CONVERGENCE ANALYSIS TO THE DISCRETE EIGENFUNCTION APPROACH FOR FRACTIONAL LAPLACIAN

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ABSTRACT. We propose a solution to the solver accuracy issue of discrete eigenfunction approach for computing solutions to the fractional Laplacian problem. Our approach utilized a reformulation of Kato's formula to design numerical algorithms and do the convergence analysis. With this approach, we can understand the error in the discrete eigenfunction solution to fractional PDE without requiring the knowledge of eigenvalues of discrete operator.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^d$ be a physical domain with Lipschitz boundary. We are primarily interested in the following fractional Poisson problem with homogeneous Dirichlet boundary conditions,

$$(-\Delta)^s u = f, \qquad x \in \Omega, \tag{1a}$$

$$u = 0, \qquad x \in \partial\Omega, \tag{1b}$$

where $s \in (0, 1)$.

Before looking at the fractional Laplacian $(-\Delta)^s$, if we just stare at Δ the classical Laplacian, we will recognize that it is one of the most frequently seen differential operators which are local operators. This is trivial because it is possible to compute the values of output functions (derivative functions) using only knowledge of the values of input functions in an arbitrarily small neighborhood of a point. But the fractional Laplacian $(-\Delta)^s$, as a nonlocal operator, which maps functions on a topological space to functions, in such a way that the value of the output function at a given point cannot be determined solely from the values of the input function in any neighborhood of any point, is given by

$$(-\Delta)^s u = c_{d,s} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{\mid x - y \mid^{d+2s}} dy, \qquad c_{d,s} = \frac{4^s \Gamma(d/2 + s)}{\pi^{d/2} \mid \Gamma(-s) \mid}.$$

The coefficient $c_{d,s}$ involves the Gamma function and serves as a normalizing factor. Thus as a nonlocal operator, the fractional Laplacian $(-\Delta)^s$ has received much attention and played an important role due to the wide range of applications. While in this report, instead of this integral definition, we will focus more on the definition of the fractional Laplacian in spectral sense from the purpose of computation and convergence analysis.

Differential equations involving fractional derivative powers have gained in popularity in recent years. These non-classical differential equations have shown potential to model nonlocal and time-delay effects, making them good candidates for modeling hysteric and

globally-coupled phenomena. For example, fractional differential equations have recently been used to model fluid mechanics, arterial blood flow, cardiac ischemia, and are been used as ingredients in image denoising and image segmentation [17, 14, 7, 10, 1, 2]. Also, fractional PDEs have shown tremendous potential to model applications in geophysics [19] and manifold learning [3]. The details about some of above applications are described here:

In [17], the fractional Laplacian plays an important role in constructing fractional-order viscoelastic models in a one-dimensional blood flow solver. The use of fractional-order models is motivated by recent experimental studies indicating that such models provide a new flexible alternative to fitting biological tissue data. This is attributed to their inherent ability to control the interplay between elastic energy storage and viscous dissipation by tuning a single parameter, the fractional order, as well as to account for a continuous viscoelastic relaxation spectrum. Perform simulations using four viscoelastic parameter data-sets aiming to compare different viscoelastic models and highlight the important role played by the fractional order. The use of a fractional Laplacian operator in the monodomain equation to account for the complex heterogeneous structures in heart tissue,

In [10], Many heart problems are linked to disturbances in cardiac electrical activity, such as wave re-entry caused by ischaemia. In terms of mathematical modelling, the monodomain equation is widely used to model electrical activity in the heart. A fractional Laplacian operator in the monodomain equation is used to account for the complex heterogeneous structures in heart tissue. In this work we consider how to extend this approach to apply to hearts with regions of damaged tissue. This requires the use of a fractional Laplacian operator whose fractional order varies spatially. Results are presented for several test problems in one dimension, demonstrating the effects of different fractional orders in regions of healthy and damaged tissue. (UQ)

There are already several numerical algorithms for computing the solution to differential equations involving fractional derivatives, such as the discrete eigenfunction approach, extension procedure approach, integral approach and etc. The problem of all these approaches is that we don't know if these methods are efficient or not due to the disability of convergence analysis for this fractional Laplace problem. In this report, we will review some of approaches and propose a strategy for the analysis via the reformulation of discretization of fractional Laplacian with the help of Kato's formula and numerically enhance the analysis. All of the proposed methods here apply to the *spectral* definition of the fractional Laplacian. We define a proper Sobolev space in the weak formulation and show the existence and uniqueness of the fractional poisson problem. Since we have few tools in convergence analysis for fractional laplace problem, we propose theoretical results for the convergence analysis using Kato's integral representation formula, which is a more analysis-friendly discretization. With this reformulation, the fractional laplace problem can be converted to the singular perturbation problem, for which we have much more techniques to analyze. Then we show part of the proposals numerically by doing experiments for distinct fractional orders s. Our expectation is to verify the rate of convergence can be measured as a function of the fractional order. We will perform numerical experiments for different types of given functions f in higher dimensions to illustrate this in the future work.

This report is structured as follow. Section 2 firstly gives necessary notations and prove the well-posedness of the problem. In section 3, we introduce the discrete eigenfunction approach and numerical algorithms for computing the solution to differential equations involving fractional derivatives. Section 4 describes another method called integral approach, which involves the Kato's formula. Section 5 provides the discrete eigenfunction approach to the fractional PDE problem via Kato's formula, which gives the convergence results for the discrete eigenfunction algorithm without knowing the eigenvalues of the discrete operators. This can be achieved because we recognize that the discrete eigenfunction approach is equivalent to a more analysis-friendly discretization through an expression for fractional operators call Kato's formula. Section 6 provides some of the numerical experiments to illustrate some results from section 5 and post some unanswered questions.

2. BACKGROUND OF THE FRACTIONAL LAPLACE PROBLEM

2.1. **Notation and setup.** Let $\Omega \subset \mathbb{R}^d$ be a physical domain with Lipschitz boundary. We are primarily interested in $d = 2, 3$. We define

$$L^2(\Omega) \coloneqq \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \|u\| < \infty \right\}, \qquad \|u\|^2 \coloneqq \langle u, u \rangle, \qquad \langle u, v \rangle \coloneqq \int_\Omega u(x)v(x)\mathrm{d}x.$$

We will write $L^2 = L^2(\Omega)$ when the context is clear. The operator $(-\Delta)^s$ in probelem (1) is defined in the spectral sense. In order to this clear, we introduce the eigendecomposition of $(-\Delta)$: there is a countable sequence of non-decreasing positive numbers $\{\lambda_n\}_{n=1}^\infty$, i.e., $0 < \lambda_1 < \lambda_2 < \cdots$, along with an associated sequence of functions $\{\phi_n\}_{n=1}^\infty$ such that,

$$-\Delta\phi_n = \lambda_n\phi_n, \qquad n \in N. \tag{2}$$

The eigenfunctions $\{\phi_n\}_{n=1}^\infty$ are L^2-orthogonal and complete, so that

$$u \in L^2 \implies u = \sum_{n=1}^{\infty} \widehat{u}_n\phi_n, \tag{3}$$

where, if we futher assume that $\{\phi_n\}_{n=1}^\infty$ is an ortho*normal* set, then,

$$\widehat{u}_n = \langle u, \phi_n \rangle .$$

Therefore, under the assumption that u has the expansion (3), then (2) implies that the following formal sum is true:

$$-\Delta u = \sum_{n=1}^{\infty} \lambda_n \widehat{u}_n \phi_n.$$

Based on this summation, the *spectral* definition of the fractional Laplacian is

$$(-\Delta)^s u \coloneqq \sum_{n=1}^{\infty} \lambda_n^s \widehat{u}_n \phi_n, \qquad s \in (0, 1). \tag{4}$$

The definition above is the starting point for developing algorithms.

