ENGG 5403 Homework Design Problem 2

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April 25 2019

1 Problem Overview and background

In this design problem, we need to design the controller for an unmanned helicopter fight control system. The modelling part of the Helion has been completed and the state space representation has been given. What is required is that we should use either the H_2 optimal or the H_∞ sub-optimal control law to design a measurement feedback control law that meet the following specifications:

1. The overall system is stabilized;
2. The dynamics of the helicopter will be driven to the hovering state, i.e. all state variables are to be driven to 0, as quickly as possible from the given initial condition.

The proposed H_2 and H_∞ controller has been widely used in the past decades. In this assignment, H_2 optimal controller is used as it has the following advantages over the H_∞ controller:

1. We are able to get the optimal control law for H_2 controller, but only a sub-optimal control law for H_∞ controller;
2. The optimal value $\gamma^\star$ of H_2 controller can be computed easily, while $\gamma^\star$ of H_∞ controller has no closed form solution, can only be computed numerically;
3. By using the small perturbation method, the resultant H_∞ control law will cause huge control signal to the system;
4. H_2 controller is relatively easier to design and be computed by solving simple Riccati equation (ARE).

2 Model of HeLion

According to the given parameters, the unmanned aerial vehicle (UAV) platform HeLion has a linearized state space model which is described as:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0.0009 & 0 & 0 \\ 0 & 0 & 0 & 0.992 & 0 & 0 & -0.0389 & 0 & 0 \\ 0 & 0 & -0.0302 & -0.0056 & -0.0003 & 585.1165 & 11.4448 & -59.529 & 0 \\ 0 & 0 & 0 & -0.0707 & 267.7499 & -0.0003 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -3.3607 & 2.2223 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2.4483 & -3.3607 & 0 & 0 & 0 \\ 0 & 0 & 0.0579 & 0.0108 & 0.0049 & 0.0037 & -21.9557 & 114.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0.389 & 0 & 0 & 0.992 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 43.3635 \\ 0 & 0 & 0 \\ 0.2026 & 2.5878 & 0 \\ 2.5878 & -0.0663 & 0 \\ 0 & 0 & -83.1883 \\ 0 & 0 & -3.8500 \\ 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In addition, in the second part of the simulation, a disturbance of wind gust disturbance will be considered in the problem formulation. Therefore the dynamical equation can be modified as:

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= Cx \end{aligned} \tag{2}$$

where

And the problem formulation can be further considered as

$$\begin{aligned} \dot{x} &= Ax + \begin{bmatrix} B & E \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} \\ y &= Cx \end{aligned} \tag{3}$$

which will facilitate the simulation and make it easier for us to take all the disturbance into consideration.

3 The H_2 Controller Design

The initial condition in our simulation is

$$x_0 = \begin{bmatrix} 0 \\ -0.1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix}$$

Physically, it means the helicopter is commanded to hover from a pitch angle of -0.1 rad (a nose-down angle of 5.7 degrees) and yaw angle of 0.1 rad (5.7 degrees). In our simulation, we are required to show that all state variables are able to be driven to 0 from an initial condition.

Other design considerations, such as frequency domain requirements on gain and phase margins, are to be ignored. And due to physical limitation and safety, they will have to be kept within the following limits:

$$|\delta_{lat}| < 0.35 \quad |\delta_{lon}| < 0.35 \quad |\delta_{ped}| < 0.4$$

Now let us consider the UAV system written in:

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= C_1 x + D_1 w \\ z &= C_2 x + D_2 u \end{aligned}$$

where w is the disturbance to the system, y is the measured output of the system, and z is the output to be controlled of the system. Also, we have the controller dynamic given by:

$$\begin{aligned} \dot{v} &= A_c v + B_c y \\ u &= C_c v + D_c y \end{aligned}$$

Here, A_c, B_c, C_c and D_c are the system matrices of the controller. In order to design the controller, we need to satisfy 4 conditions:

