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Asymptotic Behavior of Least Squares Estimator for Nonlinear Autoregressive Models

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Appendix A Proofs of Theorems 1–2

It is obvious that to show Theorems 1–2, it suffices to prove

Proposition 1. Under Assumptions A1' and A2, let θ be a random variable independent of $\{w_t\}_{t\geqslant 1}$.Then, there is a constant $M_\phi > 0$ depending only on ϕ such that for any $C_\phi > M_\phi$ and $M, K > 0$,

$$\liminf_{t\to+\infty}\frac{\lambda_{\min}(t+1)}{N_t(M)} > 0 \quad \text{a.s. on } \Omega(M)\cap\{\|\theta\|\leqslant K\}. \tag{A1}$$

Appendix A.1 Proof of Proposition 1

Following the idea of [2], for every $x\in\mathbb{R}^m$ with $\|x\|=1$, we construct a set $\mathcal{S}\triangleq\prod_{i=1}^n\bigcup_{j=1}^{p_i}S_i^j(q)$ with disjoint open intervals $\{S_i^j(q): j=1,\ldots,p_i\}$ such that

$$\ell\left(\{y\in\mathcal{S}: |\phi^\tau(y)x|>0\}\right)>0 \quad \text{and} \quad \overline{\mathcal{S}}\subset\prod_{i=1}^n E_i. \tag{A2}$$

Define

$$U_x(\delta)\triangleq\{y: |\phi^\tau(y)x|>\delta\}\cap\mathcal{S},\quad \delta>0. \tag{A3}$$

Next, let $\{d_k\}_{k=1}^{2n}$ be a sequence of numbers and for $k\in[n+1,2n]$ define

$$\varsigma_k\triangleq d_k - x^\tau\phi(d_{k-1},\ldots,d_{k-n}),\quad x\in\mathbb{R}^m. \tag{A4}$$

Denote $y=(d_n,\ldots,d_1)^\tau$ and $\varsigma=(\varsigma_{2n},\ldots,\varsigma_{n+1})^\tau$. Evidently, (A4) implies that there is a function $g:\mathbb{R}^{2n+m}\to\mathbb{R}^n$ such that

$$(d_{2n},\ldots,d_{n+1})^\tau = g(\varsigma,y,x). \tag{A5}$$

We take δ in (A3) according to the following lemma.

Lemma 1. Under Assumption A2, the following two statements hold:
(i) given $y\in\mathbb{R}^n$, $x\in\mathbb{R}^m$ and a box $O=\prod_{i=1}^n I_i$ with $\{I_i\}_{i=1}^n$ being some intervals, then

$$\ell(\{\varsigma: g(\varsigma,y,x)\in O\})=\ell(O); \tag{A6}$$

(ii) for any constants $M,K>0$, there is a $\delta^*>0$ such that $\inf_{\|z\|=1,\|y\|\leqslant M,\|x\|\leqslant K}\ell(\{\varsigma:|\phi^\tau(g(\varsigma,y,x))z|>\delta^*, g(\varsigma,y,x)\in\mathcal{S}\})>0$.

Proof. (i) Note that in view of (A4), $d_k=\varsigma_k+o_{k-1}$, $k=n+1,\ldots,2n$, where $o_{k-1}\in\mathbb{R}$ is a point determined by ς_{k-1}, y and x (for $k=n+1$, ς_n does not exist and o_n depends only on y and x). So, $\{\varsigma:\varsigma+o_{k-1}\in I_k\}=I_k-o_{k-1}$ is an interval with length $|I_k|$. By the definition of the Lebesgue measure in $\mathbb{R}^n$, it is straightforward that $\ell(\{\varsigma: g(\varsigma,y,x)\in O\})=\prod_{k=1}^n|I_k|=\ell(O)$.
(ii) Suppose (ii) is false. Then for each integer $k\geqslant1$, we can take some $(z(k),y(k),x(k))$ with $\|z(k)\|=1$ in $\overline{B(0,1)}\times\overline{B(0,M)}\times\overline{B(0,K)}\subset\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m$ such that

$$\ell(\{\varsigma:|\phi^\tau(g(\varsigma,y(k),x(k)))z(k)|>\frac{1}{k}, g(\varsigma,y(k),x(k))\in\mathcal{S}\})<\frac{1}{k}. \tag{A7}$$

Hence there is a subsequence $\{z(k_r),y(k_r),x(k_r)\}_{r\geqslant1}$ and an accumulation point (z^*,y^*,x^*) satisfying

$$\lim_{r\to+\infty}(x(k_r),y(k_r),z(k_r))=(x^*,y^*,z^*). \tag{A8}$$

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Clearly, $\|z^*\| = 1$, $\|y^*\| \leqslant M$, $\|x^*\| \leqslant K$. If $\ell(\{\varsigma : |\phi^\tau(g(\varsigma, y^*, x^*))z^*| > 0, g(\varsigma, y^*, x^*) \in \mathcal{S}\}) = 0$, then $\phi^\tau(y)z^* \equiv 0$ for all $y \in \mathcal{S}$ due to (A4), (A5) and the continuity of ϕ. It contradicts to (A2). Consequently, for any $\{\mathcal{S}_k\}_{k\geqslant 1}$ satisfying $\mathcal{S}_k \subset \mathcal{S}_{k+1}$ and $\lim_{k\to+\infty} \mathcal{S}_k = \mathcal{S}$, we have

$$\begin{aligned}&\lim_{k\to+\infty} \ell(\{\varsigma : |\phi^\tau(g(\varsigma, y^*, x^*))z^*| > \frac{1}{k}, g(\varsigma, y^*, x^*) \in \mathcal{S}_k\})\\ =\ & \ell(\{\varsigma : |\phi^\tau(g(\varsigma, y^*, x^*))z^*| > 0, g(\varsigma, y^*, x^*) \in \mathcal{S}\}) > 0.\end{aligned}$$

