CS 1501

Network Flow

Defining network flow

- Consider a directed, weighted graph G(V, E)
  - Weights are applied to edges to state their *capacity*
    - c(u, w) is the capacity of edge (u, w)
    - if there is no edge from u to w, c(u, w) = 0
- Consider two vertices, a *source s* and a *sink t*

Flow

- Let the f(u, w) be the amount of flow being carried along the edge (u, w)
- Some rules on the flow running through an edge:
  - $\forall (u, w) \in E \; f(u, w) <= c(u, w)$
  - $\forall u \in (V - \{s,t\}) \; (\Sigma_{w \in V} f(w, u) - \Sigma_{w \in V} f(u, w)) = 0$

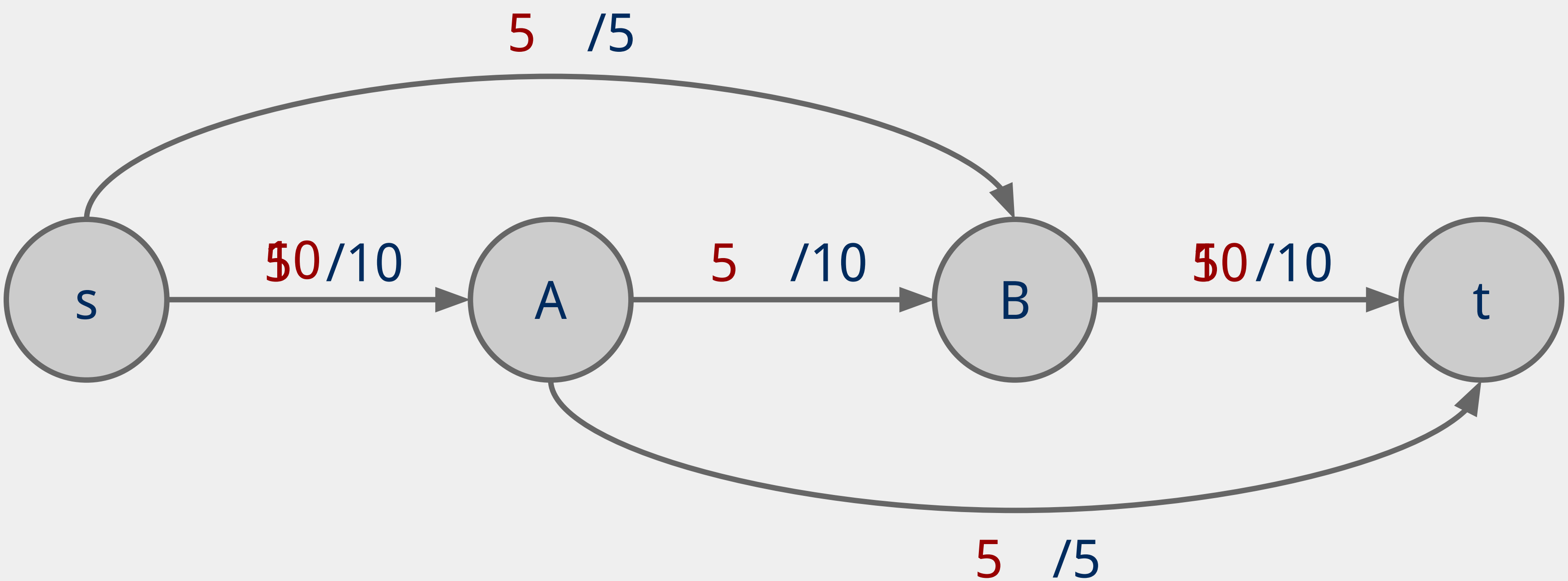
The Max Flow Problem

Given a graph G, a source vertex s, and a sink vertex t, find the maximum possible flow rate from s to t

