demo_ou

August 31, 2025

1 Ornstein–Uhlenbeck (OU) Process for Mean-Reverting Spreads

1.1 1. OU in Continuous Time (from AR(1) intuition)

Start with a discrete mean-reverting spread y_t:

$$y_{t+\Delta} = (1 - \kappa\Delta)\, y_t + \kappa\Delta\, \theta + \sigma\sqrt{\Delta}\, \varepsilon_t, \tag{1}$$

with $\varepsilon_t \sim \mathcal{N}(0, 1)$.

Let $\Delta \to 0$. This converges to the **Ornstein–Uhlenbeck (OU)** SDE:

$$dX_t = \kappa(\theta - X_t)\, dt + \sigma\, dW_t, \tag{2}$$

where $\kappa > 0$ is the speed of mean reversion, θ the long-run mean, and σ the volatility.

Solution and properties:

$$X_t = \theta + (X_0 - \theta)e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)}\, dW_s, \tag{3}$$

$$\mathbb{E}[X_t] = \theta + (X_0 - \theta)e^{-\kappa t}, \tag{4}$$

$$\mathrm{ACF}(\tau) = e^{-\kappa\tau}, \quad \text{half-life } \tau_{1/2} = \frac{\ln 2}{\kappa}. \tag{5}$$

1.2 2. How it Appears for the Spread Between Two Stocks

Let two stocks follow correlated GBMs:

$$\frac{dS_1}{S_1} = \mu_1\, dt + \sigma_1\, dW_1, \qquad \frac{dS_2}{S_2} = \mu_2\, dt + \sigma_2\, dW_2, \qquad dW_1\, dW_2 = \rho\, dt. \tag{6}$$

Log prices:

$$d \log S_i = \left(\mu_i - \tfrac{1}{2}\sigma_i^2\right) dt + \sigma_i\, dW_i. \tag{7}$$

Define a **log-spread**:

$$y_t = \log S_{1,t} - \beta \log S_{2,t} - \theta. \tag{8}$$

Then:

$$dy_t = \Big(\mu_1 - \beta\mu_2 - \tfrac{1}{2}(\sigma_1^2 - \beta\sigma_2^2)\Big)dt + \sigma_1\, dW_1 - \beta\sigma_2\, dW_2. \tag{9}$$

To achieve mean reversion, impose an **error-correction condition**:

$$\mu_1 - \beta\mu_2 = -\kappa y_t + \tfrac{1}{2}(\sigma_1^2 - \beta\sigma_2^2). \tag{10}$$

Substitute:

$$dy_t = -\kappa y_t\, dt + \sigma_1\, dW_1 - \beta\sigma_2\, dW_2. \tag{11}$$

Define effective spread volatility:

$$\sigma_y^2 = \sigma_1^2 + \beta^2\sigma_2^2 - 2\beta\rho\sigma_1\sigma_2, \tag{12}$$

and a new Brownian motion $d\widetilde{W}_t$:

$$d\widetilde{W}_t = \frac{\sigma_1\, dW_1 - \beta\sigma_2\, dW_2}{\sigma_y}. \tag{13}$$

Finally:

$$dy_t = -\kappa y_t\, dt + \sigma_y\, d\widetilde{W}_t, \tag{14}$$

which is exactly an OU process.

1.3 3. Why This is Useful

- Connects **econometrics** (cointegration) with **stochastic calculus** (OU).
- Closed-form transition density enables **fast MLE**.
- Parameters κ, θ, σ_y give interpretable dials for trading design.
- OU models are mean-reverting — important in finance because many quantities aren't well modeled by GBM drift-to-infinity:
  - Interest rates: Vasicek model is essentially OU.
  - Volatility dynamics (Heston's variance process is a square-root diffusion, but OU intuition applies).
  - Pairs trading / spreads: log-price differences between cointegrated assets revert to equilibrium.

1.4 4. Practical Modeling Pipeline for a Pairs Spread

1. **Pick a pair / basket.**
2. **Estimate β and θ** via cointegration.
3. **Form the spread** $y_t = \log S_{1,t} - \beta \log S_{2,t} - \theta$ and test stationarity.

