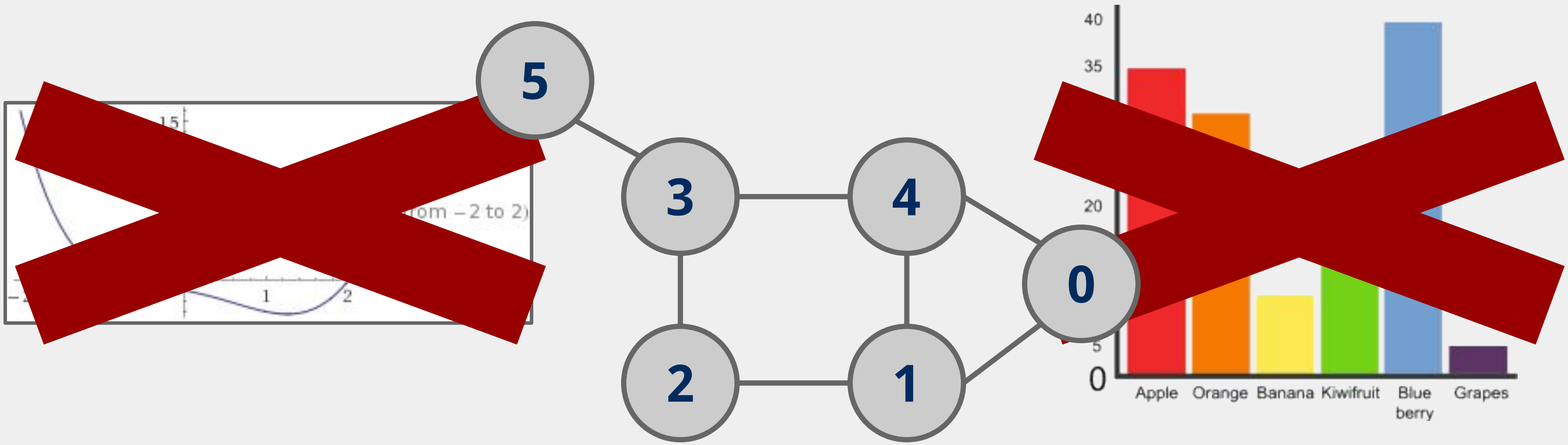
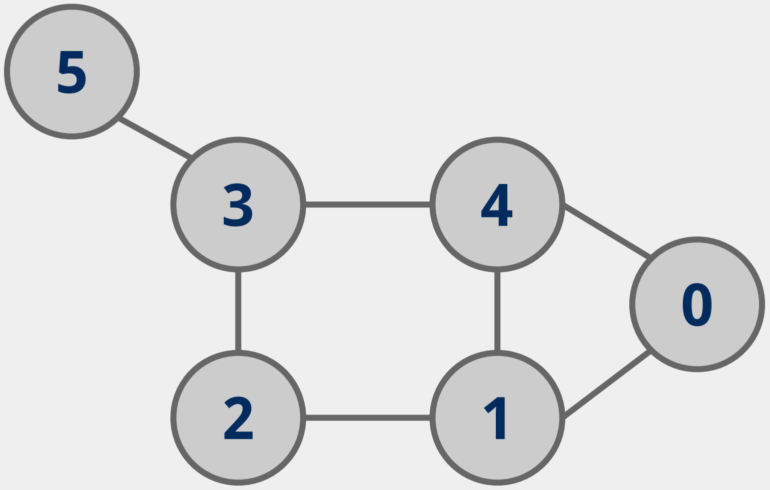
CS 1501

