MATH2015 Tut 6.5

1a) $M = \mathbb{Z},\ a*b = a-b$

Closure: $\forall a,b \in \mathbb{Z},\ a-b \in \mathbb{Z}$, so $*$ closed in $\mathbb{Z}$.

Commutative: $a-b \neq b-a\ \forall a,b \in \mathbb{Z}$ so $*$ is not commutative in $\mathbb{Z}$.

Associative: $(a-b)-c \neq a-(b-c)$ so $*$ is not associative in $\mathbb{Z}$

Unity: $a-0 \neq 0-a$ so 0 is not unity for $*$ in $\mathbb{Z}$

Also since $a-c \neq c-a$ then $2a = 2c \Rightarrow a = c$ for each $a \in \mathbb{Z}$ so no unity in $\mathbb{Z}$ under $*$

Units: No unity, hence no units.

b) $M = \mathbb{R},\ a*b = a+b-ab$

Closure: $\forall a,b \in \mathbb{R},\ a+b-ab \in \mathbb{R}$, so $*$ closed in $\mathbb{R}$.

Commutative: $a*b = a+b-ab = b+a-ba$ since $\mathbb{R}$ is commutative

$= b*a \in \mathbb{R}$ so $*$ is not commutative in $\mathbb{R}$.

Associative:
$$
\begin{aligned}
(a*b)*c &= (a+b-ab)*c = (a+b-ab)+c-(a+b-ab)c \\
&= a+b+c-ab-ac-bc+abc \\
&= a+(b+c-bc)-a(b+c-bc) \\
&= a*(b*c) \quad \text{so } * \text{ is associative in } \mathbb{R}
\end{aligned}
$$

Unity:

$\forall a,c : c*a = a \Rightarrow c+a-ca = a \Rightarrow c-ac = 0 \Rightarrow c(1-a) = 0 \Rightarrow c = 0$

i.e. $0*a = 0+a-0a = a = a+0-a0 = a*0$ so 0 is unity for $*$ in $\mathbb{R}$

Units: $a*b = 0 \Rightarrow a+b-ab = 0 \Rightarrow b(1-a) = -a \Rightarrow b = \dfrac{a}{a-1} \in \mathbb{R}$, if $a \neq 1$

Check: $a*\left(\dfrac{a}{a-1}\right) = a+\left(\dfrac{a}{a-1}\right)-a\left(\dfrac{a}{a-1}\right) = \dfrac{a(a-1)+a-a^2}{a-1} = 0$

Therefore $a^{-1} = \dfrac{a}{a-1}\ \forall a \neq 1$

c) $M = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, with $(x,y,z)*(x',y',z') = (xx', xy'+yz', zz')$

Closure: $(xx', xy'+yz', zz') \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, so $*$ closed in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

Commutative: $xy'+yz' \neq yx'+zy'$ so not commutative

Associative:
$$
\begin{aligned}
[(x,y,z)*(x',y',z')]*(x'',y'',z'') &= (xx', xy'+yz', zz')*(x'',y'',z'') \\
&= (xx'x'', xx'y''+(xy'+yz')z'', zz'z'') \\
&= (xx'x'', xx'y''+xy'z''+yz'z'', zz'z'')
\end{aligned}
$$

And $\quad (x,y,z)*[(x',y',z')*(x'',y'',z'')] = (x,y,z)*(x'x'',x'y''+y'z'',z'z'')$

$$= (xx'x'', x(x'y''+y'z'')+yz'z'', zz'z'')$$
$$= (xx'x'', xx'y''+xy'z''+yz'z'', zz'z'')$$

so * associative

Unity: $(x,y,z)*(a,b,c) = (xa, xb+yc, zc) = (x,y,z)$

$\Rightarrow x = xa, y = xb+yc, z = zc \Rightarrow a = 1, c = 1$ and

$y = xb + y \Rightarrow b = 0$, so unity is $(1,0,1)$ in $\mathrm{R}\times\mathrm{R}\times\mathrm{R}$

Units: $(x,y,z)*(a,b,c) = (xa, xb+yc, zc) = (1,0,1)$

$\Rightarrow 1 = xa, 0 = xb+yc, 1 = zc \Rightarrow a = \frac{1}{x}, c = \frac{1}{z}$ and

$b = \frac{-yc}{x} = \frac{-y}{xz}$ for $x \neq 0$ and $z \neq 0$

So $(x,y,z)^{-1} = \left(\frac{1}{x},\frac{-y}{xz},\frac{1}{z}\right)$, for $x \neq 0$ and $z \neq 0$.

Check: $(x,y,z)*\left(\frac{1}{x},\frac{-y}{xz},\frac{1}{z}\right) = \left(x\left(\frac{1}{x}\right), x\left(\frac{-y}{xz}\right)+y\left(\frac{1}{z}\right), z\left(\frac{1}{z}\right)\right) = (1,0,1)$

Units for $M = \mathrm{R}\times\mathrm{R}\times\mathrm{R}$ under $*\{x,y,z \mid x \neq 0, z \neq 0, x,y,z \in \mathrm{R}.\}$

2i) u a unit in $M \Rightarrow \exists\, v \in M$ such that $uv = vu = 1$

$au = bu \Rightarrow auv = buv \Rightarrow a1 = b1 \Rightarrow a = b$

2ii) Let $M = \{1, u, a_3, a_4, \ldots, a_n\}$ be a set of n distinct elements.

Consider $uM = \{u1, uu, ua_3, ua_4, \ldots, ua_n\} = \{u, u^2, ua_3, ua_4, \ldots, ua_n\}$

M is closed under its operation so $1 \in \{u, u^2, ua_3, ua_4, \ldots, ua_n\}$. Then

$1 = u$ or $1 = u^2$ or $1 = ua_j$ for some $j = 2,3,\ldots n$, therefore in each of these cases

u is a unit.

If $1 \notin \{u, u^2, ua_3, ua_4, \ldots, ua_n\}$ then the set contains a repetition so

$ua_i = ua_j \Rightarrow a_i = a_j$ for some $i,j = 1,2,3,\ldots n$ which is impossible as all elements are distinct.

3i) $u^{-1} = v^{-1} \Rightarrow uu^{-1} = uv^{-1} \Rightarrow 1 = uv^{-1} \Rightarrow 1v = uv^{-1}v \Rightarrow v = u1 = u$

ii) $ua = au \Rightarrow u^{-1}ua = u^{-1}au \Rightarrow a = u^{-1}au$

$\Rightarrow au^{-1} = u^{-1}auu^{-1} \Rightarrow au^{-1} = u^{-1}a \;\forall a \in M$

iii) $uv = vu \Rightarrow vu^{-1} = u^{-1}v$

$\Rightarrow v^{-1}(vu^{-1})v^{-1} = v^{-1}(u^{-1}v)v^{-1}$

$\Rightarrow u^{-1}v^{-1} = v^{-1}u^{-1}$