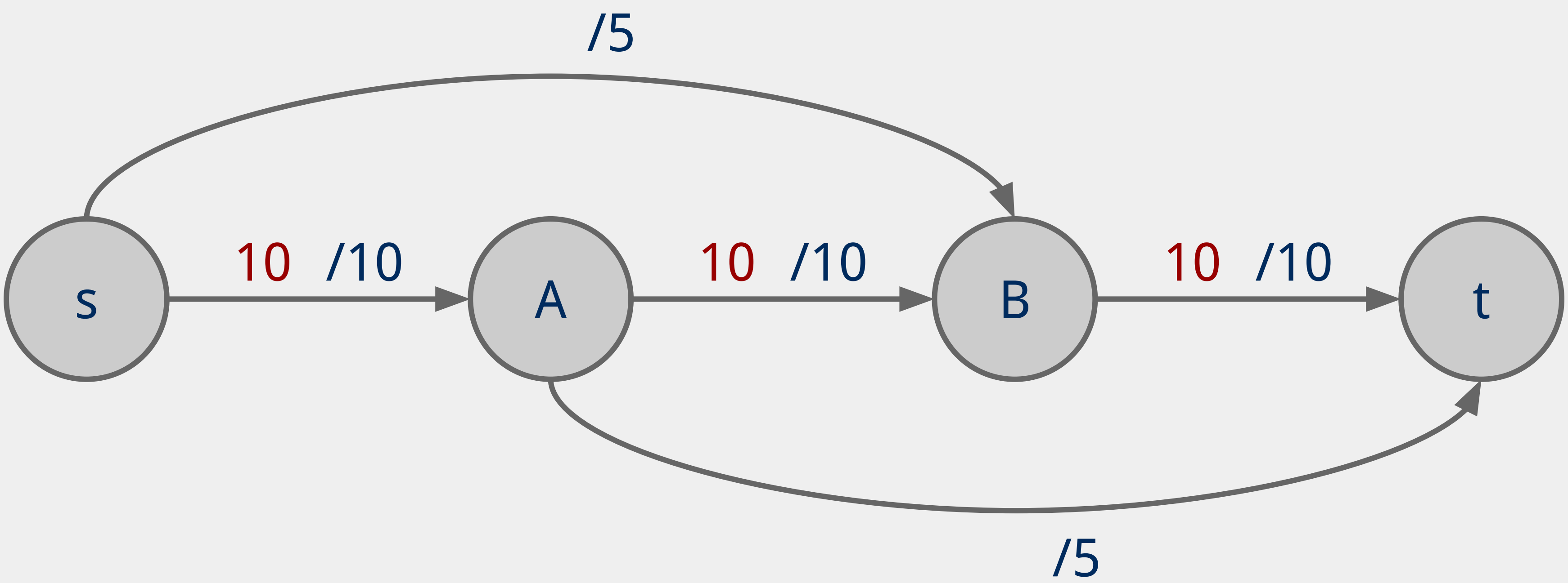
Ford Fulkerson

- Let all edges in G have an allocated flow of 0
- While there is path *p* from s to t in G s.t. all edges in p have some *residual capacity* (i.e., $\forall (u, w) \in p\ f(u, w) < c(u, w)$):
    - (Such a path is called an *augmenting path*)
    - Compute the residual capacity of each edge in p
        - Residual capacity of edge (u, w) is c(u, w) - f(u, w)
    - Find the edge with the minimum residual capacity in p
        - We'll call this residual capacity *new_flow*
    - Increment the flow on all edges in p by new_flow

