Nelson & Siegel (1987) equations

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Nelson & Siegel (1987) use (the solution of) the nonhomogeneous second order linear differential equation below

$$\ddot{r}(m) + \left[\frac{\tau_1 + \tau_2}{\tau_1 \tau_2}\right]\dot{r}(m) + \frac{1}{\tau_1 \tau_2} r(m) = a_0. \tag{1}$$

First we will assume $\tau_1 \neq \tau_2$ and $r(m) = e^{pm}$ such as

$$\begin{aligned} \dot{r} &= pe^{pm}, \\ \ddot{r} &= p^2 e^{pm}, \end{aligned}$$

therefore

$$\begin{aligned} p^2 e^{pm} + \left[\frac{\tau_1 + \tau_2}{\tau_1 \tau_2}\right] pe^{pm} + \frac{1}{\tau_1 \tau_2} e^{pm} &= 0, \\ e^{pm}\left(p^2 + \left[\frac{\tau_1 + \tau_2}{\tau_1 \tau_2}\right] p + \frac{1}{\tau_1 \tau_2}\right) &= 0, \\ p^2 + \left[\frac{\tau_1 + \tau_2}{\tau_1 \tau_2}\right] p + \frac{1}{\tau_1 \tau_2} &= 0, \end{aligned} \tag{2}$$

a well known functional form, where p is a constant and (2) is the corresponding characteristic polynomial of (1). We can use the quadratic formula to find the roots of (2)

$$\begin{aligned} \Delta &= \left[\frac{\tau_1 + \tau_2}{\tau_1 \tau_2}\right] - 4\frac{1}{\tau_1 \tau_2} = \frac{(\tau_1 + \tau_2)^2 - 4(\tau_1 \tau_2)}{(\tau_1 \tau_2)^2} = \frac{(\tau_1 - \tau_2)^2}{(\tau_1 \tau_2)^2}, \\ p &= -\frac{\frac{\tau_1 + \tau_2}{\tau_1 \tau_2} \pm \sqrt{\Delta}}{2}, \\ p &= -\frac{\frac{\tau_1 + \tau_2}{\tau_1 \tau_2} \pm \frac{\tau_1 - \tau_2}{\tau_1 \tau_2}}{2}, \quad p_1 = -\frac{1}{\tau_1}, \quad \text{and} \quad p_2 = -\frac{1}{\tau_2}. \\ r_1 &= e^{-m/\tau_1}, \quad \text{and} \quad r_2 = e^{-m/\tau_2}. \end{aligned}$$

We found two solutions. Are they the more general solutions? Not yet. If p_1 and p_2 are **real and unequal** roots of the characteristic equation, which means $\Delta > 0$, and if k_1 and k_2 are constants, a general solution of the corresponding homogeneous equation is

$$r(m) = \beta_0 + \beta_1 e^{-m/\tau_1} + \beta_2 e^{-m/\tau_2}, \tag{3}$$

where β_1 and β_2 are constants, and $\beta_0{=}a_0\tau_1\tau_2$ is a particular solution of the nonhomogeneous differential equation (1). The equation (3) is presented in the beginning by Nelson & Siegel (1987). We need three initials conditions to solve this differential equation. Check it ■

On the other hand, a more parsimonious model is given by the solution equation for the case of equal roots. We assume $\tau_1 = \tau_2 = \tau$, then (1) and (2) become respectively

$$\ddot{r} + \frac{2}{\tau}\dot{r} + \frac{1}{\tau^2}r = a_0, \tag{4}$$

$$p^2 + \frac{2\tau}{\tau^2}p + \frac{1}{\tau^2} = 0,$$

$$\Delta = 0, \quad \text{and} \quad p = -\frac{1}{\tau}.$$

Thus $r(m) = e^{pm}$ is not a solution as seen before. Check it with initials conditions. We can use a technique called reduction of order. Then we guess a second solution.