Graphs


5
3
4
0
2
1
40
35
20
0
Apple
Orange
Banana
Kiwifruit
Blue
berry
Grapes

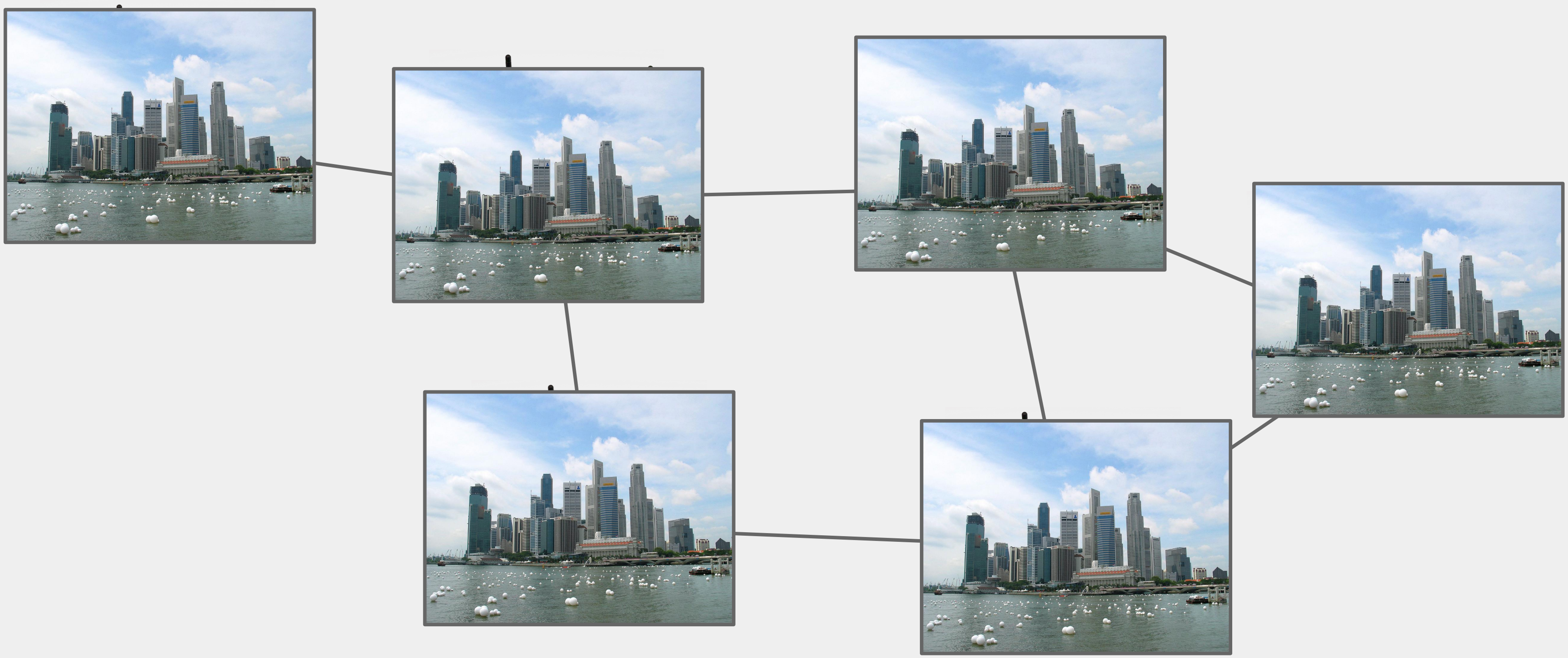
Graphs

- A graph G = (V, E)
  - Where V is a set of vertices
  - E is a set of edges connecting vertex pairs
- Example:
  - V = {0, 1, 2, 3, 4, 5}
  - E = {(0, 1), (0, 4), (1, 2), (1, 4), (2, 3), (3, 4), (3, 5)}



Why?

- Can be used to model many different scenarios



Some definitions

- Undirected graph
  - Edges are unordered pairs: (A, B) == (B, A)
- Directed graph
  - Edges are ordered pairs: (A, B) != (B, A)
- Adjacent vertices, or neighbors
  - Vertices connected by an edge