Define a finite-dimensional function space V with $\dim V = N < \infty$. If $\{\chi_j\}_{j=1}^N$ denotes any basis for this space, then we define mass and stiffness matrices $\boldsymbol{M} \in \mathbb{R}^{N\times N}$ and $\boldsymbol{S} \in \mathbb{R}^{N\times N}$ as,

$$(\boldsymbol{M})_{j,k} = \langle \chi_k, \chi_j \rangle, \qquad (\boldsymbol{S})_{j,k} = \sum_{q=1}^{d} \left\langle \frac{\partial \chi_k}{\partial x_q}, \frac{\partial \chi_j}{\partial x_q} \right\rangle, \tag{5}$$

The mass and stiffness matrices correspond to variational forms of the identity and Laplacian operator, i.e., let $v, w \in V$ have expansions

$$v = \sum_{n=1}^{N} v_n \chi_n, \qquad w = \sum_{n=1}^{N} w_n \chi_n,$$

which can be represented by the coordinate vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^N$. Then,

$$\langle v, w \rangle = \boldsymbol{w}^T \boldsymbol{M} \boldsymbol{v}, \qquad \langle \Delta v, w \rangle = \boldsymbol{w}^T \boldsymbol{S} \boldsymbol{v},$$

where the second equality uses the Divergence Theorem and the homogeneous Dirichlet boundary conditions as an intermediate step.

The mass matrix also allows us to compute L^2 norms from vectors in $\mathbb{R}^N$. Let $\boldsymbol{c} \in \mathbb{R}^N$ have entries that correspond to expansion coefficients defining an element of V:

$$\boldsymbol{c} \in \mathbb{R}^N \iff c = \sum_{n=1}^{N} c_n v_n \in V.$$

Then we observe that

$$\|c\|^2 = \sum_{n,m=1}^{N} c_n c_m (\boldsymbol{M})_{n,m} = \boldsymbol{c}^T \boldsymbol{M} \boldsymbol{c}.$$

2.2. **Existence and uniqueness of the fractional Poisson problem.** For the Poisson problem (1), by defining the Sobolev space and its dual space in the weak formulation properly, we can show the uniqueness of the fractional poisson problem by Lax-Milgram theorem.

Define the Sobolev space

$$H_0^s := \{ f \in L^2(\Omega) \mid f = \sum_{j=1}^{\infty} \widehat{f}_j \phi_j, \ \sum_{j=1}^{\infty} |\widehat{f}_j|^2 \lambda_j^s < \infty, \ f|_{\partial\Omega} = 0 \}. \tag{6}$$

where λ_j, ϕ_j are defined in (2) and its dual space

$$H^{-s} := \{ f \in L^2(\Omega) \mid f = \sum_{j=1}^{\infty} \widehat{f}_j \phi_j, \ \sum_{j=1}^{\infty} |\widehat{f}_j|^2 \lambda_j^{-s} < \infty \}. \tag{7}$$

Define the bilinear form on H_0^s as

$$B(u, v) = \langle (-\Delta)^s u, v \rangle = \langle f, v \rangle, \forall v \in H_0^s. \tag{8}$$

We can prove that H^{-s} is the dual space H^{-s} of H_0^s by showing the boundness of the functionals in the dual space. Let $f \in H_0^s$, then for $l \in H^{-s}$, we have

$$\begin{aligned}
|\langle l, f\rangle_{L^2}| &= |\left\langle \sum_{j=1}^{\infty} \widehat{l}_j \phi_j, \sum_{j=1}^{\infty} \widehat{f}_j \phi_j \right\rangle_{L^2}| \\
&= |\sum_{j=1}^{\infty} \widehat{l}_j \widehat{f}_j| \\
&= |\sum_{j=1}^{\infty} \widehat{l}_j \lambda_j^{-s} \lambda_j^s \widehat{f}_j| \\
&\leq (\sum_{j=1}^{\infty} |\widehat{l}_j \lambda_j^{-s}|^2)^{\frac{1}{2}} (\sum_{j=1}^{\infty} |\widehat{f}_j \lambda_j^s|^2)^{\frac{1}{2}} \\
&= \|l\|_{H^{-s}} \|f\|_{H_0^s}
\end{aligned}$$

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a physical domain with Lipschitz boundary. Assume $s > \frac{1}{2}$, then for any $f \in H^{-s}$, there is a unique solution $u \in H_0^s$ to* (8).

Proof. We firstly prove the continuity and coercivity in Lax-Milgram theorem.

The bilinear form on H_0^s can be obtained by Garlerkin as

$$\begin{aligned}
B(u,u) &= \langle (-\Delta)^s u, u\rangle \\
&= \left\langle \sum_{j=1}^{\infty} \lambda_j^s \widehat{u}_j \phi_j, \sum_{j=1}^{\infty} \widehat{u}_j \phi_j \right\rangle \\
&= \sum_{j=1}^{\infty} \lambda_j^s |\widehat{u}_j|^2 \\
&= \|u\|_{H_0^s}^2
\end{aligned}$$

Then there for $a = 1, b = 1$, we have:

$$\sup_{u \in H_0^s} |B(u,u)| \leq a\|u\|_{H_0^s}^2, \forall u \in H_0^s \tag{9}$$

and

$$\inf_{u \in H_0^s} |B(u,u)| \geq b\|u\|_{H_0^s}^2, \forall u \in H_0^s \tag{10}$$

Now we can use the Lax-Milgram theorem to show the uniqueness of u in (8). □

Note the Sobolev space H_0^s defined in (6) is in general distinct from the classical fractional Sobolev space $\tilde{H}_0^s$, defined in [16] as

$$\tilde{H}_0^s(\Omega) := \{f \in L^2(\Omega) \mid \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}} dxdy < \infty\}$$

For $s > \frac{1}{2}$, we can draw an equivalence between H_0^s and $\tilde{H}_0^s$, see [6, 4] for an overview and [16, Thm. 7.7 of Ch. 1] for details.

We also need to mention that $s > \frac{1}{2}$ is required for $f|_{\partial\Omega} = 0$ in the classical sense based on trace theorems. However, appropriate trace-type theorems for $s \leq \frac{1}{2}$ can be defined so that the fractional Poisson problem is still well-posed.

3. DISCRETE EIGENFUNCTION METHOD AND ALGORITHM

Define a finite-dimensional function space V with $\dim V = N < \infty$. If $\{\chi_j\}_{j=1}^N$ denotes any basis for this space, then we define mass and stiffness matrices $\boldsymbol{M} \in \mathbb{R}^{N\times N}$ and $\boldsymbol{S} \in \mathbb{R}^{N\times N}$ as,

$$(\boldsymbol{M})_{j,k} = \langle \chi_k, \chi_j \rangle, \qquad (\boldsymbol{S})_{j,k} = \sum_{q=1}^{d} \left\langle \frac{\partial \chi_k}{\partial x_q}, \frac{\partial \chi_j}{\partial x_q} \right\rangle, \tag{11}$$

The mass and stiffness matrices correspond to variational forms of the identity and Laplacian operator, i.e., let $v, w \in V$ have expansions

$$v = \sum_{n=1}^{N} v_n \chi_n, \qquad w = \sum_{n=1}^{N} w_n \chi_n,$$

which can be represented by the coordinate vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^N$. Then,

$$\langle v, w \rangle = \boldsymbol{w}^T \boldsymbol{M} \boldsymbol{v}, \qquad \langle \Delta v, w \rangle = \boldsymbol{w}^T \boldsymbol{S} \boldsymbol{v},$$

where the second equality uses the Divergence Theorem and the homogeneous Dirichlet boundary conditions as an intermediate step.