1. D_2 is of maximal column rank;
2. The subsystem (A, B, C_2, D_2) has no invariant zeros on the imaginary axis;
3. D_1 is of maximal row rank;
4. The subsystem (A, E, C_1, D_1) has no invariant zeros on the imaginary axis;

However, for our case in this assignment, we can easily see that the system does not satisfy the conditions, as both D_1 and D_2 are equal to 0 matrix. This is refer as a singular case. In literature, for a general system which the regularity conditions are not satisfied, it can be solved by using a perturbation approach. In this approach, we define a new controlled output:

$$\widetilde{z} = \begin{bmatrix} z \\ \varepsilon x \\ \varepsilon u \end{bmatrix} = \begin{bmatrix} C_2 \\ \varepsilon I \\ 0 \end{bmatrix} x + \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I \end{bmatrix} u$$

and new matrices associated with the disturbance inputs:

$$\begin{aligned} \tilde{E} &= [E \quad \varepsilon I \quad 0] \\ \tilde{D}_1 &= [D_1 \quad 0 \quad \varepsilon I] \end{aligned}$$

Then, the singular output feedback system will then transform to the following perturbed regular system with sufficiently small ε:

$$\begin{aligned} \dot{x} &= Ax + Bu + \tilde{E}\tilde{w} \\ y &= C_1 x + D_1 \tilde{w} \\ \tilde{z} &= \tilde{C}_2 x + \tilde{D}_2 u \end{aligned}$$

Here, intuitively, we let $\tilde{w}$ to be the combination of the disturbance w as the last 3 variables in $\tilde{w}$, and random measurement Gaussian noise as the last 12 variables. The Gaussian noise has a mean of 0 and standard deviation of 0.1. The invariant zero of the systems can be checked by using the Matlab command $tzero(A, B, C_2, D_2)$. It turns out that there is no invariant zero for both the perturbed subsystems, thus assumption 2 and 4 above become valid. Next, the controller is designed by first solving the Riccati equations:

$$\begin{aligned} A^T P + PA + C_2^T C_2 - (PB + C_2^T D_2)(D_2^T D_2)^{-1}(D_2^T C_2 + B_T P) = 0 \\ QA^T + AQ + EE^T - (QC_1^T + ED_1^T)(D_1 D_1^T)^{-1}(D_1 E^T + C_1 Q) = 0 \end{aligned}$$

to obtain 2 positive definite P and Q matrices. Matlab command *h2care* from the MIMO toolkit developed by the course lecturer, Professor Ben M. Chen is used to compute the matrix P and Q. Positive definiteness can be shown by finding the eigenvalue of the matrices using Matlab command *eig(P)*. Here, eigenvalues of P are

$$0.0622, 0.0431, 0.0107, 0.0107, 0.0105, 0.0026, 0.0001, 0.0002, 0.0003$$

and eigenvalues of Q are

$$0.0122, 0.0101, 0.0087, 0.0085, 0.0011, 0.0004, 0.0001, 0.0000, 0.0000$$

As shown here, matrix P is positive definite and matrix Q is positive semi-definite. Upon obtaining the matrix P and Q, the H_2 optimal output feedback controller can be computed as follows:

$$\begin{aligned} \ddot{v} &= (A + BF + KC_1)v - Ky \\ u &= Fv \end{aligned}$$

where $F = -(\tilde{D}_2^T \tilde{D}_2)^{-1}(\tilde{D}_2^T \tilde{C}_2 + B^T P)$ and $K = -(QC_1^T + E\tilde{D}_1^T)(\tilde{D}_1 \tilde{D}_1^T)^{-1}$
Here, we have:

$$A_c = A + BF + KC_1$$
$$B_c = -K$$
$$C_c = F$$
$$D_c = 0$$

and the closed-loop transfer function from w to z will then be given by

$$T(s) = C_{cl}(sI - A_{cl})^{-1}B_{cl} + D_{cl}$$

where

$$A_{cl} = \begin{bmatrix} A & BC_c \\ B_c C_1 & A_c \end{bmatrix}$$

$$B_{cl} = \begin{bmatrix} \tilde{E} \\ B_c \tilde{D}_1 \end{bmatrix}$$

$$C_{cl} = \begin{bmatrix} \tilde{C}_2 & \tilde{D}_2 C_c \end{bmatrix}$$

$$D_{cl} = 0$$

To check the stability of the closed-loop system, Matlab command *eig(A_{cl})* is used to find eigenvalues of A_{cl}:

$$-93.6476 + 0.0000i, -22.6826 + 30.8785i, -22.6826 - 30.8785i, -2.1318 + 34.0488i,$$
$$-2.1318 - 34.0488i, -15.6237 + 22.3603i, -15.6237 - 22.3603i, -1.3631 + 16.3053i,$$
$$-1.3631 - 16.3053i, -15.9141 + 0.0000i, -7.2051 + 0.0000i, -5.3713 + 0.0000i,$$
$$-1.7407 + 0.0000i, -0.0230 + 0.0000i, -1.0001 + 0.0000i, -0.9674 + 0.0000i,$$
$$-0.9357 + 0.0000i, -0.9293 + 0.0000i$$

The eigenvalues are all in the left half complex plane, which means the closed-loop linearized system is **asymptotically stable**. Lastly, the optimal value γ_2^* is computed as

$$\gamma_2^* = trace(E^T PE) + trace[(A^T P + PA + \tilde{C}_2^T \tilde{C}_2)Q]^{1/2} = 0.0195$$

4 Simulation Results

Since the UAV model parameters and controller gains have already derived in the previous section, simulation can be done by using Matlab and Simulink. In this section, simulation results will be provided for closed-loop system with input and measurement noise, together with a wind gust disturbance in the form of sine wave to the velocity channels. Fig. 1 shows the Simulink block diagram of the controller design in this part. Here, the random noise added to the input and the measurement output are in Normal distribution with mean 0 and standard deviation 0.1. The simulation results in Fig. 2 to Fig. 4 has

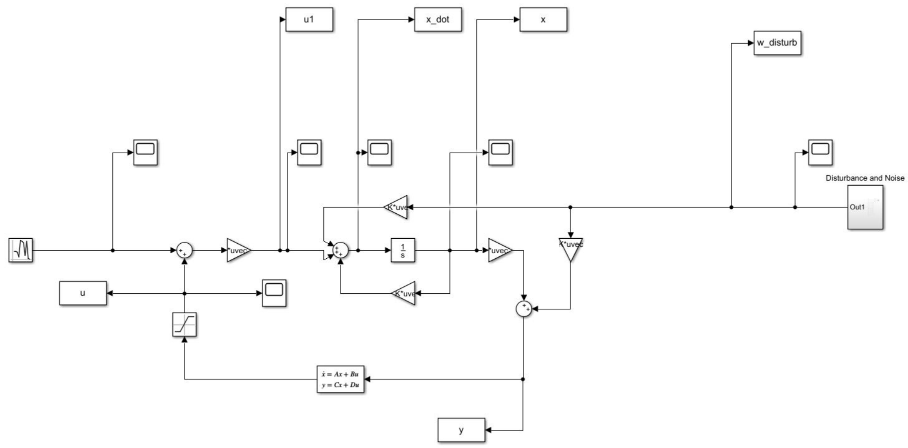

Figure 1: Simulink block diagram of the H_2 controller design

verified that this H_2 controller is able to give good responses to the closed-loop system.

In detail, show the ϕ, θ, Ψ, p, q, r of the UAV in simulation from time t = 0 to t = 40 secs. However, at time t = 10 to t = 30 sec, the UAV is undergoing wind gust disturbances. Based on the plotted graphs, it seems that the controller designed here provide better disturbance rejection as compared to the previous assignment, where LQG/LQR controller was used. Here, although there are still defection from the equilibrium state during the period of disturbance, it shows a much lower amplitude as compared to the previous assignment.

