demo_barrier

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1 Barrier Options Simulation

1.0.1 What is a Barrier Option?

Barrier options are a class of **exotic options** whose payoff depends not only on the terminal underlying price S_T, but also on whether the underlying has **touched a pre-specified barrier** level H during the option's life.

- **Knock-out options:** become worthless if the barrier is breached. – **The One that we focus**
  - *Example:* Down-and-out call – pays $(S_T - K)^+$ **only if** the asset price never falls below the barrier H.
- **Knock-in options:** only come into existence if the barrier is breached.
  - *Example:* Up-and-in call – worthless unless the asset price touches an upper barrier U.
- **Scope: In this demo we focus on knock-out options (knock-in can be obtained by parity relations)**

1.0.2 Why they matter in markets

- **Cost reduction:** Cheaper than vanilla options, since the barrier condition reduces the likelihood of payout.

- **Structured products:** Widely embedded in retail and institutional notes to tailor payoffs (e.g. autocallables).

- **Trading desks:** Standard instruments on exotics desks; accurate pricing requires handling continuous monitoring and path-dependence.

1.0.3 Key Idea

- **Knock-out options** (e.g. down-and-out call) are deactivated if the asset ever hits a barrier during $[0, T]$.
- True survival event:

$$\min_{0 \le t \le T} S_t > H. \tag{1}$$

1.0.4 Problem: Discrete Simulation

- Monte Carlo usually samples on a grid $0 = t_0 < \cdots < t_M = T$.
- Naïve check: survive if $S_{t_i} > H$ for all grid points.
- **Issue:** path may dip below H between sampling times → **missed knock-outs**.

- Result: **upward bias** (price too high) for knock-out options.

1.0.5 Brownian Bridge Correction

- Condition on endpoints $X_{t_i}, X_{t_{i+1}}$ with $X = \ln S$.
- Between them, the process is a **Brownian bridge**.
- Probability of hitting a lower barrier $h = \ln H$ in $[t_i, t_{i+1}]$:

$$p_i = \exp\left(-\frac{2\,(X_{t_i} - h)(X_{t_{i+1}} - h)}{\sigma^2 \Delta t_i}\right), \tag{2}$$

  if both endpoints are above h, else $p_i = 1$.

1.0.6 Corrected Estimators

1. **Survival weight method (lower variance):**

$$\widehat{C} = e^{-rT}\,\frac{1}{N}\sum_{n=1}^{N}(S_T^{(n)} - K)^+ \prod_i (1 - p_i^{(n)}). \tag{3}$$

2. **Bernoulli knock-out method (more intuitive):**
   - For each interval, flip a coin with probability p_i.
   - If any hit occurs $\to$ payoff $= 0$, else payoff $= (S_T - K)^+$.

[4]:
```
import math
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

import sys, os
project_root = os.path.abspath(os.path.join(os.getcwd(), '..'))
sys.path.append(project_root)
```

[5]:
```
%cd $project_root
```

```
/Users/zhaoyub/Documents/Tradings/option-mini-lab
```

```
/Users/zhaoyub/Library/Python/3.12/lib/python/site-
packages/IPython/core/magics/osm.py:417: UserWarning: This is now an optional
IPython functionality, setting dhist requires you to install the `pickleshare`
library.
  self.shell.db['dhist'] = compress_dhist(dhist)[-100:]
```

[6]:
```
%load_ext autoreload
```

```
The autoreload extension is already loaded. To reload it, use:
  %reload_ext autoreload
```

[14]:
```
from src.barrier import price_barrier_ko_call, barrier_survival_weights_bridge
from src.gbm import simulate_gbm_paths
```

1.0.7 Basic Example: Down-and-Out Call

We price a down-and-out call option with the following parameters: - Spot price (S_0 = 100) - Strike (K = 100) - Barrier (H = 90) - Risk-free rate (r = 2%) - Volatility (= 20%) - Maturity (T = 1) year - Paths: (200,000) - Steps: 64

[9]:
```
price, se = price_barrier_ko_call(
    S0=100, K=100, r=0.02, sigma=0.2, T=1.0,
    barrier=90.0, kind="down",
    n_steps=64, n_paths=200_000,
    method="bridge_weight",
)
print(f"Down-and-out Call Price: {price:.4f} ± {1.96*se:.4f} (95% CI)")
```

```
Down-and-out Call Price: 7.3107 ± 0.0588 (95% CI)
```

1.0.8 Comparison: Barrier Removed (Vanilla Call)

If we push the barrier very low (e.g. $H \to 0$), the knock-out option should reduce to the vanilla Black–Scholes call.