The mass matrix also allows us to compute L^2 norms from vectors in $\mathbb{R}^N$. Let $\boldsymbol{c} \in \mathbb{R}^N$ have entries that correspond to expansion coefficients defining an element of V:

$$\boldsymbol{c} \in \mathbb{R}^N \iff c = \sum_{n=1}^{N} c_n v_n \in V.$$

Then we observe that

$$\|c\|^2 = \sum_{n,m=1}^{N} c_n c_m (\boldsymbol{M})_{n,m} = \boldsymbol{c}^T \boldsymbol{M} \boldsymbol{c}.$$

One numerical scheme for solving (1) is a so-called discrete eigenfunction method [11, 12, 20, 18], and uses a truncated and inverted version of (4). We provide a short derivation of this algorithm below.

The formal inverse of (4) applied to the known function f on the right-hand side of (1) is

$$u = (-\Delta)^{-s} f = \sum_{n=1}^{\infty} \lambda_n^{-s} \widehat{f}_n \phi_n, \qquad \widehat{f}_n = \langle f, \phi_n \rangle.$$

Since $\lambda_n \uparrow \infty$, then one supposes that this strategy could be accurate if approximations to the eigenvalues λ_n and eigenfunctions ϕ_n could be computed. The general strategy to compute such an approximate eigendecomposition of the continuous operator $(-\Delta)$ is to use the spectrum of the finite element discretized problem.

The (continuous) eigenvalue problem

$$-\Delta u = \lambda u, \tag{12}$$

defining (λ_n, ϕ_n) can be discretized using the mass and stiffness matrix introduced above. Using the ansatz

$$u \simeq u_V = \sum_{n=1}^{N} \widehat{u}_n \chi_n \in V,$$

then substituting this into (12) and imposing that the residual of this equation vanish on V yields the equation

$$\boldsymbol{S}\boldsymbol{u} = \lambda \boldsymbol{M}\boldsymbol{u},$$

where $\boldsymbol{u} \in \mathbb{R}^N$ contains the vector of coordinates of $u_V \in V$. We seek N linearly independent vectors $\{\boldsymbol{\phi}_n^N\}_{n=1}^N$ and scalars $\{\lambda_n^N\}_{n=1}^N$ that satisfy the generalized eigenvalue problem above. Since $\{\chi_n\}_{n=1}^N$ is a basis for V, then $\boldsymbol{M}$ is invertible.

Lemma 3.1. *Let $\boldsymbol{M}$ and $\boldsymbol{S}$ the mass and stiffness matrices, respectively, defined in* (11)*. Consider the generalized eigenvalue problem for the eigenvalue λ and the vector $\boldsymbol{\psi}$,*

$$\boldsymbol{M}^{-1}\boldsymbol{S}\boldsymbol{\psi} = \lambda\boldsymbol{\psi}.$$

There are N linearly independent eigenpairs $\left\{\left(\lambda_n \boldsymbol{\phi}_n^N\right)\right\}_{n=1}^N$ that satisfy the above equation, with $\lambda_n^N \in (0, \infty)$ for all n. Furthermore, define vectors $\{\boldsymbol{p}_n^N\}_{n=1}^N$ via the relation

$$\boldsymbol{P} = \left[\boldsymbol{p}_1^N,\ \boldsymbol{p}_2^N,\ \ldots,\ \boldsymbol{p}_N^N\right] \coloneqq \sqrt{\boldsymbol{M}}\left[\boldsymbol{\phi}_1^N,\ \boldsymbol{\phi}_2^N,\ \ldots,\ \boldsymbol{\phi}_N^N\right]. \tag{13}$$

Then $\left\{\left(\lambda_n^N, p_n^N\right)\right\}_{n=1}^N$ are eigenpairs for the symmetric positive-definite matrix $\boldsymbol{A}$ defined as

$$\boldsymbol{A} \coloneqq \sqrt{\boldsymbol{M}^{-1}}\boldsymbol{S}\sqrt{\boldsymbol{M}^{-1}}, \tag{14}$$

and so $\left[\boldsymbol{p}_1^N,\ \boldsymbol{p}_2^N,\ \ldots,\ \boldsymbol{p}_N^N\right]$ are mutually orthogonal. Thus there exist a diagonal matrix $\boldsymbol{K} = \boldsymbol{diag}(\frac{1}{k1}, \ldots, \frac{1}{k_N})$ such that $\boldsymbol{P}\boldsymbol{K}$ is an orthogonal matrix, where k_i is the square root of the sum of squares of elements of the i-th column of $\boldsymbol{P}$.

The eigenvectors that solve the above problem can be used to define functions in V:

$$\phi_n \coloneqq \sum_{j=1}^{N} (\boldsymbol{\phi}_n^N/k_n)_j \chi_j \in V. \tag{15}$$

By (13), these functions are L^2-orthonormal:

$$\langle \phi_n, \phi_m \rangle = \left(\boldsymbol{\phi}_m^N/k_m\right)^T \boldsymbol{M}\left(\boldsymbol{\phi}_n^N/k_n\right) = \left(\boldsymbol{p}_m^N/k_m\right)^T \sqrt{\boldsymbol{M}^{-1}}\boldsymbol{M}\sqrt{\boldsymbol{M}^{-1}}\left(\boldsymbol{p}_n^N/k_n\right) = \delta_{n,m},$$

where we have used the fact that $\boldsymbol{M}$ is symmetric and positive definite, and that $\boldsymbol{P}\boldsymbol{K} = \sqrt{\boldsymbol{M}}\boldsymbol{\Phi}\boldsymbol{K}$ is an orthogonal matrix.

These functions are eigenfunctions of this "discretized" Laplacian operator, which can be used to solve the fractional PDE (1). The algorithm is as follows: given any $f \in L^2$, compute the entries

$$f_n \coloneqq \langle f, \chi_n \rangle, \qquad \boldsymbol{f} = (f_1,\ f_2,\ \ldots,\ f_N)^T, \tag{16}$$

so that $\boldsymbol{M}^{-1}\boldsymbol{f}$ gives coordinates of the V-projection of f in the basis $\{\chi_j\}_{j=1}^N$. Next, we compute the coordinates of this V-projection of f in the basis $\{\phi_n\}_{n=1}^N$. The relation (15) shows that if $\boldsymbol{c}_\phi \in \mathbb{R}^N$ is a vector of ϕ_n-coordinates for some element in V, then the corresponding vector $\boldsymbol{c}_\chi$ of coordinates in the basis χ_n is

$$\sum_{n=1}^N c_{\chi,n}\chi_n = \sum_{n=1}^N c_{\phi,n}\phi_n \quad \Longleftrightarrow \quad \boldsymbol{c}_\chi = \boldsymbol{\Phi}\boldsymbol{K}\boldsymbol{c}_\phi.$$

Therefore, the vector

$$(\boldsymbol{\Phi}\boldsymbol{K})^{-1}\boldsymbol{M}^{-1}\boldsymbol{f}$$

contains the ϕ-coordinates of the V-projection of f. The next step corresponds to multiplying each component of this vector by the appropriate power of an eigenvalue. Since we wish to apply $(-\Delta)^{-s}$, we multiply λ_n^{-s} to element n of the vector above. Finally, we transform back into χ-coordinates via the map $\boldsymbol{\Phi}$. Thus, the solution u_V computed via the discrete eigenfunction method is

$$u_V = \sum_{j=1}^N u_n\chi_n, \qquad \boldsymbol{u} \coloneqq (u_1,\ u_2,\ ,\ldots,\ u_N)^T \in \mathbb{R}^N, \tag{17a}$$

with

$$\boldsymbol{u} = \boldsymbol{\Phi}\boldsymbol{K}\boldsymbol{\Lambda}^{-s}(\boldsymbol{\Phi}\boldsymbol{K})^{-1}\boldsymbol{M}^{-1}\boldsymbol{f}, \qquad \boldsymbol{\Lambda} \coloneqq \operatorname{diag}\left(\lambda_1^N, \ldots, \lambda_N^N\right), \tag{17b}$$

and $\boldsymbol{\Lambda}^{-s}$ denotes componentwise exponentiation on the diagonal.