4. **Calibrate OU** using MLE:

$$y_{t+\Delta} \,|\, y_t \sim \mathcal{N}\Big(\theta + (y_t - \theta)e^{-\kappa\Delta},\; \frac{\sigma_y^2}{2\kappa}(1 - e^{-2\kappa\Delta})\Big). \tag{15}$$

   - Maximize likelihood to obtain $\hat{\kappa}, \hat{\theta}, \hat{\sigma}_y$.
   - Compute half-life $\ln 2/\hat{\kappa}$.

5. **Signal & execution.**
   - Enter trades when standardized spread $z_t = (y_t - \hat{\theta})/\hat{\sigma}_{\text{st}}$ exceeds thresholds.
   - Exit when z_t reverts to 0.
   - Size positions by variance forecast and hedge ratio β.

[3]:
```
import math
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

import sys, os
project_root = os.path.abspath(os.path.join(os.getcwd(), '..'))
sys.path.append(project_root)
```

[4]:
```
%cd $project_root
```

```
/Users/zhaoyub/Documents/Tradings/option-mini-lab
```

```
/Users/zhaoyub/Library/Python/3.12/lib/python/site-
packages/IPython/core/magics/osm.py:417: UserWarning: This is now an optional
IPython functionality, setting dhist requires you to install the `pickleshare`
library.
  self.shell.db['dhist'] = compress_dhist(dhist)[-100:]
```

[5]:
```
%load_ext autoreload
```

```
The autoreload extension is already loaded. To reload it, use:
  %reload_ext autoreload
```

2 Ornstein–Uhlenbeck (OU): Simulation · MLE Calibration · Diagnostics

Why it matters (finance): - Many quantities mean-revert rather than drift to infinity (e.g., short rates à la Vasicek, cointegrated spreads). - The OU process is Gaussian, stationary, and interpretable: the **half-life** $t_{1/2} = \ln 2/\kappa$ is a direct measure of reversion speed.

Model

$$dX_t = \kappa(\mu - X_t)\,dt + \sigma\,dW_t, \quad X_{t+\Delta} \mid X_t \sim \mathcal{N}\Big(\mu + (X_t - \mu)e^{-\kappa\Delta},\; \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta})\Big). \tag{16}$$

OU AR(1) mapping

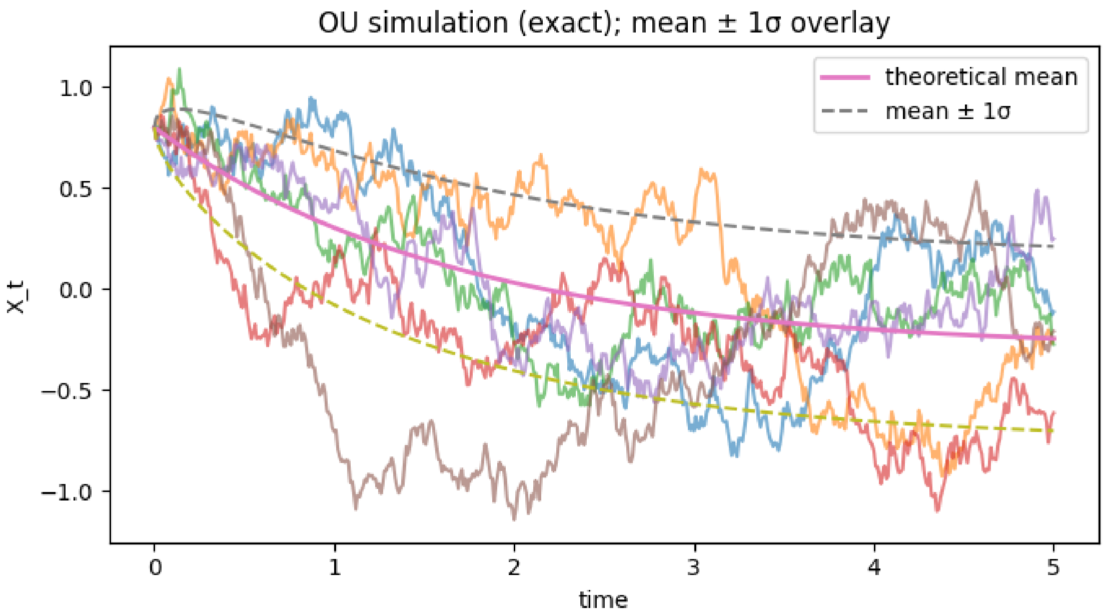
$$X_{n+1} = a + bX_n + \varepsilon_n, \quad b = e^{-\kappa\Delta},\ a = \mu(1-b),\ \varepsilon_n \sim \mathcal{N}\Big(0,\ \tfrac{\sigma^2}{2\kappa}(1-b^2)\Big). \tag{17}$$