Ford Fulkerson example



Another Ford Fulkerson example



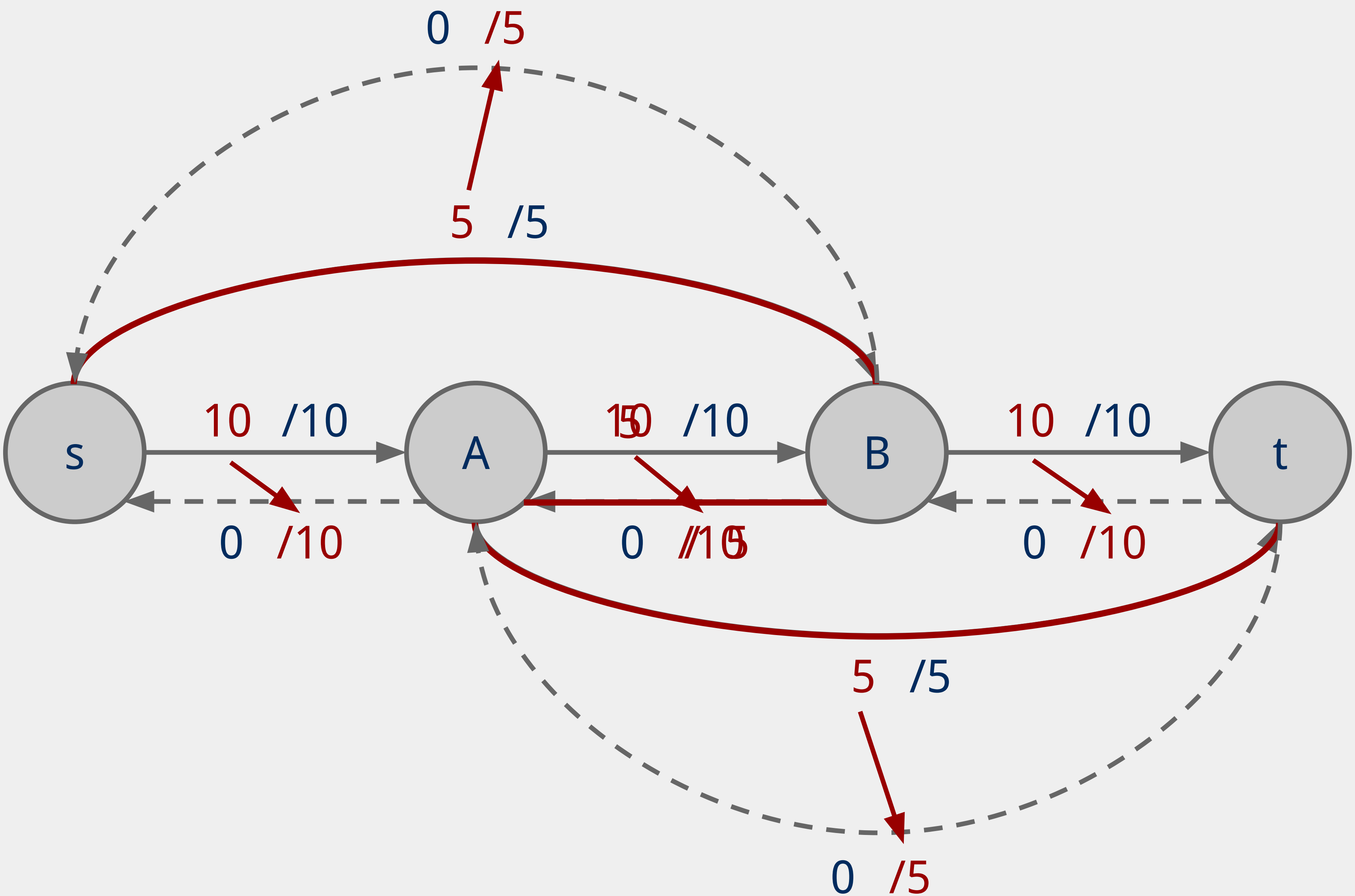
Expanding on residual capacity

- To find the max flow we will have need to consider re-routing flow we had previously allocated
  - This means, when finding an augmenting path, we will need to look not only at the edges of G, but also at *backwards edges* that allow such re-routing
    - For each edge $(u, w) \in E$, a backwards edge (w, u) must be considered during pathfinding if $f(u, w) > 0$
      - The capacity of a backwards edge (w, u) is equal to $f(u, w)$

