PC 5228 FINAL PROJECT

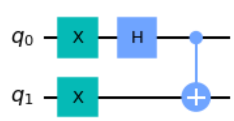
Song Zichen

November 27, 2023

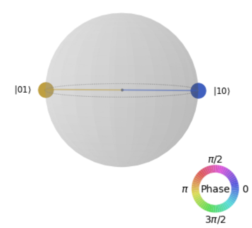
1 Quantum computer test of local realism

1.1 Prepare the singlet state$|\psi_-\rangle$

The quantum circuit created to prepare the singlet state $|\psi_-\rangle$ is shown in Fig. 1a. Starting with the state $|00\rangle$, it is first flipped by an X gate to the state $|11\rangle$. After applying the Hadamard gate, the state becomes $\frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$. Finally, applying the CNOT gate results in the state $|\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. In addition, the quantum state sphere is illustrated in Fig. 1b (note that Qiskit uses a reversed qubit order[1]).



(a) $|\psi_-\rangle$ quantum circuit



(b) $|\psi_-\rangle$ quantum state sphere

Figure 1: Sub-fig(a) shows the process of preparing the state$|\psi_-\rangle$: First, an X gate is applied to both qubits in the quantum circuit. Next, the Hadamard gate was applied to q_0, along with a CNOT gate. Sub-fig(b) illustrates the quantum state sphere representation of $|\psi_-\rangle$.

1.2 Measurements and calculate correlation functions

After preparing the singlet state, we need to choose measurement bases for Alice and Bob to perform measurements. As shown in Fig. 2, Alice was measured on the Z axis and X axis, while Bob was measured on the $(Z+X)/\sqrt{2}$ axis and $(Z-X)/\sqrt{2}$ axis. To measure Bob in these bases, we applied a U3 gate, which is the most general form of a single-qubit rotation gate in Qiskit. It is defined by three parameters: θ, ϕ, and λ. The matrix representation of the U3 gate is given by[1]:

$$U3(\theta,\phi,\lambda) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -e^{i\lambda}\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) & e^{i(\phi+\lambda)}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{1}$$

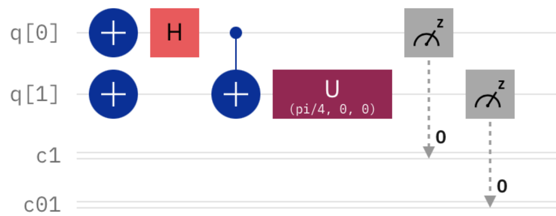
[1] https://qiskit.org/documentation/tutorials/circuits/1_getting_started_with_qiskit.html

To determine the exact values of θ, ϕ, and λ, we need to calculate the eigenvalues and eigenvectors of $(Z+X)/\sqrt{2}$ and $(Z-X)/\sqrt{2}$. This will provide us with the matrix representation of the Unitary gates.

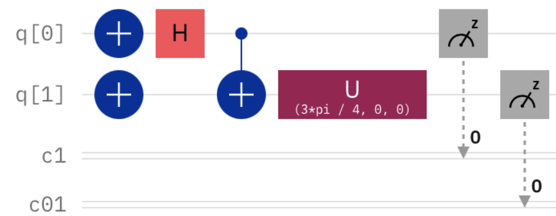
$$U_{(Z+X)/\sqrt{2}} = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1 & \sqrt{2}-1 \\ 1-\sqrt{2} & 1 \end{pmatrix} \tag{2}$$

$$U_{(Z-X)/\sqrt{2}} = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1 & -\sqrt{2}-1 \\ \sqrt{2}+1 & 1 \end{pmatrix} \tag{3}$$

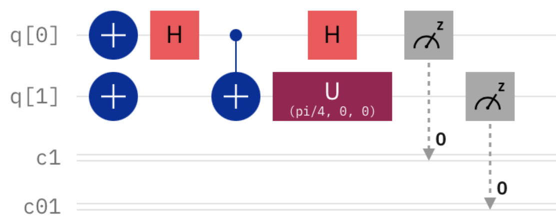
The corresponding angles for these gates are U3($\pi/4$, 0, 0) and U3($3\pi/4$,0,0). After applying the necessary gates to each qubit, we performed measurements on each qubit and stored the results in the corresponding classical bits. We ran this circuit on a physical IBM quantum computer multiple times to gather statistical data.