[8]:
```
import numpy as np
import matplotlib.pyplot as plt

from src.ou import (
    OUParams, simulate_ou_exact, simulate_ou_euler,
    fit_ou_mle, ou_residuals, acf, qq_data, half_life
)

rng = np.random.default_rng(12345678)
```

2.1 Simulate OU paths (exact vs Euler)

[16]:
```
# True parameters
true = OUParams(kappa=0.6, mu=-0.3, sigma=0.5)

# Grid
dt   = 0.01
T    = 5.0
steps = int(T / dt)
t = np.linspace(0, T, steps+1)

# Simulate
x0 = 0.8
n_paths = 50

X_exact = simulate_ou_exact(n_paths, steps, dt, true, x0, rng=rng)
X_euler = simulate_ou_euler(n_paths, steps, dt, true, x0, rng=rng)

# Theoretical mean and std envelope
mean_the = true.mu + (x0 - true.mu) * np.exp(-true.kappa * t)
var_the  = (true.sigma**2) / (2 * true.kappa) * (1 - np.exp(-2 * true.kappa *
 ↪t))
std_the  = np.sqrt(var_the)

# Plot a few exact paths + theory envelope
plt.figure(figsize=(7,4))
for i in range(min(n_paths, 6)):
    plt.plot(t, X_exact[i], alpha=0.6)
plt.plot(t, mean_the, linewidth=2, label="theoretical mean")
plt.plot(t, mean_the + std_the, linestyle="--", label="mean ± 1 ")
plt.plot(t, mean_the - std_the, linestyle="--")
plt.xlabel("time")
plt.ylabel("X_t")
```

```python
plt.title("OU simulation (exact); mean ± 1  overlay")
plt.legend()
plt.tight_layout()
plt.show()
```



2.2 Euler bias illustration

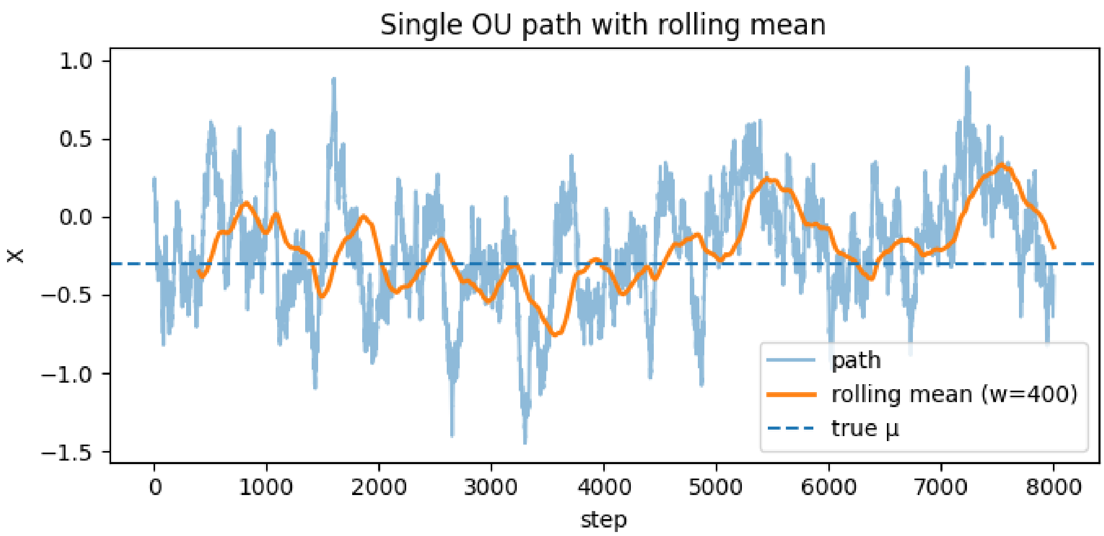
[10]:
```python
# Compare final-time distribution mean/var vs theory
def summarize(arr):
    return float(arr.mean()), float(arr.var(ddof=1))

end_exact = X_exact[:, -1]
end_euler = X_euler[:, -1]

m_e, v_e = summarize(end_exact)
m_u, v_u = summarize(end_euler)

print("T = {:.2f}, theory: mean={:.4f}, var={:.4f}".format(T, mean_the[-1],␣
 ↪var_the[-1]))
print("Exact : mean={:.4f}, var={:.4f}".format(m_e, v_e))
print("Euler : mean={:.4f}, var={:.4f}".format(m_u, v_u))
```

```
T = 5.00, theory: mean=-0.2452, var=0.2078
Exact : mean=-0.4840, var=0.0953
Euler : mean=-0.2504, var=0.1704
```