Using Lemma 3.1 we can rewrite the expression above. Since $\boldsymbol{P}\boldsymbol{K} = \sqrt{\boldsymbol{M}}\boldsymbol{\Phi}\boldsymbol{K}$ is an orthogonal matrix, then $\boldsymbol{u}$ can be computed via:

$$\begin{aligned}
\boldsymbol{u} &= \boldsymbol{\Phi}\boldsymbol{K}\boldsymbol{\Lambda}^{-s}\boldsymbol{K}^{-1}\boldsymbol{\Phi}^{-1}\boldsymbol{M}^{-1}\boldsymbol{f}, \\
&= \boldsymbol{\Phi}\boldsymbol{K}\boldsymbol{\Lambda}^{-s}\boldsymbol{K}^{-1}\boldsymbol{P}^{-1}\sqrt{\boldsymbol{M}^{-1}}\boldsymbol{f}, \\
&= \boldsymbol{\Phi}\boldsymbol{K}\boldsymbol{\Lambda}^{-s}(\boldsymbol{P}\boldsymbol{K})^{-1}\sqrt{\boldsymbol{M}^{-1}}\boldsymbol{f} \\
&= \boldsymbol{\Phi}\boldsymbol{K}\boldsymbol{\Lambda}^{-s}(\boldsymbol{P}\boldsymbol{K})^{T}\sqrt{\boldsymbol{M}^{-1}}\boldsymbol{f} \\
&= \boldsymbol{\Phi}\boldsymbol{K}\boldsymbol{\Lambda}^{-s}\boldsymbol{K}^{T}(\sqrt{\boldsymbol{M}^{-1}}P)^{T}\boldsymbol{f} \\
&= \boldsymbol{\Phi}\boldsymbol{K}\boldsymbol{\Lambda}^{-s}\boldsymbol{K}^{T}\boldsymbol{\Phi}^{T}\boldsymbol{f}, \qquad (18)
\end{aligned}$$

showing that one need not compute $\boldsymbol{\Phi}^{-1}$. Algorithm 1 summarizes the steps required to compute u_V via the discrete eigenfunction approach.

While this algorithm is simple to mathematically understand and implement, there are two drawbacks to this approach:

Algorithm 1 Discrete Eigenfunction Algorithm

Input: Finite element space V and basis functions $\{\chi_n\}_{n=1}^N$, fractional order s, source function $f \in L^2$

Output: Vector of expansion coefficients $\boldsymbol{u}$ defining $u_V \in V$, an approximate solution to (1).

1: Compute $\left\{\left(\lambda_n^N, \boldsymbol{\phi}_n^N\right)\right\}_{n=1}^N$ as eigenpairs of the matrix $\boldsymbol{M}^{-1}\boldsymbol{S}$.
2: Assemble $\boldsymbol{\Phi}$ as in (15) and $\boldsymbol{\Lambda}$ as in (18).
3: Compute $\boldsymbol{f}$ from f via (16).
4: **Return** $\boldsymbol{u}$ computed from (18).

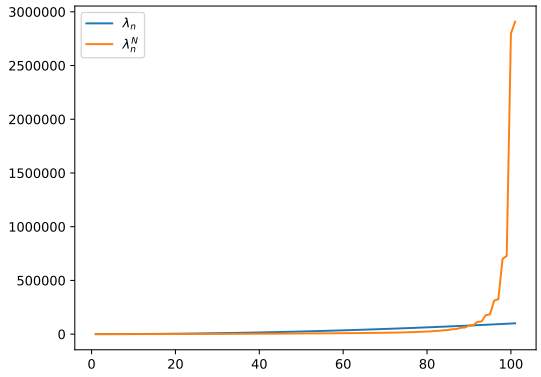


Figure 1. eigenvalues computed by continuous and discretized operators, $s = 0.8$

- *Computational feasibility* – Computing the spectrum of $\boldsymbol{M}^{-1}\boldsymbol{S}$ can be expensive when the dimension of the finite element space N is very large. It is common in practical scenarios to have $N \gtrsim 10^6$, so that direct application of eigenvalue solvers is challenging. Much recent work [20, 18] has gone into ameliorating this drawback, so that modern computational packages now have the ability to compute the spectrum of $\boldsymbol{M}^{-1}\boldsymbol{S}$ even if N is very large.
- *Solver Accuracy* – It is difficult to understand the accuracy of the solution u_V computed via Algorithm 1. The main challenge is understanding how accurately the discrete eigenpairs $\left\{\left(\lambda_n^N, \boldsymbol{\phi}_n^N\right)\right\}_{n=1}^N$ of the matrix $\boldsymbol{M}^{-1}\boldsymbol{S}$ reflect the continuous eigenpairs $\{(\lambda_n, \phi_n)\}_{n=1}^N$ of the operator $-\Delta$. It is well-known for finite element discretiztions that much of the "upper" part of the discrete spectrum is essentially "wrong", and this lack of accuracy in the spectrum makes the accuracy of the computed u_V difficult to understand. This is an open problem that we seek to address in this paper.

For the second issue, we use an easy example to illustrate the problem: We set the fractional Laplacian problem in a cheating way: let $f = \sin(n\pi x)$ and based on the eigendecomposition of continuous operator $-\Delta$, we can derive the countable sequence of non-decreasing positive numbers $\{\lambda_n\}_{n=1}^\infty$: $\lambda_n = n\pi^2$. Then comparing these eigenvalues

with those computed using the spectral expansion of the associated discretized operators in Figure 1, we can find that a huge gap occurs when n is large enough.

This shows that for finite element discretions that much of the "upper" part of the discrete spectrum is essentially "wrong", and this lack of accuracy in the spectrum makes the accuracy of computed numerical solution difficult to understand. This is an open problem that we seek to address in this report. This problem motivates us to use some efficient high-performance scalable solvers like Nektar++ to compute eigenpairs accurately in the future work.

Note that this method is not the algorithm we are proposing in this report. In the remainder of this paper, we provide a solution to the second issue above by deriving convergence results for the discrete eigenfunction algorithm. The crux of the idea is to recongize that the discrete eigenfunction approach is equivalent to a more analysis-friendly discretization through an expression for fractional operators called Kato's formula.[1]

4. Integral operator approach

Given data $f \in L^2$, our main goal is to compute the solution u to (1) for arbitrary $s \in (0,1)$. Notationally, we omit showing explicit dependence of u on the spatial variable x, and only show dependence on the fractional order s, which is a parameter. This convention will be used in the remainder of this report when considering solutions to parameterized PDE's: notational dependence on parameters will be explicit, but that on the spatial variable x will be implicit. Therefore, we let $u(s) \in L^2$ denote the solution u to (1) for a fixed value of s. We will be interested in developing a computational algorithm for computing the family or manifold of solutions

$$U \coloneqq \{u(s) \mid s \in (0,1)\}$$

4.1. Kato's formula.

The method proposed here is called an integral operator approach, which writes the solution as a type of Dunford-Taylor integral involving the resolvent of the classical operator, see [13, 8, 5].

The following remarkable result provides an appealing formula for the solution u to (1):

$$u(s) = \beta(s) \int_{-\infty}^{\infty} e^{(1-s)y}(-\Delta + e^y)^{-1} f \, dy, \qquad \beta(s) \coloneqq \frac{\sin \pi s}{\pi}$$

which is a reformulation of Kato's formula [Theorem 2 with $\lambda = 0$] [13]. This representation was first exploited in [8] for designing numerical algorithms, and is derived via a special kind of Dunford-Taylor integral. To write the above more explicitly, define $q(y)$, for fixed $y \in \mathbb{R}$, as the solution to the classical $y-$parameterized PDE,

$$(-\Delta)q(y) + e^y q(y) = f, \qquad x \in \Omega, \tag{19a}$$

$$u = 0, \qquad x \in \partial\Omega, \tag{19b}$$

[1] Kato's formula in the context of fractional PDE's has recently also gone by the name of the Balakrishnan formula or a Dunford-Taylor formula [8, 15].

