|C_1-EC_1\right\|^2$, we have

$$\begin{aligned}
& \mathbf{Var}\left[e_{(k)}^T\left(P_{k,s}-P_{k,S}\right)P\left(w^*\right)e_{(k)}\middle| X_{(k)}\right] \\
\leq \; & \left\|\left(P_{k,s}-P_{k,S}\right)P\left(w^*\right)\right\|_F^2\mathbf{E}e_{k,i}^4 \\
= \; & O\left(p_S\right),
\end{aligned}$$

and then

$$\begin{aligned}
\mathbf{E}\left\|C_1-EC_1\right\|^2 &= \sum_{s=1}^{S-1}\mathbf{Var}\left[e_{(k)}^T\left(P_{k,s}-P_{k,S}\right)P\left(w^*\right)e_{(k)}\right] \\
&= O\left(Sp_S\right). \qquad (53)
\end{aligned}$$

Similar to (53), we see that

$$\mathbf{E}\left\|D_2\right\|^2 = O\left(Sp_S^2\right). \tag{54}$$

Combining (51)-(54), we get (37).

To prove (38), we calculate

$$\begin{aligned}
\nabla^2 C_k\left(w_0^*\right) &= \frac{2}{n}\left\{\left(X_{k,s_1}\hat{\beta}_{k,s_1} - X_{k,S}\hat{\beta}_{k,S}\right)^T\left(X_{k,s_2}\hat{\beta}_{k,s_2} - X_{k,S}\hat{\beta}_{k,S}\right)\right\}_{1\le s_1,s_2\le S-1} \\
&= \frac{2}{n}\left\{Y_k^T\left(P_{k,s_1} - P_{k,S}\right)^T\left(P_{k,s_2} - P_{k,S}\right)Y_k\right\}_{1\le s_1,s_2\le S-1} \\
&= \frac{2}{n}\left\{\left(\boldsymbol{\mu}_k + e_{(k)}\right)^T\left(P_{k,S} - P_{k,\max\{s_1,s_2\}}\right)\left(\boldsymbol{\mu}_k + e_{(k)}\right)\right\}_{1\le s_1,s_2\le S-1}.
\end{aligned}$$

With the help of Theorem 2 of Whittle (1960), it can be claimed that

$$\begin{aligned}
&\mathbf{Var}\left[e_{(k)}^T\left(P_{k,s_1} - P_{k,S}\right)^T\left(P_{k,s_2} - P_{k,S}\right)e_{(k)}\right] \\
=\ &\mathbf{E}\left(\mathbf{Var}\left[e_{(k)}^T\left(P_{k,s_1} - P_{k,S}\right)^T\left(P_{k,s_2} - P_{k,S}\right)e_{(k)}\middle| X_{(k)}\right]\right) \\
=\ &O\left(tr\left[\left(P_{k,s_1} - P_{k,S}\right)^2\left(P_{k,s_2} - P_{k,S}\right)^2\right]\right) = O\left(p_S\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}\left[e_{(k)}^T\left(P_{k,s_1} - P_{k,S}\right)^T\left(P_{k,s_2} - P_{k,S}\right)\boldsymbol{\mu}_k\right]^2 \\
=\ &\mathbf{E}\left[\mathbf{E}\left\{\left|e_{(k)}^T\left(P_{k,\max\{s_1,s_2\}} - P_{k,S}\right)\boldsymbol{\mu}_k\right|^2\middle| X_{(k)}\right\}\right] = O\left(np_S\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbf{Var}\left[\left(\boldsymbol{\mu}_k + e_{(k)}\right)^T\left(P_{k,S} - P_{k,\max\{s_1,s_2\}}\right)\left(\boldsymbol{\mu}_k + e_{(k)}\right)\right] \\
=\ &\mathbf{E}\left[\boldsymbol{\mu}_k^T\left(P_{k,S} - P_{k,\max\{s_1,s_2\}}\right)\boldsymbol{\mu}_k - \mathbf{E}\left\{\boldsymbol{\mu}_k^T\left(P_{k,S} - P_{k,\max\{s_1,s_2\}}\right)\boldsymbol{\mu}_k\right\}\right. \\
&\quad + e_{(k)}^T\left(P_{k,S} - P_{k,\max\{s_1,s_2\}}\right)e_{(k)} - \mathbf{E}\left\{e_{(k)}^T\left(P_{k,S} - P_{k,\max\{s_1,s_2\}}\right)e_{(k)}\right\} \\
&\quad \left. + 2\boldsymbol{\mu}_k^T\left(P_{k,S} - P_{k,\max\{s_1,s_2\}}\right)e_{(k)}\right]^2 \\
\le\ &3\left(\mathbf{Var}\left[\boldsymbol{\mu}_k^T\left(P_{k,S} - P_{k,\max\{s_1,s_2\}}\right)\boldsymbol{\mu}_k\right] + \mathbf{Var}\left[e_{(k)}^T\left(P_{k,S} - P_{k,\max\{s_1,s_2\}}\right)e_{(k)}\right]\right. \\
&\quad \left. + 4\mathbf{E}\left\{\boldsymbol{\mu}_k^T\left(P_{k,S} - P_{k,\max\{s_1,s_2\}}\right)e_{(k)}\right\}^2\right) \\
=\ &O\left(np_S\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbf{E}\left\|\nabla^2 C_k\left(w_0^*\right) - \nabla^2\mathbf{E}\left\{C_k\left(w_0^*\right)\right\}\right\|_2^2 \le \mathbf{E}\left\|\nabla^2 C_k\left(w_0^*\right) - \nabla^2\mathbf{E}\left\{C_k\left(w_0^*\right)\right\}\right\|_F^2 \\
=\ &\frac{2}{n^2}\sum_{s_1=1}^{S-1}\sum_{s_2=1}^{S-1}\mathbf{Var}\left[\left(\boldsymbol{\mu}_k + e_{(k)}\right)^T\left(P_{k,S} - P_{k,\max\{s_1,s_2\}}\right)\left(\boldsymbol{\mu}_k + e_{(k)}\right)\right] = O\left(n^{-1}S^2p_S\right),
\end{aligned}$$

which completes the proof of (38). ■

Lemma 3 *Under Conditions 1-4, we have*

$$\mathbf{E}\left\|\hat{W}_{k,0}-w_0^*\right\|^2=O\left(\frac{Sp_S(S+\sigma_n^2)}{\lambda_n^2 n}\right),$$

and then

$$\mathbf{E}\left\|\hat{W}_{k}-w^*\right\|^2=O\left(\frac{S^2p_S(S+\sigma_n^2)}{\lambda_n^2 n}\right).$$

Proof Recalling the events $\mathcal{E}_1$ and $\mathcal{E}_2$, we define the event $\mathcal{E}\triangleq\mathcal{E}_1\cap\mathcal{E}_2$. In view of Lemma 1, we get

$$\begin{aligned}\mathbf{E}\left\|\hat{W}_{k,0}-w_0^*\right\|^2&=\mathbf{E}\left[1_{(\mathcal{E})}\left\|\hat{W}_{k,0}-w_0^*\right\|^2\right]+\mathbf{E}\left[1_{(\mathcal{E}^c)}\left\|\hat{W}_{k,0}-w_0^*\right\|^2\right]\\ \leq&\ \frac{4\mathbf{E}\left[1_{(\mathcal{E})}\left\|\nabla C_k\left(w_0^*\right)\right\|^2\right]}{\left(1-\rho\right)^2\lambda_n^2}+2\mathbf{P}\left(\mathcal{E}^c\right)\leq\frac{4\mathbf{E}\left\|\nabla C_k\left(w_0^*\right)\right\|^2}{\left(1-\rho\right)^2\lambda_n^2}+2\mathbf{P}\left(\mathcal{E}^c\right).\end{aligned}\tag{55}$$

From Lemma 2 and some direct calculations, we obtain

$$\begin{aligned}\mathbf{P}(\mathcal{E}^c)&=\mathbf{P}(\mathcal{E}_1^c\cup\mathcal{E}_2^c)\leq\mathbf{P}(\mathcal{E}_1^c)+\mathbf{P}(\mathcal{E}_2^c)\\ &\leq\frac{\mathbf{E}\left\|\nabla^2C_k(w_0^*)-\nabla^2R_0\left(w_0^*\right)\right\|_2^2}{\rho^2\lambda_n^2}+\frac{4\mathbf{E}\left\|\nabla C_k\left(w_0^*\right)\right\|^2}{\left(1-\rho\right)^2\lambda_n^2\delta_\rho^2}\\ &=O\left(Sp_S\lambda_n^{-2}n^{-1}(S+\sigma_n^2)\right),\end{aligned}\tag{56}$$

which together with (55) leads to

$$\mathbf{E}\left\|\hat{W}_{k,0}-w_0^*\right\|^2=O\left(Sp_S\lambda_n^{-2}n^{-1}(S+\sigma_n^2)\right).$$

This completes the proof of Lemma 3. ■

Lemma 4 *Under Condition 2, for any random variable a with $\|a\|^2\leq 2$, we have*

$$\mathbf{E}\left[\max_{s=1,\ldots,S}\left(x_{(i)}^T\Pi_s^T\beta_{\star,s}\right)^2\|a\|^2\right]=O\left(S^{2/(\eta+2)}\left(\mathbf{E}\|a\|^2\right)^{\eta/(\eta+2)}\right).$$

Proof Define the event

$$\mathcal{E}_3=\left\{\max_{s=1,\ldots,S}\left(x_{(i)}^T\Pi_s^T\beta_{\star,s}\right)^2\leq\left(\mathbf{E}\|a\|^2/S\right)^{-1/(\eta+2)}\right\},$$

then

$$\begin{aligned}&\mathbf{E}\left[\max_{s=1,\ldots,S}\left(x_{k,1}^T\Pi_s^T\beta_{\star,s}\right)^2\|a\|^2\right]\\ \leq&\ \mathbf{E}\left[1_{(\mathcal{E}_3)}\max_{s=1,\ldots,S}\left(x_{k,1}^T\Pi_s^T\beta_{\star,s}\right)^2\|a\|^2\right]+2S\mathbf{E}\left[1_{(\mathcal{E}_3^c)}\left(x_{k,1}^T\Pi_s^T\beta_{\star,s}\right)^2\right]\\ \leq&\ \left(\mathbf{E}\|a\|^2/S\right)^{-2/(\eta+2)}\mathbf{E}\|a\|^2+2S\left(\mathbf{E}\|a\|^2/S\right)^{\eta/(\eta+2)}\\ =&\ O\left(S^{2/(\eta+2)}\left(\mathbf{E}\|a\|^2\right)^{\eta/(\eta+2)}\right).\end{aligned}\tag{57}$$