Therefore, for some $h \geqslant 1$,

$$\ell(\{\varsigma : |\phi^\tau(g(\varsigma, y^*, x^*))z^*| > \frac{1}{h}, g(\varsigma, y^*, x^*) \in \mathcal{S}_h\}) > 0. \tag{A9}$$

Note that all points $\{y(k_r), x(k_r)\}_{r\geqslant 1}$ are restricted to $\overline{B(0,M)} \times \overline{B(0,K)}$, (A4) and (A5) then indicate that there is a compact set O' such that $\{\varsigma : g(\varsigma, y(k_r), x(k_r)) \in \mathcal{S}\} \subset O'$. Further, g and ϕ are continuous due to (A4), (A5) and Assumption A2(i), hence (A8) shows $\lim\limits_{r\to\infty} \sup\limits_{\varsigma\in O'} \|g(\varsigma, y^*, x^*) - g(\varsigma, y(k_r), x(k_r))\| = 0$ and $\lim\limits_{r\to\infty} \sup\limits_{\varsigma\in O'} \|\phi^\tau(g(\varsigma, y^*, x))z^* - \phi^\tau(g(\varsigma, y(k_r), x(k_r)))z(k_r)\| = 0$.

As a consequence, for all sufficiently large r,

$$\begin{aligned}&\ell(\{\varsigma : |\phi^\tau(g(\varsigma, y^*, x^*))z^*| > \frac{1}{h}, g(\varsigma, y^*, x^*) \in \mathcal{S}_h\})\\ <\ & \ell(\{\varsigma : |\phi^\tau(g(\varsigma, y(k_r), x(k_r)))z(k_r)| > \frac{1}{k_r}, g(\varsigma, y(k_r), x(k_r)) \in \mathcal{S}\}) < \frac{1}{k_r},\end{aligned}$$

which contradicts to (A9) by letting $r \to +\infty$. Lemma 1 follows. □

Remark 1. In Lemma 1, Assumption A2 can be weaken to Assumption A2' when $n = 1$. Statement (i) is trivial. For (ii), note that (A2) still holds by Assumption A2'. But, (A4), (A7) and (A9) yield that for all sufficiently large r,

$$\begin{aligned}\frac{1}{k_r}\ &>\ \ell(\{\varsigma : |\phi^\tau(g(\varsigma, y(k_r), x(k_r)))z(k_r)| > \frac{1}{k_r}, g(\varsigma, y(k_r), x(k_r)) \in \mathcal{S}\})\\ &=\ \ell(\{y : |\phi^\tau(y)z(k_r)| > \frac{1}{k_r}, y \in \mathcal{S}\}) \geqslant \ell(\{y \in \mathcal{S} : |\phi^\tau(y)z^*| > \frac{1}{k_r} + \frac{1}{h}\}),\end{aligned}$$

where $\{z(k_r), y(k_r), x(k_r)\}_{r\geqslant 1}$ is defined in the proof of Lemma 1. Letting $r \to +\infty$ in the above inequality infers

$$0\ \geqslant\ \lim_{r\to+\infty} \ell(\{y \in \mathcal{S} : |\phi^\tau(y)z^*| > \frac{1}{k_r} + \frac{1}{h}\}) = \ell(\{y \in \mathcal{S} : |\phi^\tau(y)z^*| > \frac{1}{h}\}),$$

which contradicts to (A9).

Fix two positve numbers M and K and let δ^* be constructed in Lemma 1(ii). Now, for every unit vector $x \in \mathbb{R}^m$, define $U_x \triangleq U_x(\delta^*)$.

For the next lemma, fix a closed box $O = \prod_{i=1}^n I_i \in \mathbb{R}^n$ and a positive integer r. Equally divide each I_i into r closed intervals $\{I_{i,j}\}_{j=1}^r$ so that $\mathrm{int}(I_{i,j}) \cap \mathrm{int}(I_{i,j'}) = \emptyset$ if $j \neq j'$. We thus obtain r^n small closed boxes $\prod_{i=1}^n \{I_{i,j}\}_{j=1}^r$, which are denoted by $\mathcal{T}(O, r)$. It is easy to see that for any distinct boxes $U, U' \in \mathcal{T}(O, r)$, $\mathrm{int}(U) \cap \mathrm{int}(U') = \emptyset$. Define

$$\mathcal{T}_\delta(O, r) \triangleq \left\{U \in \mathcal{T}(O, r) : \mathcal{B}(\delta) \cap \overline{\mathcal{S}} \cap U \neq \emptyset\right\}, \tag{A10}$$

where $\mathcal{B}(\delta) \triangleq \partial(\{y : \phi^\tau(y)x > \delta\})$. Let $\mathcal{K}_\delta(O, x, r) \triangleq |\mathcal{T}_\delta(O, r)|$.