Then u is given by

$$u(s) = \beta(s) \int_{-\infty}^{\infty} q(y) e^{(1-s)y} dy \tag{20}$$

This representation reveals that u is actually just an integral of solutions q to classical Laplace-type problems. A solution method employing a discretization of the above formula then only requires solves of classical local PDE's in order to solve the nonlocal problem (1). The straightforward way to compute the solution via (20) is to approximate the integral with a quadrature rule. This would require computing solutions $q(y)$ to the PDE (1) for many values of the parameter y.

More precisely, let $\{y_m, \tau_m\}$ be a quadrature rule for approximating the integral in (20). Then we can approximate the solution $u(s)$ as

$$u(s) \approx u_M(s) = \sum_{\sigma \in \{+,-\}} \beta^\sigma(s) \sum_{m=1}^{M} \tau_m w^\sigma(y_m / s^\sigma) e^{-y_m}$$

One the need only to compute the ensemble of functions $\{q(y_m)\}_{m=1}^M$, which are solutions to classical PDE's in order to approximate the solution to the fractional problem. The observation we make is that the approach above requires approximately M times the work of a classical problem; when M is large (which can be required when s is small), this may become computationally prohibitive.

4.2. **Fractional Laplace solutions via integral formulation.** Our approach to solving (1) uses Kato's formula [Theorem 2 with $\lambda = 0$] [13]. Our starting point is the version of this formula as written in Lemma 3.1 of [9]. Assume $f \in L^2$. The solution $u(s)$ to (1) can be expressed as

$$u(s) = \sum_{\sigma \in \{+,-\}} \beta^\sigma(s) \int_0^{\infty} w^\sigma(y/s^\sigma) e^{-y} \mathrm{d}y, \tag{21}$$

where $\beta^\pm$ are the constants

$$\beta^\pm(s) \coloneqq \frac{\sin(\pi s^\pm)}{\pi s^\pm} = \mathrm{sinc}(s^\pm), \qquad\qquad s^\pm \coloneqq \frac{1}{2} \pm \left(s - \frac{1}{2} \right).$$

Proof.

$$
\begin{aligned}
u(s) &= \beta(s)\int_{-\infty}^{0} e^{(1-s)y} w(1,e^y)dy + \beta(s)\int_0^{\infty} e^{(1-s)y} w(1,e^y)dy \\
&= \beta(1-s)\int_0^{\infty} e^{-(1-s)y} w(1,e^y)dy + \beta(s)\int_0^{\infty} e^{(1-s)y}(-\Delta+e^y)^{-1} f dy \\
&= \beta(1-s)\int_0^{\infty} e^{-z} w^-(\frac{z}{1-s})\frac{1}{1-s}dz + \beta(s)\int_0^{\infty} e^{(1-s)y}e^{-y}(e^{-y})^{-1}(-\Delta+e^y)^{-1} f dy \\
&= \beta^-(s)\int_0^{\infty} w^-(\frac{y}{s^-})e^{-y}dy + \beta(s)\int_0^{\infty} e^{-z} w^+(\frac{z}{s})\frac{1}{s}dz \\
&= \beta^-(s)\int_0^{\infty} w^-(\frac{y}{s^-})e^{-y}dy + \beta^+(s)\int_0^{\infty} w^+(\frac{y}{s^+})e^{-y}dy \\
&= \sum_{\sigma\in\{+,-\}} \beta^\sigma(s)\int_0^{\infty} w^\sigma(y/s^\sigma)e^{-y}\mathrm{d}y
\end{aligned}
$$

here we use the fact that $\beta(s) = \frac{\sin(\pi s)}{\pi} = \beta(1-s)$ and many times of change of variables □

The functions $w^\pm(y) \in L^2$ for each fixed $y \in [0,\infty)$, are solutions to the following *classical* PDE's in the interior of Ω:

$$
\left(-\Delta + e^{-y}\right) w^-(y) = f, \qquad\qquad \left(-e^{-y}\Delta + 1\right) w^+(y) = f, \tag{22}
$$

both supplemented with homogeneous Dirichlet boundary conditions on $\partial\Omega$. The utility of (21) is that, if one discretizes the y integral with quadrature, then given a quadrature rule $\{y_m, \tau_m\}_{m=1}^M$, the solution to a fractional PDE $u(s)$ can be computed simply by repeatedly computing solutions to *classical* PDE's as $u(s) \approx u_M(s) = \sum_{\sigma\in\{+,-\}} \beta^\sigma(s) \sum_{m=1}^{M} \tau_m w^\sigma(y_m/s^\sigma)e^{-y_m}$. This formula will also allow us to derive error estimates. However, our first step is to describe a discrete eigenfunction algorithm for solving the fractional PDE (1) that uses (21).

5. DISCRETE EIGENFUNCTION SOLUTION VIA KATO'S FORMULA

5.1. **Discrete eigenfunction solution.** We first compute discrete solutions $w_V^\pm(y)$ to the PDEs (22) in the standard way. With a basis $\{\chi_n\}_{n=1}^N$ for V as before, we adopt the ansatzes $w_V^\pm(y)$ and define the corresponding vectors $\boldsymbol{w}^\pm(y)$ as,

$$
w_V^\pm(y) = \sum_{n=1}^{N} w_n^\pm(y)\chi_n, \qquad\qquad \boldsymbol{w}(y)^\pm \coloneqq \left(w_1^\pm(y),\ \ldots,\ w_N^\pm(y)\right)^T \in \mathbb{R}^N. \tag{23}
$$

Using these in (22), the Galerkin discretization of those equations are

$$
\left(\boldsymbol{S} + e^{-y}\boldsymbol{M}\right)\boldsymbol{w}^-(y) = \boldsymbol{f}, \qquad\qquad \left(e^{-y}\boldsymbol{S} + \boldsymbol{M}\right)\boldsymbol{w}^+(y) = \boldsymbol{f}. \tag{24}
$$

Both of these linear algebra equations are uniformly well-posed for $y \in [0,\infty)$, and hence $w_V^\pm(y) \in V$ are well-defined functions. Our main result in this section is that if we use $w_V^\pm(y)$ in place of $w^\pm(y)$ in (21), we recover the discrete eigenfunction solution u_V.

Theorem 5.1. *With u_V the discrete eigenfunction solution as defined in* (17)*, then*

$$u_V(s) = \sum_{\sigma\in\{+,-\}} \beta^\sigma(s) \int_0^\infty w_V^\sigma(y/s^\sigma) e^{-y} \mathrm{d}y.$$

Proof. First, since $w_V^\sigma \in V$, then clearly the right-hand side of Kato's formula is an element of V. Therefore, we can write

$$\sum_{\sigma\in\{+,-\}} \beta^\sigma(s) \int_0^\infty w_V^\sigma(y/s^\sigma) e^{-y} \mathrm{d}y =: \sum_{n=1}^N \widetilde{u}_n \chi_n,$$

for some coefficients $\widetilde{\boldsymbol{u}} = (\widetilde{u}_1, \ldots, \widetilde{u}_N)^T \in \mathbb{R}^N$. Since $\{\chi_n\}_{n=1}^N$ are linearly independent, we therefore have the vector-valued equality

$$\widetilde{\boldsymbol{u}}(s) = \sum_{\sigma\in\{+,-\}} \beta^\sigma(s) \int_0^\infty \boldsymbol{w}^\sigma(y/s^\sigma) e^{-y} \mathrm{d}y. \tag{25}$$

The proof will be complete if we can show that $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$, with $\boldsymbol{u}$ the discrete eigenfunction expansion coefficients as given in (18).