$$r = v(m)e^{-m/\tau}, \tag{5}$$

$$\dot{r} = \dot{v}e^{-m/\tau} - \frac{1}{\tau}ve^{-m/\tau}, \tag{6}$$

$$\begin{aligned} \ddot{r} =& \ddot{v}e^{-m/\tau} - \frac{1}{\tau}\dot{v}e^{-m/\tau} - \frac{1}{\tau}\dot{v}e^{-m/\tau} + \frac{1}{\tau^2}ve^{-m/\tau}, \\ =& \ddot{v}e^{-m/\tau} - \frac{2}{\tau}\dot{v}e^{-m/\tau} + \frac{1}{\tau^2}ve^{-m/\tau}, \end{aligned} \tag{7}$$

where $v(m)$ is an arbitrary function. Plugging (5)-(7) in (4) leads to the equation

$$\begin{aligned} \ddot{v}e^{-m/\tau} - \frac{2}{\tau}\dot{v}e^{-m/\tau} + \frac{1}{\tau^2}ve^{-m/\tau} + \frac{2}{\tau}\left[\dot{v}e^{-m/\tau} - \frac{1}{\tau}ve^{-m/\tau}\right] + \frac{1}{\tau^2}ve^{-m/\tau} =& 0, \\ e^{-m/\tau}\left(\ddot{v} - \frac{2}{\tau}\dot{v} + \frac{1}{\tau^2}v + \frac{2}{\tau}\dot{v} - \frac{2}{\tau^2}v + \frac{1}{\tau^2}v\right) =& 0, \\ e^{-m/\tau}\ddot{v} =& 0, \\ \ddot{v} =& 0, \end{aligned}$$

therefore

$$\dot{v} = C_2,$$

$$v = mC_2 + C_1. \tag{8}$$

Our solution is some arbitrary function $v(m)$ times $e^{-m/\tau}$. Thus plugging (8) in (5)

$$\begin{aligned} r =& v(m)e^{-m/\tau}, \\ =&(mC_2 + C_1)e^{-m/\tau}, \\ =&C_2 m e^{-m/\tau} + C_1 e^{-m/\tau}. \end{aligned}$$

If p_1 and p_2 are **real and equal** roots of the characteristic equation, which means $\Delta = 0$, a general solution of the corresponding homogeneous equation is

$$r(m) = \beta_0 + \beta_1 e^{-m/\tau} + \beta_2 (m/\tau) e^{-m/\tau}, \tag{9}$$

where $C_2{=}\beta_2(1/\tau)$, $C_1{=}\beta_1$, and $\beta_0{=}a_0\tau_1\tau_2$ is a particular solution of the nonhomogeneous differential equation (4). Then we have the **forward rate curve** in Nelson & Siegel (1987) notation - equation (1) in the paper. ■

Before derive the equation (2) in Nelson & Siegel (1987), we have to fix ideas and establish notation by introducing three key theoretical constructs and the relationships among them: the discount curve, the forward curve, and the yield curve. Let $P_t(m)$ denote the price of a m-period discount bond, i.e., the present value at time t of \$1 receivable m periods ahead, and let $R_t(m)$ denote its continuously compounded zero-coupon nominal yield to maturity. From the yield curve we obtain the discount curve,

$$P_t(m) = e^{-mR_t(m)},$$

and from the discount curve we obtain the instantaneous (nominal) forward rate curve (after some transformations),

$$\begin{aligned} \ln P_t(m) &= \ln e^{-mR_t(m)}, \\ \ln P_t(m) &= -mR_t(m) \ln e, \\ \frac{\mathrm{d}}{\mathrm{d}m}[\ln P_t(m)] &= \frac{\mathrm{d}}{\mathrm{d}m}[mR_t(m)], \\ -\frac{\dot{P}_t(m)}{P_t(m)} &= R_t(m) + m\dot{R}_t(m), \\ r_t(m) &= -\frac{\dot{P}_t(m)}{P_t(m)}. \end{aligned}$$

The relationship between the yield to maturity and the forward rate is therefore

$$R_t(m) = \frac{1}{m}\int_0^m r_t(x)dx, \tag{10}$$

which implies that the zero-coupon yield is an equally-weighed average of forward rates. Given the yield curve or forward curve, we can price any coupon bond as the sum of the present values of future coupon and principal payments.