[10]:
```
price_no_barrier, se_no_barrier = price_barrier_ko_call(
    S0=100, K=100, r=0.02, sigma=0.2, T=1.0,
    barrier=1e-6, kind="down",  # essentially inactive barrier
    n_steps=64, n_paths=200_000,
)
print(f"Vanilla-equivalent Call Price: {price_no_barrier:.4f}")
```

```
Vanilla-equivalent Call Price: 8.9001
```

1.0.9 Sanity Check: Immediate Knock-Out

If the initial spot is already below the barrier, the option value must be zero.

[11]:
```
price_ko, se_ko = price_barrier_ko_call(
    S0=85, K=100, r=0.02, sigma=0.2, T=1.0,
    barrier=90.0, kind="down",
    n_steps=64, n_paths=100_000,
)
print(f"Price with S0 below barrier: {price_ko:.4f}")
```

```
Price with S0 below barrier: 0.0000
```

1.0.10 Grid Invariance Check

Thanks to the Brownian-bridge correction, the estimated price should not depend on the number of time steps (unlike the naïve discrete approach).

[12]:
```
for steps in [8, 32, 128, 512]:
    price, se = price_barrier_ko_call(
        S0=100, K=100, r=0.02, sigma=0.2, T=1.0,
```

```
        barrier=90.0, kind="down",
        n_steps=steps, n_paths=100_000,
    )
    print(f"Steps={steps:<4d} → Price: {price:.4f} ± {se:.4f}")
```

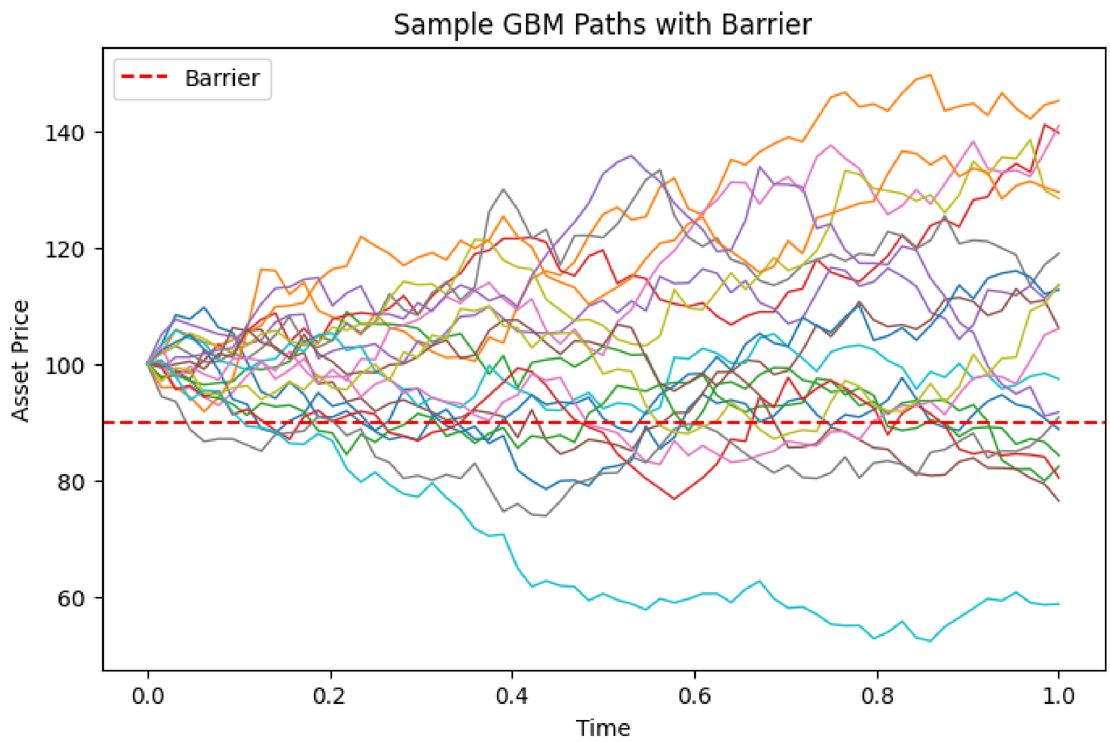
```
Steps=8    → Price: 7.2591 ± 0.0415
Steps=32   → Price: 7.2693 ± 0.0422
Steps=128  → Price: 7.2926 ± 0.0426
Steps=512  → Price: 7.2864 ± 0.0427
```

1.1 Visualization

1.1.1 Simulated Paths with Barrier

[15]:
```
S0, K, r, sigma, T = 100, 100, 0.02, 0.2, 1.0
barrier = 90.0

t, paths = simulate_gbm_paths(S0, mu=r, sigma=sigma, T=T, n_steps=64,␣
 ↪n_paths=20, antithetic=False)
plt.figure(figsize=(8,5))
for i in range(paths.shape[0]):
    plt.plot(t, paths[i], lw=1)
plt.axhline(barrier, color="red", ls="--", label="Barrier")
plt.title("Sample GBM Paths with Barrier")
plt.xlabel("Time")
plt.ylabel("Asset Price")
plt.legend()
plt.show()
```