The first step is to show that $\widetilde{\boldsymbol{u}}$ satisfies a relation that is very similar to that for $\boldsymbol{u}$. Recalling the definition of $\boldsymbol{A}$ in (14), since the orthogonal matrix $\boldsymbol{PK}$ is comprised of the eigenvectors of $\boldsymbol{A}$, then

$$\boldsymbol{A} = (\boldsymbol{PK})\boldsymbol{\Lambda}(\boldsymbol{PK})^T$$

Thus for $\boldsymbol{w}^+$ we have

$$\begin{aligned}
\boldsymbol{w}^+(y/s^+) &= \left(e^{-y/s^+}\boldsymbol{S} + \boldsymbol{M}\right)^{-1} \boldsymbol{f} \\
&= \sqrt{\boldsymbol{M}^{-1}} \left(e^{-y/s^+}\boldsymbol{A} + \boldsymbol{I}\right)^{-1} \sqrt{\boldsymbol{M}^{-1}} \boldsymbol{f} \\
&= \sqrt{\boldsymbol{M}^{-1}} \left(e^{-y/s^+}(\boldsymbol{PK})\boldsymbol{\Lambda}(\boldsymbol{PK}^T) + (\boldsymbol{PK})(\boldsymbol{PK})^T\right)^{-1} \sqrt{\boldsymbol{M}^{-1}} \boldsymbol{f} \\
&= \sqrt{\boldsymbol{M}^{-1}}(\boldsymbol{PK}) \left(e^{-y/s^+}\boldsymbol{\Lambda} + \boldsymbol{I}\right)^{-1} (\boldsymbol{PK})^T \sqrt{\boldsymbol{M}^{-1}} \boldsymbol{f} \\
&= \boldsymbol{\Phi K} \left(e^{-y/s^+}\boldsymbol{\Lambda} + \boldsymbol{I}\right)^{-1} \boldsymbol{K}^T \boldsymbol{\Phi}^T \boldsymbol{f}.
\end{aligned}$$

A similar computation can be done for $\boldsymbol{w}^-(y/s^-)$, resulting in the following equations for the vectors $\boldsymbol{w}^\pm(y/s^\pm)$ defined in (24):

(26)

$$\boldsymbol{w}^+(y/s^+) = \boldsymbol{\Phi K} \left(e^{-y/s^+}\boldsymbol{\Lambda} + \boldsymbol{I}\right)^{-1} \boldsymbol{K}\boldsymbol{\Phi}^T \boldsymbol{f}, \quad \boldsymbol{w}^-(y/s^-) = \boldsymbol{\Phi K} \left(\boldsymbol{\Lambda} + e^{-y/s^-}\boldsymbol{I}\right)^{-1} \boldsymbol{K}^T \boldsymbol{\Phi}^T \boldsymbol{f}.$$

By using the formulas (26) for the coordinate vectors $\boldsymbol{w}^\pm$ in (25), we conclude that $\widetilde{\boldsymbol{u}}$ is given by

$$\widetilde{\boldsymbol{u}} = \boldsymbol{\Phi K L K}^T \boldsymbol{\Phi}^T \boldsymbol{f}, \tag{27}$$

where the diagonal matrix $\boldsymbol{L}$ has entries,

$$(\boldsymbol{L})_{n,n} = L(\lambda_n^N, s), \tag{28}$$

and the function L is defined as

$$L(\lambda, s) \coloneqq \beta^+(s) \int_0^\infty \frac{e^{-y}}{\lambda e^{-y/s^+} + 1} \mathrm{d}y + \beta^-(s) \int_0^\infty \frac{e^{-y}}{\lambda + e^{-y/s^-}} \mathrm{d}y. \tag{29}$$

The second step of the proof involves computing $L(\lambda, s)$. The main result is that

$$L(\lambda, s) = \lambda^{-s} \tag{30}$$

for any $s \in (0, 1)$ and $\lambda > 0$. This is a relatively straightforward but technical computation, which we show in detail in Lemma A.1 in the appendix.

With (30), we have that $\boldsymbol{L} \equiv \boldsymbol{\Lambda}^{-s}$. By comparing (27) with (18), we conclude $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$, completing the proof. □

Remark 5.1. The identity (30) is fully expected, even without going through the proof. From the defintion of the fractional Laplacian in (4), if f is chosen as an eigenfunction of the Laplacian then the solution u to (1) is given by:

$$f = \phi_n \quad \Longrightarrow \quad u = \lambda_n^{-s} \phi_n. \tag{31a}$$

Furthermore, with f chosen in this way, the solutions to (23) are likewise straightforward:

$$w^+ = (1 + \lambda_n e^{-y/s^+})^{-1} \phi_n, \qquad w^- = (\lambda_n + e^{-y/s^-})^{-1} \phi_n.$$

By using these expressions directly in Kato's formula (21), we obtain

$$u = L(\lambda_n, s) \phi_n. \tag{31b}$$

In order to reconcile the two expressions for u in (31), we must have $L(\lambda_n, s) = \lambda_n^{-s}$.

We have therefore established that the discrete eigenfunction method is equivalent to computing the solutions $w_V^\pm$ and substituting these Kato's formula. To understand the error committed by the discrete eigenfunction solution, we therefore need only understand the error committed in Kato's formula by repacling $w^\pm$ with $w_V^\pm$.

5.2. **A proposal for convergence analysis.** In this section we provide convergence analysis for the discrete eigenfunction algorithm solution u_V; this analysis is provided through the Kato integral representation formula in Theorem 5.1. The first step is to leverage a stability property of the formula (21) that was noted in [9].

Lemma 5.1. *Assume $f \in L^2$. Then*

$$\sup_{s \in (0,1)} \|u(s)\| \leq \frac{4}{\pi} \max_{\sigma \in \{+,-\}} \sup_{y \geq 0} \|w^\sigma(y)\|.$$

Proof. Taking the L^2 norm in (21) and using the triangle inequality yields

$$\begin{aligned}\|u(s)\| &\leq \beta^- \int_0^\infty \|w^-(y/s^-)\| e^{-y} dy + \beta^+ \int_0^\infty \|w^+(y/s^+)\| e^{-y} dy \\ &\leq \max_{\sigma\in\{+,-\}} \sup_{y\geq 0} \|w^\sigma(y)\| \int_0^\infty e^{-y}(\beta^-(s)+\beta^+(s)) \\ &= \max_{\sigma\in\{+,-\}} \sup_{y\geq 0} \|w^\sigma(y)\| \frac{\sin(\pi s)}{\pi s(1-s)} \\ &\leq \frac{4}{\pi} \max_{\sigma\in\{+,-\}} \sup_{y\geq 0} \|w^\sigma(y)\|.\end{aligned}$$

□

The utility of this stability estimate is that it connects accurcy of $w_V^\pm$ to that of u_V.