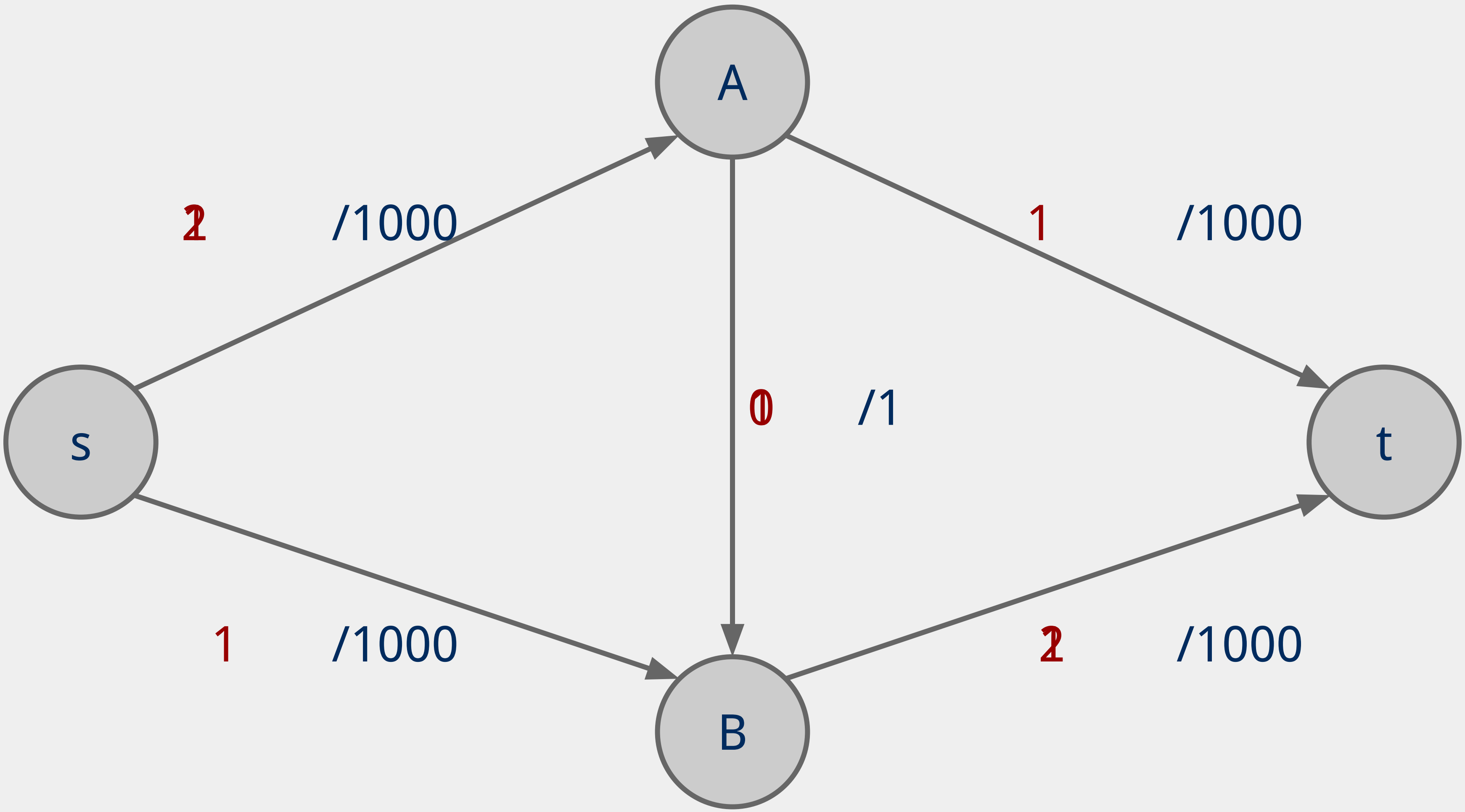
The residual graph

- We will perform searches for an augmenting path not on G, but on a residual graph built using the current state of flow allocation on G
- The residual graph is made up of:
  - V
  - An edge for each $(u, w) \in E$ where $f(u, w) < c(u, w)$
    - (u, w)'s mirror in the residual graph will have 0 flow and a capacity of $c(u, w) - f(u, w)$
  - A backwards edge for each $(u, w) \in E$ where $f(u, w) > 0$
    - (u, w)'s backwards edge has a capacity of $f(u, w)$
    - All backwards edges have 0 flow