Lemma 2. There is a constant $C > 0$ such that for any closed box $O = \prod_{i=1}^n I_i$, non-zero vector $x \in \mathbb{R}^m$, $\delta \in \mathbb{R}$ and integer $r \geqslant 1$,

$$\mathcal{K}_\delta(O, x, r) \leqslant Cr^{n-1}. \tag{A11}$$

Proof. Denote $A(g) \triangleq \{x : g(x) = 0\}$ for function g. For $i \in [1, n]$, let $(\phi^{(i)})' = (f'_{i1}, \ldots, f'_{im_i})^\tau$ and

$$\begin{cases} K_i = \mathrm{int}(A(x_i^\tau(\phi^{(i)})')) \cap \left(\bigcup_{j=1}^{p_i} \overline{S_i^j(q)}\right) \\ L_i = (A(x_i^\tau(\phi^{(i)})') \cap \left(\bigcup_{j=1}^{p_i} \overline{S_i^j(q)})\right) \setminus K_i \end{cases}. \tag{A12}$$

We prove (A11) by induction. For $n = 1$, let $O = I_1$ be a closed box. By [2, Lemma B.10], it is easy to check that

$$\left|\mathcal{B}(\delta) \cap \bigcup_{j=1}^{p_1} S_1^j(q)\right| \leqslant 2p_1(|L_1| + 2) < +\infty. \tag{A13}$$

Moreover, since $\mathcal{B}(\delta) \cap (\bigcup_{j=1}^{p_1} \overline{S_1^j(q)}) \subset \mathcal{B}(\delta) \cap (\bigcup_{j=1}^{p_1} S_1^j(q)) \cup \partial(\bigcup_{j=1}^{p_1} S_1^j(q))$, it follows that $\mathcal{K}_\delta(O, x, r) \leqslant 2|\mathcal{B}(\delta) \cap (\bigcup_{j=1}^{p_1} \overline{S_1^j(q)})| \leqslant 4p_1(|L_1| + 2) + 4p_1$. Hence, (A11) is true for $n = 1$ by taking $C = 4p_1(|L_1| + 2) + 4p_1$.

Now, suppose (A11) holds for $n = k$ with some $k \geqslant 1$. Let us consider the case where $n = k + 1$. Take a closed box $O = \prod_{i=1}^{k+1} I_i \in \mathbb{R}^{k+1}$, and let $\mathcal{T}(O, r)$ be the set of the r^{k+1} disjoint refined boxes. These boxes correspond to two sets

$$\mathcal{T}^1 = \prod_{i=1}^k \{I_{i,j}\}_{j=1}^r \quad \text{and} \quad \mathcal{T}^2 = \{I_{k+1,j}\}_{j=1}^r.$$

Write vector $x = \mathrm{col}\{x_1, \ldots, x_{k+1}\} \neq \mathbf{0}$. First, assume there is an index $l \in [1, k+1]$ such that $x_l = \mathbf{0}$. Without loss of generality, let $l = k+1$, then

$$\begin{aligned}&\mathcal{B}(\delta) \cap \prod_{i=1}^{k+1} \bigcup_{j=1}^{p_i} \overline{S_i^j(q)} \cap O \\ \subset\ & \left(\partial\left(\left\{z \in \mathbb{R}^k : \sum_{i=1}^{k} x_i \phi^{(i)}(z_i) > \delta\right\}\right) \cap \prod_{i=1}^{k} \bigcup_{j=1}^{p_i} \overline{S_i^j(q)} \cap \prod_{i=1}^{k} I_i\right) \times I_{k+1}. \end{aligned} \tag{A14}$$

where $z = (z_1, \ldots, z_k)^\tau \in \mathbb{R}^k$. By applying the induction assumption for $n = k$ and for the refined boxes in $\mathcal{T}^1$, there is a constant $C > 0$ such that $\mathcal{K}_\delta\left(\prod_{i=1}^{k} I_i, \mathrm{col}\{x_1, \ldots, x_k\}, r\right) \leqslant Cr^{k-1}$, which, together with (A14) and $\mathcal{T}(O, a) = \mathcal{T}^1 \times \mathcal{T}^2$, yields $\mathcal{K}_\delta(O, x, r) \leqslant Cr^k$. This is exactly (A11) for $n = k+1$.

So, let $x_i \neq \mathbf{0}$ for all $i \in [1, k+1]$. For any $B \in \mathcal{T}^1$, define set

$$Z(B) \triangleq \{z_{k+1} \in I_{k+1} : (B \times z_{k+1}) \cap \mathcal{B}(\delta) \cap \prod_{i=1}^{k+1} \bigcup_{j=1}^{p_i} \overline{S_i^j(q)} \neq \emptyset\}.$$

Observe that $Z(B)$ is a closed set, then $\partial Z(B) \subset Z(B)$. Define

$$\begin{cases} \mathcal{Z}_1(B) \triangleq \{I_{k+1,j} \in \mathcal{T}^2 : Z(B) \cap I_{k+1,j} \neq \emptyset\} \\ \mathcal{Z}_2(B) \triangleq \{I_{k+1,j} \in \mathcal{T}^2 : \partial Z(B) \cap I_{k+1,j} \neq \emptyset\} \end{cases}.$$