■

Lemma 5 *Under Condition 3, for $s = 1, 2, \ldots, S$, we have*

$$\mathbf{E}\left[\left\{\hat{\beta}_{k,s} - \beta_{\star,s}\right\}^T \Pi_s X_k^T X_k \Pi_s^T \left\{\hat{\beta}_{k,s} - \beta_{\star,s}\right\}\right] \leq p_s(\sigma_n^2 + \sigma^2). \tag{58}$$

Proof Since

$$\hat{\beta}_{k,s} - \beta_{\star,s} \;=\; \left(X_{k,s}^T X_{k,s}\right)^{-1} X_{k,s}^T \left(b_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s + e_{(k)}\right),$$

we obtain

$$\begin{aligned}
& \mathbf{E}\left[\left\{\hat{\beta}_{k,s} - \beta_{\star,s}\right\}^T \Pi_s X_k^T X_k \Pi_s^T \left\{\hat{\beta}_{k,s} - \beta_{\star,s}\right\}\right] \\
= \;& \mathbf{E}\left[\left(b_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s + e_{(k)}\right)^T X_{k,s}\left(X_{k,s}^T X_{k,s}\right)^{-1} X_{k,s}^T \left(b_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s + e_{(k)}\right)\right] \\
= \;& \mathbf{E}\left[\left(b_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s\right)^T P_{k,s}\left(b_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s\right)\right] + \mathbf{E}\left[e_{(k)}^T P_{k,s} e_{(k)}\right] \\
\leq \;& p_s \mathbf{E}\left[\lambda_{\max}\left(\mathbf{E}\left(\left(b_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s\right)\left(b_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s\right)^T | X_{k,s}\right)\right)\right] + \sigma^2 p_s, \\
= \;& p_s(\sigma_n^2 + \sigma^2).
\end{aligned} \tag{59}$$

Then Lemma 5 follows. ■

Lemma 6 *Under Conditions 3 and 6, for $s = 1, 2, \ldots, S$, we have*

$$\mathbf{E}\left[\left\{\hat{\beta}_{k,s} - \beta_{\star,s}\right\}^T \Pi_s \Sigma_s \Pi_s^T \left\{\hat{\beta}_{k,s} - \beta_{\star,s}\right\}\right] = O\left(\frac{p_s\sigma_n^2}{n}\right). \tag{60}$$

Proof Denote $\overline{P}_{k,s} = X_{k,s}(X_{k,s}^T X_{k,s})^{-1}\Sigma_s(X_{k,s}^T X_{k,s})^{-1}X_{k,s}^T$, then it follows that $tr[\overline{P}_{k,s}] = tr[(X_{k,s}^T X_{k,s})^{-1}\Sigma_s]$, and hence

$$\begin{aligned}
& \mathbf{E}\left[\left\{\hat{\beta}_{k,s} - \beta_{\star,s}\right\}^T \Pi_s \Sigma_s \Pi_s^T \left\{\hat{\beta}_{k,s} - \beta_{\star,s}\right\}\right] \\
= \;& \mathbf{E}\left[\left(b_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s + e_{(k)}\right)^T \overline{P}_{k,s}\left(b_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s + e_{(k)}\right)\right] \\
= \;& \mathbf{E}\left[\left(b_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s\right)^T \overline{P}_{k,s}\left(b_{k,s} - X_{k,s}\Sigma_s^{-1}\gamma_s\right)\right] + \mathbf{E}\left[e_{(k)}^T \overline{P}_{k,s} e_{(k)}\right] \\
\leq \;& \mathbf{E}\left\{tr\left[\left(X_{k,s}^T X_{k,s}\right)^{-1}\Sigma_s\right]\lambda_{\max}\left(\Sigma_{\infty|s}\right)\right\} + \sigma^2 \mathbf{E}\left\{tr\left[\left(X_{k,s}^T X_{k,s}\right)^{-1}\Sigma_s\right]\right\} \\
= \;& O\left(\frac{p_s\sigma_n^2}{n}\right).
\end{aligned}$$

■

Appendix B. Proofs of Theorems

Proof of Theorem 1

To obtain the bound of $\mathbf{E}\left\|\overline{w}_0 - w_0^*\right\|^2$, we first show that the function $C_k(w_0)$ behaves similarly to the risk function $R_0\left(w_0\right)$ in the neighborhood of the point w_0^* under the two events $\mathcal{E}_1$ and $\mathcal{E}_2$. Intuitively, $R_0\left(w_0\right)$ is locally strongly convex, so the minimizer $\hat{W}_{k,0}$ of $C_k\left(w_0\right)$ will be close to w_0^*. Hence our idea is to show that the events $\mathcal{E}_1$ and $\mathcal{E}_2$ hold with high probability, which will guarantee the closeness of $\hat{W}_{k,0}$ and w_0^*.

From the definition of $\bar{w}_0$, it is seen that

$$\begin{aligned}
\mathbf{E}\left\|\overline{w}_0 - w_0^*\right\|^2 &= \mathbf{E}\left\|\frac{1}{K}\sum_{k=1}^{K}\hat{W}_{k,0} - w_0^*\right\|^2 \\
&= \frac{1}{K^2}\mathbf{E}\left\{\sum_{k=1}^{K}\left\|\hat{W}_{k,0} - w_0^*\right\|^2 + \sum_{k\neq j}\left\langle \hat{W}_{k,0} - w_0^*, \hat{W}_{j,0} - w_0^*\right\rangle\right\} \\
&= \frac{1}{K^2}\sum_{k=1}^{K}\mathbf{E}\left\|\hat{W}_{k,0} - w_0^*\right\|^2 + \frac{1}{K^2}\sum_{k\neq j}\left\langle \mathbf{E}\left(\hat{W}_{k,0} - w_0^*\right), \mathbf{E}\left(\hat{W}_{j,0} - w_0^*\right)\right\rangle \\
&= \frac{1}{K}\mathbf{E}\left\|\hat{W}_{1,0} - w_0^*\right\|^2 + \frac{K(K-1)}{K^2}\left\|\mathbf{E}\left(\hat{W}_{1,0} - w_0^*\right)\right\|^2 \\
&\leq \frac{1}{K}\mathbf{E}\left\|\hat{W}_{1,0} - w_0^*\right\|^2 + \left\|\mathbf{E}\left(\hat{W}_{1,0} - w_0^*\right)\right\|^2, \qquad (61)
\end{aligned}$$

where the third equality is from the fact that the weights $\hat{W}_{k,0}$ and $\hat{W}_{j,0}$ are independent. The upper bound in (61) illuminates the path for the remainder of our proof: We only need to bound $\mathbf{E}\left\|\hat{W}_{1,0} - w_0^*\right\|^2$ and $\left\|\mathbf{E}\left(\hat{W}_{1,0} - w_0^*\right)\right\|^2$.

Noting that Lemma 3 gives the bound on $\mathbf{E}\left\|\hat{W}_{1,0} - w_0^*\right\|^2$, we derive the bound on $\left\|\mathbf{E}\left(\hat{W}_{1,0} - w_0^*\right)\right\|^2$ below. With the fact that $\triangledown C_1(\hat{W}_{1,0}) = 0$, and the Taylor series expansion of $\triangledown C_1(\hat{W}_{1,0})$ at w_0^*, we have

$$0 = \triangledown C_1\left(\hat{W}_{1,0}\right) = \triangledown C_1\left(w_0^*\right) + \triangledown^2 C_1\left(w_0'\right)\left(\hat{W}_{1,0} - w_0^*\right),$$

where $w_0' = \kappa w_0^* + (1-\kappa)\hat{W}_{1,0}$ for some $\kappa \in [0,1]$. Clearly, this is equivalent to

$$0 = \triangledown C_1\left(w_0^*\right) + \left[\triangledown^2 C_1\left(w_0^*\right) - \triangledown^2 R_0\left(w_0^*\right)\right]\left(\hat{W}_{1,0} - w_0^*\right) + \triangledown^2 R_0\left(w_0^*\right)\left(\hat{W}_{1,0} - w_0^*\right). \qquad (62)$$

By Condition 1, we can set $\Sigma = \triangledown^2 R_0\left(w_0^*\right)$ and $\Sigma^{-1} = \left[\triangledown^2 R_0\left(w_0^*\right)\right]^{-1}$. Multiplying both sides of (62) by Σ^{-1}, we obtain

$$\hat{W}_{1,0} - w_0^* = -\Sigma^{-1}\triangledown C_1\left(w_0^*\right) + \Sigma^{-1}\left[\triangledown^2 R_0\left(w_0^*\right) - \triangledown^2 C_1\left(w_0^*\right)\right]\left(\hat{W}_{1,0} - w_0^*\right). \qquad (63)$$

Therefore, by Lemmas 2 and 3, it is seen that

$$\begin{aligned}\left\|\mathbf{E}\left(\hat{W}_{1,0}-w_0^*\right)\right\| &= \left\|\mathbf{E}\left\{\Sigma^{-1}\left(\nabla^2 R_0\left(w_0^*\right)-\nabla^2 C_1\left(w_0^*\right)\right)\left(\hat{W}_{1,0}-w_0^*\right)\right\}\right\| \\ &\leq \mathbf{E}\left\|\Sigma^{-1}\left(\nabla^2 R_0\left(w_0^*\right)-\nabla^2 C_1\left(w_0^*\right)\right)\left(\hat{W}_{1,0}-w_0^*\right)\right\| \\ &\leq \left(\mathbf{E}\left\|\Sigma^{-1}\left(\nabla^2 R_0\left(w_0^*\right)-\nabla^2 C_1\left(w_0^*\right)\right)\right\|^2\right)^{1/2}\left(\mathbf{E}\left\|\left(\hat{W}_{1,0}-w_0^*\right)\right\|^2\right)^{1/2} \\ &= O\left(S^{3/2}(S+\sigma_n^2)^{1/2}p_S\lambda_n^{-2}n^{-1}\right). \end{aligned} \tag{64}$$

By combining Lemma 5, Condition 4 and

$$|\lambda_n-\overline{\lambda}_S| \leq O\left(Sn^{-1/2}\max_{s=1,\ldots,S}\mathbf{E}^{1/2}\left[\left\{\hat{\beta}_{k,s}-\beta_{\star,s}\right\}^T\Pi_s X_k^T X_k \Pi_s^T\left\{\hat{\beta}_{k,s}-\beta_{\star,s}\right\}\right]\right), \tag{65}$$

it follows that $\lambda_n=\overline{\lambda}_S+o(\overline{\lambda}_S)$, which together with (64) leads to

$$\begin{aligned}\mathbf{E}\left\|\overline{w}_0-w_0^*\right\|^2 &\leq \frac{1}{K}\mathbf{E}\left\|\hat{W}_{1,0}-w_0^*\right\|^2+\left\|\mathbf{E}\left(\hat{W}_{1,0}-w_0^*\right)\right\|^2 \\ &= O\left(\frac{Sp_S(S+\sigma_n^2)}{Kn\overline{\lambda}_S^2}\right)+O\left(\frac{S^3p_S^2(S+\sigma_n^2)}{n^2\overline{\lambda}_S^4}\right).\end{aligned}$$

Theorem 1 is proved.