Residual graph example



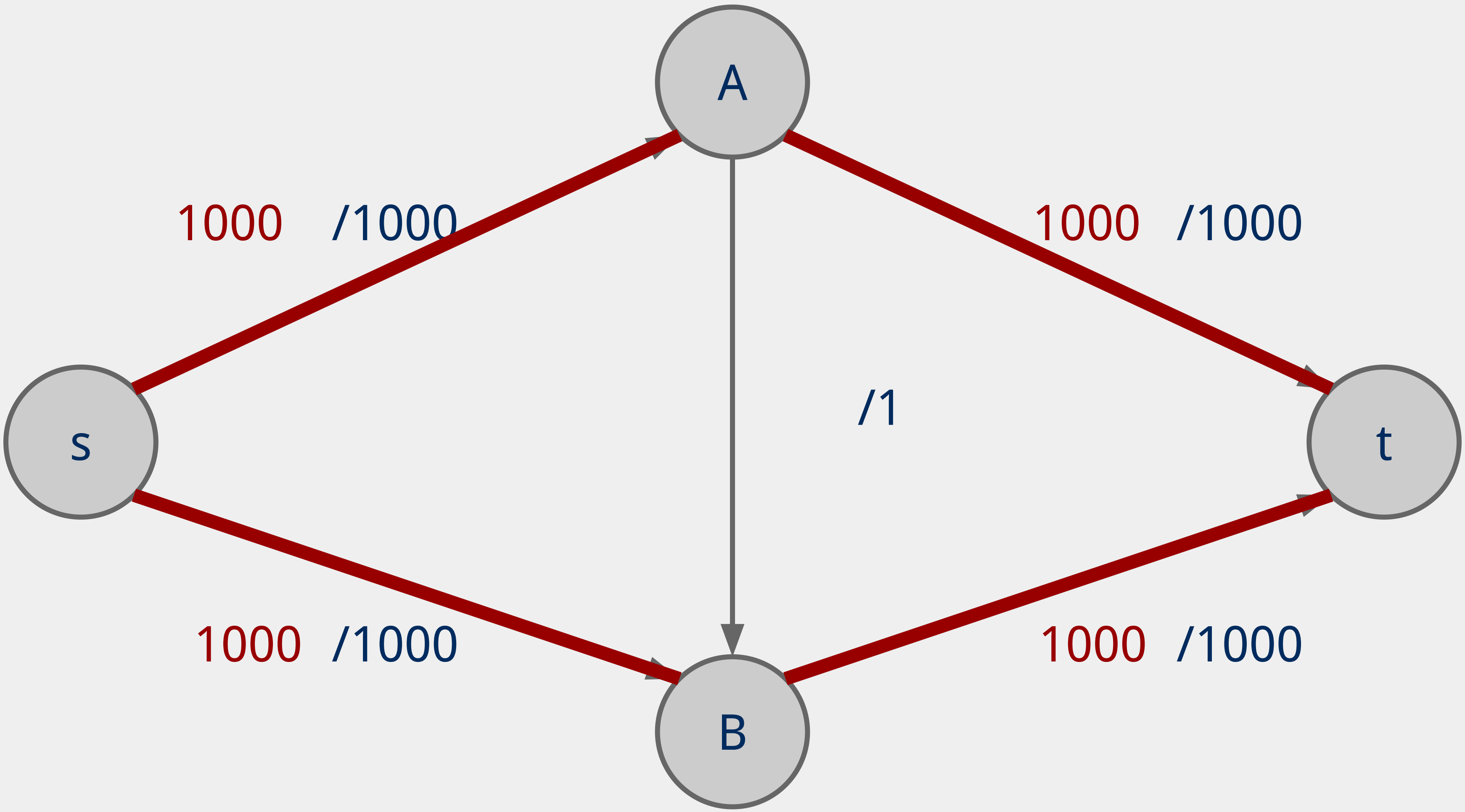
Another example



Edmonds Karp

- How the augmenting path is chosen affects the performance of the search for max flow
- Edmonds and Karp proposed a shortest path heuristic for Ford Fulkerson
  - Use BFS to find augmenting paths

Another example



But our flow graph is weighted...

- Edmonds-Karp only uses BFS
  - Used to find spanning trees and shortest paths for *unweighted* graphs
  - Why do we not use some measure of priority to find augmenting paths?

Implementation concerns

- Representing the graph:
    - Similar to a directed graph
    - Can store an adjacency list of directed edges
        - Actually, more than simply directed edges
            - Flow edges

Flow edge implementation

- For each edge, we need to store:
  - Start point, the from vertex
  - End point, the to vertex
  - Capacity
  - Flow
  - Residual capacities
    - For forwards and backwards edges

FlowEdge class

```
class FlowEdge:
    def __init__(self, v, w, c):
        self.v = v              # from
        self.w = w              # to
        self.capacity = c      # capacity
        self.flow = 0           # flow
    …
    def residualCapacityTo(self, vertex):
        if vertex == self.v:
            return self.flow
        elif vertex == self.w:
            return self.capacity - self.flow
    …
```

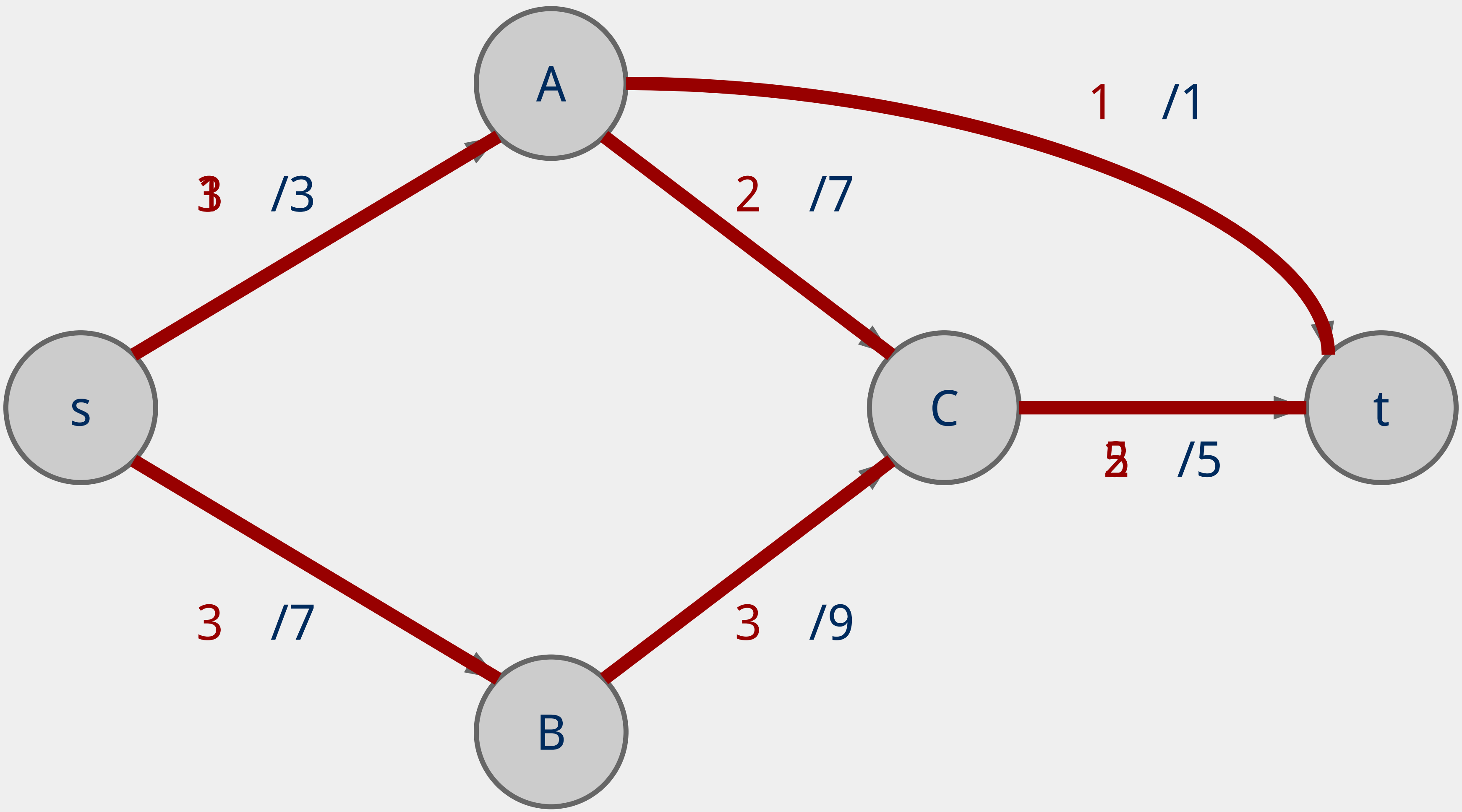
BFS search for an augmenting path

```
edgeTo = [None for i in range(len(self.adj_list))]
marked = [False for i in range(len(self.adj_list))]
q = [s]
marked[s] = True
while len(q) > 0:
    vertex = q.pop(0)
    for edge in self.adj_list[vertex]:
        w = edge.other(vertex)
        if edge.residualCapacityTo(w) > 0:
            if not marked[w]:
                edgeTo[w] = edge;
                marked[w] = True;
                q.append(w);
```