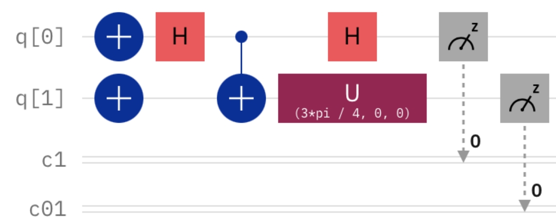
(a)



(b)



(c)



(d)

Figure 2: Measurement circuit. Sub-figure (a) refers to measurements on the Z axis and the $(Z+X)/\sqrt{2}$ axis; (b) refers to measurements on the Z axis and the $(Z-X)/\sqrt{2}$ axis; (c) refers to measurements on the X axis and the $(Z+X)/\sqrt{2}$ axis; (d) refers to measurements on the X axis and the $(Z-X)/\sqrt{2}$ axis.

1.3 Check the Bell-CHSH inequality

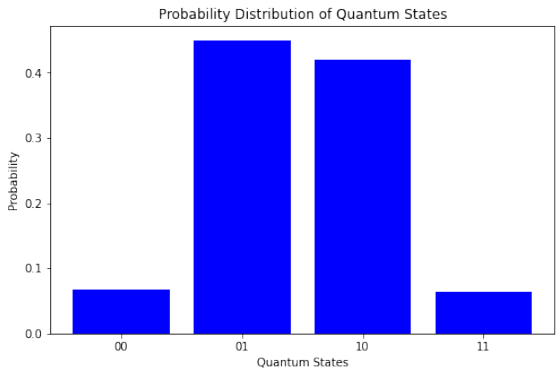
The Bell-CHSH inequilty give by equation:

$$|S| = |C(\hat{z}, \frac{1}{\sqrt{2}}(\hat{z}-\hat{x})) + C(\hat{x}, \frac{1}{\sqrt{2}}(\hat{z}-\hat{x})) + C(\hat{z}, \frac{1}{\sqrt{2}}(\hat{z}+\hat{x})) - C(\hat{x}, \frac{1}{\sqrt{2}}(\hat{z}+\hat{x}))| \tag{4}$$

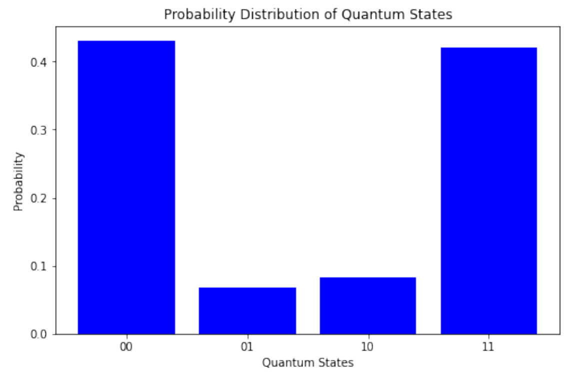
If $|S| \leq \pm 2$, then local realism holds; otherwise, it is violated. In this case, we first ran the circuit shown in Fig. 2 on an IBM Quantum Computer. Each experiment was repeated 2048 times, and the statistical results, as shown in Fig. 3, were obtained. Then, by plugging the data into Eq. 4, we get:

$$2 < |S| = 2.818 < 2\sqrt{2}(2.828) \tag{5}$$

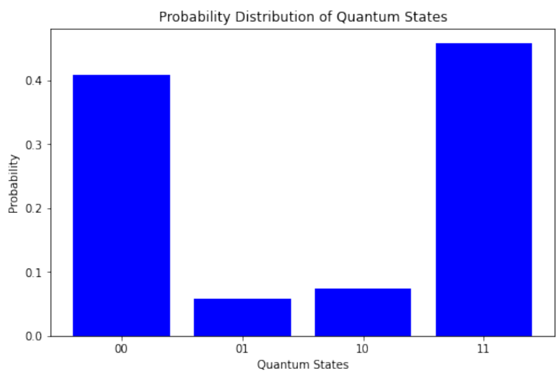
Thus, we have essentially demonstrated that when the initial qubit is $|\psi_{-}\rangle$, this circuit violates the Bell-CHSH inequality.