Lemma 5.2. *Assume $f \in L^2$. Then*

$$\sup_{s\in(0,1)} \|u(s) - u_V(s)\| \leq \frac{4}{\pi} \max_{\sigma\in\{+,-\}} \sup_{y\geq 0} \|w_V^\sigma(y) - w^\sigma(y)\|.$$

Proof. Taking the L^2 norm of $u(s)$ and $u_V(s)$ and using the triangle inequality yields

$$\begin{aligned}\|u(s) - u_V(s)\| &\leq \beta^-(s) \int_0^\infty \|w^-(y/s^-) - w_V^-(y/s^-)\| e^{-y} dy + \beta^+(s) \int_0^\infty \|w^+(y/s^+) - w_V^+(y/s^+)\| e^{-y} dy \\ &\leq \max_{\sigma\in\{+,-\}} \sup_{y\geq 0} \|w_V^\sigma(y) - w^\sigma(y)\| \int_0^\infty e^{-y} dy (\beta^-(s)+\beta^+(s)) \\ &= \max_{\sigma\in\{+,-\}} \sup_{y\geq 0} \|w_V^\sigma(y) - w^\sigma(y)\| \frac{\sin(\pi s)}{\pi s(1-s)} \\ &\leq \frac{4}{\pi} \max_{\sigma\in\{+,-\}} \sup_{y\geq 0} \|w_V^\sigma(y) - w^\sigma(y)\|.\end{aligned}$$

□

Thus, in order to understand the error in the discrete eigenfunction solution u_V to a fractional PDE, we need only understand the errors committed by classical finite element approximations $w_V^\pm$ to the classical PDEs (22). In this case, we don't have to require the knowledge of the eigenvalues, even extremal eigenvalues of the discrete operators. Since we already have the fact that the errors committed by classical finite element approximations $w_V^\pm$ to the classical PDEs (22) can be bounded in L_2 norm, in the next step, we are aiming to explore the convergence analysis in H_0^1 norm mathematically, which for now can be tested through devised numerical experiments. Our expectation is that maybe the error bound can has a relationship with the fractional order s. We'll devise the numerical experiments and show part of the results and open questions in next section.

6. NUMERICAL EXPERIMENTS

We assume that the given function $f(x)$ satisfies the boundary condition (We shall consider and talk about the cases when $f(x)$ does not satisfy the boundary condition). Let $f(x) = \sin(n\pi x)$ where n is actually the frequency of the given function, then based on the spectral expansion mentioned in section 2, we can easily have the true solution $u = (\lambda_n)^{-s} \sin(n\pi x)$, where λ_n is the eigenvalues of the discretized fractional Laplacian and s is the fractional order. We do this numerical test for different sequences and fractional orders for one-dimensional case.

We can observe from Figure 2 that as the frequency increase, we need higher degree of function space to achieve the same accuracy. This is reasonable because larger frequency yields stronger wiggles, which means we need higher order functions in function space to approximate smoothly.

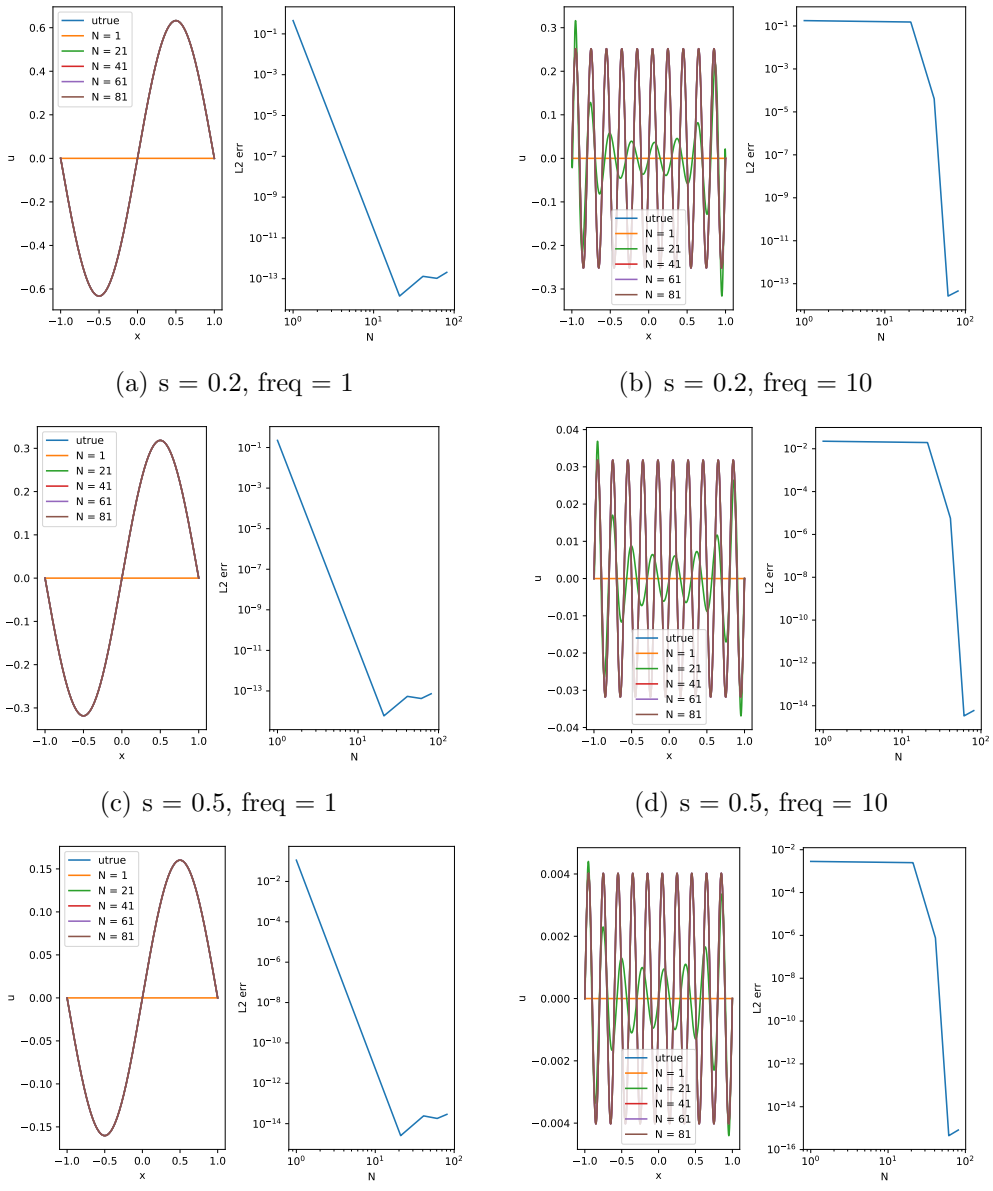
Since $s \in (0, 1)$ is the fractional order, when s is close to 1, then the problem is nearly a problem involving the classical Laplacian Δ, not the fractional Laplacian. So it seems plausible that cases with larger s perform better in convergence, but we still need more numerical experiments to illustrate this. When s is close to 0, it's okay if we still have the assumption that $f(x)$ satisfies the boundary condition. If the assumption is not hold, then the problems turns out to be a singular perturbation problem since f does not satisfy the boundary condition while the solution u has to. We will discuss this case in the future work.

7. Summary

The most contribution of this project for now is that we provide a solution to the accuracy issue of spectral method (discrete eigenfunction approach) by deriving the convergence results for the discrete eigenfunction algorithm. We achieve this by recognizing that the discrete eigenfunction approach is equivalent to a more friendly discretization through an expression for fractional operators called Kato's formula.

For the future work, we can do:

- study the case when $f(x)$ does not satisfy the boundary condition, for example, let $f(x)$ is an indicator function on [-0.5, 0.5] on physical domain on [-1,1] (can compute the true solution via Fourier Series coefficients). This will make our problem become more general.
- study the case when the fractional order is close to zero under the assumption that $f(x)$ does not satisfy the boundary condition, check if the convergence would perform worse than satisfying the assumption. It is a very interesting question involving the singular perturbation problems.
- extend one dimensional to two and three dimensional cases and do the previous numerical experiments to verify if the rate of convergence can be measured as a function of the fractional order s. In other word, prove numerically that the parameter s has an influence on the rate of convergence.
- show the error analysis / rate of convergence theoretically.