Proof of Theorem 2

Noting that

$$\begin{aligned}& NL_N(w) \\ =& \sum_{k=1}^K\left\|\hat{\boldsymbol{\mu}}_k-X_k\beta_\star(w)+X_k\beta_\star(w)-\boldsymbol{\mu}_k\right\|^2 \\ =& \sum_{k=1}^K\left\{\left\|\hat{\boldsymbol{\mu}}_k-X_k\beta_\star(w)\right\|^2+\left\|X_k\beta_\star(w)-\boldsymbol{\mu}_k\right\|^2+2\left\langle\hat{\boldsymbol{\mu}}_k-X_k\beta_\star(w),X_k\beta_\star(w)-\boldsymbol{\mu}_k\right\rangle\right\} \\ =& \sum_{k=1}^K\left\|\hat{\boldsymbol{\mu}}_k-X_k\beta_\star(w)\right\|^2+L_{N,\star}(w)+2\sum_{k=1}^K\left\langle\hat{\boldsymbol{\mu}}_k-X_k\beta_\star(w),X_k\beta_\star(w)-\mu_k\right\rangle, \end{aligned} \tag{66}$$

and then we have

$$\begin{aligned}& NR_N(w) \\ =& NR_N^*(w)+\sum_{k=1}^K\mathbf{E}\left\|\hat{\boldsymbol{\mu}}_k-X_k\beta_\star(w)\right\|^2+2\sum_{k=1}^K\mathbf{E}\left\langle\hat{\boldsymbol{\mu}}_k-X_k\beta_\star(w),X_k\beta_\star(w)-\boldsymbol{\mu}_k\right\rangle \\ \leq& NR_N^*(w)+\sum_{k=1}^K\mathbf{E}\left\|\hat{\boldsymbol{\mu}}_k-X_k\beta_\star(w)\right\|^2+2\sqrt{NR_N^*(w)\sum_{k=1}^K\mathbf{E}\left\|\hat{\boldsymbol{\mu}}_k-X_k\beta_\star(w)\right\|^2},\end{aligned}$$

and

$$\left|\frac{R_N(w)-R_N^*(w)}{R_N^*(w)}\right| \le \frac{\sum_{k=1}^K \mathbf{E}\left\|\hat{\boldsymbol{\mu}}_k - X_k\beta_\star(w)\right\|^2}{NR_N^*(w)} + 2\sqrt{\frac{\sum_{k=1}^K \mathbf{E}\left\|\hat{\boldsymbol{\mu}}_k - X_k\beta_\star(w)\right\|^2}{NR_N^*(w)}}.$$

So we need only to prove

$$\sup_{w\in Q}\frac{\sum_{k=1}^K \mathbf{E}\left\|\hat{\boldsymbol{\mu}}_k - X_k\beta_\star(w)\right\|^2}{NR_N^*(w)} = o(1). \tag{67}$$

Since

$$\begin{aligned}\mathbf{E}\left\|\hat{\boldsymbol{\mu}}_k - X_k\beta_\star(w)\right\|^2 &= \mathbf{E}\left\|\sum_{s=1}^S w_s X_k\Pi_s^T\left(\hat{\beta}_{k,s}-\beta_{\star,s}\right)\right\|^2\\ &\le \max_{s=1,\ldots,S}\mathbf{E}\left\|X_k\Pi_s^T\left(\hat{\beta}_{k,s}-\beta_{\star,s}\right)\right\|^2,\end{aligned}$$

it is sufficient to prove that

$$\frac{1}{n}\max_{s=1,\ldots,S}\mathbf{E}\left\|X_k\Pi_s^T\left(\hat{\beta}_{k,s}-\beta_{\star,s}\right)\right\|^2 = o\left(\inf_{w\in Q}R_N^*(w)\right). \tag{68}$$

By the definitions of $\hat{\beta}_{k,s}$ and $\beta_{\star,s}$ and Lemma 5, it can be seen that

$$\max_{s=1,\ldots,S}\mathbf{E}\left\|X_k\Pi_s^T\left(\hat{\beta}_{k,s}-\beta_{\star,s}\right)\right\|^2 \le p_S(\sigma_n^2+\sigma^2).$$

So with the help of Condition 5, (68) holds.

Now from (13), we have

$$R_N(w^*) = R_N^*(w^*) + o\left(R_N^*(w^*)\right) = \xi_{\star,N} + o(\xi_{\star,N}).$$

This completes the proof of Theorem 2.

Proof of Theorem 3

By applying Lemmas 3 and 6, (64) and Theorem 1, we obtain

$$\begin{aligned}
& \mathbf{E}\left\|\Sigma_S^{1/2}\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}\right\|^2 \\
= & \mathbf{E}\left\|\Sigma_S^{1/2}\left\{\frac{1}{K}\sum_{k=1}^{K}\hat{\beta}_k\left(\hat{W}_k\right)-\beta_\star\left(w^*\right)\right\}\right\|^2 \\
\leq & \frac{1}{K}\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\hat{\beta}_1\left(\hat{W}_1\right)-\beta_\star\left(w^*\right)\right\}\right\|^2+\frac{K(K-1)}{K^2}\left\|\mathbf{E}\left[\Sigma_S^{1/2}\left\{\hat{\beta}_1\left(\hat{W}_1\right)-\beta_\star\left(w^*\right)\right\}\right]\right\|^2 \\
\leq & 2\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\hat{\beta}_1\left(\hat{W}_1\right)-\beta_\star\left(\hat{W}_1\right)\right\}\right\|^2+\frac{2}{K}\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\beta_\star\left(\hat{W}_1\right)-\beta_\star\left(w^*\right)\right\}\right\|^2 \\
& +\frac{2(K-1)}{K}\left\|\mathbf{E}\left[\Sigma_S^{1/2}\left\{\beta_\star\left(\hat{W}_1\right)-\beta_\star\left(w^*\right)\right\}\right]\right\|^2 \\
\leq & 2\max_{s=1,\ldots,S}\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\hat{\beta}_{1,s}-\beta_{\star,s}\right\}\right\|^2+\frac{2}{K}\left(\sum_{s=1}^{S}\beta_{\star,s}^T\Pi_s\Sigma_S\Pi_s^T\beta_{\star,s}\right)\mathbf{E}\left\|\hat{W}_1-w^*\right\|^2 \\
& +\frac{2(K-1)}{K}\left(\sum_{s=1}^{S}\beta_{\star,s}^T\Pi_s\Sigma_S\Pi_s^T\beta_{\star,s}\right)\left\|\mathbf{E}\left[\hat{W}_1-w^*\right]\right\|^2 \\
= & O\left(\frac{p_S\sigma_n^2}{n}+\frac{S^3p_S(S+\sigma_n^2)}{K\overline{\lambda}_S^2n}+\frac{S^5p_S^2\left(S+\sigma_n^2\right)}{n^2\overline{\lambda}_S^4}\right),
\end{aligned} \tag{69}$$

and

$$\begin{aligned}
& \mathbf{E}\left\|\Sigma_S^{1/2}\left\{\overline{\tilde{\beta}}-\beta_*\left(w^*\right)\right\}\right\|^2 \\
= & \mathbf{E}\left\|\Sigma_S^{1/2}\left\{\sum_{s=1}^{S}\overline{w}_s\Pi_s^T\tilde{\beta}_s-\sum_{s=1}^{S}w_s^*\Pi_s^T\beta_{\star,s}\right\}\right\|^2 \\
= & \mathbf{E}\left\|\Sigma_S^{1/2}\left\{\sum_{s=1}^{S}\overline{w}_s\Pi_s^T\left(\tilde{\beta}_s-\beta_{\star,s}\right)+\sum_{s=1}^{S}\left(\overline{w}_s-w_s^*\right)\Pi_s^T\beta_{\star,s}\right\}\right\|^2 \\
\leq & 2\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\sum_{s=1}^{S}\overline{w}_s\Pi_s^T\left(\tilde{\beta}_s-\beta_{\star,s}\right)\right\}\right\|^2+2\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\sum_{s=1}^{S}\left(\overline{w}_s-w_s^*\right)\Pi_s^T\beta_{\star,s}\right\}\right\|^2 \\
\leq & 2\max_{s=1,\ldots,S}\left(\frac{1}{K}\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\hat{\beta}_s-\beta_{\star,s}\right\}\right\|^2+\left\|\mathbf{E}\left[\Sigma_S^{1/2}\left\{\hat{\beta}_s-\beta_{\star,s}\right\}\right]\right\|^2\right) \\
& +2\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\sum_{s=1}^{S}\left(\overline{w}_s-w_s^*\right)\Pi_s^T\beta_{\star,s}\right\}\right\|^2 \\
= & O\left(m_S^2\right)+O\left(\frac{S^3p_S(S+\sigma_n^2)}{Kn\overline{\lambda}_S^2}\right)+O\left(\frac{S^5p_S^2(S+\sigma_n^2)}{n^2\overline{\lambda}_S^4}\right).
\end{aligned}$$

Thus, Theorem 3 is proved.