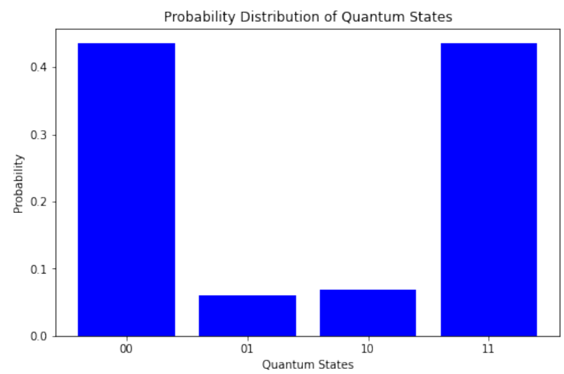
(a) Measured on Z, $(Z+X)/\sqrt{2}$ axis



(b) Measured on Z, $(Z-X)/\sqrt{2}$ axis



(c) Measured on X, $(Z+X)/\sqrt{2}$ axis



(d) Measured on X, $(Z-X)/\sqrt{2}$ axis

Figure 3: Histograms showing the probability distribution of quantum states. The horizontal axis represent quantum states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, every experiment running 2048 times. Sub-figure (a) presents the probability distribution of the circuit in Fig. 2a, (b) refers to the circuit in Fig. 2b, (c) pertains to Fig. 2c, and (d) corresponds to Fig. 2d.

1.4 Results Discuss

It's worth noting that when I first calculated it (repeating the process 1024 times), the result was slightly larger than $2\sqrt{2}$, exceeding the maximal violation of the Bell-CHSH inequality. We believe this can be attributed to several factors:

1. Statistical Fluctuations: With a lower number of shots (e.g., 1024), statistical fluctuations have a more significant impact. This can lead to deviations from the true behavior of the quantum system[4]. Increasing the number of shots to 2048 can provide more data, reducing statistical errors and yielding results closer to theoretical predictions.

2. Quantum Decoherence and Noise: Quantum computers are subject to noise and decoherence, which affect the state of qubits and, consequently, the experimental outcomes[3]. As the number of experimental runs increases, the statistical significance of these effects might become more evident, causing variations in the results.

2 Two-qubit teleportation

2.1 Quantum circuit

To demonstrate two-qubit teleportation, we designed a quantum circuit as shown in Fig. 4. In this circuit, q_0, q_1, q_2, and q_3 belong to Alice, while q_4 and q_5 belong to Bob, who will use them to 'receive' the teleported state. First, we need to create a generalized Bell state $|g_1\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$, which is shared by Alice and Bob. We place a Hadamard gate on q_2 and q_3, followed by a CNOT gate, similar to the first question's solution. Next, Alice performs a joint measurement on the two qubits she intends to teleport (q_0 and q_1), along with the two qubits from the G-state that she shares with Bob (q_2 and q_3)[2].

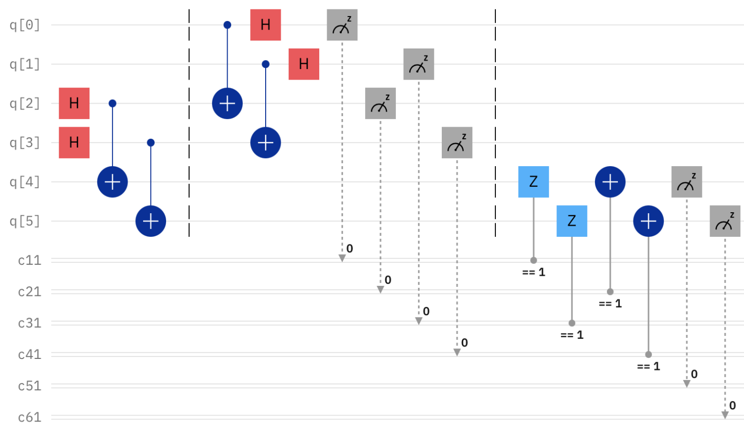


Figure 4: Quantum circuit for two-qubit teleportation. There are six qubit lines and six classical lines. The state Alice wishes to teleport is registered on q_0 and q_1. She shares q_2 and q_3 with Bob, while q_4 and q_5 belong to Bob for receiving the state. We use classical lines to record the measurement outcomes. For example, c_{11} (where the last '1' denotes the number of bits) stores the measurement of q_0. If this outcome is 1, then a Z gate is applied to the state of q_4.