Since any interval in $\mathcal{Z}_1(B) \setminus \mathcal{Z}_2(B)$ must be contained in $Z(B)$,

$$|\mathcal{Z}_1(B)| - |\mathcal{Z}_2(B)| = |\mathcal{Z}_1(B) \setminus \mathcal{Z}_2(B)| \leqslant \frac{r}{|I_{k+1}|} \ell(Z(B)).$$

At the same time, $\sum_{B \in \mathcal{T}^1} \ell(Z(B)) = \sum_{B \in \mathcal{T}^1} \int_{\mathbb{R}} I_{Z(B)} dz_{k+1} = \int_{I_{k+1}} \sum_{B \in \mathcal{T}^1} I_{Z(B)} dz_{k+1}$, therefore

$$\mathcal{K}_\delta(O, x, r) = \sum_{B \in \mathcal{T}^1} |\mathcal{Z}_1(B)| \leqslant \frac{r}{|I_{k+1}|} \int_{I_{k+1}} \sum_{B \in \mathcal{T}^1} I_{Z(B)} dz_{k+1} + \sum_{B \in \mathcal{T}^1} |\mathcal{Z}_2(B)|. \tag{A15}$$

The last step is to estimate the term in (A15). Since the argument is involved, it is included in Appendix A.2. In light of Lemmas 5 and 6, when $n = k+1$, there are two constants $C_1, C_2 > 0$ depending only on ϕ such that $\mathcal{K}_\delta(O, x, r) \leqslant (C_1 + C_2)\, r^k$. The proof is thus completed. □

By applying Lemma 2, we can find a constant $C_0 > 0$ depending only on ϕ such that

$$|\{U \in \mathcal{T}(O, r) : \partial(U_x) \cap U \neq \emptyset\}| \leqslant C_0 r^{n-1}. \tag{A16}$$

Now, we estimate the frequency of $\{Y_t\}_{t \geqslant 1}$, where $Y_i \triangleq (y_{i+n-1}, \ldots, y_i)^\tau$, falling into U_x. For this, define a random process g_x by

$$g_x(i) \triangleq I_{\{Y_i \in U_x\}} - P(Y_i \in U_x | \mathcal{F}_{i-1}^y), \quad i \geqslant 1,$$

where $\mathcal{F}_{i-1}^y \triangleq \sigma\{\theta, y_0, \ldots, y_{i-1}\}$. By modifying the proof of [2, Lemma B.12] slightly, we can obtain

Lemma 3. For any $\epsilon > 0$, there is a class $\mathcal{G}_\epsilon$ such that
(i) each element of $\mathcal{G}_\epsilon$, denoted by g_ϵ, is a random series $\{g_\epsilon(i)\}_{i \geqslant 1}$ with the form

$$g_\epsilon(i) = I_{\{Y_i \in U_\epsilon\}} - P(Y_i \in U_\epsilon | \mathcal{F}_{i-1}^y) - \epsilon, \quad i \geqslant 1, \tag{A17}$$

where U_ϵ is a set in $\mathbb{R}^n$;
(ii) $\mathcal{G}_\epsilon$ contains a lower process g_ϵ to each g_x in the sense that

$$g_\epsilon(i) \leqslant g_x(i) \quad \forall i \geqslant 1. \tag{A18}$$

Proof of Proposition 1. First, recall the definition of U_x, for any $x \in \mathbb{R}^m$ with $\|x\| = 1$, Lemma 1(ii) and Assumption A1' yield

$$\begin{aligned} &P(Y_i \in U_x | \mathcal{F}_{i-1}^y) I_{\{\|Y_{i-n}\| \leqslant M, \|\theta\| \leqslant K\}} = P(Y_i \in \{y : |\phi^\tau(y) x| > \delta^*\} \cap \mathcal{S} | \mathcal{F}_{i-1}^y) \cdot I_{\{\|Y_{i-n}\| \leqslant M, \|\theta\| \leqslant K\}} \\ \geqslant\ & \inf_{\|x\|=1, \|y\| \leqslant M, \|z\| \leqslant K} \ell(\{\varsigma : |\phi^\tau(g(\varsigma, y, z)) x| > \delta^*, g(\varsigma, y, z) \in \mathcal{S}\}) \cdot \left(\inf_{s \in [-S', S']} \rho(s)\right)^n I_{\{\|Y_{i-n}\| \leqslant M, \|\theta\| \leqslant K\}} \\ \triangleq\ & C_P I_{\{\|Y_{i-n}\| \leqslant M, \|\theta\| \leqslant K\}}, \end{aligned} \tag{A19}$$

where $S' = K \sup_{\|y\| \leqslant M + R'} \|\phi(y)\| + R'$ and $R' \triangleq \max_{1 \leqslant i \leqslant n} \mathrm{dist}\left(0, \bigcup_{j=1}^{p_i} S_i^j(q)\right)$.