2.3 MLE calibration from one long path

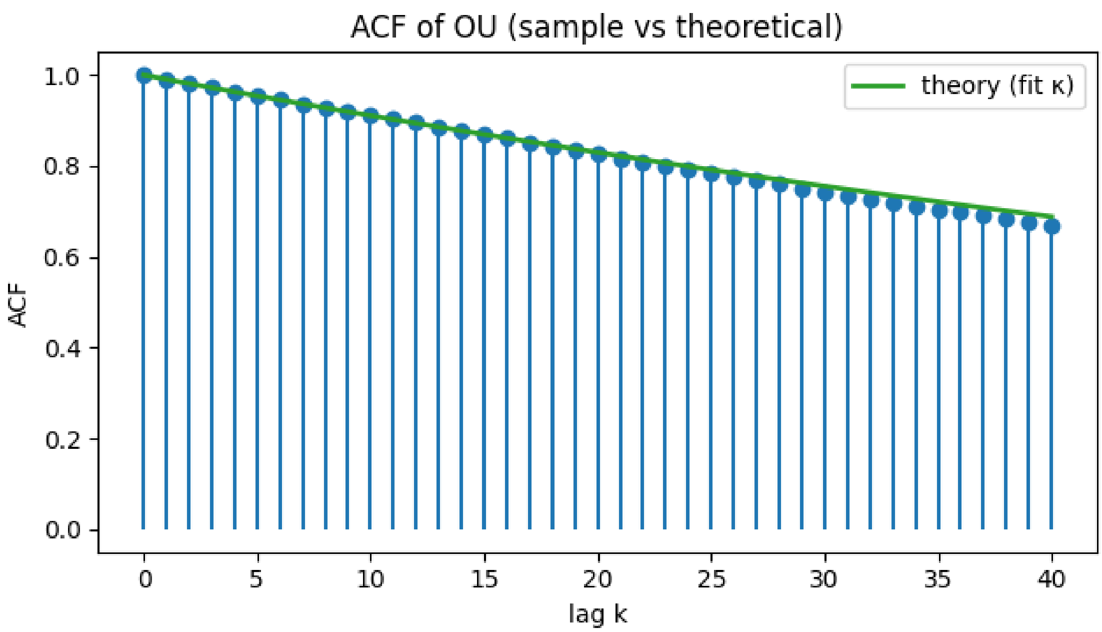
[11]:
```python
# Use one long exact path for calibration
x = simulate_ou_exact(1, 8000, dt=0.01, params=true, x0=0.2, rng=rng)[0]

fit = fit_ou_mle(x, dt=0.01)
print("Estimated OU params")
print(f"  kappâ = {fit.kappa:.4f}  (true {true.kappa:.4f})")
print(f"  mû    = {fit.mu:.4f}     (true {true.mu:.4f})")
print(f"  sigmâ = {fit.sigma:.4f}  (true {true.sigma:.4f})")
print("\nAR(1) view")
print(f"  â = {fit.a:.6f},  b̂ = {fit.b:.6f},   _ˆ = {fit.sigma_eps:.6f}")
print(f"  s.e.(â) = {fit.stderr_a:.6f}, s.e.(b̂) = {fit.stderr_b:.6f}")

# Visual check: sample vs theory mean/var on this single path (rolling)
w = 400  # rolling window to smooth the single path
roll_mean = np.convolve(x, np.ones(w)/w, mode="valid")
plt.figure(figsize=(7,3.5))
plt.plot(x, alpha=0.5, label="path")
plt.plot(np.arange(w-1, len(x)), roll_mean, linewidth=2, label=f"rolling mean␣
↪(w={w})")
plt.axhline(true.mu, linestyle="--", label="true  ")
plt.xlabel("step")
plt.ylabel("X")
plt.title("Single OU path with rolling mean")
plt.legend()
plt.tight_layout()
plt.show()
```

```
Estimated OU params
  kappâ = 0.9348  (true 0.6000)
  mû    = -0.2129     (true -0.3000)
  sigmâ = 0.5002  (true 0.5000)

AR(1) view
  â = -0.001981,  b̂ = 0.990696,   _ˆ = 0.049789
  s.e.(â) = 0.000638, s.e.(b̂) = 0.001517
```