Each FlowEdge object is stored in the adjacency list twice:

Once for its forward edge
Once for its backwards edge

An example to review



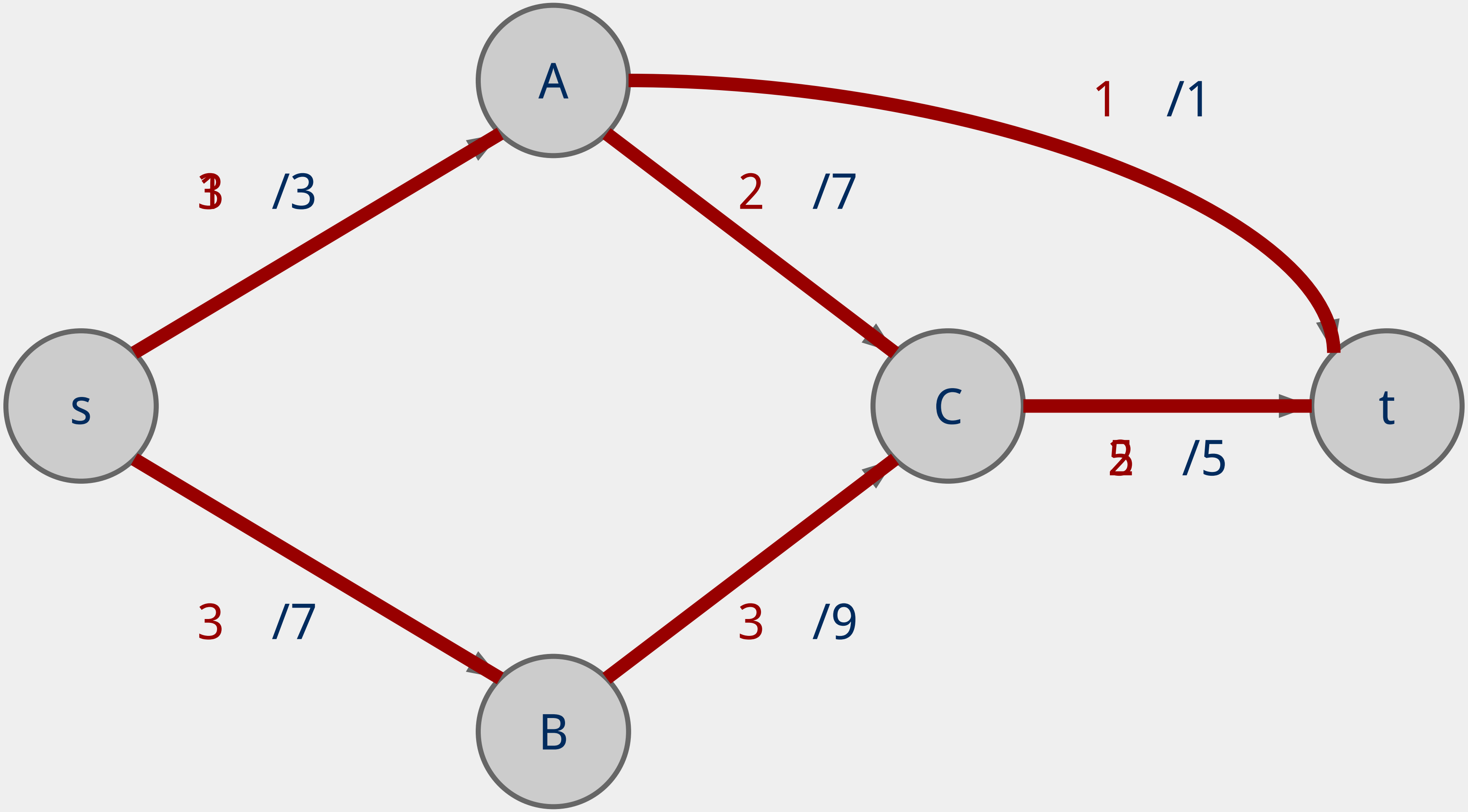
The Min Cut Problem

Given a graph G, a source vertex s, and a sink vertex t, find a set of edges that, if removed from the graph would separate s from t with the minimum possible sum of their edge weights