Next, note that for any $\epsilon > 0$ and $g_\epsilon \in \mathcal{G}_\epsilon$, $\{g_\epsilon(i) + \epsilon, \mathcal{F}_i^y\}_{i \geqslant 1}$ is a martingale difference sequence. Taking account of [1, Theorem 2.8],

$$\lim_{t \to +\infty} \frac{\sum_{i=1}^{t} I_{\{\|Y_{i-n}\| \leqslant M\}} (g_\epsilon(i) + \epsilon)}{N_t(M)} = 0, \quad \text{a.s. } on \quad \Omega(M),$$

where $\Omega(M)$ is defined in Theorem 1. Thanks to the finite number of U_ϵ constrained in $\mathcal{S}$, it gives

$$\lim_{t\to+\infty}\inf_{U_\epsilon\subset\mathcal{S}}\frac{1}{N_t(M)}\sum_{i=1}^{t} I_{\{\|Y_{i-n}\|\leqslant M\}}g_\epsilon(i)=-\epsilon,\quad \text{a.s. on}\quad \Omega(M).$$

As a result, Lemma 3(ii) infers that for some $g_\epsilon^x\in\mathcal{G}_\epsilon$,

$$\begin{aligned}\liminf_{t\to+\infty}\inf_{\|x\|=1}\frac{1}{N_t(M)}\sum_{i=1}^{t} I_{\{\|Y_{i-n}\|\leqslant M\}}g_x(i) &\geqslant \liminf_{t\to+\infty}\inf_{\|x\|=1}\frac{1}{N_t(M)}\sum_{i=1}^{t} I_{\{\|Y_{i-n}\|\leqslant M\}}g_\epsilon^x(i)\\ &\geqslant \liminf_{t\to\infty}\inf_{U_\epsilon\subset\mathcal{S}}\frac{1}{N_t(M)}\sum_{i=1}^{t} I_{\{\|Y_{i-n}\|\leqslant M\}}g_\epsilon(i)\\ &= -\epsilon,\quad \text{a.s. on}\quad \Omega(M).\end{aligned}$$

Further, by the arbitrariness of ϵ, we obtain that on $\Omega(M)$

$$\liminf_{t\to+\infty}\inf_{\|x\|=1}\frac{1}{N_t(M)}\sum_{i=1}^{t} I_{\{\|Y_{i-n}\|\leqslant M\}}g_x(i)\geqslant 0\quad \text{a.s.}. \tag{A20}$$

Finally, by (A19)–(A20), for sufficiently small ϵ, there is a positive random integer T such that for any unit vector $x\in\mathbb{R}^m$ and all $t>T$, $\frac{1}{N_t(M)}\sum_{i=1}^t I_{\{\|Y_{i-n}\|\leqslant M\}}I_{\{Y_i\in U_x\}}>\frac{1}{N_t(M)}\sum_{i=1}^t I_{\{\|Y_{i-n}\|\leqslant M\}}P(Y_i\in U_x|\mathcal{F}_{i-1}^y)-\frac{C_P}{2}\geqslant\frac{C_P}{2}$, a.s. on $\Omega(M)\cap\{\|\theta\|\leqslant K\}$. Hence, we select C_ϕ satisfies $C_\phi>nR'$ and $U_x\subset\overline{B(0,C_\phi)}$, for sufficiently large t,

$$\begin{aligned}\lambda_{\min}(t+1) &= \inf_{\|x\|=1}x^\tau\left(I_m+\sum_{i=0}^{t}\phi_i\phi_i^\tau\right)x\\ &\geqslant \sum_{i=1}^{t-n+1}I_{\{Y_i\in U_x\}}(\phi^\tau(Y_i)x)^2\geqslant(\delta^*)^2\sum_{i=1}^{t-n+1}I_{\{Y_i\in U_x\}}\\ &\geqslant \frac{(\delta^*)^2C_P}{2}(N_t(M)-n),\quad \text{a.s. on}\quad \Omega(M)\cap\{\|\theta\|\leqslant K\}.\end{aligned}$$

Proposition 1 is thus proved. □

Appendix A.2 Proof of (A15)

In this section, we follow the definitions and symbols in the proof of Lemma 2 and complete the estimation details of (A15). To this end, define $\mathbb{S}_i\triangleq\bigcup_{j=1}^{p_i}\overline{S_i^j(q)}$, $i\in[1,n]$,

$$\begin{aligned}I_{k+1}^* &\triangleq \left\{z_{k+1}:\left(\prod_{i=1}^{k}I_i\times z_{k+1}\right)\cap\mathcal{B}(\delta)\cap\left(\prod_{i=1}^{k}K_i\times z_{k+1}\right)\neq\emptyset\right\}\\ &\quad\cap I_{k+1}\cap\left(\bigcup_{j=1}^{p_{k+1}}\overline{S_{k+1}^j(q)}\right),\quad k\geqslant 1\\ \mathcal{T}^3 &\triangleq \{A\in\mathcal{T}^2: A\cap I_{k+1}^*\neq\emptyset\},\\ \mathcal{T}^4 &\triangleq \left\{B\in\mathcal{T}^1:\bigcup_{i=1}^{k}\{z:z_i\in L_i\}\cap B\neq\emptyset\right\},\end{aligned}$$

where $\prod_{i=1}^{k+1}I_i=O$ is the given closed box in the proof of Lemma 2.