(a) s = 0.2, freq = 1

(b) s = 0.2, freq = 10

(c) s = 0.5, freq = 1

(d) s = 0.5, freq = 10

(e) s = 0.8, freq = 1

(f) s = 0.8, freq = 10

Figure 2. convergence for different sequences $\{1, 10\}$ and fractional orders $\{0.2, 0.5, 0.8\}$

Appendix A. Technical lemmas

This secton collects some technical results that are required in the main portion of the manuscript.

Lemma A.1. *For any* $s \in (0, 1)$ *and* $\lambda > 0$,

$$L(\lambda, s) = \lambda^{-s},$$

where $L(\lambda, s)$ *is defined in* (29).

Proof. We introduce the following constant that will be useful in this proof:

$$\beta_0(s) \coloneqq s_\pm \beta_\pm(s) = \frac{\sin(\pi s_\pm)}{\pi} = \frac{\sin(\pi s)}{\pi}.$$

Now starting from (29), we make substitutions in the integrals:

$$\begin{aligned}
L(\lambda, s) &= \beta_+(s) \int_0^\infty \frac{e^{-y}}{\lambda e^{-y/s_+} + 1} \mathrm{d}y + \beta_-(s) \int_0^\infty \frac{e^{-y}}{\lambda + e^{-y/s_-}} \mathrm{d}y \\
&= \beta_0(s) \int_0^\infty \frac{e^{-s_+ y}}{\lambda e^{-y} + 1} \mathrm{d}y + \beta_0(s) \int_0^\infty \frac{e^{-s_- y}}{\lambda + e^{-y}} \mathrm{d}y \\
&= \beta_0(s) \int_0^\infty \frac{e^{-sy}}{\lambda e^{-y} + 1} \mathrm{d}y + \beta_0(s) \int_0^\infty \frac{e^{sy}}{\lambda e^{y} + 1} \mathrm{d}y \\
&= \beta_0(s) \int_{-\infty}^\infty \frac{e^{-sy}}{\lambda e^{-y} + 1} \mathrm{d}y \\
&= \beta_0(s) \int_{-\infty}^\infty \frac{e^{-sy}}{e^{-y + \log\lambda} + 1} \mathrm{d}y \\
&= \beta_0(s) \int_{-\infty}^\infty \frac{e^{-sy - s\log\lambda}}{e^{-y} + 1} \mathrm{d}y \\
&= \lambda^{-s} \beta_0(s) \int_{-\infty}^\infty \frac{e^{-sy}}{e^{-y} + 1} \mathrm{d}y
\end{aligned}$$

Now define

$$\begin{aligned}
c(s) &= \beta_0(s) \int_{-\infty}^\infty \frac{e^{-sy}}{e^{-y} + 1} \mathrm{d}y \\
&= \beta_0(s) \int_{-\infty}^0 \frac{e^{-sy}}{e^{-y} + 1} \mathrm{d}y + \beta_0(s) \int_0^\infty \frac{e^{-sy}}{e^{-y} + 1} \mathrm{d}y.
\end{aligned}$$

We are left to show that $c(s) = 1$ to reach the conclusion.

To achieve this, we will need the digamma function ψ, i.e., the logarithmic derivative of the Euler Gamma function Γ:

$$\psi(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log \Gamma(t),$$

which has the following integral representation:

$$\psi(z+1) = -\gamma + \int_0^1 \frac{1 - t^z}{1 - t} \mathrm{d}t \tag{32}$$

From (32) we derive an identity for the digamma function, which holds for any $z, w > 0$:

$$\psi(z+1) - \psi(w+1) = \int_0^1 \frac{t^w - t^z}{1 - t} \mathrm{d}t. \tag{33}$$

Now in our formula for $c(s)$ we rewrite the integral over the positive real line as

$$\begin{aligned}\beta_0(s)\int_0^\infty \frac{e^{-sy}}{e^{-y}+1}\mathrm{d}y &= \beta_0(s)\int_0^\infty \frac{e^{(1-s)y}}{1+e^y}\mathrm{d}y \\ &= \beta_0(s)\int_{-\infty}^0 \frac{e^{-(1-s)y}}{e^{-y}+1}\mathrm{d}y,\end{aligned}$$

so that $c(s)$ has the form

$$c(s) = \beta_0(s)\left[\int_{-\infty}^0 \frac{e^{-sy}}{e^{-y}+1}\mathrm{d}y + \int_{-\infty}^0 \frac{e^{-(1-s)y}}{e^{-y}+1}\mathrm{d}y\right].$$

Therefore, we need only concentrate on one of these integrals; we focus on the first integral, and can compute the second from the first by making the substitution $s \leftarrow 1-s$. Making the substitution $x = \exp(y)$ for the first integral above, followed by $t = x^2$, we have

$$\begin{aligned}\int_{-\infty}^0 \frac{e^{-sy}}{e^{-y}+1}\mathrm{d}y &= \int_0^1 \frac{x^{-s}}{1+x}\mathrm{d}x \\ &= \int_0^1 \frac{x^{-s}(1-x)}{1-x^2}\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 \frac{t^{-(s+1)/2}-t^{-s/2}}{1-t}.\end{aligned}$$

Combining the above with (33), we obtain

$$\int_{-\infty}^0 \frac{e^{-sy}}{e^{-y}+1}\mathrm{d}y = \frac{1}{2}\left[\psi\left(-\frac{s}{2}+1\right) - \psi\left(-\frac{s}{2}+\frac{1}{2}\right)\right].$$

$$\int_{-\infty}^0 \frac{e^{-sy}}{e^{-y}+1}\mathrm{d}y = \frac{1}{2}\left[\psi\left(-\frac{s}{2}+\frac{1}{2}\right) - \psi\left(\frac{s}{2}\right)\right].$$

Using this identity in the formula for $c(s)$, we obtain

$$\begin{aligned}c(s) &= \beta_0(s)\left[\int_{-\infty}^0 \frac{e^{-sy}}{e^{-y}+1}\mathrm{d}y + \int_{-\infty}^0 \frac{e^{-(1-s)y}}{e^{-y}+1}\mathrm{d}y\right] \\ &= \frac{1}{2}\beta_0(s)\left[\psi\left(-\frac{s}{2}+1\right) - \psi\left(\frac{s}{2}\right) + \psi\left(-\frac{s}{2}+\frac{1}{2}\right) - \psi\left(-\frac{s}{2}+\frac{1}{2}\right)\right].\end{aligned}$$

Exercising the digamma function reflection property,

$$\psi(1-z) - \psi(z) = \pi\cot(\pi z),$$

we obtain

$$\begin{aligned}
c(s) &= \frac{1}{2}\beta_0(s)\left[\pi\cot\left(\frac{\pi s}{2}\right)+\pi\cot\left(\pi\left(\frac{1}{2}-\frac{s}{2}\right)\right)\right] \\
&= \frac{1}{2}\sin(\pi s)\left[\pi\cot\left(\frac{\pi s}{2}\right)+\pi\cot\left(\pi\left(\frac{1}{2}-\frac{s}{2}\right)\right)\right] \\
&= \cos\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi s}{2}\right)\left[\frac{\cos\left(\frac{\pi s}{2}\right)}{\sin\left(\frac{\pi s}{2}\right)}+\frac{\cos\left(\pi\left(\frac{1}{2}-\frac{s}{2}\right)\right)}{\sin\left(\pi\left(\frac{1}{2}-\frac{s}{2}\right)\right)}\right] \\
&= \cos\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi s}{2}\right)\left[\frac{\cos\left(\frac{\pi s}{2}\right)}{\sin\left(\frac{\pi s}{2}\right)}+\frac{\sin\left(\frac{\pi s}{2}\right)}{\cos\left(\frac{\pi s}{2}\right)}\right] \\
&= \cos^2\left(\frac{\pi s}{2}\right)+\sin^2\left(\frac{\pi s}{2}\right) \\
&= 1
\end{aligned}$$

□

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