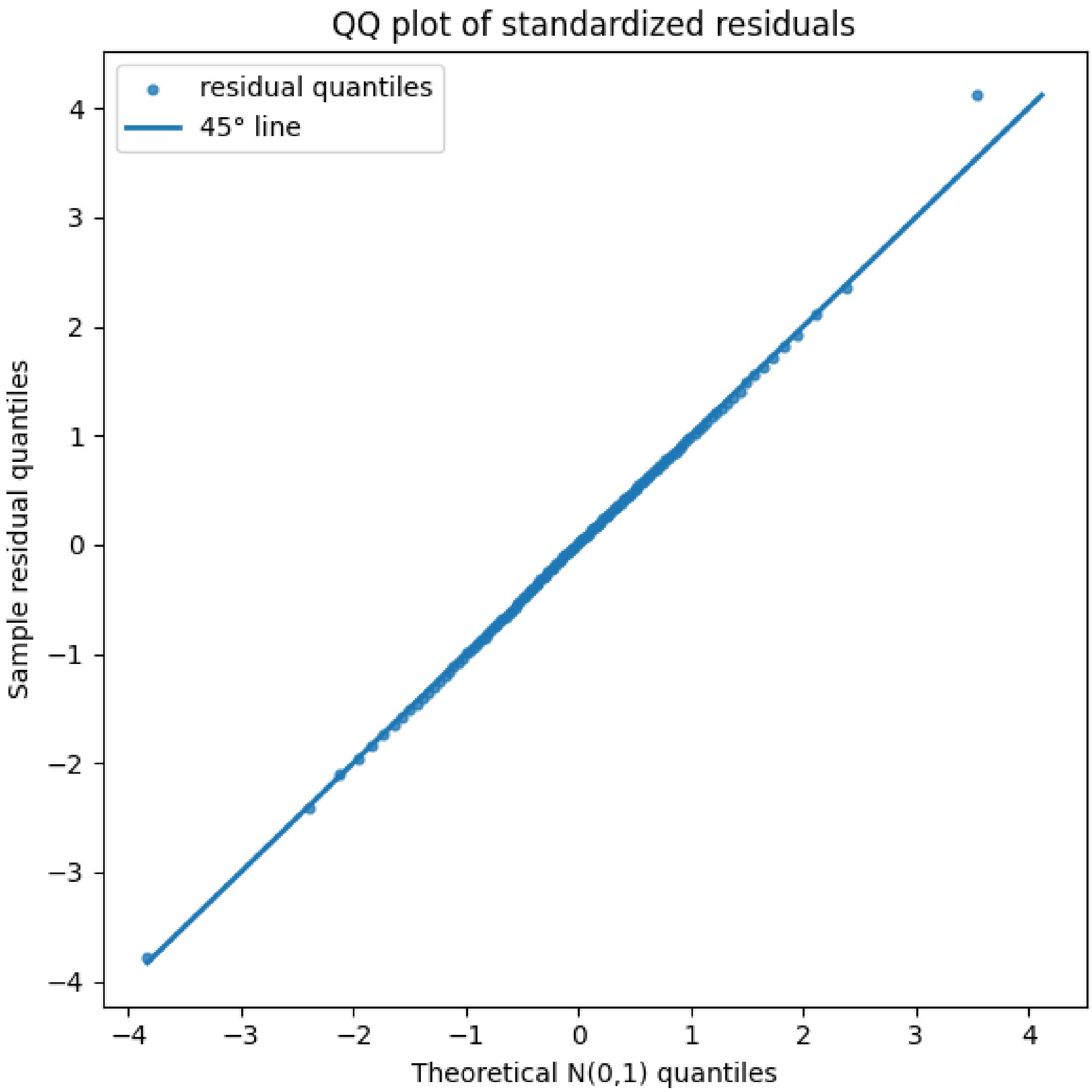
2.4 ACF: empirical vs theoretical

[13]:
```
max_lag = 40
r = acf(x, max_lag=max_lag)

# Theoretical discrete ACF: rho(k) = (e^{-kappa dt})^k
rho_the = np.exp(-fit.kappa * np.arange(max_lag+1) * 0.01)

plt.figure(figsize=(6.5,3.8))
plt.stem(range(max_lag+1), r, linefmt='-', markerfmt='o', basefmt=' ')
plt.plot(range(max_lag+1), rho_the, linewidth=2, label="theory (fit  )")
plt.xlabel("lag k")
plt.ylabel("ACF")
plt.title("ACF of OU (sample vs theoretical)")
plt.legend()
plt.tight_layout()
plt.show()
```



2.5 QQ-plot of standardized residuals

[14]:
```
# One-step residuals (AR(1) fitted form)
res = ou_residuals(x, fit)
z = (res - res.mean()) / res.std(ddof=1)

theo_q, samp_q = qq_data(z, n_points=120)

plt.figure(figsize=(6,6))
plt.scatter(theo_q, samp_q, s=12, alpha=0.8, label="residual quantiles")
lims = [min(theo_q.min(), samp_q.min()), max(theo_q.max(), samp_q.max())]
plt.plot(lims, lims, linewidth=2, label="45° line")
plt.xlabel("Theoretical N(0,1) quantiles")
plt.ylabel("Sample residual quantiles")
plt.title("QQ plot of standardized residuals")
plt.legend()
plt.tight_layout()
plt.show()
```



2.6 Half-life: theory vs empirical

[15]:
```
# Theoretical half-life from fitted kappa
t12_the = half_life(fit.kappa)