Lemma 4. The cardinals of $I_{k+1}^*,\mathcal{T}^3$ and $\mathcal{T}^4$ are bounded by

$$|I_{k+1}^*| \leqslant (2p_{k+1}(|L_{k+1}|+2)+2)\prod_{i=1}^{k}(|L_i|+p_i), \tag{A21}$$

$$|\mathcal{T}^3| \leqslant 2(2p_{k+1}(|L_{k+1}|+2)+2)\prod_{i=1}^{k}(|L_i|+p_i),$$

$$|\mathcal{T}^4| \leqslant 2r^{k-1}\sum_{i=1}^{k}|L_i|, \tag{A22}$$

Proof. By the definitions of $\mathcal{T}^3$ and $\mathcal{T}^4$, $\mathcal{T}^3\leqslant 2|I_{k+1}^*|$ and (A22) is trivial. So, it suffices to show (A21). For this, recall the definitions of K_i and L_i, then for each $i\in[1,n]$, there is a set $\mathcal{P}_i$ consisting of some disjoint intervals such that $|\mathcal{P}_i|\leqslant|L_i|+p_i$ and $\bigcup_{I\in\mathcal{P}_i}I=K_i$. As a result, $|\prod_{i=1}^k\mathcal{P}_i|\leqslant\prod_{i=1}^k(|L_i|+p_i)$. For each box $B\in\prod_{i=1}^k\mathcal{P}_i$, denote $I_{k+1}^*(B)=\{z_{k+1}:(\prod_{i=1}^kI_i\times z_{k+1})\cap\mathcal{B}(\delta)\cap(B\times z_{k+1})\neq\emptyset\}\cap I_{k+1}\cap\mathbb{S}_{k+1}$. Since $B\subset\prod_{i=1}^kK_i$, it is evident that

$$\sum_{i=1}^{k}x_i^\tau\phi^{(i)}\equiv\text{constant}\qquad\text{on } B. \tag{A23}$$

So, for any $z_{k+1} \in I_{k+1}^*(B)$, arbitrarily taking a $(z_1, \ldots, z_k)^\tau \in \mathrm{int}(B)$ infers $(z_1, \ldots, z_{k+1})^\tau \in \mathcal{B}(\delta)$. Let $\{(z_{1,j}, \ldots, z_{k+1,j})^\tau\}_{j=1}^{+\infty}$ be a sequence of points in $(\mathrm{int}(B) \times E_{k+1}) \cap \{y : \phi^\tau(y)x > \delta\}$ and tend to $(z_1, \ldots, z_{k+1})^\tau$. Then, $\lim_{j\to+\infty} \|z_{k+1,j} - z_{k+1}\| = 0$ and

$$x_{k+1}^\tau \phi^{(k+1)}(z_{k+1,j}) \ > \ \delta - \sum_{i=1}^{k} x_i^\tau \phi^{(i)}(z_{i,j}) = \delta - \sum_{i=1}^{k} x_i^\tau \phi^{(i)}(z_i). \tag{A24}$$

Denote $\bar{\delta} = \delta - \sum_{i=1}^k x_i^\tau \phi^{(i)}(z_i)$, so (A24) implies $z_{k+1} \in \partial(\{z : x_{k+1}^\tau \phi^{(k+1)}(z) > \bar{\delta}\}) \cap \mathbb{S}_{k+1}$, Therefore, applying Lemma A.3(iii), $|I_{k+1}^*(B)| \leqslant |\partial(\{z : x_{k+1}^\tau \phi^{(k+1)}(z) > \bar{\delta}\}) \cap \mathbb{S}_{k+1}| \leqslant 2p_{k+1}(|L_{k+1}| + 2) + 2$, and thus $|I_{k+1}^*| \leqslant (2p_{k+1}(|L_{k+1}| + 2) + 2) \left|\prod_{i=1}^k \mathcal{P}_i\right| \leqslant (2p_{k+1}(|L_{k+1}| + 2) + 2) \prod_{i=1}^k (|L_i| + p_i)$, which completes the proof. □

Lemma 5. Let Lemma 2 hold with $n = k$. Then, there is a constant $C_1 > 0$ depending only on ϕ such that

$$\frac{r}{|I_{k+1}|} \int_{I_{k+1}} \sum_{B \in \mathcal{T}^1} I_{\mathcal{Z}(B)} dz_{k+1} \leqslant C_1 r^k. \tag{A25}$$

Proof. Denote $\phi' = \mathrm{col}\{\phi^{(1)}, \ldots, \phi^{(k)}\}$, $x' = \mathrm{col}\{x_1, \ldots, x_k\}$ and $z = (z_1, \ldots, z_k)^\tau$. Given $z_{k+1} \in I_{k+1}$, define $\delta' \triangleq \delta - \phi^{(k+1)}(z_{k+1})x_{k+1}$. Then,

$$\begin{aligned} &\{z : (z_1, \ldots, z_{k+1})^\tau \in \mathcal{B}(\delta)\} \cap \prod_{i=1}^k A^c(x_i^\tau (\phi^{(i)})') \cap \prod_{i=1}^k \mathbb{S}_i \\ = \ &\partial(\{z : (\phi')^\tau(z)x' > \delta'\}) \cap \prod_{i=1}^k A^c(x_i^\tau (\phi^{(i)})') \cap \prod_{i=1}^k \mathbb{S}_i. \end{aligned}$$

In addition, by the definition of $\{L_i, K_i\}_{i=1}^n$ in (A12), $(\prod_{i=1}^k A^c(x_i^\tau (\phi^{(i)})'))^c = (\bigcup_{i=1}^k \{z : z_i \in L_i\}) \cup \prod_{i=1}^k K_i$, so we arrive at

$$\begin{aligned} &\{z : (z_1, \ldots, z_{k+1})^\tau \in \mathcal{B}(\delta)\} \cap \left(\prod_{i=1}^k \bigcup_{j=1}^{p_i} \overline{S_i^j(q)}\right) \\ \subset \ &\partial(\{z : (\phi')^\tau(z)x' > \delta'\}) \cap \prod_{i=1}^k \mathbb{S}_i \cup \bigcup_{i=1}^k \{z : z_i \in L_i\} \cup \prod_{i=1}^k K_i. \end{aligned}$$

