Line Integrals for Conservative Vector Fields

Let $\vec{F}(x,y,z)$ be a conservative vector field $\Rightarrow$ There is a potential $\phi(x,y,z)$ such that

$$\nabla\phi = \vec{F}$$

The line integral $\int_C \vec{F}\cdot d\vec{R}$ for $\vec{F} = \nabla\phi$ can be simplified as follows $\Rightarrow$

$$\int_C \vec{F}\cdot d\vec{R} = \int_C \nabla\phi \cdot d\vec{R}$$

Hence $\Rightarrow$

$$\int_C \vec{F}\cdot d\vec{R} = \int_C \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle \cdot \langle dx, dy, dz\rangle$$

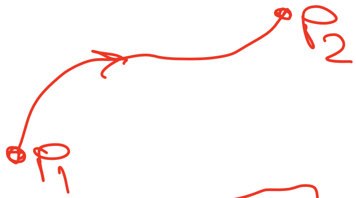
$$= \int_C \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

$$= \int_C \left(\frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt}\right)dt$$

$$= \int_C \frac{d\varphi}{dt}dt = \int_C d\varphi$$

Hence:

If C is a path from P_1 to P_2

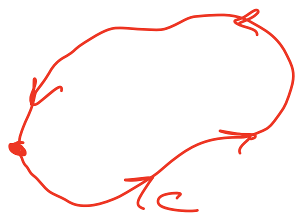


$\Rightarrow$ For Conservative Field $\vec{F} = \nabla \phi$

we write

$$\boxed{\int_C \vec{F} \cdot d\vec{R} = \int_C d\phi = \phi\Big|_{P_2} - \phi\Big|_{P_1}}$$

If C is a closed path



For $\vec{F} = \nabla \phi \Rightarrow$

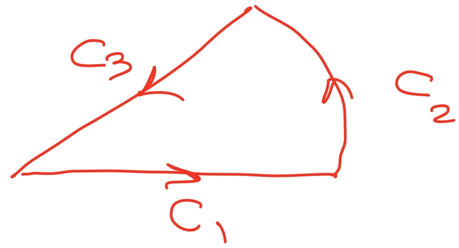
$$\boxed{\int_C \vec{F} \cdot d\vec{R} = 0}$$

Example:

Let $\vec{F}(x,y) = \langle y, x \rangle$, show that $\vec{F}$ is a conservative field with $\phi = xy$.

Compute $\int_C \vec{F} \cdot d\vec{R}$ where C is the trajectory given by

$C = C_1 \cup C_2 \cup C_3$



Solution:

1) $$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix}$$

$$= \langle 0, 0, 1-1 \rangle$$

$$= \langle 0, 0, 0 \rangle$$

$\Rightarrow$ $\vec{F}$ is a Conservative Vector Field or a gradient Vector Field. $\Rightarrow$

There is potential Φ such that

$$\nabla \Phi = \vec{F}$$

2) To find Φ $\Rightarrow$ as done befor

$$\left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right\rangle = \langle y, x, 0 \rangle$$

Hence

(4)

$$\frac{\partial \phi}{\partial x} = y \quad \text{——— (1)}$$

$$\frac{\partial \phi}{\partial y} = x \quad \text{——— (2)}$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{——— (3)}$$

From (1) $\Rightarrow$ $\phi = xy + f(y,z)$

Using this ϕ and substitute in (2) $\Rightarrow$

$$x + \frac{\partial f}{\partial y} = x$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \quad f = g(z)$$

Hence $\phi = xy + g(z)$

Using this ϕ and substituting in (3)

$$\frac{dg}{dz} = 0$$

$$\Rightarrow \quad g = \text{Constant} = A$$

Hence $\boxed{\phi = xy + A}$

3) Now, since $\bar{F}$ is conservative and $\nabla\phi = \bar{F}$ $\Rightarrow$

$$\int_C \bar{F}\cdot d\bar{R} = \phi\Big|_{P_1} - \phi\Big|_{P_2}$$

But $P_1 \equiv P_2$ since the curve is a closed trajectory $\Rightarrow$

$$\int_C \bar{F}\cdot d\bar{R} = 0$$

as found earlier

Example:

Let $\bar{F} = \langle 3+2xy,\ x^2-3y^2,\ 0\rangle$

1) Show that $\bar{F}$ is a gradient vector field

2) Find the corresponding potential ϕ

3) Evaluate $\int_C \bar{F}\cdot d\bar{R}$ where C is the path C: $\bar{R}(t) = \langle e^t \sin t,\ e^t \cos t,\ 0\rangle$ and $0 \leq t \leq \pi$

Solution:

1) $$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3+2xy & x^2-3y^2 & 0 \end{vmatrix}$$

$$\nabla \times \vec{F} = \langle 0, 0, 2x - 2x \rangle = \langle 0,0,0 \rangle$$

Hence $\vec{F}$ is conservative field

2) To find $\phi \Rightarrow \nabla \phi = \vec{F} \Rightarrow$

$$\frac{\partial \phi}{\partial x} = 3 + 2xy \quad \text{——} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = x^2 - 3y^2 \quad \text{——} \quad (2)$$

From (1) $\Rightarrow$ $\phi = 3x + x^2 y + f(y,z)$

substituting in (2)

$$x^2 + \frac{\partial f}{\partial y} = x^2 - 3y^2$$

$$\frac{\partial f}{\partial y} = -3y^2 \Rightarrow$$

$$f(y,z) = -y^3 + g(z)$$

Hence $\Rightarrow$

$$\phi = 3x + x^2 y - y^3 + g(z)$$

But $\frac{\partial \phi}{\partial z} = 0$ since $\vec{F} = \langle 3+2xy, x^2-3y^2, 0 \rangle$

$$\Rightarrow \boxed{\phi = 3x + x^2 y - y^3 + K}$$

3) Since $\bar{F}$ is conservative ⑦

$$\int_C \bar{F}\cdot d\bar{R} = \phi\Big|_{P_2} - \phi\Big|_{P_1}$$

Since $C: \bar{R}(t) = \langle e^t \sin t, e^t \cos t, 0\rangle$
$0 \leq t \leq \pi$

Hence
P_1 is $(0, 1, 0)$
P_2 is $(0, -e^{\pi}, 0)$

$$\int_C \bar{F}\cdot d\bar{R} = \left(0 + 0 - (-e^{\pi})^3 + K\right) - (0 + 0 - 1 + K)$$

$$= e^{3\pi} + 1$$

For practice:

Given $\bar{F} = \langle y^2, 2xy + e^{3z}, 3ye^{3z}\rangle$

1) Verify that $\bar{F}$ is a conservative vector field.

2) Find its potential ϕ

3) Evaluate $\int_C \bar{F}\cdot d\bar{R}$ where C is any closed trajectory

Ans: $\phi = xy^2 + ye^{3z} + K$

Triple Integrals

$$\iiint_Q f(x,y,z)\,dV$$

dV is element of volume

$$dV = \begin{cases} \text{Cartesian form} \\ \text{Cylindrical form} \\ \text{Spherical form.} \end{cases}$$

Q is a region in R^3

This topic is very important and will be covered in the coming weeks.