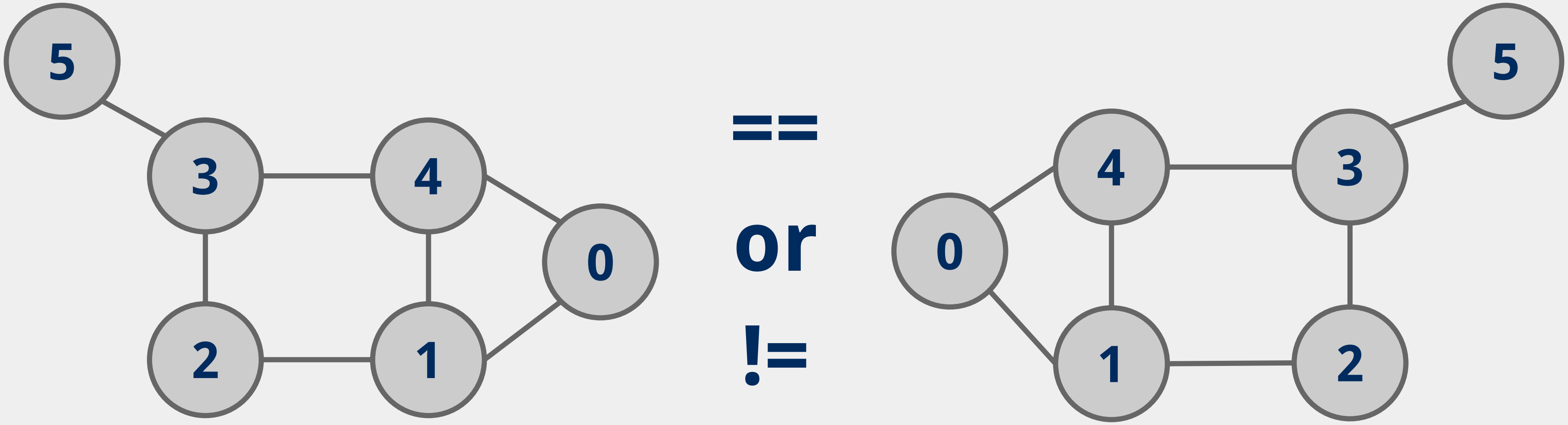
Graph sizes

- Let v = |V|, and e = |E|
- Given v, what are the minimum/maximum sizes of e?
  - Minimum value of e?
    - Definition doesn't necessitate that there are any edges…
    - So, 0
  - Maximum of e?
    - Depends…
      - Are self edges allowed?
      - Directed graph or undirected graph?
    - In this class, we'll assume directed graphs have self edges while undirected graphs do not

More definitions

- A graph is considered *sparse* if:
  - e <= v lg v
- A graph is considered *dense* as it approaches the maximum number of edges
  - I.e., e == MAX - ε
- A *complete* graph has the maximum number of edges

Question:



- ?

Even more definitions

- Path
  - A sequence of adjacent vertices
- Simple Path
  - A path in which no vertices are repeated
- Simple Cycle
  - A simple path with the same first and last vertex
- Connected Graph
  - A graph in which a path exists between all vertex pairs
- Connected Component
  - Connected subgraph of a graph
- Acyclic Graph
  - A graph with no cycles
- Tree
  - ?
  - A connected, acyclic graph
    - Has exactly v-1 edges

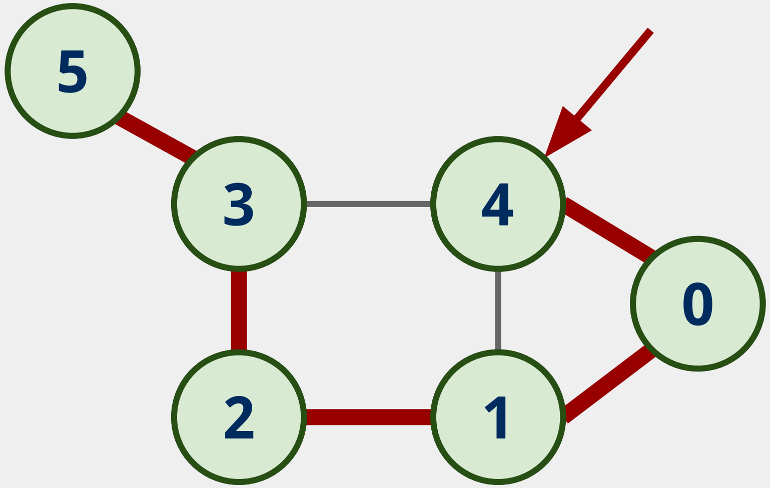
Graph traversal

- What is the best order to traverse a graph?
- Two primary approaches:
  - Depth-first search (DFS)
    - "Dive" as deep as possible into the graph first
    - Branch when necessary
  - Breadth-first search (BFS)
    - Search all directions evenly
      - I.e., from i, visit all of i's neighbors, then all of their neighbors, etc.