# Empirical half-life estimate:
# Estimate rho(1) from data, then kappa_emp = -ln(rho(1))/dt, t12_emp = ln(2)/
↪kappa_emp.
rho1 = acf(x, max_lag=1)[1]
rho1 = np.clip(rho1, 1e-6, 1-1e-6)  # numerical guard
kappa_emp = -np.log(rho1) / 0.01
t12_emp = np.log(2.0) / kappa_emp
```

```
print(f"Half-life (theory from fit  ): {t12_the:.3f}")
print(f"Half-life (empirical from  (1)): {t12_emp:.3f}")
```

```
Half-life (theory from fit  ): 0.741
Half-life (empirical from  (1)): 0.739
```

2.6.1 OU, AR(1) recap

Given sampling step Δt: - $b = e^{-\kappa \Delta t}$ controls the ACF slope: $\rho(k) = b^k$. - $a = \mu(1-b)$ ensures the process reverts to μ. - Shock variance per step: $\sigma_\varepsilon^2 = \frac{\sigma^2}{2\kappa}(1-b^2)$.

MLE via OLS on $X_{t+1} = a + bX_t + \varepsilon_t$ gives $(\hat{a}, \hat{b})$, then back out $\hat{\kappa}, \hat{\mu}, \hat{\sigma}$.

2.7 Takeaways

- OU is a **continuous-time AR(1)** with exponential ACF.
- Exact simulation matches theoretical mean/variance at any step size.
- OLS on the AR(1) provides **Gaussian MLE** for (κ, μ, σ).
- Diagnostics (ACF + QQ) support **stationarity** and **Gaussian residuals**.
- **Half-life** $t_{1/2} = \ln 2/\kappa$ offers a direct, interpretable speed of mean reversion.

[ ]:
```
! jupyter nbconvert --to pdf notebooks/demo_ou.ipynb
```

[ ]:
```
```