How do we find the min st-cut?

- We could examine residual graphs
  - Specifically, try and allocate flow in the graph until we get to a residual graph with no existing augmenting paths
    - A set of saturated edges will make a minimum st-cut

Min cut example



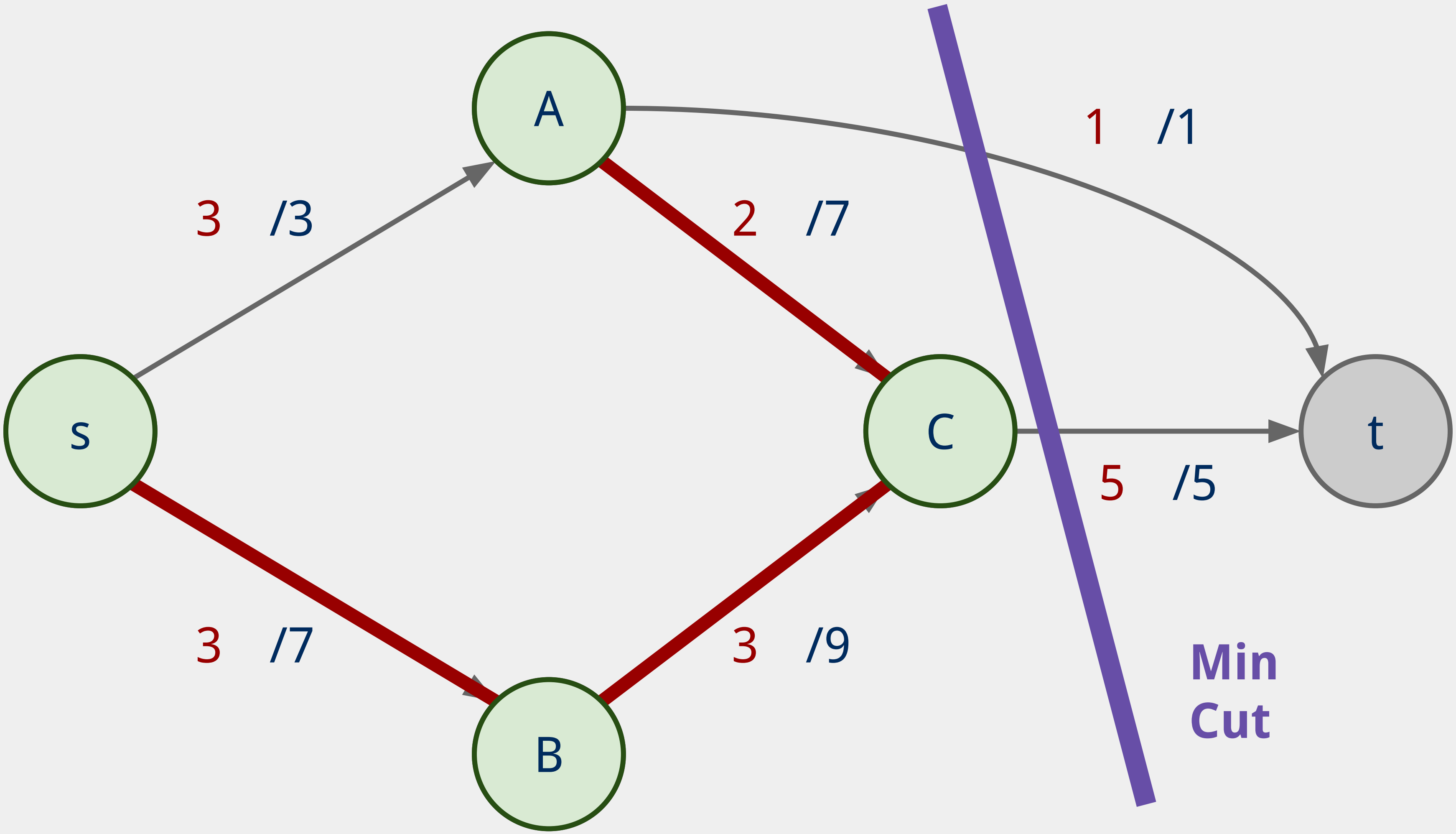
Max flow == min cut

- A special case of duality
  - I.e., you can look at an optimization problem from two angles
    - In this case to find the maximum flow or minimum cut
  - In general, dual problems do not have to have equal solutions
    - The differences in solutions to the two ways of looking at the problem is referred to as the *duality gap*
      - If the duality gap = 0, strong duality holds
        - Max flow/min cut uphold strong duality
      - If the duality gap > 0, weak duality holds

