Zaki Glenroy Lindo, Assignment 4

Assignment 4: Due Monday October 2, 2017, no later than 8pm.

$p.38-39$ # 42, 43, 46, 60 , 66

#42. Prove or disprove that $H = \{0, 2, 4, 6, 16\}$ is a subgroup of $(\mathbb{Z}_{18}, +_{18})$

Counter Example:

Let $x = 2$.
Let $y = 6$.

As we defined H, $x, y \in H$. However, $x * y = 2 +_{18} 6 = 8$. $8 \notin H$, therefore, H is not closed under its operation, $+_{18}$. Thus, H is not a subgroup of $(\mathbb{Z}_{18}, +_{18})$.

#43. Prove or disprove that $H = \{(0, 0), (0, 2), (1, 0), (1, 2)\}$ is a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_4$under the operation using $+_2$ in the first coordinate and $+_4$ in the second coordinate.

$+_2 \times +_4$	(0,0)	(0,2)	(1,0)	(1,2)
(0,0)	(0,0)	(0,2)	(1,0)	(1,2)
(0,2)	(0,2)	(0,0)	(1,2)	(1,0)
(1,0)	(1,0)	(1,2)	(0,0)	(0,2)
(1,2)	(1,2)	(1,0)	(0,2)	(0,0)

Proof: We want to show that H is a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_4$ under the operation using $+_2$ in the first coordinate and $+_4$ in the second coordinate. We need to show $H \neq \emptyset^1$, H is closed under the operation^2, and H is closed under inverses^3:
1) ($H \neq \emptyset$): As $H = \{(0, 0), (0, 2), (1, 0), (1, 2)\}$, there are elements in H. Thus, $H \neq \emptyset$.

2) (H is closed under the operation): We want to show that for all $x, y \in H$, $x * y \in H$. Looking at the Cayley Table, we can see that for all $x, y \in H$, $x * y \in H$. Thus, H is closed under the operation.

3) (H is closed under inverses): We want to show that for all $a \in H$, $a^{-1} \in H$.

a=(0,0):
Looking at the Cayley Table, we see that (0,0) $*$ (0,0) = (0,0). As (0,0) is

the identity of H(as seen on the Cayley Table), (0,0) is the inverse of (0,0). Thus, $(0,0)^{-1} \in H$.

a=(0,2):
Looking at the Cayley Table, we see that (0,2) $*$ (0,2) = (0,0). As (0,0) is the identity of H(as seen on the Cayley Table), (0,2) is the inverse of (0,2). Thus, $(0,2)^{-1} \in H$.

a=(1,0):
Looking at the Cayley Table, we see that (1,0) $*$ (1,0) = (0,0). As (0,0) is the identity of H(as seen on the Cayley Table), (1,0) is the inverse of (1,0). Thus, $(1,0)^{-1} \in H$.

a=(1,2):
Looking at the Cayley Table, we see that (1,2) $*$ (1,2) = (0,0). As (0,0) is the identity of H(as seen on the Cayley Table), (1,2) is the inverse of (1,2). Thus, $(1,2)^{-1} \in H$.

As for each element $a \in H$, $a^{-1} \in H$, H is closed under inverses.

Hence, as $H \neq \emptyset$[1], H is closed under the operation[2], and H is closed under inverses[3], H is a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_4$ under the operation using $+_2$ in the first coordinate and $+_4$ in the second coordinate.

#46. Determine if the set H = $\left\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\right\}$ is a subgroup of $(\mathbb{Q}^*, \cdot)$ (nonzero rational numbers under multiplication). Either prove that it is or give a specific example of how it fails.

Counter Example:

Let $n = 12$. We know that for 12, $\frac{1}{12} \in H$.
Under the usual multiplication on nonzero rational numbers, $(\frac{1}{12})^{-1} = 12$. However, $12 \notin H$. Thus, H is not closed under inverses.

As H is not closed under inverses, H is not a subgroup of $(\mathbb{Q}^*, \cdot)$

#60. Determine if the function $f : M_2(\mathbb{R}) \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}\right) = a+b$ is a homomorphism. Note that $M_2(\mathbb{R})$ is a group under matrix addition and $\mathbb{R}$ is a group under usual real number addition.

Proof: We want to show that $f : M_2(\mathbb{R}) \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}\right) = a + b$ is a homomorphism. That is, for every $x, y \in M_2(\mathbb{R})$, $f(x+y) = f(x) + f(y)$. Let $x = \begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}$ and let $y = \begin{bmatrix} \text{e} & \text{f} \\ \text{g} & \text{h} \end{bmatrix}$.

We want to show that $f\left(\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix} + \begin{bmatrix} \text{e} & \text{f} \\ \text{g} & \text{h} \end{bmatrix}\right) = f\left(\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}\right) + f\left(\begin{bmatrix} \text{e} & \text{f} \\ \text{g} & \text{h} \end{bmatrix}\right)$.

Now, $f\left(\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix} + \begin{bmatrix} \text{e} & \text{f} \\ \text{g} & \text{h} \end{bmatrix}\right) = f\left(\begin{bmatrix} \text{a+e} & \text{b+f} \\ \text{c+g} & \text{d+h} \end{bmatrix}\right)$ by Matrix multiplication. By f, $f\left(\begin{bmatrix} \text{a+e} & \text{b+f} \\ \text{c+g} & \text{d+h} \end{bmatrix}\right) = (a+e) + (b+f)$. By associativity and commutativity of addition of real numbers, $(a+e)+(b+f) = a+b+e+f$. Thus, $f\left(\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix} + \begin{bmatrix} \text{e} & \text{f} \\ \text{g} & \text{h} \end{bmatrix}\right) = a+b+e+f$

Now, $f\left(\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}\right) + f\left(\begin{bmatrix} \text{e} & \text{f} \\ \text{g} & \text{h} \end{bmatrix}\right) = (a+b) + (e+f)$. By associativity of addition of real numbers, $(a+b)+(e+f) = a+b+e+f$. Thus, $f\left(\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}\right) + f\left(\begin{bmatrix} \text{e} & \text{f} \\ \text{g} & \text{h} \end{bmatrix}\right) = a+b+e+f$

Therefore, $f\left(\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix} + \begin{bmatrix} \text{e} & \text{f} \\ \text{g} & \text{h} \end{bmatrix}\right) = f\left(\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}\right) + f\left(\begin{bmatrix} \text{e} & \text{f} \\ \text{g} & \text{h} \end{bmatrix}\right)$.

Hence, the function $f : M_2(\mathbb{R}) \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}\right) = a + b$ is a homomorphism.

#66. Suppose $f : G \to K$ and $g : K \to H$ are homomorphisms of groups G, K, and H. Prove that the function $g \circ f$ is a homomorphism from G to H.

Proof: Let $a, b \in G$. We want to show that $g \circ f(ab) = g \circ f(a) g \circ f(b)$. By composition of functions, $g \circ f(ab) = g(f(ab))$. Since f is a homomorphism, $g(f(ab)) = g(f(a)f(b))$. Since $f : G \to K$ and $g : K \to H$, $f(a), f(b) \in K$. Since g is a homomorphism, $g(f(a)f(b)) = g(f(a))g(f(b))$. By composition of functions, $g(f(a)) = g \circ f(a)$. Similarly, $g(f(b)) = g \circ f(b)$. Thus, $g \circ f(ab) = g \circ f(a) g \circ f(b)$. Therefore, $g \circ f$ is a homomorphism from $G \to H$.