Proof of Theorem 4

By Theorem 3 and Condition 5, we have

$$\begin{aligned}
& \mathbf{E}\left(\overline{\mu}_v-\mu_v\right)^2 \\
= \;& \xi_{\star,N}+2\mathbf{E}\left[\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}^T x_{v,S}\left(x_{v,S}^T\beta_\star\left(w^*\right)-\mu_v\right)\right] \\
& +\mathbf{E}\left[\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}^T x_{v,S}x_{v,S}^T\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}\right] \\
\leq \;& \xi_{\star,N}\left(1+2\sqrt{\xi_{\star,N}^{-1}\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}\right\|^2}+\xi_{\star,N}^{-1}\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}\right\|^2\right) \\
= \;& \xi_{\star,N}\left\{1+O\left(\sqrt{\frac{p_S\sigma_n^2}{n\xi_{\star,N}}}\right)+O\left(\sqrt{\frac{S^3p_S(S+\sigma_n^2)}{Kn\overline{\lambda}_S^2\xi_{\star,N}}}\right)+O\left(\sqrt{\frac{S^5p_S^2(S+\sigma_n^2)}{n^2\overline{\lambda}_S^4\xi_{\star,N}}}\right)\right\}^2,
\end{aligned}$$

and

$$\begin{aligned}
& \mathbf{E}\left(\overline{\widetilde{\mu}}_v-\mu_v\right)^2 \\
\leq \;& \xi_{\star,N}\left(1+2\sqrt{\xi_{\star,N}^{-1}\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\overline{\widetilde{\beta}}-\beta_\star\left(w^*\right)\right\}\right\|^2}+\xi_{\star,N}^{-1}\mathbf{E}\left\|\Sigma_S^{1/2}\left\{\overline{\widetilde{\beta}}-\beta_\star\left(w^*\right)\right\}\right\|^2\right) \\
= \;& \xi_{\star,N}\left\{1+O\left(\sqrt{\frac{m_s^2}{\xi_{\star,N}}}\right)+O\left(\sqrt{\frac{S^3p_S(S+\sigma_n^2)}{Kn\overline{\lambda}_S^2\xi_{\star,N}}}\right)+O\left(\sqrt{\frac{S^5p_S^2(S+\sigma_n^2)}{n^2\overline{\lambda}_S^4\xi_{\star,N}}}\right)\right\}^2.
\end{aligned}$$

Hence, Theorem 4 holds.

Proof of Theorem 5

We first show (19). By Lemmas 4 and 5, Theorem 1, and noting that $\lambda_n=\overline{\lambda}_S+o(\overline{\lambda}_S)$, it is seen that

$$\begin{aligned}
& \mathbf{E}\left\|X_k\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}\right\|^2 \\
= \;& \frac{1}{K^2}\mathbf{E}\left\|X_k\sum_{j=1}^K\left\{\hat{\beta}_j\left(\hat{W}_j\right)-\beta_\star\left(w^*\right)\right\}\right\|^2 \\
\leq \;& \frac{1}{K}\sum_{j=1}^K\mathbf{E}\left\|X_k\left\{\hat{\beta}_j\left(\hat{W}_j\right)-\beta_\star\left(w^*\right)\right\}\right\|^2 \\
= \;& \frac{1}{K}\mathbf{E}\left\|X_k\left\{\hat{\beta}_k\left(\hat{W}_k\right)-\beta_\star\left(w^*\right)\right\}\right\|^2+\frac{1}{K}\mathbf{E}_{j\neq k}\left\|X_k\left\{\hat{\beta}_j\left(\hat{W}_j\right)-\beta_\star\left(w^*\right)\right\}\right\|^2 \\
= \;& \frac{1}{K}\mathbf{E}\left\|X_k\left\{\hat{\beta}_k\left(\hat{W}_k\right)-\beta_\star\left(w^*\right)\right\}\right\|^2+\frac{K-1}{K}\mathbf{E}\left\|n^{1/2}\Sigma_S^{1/2}\left\{\hat{\beta}_j\left(\hat{W}_j\right)-\beta_\star\left(w^*\right)\right\}\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2}{K}\mathbf{E}\left\|X_k\left\{\hat{\beta}_k\left(\hat{W}_k\right)-\beta_\star\left(\hat{W}_k\right)\right\}\right\|^2+\frac{2}{K}\mathbf{E}\left\|X_k\left\{\beta_\star\left(\hat{W}_k\right)-\beta_\star\left(w^*\right)\right\}\right\|^2\\
&\quad+\frac{2\left(K-1\right)}{K}\mathbf{E}\left\|n^{1/2}\Sigma_S^{1/2}\left\{\hat{\beta}_j\left(\hat{W}_j\right)-\beta_\star\left(\hat{W}_j\right)\right\}\right\|^2\\
&\quad+\frac{2\left(K-1\right)}{K}\mathbf{E}\left\|n^{1/2}\Sigma_S^{1/2}\left\{\beta_\star\left(\hat{W}_j\right)-\beta_\star\left(w^*\right)\right\}\right\|^2\\
&\leq \frac{2}{K}\max_{s=1,\ldots,S}\mathbf{E}\left\|X_{k,s}\left\{\hat{\beta}_{k,s}-\beta_{\star,s}\right\}\right\|^2 n^{1/2}\Sigma_S^{1/2}+\frac{2S}{K}\mathbf{E}\left\{\sum_{s=1}^{S}\left\|X_{k,s}\beta_{\star,s}\right\|^2\left(\hat{w}_{k,s}-w_s^*\right)^2\right\}\\
&\quad+\frac{2n\left(K-1\right)}{K}\max_{s=1,\ldots,S}\mathbf{E}\left\|\Sigma_S^{1/2}\Pi_s^T\left\{\hat{\beta}_{k,s}-\beta_{\star,s}\right\}\right\|^2\\
&\quad+\frac{2n\left(K-1\right)}{K}\mathbf{E}\left(\left\{\sum_{s=1}^{S}\beta_{\star,s}^T\Pi_s\Sigma_S\Pi_s^T\beta_{\star,s}\right\}\left\|\hat{W}_k-w^*\right\|^2\right)\\
&= O\left(\frac{p_S\sigma_n^2}{K}\right)+\frac{2Sn}{K}\mathbf{E}\left[\max_{s=1,\ldots,S}\left(x_{(k,1)}^T\Pi_s^T\beta_{\star,s}\right)^2\left\|\hat{W}_k-w^*\right\|^2\right]\\
&\quad+O\left(p_S\sigma_n^2\right)+O\left(S^3p_S\lambda_n^{-2}\left(S+\sigma_n^2\right)\right),\\
&= O\left(S^{\eta+4/(\eta+2)}nK^{-1}\left(S^2p_Sn^{-1}\overline{\lambda}_S^{-2}\left(S+\sigma_n^2\right)\right)^{\eta/(\eta+2)}+S^3p_S\overline{\lambda}_S^{-2}\left(S+\sigma_n^2\right)\right), \qquad (70)
\end{aligned}
$$

where $x_{(k,1)}=X_k^T\epsilon$ with $\epsilon=(1,0,\ldots,0)$ being a n dimensional column vector. Therefore, observing that

$$
\begin{aligned}
\mathbf{E}\left(\overline{\mu}_v-\mu_v\right)^2 &= \xi_{\star,N}+2\mathbf{E}\left[\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}^T x_{v,S}\left(x_{v,S}^T\beta_\star\left(w^*\right)-\mu_v\right)\right]\\
&\quad+\mathbf{E}\left[\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}^T x_{v,S}x_{v,S}^T\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}\right],
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{1}{N}\sum_{k=1}^{K}\mathbf{E}\left\|X_k\overline{\beta}-\mu_k\right\|^2\\
&= \xi_{\star,N}+\frac{2}{N}\sum_{k=1}^{K}\mathbf{E}\left[\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}^T X_k^T\left(X_k\beta_\star\left(w^*\right)-\mu_k\right)\right]+\frac{1}{N}\sum_{k=1}^{K}\mathbf{E}\left\|X_k\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}\right\|^2\\
&\leq \xi_{\star,N}\left(1+2\sqrt{\frac{\sum_{k=1}^{K}\mathbf{E}\left\|X_k\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}\right\|^2}{N\xi_{\star,N}}}+\frac{\sum_{k=1}^{K}\mathbf{E}\left\|X_k\left\{\overline{\beta}-\beta_\star\left(w^*\right)\right\}\right\|^2}{N\xi_{\star,N}}\right)\\
&\leq \xi_{\star,N}\left\{1+O\left(\sqrt{\xi_{\star,N}^{-1}\cdot\left(\frac{S^3p_S\left(S+\sigma_n^2\right)}{n\overline{\lambda}_S^2}+\frac{S^{\frac{\eta+4}{\eta+2}}}{K}\left(\frac{S^2p_S\left(S+\sigma_n^2\right)}{n\overline{\lambda}_S^2}\right)^{\frac{\eta}{\eta+2}}\right)}\right)\right\}^2.
\end{aligned}
$$