Consequently, for any $z_{k+1} \in A \in \mathcal{T}^2 \setminus \mathcal{T}^3$ and $B \in \mathcal{T}^1 \setminus \mathcal{T}^4$,

$$\{z : (z_1, \ldots, z_{k+1})^\tau \in \mathcal{B}(\delta)\} \cap \prod_{i=1}^k \mathbb{S}_i \cap B \subset \partial(\{z : (\phi')^\tau(z)x' > \delta'\}) \cap \prod_{i=1}^k \mathbb{S}_i \cap B.$$

Now, for $\partial(\{z : (\phi')^\tau(z)x' > \delta'\}) \cap \prod_{i=1}^k \mathbb{S}_i$ and $\mathcal{T}^1$, applying Lemma 2 with $n = k$ leads to

$$\sum_{B \in \mathcal{T}^1 \setminus \mathcal{T}^4} I_{Z(B)}(z_{k+1}) \leqslant Cr^{k-1}. \tag{A26}$$

Based on (A26), it is readily to compute

$$\begin{aligned} &\int_{I_{k+1}} \sum_{B \in \mathcal{T}^1} I_{\mathcal{Z}(B)} dz_{k+1} = \sum_{A \in \mathcal{T}^2} \int_A \sum_{B \in \mathcal{T}^1} I_{\mathcal{Z}(B)} dz_{k+1} \\ \leqslant \ &\sum_{A \in \mathcal{T}^2 \setminus \mathcal{T}^3} \int_A \sum_{B \in \mathcal{T}^1} I_{\mathcal{Z}(B)} dz_{k+1} + \sum_{A \in \mathcal{T}^3} \int_A r^k dz_{k+1} \\ = \ &\sum_{A \in \mathcal{T}^2 \setminus \mathcal{T}^3} \int_A \sum_{B \in \mathcal{T}^1 \setminus \mathcal{T}^4} I_{\mathcal{Z}(B)} dz_{k+1} + \sum_{A \in \mathcal{T}^2 \setminus \mathcal{T}^3} \int_A \sum_{B \in \mathcal{T}^4} I_{\mathcal{Z}(B)} dz_{k+1} + r^k \cdot \frac{|I_{k+1}|}{r} \cdot |\mathcal{T}^3| \\ \leqslant \ &\int_{I_{k+1}} Cr^{k-1} dz_{k+1} + \sum_{B \in \mathcal{T}^4} \int_{I_{k+1}} 1 dz_{k+1} + r^{k-1}|I_{k+1}||\mathcal{T}^3| \\ \leqslant \ &((C + |\mathcal{T}^3|)r^{k-1} + |\mathcal{T}^4|)|I_{k+1}|. \end{aligned}$$

The result follows from Lemmas 4 and A.3(ii). □

Lemma 6. There is a constant $C_2 > 0$ depends only on ϕ such that $\sum_{B \in \mathcal{T}^1} |\mathcal{Z}_2(B)| \leqslant C_2 r^k$.

Proof. Let

$$\mathcal{T}^5 \triangleq \left\{\prod_{i=1}^k I_i' \in \mathcal{T}^1 : \partial\left(\bigcup_{j=1}^{p_i} S_i^j(q)\right) \cap I_i' \neq \emptyset \text{ for some } i \in [1, k]\right\}.$$

Clearly, $|\mathcal{T}^5| \leqslant 4r^{k-1} \sum_{i=1}^k p_i$. Hence,

$$\sum_{B \in \mathcal{T}^1} |\mathcal{Z}_2(B)| \leqslant \sum_{B \in \mathcal{T}^1 \setminus (\mathcal{T}^5 \cup \mathcal{T}^4)} |\mathcal{Z}_2(B)| + r|\mathcal{T}^4| + 4r^k \sum_{i=1}^k p_i. \tag{A27}$$

It suffices to estimate the first term in the right hand side of (A27). To this end, take a set $B=\prod_{i=1}^{k} I_i' \in \mathcal{T}^1 \setminus (\mathcal{T}^5 \cup \mathcal{T}^4)$ and let $z_{k+1} \in \partial Z(B) \cap \mathrm{int}(I_{k+1})$. Select a point $(z_1,\ldots,z_k)^\tau \in B$ that

$$
\begin{aligned}
& \mathrm{dist}((z_1,\ldots,z_{k+1})^\tau, \prod_{i=1}^{k} \partial(I_i') \times z_{k+1}) \\
= & \min_{y \in \mathcal{B}(\delta) \cap \prod_{i=1}^{k+1} \mathbb{S}_i \cap (B \times z_{k+1})} \mathrm{dist}(y, \prod_{i=1}^{k} \partial(I_i') \times z_{k+1}).
\end{aligned} \tag{A28}
$$