1.1.2 Terminal Distribution with vs without Barrier

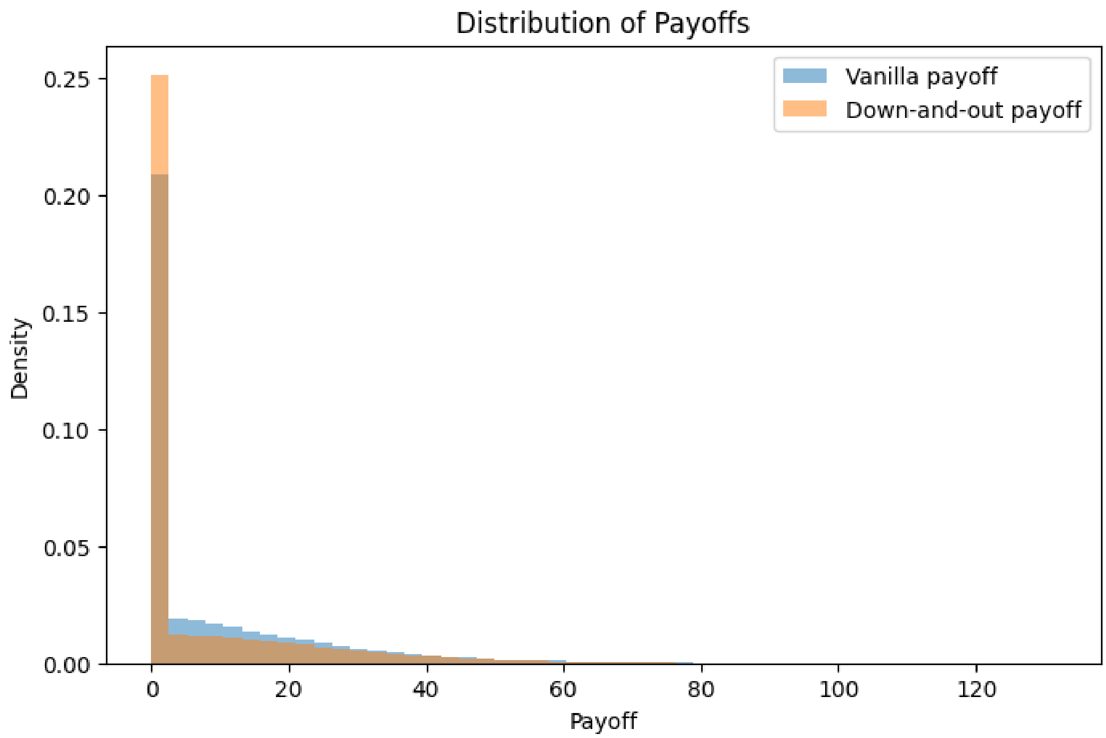
[16]:
```
n_paths = 200_000
t, paths = simulate_gbm_paths(S0, mu=r, sigma=sigma, T=T, n_steps=64,␣
    ↪n_paths=n_paths)

# Payoffs without barrier
payoff_vanilla = np.maximum(paths[:,-1] - K, 0)

# Survival weights (Brownian bridge)
w = barrier_survival_weights_bridge(paths, sigma=sigma, T=T, barrier=barrier,␣
    ↪kind="down")
payoff_barrier = np.maximum(paths[:,-1] - K, 0) * w

plt.figure(figsize=(8,5))
plt.hist(payoff_vanilla, bins=50, alpha=0.5, label="Vanilla payoff",␣
    ↪density=True)
plt.hist(payoff_barrier, bins=50, alpha=0.5, label="Down-and-out payoff",␣
    ↪density=True)
plt.title("Distribution of Payoffs")
plt.xlabel("Payoff")
plt.ylabel("Density")
```

```
plt.legend()
plt.show()
```



1.2 Summary

- Discrete monitoring → upward bias in knock-out pricing.
- Brownian-bridge correction fixes this by computing crossing probabilities between grid points.
- The corrected estimator is grid-invariant, consistent with vanilla in the barrier $\to 0/\infty$ limit, and zero when S_0 starts beyond the barrier.
- Visualizations confirm: the knock-out option has a thinner payoff distribution compared to vanilla because many paths are eliminated.

[ ]:
```
! jupyter nbconvert --to pdf notebooks/demo_barrier.ipynb
```

[ ]:
```

```