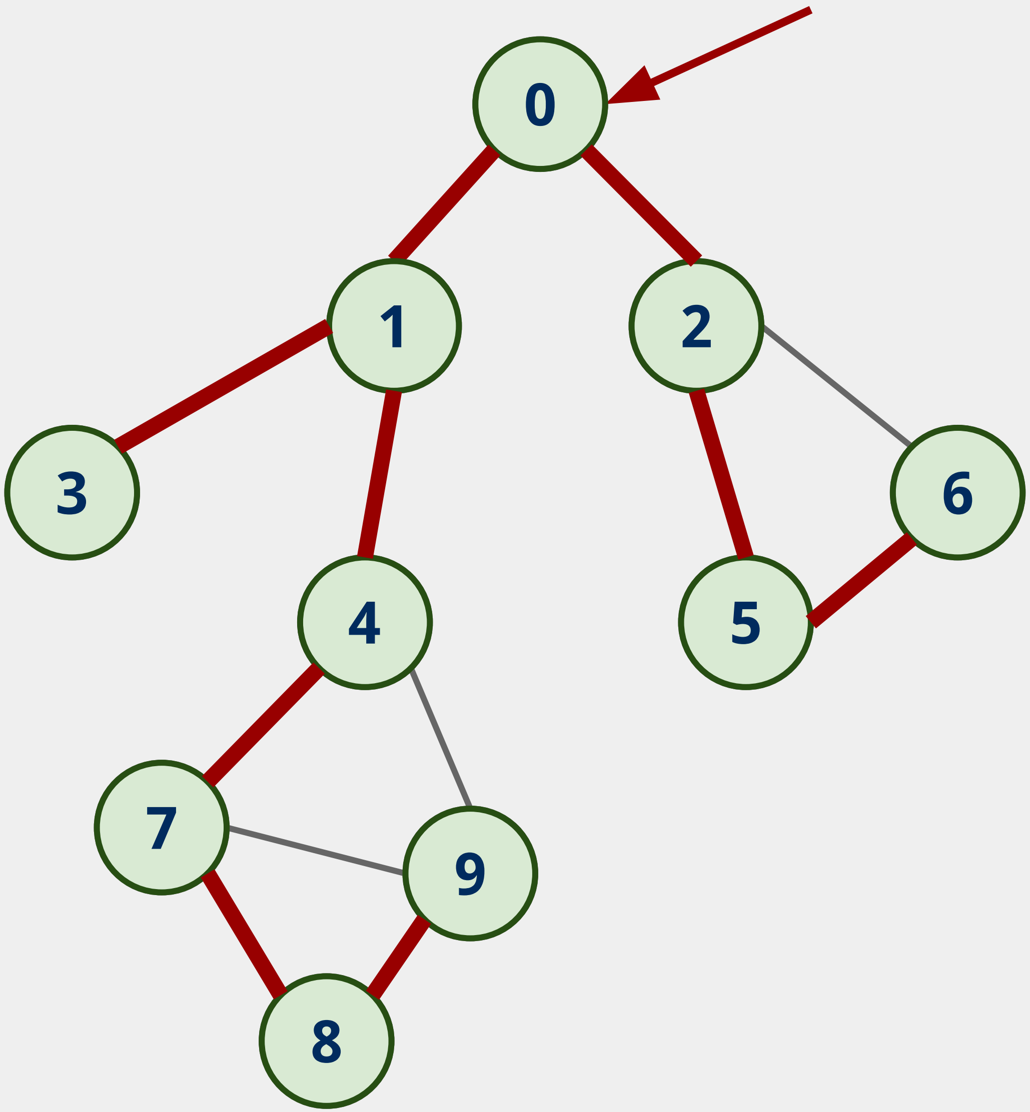
DFS

- Already seen and used this throughout the term
  - For tries…
  - For Huffman encoding…
- Can be easily implemented recursively
  - For each vertex, visit first unseen neighbor
  - Backtrack at deadends (i.e., vertices with no unseen neighbors)
    - Try next unseen neighbor after backtracking

