Gradients in Logistic Regression and Perceptrons Supplement

CS114B Lab 4

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Gradients in Logistic Regression

- Cross-entropy loss $L(\hat{y}, y) = -[y \log \hat{y} + (1 - y) \log(1 - \hat{y})]$

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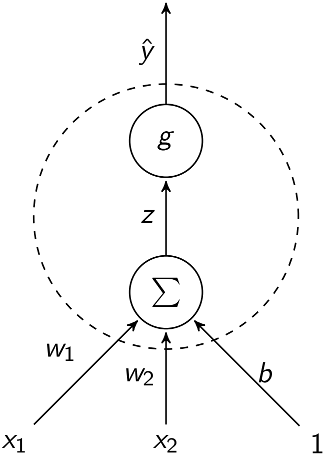
Graphical Representation of a Linear Classifier (1)



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- $$\frac{\partial L}{\partial w_j} = \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z} \frac{\partial z}{\partial w_j}$$

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- $\dfrac{\partial L}{\partial w_j} = \dfrac{\partial L}{\partial \hat{y}} \dfrac{\partial \hat{y}}{\partial z} x_j$
  - $\dfrac{\partial z}{\partial w_j} = x_j$

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- $\dfrac{\partial L}{\partial w_j} = \dfrac{\partial L}{\partial \hat{y}} \hat{y}(1 - \hat{y}) x_j$
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  - For the logistic function: $\dfrac{\partial \hat{y}}{\partial z} = \hat{y}(1 - \hat{y})$

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- $\frac{\partial L}{\partial w_j} = \frac{\hat{y} - y}{\hat{y}(1-\hat{y})}\hat{y}(1-\hat{y})x_j$
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  - For the logistic function: $\frac{\partial \hat{y}}{\partial z} = \hat{y}(1-\hat{y})$
  - For the cross-entropy loss: $\frac{\partial L}{\partial \hat{y}} = \frac{\hat{y} - y}{\hat{y}(1-\hat{y})}$

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  - For the cross-entropy loss: $\frac{\partial L}{\partial \hat{y}} = \frac{\hat{y} - y}{\hat{y}(1-\hat{y})}$
- $\frac{\partial L}{\partial b} = \frac{\hat{y} - y}{\hat{y}(1-\hat{y})}\hat{y}(1-\hat{y})(1) = \hat{y} - y$
  - $\frac{\partial z}{\partial b} = 1$
  - ...

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  - Problem: $\dfrac{\partial \hat{y}}{\partial z} = 0$ almost everywhere
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  - Solution: consider $\dfrac{\partial L}{\partial z}$ directly

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- Looking at the graph: $\dfrac{\partial L}{\partial w_j} = \color{blue}{\dfrac{\partial L}{\partial z}}\color{black}{\dfrac{\partial z}{\partial w_j}}$

Gradients in Perceptrons

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