As shown in Fig. 2 and Fig. 3, during the period of disturbance, the angle deviate with a small magnitude. In steady state, the system is able to achieve asymptotic stable again as all state variables nearly converges to 0. However, because of the noise input, all state variables will not converge exactly to zero. And note that Ψ will deviate a bit in the short term 0-40s, but it will still converge in the long term as shown in Fig. 4.

Lastly, Fig. 5 shows the control input to the UAV. Note that here, the $1st$ control input, δ_{lat} is saturated (reached it's limit) due to countering the effect of the disturbance. We can see that now with this H_2 controller design, the system is trying to achieve the equilibrium state again by applying large control action to the δ_{lat} input, which will require a larger control input effort to reject the disturbance.

5 Conclusion

In conclusion, as can be seen from the results, H_2 optimal controller is able to reject disturbance fairly well. Intuitively, H_2 control law is a type of optimal control as it minimize the H_2 norm of the transfer function T. In other word, the

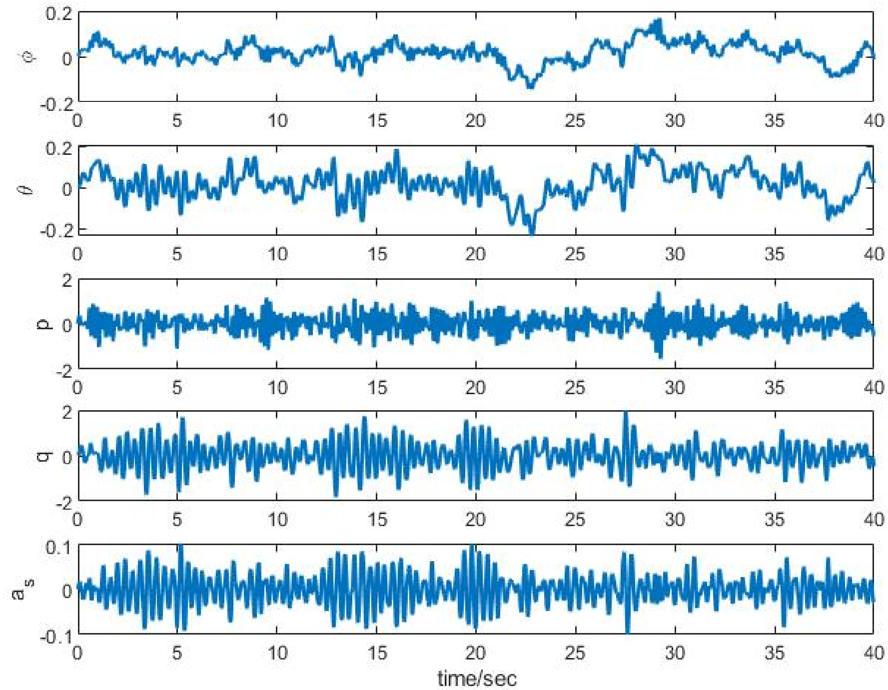

Figure 2: States of the UAV

H_2 norm is the total energy corresponding to the impulse response of the transfer function from w to z, and thus minimizing it is equivalent to the minimization of the total energy from the disturbance w to the controlled output z. H_2 controller is able to achieve this target by reaching the optimal value, γ_2^*=0.0195, in this assignment.