DFS example



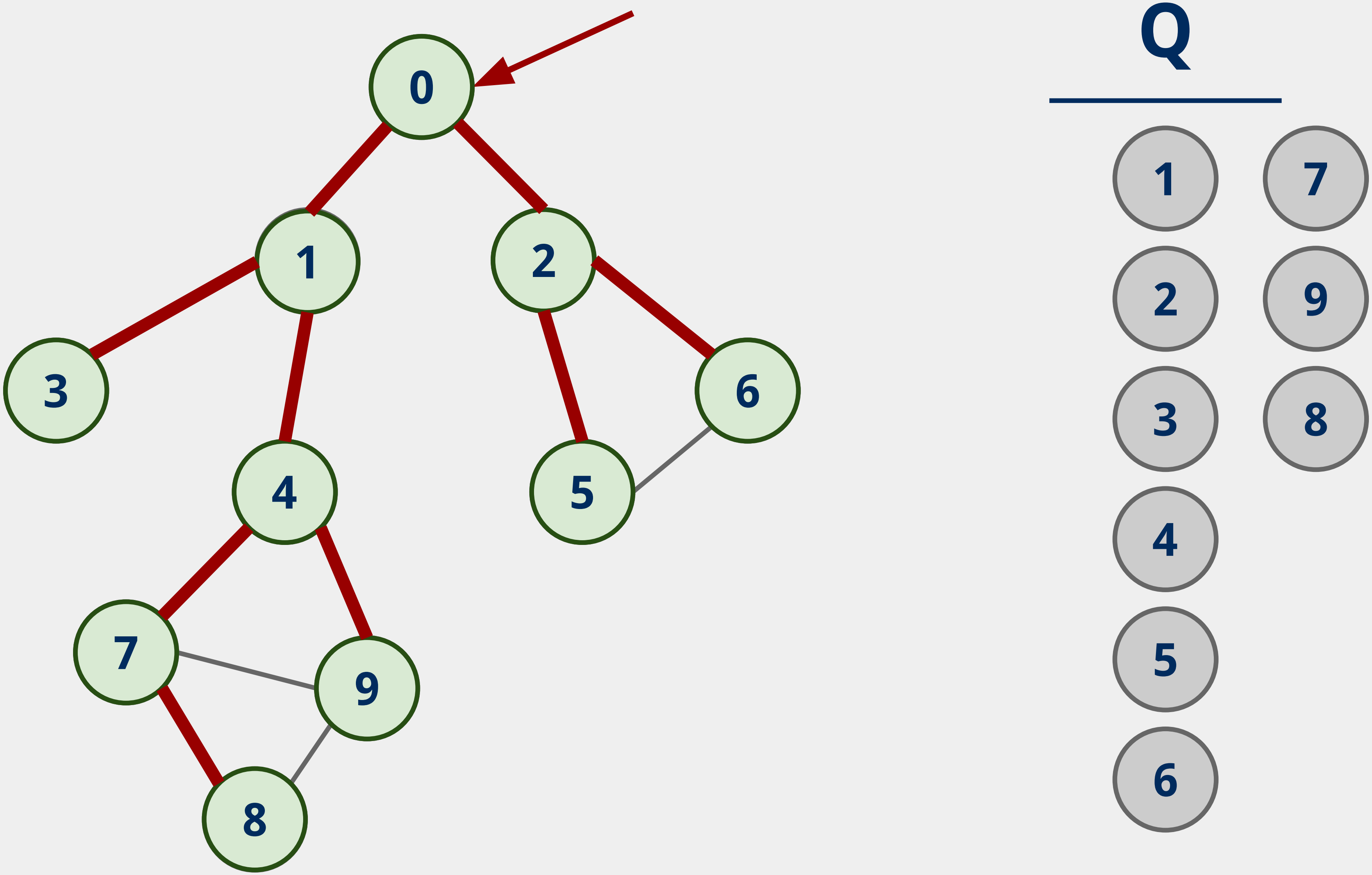
DFS example 2



BFS

- Can be easily implemented using a queue
  - For each vertex visited, add all of its neighbors to the queue
    - Vertices that have been seen but not yet visited are said to be the *fringe*
  - Pop head of the queue to be the next visited vertex
- See example

BFS example



The Shortest Path

Given a graph G, and two vertices u and w, find the shortest path from u to w

What's the runtime?

- At a high level, DFS and BFS have the same runtime
  - Each vertex must be seen and then visited, but the order will differ between these two approaches
- How do we represent the graph in our code?
  - How will the representation of the graph affect the runtimes of of these traversal algorithms?