Similarly, we can derive

$$
\begin{aligned}
& \mathbf{E}\left\|X_k\left\{\overline{\tilde{\beta}}-\beta_\star\left(w^*\right)\right\}\right\|^2 \\
\leq\ & 2\mathbf{E}\left\{\left\|X_k\left\{\sum_{s=1}^S \overline{w}_s \Pi_s^T\left(\tilde{\beta}_s-\beta_{\star,s}\right)\right\}\right\|^2+\left\|X_k\left\{\sum_{s=1}^S\left(\overline{w}_s-w_s^*\right)\Pi_s^T\beta_{\star,s}\right\}\right\|^2\right\} \\
=\ & 2\mathbf{E}\left\|X_k\left\{\sum_{s=1}^S\left(\frac{1}{K}\sum_{k=1}^K \hat{w}_{k,s}\right)\Pi_s^T\left(\frac{1}{K}\sum_{k=1}^K\hat{\beta}_{k,s}-\beta_{\star,s}\right)\right\}\right\|^2 \\
& +2\mathbf{E}\left\|X_k\left\{\sum_{s=1}^S\left(\frac{1}{K}\sum_{k=1}^K \hat{w}_{k,s}-w_s^*\right)\Pi_s^T\beta_{\star,s}\right\}\right\|^2 \\
\leq\ & 2\max_{s=1,\ldots,S}\mathbf{E}\left\|X_k\left\{\Pi_s^T\left(\frac{1}{K}\sum_{k=1}^K\hat{\beta}_{k,s}-\beta_{\star,s}\right)\right\}\right\|^2+2nS\mathbf{E}\left[\max_{s=1,\ldots,S}\left(x_{k,1}^T\Pi_s^T\beta_{\star,s}\right)^2\left\|\overline{w}-w^*\right\|^2\right] \\
\leq\ & 2\max_{s=1,\ldots,S}\left\{\frac{K-1}{K^2}\mathbf{E}_{q\neq k}\left\|X_k\left\{\Pi_s^T\left(\hat{\beta}_{q,s}-\beta_{\star,s}\right)\right\}\right\|^2+\frac{1}{K^2}\mathbf{E}\left\|X_k\left\{\Pi_s^T\left(\hat{\beta}_{k,s}-\beta_{\star,s}\right)\right\}\right\|^2\right\} \\
& +2\max_{s=1,\ldots,S}\left(\mathbf{E}\left[X_k\left\{\Pi_s^T\left(\hat{\beta}_{k,s}-\beta_{\star,s}\right)\right\}\right]\right)^2+O\left(S^{\frac{\eta+4}{\eta+2}}n\left(\mathbf{E}\left\|\overline{w}-w^*\right\|^2\right)^{\frac{\eta}{\eta+2}}\right) \\
\leq\ & O\left(\frac{p_S\sigma_n^2}{K}\right)+O(n\overline{m}_S^2)+O\left(S^{\frac{\eta+4}{\eta+2}}n\left(\frac{S^2p_S\left(S+\sigma_n^2\right)}{Kn\lambda_S^2}+\frac{S^4p_S^2\left(S+\sigma_n^2\right)}{n^2\lambda_S^4}\right)^{\frac{\eta}{\eta+2}}\right) \\
=\ & O(n\overline{m}_S^2)+O\left(S^{\frac{\eta+4}{\eta+2}}n\left(\frac{S^2p_S\left(S+\sigma_n^2\right)}{Kn\overline{\lambda}_S^2}+\frac{S^4p_S^2\left(S+\sigma_n^2\right)}{n^2\overline{\lambda}_S^4}\right)^{\frac{\eta}{\eta+2}}\right).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
& \frac{1}{N}\sum_{k=1}^K\mathbf{E}\left\|X_k\overline{\tilde{\beta}}-\mu_k\right\|^2 \\
=\ & \xi_{\star,N}+\frac{2}{N}\sum_{k=1}^K\mathbf{E}\left[\left\{\overline{\tilde{\beta}}-\beta_\star\left(w^*\right)\right\}^T X_k^T\left(X_k\beta_\star\left(w^*\right)-\mu_k\right)\right] \\
& +\frac{1}{N}\sum_{k=1}^K\mathbf{E}\left[\left\{\overline{\tilde{\beta}}-\beta_\star\left(w^*\right)\right\}^T X_k^TX_k\left\{\overline{\tilde{\beta}}-\beta_\star\left(w^*\right)\right\}\right] \\
\leq\ & \xi_{\star,N}\left(1+2\sqrt{\frac{\sum_{k=1}^K\mathbf{E}\left\|X_k\left\{\overline{\tilde{\beta}}-\beta_\star\left(w^*\right)\right\}\right\|^2}{N\xi_{\star,N}}}+\frac{\sum_{k=1}^K\mathbf{E}\left\|X_k\left\{\overline{\tilde{\beta}}-\beta_\star\left(w^*\right)\right\}\right\|^2}{N\xi_{\star,N}}\right) \\
\leq\ & \xi_{\star,N}\left\{1+O\left(\sqrt{\xi_{\star,N}^{-1}\cdot\left(\overline{m}_S^2+S^{\frac{\eta+4}{\eta+2}}\left(\frac{S^2p_S\left(S+\sigma_n^2\right)}{Kn\overline{\lambda}_S^2}+\frac{S^4p_S^2\left(S+\sigma_n^2\right)}{n^2\overline{\lambda}_S^4}\right)^{\frac{\eta}{\eta+2}}\right)}\right)\right\}^2.
\end{aligned}
$$

Thus, (20) also holds.

Proof of Theorem 7

We first consider (29). Since Θ is compact, there is $\theta^* \in \Theta$ such that

$$\theta^* \triangleq \operatorname*{argmax}_{\theta\in\Theta} \overline{MSE}, \tag{71}$$

the according $\overline{MSE}$, $\bar{\lambda}_S$,λ_n are denoted by $\overline{MSE}_{\theta^*}$,$\bar{\lambda}_S(\theta^*)$, $\lambda_n(\theta^*)$, respectively. For the model with parameter θ^*, by $\sigma_n^2 = o(n)$ and the definition of Θ, it is easy to check Condition 1 and 4 hold, and by C_r inequality, $\sup_{j\geq 1}\mathbf{E}|x_{k,i,j}|^q < \infty$ and $\|\theta^*\| \leq \varepsilon_3$ can deduce that Condition 2 holds. Now, Conditions 1–4 and 6 are all holds for the model with parameter θ^*, by Theorem 5,

$$\frac{\overline{MSE}_{\theta^*}}{\inf_{w\in Q} R_N^\star(w)} = 1 + O\left(\frac{\sigma_n^2}{n\bar{\lambda}_S^2(\theta^*)} + \frac{1}{K}\left(\frac{\sigma_n^2}{n\bar{\lambda}_S^2(\theta^*)}\right)^{\frac{q-2}{q}}\right), \tag{72}$$

note that $\lambda_n(\theta^*) = \bar{\lambda}_S^2(\theta^*) + o(\bar{\lambda}_S^2(\theta^*))$, which together with (72) leads to (29). In a similar manner, we can show (30).

Next, we focus on (31) and (32). By Lemma 6, it can be verified that

$$\begin{aligned}&\sup_{W_1\in Q, W_2\in Q,\ldots,W_K\in Q}\sup_{\theta\in\Theta}\left(\overline{Mse}\,(W_1,W_2,\ldots,W_K) - \frac{1}{N}\sum_{k=1}^K \mathbf{E}\left\|X_k\frac{1}{K}\sum_{k=1}^K\beta_\star(W_k) - \boldsymbol{\mu}_k\right\|^2\right)\\ &\quad = O\left(\frac{\sigma_n^2}{n}\right)\end{aligned}$$

and

$$\begin{aligned}&\sup_{W_1\in Q, W_2\in Q,\ldots,W_K\in Q}\sup_{\theta\in\Theta}\left(\widetilde{Mse}\,(W_1,W_2,\ldots,W_K) - \frac{1}{N}\sum_{k=1}^K \mathbf{E}\left\|X_k\frac{1}{K}\sum_{k=1}^K\beta_\star(W_k) - \boldsymbol{\mu}_k\right\|^2\right)\\ &\quad = O\left(\frac{\sigma_n^2}{n}\right).\end{aligned}$$

Note that

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^K \mathbf{E}\left\|X_k\frac{1}{K}\sum_{k=1}^K\beta_\star(W_k) - \boldsymbol{\mu}_k\right\|^2 &= \frac{1}{N}\sum_{k=1}^K \mathbf{E}\left\|X_k\sum_{s=1}^S\left(\frac{1}{K}\sum_{k=1}^K w_{k,s}\right)\Pi_s^T\beta_{\star,s} - \boldsymbol{\mu}_k\right\|^2\\ &= \frac{1}{N}\sum_{k=1}^K \mathbf{E}\left\|X_k\beta_\star\left(\frac{1}{K}\sum_{k=1}^K W_k\right) - \boldsymbol{\mu}_k\right\|^2.\end{aligned}$$

then

$$\begin{aligned}
& \inf_{W_1\in Q, W_2\in Q,\ldots,W_K\in Q} \sup_{\theta\in\Theta} \overline{Mse}\,(W_1, W_2, \ldots, W_K) \\
\leq\ & \inf_{W_1\in Q, W_2\in Q,\ldots,W_K\in Q} \sup_{\theta\in\Theta} \frac{1}{N}\sum_{k=1}^{K} \mathbf{E}\left\| X_k \frac{1}{K}\sum_{k=1}^{K}\beta_\star(W_k) - \boldsymbol{\mu}_k \right\|^2 + O\left(\frac{\sigma_n^2}{n}\right) \\
=\ & \inf_{W_1\in Q, W_2\in Q,\ldots,W_K\in Q} \sup_{\theta\in\Theta} \frac{1}{N}\sum_{k=1}^{K} \mathbf{E}\left\| X_k \beta_\star\left(\frac{1}{K}\sum_{k=1}^{K} W_k\right) - \boldsymbol{\mu}_k \right\|^2 + O\left(\frac{\sigma_n^2}{n}\right) \\
=\ & \inf_{w\in Q} \sup_{\theta\in\Theta} R_N^\star(w) + O\left(\frac{\sigma_n^2}{n}\right) \\
\leq\ & \left(1 + O\left(\frac{\sigma_n^2}{n}\right)\right) \inf_{w\in Q} \sup_{\theta\in\Theta} R_N^\star(w), \qquad (73)
\end{aligned}$$

which confirms (31), the last inequality is obtained from the definition of S_2. Similarly, we can imitate above process to prove (32).