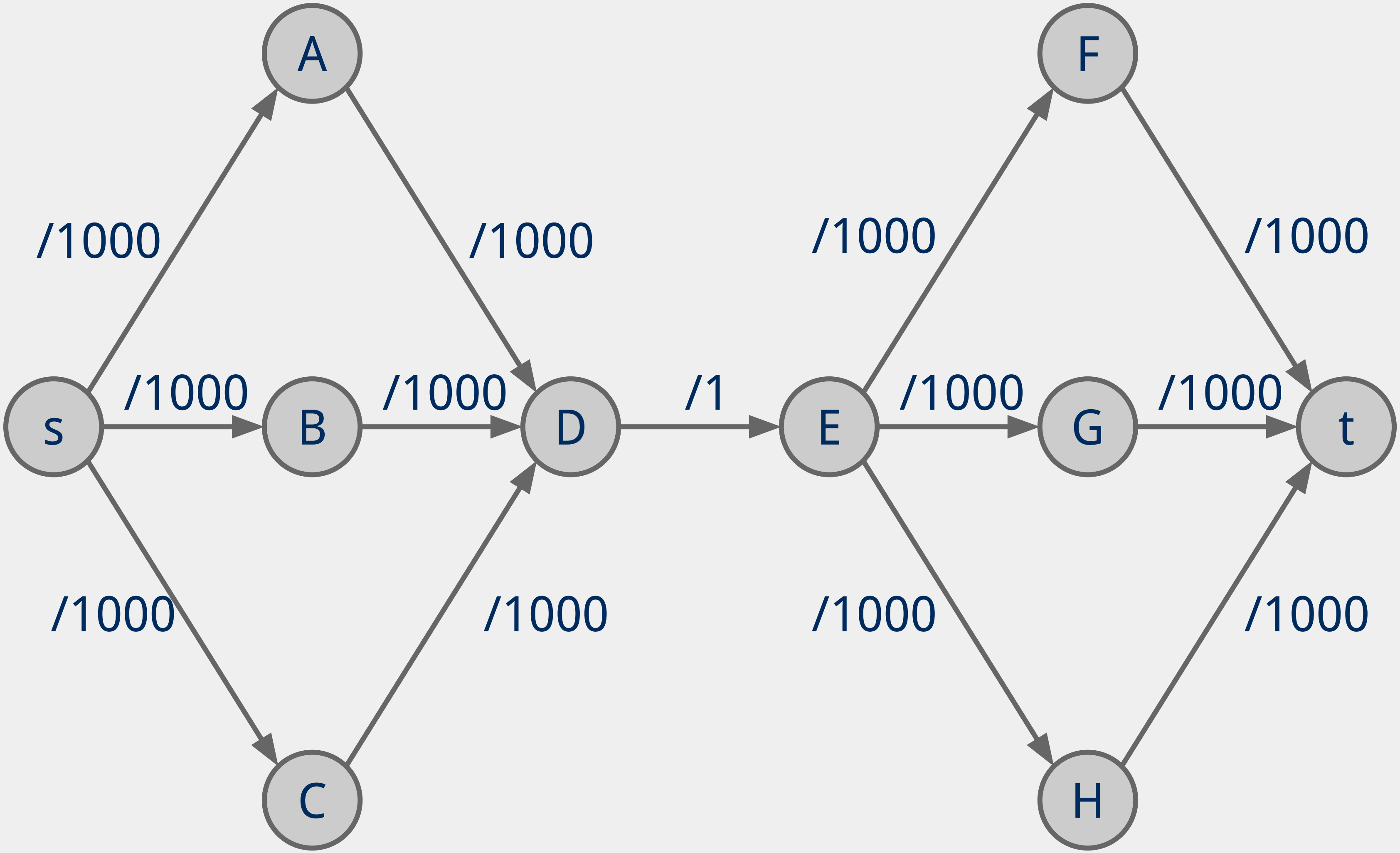
Determining a minimum st-cut

- First, run Ford Fulkerson to produce a residual graph with no further augmenting paths
- The last attempt to find an augmenting path will visit every vertex reachable from s
  - Edges with only one endpoint in this set comprise a minimum st-cut

Determining the min cut



Will max flow/min cut always be near s/t?



Max flow / min cut on unweighted graphs

- Is it possible?
- How would we measure the Max flow / min cut?
- What would an algorithm to solve this problem look like?

Unweighted network flow