Representing graphs

- Trivially, graphs can be represented as:
  - List of vertices
  - List of edges
- Performance?
  - Assume we're going to be analyzing static graphs
    - I.e., no insert and remove
  - So what operations should we consider?

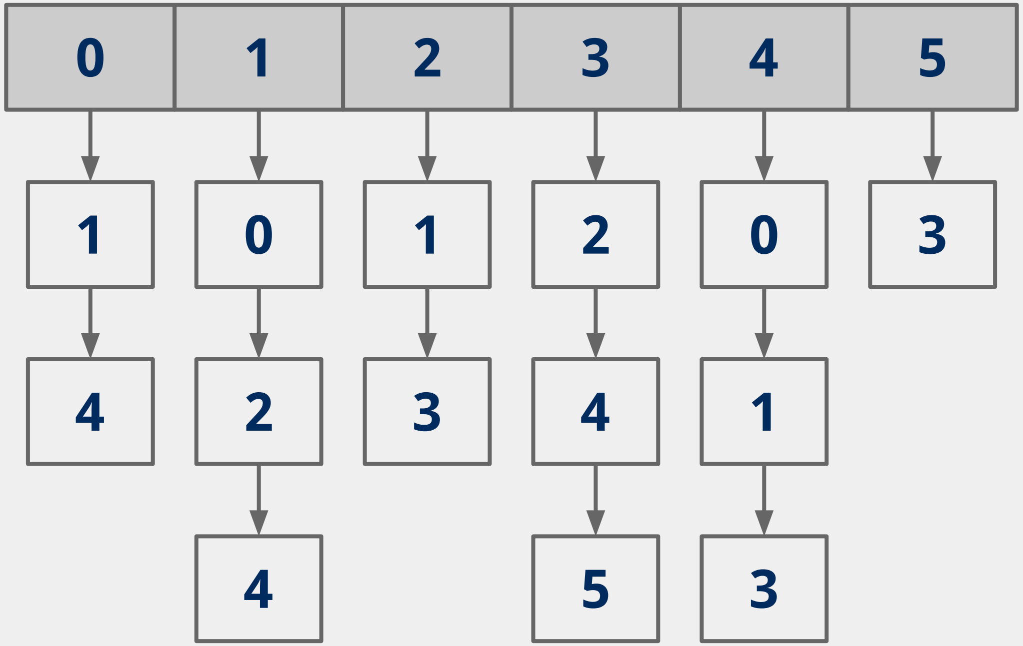
Using an adjacency matrix

- Rows/columns are vertex labels
  - M[i][j] = 1 if $(i, j) \in E$
  - M[i][j] = 0 if $(i, j) \notin E$

	0	1	2	3	4	5
0	0	1	0	0	1	0
1	1	0	1	0	1	0
2	0	1	0	1	0	0
3	0	0	1	0	1	1
4	1	1	0	1	0	0
5	0	0	0	1	0	0

Adjacency lists

- Array of neighbor lists
  - A[i] contains a list of the neighbors of vertex i



Analysis of graph traversals revisited

- Runtime of BFS using an adjacency matrix?
- Runtime of BFS using an adjacency list?
- Runtime of DFS using an adjacency matrix?
- Runtime of DFS using an adjacency list?

Comparison of graph representations

- Where would we want to use adjacency lists vs adjacency matrices?
  - What about the list of vertices/list of edges approach?

DFS and BFS should be called from a wrapper function

- If the graph is connected:
  - dfs()/bfs() is called only once and returns a *spanning tree*
- Else:
  - A loop in the wrapper function will have to continually call dfs()/bfs() while there are still unseen vertices
  - Each call will yield a spanning tree for a connected component of the graph

Traversal orders

-

Biconnected graphs

- A *biconnected graph* has at least 2 distinct paths (no common edges or vertices) between all vertex pairs
- Any graph that is not biconnected has one or more *articulation points*
  - Vertices, that, if removed, will separate the graph
- Any graph that has no articulation points is biconnected
  - Thus we can determine that a graph is biconnected if we look for, but do not find any articulation points

