Linear Classifiers (Part 3)

CS114B Lab 4

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Multi-Class Classification

- Two-class (binary) classification
  - Compute "score" $z = \theta \cdot \mathbf{x}$ (or $\mathbf{w} \cdot \mathbf{x} + b$)
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    - If $\hat{y}$ is interpreted as the probability of (or indicator for) one class, $1 - \hat{y}$ is the probability of (indicator for) the other class
- Multi-class (multinomial) classification
  - Compute a vector of scores $\mathbf{z} = \Theta \cdot \mathbf{x}$ (or $\mathbf{W} \cdot \mathbf{x} + \mathbf{b}$)
  - Compute decision $\hat{y}$ as a function of $\mathbf{z}$

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- A matrix is a rectangular array of scalars, that can be used to define a linear map

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    - More intuitive (input $\rightarrow$ output)
    - Aligns with the convention in (mini)batch training that the first dimension is the batch size ("feature vectors are stacked row-wise")

General Advice

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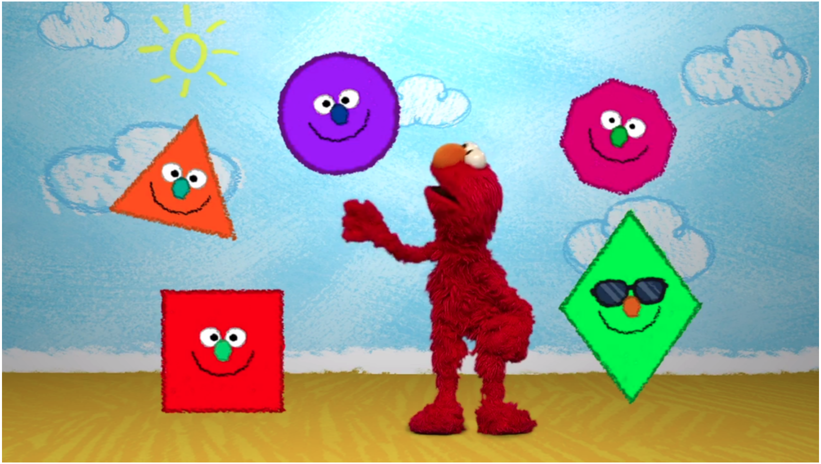

Source

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  - What if there is a tie?
    - Do whatever `numpy.argmax` does

Naïve Bayes as a Linear Classifier

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- Suppose we observe a document d. What is the most likely class $\hat{c}$?
- $P(c|d) = \dfrac{P(d|c)P(c)}{P(d)}$
  - Bayes' Rule
- $\hat{c} = \underset{c \in C}{\operatorname{argmax}}\, P(d|c)P(c)$
  - $P(d)$ is the same for each class
- $\hat{c} = \underset{c \in C}{\operatorname{argmax}}\, P(c) \prod_{i \in \text{positions}} P(w_i|c)$
  - Bag of words assumption, Naïve Bayes assumption
- $\hat{c} = \underset{c \in C}{\operatorname{argmax}} \log P(c) + \sum_{i \in \text{positions}} \log P(w_i|c)$
  - If $xy = z$, then $\log(x) + \log(y) = \log(z)$

Naïve Bayes as a Linear Classifier

- $\hat{c} = \operatorname*{argmax}_{c \in C} \sum_{w \in |V|} \left[(\log P(w|c))(\text{count}(w,d)) \right] + \log P(c)$
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$$\begin{aligned} \hat{c} &= \underset{c \in C}{\operatorname{argmax}} \sum_{w \in |V|} \ell_{cw} x_w + p_c \\ &= \underset{c \in C}{\operatorname{argmax}} (\ell_c \cdot \mathbf{x} + p_c) \end{aligned}$$

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  - Update the parameters using gradient descent

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    - If $\hat{y} = y$, $L = 0$
    - If $\hat{y} \neq y$, $L > 0$

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  - "Slope of a function's graph"
- The derivative of f with respect to x is denoted $\dfrac{df}{dx}$, f', etc.

Derivatives

- Rules of differentiation
  - Constant rule: If $f(x)$ is constant, then $f'(x) = 0$
  - Sum rule: $(f+g)' = f' + g'$
  - Product rule: $(fg)' = f'g + fg'$
  - Power rule: If $f(x) = x^r$, then $f'(x) = rx^{r-1}$
  - Chain rule: If $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$ (or $\frac{dh}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$)

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- Anything more complicated than this, we will tell you what the derivative is

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Gradients

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$$\nabla F = \begin{bmatrix} \dfrac{\partial F}{\partial x_1} \\ \vdots \\ \dfrac{\partial F}{\partial x_n} \end{bmatrix}$$

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    - $\approx$ slope of loss function

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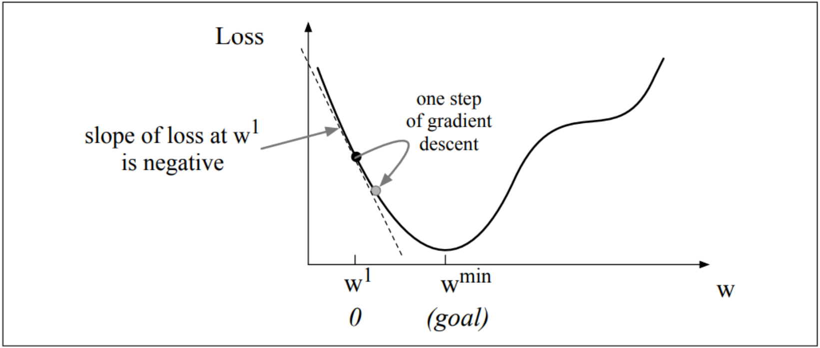

Figure 5.4 The first step in iteratively finding the minimum of this loss function, by moving w in the reverse direction from the slope of the function. Since the slope is negative, we need to move w in a positive direction, to the right. Here superscripts are used for learning steps, so w^1 means the initial value of w (which is 0), w^2 at the second step, and so on.