Third, after receiving the four classical messages from Alice, Bob can apply the appropriate unitary operation. If the measurement outcome of q_0 is 1, then he should apply a Z gate on q_4. Similarly, if the measurement outcome of q_1 is 1, then he should apply an X gate on q_4 (and the same applies for q_5). The protocol is thus completed. By measuring q_4 and q_5 in c_{51} and c_{61}, respectively, we can determine whether Alice has successfully teleported her arbitrary two-qubit state.

2.2 Test the circuit

2.2.1 $|\psi_1\rangle = \sqrt{0.1}\,|00\rangle + \sqrt{0.2}\,|01\rangle + \sqrt{0.3}\,|10\rangle + \sqrt{0.4}\,|11\rangle$

To test the circuit, we first initialize the state of the first two qubits to $|\psi_1\rangle$, and then apply the circuit demonstrated in Fig. 4. Next, we run the circuit on the simulator (Qiskit AerSimulator) 1024 times to obtain the test results. To test whether the teleportation is successful, it is necessary to check the final states of q[4] and q[5] (Bob) and compare them to the initial states (Alice). Specifically, we sum the counts of $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ in the first two numbers of the state in the measurement outcome (as shown on the horizontal axis of Fig. 5), and then divide each by the number of shots (1024). The calculation outcome is:

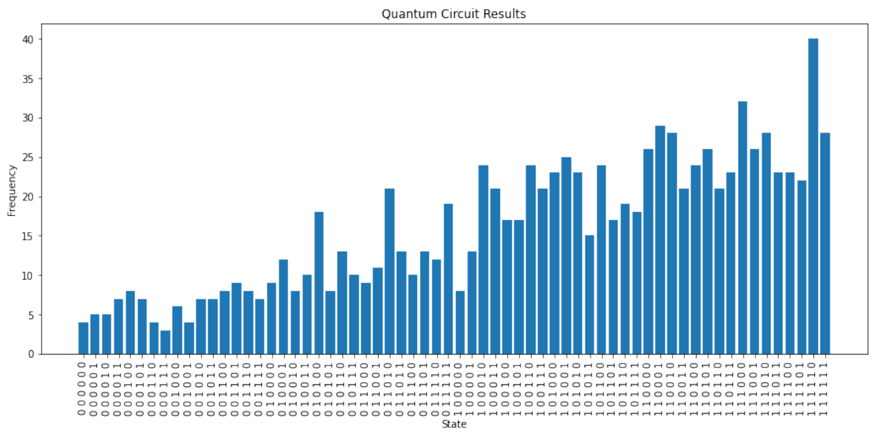


Figure 5: Histogram of $|\psi_1\rangle$ test result. The horizontal axis represent quantum states (reversed qubit order).

State	Simulation	Expect
$\|00\rangle$	0.097	0.1
$\|01\rangle$	0.216	0.2
$\|10\rangle$	0.263	0.3
$\|11\rangle$	0.425	0.4

Table 1: First test result

2.2.2 $|\psi_2\rangle = \sqrt{0.1}\,|00\rangle + \sqrt{0.1}\,|01\rangle\,\sqrt{0.8}\,|11\rangle$

Similarly, we use the same way to test the teleportation of the state $|\psi_2\rangle$, the experiment outcome are shown in Fig. 6 and Table. 2.