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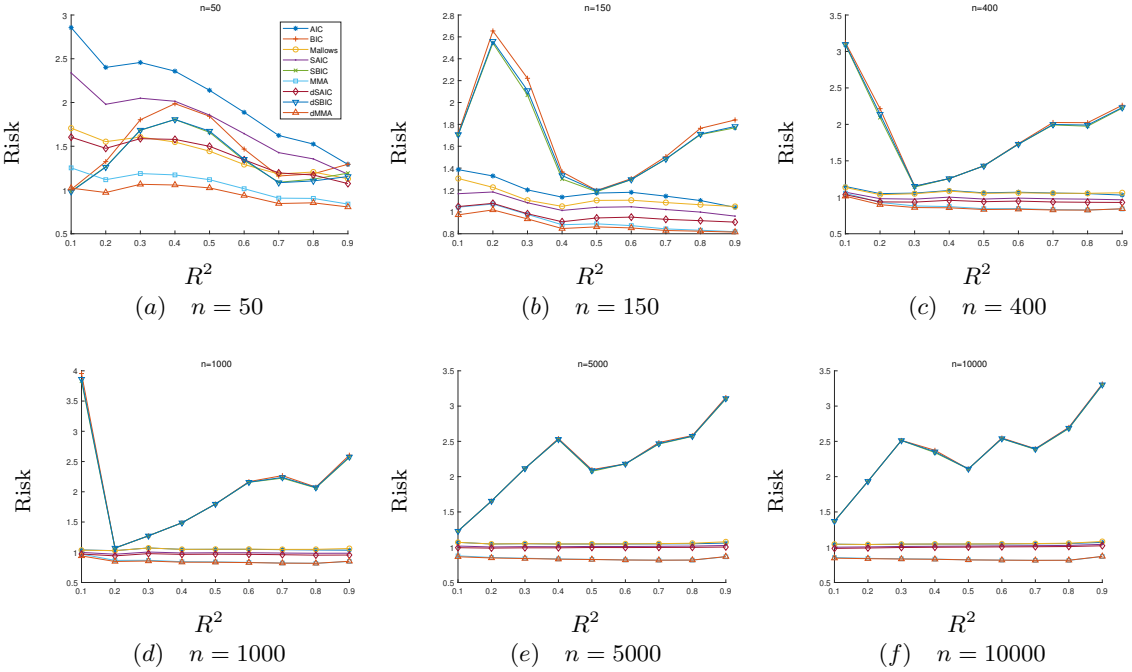

(a) $n = 50$ (b) $n = 150$ (c) $n = 400$

(d) $n = 1000$ (e) $n = 5000$ (f) $n = 10000$

Figure 1: In-sample risk results with $\alpha = 0.5$ and $K = 2$ in Section 4.2.

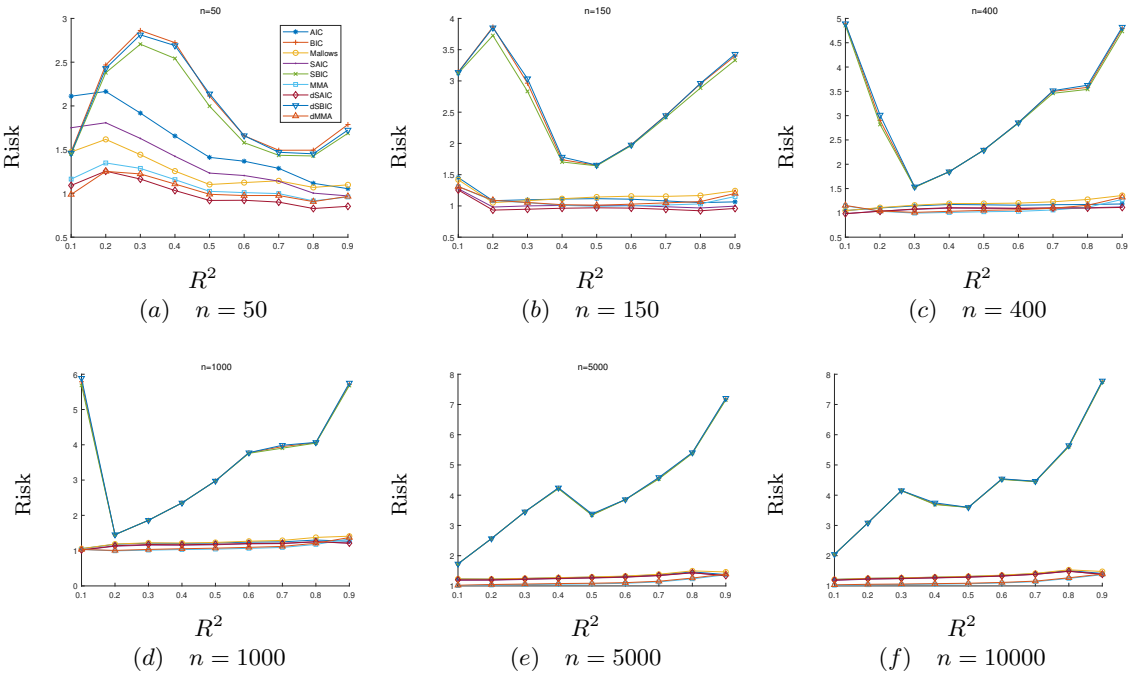

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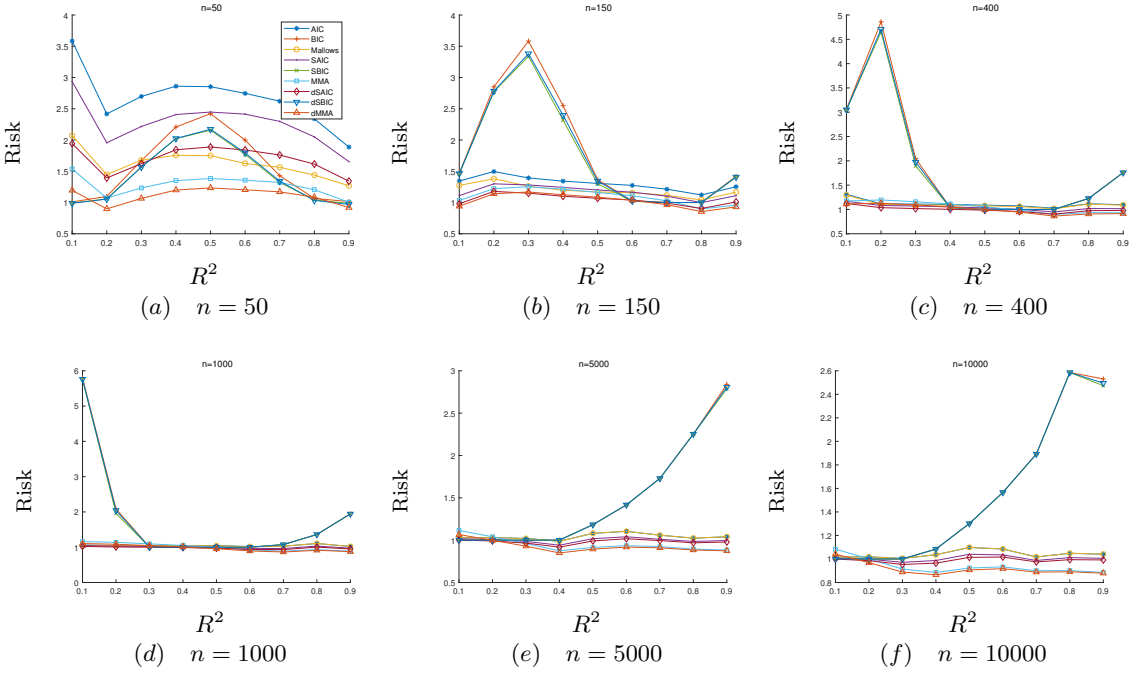
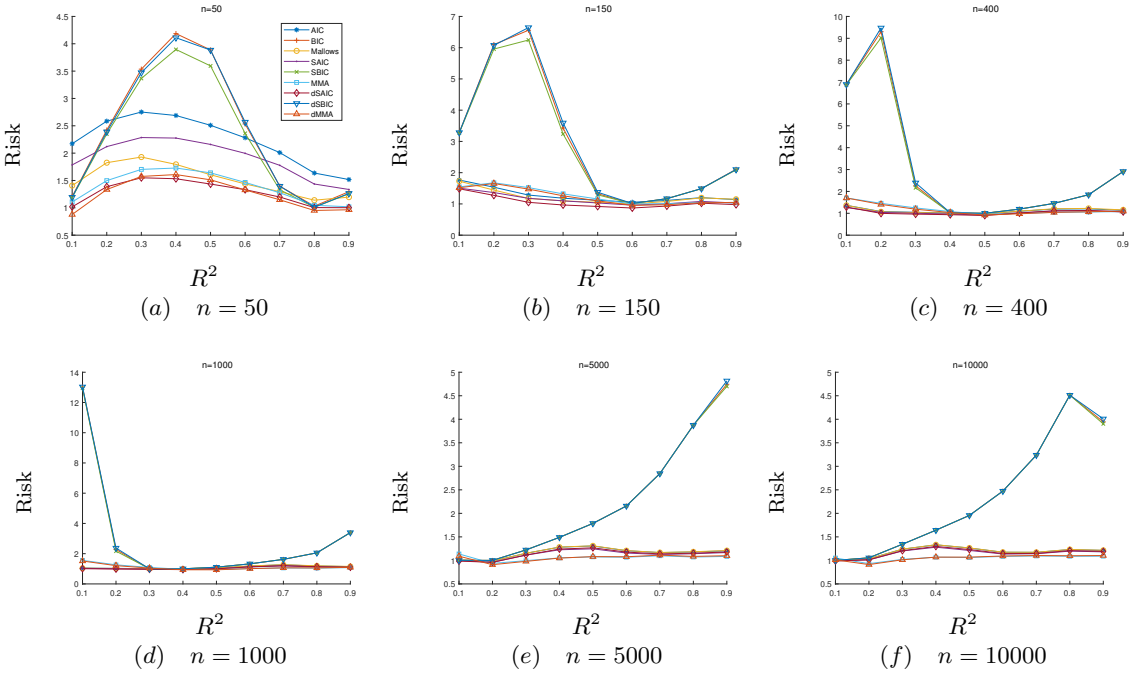
Figure 2: In-sample risk results with $\alpha = 0.5$ and $K = 5$ in Section 4.2.



