Proofs and Supplementary Material: Unified Characterization and Precoding for Non-Stationary Channels

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Instructions: Equations (1)–(34) and the references [1]–[23] refer to the equations and references from the main manuscript ("Unified Characterization and Precoding for Non-Stationary Channels" accepted for publication at IEEE ICC 2022), respectively. This document provides the supplementary material including a comprehensive related work, the complete proofs and extended evaluation results to support the main manuscript.

Appendix A
Related work

We categorize the related work into three categories:

Characterization of Non-Stationary Channels: Wireless channel characterization in the literature typically require several local and global (in space-time dimensions) higher order statistics to characterize or model non-stationary channels, due to their time-varying statistics. These statistics cannot completely characterize the non-stationary channel, however are useful in reporting certain properties that are required for the application of interest such as channel modeling, assessing the degree of stationarity etc. Contrarily, we leverage the 2-dimensional eigenfunctions that are decomposed from the most generic representation of any wireless channel as a spatio-temporal channel kernel. These spatio-temporal eigenfunctions can be used to extract any higher order statistics of the channel as demonstrated in Section III, and hence serves as a complete characterization of the channel. Furthermore, since this characterization can also generalize to stationary channels, it is a unified characterization for any wireless channel. Beyond characterizing the channel, these eigenfunctions are the core of the precoding algorithm.

Precoding Non-Stationary Channels: Although precoding non-stationary channels is unprecedented in the literature [7], we list the most related literature for completeness. The challenge in precoding non-stationary channels is the lack of accurate models of the channel and the (occasional) CSI feedback does not fully characterize the non-stationarities in its statistics. This leads to suboptimal performance using state-of-the-art precoding techniques like Dirty Paper Coding which assume that complete and accurate knowledge of the channel is available, while the CSI is often outdated in non-stationary channels. While recent literature present attempt to deal with imperfect CSI by modeling the error in the CSI [24], [25], [26], [27], [28], [29], [30], [31], they are limited by the assumption the channel or error statistics are stationary or WSSUS at best. Another class of literature, attempt to deal with the impact of outdated CSI [32], [33] in time-varying channels by quantifying this loss or relying statistical CSI. These methods are not directly suitable for non-stationary channels, as the time dependence of the statistics may render the CSI (or its statistics) stale, consequently resulting in precoding error.

Space-Temporal Precoding: While, precoding has garnered significant research, spatio-temporal interference is typically treated as two separate problems, where spatial precoding at the transmitter aims to cancel inter-user and inter-antenna interference, while equalization at the receiver mitigates inter-carrier and inter-symbol interference. Alternately, [34] proposes to modulate the symbols such that it reduces the cross-symbol interference in the delay-Doppler domain, but requires equalization at the receiver to completely cancel such interference in practical systems. Moreover, this approach cannot completely minimize the joint spatio-temporal interference that occurs in non-stationary channels since their statistics depend on the time-frequency domain in addition to the delay-Dopper domain (explained in Section II). While spatio-temporal block coding techniques are studied in the literature [1] they add redundancy and hence incur a communication overhead to mitigate interference, which we avoid by precoding. These techniques are capable of independently canceling the interference in each domain, however are incapable of mitigating interference that occurs in the joint spatio-temporal domain in non-stationary channels. We design a joint spatio-temporal precoding that leverages the extracted 2-D eigenfunctions from non-stationary channels to mitigate interference that occurs on the joint space-time dimensions, which to the best of our knowledge is unprecedented in the literature.

Appendix B
Proofs on Unified Characterization

A. *Proof of Lemma 1: Generalized Mercer's Theorem*

Proof. Consider a 2-dimensional process $K(t,t') \in L^2(Y \times X)$, where $Y(t)$ and $X(t')$ are square-integrable zero-mean random processes with covariance function K_Y and K_X, respecly. The projection of $K(t,t')$ onto $X(t')$ is obtained as in (35),

$$C(t) = \int K(t,t')X(t')\, dt' \tag{35}$$

Using *Karhunen–Loève Transform* (KLT), $X(t')$ and $C(t)$ are both decomposed as in (36) and (37),

$$X(t') = \sum_{i=1}^{\infty} x_i \phi_i(t') \tag{36}$$

$$C(t) = \sum_{j=1}^{\infty} c_j \psi_j(t) \tag{37}$$

where x_i and c_j are both random variables with $\mathbb{E}\{x_i x_{i'}\}=\lambda_{x_i}\delta_{ii'}$ and $\mathbb{E}\{c_j c_{j'}\}=\lambda_{c_j}\delta_{jj'}$. $\{\lambda_{x_i}\}$, $\{\lambda_{x_j}\}$ $\{\phi_i(t')\}$ and $\{\psi_j(t)\}$ are eigenvalues and eigenfuncions, respectively.