State	Simulation	Expect
$\|00\rangle$	0.106	0.1
$\|01\rangle$	0.093	0.1
$\|10\rangle$	0.0	0.0
$\|11\rangle$	0.801	0.8

Table 2: Second test result

2.3 Result discuss

The experiment was conducted in two sets, with each set testing the teleportation of different quantum states. The results are summarized as follows:

Test state $|\psi_1\rangle$

- State $|00\rangle$: Deviation of 0.003. This small deviation indicates a high accuracy in teleportation for this state.
- State $|01\rangle$: Deviation of 0.016. Slightly larger, yet still within an acceptable range, suggesting reasonably accurate teleportation.

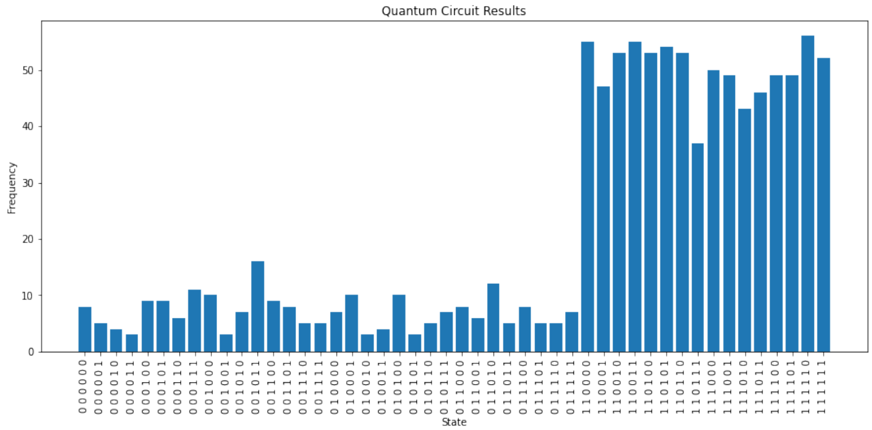


Figure 6: Histogram of $|\psi_2\rangle$ test result.

- State $|10\rangle$: Deviation of 0.037. This is the largest deviation in this set but can be considered within a tolerable range given the complexities of quantum teleportation.
- State $|11\rangle$: Deviation of 0.025. Indicates good accuracy, with only a minor discrepancy from the expected value.

Test state $|\psi_2\rangle$

- State $|00\rangle$: Deviation of 0.006. Shows a very close match to the expected outcome, reflecting successful teleportation.
- State $|01\rangle$: Deviation of 0.007. Also indicates a close match and successful teleportation.
- State $|10\rangle$: Deviation is 0 (Because the coefficient of $|10\rangle$ in initial state is 0).
- State $|11\rangle$: Minimal deviation of 0.001. This near-perfect accuracy further demonstrates the effectiveness of the teleportation protocol.

Besides, fidelity for $|\psi_2\rangle$ and $|\psi_2\rangle$ are approximately 0.9995 and 0.9999. Fidelities near 1 suggest that the teleported states closely match the expected states, demonstrating the success of our teleportation protocol. To sum up, while there are minor discrepancies, they are within the expected range for quantum teleportation experiments, likely due to inherent quantum noise and imperfections in the simulation process. Overall, the protocol demonstrates a high level of fidelity in 2-qubit teleportation.

A Bell CHSH Inequality

```
# 1. Create bell state
from qiskit import QuantumRegister, ClassicalRegister, QuantumCircuit
from numpy import pi

qc = QuantumRegister(2, 'q')
circuit = QuantumCircuit(qc)

circuit.x(qc[0])
circuit.x(qc[1])
circuit.h(qc[0])
circuit.cx(qc[0], qc[1])
circuit.draw('mpl')

# 2. Get new state sphere
from qiskit.quantum_info import Statevector
from qiskit.visualization import plot_state_qsphere
sv = Statevector.from_label('00')
new_sv = sv.evolve(circuit)
print(new_sv)
plot_state_qsphere(new_sv.data)

# 3. Make measurement on different axis
#for example: Measure on $Z$ axis and $(Z+X)/\sqrt{2}$ axis
import matplotlib.pyplot as plt
    # Given data form IBM Quantum Computer
    data = {
    "00": 0.0712890625,
    "01": 0.4306640625,
    "10": 0.42041015625,
    "11": 0.07763671875,
    }

    # Extracting keys and values for plotting
    states = list(data.keys())
    probabilities = list(data.values())

    # Creating the bar chart
    plt.figure(figsize=(8, 5))
    plt.bar(states, probabilities, color='blue')
    plt.xlabel('Quantum States')
    plt.ylabel('Probability')
    plt.title('$Z$ and $(Z+X)/\sqrt{2}$')
    plt.show()

    #correlation coefficient
    #E1=E(Z,(Z+X)/\sqrt{2})
    E1 = (data["00"]+data["11"])-(data["01"]+data["10"])
    print(E1)

# 3. Check the Bell-CHSH inequality
-E1+E2+E3+E4
```