(a) $n = 50$ (b) $n = 150$ (c) $n = 400$

(d) $n = 1000$ (e) $n = 5000$ (f) $n = 10000$

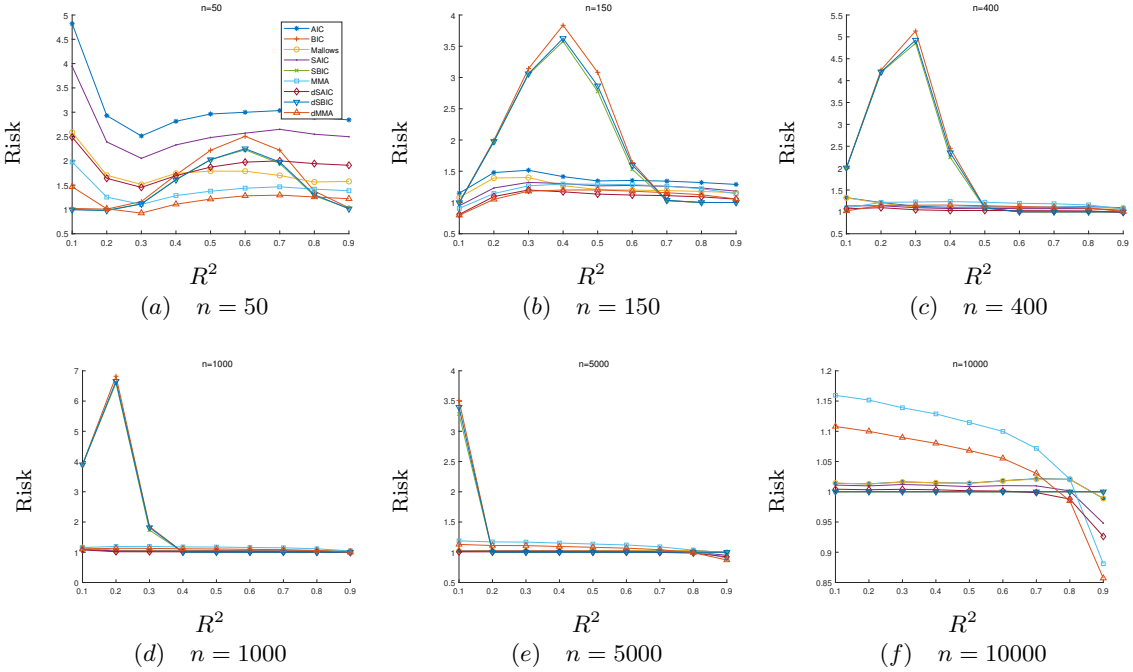
Figure 3: In-sample risk results with $\alpha = 1$ and $K = 2$ in Section 4.2.



(a) $n = 50$ (b) $n = 150$ (c) $n = 400$

(d) $n = 1000$ (e) $n = 5000$ (f) $n = 10000$

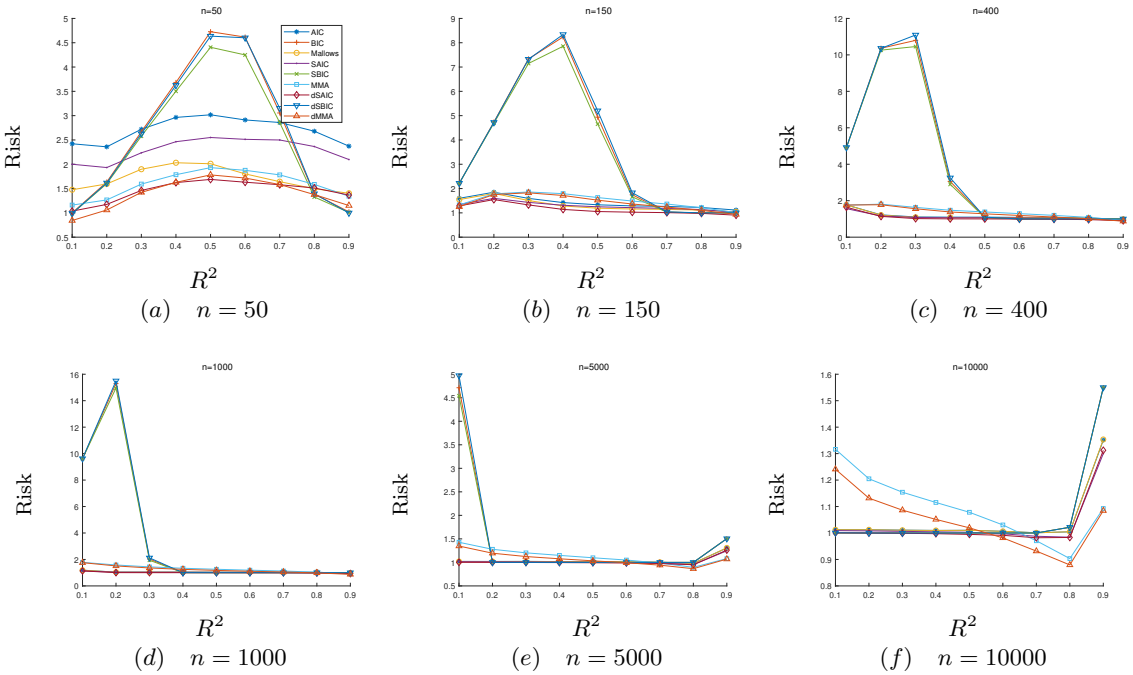
Figure 4: In-sample risk results with $\alpha = 1$ and $K = 5$ in Section 4.2.



(a) $n = 50$ (b) $n = 150$ (c) $n = 400$

(d) $n = 1000$ (e) $n = 5000$ (f) $n = 10000$

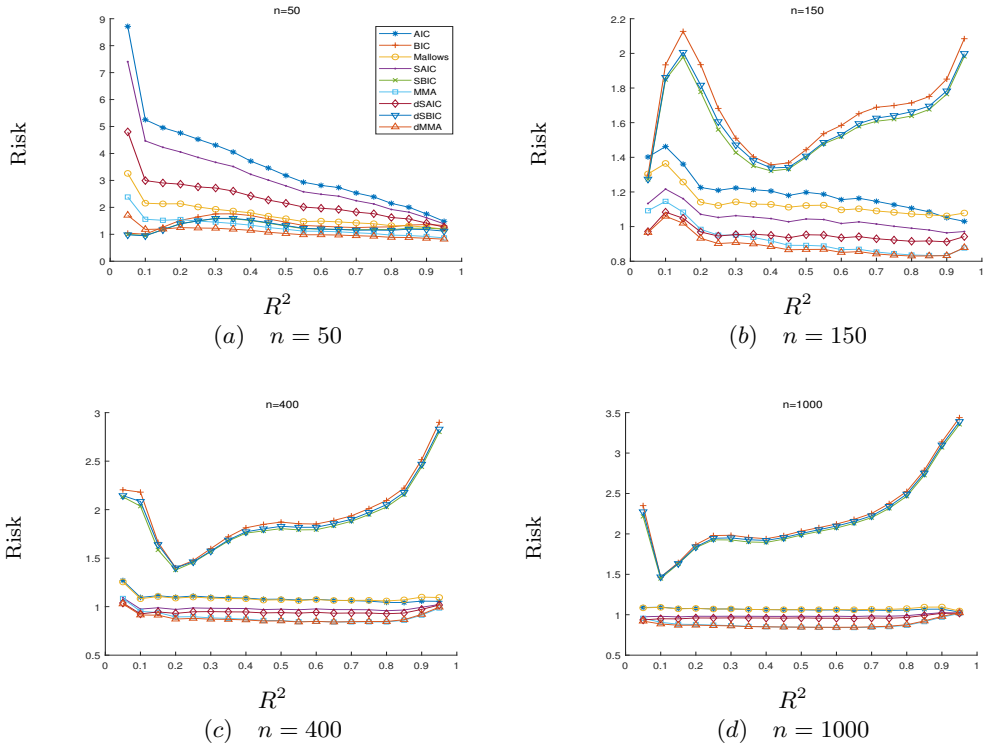
Figure 5: In-sample risk results with $\alpha = 1.5$ and $K = 2$ in Section 4.2.



(a) $n = 50$ (b) $n = 150$ (c) $n = 400$

(d) $n = 1000$ (e) $n = 5000$ (f) $n = 10000$

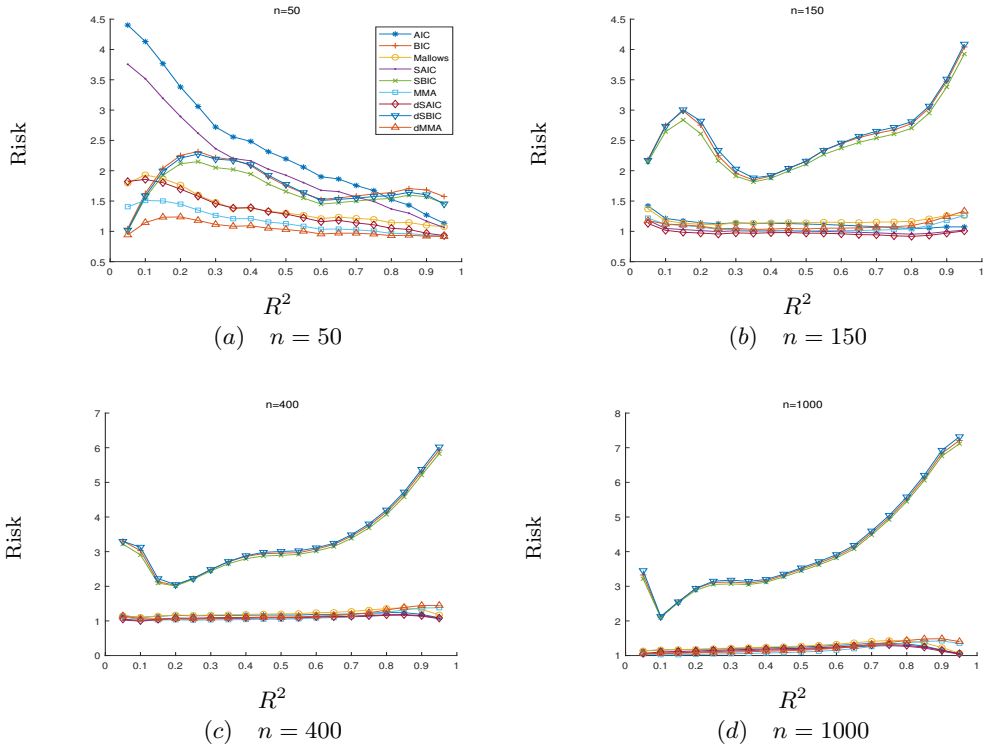
Figure 6: In-sample risk results with $\alpha = 1.5$ and $K = 5$ in Section 4.2.