Clearly, $B \in \mathcal{T}^1 \setminus (\mathcal{T}^5 \cup \mathcal{T}^4)$ implies that for each $i=1,\ldots,k$, $\mathrm{int}(I_i') \subset \bigcup_{j=1}^{p_i} S_i^j(q)$ and $\mathrm{int}(I_i') \cap L_i = \emptyset$. We consider the following two cases:

Case 1: $(z_1,\ldots,z_k)^\tau \notin \prod_{i=1}^{k} \partial(I_i')$. Then, there is an integer $i \in [1,k]$ such that $z_i \in \mathrm{int}(I_i')$. By (A28), $z_i \notin K_i \cap \mathrm{int}(I_i')$. Otherwise, there is a $\rho > 0$ such that $x_i^\tau(\phi^{(i)})' \equiv 0$ on $[z_i-\rho, z_i+\rho] \subset \mathrm{int}(I_i')$. Similar to (A23)–(A24), for any $z_i' \in [z_i-\rho, z_i+\rho]$, $(z_1,\ldots,z_{i-1},z_i',z_{i+1},\ldots,z_{k+1})^\tau \in \mathcal{B}(\delta) \cap \prod_{i=1}^{k+1} \mathbb{S}_i \cap (B \times z_{k+1})$. Then, $\min\{\mathrm{dist}((z_1,\ldots,z_{i-1},z_i-\rho,z_{i+1},\ldots,z_{k+1})^\tau, \prod_{i=1}^{k} \partial(I_i') \times z_{k+1}), \mathrm{dist}((z_1,\ldots,z_{i-1},z_i+\rho,z_{i+1},\ldots,z_{k+1})^\tau, \prod_{i=1}^{k} \partial(I_i') \times z_{k+1})\} < \mathrm{dist}((z_1,\ldots,z_{k+1})^\tau, \prod_{i=1}^{k} \partial(I_i') \times z_{k+1})$, which contradicts to (A28).

Now, since $z_i \notin K_i \cap \mathrm{int}(I_i')$ and $B \notin \mathcal{T}^4$, it yields that $x_i^\tau(\phi^{(i)})'(z_i) \neq 0$. We claim

$$
z_{k+1} \in \bigcup_{j=1}^{p_{k+1}} \partial(S_{k+1}^j(q)). \tag{A29}
$$

Otherwise, $z_{k+1} \in \bigcup_{j=1}^{p_{k+1}} S_{k+1}^j(q)$. By the *Implicit function theorem*, there is a sufficiently small $\eta > 0$ such that for any $z_{k+1}' \in (z_{k+1}-\eta, z_{k+1}+\eta)$, a point $z_i' \in \mathrm{int}(I_i)$ exists and $(z_1,\ldots,z_{i-1},z_i',z_{i+1},\ldots,z_k,z_{k+1}')^\tau \in \mathcal{B}(\delta) \cap \prod_{i=1}^{k+1} \mathbb{S}_i$. This means $z_{k+1} \in \mathrm{int}(Z(B))$, which is impossible due to $z_{k+1} \in \partial Z(B)$. Hence (A29) holds.

Case 2: $(z_1,\ldots,z_k)^\tau \in \prod_{i=1}^{k} \partial(I_i')$. Since $z_{k+1} \in \partial(Z(B))$, $x_{k+1}^\tau \phi^{(k+1)}$ cannot be a constant on any neighbourhood of z_k. So,

$$
z_{k+1} \in \partial(\{z : x_{k+1}^\tau \phi^{(k+1)}(z) \neq \bar{\delta}\}) \cap \left(\bigcup_{j=1}^{p_{k+1}} S_{k+1}^j(q) \right) \cup \left(\bigcup_{j=1}^{p_{k+1}} \partial(S_{k+1}^j(q)) \right), \tag{A30}
$$

where $\bar{\delta} = \delta - \sum_{i=1}^{k} x_i^\tau \phi^{(i)}(z_i)$.

Combining the above two cases, $z_{k+1} \in \partial(Z(B)) \cap \mathrm{int}(I_{k+1})$ implies (A30). Taking the case $z_{k+1} \in \partial(I_{k+1})$ into consideration, we obtain

$$
\partial(Z(B)) \subset \partial(\{y \in \mathbb{R} : x_{k+1}^\tau \phi^{(k+1)}(y) \neq \bar{\delta}\}) \cap \left(\bigcup_{j=1}^{p_{k+1}} S_{k+1}^j(q) \right) \cup \left(\bigcup_{j=1}^{p_{k+1}} \partial(S_{k+1}^j(q)) \right) \cup \partial(I_{k+1}),
$$

which, together with the fact $|\partial(\{z : x_{k+1}^\tau \phi^{(k+1)}(z) \neq \bar{\delta}\}) \cap (\bigcup_{j=1}^{p_{k+1}} S_i^j(q))| \leqslant 4p_{k+1}(|L_{k+1}|+2)$ from (A13), leads to $|\mathcal{Z}_2(B)| \leqslant 2|\partial(Z(B))| \leqslant 8p_{k+1}(|L_{k+1}|+2) + 4p_{k+1} + 4$. Now, in view of (A27), we derive

$$
\sum_{B \in \mathcal{T}^1} |\mathcal{Z}_2(B)| \leqslant (8p_{k+1}(|L_{k+1}|+2) + 4p_{k+1} + 4)r^k + |\mathcal{T}^4| r + 4r^k \sum_{i=1}^{k} p_i,
$$

which yields the result by Lemma 4. □

References

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