In practice, yield curves, discount curves and forward curves are not observed. Instead, they must be estimated from observed bond prices. At any given time, we can have a large set of (Fama-Bliss unsmoothed for instance) yields, to which we fit a parametric curve for purposes of modeling and forecasting. We are studying the forward rate curve functional form in Nelson & Siegel (1987). Plugging (9) in (10) the correspondent yield curve is

$$R(m) = \frac{1}{m}\int_0^m \beta_0 + \beta_1 e^{-x/\tau} + \beta_2 (x/\tau) e^{-x/\tau} \mathrm{d}x,$$

$$= \underbrace{\frac{1}{m}\int_0^m \beta_0 \mathrm{d}x}_{\text{A}} + \underbrace{\frac{1}{m}\int_0^m \beta_1 e^{-x/\tau} \mathrm{d}x}_{\text{B}} + \underbrace{\frac{1}{m}\int_0^m \beta_2 (x/\tau) e^{-x/\tau} \mathrm{d}x}_{\text{C}},$$

$$\text{A} = \frac{1}{m}\left\{ \left[x\beta_0 - x\beta_0 \right] \Bigg|_0^m \right\} = \frac{1}{m} m\beta_0$$

$$= \beta_0. \quad \textbf{(A1)}$$

$$\text{B} = \frac{\beta_1}{m}\int_0^m e^{-x/\tau} \mathrm{d}x = \frac{\beta_1}{m}\left\{ \left[-\tau e^{-x/\tau} + \tau e^{-x/\tau} \right] \Bigg|_0^m \right\} = -\beta_1 \frac{\tau}{m} e^{-m/\tau} + \beta_1 \frac{\tau}{m}$$

$$= \frac{\beta_1}{m/\tau} - \frac{\beta_1 e^{-m/\tau}}{m/\tau}$$

$$= \beta_1 \left[\frac{1 - e^{-m/\tau}}{m/\tau} \right]. \quad \textbf{(B1)}$$

$$
\begin{aligned}
\mathrm{C} &= \frac{\beta_2}{m}\int_0^m (x/\tau)e^{-x/\tau}\mathrm{d}x, \quad \text{integration by parts:} \quad \int f\dot{g} = fg - \int \dot{f}g, \\
&= -\frac{x}{\tau}\tau e^{-x/\tau} - \int -\frac{1}{\tau}\tau e^{-x/\tau}\mathrm{d}x = -xe^{-x/\tau} - \tau e^{-x/\tau} \\
&= \frac{\beta_2}{m}\left\{\left[-\tau e^{-x/\tau} - xe^{-x/\tau} - \left(-\tau e^{-x/\tau} - xe^{-x/\tau}\right)\right]\Bigg|_0^m\right\}, \\
&= -\beta_2\frac{\tau}{m}e^{-m/\tau} - \beta_2\frac{m}{m}e^{-m/\tau} + \beta_2\frac{\tau}{m} \\
&= \beta_2\left[\frac{1-e^{-m/\tau}}{m/\tau}\right] - \beta_2 e^{-m/\tau}. \quad \textbf{(C1)}
\end{aligned}
$$

Therefore in Nelson & Siegel (1987) notation

$$
R(m) = \beta_0 + (\beta_1 + \beta_2)\left[\frac{1-e^{-m/\tau}}{m/\tau}\right] - \beta_2 e^{-m/\tau}. \tag{11}
$$

a convenient and parsimonious three-component exponential approximation - equation (2) in the paper ■

After some transformations, the function becomes

$$
y_t(\tau) = \beta_{1,t} + \beta_{2,t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}\right) + \beta_{3,t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau} - e^{-\lambda_t\tau}\right), \tag{12}
$$

a well known functional form, where $\beta_{1,t}$, $\beta_{2,t}$, and $\beta_{3,t}$ are latent dynamic factors, τ is maturity, and λ_t is the parameter which governs the exponential decay rate.

References

NELSON, C. R.; SIEGEL, A. F. Parsimonious modeling of yield curves. *Journal of business*, JSTOR, p. 473–489, 1987.