(a) $n = 50$ (b) $n = 150$

(c) $n = 400$ (d) $n = 1000$

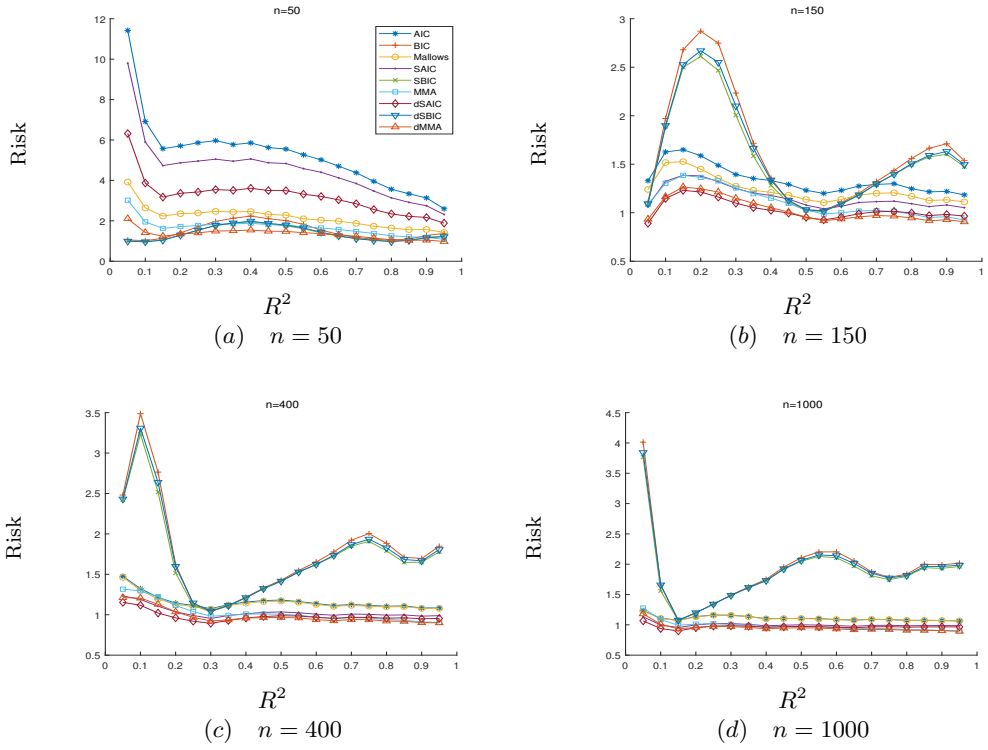
Figure 7: Out-of-sample risk results with $\alpha = 0.5$ and $K = 2$ in Section 4.3.



(a) $n = 50$ (b) $n = 150$

(c) $n = 400$ (d) $n = 1000$

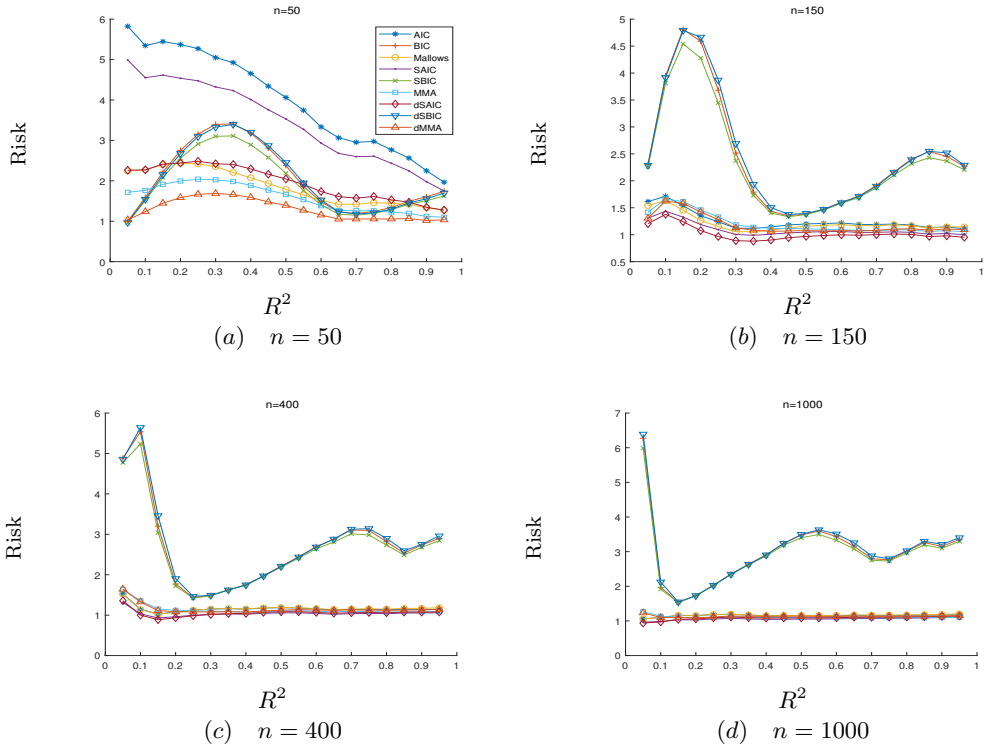
Figure 8: Out-of-sample risk results with $\alpha = 0.5$ and $K = 5$ in Section 4.3.



(a) $n = 50$ (b) $n = 150$

(c) $n = 400$ (d) $n = 1000$

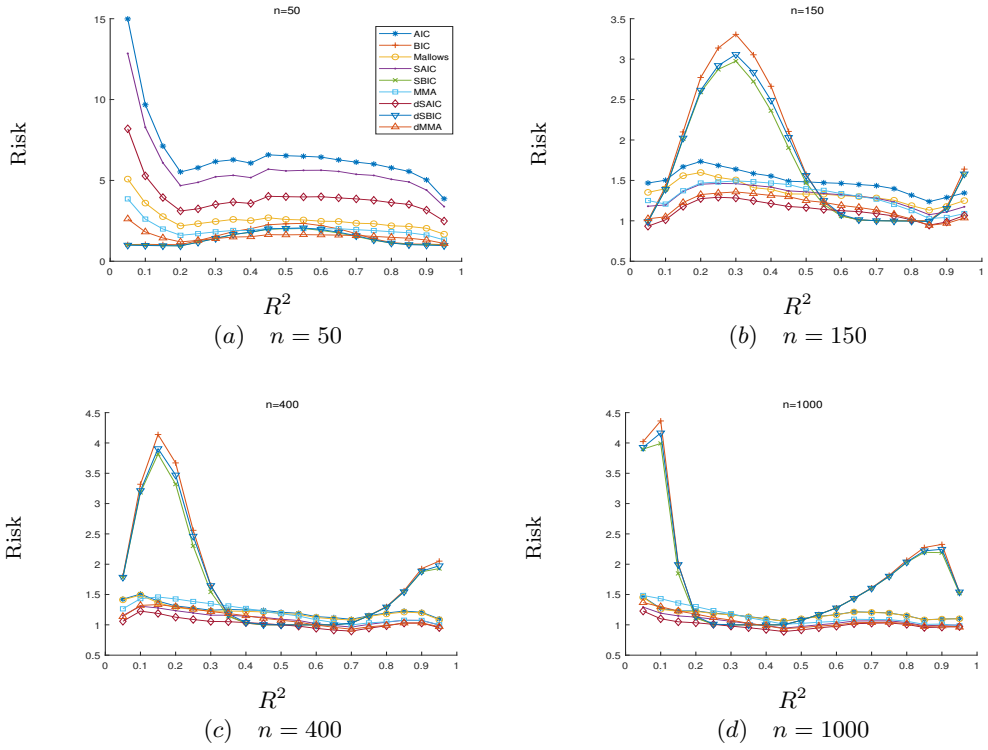
Figure 9: Out-of-sample risk results with $\alpha = 1$ and $K = 2$ in Section 4.3.



(a) $n = 50$ (b) $n = 150$

(c) $n = 400$ (d) $n = 1000$

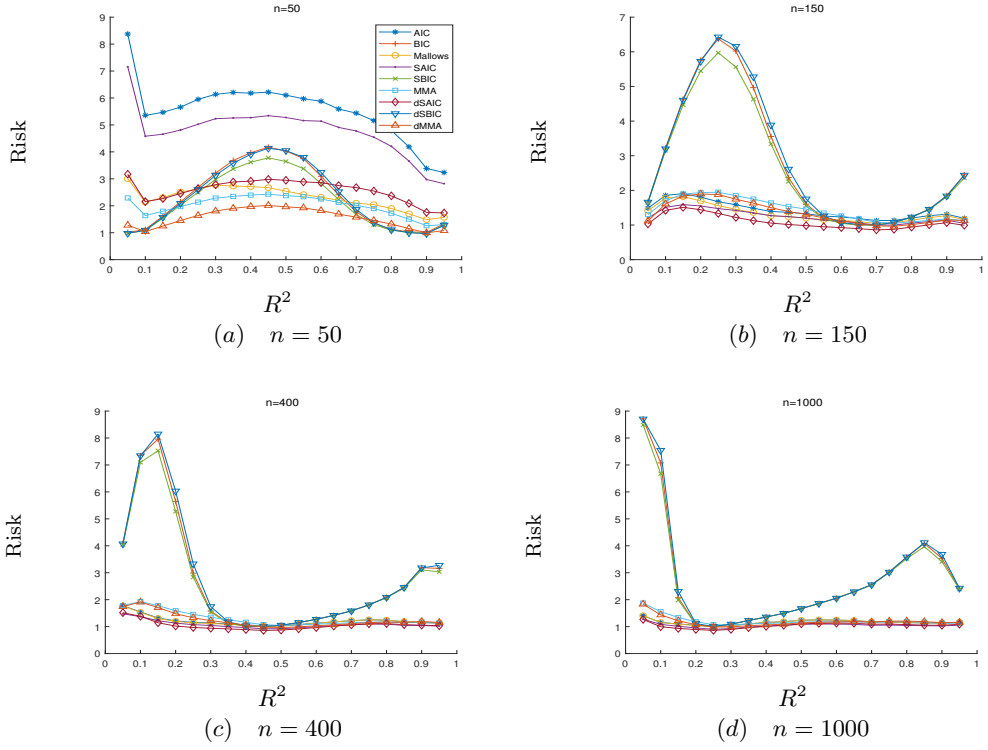
Figure 10: Out-of-sample risk results with $\alpha = 1$ and $K = 5$ in Section 4.3.



(a) $n = 50$ (b) $n = 150$

(c) $n = 400$ (d) $n = 1000$

Figure 11: Out-of-sample risk results with $\alpha = 1.5$ and $K = 2$ in Section 4.3.



(a) $n = 50$ (b) $n = 150$

(c) $n = 400$ (d) $n = 1000$

Figure 12: Out-of-sample risk results with $\alpha = 1.5$ and $K = 5$ in Section 4.3.