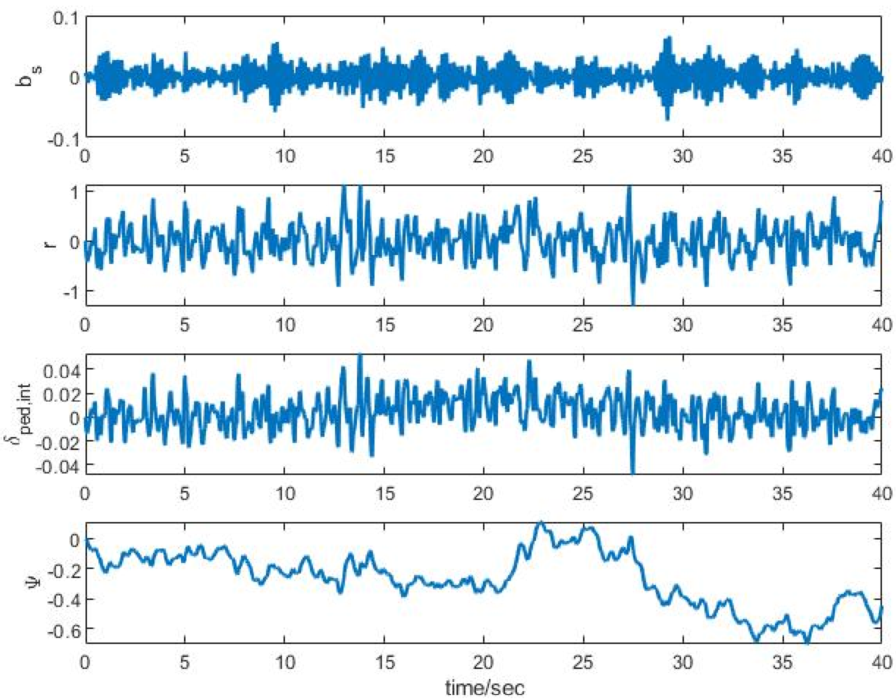

Figure 3: States of the UAV

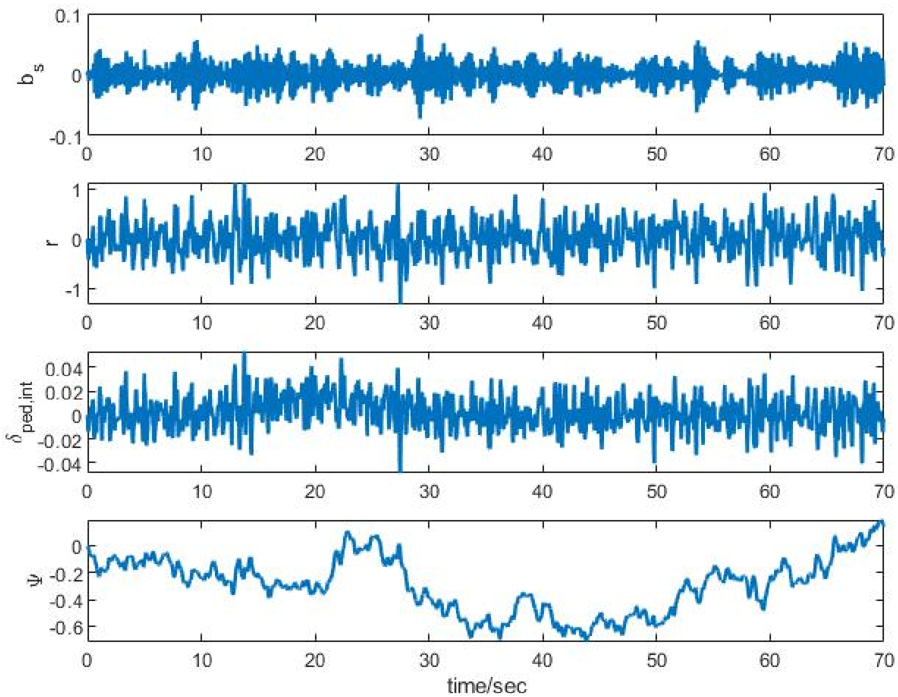

Figure 4: States of the UAV