Let us denote $n{=}i{=}j$ and $\sigma_n{=}\frac{c_n}{x_n}$, and assume that $K(t,t')$ can be expressed as in (38),

$$K(t,t') = \sum_n^\infty \sigma_n \psi_n(t)\phi_n(t') \tag{38}$$

We show that (38) is a correct representation of $K(t,t')$ by proving (35) holds under this definition. We observe that by substituting (36) and (38) into the right hand side of (35) we have that,

$$\begin{aligned}
&\int K(t,t')X(t')\ dt' \\
&= \int \sum_n^\infty \sigma_n\psi_n(t)\phi_n(t') \sum_n^\infty x_n\phi_n(t')\ dt' \\
&= \int \sum_n^\infty \sigma_n x_n \psi_n(t)|\phi_n(t')|^2 \\
&+ \sum_{n'\neq n}^\infty \sigma_n x_{n'}\psi_n(t)\phi_n(t')\phi^*_{n'}(t')\ dt' \\
&= \sum_n^\infty c_n\psi_n(t) = C(t)
\end{aligned} \tag{39}$$

which is equal to the left hand side of (35). Therefore, (38) is a correct representation of $K(t,t')$.

□

B. Proof of Theorem 1: High Order Generalized Mercer's Theorem (HOGMT

Proof. Given a 2-D process $X(\gamma_1,\gamma_2)$, the eigen-decomposition using Lemma 1 is given by,

$$X(\gamma_1,\gamma_2) = \sum_n^\infty x_n e_n(\gamma_1)s_n(\gamma_2) \tag{40}$$

Letting $\psi_n(\gamma_1,\gamma_2){=}e_n(\gamma_1)s_n(\gamma_2)$, and substituting it in (40) we have that,

$$X(\gamma_1,\gamma_2) = \sum_n^\infty x_n\phi_n(\gamma_1,\gamma_2) \tag{41}$$

where $\phi_n(\gamma_1,\gamma_2)$ are 2-D eigenfunctions with the property (42).

$$\iint \phi_n(\gamma_1,\gamma_2)\phi_{n'}(\gamma_1,\gamma_2)\ d\gamma_1\ d\gamma_2 = \delta_{nn'} \tag{42}$$

We observe that (41) is the 2-D form of KLT. With iterations of the above steps, we obtain *Higher-Order KLT* for $X(\gamma_1,\cdots,\gamma_Q)$ and $C(\zeta_1,\cdots,\zeta_P)$ as given by,

$$X(\gamma_1,\cdots,\gamma_Q) = \sum_n^\infty x_n\phi_n(\gamma_1,\cdots,\gamma_Q) \tag{43}$$

$$C(\zeta_1,\cdots,\zeta_P) = \sum_n^\infty c_n\psi_n(\zeta_1,\cdots,\zeta_P) \tag{44}$$

where $C(\zeta_1,\cdots,\zeta_P)$ is the projection of $X(\gamma_1,\cdots,\gamma_Q)$ onto $K(\zeta_1,\cdots,\zeta_P;\gamma_1,\cdots,\gamma_Q)$.

Then following similar steps as in Appendix B-A we get (45).

$$\begin{aligned}
&K(\zeta_1,\cdots,\zeta_P;\gamma_1,\cdots,\gamma_Q) \\
&= \sum_n^\infty \sigma_n\psi_n(\zeta_1,\cdots,\zeta_P)\phi_n(\gamma_1,\cdots,\gamma_Q)
\end{aligned} \tag{45}$$

□

Appendix C
Proofs on Eigenfunction based Precoding

A. Proof of Lemma 2

Proof. Using 2-D KLT as in (13), $x(u,t)$ is expressed as,

$$x(u,t) = \sum_n^\infty x_n\phi_n(u,t) \tag{46}$$

where x_n is a random variable with $E\{x_n x_{n'}\}{=}\lambda_n\sigma_{nn'}$ and $\phi_n(u,t)$ is a 2-D eigenfunction.

Then the projection of $k_H(u,t;u',t')$ onto $\phi_n(u',t')$ is denoted by $c_n(u,t)$ and is given by,

$$c_n(u,t) = \iint k_H(u,t;u',t')\phi_n(u',t')\ du'\ dt' \tag{47}$$

Using the above, (28) is expressed as,

$$||s(u,t) - Hx(u,t)||^2 = ||s(u,t) - \sum_n^\infty x_n c_n(u,t)||^2 \tag{48}$$

Let $\epsilon(x){=}||s(u,t) - \sum_n^\infty x_n\phi_n(u,t)||^2$. Then its expansion is given by,

$$\begin{aligned}
\epsilon(x) &= \langle s(u,t),s(u,t)\rangle - 2\sum_n^\infty x_n\langle c_n(u,t),s(u,t)\rangle \\
&+ \sum_n^\infty x_n^2\langle c_n(u,t),c_n(u,t)\rangle + \sum_n^\infty\sum_{n'\neq n}^\infty x_n x_{n'}\langle c_n(u,t),c_{n'}(u,t)\rangle
\end{aligned} \tag{49}$$

Then the solution to achieve minimal $\epsilon(x)$ is obtained by solving for $\frac{\partial\epsilon(x)}{\partial x_n} = 0$ as in (50).