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- Does this look familiar?

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- If $\hat{y} = 1$ and $y = 0$, then $\hat{y} - y = 1$
  - $\nabla L = \mathbf{x}$
  - $\theta_{t+1} = \theta_t - \eta \mathbf{x}$
  - Decrement weights

Gradients in Multinomial Logistic Regression

- Cross-entropy loss $L(\hat{\mathbf{y}}, \mathbf{y}) = -\sum_{k=1}^{p} y_k \log \hat{y}_k$

Gradients in Multinomial Logistic Regression

- Cross-entropy loss $L(\hat{\mathbf{y}}, \mathbf{y}) = -\sum_{k=1}^{p} y_k \log \hat{y}_k$
- Gradient ∇L becomes a matrix, where
  - $\dfrac{\partial L}{\partial w_{jk}} = (\hat{y}_k - y_k)x_j$
  - $\dfrac{\partial L}{\partial b_k} = \hat{y}_k - y_k$

Gradients in Multinomial Logistic Regression

- Cross-entropy loss $L(\hat{\mathbf{y}}, \mathbf{y}) = -\sum_{k=1}^{p} y_k \log \hat{y}_k$
- Gradient ∇L becomes a matrix, where
  - $\dfrac{\partial L}{\partial w_{jk}} = (\hat{y}_k - y_k)x_j$
  - $\dfrac{\partial L}{\partial b_k} = \hat{y}_k - y_k$
- $\nabla L = \mathbf{x} \otimes (\hat{\mathbf{y}} - \mathbf{y})$, where
  - $\otimes$ denotes the outer product

Multi-class Perceptron Learning Algorithm

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  - For other classes, do nothing

(Mini-)Batch Training

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  - What is $\mathbf{x}^T \cdot (\hat{\mathbf{y}} - \mathbf{y})$?

(Mini-)Batch Training

$$\begin{bmatrix} x_1^{(1)} & \dots & x_1^{(m)} \\ \vdots & \ddots & \vdots \\ x_n^{(1)} & \dots & x_n^{(m)} \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{y}^{(1)} - y^{(1)} \\ \vdots \\ \hat{y}^{(m)} - y^{(m)} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} (\hat{y}^{(i)} - y^{(i)}) x_1^{(i)} \\ \vdots \\ \sum_{i=1}^{m} (\hat{y}^{(i)} - y^{(i)}) x_n^{(i)} \\ \sum_{i=1}^{m} (\hat{y}^{(i)} - y^{(i)}) \end{bmatrix}$$

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$$

$$
= \begin{bmatrix} \displaystyle\sum_{i=1}^{m} \left(\frac{\partial L}{\partial w_1} \right)^{(i)} \\ \vdots \\ \displaystyle\sum_{i=1}^{m} \left(\frac{\partial L}{\partial w_n} \right)^{(i)} \\ \displaystyle\sum_{i=1}^{m} \left(\frac{\partial L}{\partial b} \right)^{(i)} \end{bmatrix}
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(Mini-)Batch Training

$$
\begin{bmatrix}
x_1^{(1)} & \dots & x_1^{(m)} \\
\vdots & \ddots & \vdots \\
x_n^{(1)} & \dots & x_n^{(m)} \\
1 & 1 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
\hat{y}^{(1)} - y^{(1)} \\
\vdots \\
\hat{y}^{(m)} - y^{(m)}
\end{bmatrix}
=
\begin{bmatrix}
\sum_{i=1}^{m} (\hat{y}^{(i)} - y^{(i)}) x_1^{(i)} \\
\vdots \\
\sum_{i=1}^{m} (\hat{y}^{(i)} - y^{(i)}) x_n^{(i)} \\
\sum_{i=1}^{m} (\hat{y}^{(i)} - y^{(i)})
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
\sum\limits_{i=1}^{m} \left(\frac{\partial L}{\partial w_1} \right)^{(i)} \\
\vdots \\
\sum\limits_{i=1}^{m} \left(\frac{\partial L}{\partial w_n} \right)^{(i)} \\
\sum\limits_{i=1}^{m} \left(\frac{\partial L}{\partial b} \right)^{(i)}
\end{bmatrix}
$$

$$
= \sum_{i=1}^{m} (\nabla L)^{(i)}
$$

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- $\nabla L = \frac{1}{m}\left(\mathbf{x}^T \cdot (\hat{\mathbf{y}} - \mathbf{y})\right)$
  - What is $\mathbf{x}^T \cdot (\hat{\mathbf{y}} - \mathbf{y})$?
    - It computes the sum of the gradients for each document i in the mini-batch!

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  - What is $\mathbf{x}^T \cdot (\hat{\mathbf{y}} - \mathbf{y})$?
    - It computes the sum of the gradients for each document i in the mini-batch!
    - Then to get the average gradient, we just divide by m