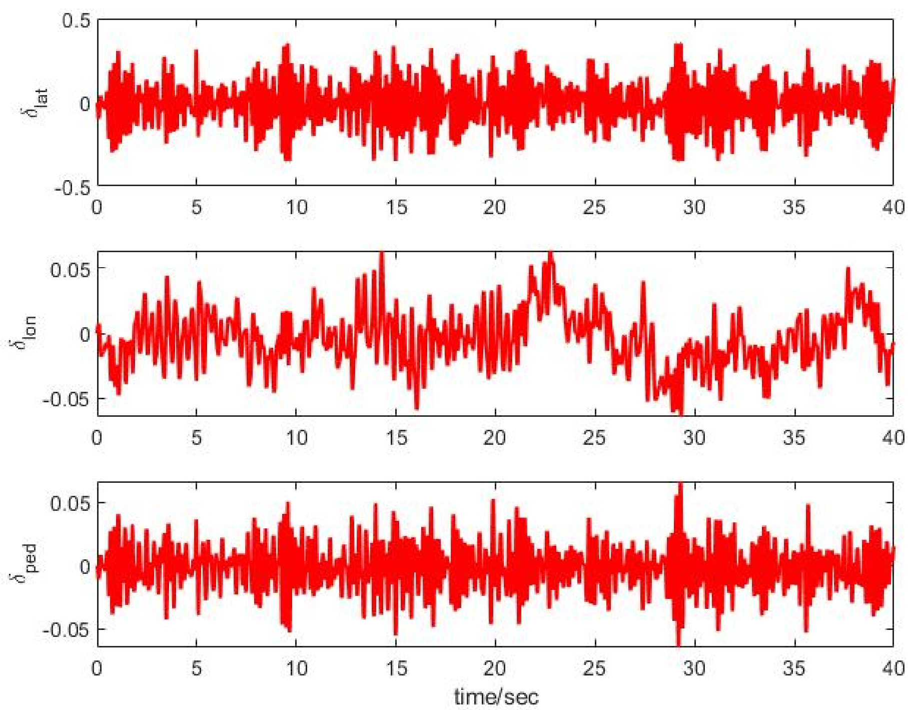

Figure 5: Control inputs of H_2 design

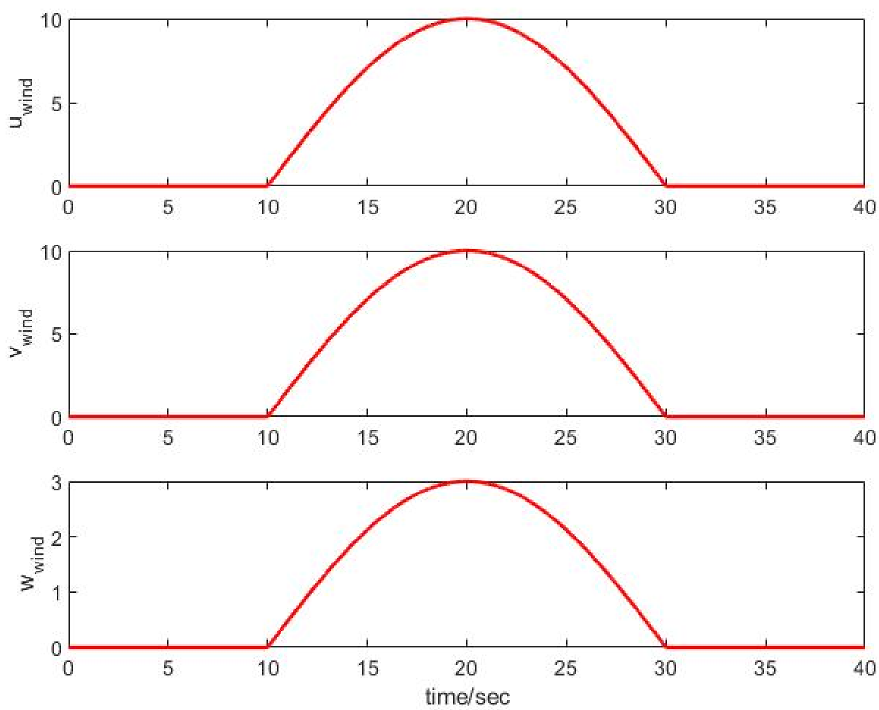

Figure 6: wind gust disturbance inputs of H_2 design