$$x_n^{opt}{=}\frac{\langle s(u,t),c_n(u,t)\rangle + \sum_{n'\neq n}^\infty x_{n'}\langle c_{n'}(u,t),c_n(u,t)\rangle}{\langle c_n(u,t),c_n(u,t)\rangle} \tag{50}$$

where $\langle a(u,t),b(u,t)\rangle{=}\iint a(u,t)b^*(u,t)\ du\ dt$ denotes the inner product. Let $\langle c_{n'}(u,t),c_n(u,t)\rangle = 0$, i.e., the projections $\{c_n(u,t)\}_n$ are orthogonal basis. Then we have a closed form expression for x^{opt} as in (51).

$$x_n^{opt}{=}\frac{\langle s(u,t),c_n(u,t)\rangle}{\langle c_n(u,t),c_n(u,t)\rangle} \tag{51}$$

Substitute (51) in (49), it is straightforward to show that $\epsilon(x){=}0$. □

B. Proof of Theorem 2: Eigenfunction Precoding

Proof. The 4-D kernel $k_H(u,t;u',t')$ is decomposed into two separate sets of eigenfunction $\{\phi_n(u',t')\}$ and $\{\psi_n(u,t)\}$

using Theorem 1 as in (30). By transmitting the conjugate of the eigenfunctions, $\phi_n(u',t')$ through the channel H, we have that,

$$
\begin{aligned}
H\phi_n^*(u',t') &= \iint k_H(u,t;u',t')\phi_n^*(u',t')\ du'\ dt' \\
&= \iint \sum_n^\infty \{\sigma_n\psi_n(u,t)\phi_n(u',t')\}\phi_n^*(u',t')\ dt'\ df' \\
&= \iint \sigma_n\psi_n(u,t)|\phi_n(u',t')|^2 \\
&+ \sum_{n'\neq n}^\infty \sigma_{n'}\psi_{n'}(u,t)\phi_{n'}(u',t')\ \phi_n^*(u',t')\ du'\ dt' \\
&= \sigma_n\psi_n(u,t) \qquad (52)
\end{aligned}
$$

where $\psi_n(u,t)$ is also a 2-D eigenfunction with the orthogonal property as in (31).

From Lemma 2, if the set of projections, $\{c_n(u,t)\}$ is the set of eigenfunctions, $\{\psi_n(u,t)\}$, which has the above orthogonal property, we achieve the optimal solution as in (51). Therefore, let $x(u,t)$ be the linear combination of $\{\phi_n^*(u,t)\}$ with coefficients $\{x_n\}$ as in (53),

$$x(u,t) = \sum_n^\infty x_n\phi_n^*(u,t) \qquad (53)$$

Then (48) is rewritten as in (54),

$$||s(u,t) - Hx(u,t)||^2 = ||s(u,t) - \sum_n^\infty x_n\sigma_n\psi(u,t)||^2 \qquad (54)$$

Therefore, optimal x_n in (51) is obtained as in (55),

$$x_n^{opt} = \frac{\langle s(u,t), \psi_n(u,t)\rangle}{\sigma_n} \qquad (55)$$

Substituting (55) in (53), the transmit signal is given by (56),

$$x(u,t) = \sum_n^\infty \frac{\langle s(u,t), \psi_n(u,t)\rangle}{\sigma_n}\phi_n^*(u,t). \qquad (56)$$

□

C. Proof of Corollary 1

Proof. First we substitute the 4-D kernel $k_H(u,t;u',t')$ with the 2-D kernel $k_H(u,u')$ in Theorem 2 which is then decomposed by the 2-D HOGMT. Then following similar steps as in Appendix C-B it is straightforward to show (34). □

Appendix D
Results on Interference

Figure 8 shows the channel response for user u=1 at t=1, t=10, t=50 and t=100, where at each instance, the response for user u=1 is not only affected by its own delay and other users' spatial interference, but also affected by other users' delayed symbols. This is the cause of joint space-time interference which necessitates joint precoding in the 2-dimensional space using eigenfunctions that are jointly orthogonal.

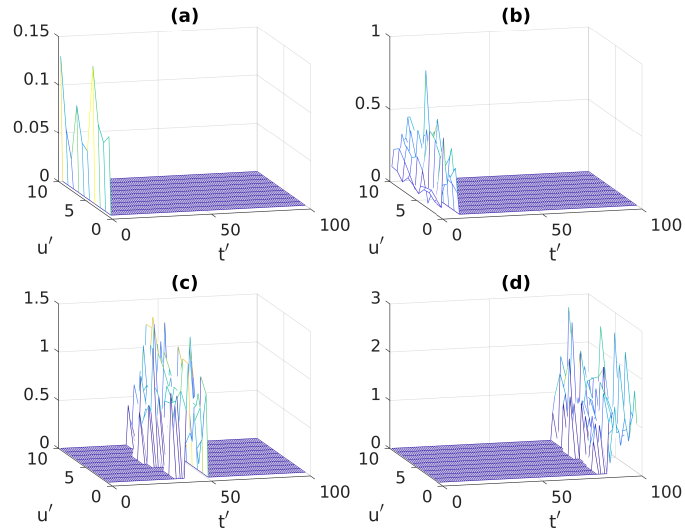

Figure 8: Kernel $k_H(u,t;u',t')$ for u=1 at a) t=1, b) t=10, c) t=50 and d) t=100.

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