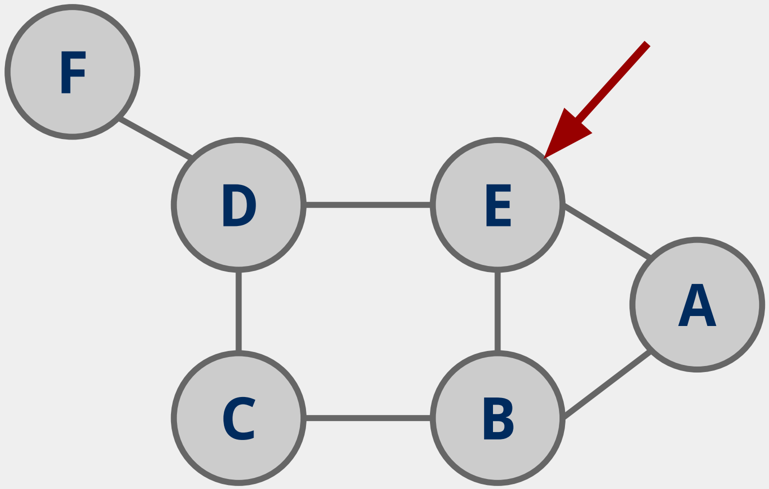
Finding articulation points

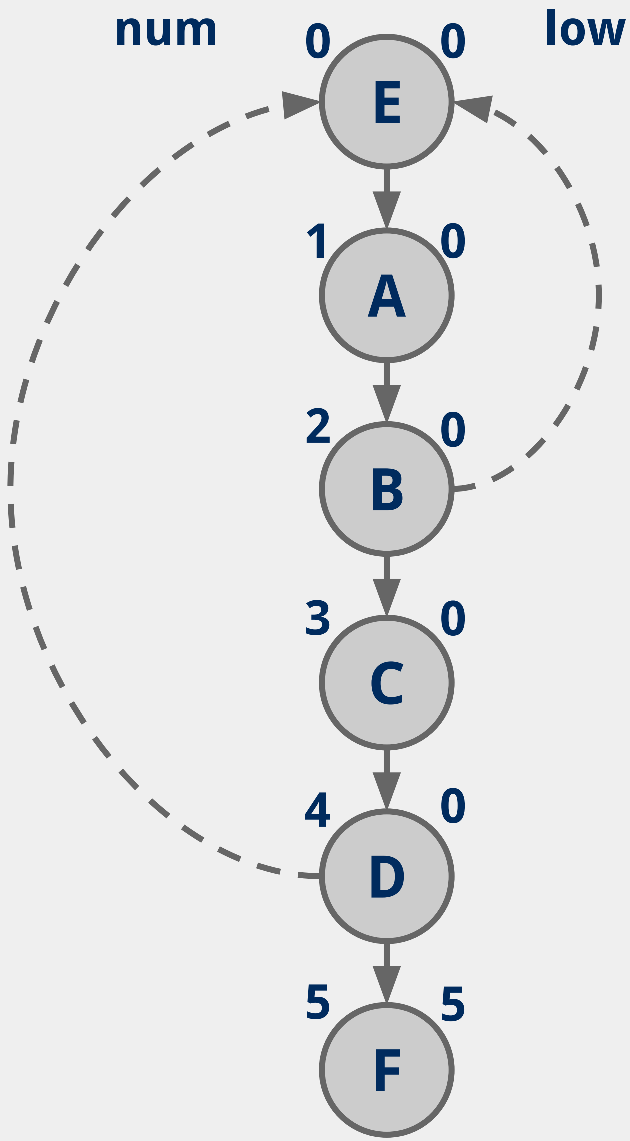
Given a graph G, find any articulation points that it contains

Finding articulation points

- Variation on DFS
- Consider building up the spanning tree
  - Have it be directed
  - Create “back edges” when considering a vertex that has already been visited in constructing the spanning tree
  - Label each vertex *v* with with two numbers:
    - num(*v*) = pre-order traversal order
    - low(*v*) = lowest-numbered vertex reachable from *v* using 0 or more spanning tree edges and then at most one back edge
      - Min of:
        - num(*v*)
        - Lowest num(*w*) of all back edges (*v*, *w*)
        - Lowest low(*w*) of all spanning tree edges (*v*, *w*)

Finding articulation points example





So where are the articulation points?

- If any (non-root) vertex v has some child w such that low(w) ≥ num(v), v is an articulation point
- What about if we start at an articulation point?
  - If the root of the spanning tree has more than one child, it is an articulation point