B Two-Qubit Telaportation Quantum Circuit Code

```python
from qiskit import QuantumRegister, ClassicalRegister, QuantumCircuit, Aer, transpile, assemble
from qiskit.visualization import plot_histogram
import matplotlib.pyplot as plt
import numpy as np
from numpy import sqrt
#Create quantum and classical registers
qreg_q = QuantumRegister(6, 'q')
creg_c1 = ClassicalRegister(1, 'c1')
creg_c2 = ClassicalRegister(1, 'c2')
creg_c3 = ClassicalRegister(1, 'c3')
creg_c4 = ClassicalRegister(1, 'c4')
creg_c5 = ClassicalRegister(1, 'c5')
creg_c6 = ClassicalRegister(1, 'c6')
#Create quantum circuit
circuit = QuantumCircuit(qreg_q, creg_c1, creg_c2, creg_c3, creg_c4, creg_c5, creg_c6)
# Initialize the state of the first two qubits
initial_state = [sqrt(0.1), sqrt(0.2), sqrt(0.3), sqrt(0.4)]
circuit.initialize(initial_state, [qreg_q[0], qreg_q[1]])
circuit.h(qreg_q[2]), circuit.h(qreg_q[3])
circuit.cx(qreg_q[2], qreg_q[4])
circuit.cx(qreg_q[3], qreg_q[5])
circuit.cx(qreg_q[0], qreg_q[2])
circuit.cx(qreg_q[1], qreg_q[3])
circuit.h(qreg_q[0]), circuit.h(qreg_q[1])
circuit.measure(qreg_q[0], creg_c1[0])
circuit.measure(qreg_q[2], creg_c2[0])
circuit.measure(qreg_q[1], creg_c3[0])
circuit.measure(qreg_q[3], creg_c4[0])
circuit.z(qreg_q[4]).c_if(creg_c1, 1)
circuit.x(qreg_q[4]).c_if(creg_c2, 1)
circuit.z(qreg_q[5]).c_if(creg_c3, 1)
circuit.x(qreg_q[5]).c_if(creg_c4, 1)
circuit.measure(qreg_q[4], creg_c5[0])
circuit.measure(qreg_q[5], creg_c6[0])
circuit.draw('mpl')

#use Qiskit AerSimulator
simulator = Aer.get_backend('qasm_simulator')
compiled_circuit = transpile(circuit, simulator)
job = simulator.run(compiled_circuit)
result = job.result()
counts = result.get_counts()
# make historgram
sorted_counts = {k: counts[k] for k in sorted(counts)}
x = list(sorted_counts.keys())
y = list(sorted_counts.values())
fig, ax = plt.subplots(figsize=(15, 6))
ax.bar(x, y)
ax.set_xticklabels(x, rotation=90)
plt.show()
```

References

[1] Marcin Musz, Marek Kuś, and Karol Życzkowski. Unitary quantum gates, perfect entanglers, and unistochastic maps. *Physical Review A*, 87(2):022111, 2013.

[2] Gustavo Rigolin. Quantum teleportation of an arbitrary two-qubit state and its relation to multipartite entanglement. *Physical Review A*, 71(3):032303, 2005.

[3] Abdullah Ash Saki, Mahabubul Alam, and Swaroop Ghosh. Study of decoherence in quantum computers: A circuit-design perspective. *arXiv preprint arXiv:1904.04323*, 2019.

[4] Andrea Solfanelli, Alessandro Santini, and Michele Campisi. Experimental verification of fluctuation relations with a quantum computer. *PRX Quantum*, 2(3):030353, 2021.